Riemann Sums

Calculus Notes

1 Introduction

A **Riemann sum** approximates the definite integral $\int_a^b f(x)\,dx$ by dividing the interval $[a,b]$ into n subintervals and summing the areas of rectangles whose heights are given by the function f.

2 Setup

- Partition: $a = x_0 < x_1 < \cdots < x_n = b$
- Subinterval width: $\Delta x = \frac{b-a}{n}$ (equal subintervals)
- Sample point in $[x_{i-1}, x_i]$: x_i^*

3 Definition

The Riemann sum is

$$S_n = \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

4 Common choices for x_i^*

Type	x_i^*
Left	x_{i-1}
Right	x_i
Midpoint	$\frac{x_{i-1}+x_i}{2}$

5 Example

Let $f(x) = x^2$ on $[0,2]$ with $n = 4$, $\Delta x = \frac{1}{2}$.

Left sum:

$$S_4^{\text{left}} = f(0)\cdot\frac{1}{2} + f(0.5)\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} + f(1.5)\cdot\frac{1}{2} = 0 + 0.125 + 0.5 + 1.125 = 1.75.$$

The exact integral is $\int_0^2 x^2\,dx = \frac{8}{3} \approx 2.667$. As $n \to \infty$, $S_n \to \int_a^b f(x)\,dx$.

6 Limit definition of the integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$