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Zero-knowledge-proofs - part 1

Engineering workshop - Dec 2025 (Ireland)

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Plan of the tutorial

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Let’s get a little deeper than usual and understand what main building blocks of ZKP looks like

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1. sketching the landscape of what we want to understand during 3-4 parts

2.

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This tutorial will focus on

1. sketching the landscape of what we want to understand during 3-4 parts

2. cover the first part in some detail **elliptic curves**

Verifiable computing vs ZKP (1)

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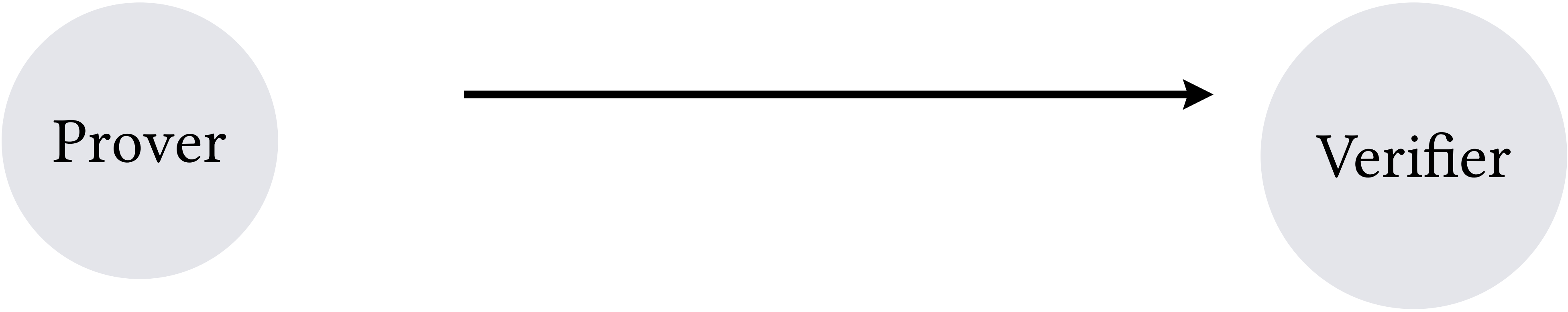
There is **asymmetry** built into those systems. It is much easy to get public key from secret. But not the other way



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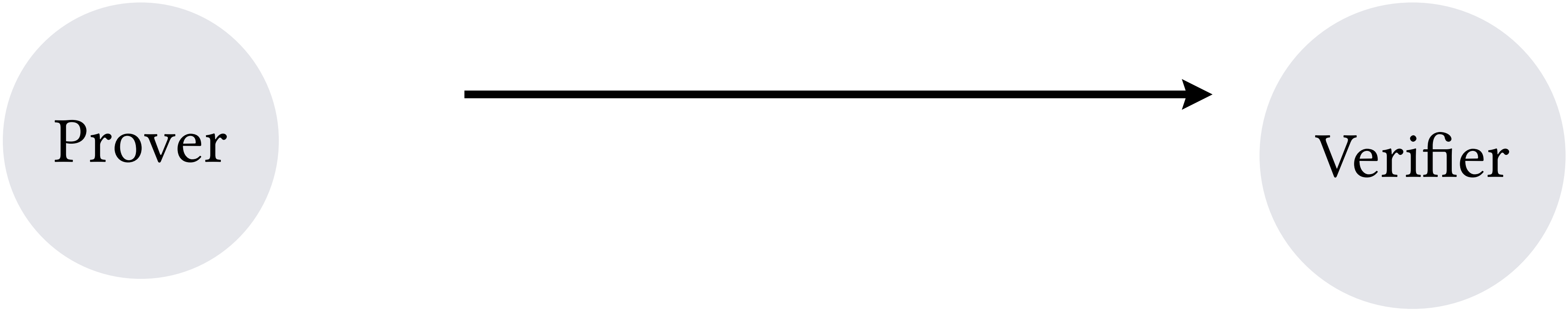

secret -> (easy) -> public

Verifiable computing vs ZKP (1)

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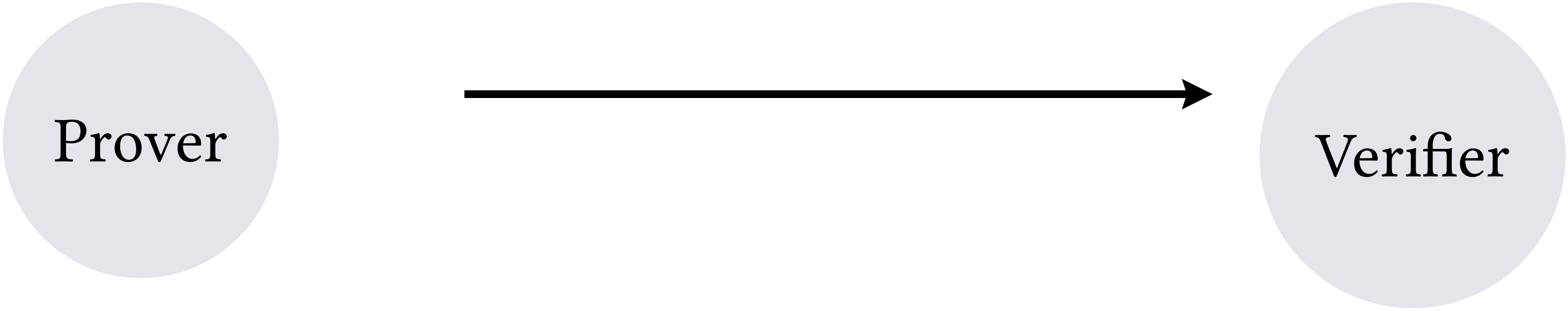

proof + public data shared

Verifiable computing vs ZKP (1)

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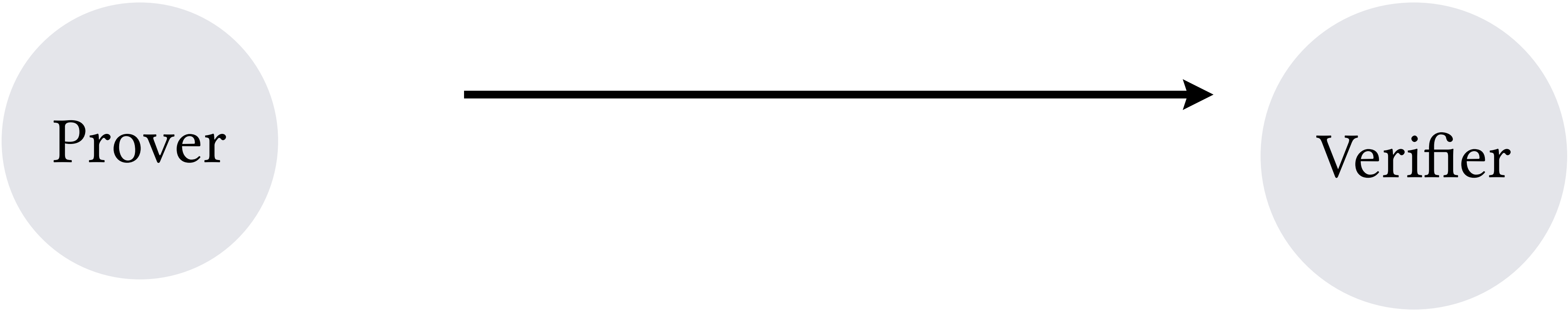

public -> (very hard) -> secret

Verifiable computing vs ZKP (1)

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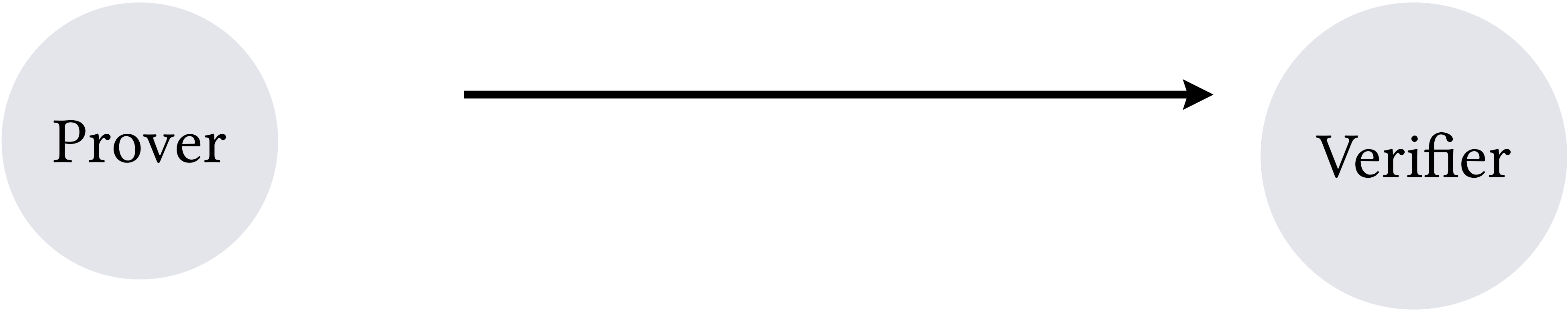
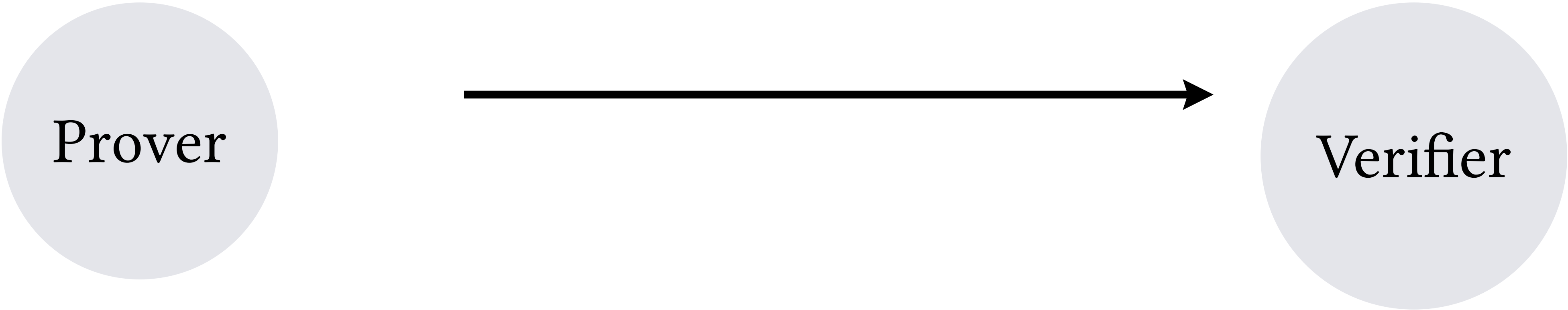


proof verified using public data

Verifiable computing vs ZKP (2)

There is **asymmetry** built into those systems. It is much quicker to verify than prove something.



