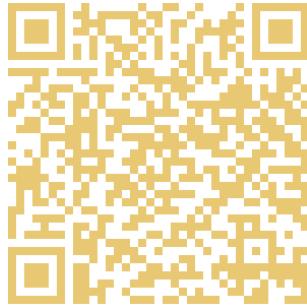


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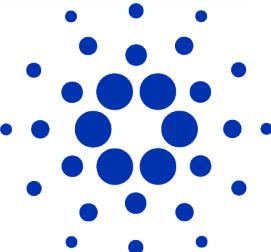
typst

$$\mathcal{L} = \nabla \cdot \mathbf{E} \quad (15)$$

The time-derivative has been chosen to be quadratic in the field tensor because we want to derive a linear field equation in which the superposition theorem holds. The action has to be a scalar, the simplest quadratic scalar is the sum of the two terms of the product given in Eq.

$$S = \frac{1}{2} \int d^3x \frac{\partial E_i}{\partial t} \frac{\partial E_i}{\partial t} \quad (16)$$

(20).



# Zero-knowledge-proofs - part 1

ENGINEERING WORKSHOP - DEC 2025  
(IRELAND)

Pawel Jakubas

# Plan of the tutorial

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Let's get a little deeper than usual and understand what main building blocks of ZKP looks like

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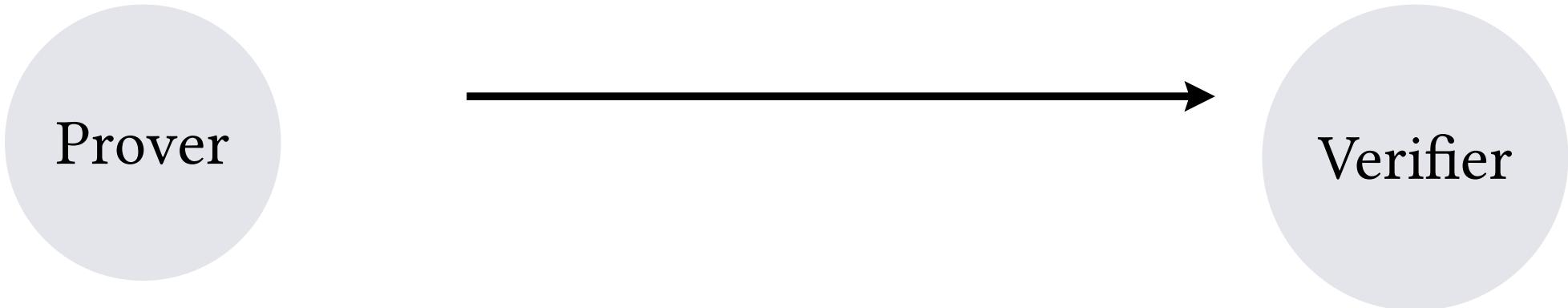
This tutorial will focus on

1. sketching the landscape of what we want to understand during 3-4 parts
2. cover the first part in some detail **elliptic curves**

# Verifiable computing vs ZKP (1)

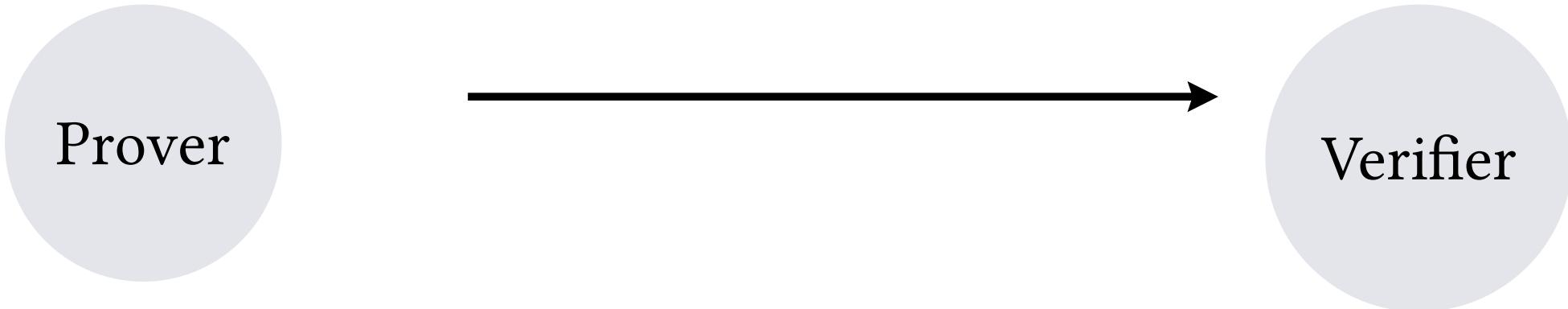
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There is **asymmetry** built into those systems. It is much easier to get public key from secret. But not the other way



