Session 9

Fields in periodic structures

Exercise 9.1 *Consider an infinitely extended planar (2D) array (at $z = 0$) made of identical sources with a given inter-element spacing dx and dy. The current distribution on the array when a given source is excited is x-directed, Gaussian (in the x and y directions), and centered on the source. It is assumed that the consecutive sources are excited with a linear phase progressions $-\psi_x$ and $-\psi_y$.*

1. *Write analytically the fields generated by the array at $z = 10\lambda$ and for $\psi_x = 30°$, $\psi_y = 30°$, $d_x = d_y = \lambda/4$.*

2. *Same question as before, but with $d_x = d_y = \lambda$.*

3. *What is the condition to avoid exciting multiple Floquet modes ?*

Exercise 9.2 *Consider a periodic grid as represented below, with $a_x = a_y = 24.17$ mm, $w_x = 10.5$ mm and $w_y = 2.42$ mm.*

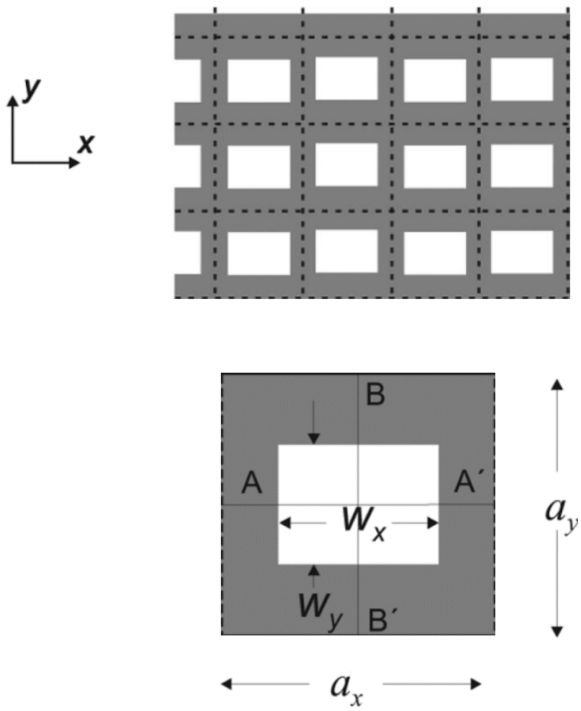

Figure 9.1 – Periodic grid

The structure is excited by a plane wave incident along z, polarized along the y direction, and with an amplitude E_0. The slot is meshed with 8 horizontal rooftop basis functions as represented below :

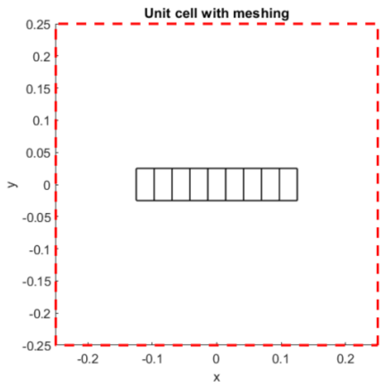

Figure 9.2 – Mesh of the slot with 8 horizontal rooftop basis functions

The MoM interaction matrix at various frequencies ([7 :17] GHz), with an electric field integral equation operator is provided for each frequency. Compute the absolute value of the transmittance of the fundamental Floquet mode at the frequencies of interest. What do you observe ?

Exercise 9.3 *Consider a 1D periodic array of point sources as represented below :*

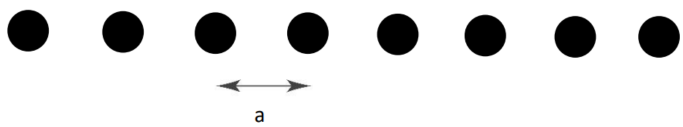

Figure 9.3 – 1D periodic array of point sources

The consecutive point sources are excited with a phase progression $-\psi$. *Express the current distribution as a set of line sources. We will see in a subsequent chapter what are the (cylindrical) waves excited by each of those sources.*

3. *To avoid exciting multiple Floquet modes, we should make sure that the spacing between two consecutive Floquet modes (along x and y) is larger than $2k$, which is the diameter of the visible region. Along the x-axis, we have :*

$$\frac{2\pi}{d_x} > 2k \Longrightarrow d_x < \frac{\lambda}{2} \tag{9.11}$$

The result is similar along the y-axis. We conclude that we should have d_x and d_y smaller than half-wavelength to avoid exciting multiple Floquet modes. We finally arrive to the usual constraint regarding the maximum spacing of elements in arrays to avoid grating lobes.