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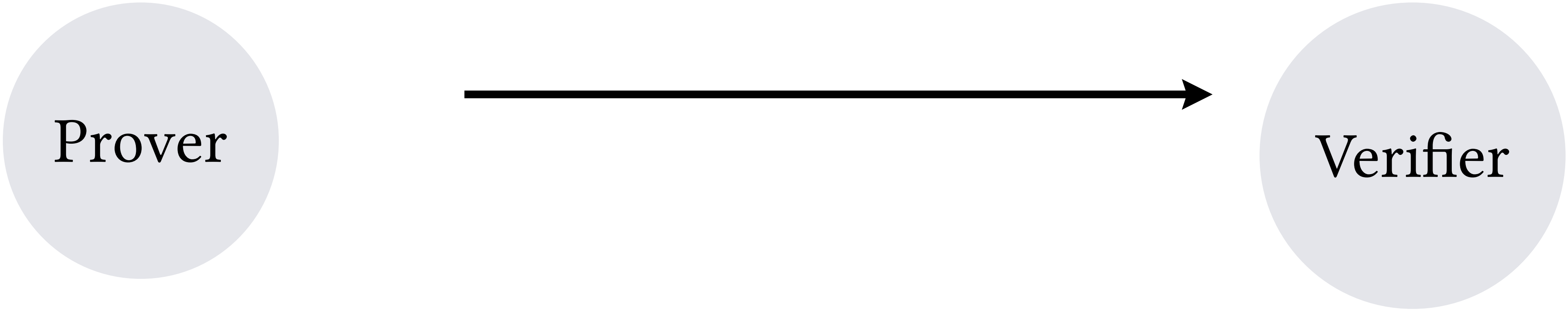

O(n) off-chain

Verifiable computing vs ZKP (2)

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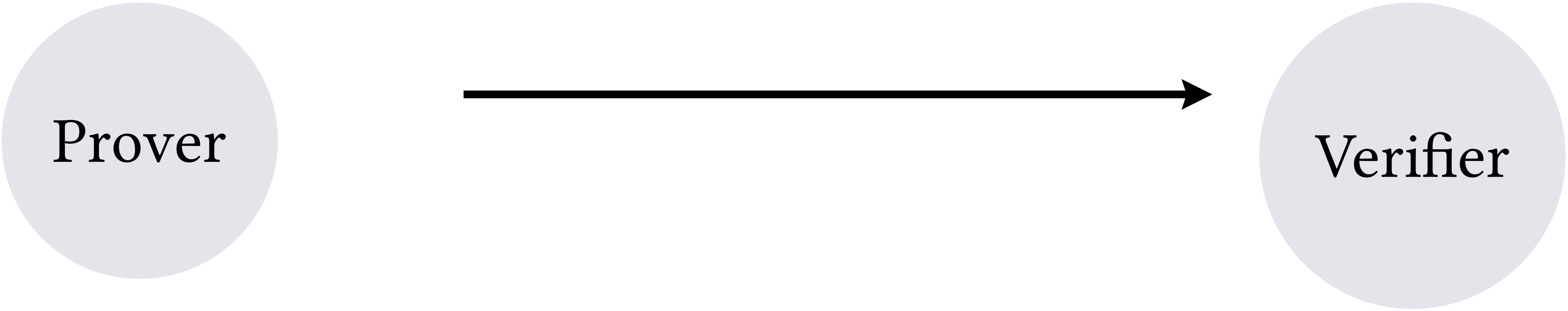
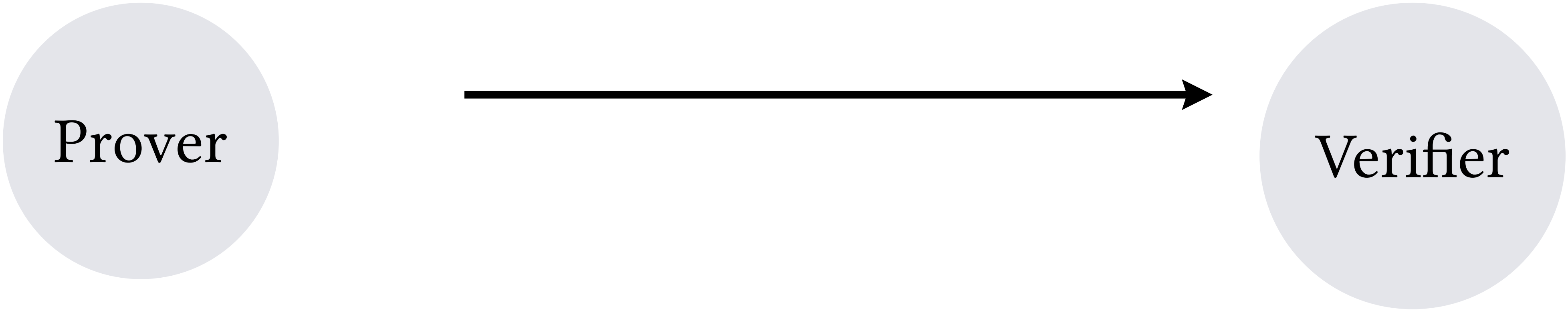


O(n) off-chain

O(log n) on-chain

ZKP (3)

Data sent to verifier is compressed, and can be hidden



ZKP (3)

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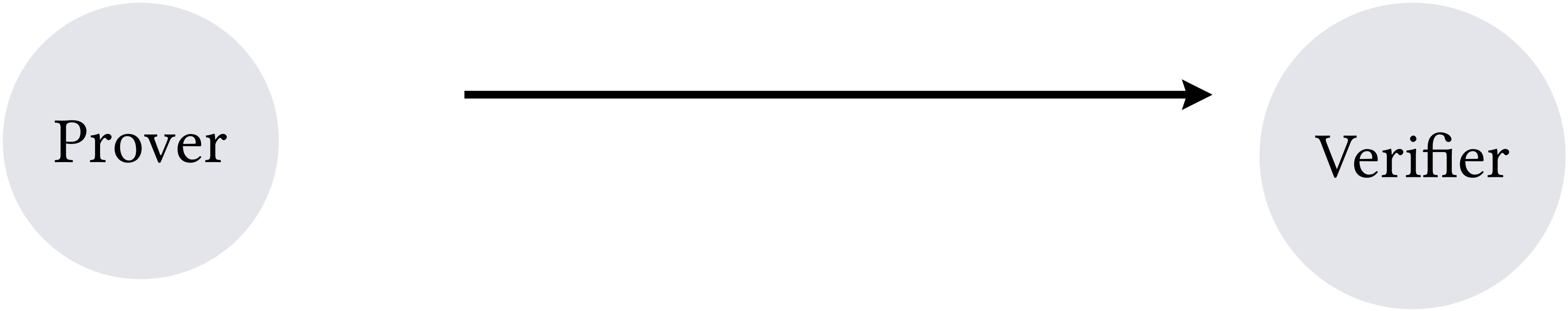

size: n

ZKP (3)

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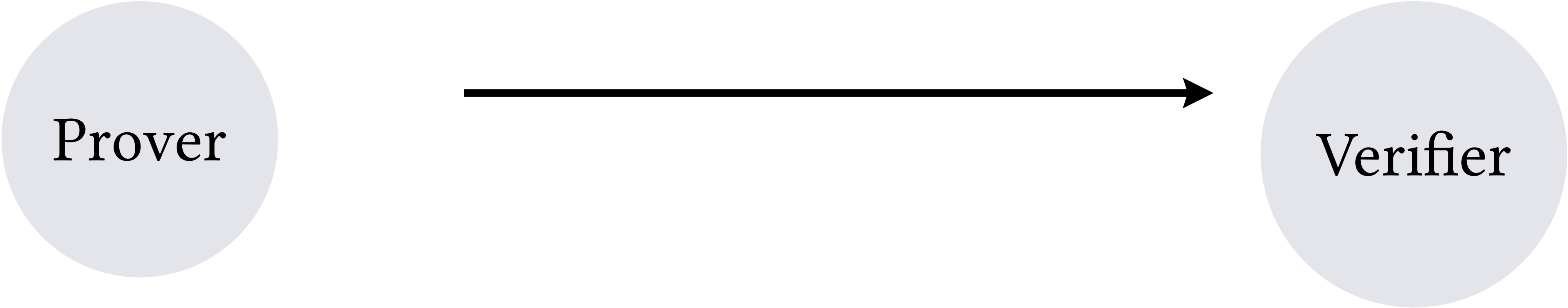
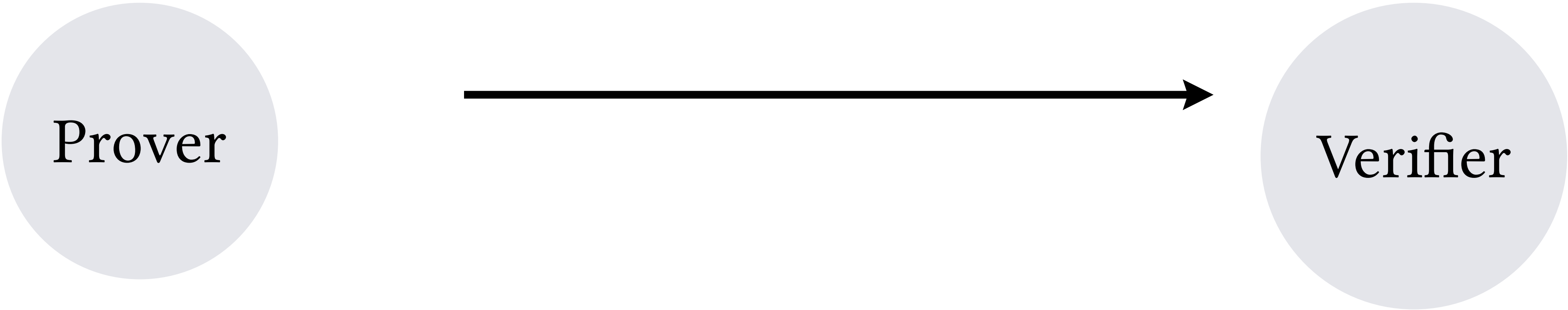


size: n

size: at least log n

ZKP (3)

Data sent to verifier is compressed, and can be hidden



size: n

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Modular arithmetics (1)

It is about integers.

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Let’s assume we arithmetics **mod 8**. It means the possible values are 0,1,2,3,4,5,6,7. if we move below or above we need to wrap up.

Modular arithmetics (1)

$$3 + 3 \bmod 8 = 6 \bmod 8$$

$$10 \bmod 8 = 2 \bmod 8$$

$$5 + 5 \bmod 8 = 2 \bmod 8$$

$$5 \cdot 5 \bmod 8 = 25 \bmod 8 = (3 \cdot 8 + 1) \bmod 8 = 1 \bmod 8$$

Modular arithmetics (1)

$$3 + 3 \bmod 8 = 6 \bmod 8$$

$$10 \bmod 8 = 2 \bmod 8$$

$$5 + 5 \bmod 8 = 2 \bmod 8$$

$$5 \cdot 5 \bmod 8 = 25 \bmod 8 = (3 \cdot 8 + 1) \bmod 8 = 1 \bmod 8$$

congruent groups

Modular arithmetics (2)

```
addition mod 8      multiplication mod 8
0 1 2 3 4 5 6 7   1 2 3 4 5 6 7
1 2 3 4 5 6 7 0   2 4 6 0 2 4 6
2 3 4 5 6 7 0 1   3 6 1 4 7 2 5
3 4 5 6 7 0 1 2   4 0 4 0 4 0 4
4 5 6 7 0 1 2 3   5 2 7 4 1 6 3
5 6 7 0 1 2 3 4   6 4 2 0 6 4 2
6 7 0 1 2 3 4 5   7 6 5 4 3 2 1
7 0 1 2 3 4 5 6
```

Modular arithmetics (2)

```
addition mod 8      multiplication mod 8          Let's solve the eq in mod 8:
0 1 2 3 4 5 6 7     1 2 3 4 5 6 7                 9(2x + 7) − 6 ≡ 2x + 6
1 2 3 4 5 6 7 0     2 4 6 0 2 4 6
2 3 4 5 6 7 0 1     3 6 1 4 7 2 5
3 4 5 6 7 0 1 2     4 0 4 0 4 0 4
4 5 6 7 0 1 2 3     5 2 7 4 1 6 3
5 6 7 0 1 2 3 4     6 4 2 0 6 4 2
6 7 0 1 2 3 4 5     7 6 5 4 3 2 1
7 0 1 2 3 4 5 6
```

Modular arithmetics (2)

```
addition mod 8
0 1 2 3 4 5 6 7
1 2 3 4 5 6 7 0
2 3 4 5 6 7 0 1
3 4 5 6 7 0 1 2
4 5 6 7 0 1 2 3
5 6 7 0 1 2 3 4
6 7 0 1 2 3 4 5
7 0 1 2 3 4 5 6
```

```
multiplication mod 8
1 2 3 4 5 6 7
2 4 6 0 2 4 6
3 6 1 4 7 2 5
4 0 4 0 4 0 4
5 2 7 4 1 6 3
6 4 2 0 6 4 2
7 6 5 4 3 2 1
```

```
Let's solve the eq in mod 8:
9(2x + 7) − 6 ≡ 2x + 6
19*2x + 19*7 - 6 ≡ 2x + 6
```

Modular arithmetics (2)

```
addition mod 8      multiplication mod 8         Let's solve the eq in mod 8:
0 1 2 3 4 5 6 7     1 2 3 4 5 6 7               9(2x + 7) − 6 ≡ 2x + 6
1 2 3 4 5 6 7 0     2 4 6 0 2 4 6               19*2x + 19*7 - 6 ≡ 2x + 6
2 3 4 5 6 7 0 1     3 6 1 4 7 2 5               38x + 133 - 6 ≡ 2x + 6  # 133 mod 8 = 5
3 4 5 6 7 0 1 2     4 0 4 0 4 0 4
4 5 6 7 0 1 2 3     5 2 7 4 1 6 3
5 6 7 0 1 2 3 4     6 4 2 0 6 4 2
6 7 0 1 2 3 4 5     7 6 5 4 3 2 1
7 0 1 2 3 4 5 6
```

Modular arithmetics (2)

```
addition mod 8      multiplication mod 8          Let's solve the eq in mod 8:
0 1 2 3 4 5 6 7     1 2 3 4 5 6 7              9(2x + 7) − 6 ≡ 2x + 6
1 2 3 4 5 6 7 0     2 4 6 0 2 4 6              19*2x + 19*7 - 6 ≡ 2x + 6
2 3 4 5 6 7 0 1     3 6 1 4 7 2 5              38x + 133 - 6 ≡ 2x + 6  # 133 mod 8 = 5
3 4 5 6 7 0 1 2     4 0 4 0 4 0 4              6x + 5 - 6 ≡ 2x + 6
4 5 6 7 0 1 2 3     5 2 7 4 1 6 3
5 6 7 0 1 2 3 4     6 4 2 0 6 4 2
6 7 0 1 2 3 4 5     7 6 5 4 3 2 1
7 0 1 2 3 4 5 6
```

Modular arithmetics (2)

```
addition mod 8     multiplication mod 8          Let's solve the eq in mod 8:
0 1 2 3 4 5 6 7    1 2 3 4 5 6 7             9(2x + 7) − 6 ≡ 2x + 6
1 2 3 4 5 6 7 0    2 4 6 0 2 4 6             19*2x + 19*7 - 6 ≡ 2x + 6
2 3 4 5 6 7 0 1    3 6 1 4 7 2 5             38x + 133 - 6 ≡ 2x + 6  # 133 mod 8 = 5
3 4 5 6 7 0 1 2    4 0 4 0 4 0 4             6x + 5 - 6 ≡ 2x + 6
4 5 6 7 0 1 2 3    5 2 7 4 1 6 3             6x + 5 - 6 + 6 ≡ 2x + 6 + 6
5 6 7 0 1 2 3 4    6 4 2 0 6 4 2
6 7 0 1 2 3 4 5    7 6 5 4 3 2 1
7 0 1 2 3 4 5 6
```

Modular arithmetics (2)

```
addition mod 8     multiplication mod 8          Let's solve the eq in mod 8:
0 1 2 3 4 5 6 7    1 2 3 4 5 6 7                 9(2x + 7) − 6 ≡ 2x + 6
1 2 3 4 5 6 7 0    2 4 6 0 2 4 6                 19*2x + 19*7 - 6 ≡ 2x + 6
2 3 4 5 6 7 0 1    3 6 1 4 7 2 5                 38x + 133 - 6 ≡ 2x + 6  # 133 mod 8 = 5
3 4 5 6 7 0 1 2    4 0 4 0 4 0 4                 6x + 5 - 6 ≡ 2x + 6
4 5 6 7 0 1 2 3    5 2 7 4 1 6 3                 6x + 5 - 6 + 6 ≡ 2x + 6 + 6
5 6 7 0 1 2 3 4    6 4 2 0 6 4 2                 6x + 5 ≡ 2x + 4  # 12 mod 8 = 4
6 7 0 1 2 3 4 5    7 6 5 4 3 2 1                 6x - 2x + 5 - 5 ≡ 2x - 2x + 4 - 5
7 0 1 2 3 4 5 6
```

Modular arithmetics (2)

```
addition mod 8      multiplication mod 8        Let's solve the eq in mod 8:
0 1 2 3 4 5 6 7     1 2 3 4 5 6 7           9(2x + 7) − 6 ≡ 2x + 6
1 2 3 4 5 6 7 0     2 4 6 0 2 4 6           19*2x + 19*7 - 6 ≡ 2x + 6
2 3 4 5 6 7 0 1     3 6 1 4 7 2 5           38x + 133 - 6 ≡ 2x + 6  # 133 mod 8 = 5
3 4 5 6 7 0 1 2     4 0 4 0 4 0 4           6x + 5 - 6 ≡ 2x + 6
4 5 6 7 0 1 2 3     5 2 7 4 1 6 3           6x + 5 - 6 + 6 ≡ 2x + 6 + 6
5 6 7 0 1 2 3 4     6 4 2 0 6 4 2           6x + 5 ≡ 2x + 4  # 12 mod 8 = 4
6 7 0 1 2 3 4 5     7 6 5 4 3 2 1           6x - 2x + 5 - 5 ≡ 2x - 2x + 4 - 5
7 0 1 2 3 4 5 6                         4x ≡ 7  # -1 mod 8 = 7
```

Now we do **NOT have multiplication inverse** for 4, ie. we cannot divide by 4 in modulo 8, ie. solve this equation We have only multiplication inverse for 1 which is 1; 3 which is 3; 5 which is 5, and 7 which is 7.

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

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addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$19(2x + 7) - 6 \equiv 2x + 6$

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$19(2x + 7) - 6 \equiv 2x + 6$

$19{*}2x + 19{*}7 - 6 \equiv 2x + 6$

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$19(2x + 7) - 6 \equiv 2x + 6$

$19*2x + 19*7 - 6 \equiv 2x + 6$

$5x + 1 - 6 \equiv 2x + 6$

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$19(2x + 7) - 6 \equiv 2x + 6$

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$5x + 1 - 6 + 6 \equiv 2x + 6 + 6$

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

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2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$$19(2x + 7) - 6 \equiv 2x + 6$$

$$19*2x + 19*7 - 6 \equiv 2x + 6$$

$$5x + 1 - 6 \equiv 2x + 6$$

$$5x + 1 - 6 + 6 \equiv 2x + 6 + 6$$

$$5x + 1 \equiv 2x + 1$$

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$19(2x + 7) - 6 \equiv 2x + 6$

$19*2x + 19*7 - 6 \equiv 2x + 6$

$5x + 1 - 6 \equiv 2x + 6$

$5x + 1 - 6 + 6 \equiv 2x + 6 + 6$

$5x + 1 \equiv 2x + 1$

$x \equiv 0$

We have solution: **{.., −22, −11, 0, 11, 22, ...}** As in each row of mult table there is 1 we have inverse for each congruence group!

Modular arithmetics (4)

addition mod 13

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
3	4	5	6	7	8	9	10	11	12	0	1	2
4	5	6	7	8	9	10	11	12	0	1	2	3
5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5
7	8	9	10	11	12	0	1	2	3	4	5	6
8	9	10	11	12	0	1	2	3	4	5	6	7
9	10	11	12	0	1	2	3	4	5	6	7	8
10	11	12	0	1	2	3	4	5	6	7	8	9
11	12	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

As in each row of mult table there is 1 we have inverse for each congruence group! => we want to work with mod **PRIME** as we want inverses!