Exercise 9.2 *The effect of the hole is replaced by equivalent magnetic currents along the $\hat{\boldsymbol{x}}$ direction in the hole. This equivalent magnetic current distribution should be found by imposing the boundary conditions in the hole, i.e. continuity of the magnetic field. It has been shown in the slides of the course that it leads to*

$$2\boldsymbol{H_{inc}} + \frac{1}{\eta^2}R(-2\boldsymbol{M}) = \frac{1}{\eta^2}R(2\boldsymbol{M}) \tag{9.12}$$

$$\Longleftrightarrow \qquad \boldsymbol{H_{inc}} = \frac{2}{\eta^2}R(\boldsymbol{M}), \tag{9.13}$$

with R the radiation operator defined as

$$\boldsymbol{E}(\boldsymbol{r}) = R(\boldsymbol{J}(\boldsymbol{r})) = -jk\eta \iint_{S'} \left(\underline{\underline{\boldsymbol{I}}} + \frac{\nabla\nabla\cdot}{k^2}\right) G(\boldsymbol{r},\boldsymbol{r'})\, \boldsymbol{J}(\boldsymbol{r'})\, dS', \tag{9.14}$$

where $\underline{\underline{\boldsymbol{I}}}$ is the unit dyad, $\boldsymbol{J}$ is the current distribution, and G is the free-space periodic Green's function defined as

$$G(\boldsymbol{r},\boldsymbol{r'}) = \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} e^{-j(\psi_x m+\psi_y n)}\frac{e^{-jkR_{mn}}}{4\pi R_{mn}}, \tag{9.15}$$

with $R_{mn} = |\boldsymbol{r} - ma\,\hat{\boldsymbol{x}} - nb\,\hat{\boldsymbol{y}} - \boldsymbol{r'}|$, where a and b are the periodicities along x and y, respectively. And in this case, there is no phase shift between the cells : $\psi_x = \psi_y = 0$. Similarly, the magnetic field is given by

$$\boldsymbol{H} = R(\boldsymbol{M})/\eta^2. \tag{9.16}$$

Now, we are interested to solve (9.13) ***to find the magnetic current $\boldsymbol{M}$****. To that aim, we will use the Method of Moments (MoM). In this method, the unknown current is decomposed into a set of basis functions. Here, we chose to decompose $\boldsymbol{M}$ into a set of $N = 8$ rooftop basis functions oriented along x on the slot (see Figure 9.1) :*

$$\boldsymbol{M} = M\,\hat{\boldsymbol{x}} = \sum_{i=1}^{N} x_i\,\boldsymbol{F_b^i}, \tag{9.17}$$

where x_i is the coefficient associated with the i-th rooftop basis function denotes as $\boldsymbol{F_b^i}$.

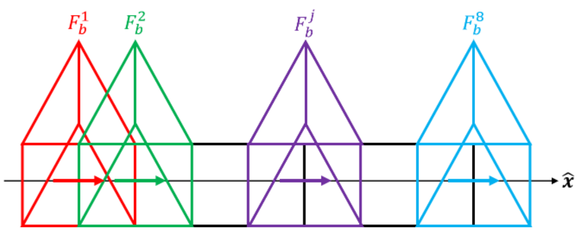

Figure 9.1 – Meshing of the slot with 8 horizontal rooftop basis functions.

Inserting (9.17) *into* (9.13) *gives*

$$\boldsymbol{H_{inc}} = \frac{2}{\eta^2}\sum_{i=1}^{N} x_i\, R(\boldsymbol{F_b^i}). \tag{9.18}$$

Then, we test the fields with a set of 8 testing functions, where $\boldsymbol{F_t^j}$ denotes the j-th testing function. Here, we chose to use the same set of functions for basis and testing functions (i.e. $\boldsymbol{F_b^i} = \boldsymbol{F_t^i}$), which gives

$$\iint \boldsymbol{F_t^j} \cdot \boldsymbol{H_{inc}} \, dS = \frac{2}{\eta^2} \sum_{i=1}^{N} x_i \iint \boldsymbol{F_t^j} \cdot R(\boldsymbol{F_b^i}) \, dS \qquad \text{for } j = 1, ..., 8, \tag{9.19}$$

which can be rewritten as

$$\boldsymbol{w_j} = \frac{2}{\eta^2} \sum_{i=1}^{N} x_i \, \boldsymbol{Z_{ij}} \qquad \text{for } j = 1, ..., 8. \tag{9.20}$$