Modular arithmetics (4)

addition mod 13

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
3	4	5	6	7	8	9	10	11	12	0	1	2
4	5	6	7	8	9	10	11	12	0	1	2	3
5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5
7	8	9	10	11	12	0	1	2	3	4	5	6
8	9	10	11	12	0	1	2	3	4	5	6	7
9	10	11	12	0	1	2	3	4	5	6	7	8
10	11	12	0	1	2	3	4	5	6	7	8	9
11	12	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

diagonal:	1	2	3	4	5	6	7	8	9	10	11	12
val:	1	4	9	3	12	10	10	12	3	9	4	1

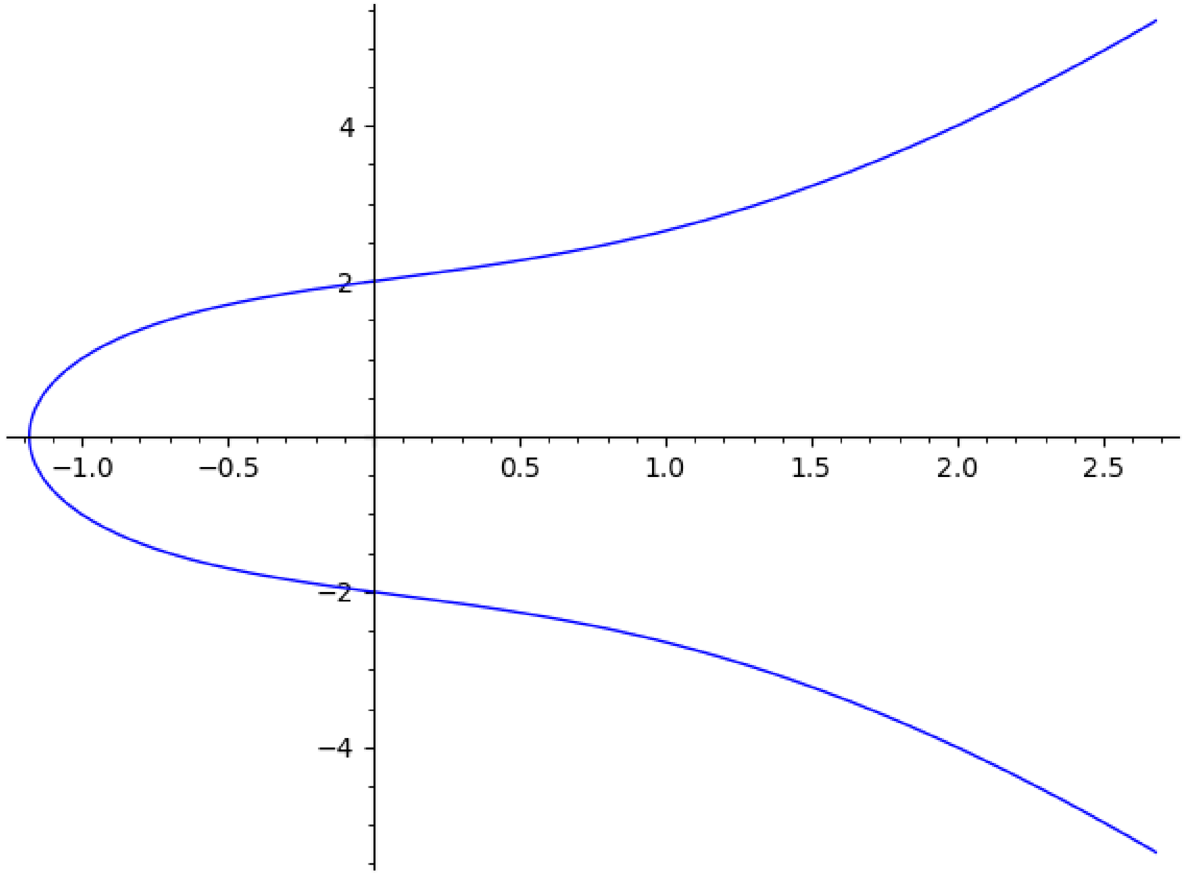
- not always square is possible within modulus
- 1, 3, 4, 9, 10, 12

  are **quadratic residues**

Elliptic curves (1)

```
sage: # Let's plot the following elliptic curve in R
sage: # y^2 == x^3 + 2x + 4 in R

sage: E = EllipticCurve([2,4]);
sage: P = E.plot()
sage: P.save("ellipticR.png")
```



Elliptic curves (2)

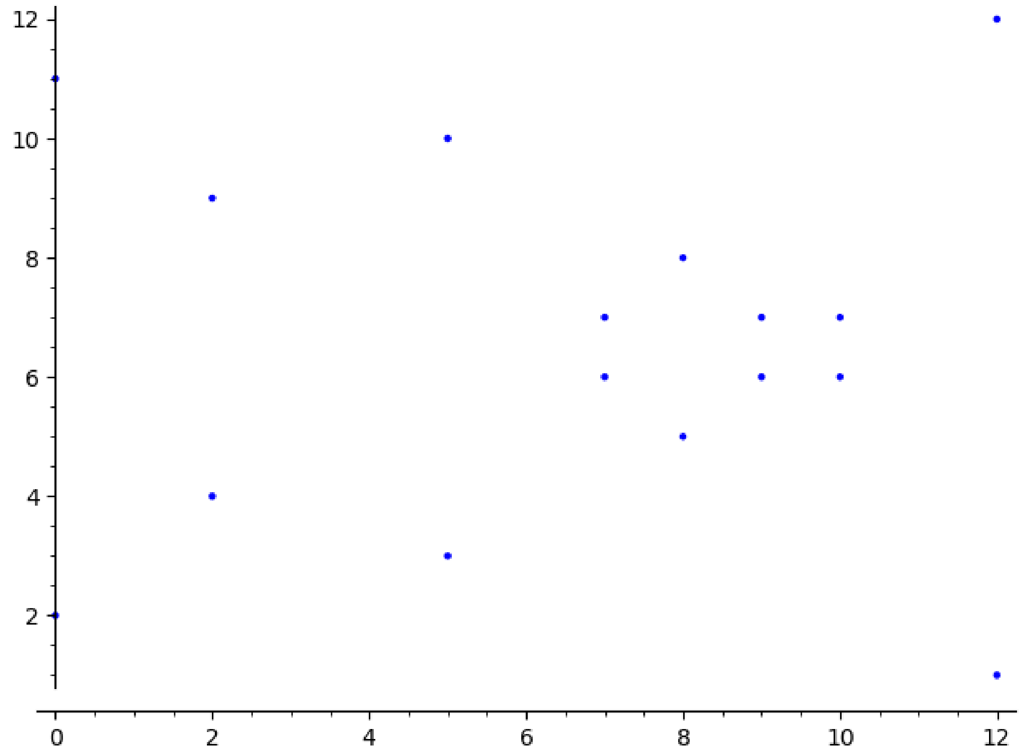
```
y^2 == x^3 + 2x + 4 in mod 13

x=0, y^2= 4 => (0,2) and (0,11) -> two points
x=1, y^2= 1 + 2 + 4 = 7 -> no point
x=2, y^2= 8 + 4 + 4 = 16 mod 13 = 3 -> (2,4) and (2,9)
x=3, y^2= 1 + 6 + 4 = 11 -> no point
x=4, y^2= 12 + 8 + 4 = 24 mod 13 = 11 -> no point
x=5, y^2= 8 + 10 + 4 = 9 -> (5,3) and (5,10)
x=6, y^2= 8 + 12 + 4 = 24 mod 13 = 11 -> no point
x=7, y^2= 5 + 14 + 4 = 23 mod 13 = 10 -> (7,6) and (7,7)
x=8, y^2= 5 + 16 + 4 = 25 mod 13 = 12 -> (8,5) and (8,8)
x=9, y^2= 1 + 18 + 4 = 23 mod 13 = 10 -> (9,6) and (9,7)
x=10, y^2= 12 + 20 + 4 = 36 mod 13 = 10 -> (10,6) and (10,7)
x=11, y^2= 5 + 22 + 4 = 31 mod 13 = 5 -> no point
x=12, y^2= 12 + 24 + 4 = 40 mod 13 = 1 -> (12,1) and (12,12)
```

For some x there are no solutions when we have mod 13, for the rest we have two!

Elliptic curves (3)

```
sage: F13=GF(13)
sage: a = F13(2)
sage: b = F13(4)
sage: # discriminant obeys condition
sage: F13(6)*(F13(4)*a^3+F13(27)*b^2) != F13(0)
True
sage: E = EllipticCurve(F13,[a,b]) # y^2 == x^3 + 2x + 4
sage: P = E(0,2) # 2^2 == 0^3 + 2*0 + 4 mod 13
sage: P.xy()
(0, 2)
sage: INF=E(0)
sage: try:
....:     INF.xy()
....: except ZeroDivisionError:
....:     pass
....:
sage: P = E.plot()
sage: P.save("elliptic13.png")
```



Modular arithmetics (5)

addition mod 13

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
3	4	5	6	7	8	9	10	11	12	0	1	2
4	5	6	7	8	9	10	11	12	0	1	2	3
5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5
7	8	9	10	11	12	0	1	2	3	4	5	6
8	9	10	11	12	0	1	2	3	4	5	6	7
9	10	11	12	0	1	2	3	4	5	6	7	8
10	11	12	0	1	2	3	4	5	6	7	8	9
11	12	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11

Addition forms a **group** as

- 0 is identity element
- addition is associative op
- addition is closed op
- each element has the inverse

Modular arithmetics (5)

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

Mult forms a **group** as

- 1 is identity element
- mult is associative op
- mult is closed op
- each element has the inverse (thanks to p being prime)

Modular arithmetics (5)

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

- 3 . 3 = 9
- 3 . 3 . 3 = 1
- 3 . 3 . 3 . 3 = 3

we can NOT generate EACH element -> 3 is not generator

Modular arithmetics (5)

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

- 8 . 8 = 12
- 8 . 8 . 8 = 5
- 8 . 8 . 8 . 8 = 1

we can NOT generate EACH element -> 8 is not generator

Modular arithmetics (5)

multiplication mod 13

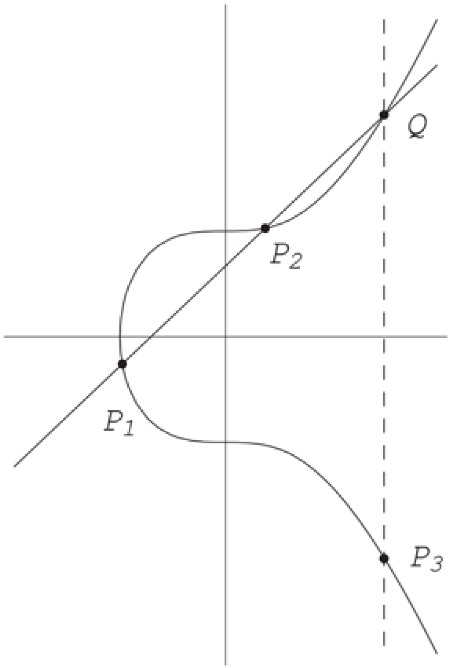
1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

- 2 . 2 = 4
- 2 . 2 . 2 = 8
- 2 . 2 . 2 . 2 = 3
- 2 . 2 . 2 . 2 . 2 = 6
- 2 . 2 . 2 . 2 . 2 . 2 = 12
- 2 . 2 . 2 . 2 . 2 . 2 . 2 = 11
- 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 = 9
- 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 = 5
- 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 = 10
- 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 = 7
- 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2 = 1

we can generate EVERY element -> 2 is **generator**, group is **cyclic**

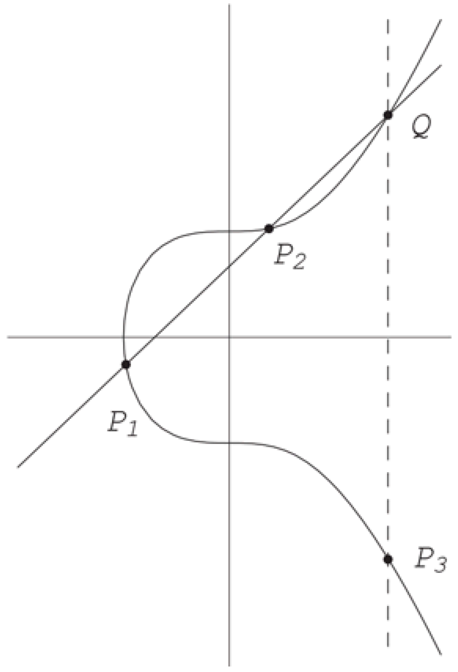
Elliptic curves (4)

Visually addition looks like here for elliptic curves with one remark: it is modulo PRIME



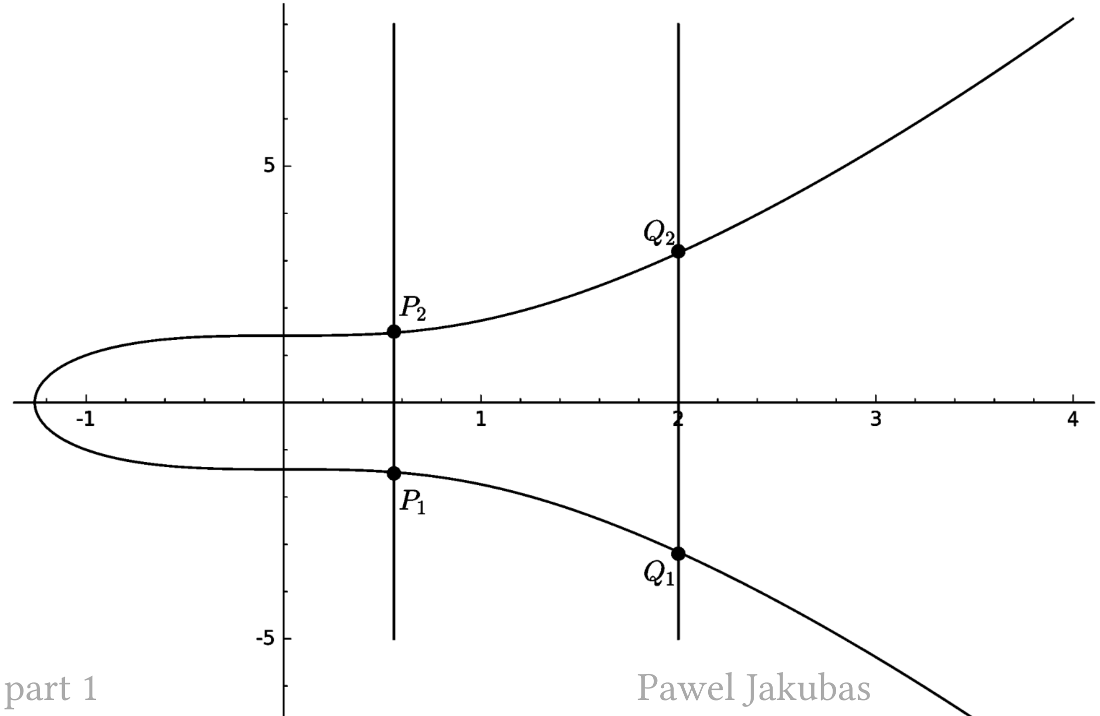
Elliptic curves (4)

It is called **chord-and-tangent** rule and visually is looks like below $P_1 + P_2 = P_3$



Elliptic curves (4)

y=INF is identity element, and because of that $P_1^{-1} = P_2$ $Q_1^{-1} = Q_2$



Elliptic curves (4)

chord-and-tangent rule algebraically is following

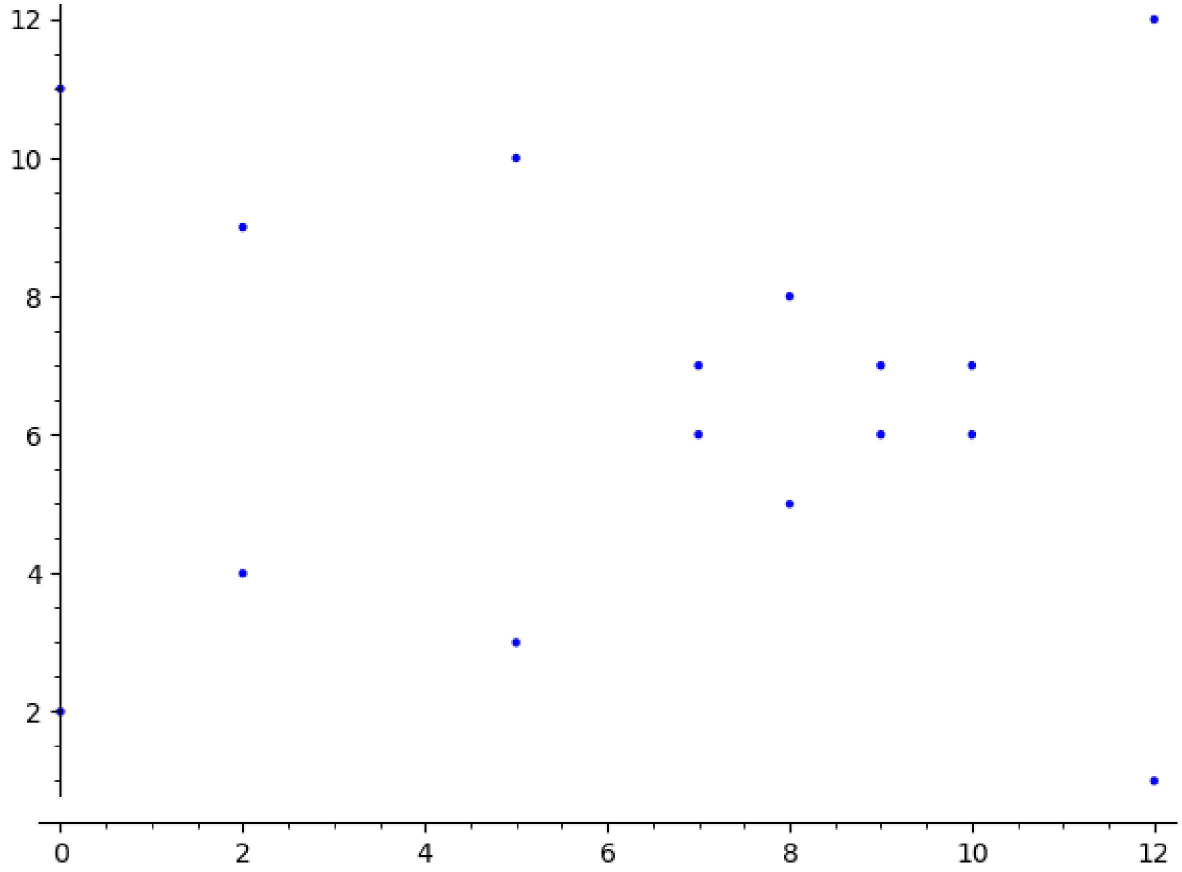
(Tangent Rule) If $P = (x, y)$ with $y \neq 0$, the group law $P \oplus P = (x', y')$ is defined as follows:

$$x' = \left(\frac{3x^2+a}{2y}\right)^2 - 2x \quad , \quad y' = \left(\frac{3x^2+a}{2y}\right)(x - x') - y$$

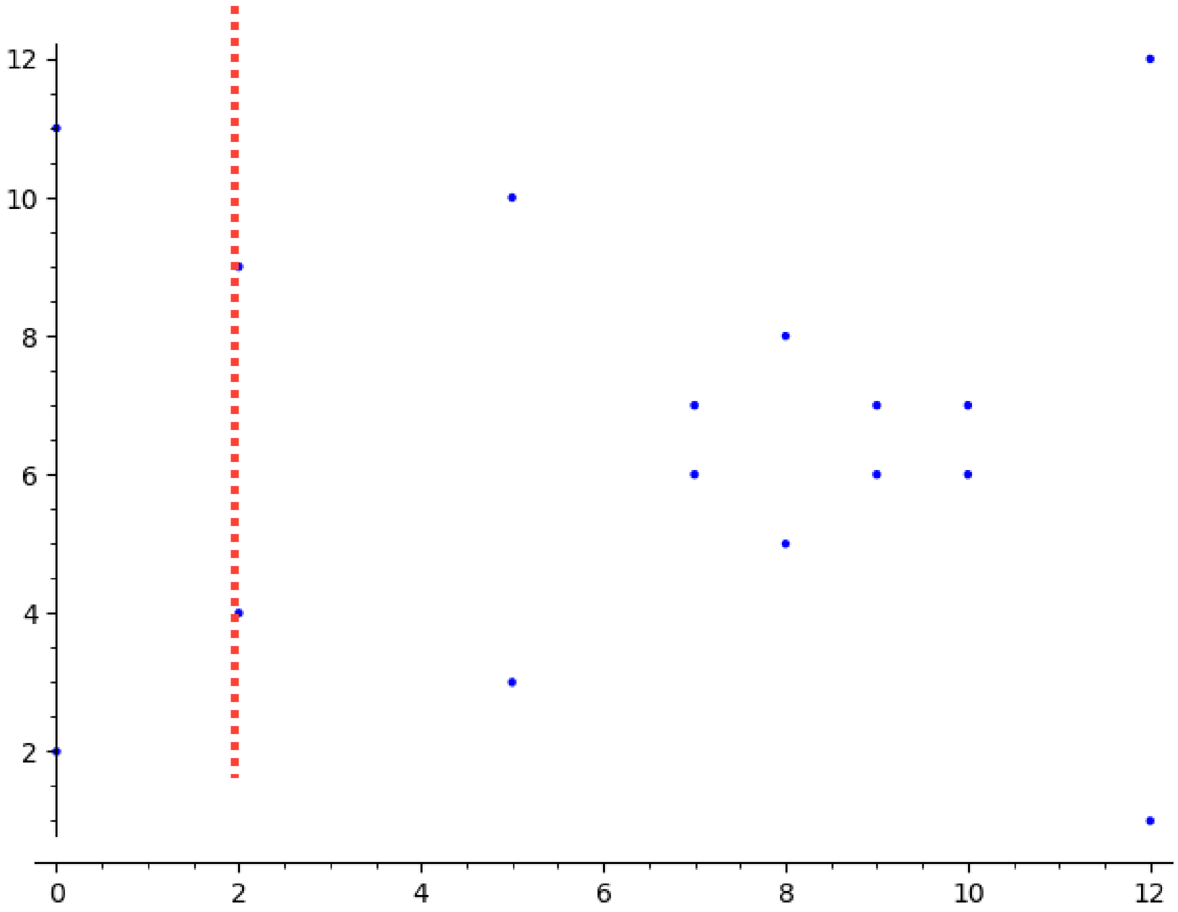
(Chord Rule) If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ such that $x_1 \neq x_2$, the group law $R = P \oplus Q$ with $R = (x_3, y_3)$ is defined as follows:

$$x_3 = \left(\frac{y_2-y_1}{x_2-x_1}\right)^2 - x_1 - x_2 \quad , \quad y_3 = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x_1 - x_3) - y_1$$