The latter can be rewritten as the following MoM linear system of equations :

$$\frac{2}{\eta^2} \, \underline{\underline{\boldsymbol{Z}}} \, \boldsymbol{x} = \boldsymbol{w}, \tag{9.21}$$

where the entries of the MoM matrix $\underline{\underline{\boldsymbol{Z}}}$ and the vector $\boldsymbol{w}$ are given by

$$\boldsymbol{Z_{ij}} = \iint \boldsymbol{F_t^j} \cdot R(\boldsymbol{F_b^i}) \, dS, \tag{9.22}$$

$$\boldsymbol{w_j} = \iint \boldsymbol{F_t^j} \cdot \boldsymbol{H_{inc}} \, dS. \tag{9.23}$$

The computation of the MoM matrix $\underline{\underline{\boldsymbol{Z}}}$ is not trivial and will be provided on Moodle (for several frequencies). Using this result, you can compute the vector $\boldsymbol{w}$ by yourself and solve the system of equations to find $\boldsymbol{x}$, which gives the magnetic current.

***Now, we are interested to compute the electric field passing through the grid**. The electric field associated with one slot is given by*

$$\boldsymbol{E_0} = \hat{\boldsymbol{n}} \times \boldsymbol{M} = \sum_{i=1}^{N} x_i \, F_b^i \, \hat{\boldsymbol{y}}. \tag{9.24}$$

Because of the 2D periodic structure, the spectral electric field is given by

$$\tilde{\boldsymbol{E}}(k_x, k_y) = \frac{(2\pi)^2}{a_x a_y} \sum_{p,q} \tilde{\boldsymbol{E}}_{\boldsymbol{0}}(k_x^p, k_y^q) \, \delta(k_x - k_x^p) \, \delta(k_y - k_y^q). \tag{9.25}$$

By inverse Fourier transform, we have that

$$\boldsymbol{E}(x, y) = \frac{1}{(2\pi)^2} \iint \tilde{\boldsymbol{E}}(k_x, k_y) \, e^{-jk_x x} \, e^{-jk_y y} \, dk_x \, dk_y. \tag{9.26}$$

As we are interested in the transmittance in the normal direction, it gives

$$\begin{aligned} E(x, y) &= \frac{1}{a_x a_y} \tilde{E}_0(k_x = 0, k_y = 0) && (9.27) \\ &= \frac{1}{a_x a_y} \iint E_0(x, y) \, dx \, dy && (9.28) \\ &= \frac{1}{a_x a_y} \sum_{i=1}^{N} x_i \iint F_b^i \, dx \, dy. && (9.29) \\ & && (9.30) \end{aligned}$$

Finally, we can compute the transmittance of the grid which is given by

$$T = \frac{E}{E_{inc}}. \tag{9.31}$$

You can therefore compute the transmittance at different frequencies and plot it as a function of the frequency.

PS : the Method of Moments (MoM) matrix $\underline{\underline{\mathbf{Z}}}$ will be provided with units Ω cm^2, so it is suggested to work with distances in cm.

Exercise 9.3 *Let us consider that all the point sources have an amplitude J_0 and that the array is oriented along the z-axis. The current of the n^{th} element is given by :*

$$J_n(x) = J_0\ \delta(x - na)\ e^{-jn\psi} \tag{9.32}$$

Then, the total current is given by :

$$J_{tot}(x) = \sum_{n=-\infty}^{+\infty} J_0\ \delta(x - na)\ e^{-jn\psi} \tag{9.33}$$

$$= \sum_{n=-\infty}^{+\infty} J_0\ \delta(x - na)\ e^{-jx\psi/a} \tag{9.34}$$

Applying the Fourier transform, we have :

$$\tilde{J}_{tot}(k_x) = \frac{2\pi}{a} \sum_{p=-\infty}^{+\infty} J_0\ \delta(k_x - k_x^p) \tag{9.35}$$

with k_x^p the Floquet wavenumbers :

$$k_x^p = p\frac{2\pi}{a} + \frac{\psi}{a} \tag{9.36}$$

Then, by inverse Fourier transform we get :

$$J_{tot}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\pi}{a} \sum_{p=-\infty}^{+\infty} J_0\ \delta(k_x - k_x^p)\ e^{-jk_x x}\ dk_x \tag{9.37}$$

$$= \frac{1}{a} \sum_{p=-\infty}^{+\infty} J_0\ e^{-jk_x^p x} \tag{9.38}$$

The total current which was expressed as a sum of discrete point sources is now expressed as a set of line sources.