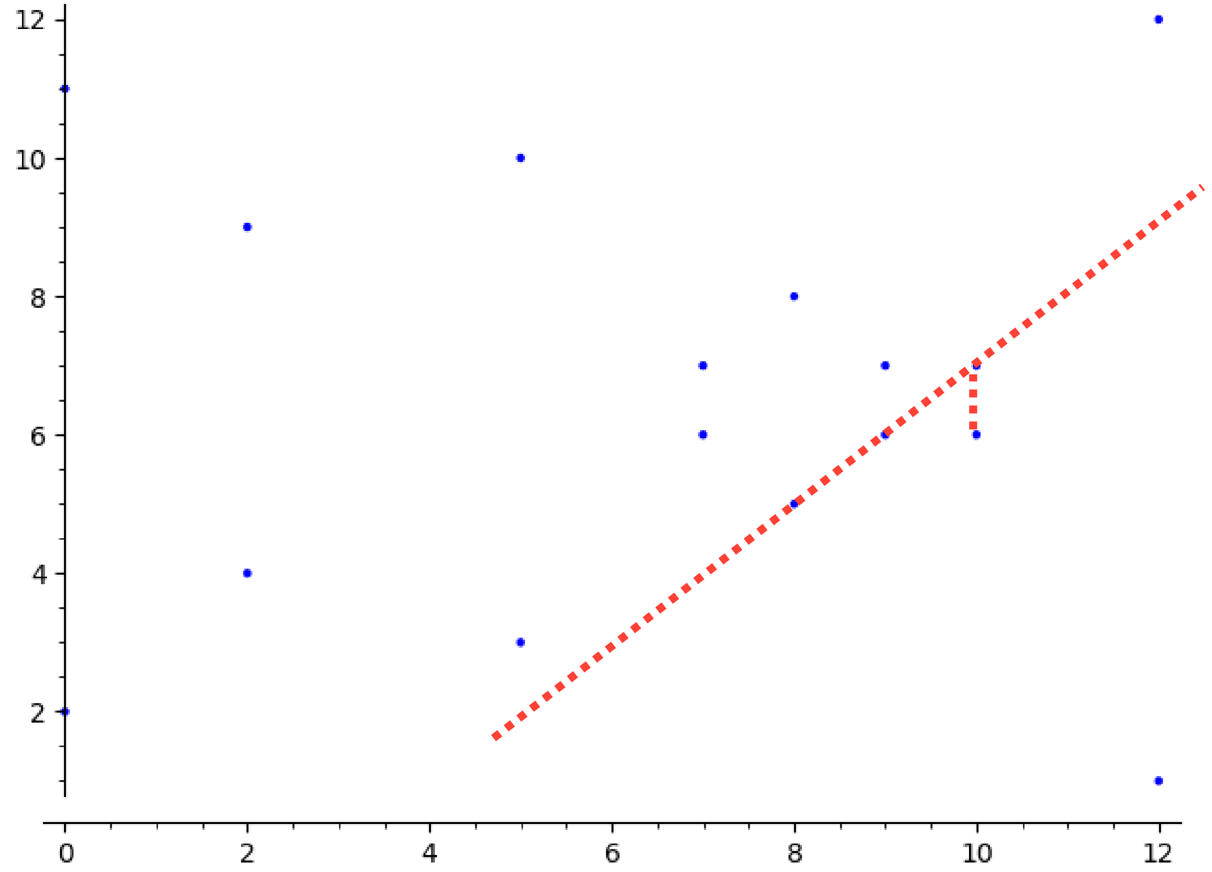
Elliptic curves (5) (mod 13)



Elliptic curves (5) (mod 13)



Elliptic curves (5) (mod 13)



Elliptic curves (6) (mod 13)

```
sage: F13 = GF(13)
sage: a = F13(2)
sage: b = F13(4)
sage: E = EllipticCurve(F13,[a,b]) # y^2 == x^3 + 2x + 4
sage: INF=E(0)
sage: E(2,4) + E(2,9) == INF
True
sage: E(8,5) + E(9,6) == E(10,7)
False
sage: E(8,5) + E(9,6) == E(10,6)
True
```

Elliptic curves (7)

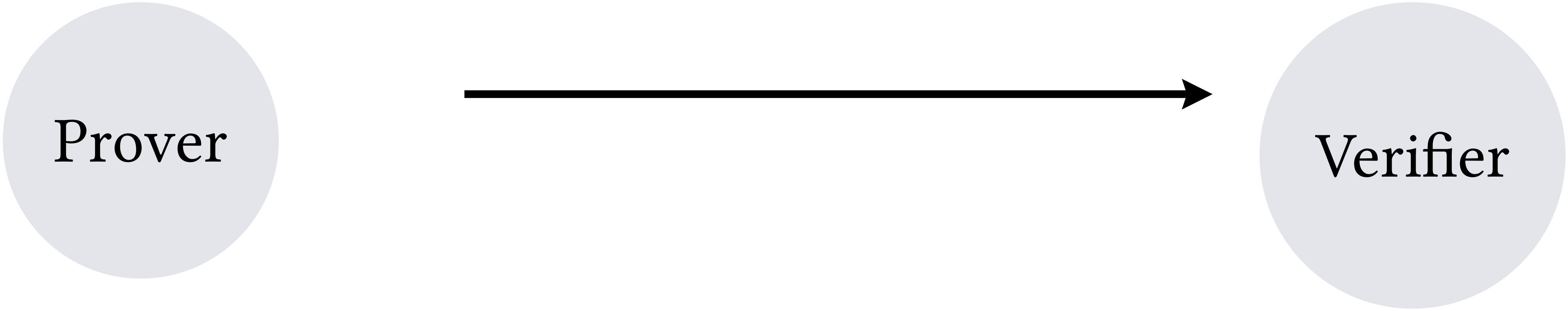
```
sage: # Bitcoin’s secp256k1 curve
sage: # p = 2^256-2^32-977
sage: p =115792089237316195423570985008687907853269984665640564039457584007908834671663
sage: p.is_prime()
True
sage: p.nbits()
256
sage: Fp = GF(p)
sage: secp256k1 = EllipticCurve(Fp,[0,7])
sage: # Base point
sage: gx= 55066263022277343669578718895168534326250603453777594175500187360389116729240L
sage: gy= 32670510020758816978083085130507043184471273380659243275938904335757337482424L
sage: G = secp256k1(Fp(gx), Fp(gy))
```

Elliptic curves (8)

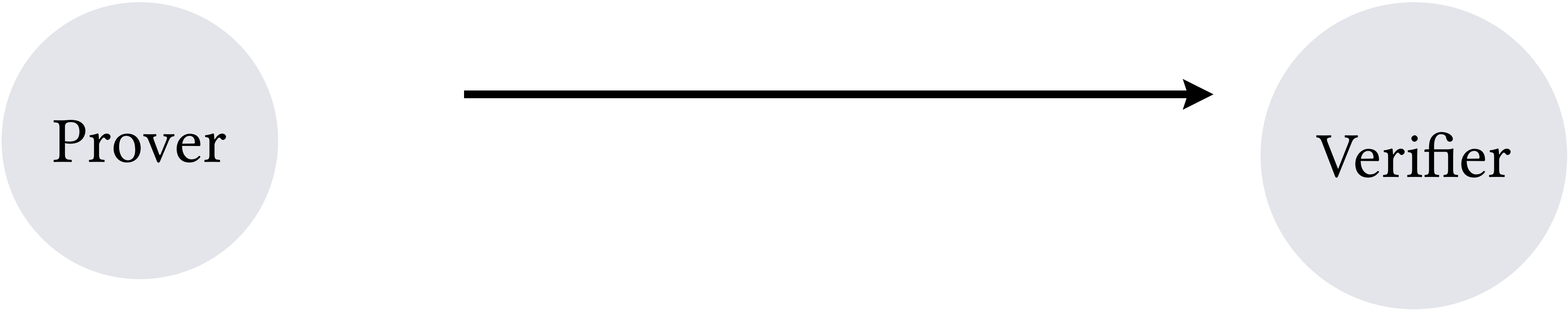
```
sage: # we have x + y = 9 to solve
sage: # PROVER provided a solution (x=2, y=7) and has the proof for it
sage: #
sage: xHidden = 2*G
sage: yHidden = 7*G
sage:
sage: # VERIFIER knows 9 which is public knowledge and gets solution hidden in POINTS
sage: rhsPoint = 9*G
sage: rhsPoint == xHidden + yHidden
True

sage: xHidden
(89565891926547004231252920425935692360644145829622209833684329913297188986597 :
12158399299693830322967808612713398636155367887041628176798871954788371653930 : 1)
```

Elliptic curves (8)



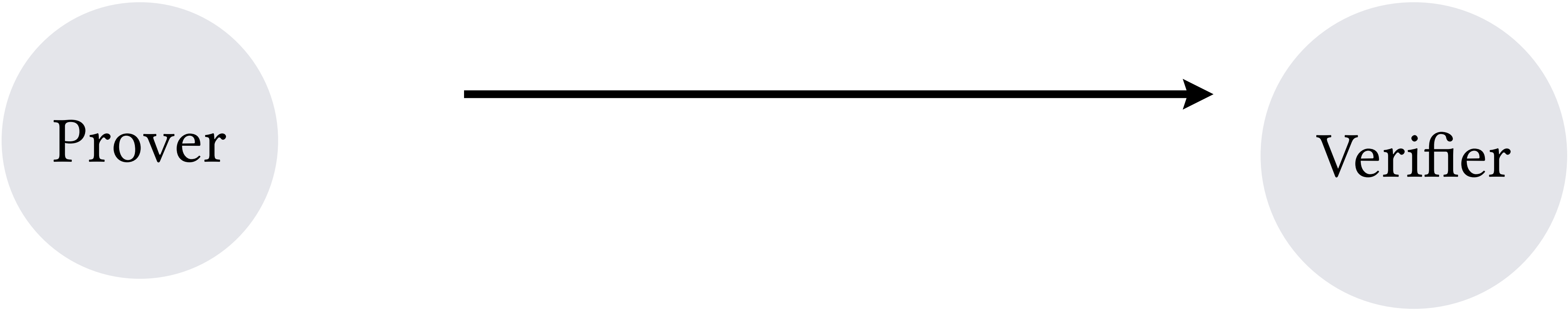
Elliptic curves (8)



x=2, y=7 and sends xG and yG

can check **xG + yG = 9G**, but cannot retrieve x and y

Elliptic curves (8)



x=2, y=7 and sends xG and yG

can check **xG + yG = 9G**, but cannot retrieve x and y

!homomorphic encryption preserves operations!

Elliptic curves (9)

At this moment we can solve problems that are linear, meaning can be expressed as set of linear expressions:

$$a_{11} * x_1 + a_{12} * x_2 + \ldots = b_1$$

$$a_{21} * x_1 + a_{22} * x_2 + \ldots = b_2$$

$$\ldots$$

$$a_{\mathrm{n}1} * x_1 + a_{\mathrm{n}2} * x_2 + \ldots = b_{\mathrm{n}}$$

but we cannot solve:

$$\mathbf{xy = 9}$$

Elliptic curves (9)

At this moment we can solve problems that are linear, meaning can be expressed as set of linear expressions:

$$a_{11} * x_1 + a_{12} * x_2 + \ldots = b_1$$

$$a_{21} * x_1 + a_{22} * x_2 + \ldots = b_2$$

$$\ldots$$

$$a_{\mathrm{n}1} * x_1 + a_{\mathrm{n}2} * x_2 + \ldots = b_{\mathrm{n}}$$

but we cannot solve:

xy = 9

=> pairings

Pairings of elliptic curves (1)

We are going to use TWO groups together
$(G_1, G_2) = G_T$
G_1 elliptic curve point => 2 numbers
G_2 elliptic curve point over an extended field in the form of polynomials **aw + b** and **a'w + b'** => 4 numbers

For $A \in G_1$, $B \in G_2$ and $C \in G_T$

G_T is multiplicative
e(A,B)=C => e(A^x, B^y)= C^{xy}

Pairings of elliptic curves (2)

G_1 has generator A
G_2 has generator B
C is pairing e(G_1, G_2)=g
xy=12
x=4 and y=3
e(4A, 3B)=C^{12}

G_1 and G_2 are elliptic groups
G_T is multiplicative group of an extensive field

$G_1 = G_2$ symmetric
$G_1 \neq G_2$ antisymmetric (used in production due to performance)

Pairings of elliptic curves (3)

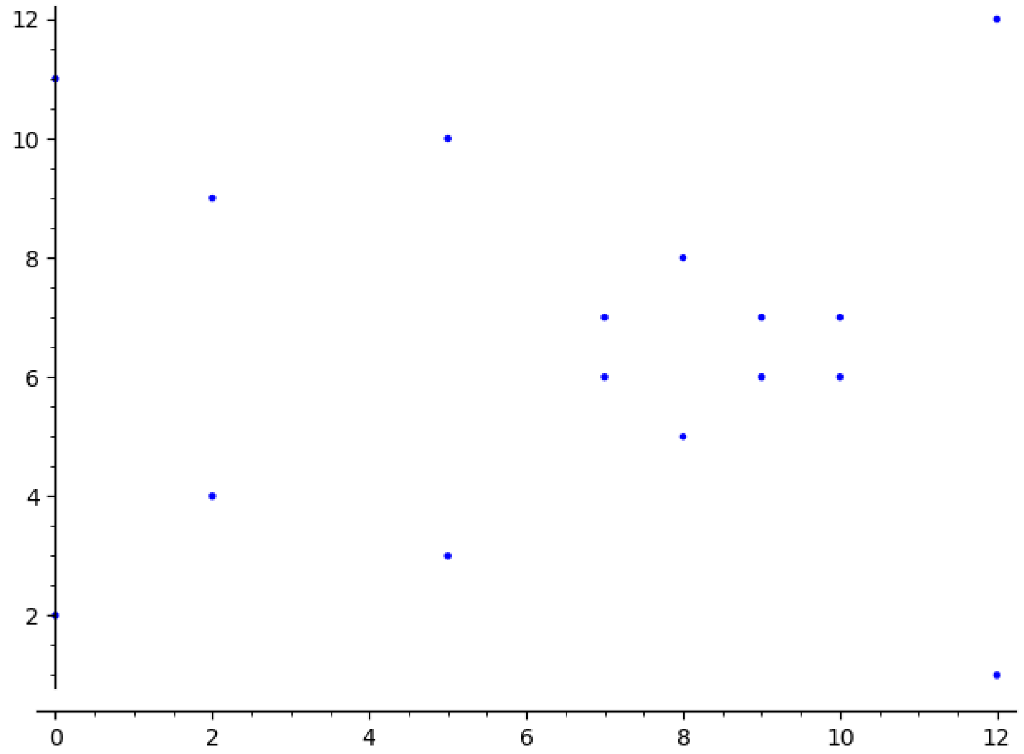
Every elliptic curve gives rise to a pairing map

BUT

not every such pairing can be efficiently computed

=>

embedding degree of a curve



Pairings of elliptic curves (3)

embedding degree of a curve **k**

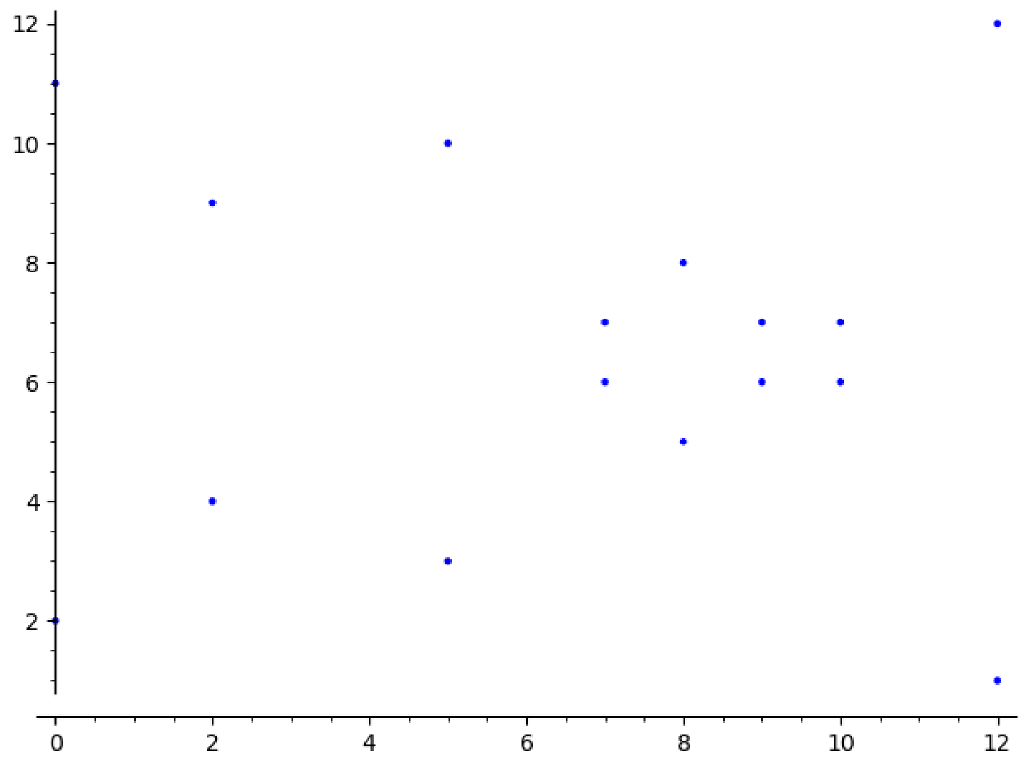
8 pairs + identity (INF) = 17

=>

order r = 17

p = 13

$$r \mid p^k - 1$$



Pairings of elliptic curves (3)

embedding degree of a curve **k**

8 pairs + identity (INF) = 17

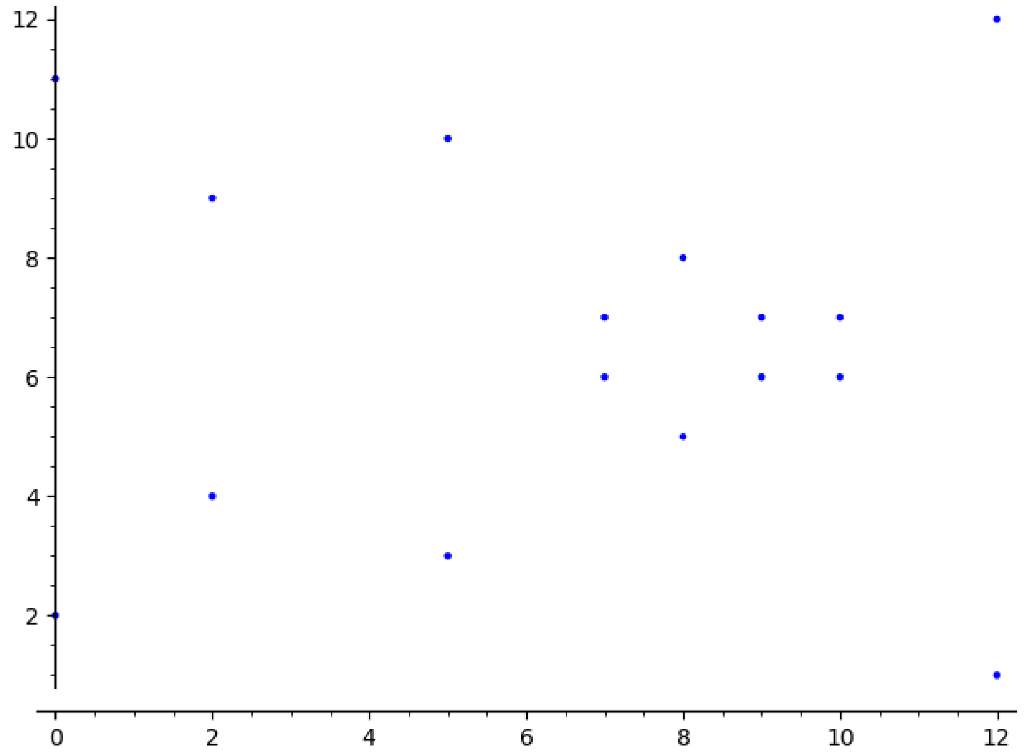
=>

order r = 17

p = 13

$r \mid p^k - 1$

$17 \mid 13^1 - 1$ <=> 17|13 NO



Pairings of elliptic curves (3)

embedding degree of a curve **k**

8 pairs + identity (INF) = 17

=>

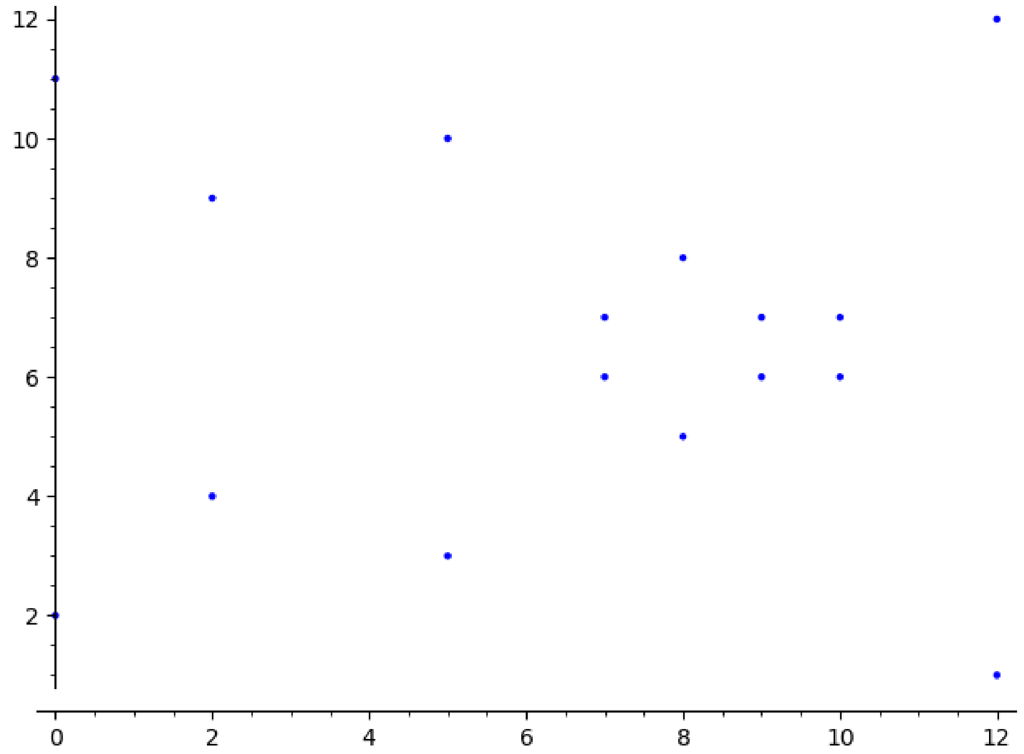
order r = 17

p = 13

$r \mid p^k - 1$

$17 \mid 13^1 - 1$ <=> 17|13 NO

$17 \mid 13^2 - 1$ <=> 17|168 NO



Pairings of elliptic curves (3)

embedding degree of a curve **k**

8 pairs + identity (INF) = 17

=>

order r = 17

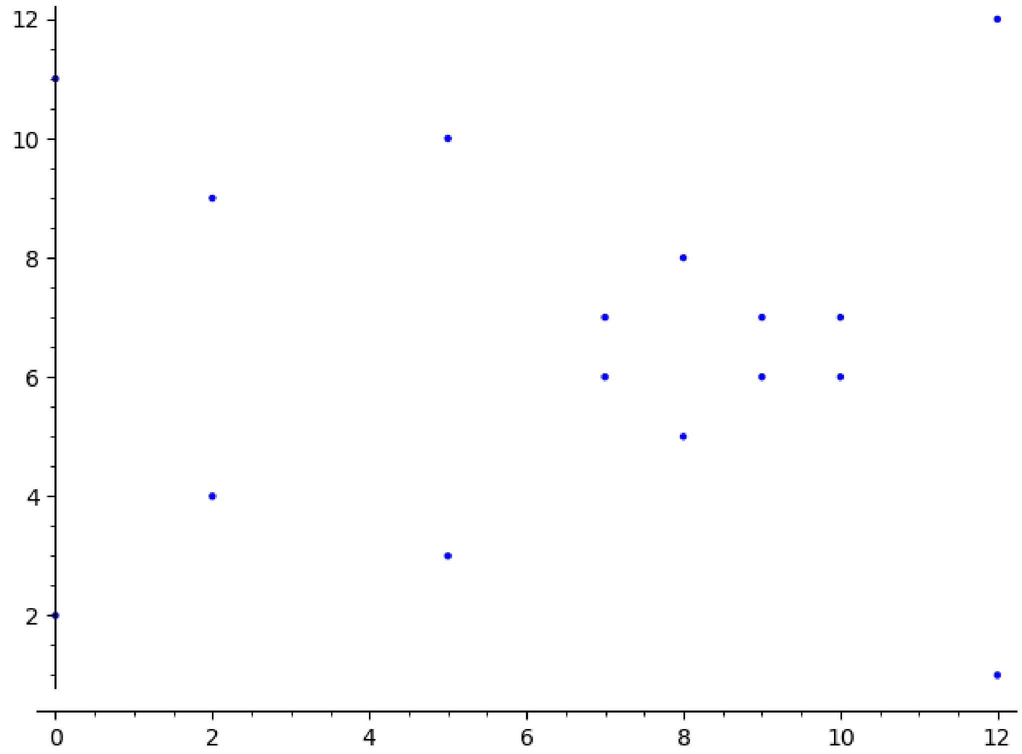
p = 13

$r \mid p^k - 1$

$17 \mid 13^1 - 1$ <=> 17|13 NO

$17 \mid 13^2 - 1$ <=> 17|168 NO

$17 \mid 13^3 - 1$ <=> 17|2196 NO



Pairings of elliptic curves (3)

embedding degree of a curve **k**

8 pairs + identity (INF) = 17

=>

order r = 17

p = 13

$r \mid p^k - 1$

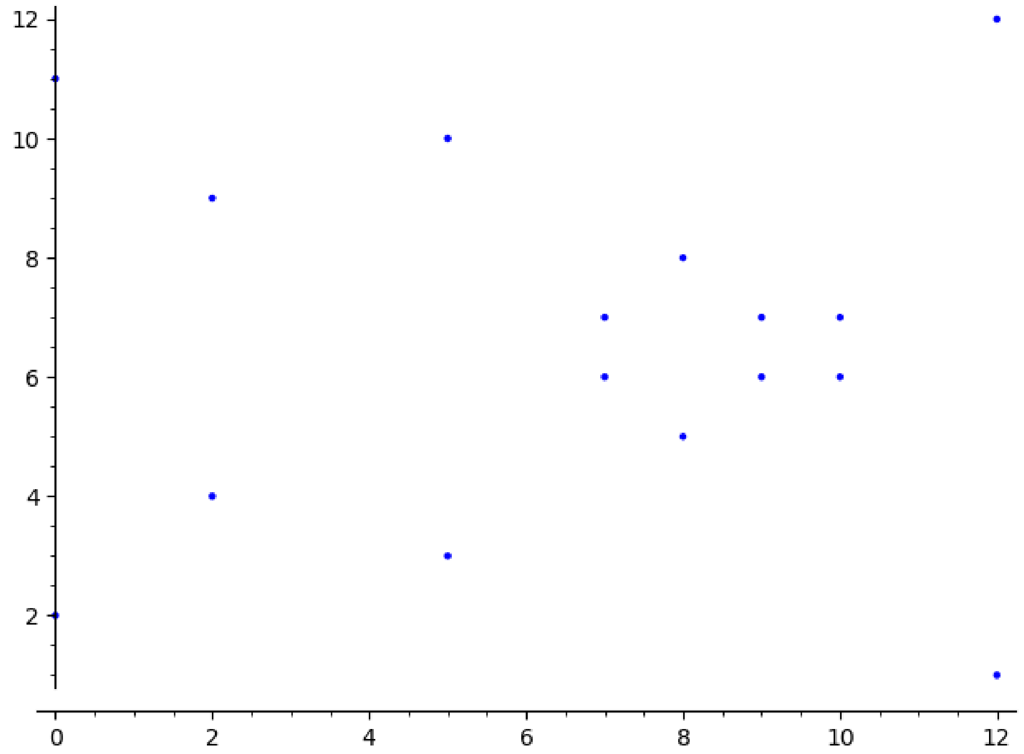
$17 \mid 13^1 - 1$ <=> 17|13 NO

$17 \mid 13^2 - 1$ <=> 17|168 NO

$17 \mid 13^3 - 1$ <=> 17|2196 NO

$17 \mid 13^4 - 1$ <=> 17|28560 YES

k=4



Pairings of elliptic curves (3)

```
sage: p=13
sage: F13=GF(p)
sage: a = F13(2)
sage: b = F13(4)
sage: E = EllipticCurve(F13,[a,b])
sage: r= E.order()
17
sage: k = 1
sage: while True:
....:     # Check if the order r divides (p^k - 1)
....:     if (p**k - 1) % r == 0:
....:         print(f"The embedding degree k is: {k}")
....:         break
....:     k += 1
....:
The embedding degree k is: 4
```

Pairings of elliptic curves (4)

```
sage: # Now again secp256k1
sage: p =115792089237316195423570985008687907853269984665640564039457584007908834671663
sage: Fp = GF(p)
sage: secp256k1 = EllipticCurve(Fp,[0,7])
sage: r= secp256k1.order()
sage: r
115792089237316195423570985008687907852837564279074904382605163141518161494337
sage: k = 1
sage: while k < 1000:
....:     if (p^k-1)%r == 0:
....:         break
....:     k=k+1
....:
sage: k
1000
```

```
sage: # in fact it very large
sage: # k =19298681539552699237261830834781317547292737984581739710086052358636024 9056
```

Pairings of elliptic curves (5)

secp256k1 is a curve that has large k

As extension field is built over p^k it means the extension field is here extremely big. =>

secp256k1 NEEDS k-many entries, each of them 256 bits

=> well not enough atoms in the observable universe out there

Pairings of elliptic curves (5)

secp256k1 is a curve that has large k

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=> well not enough atoms in the observable universe out there

secp256k1 is not pairing-friendly

we want curves with low embedding degree:

BN128 -> 12

Jubjub

BLS

That’s it! More to come in the future

Get in touch 👋

⇧My notes on GitHub