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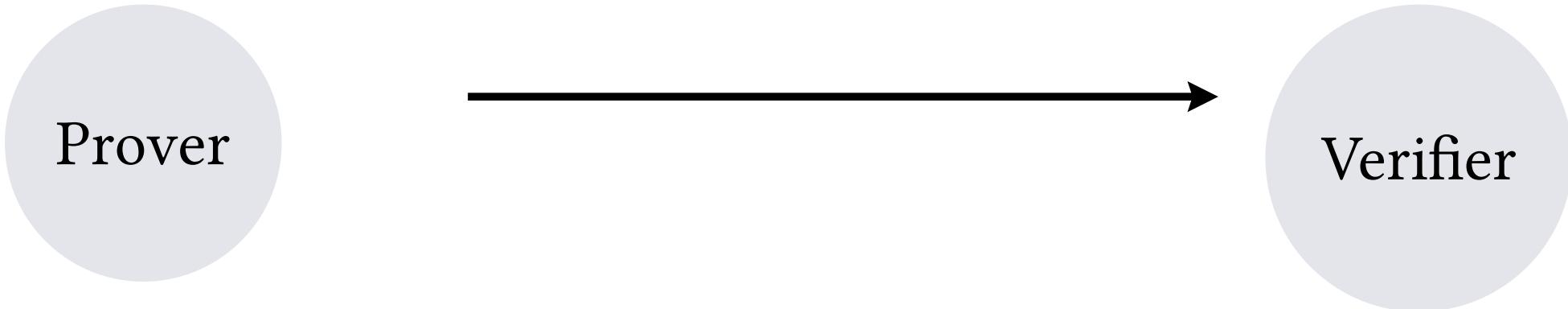
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secret -> (easy) -> public

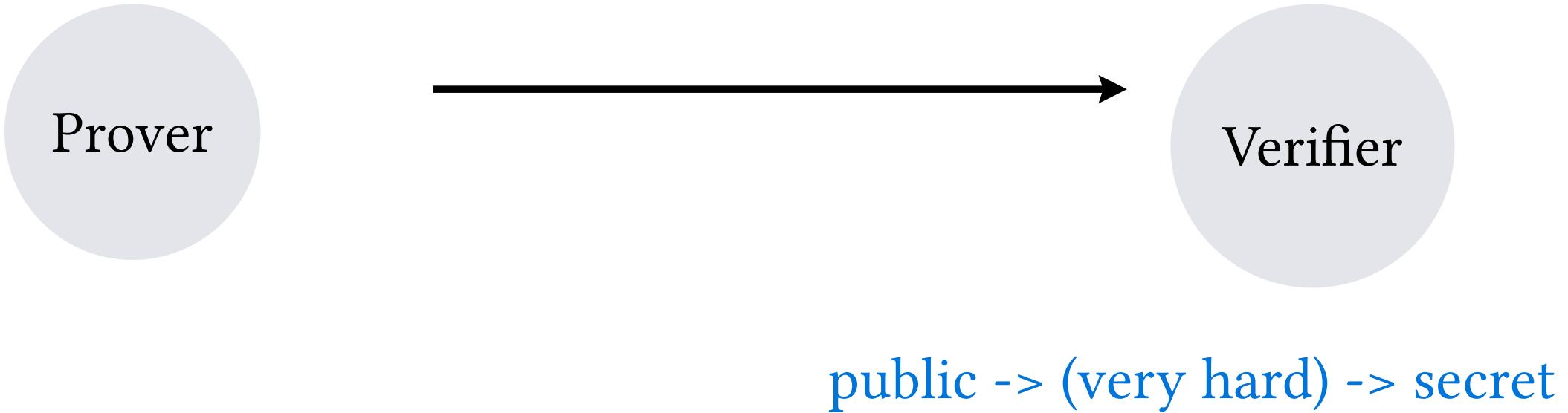
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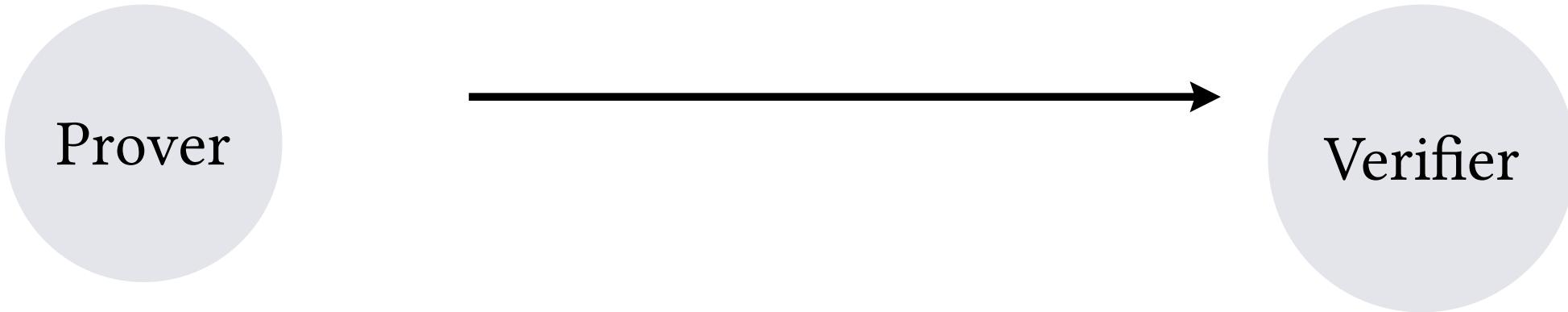
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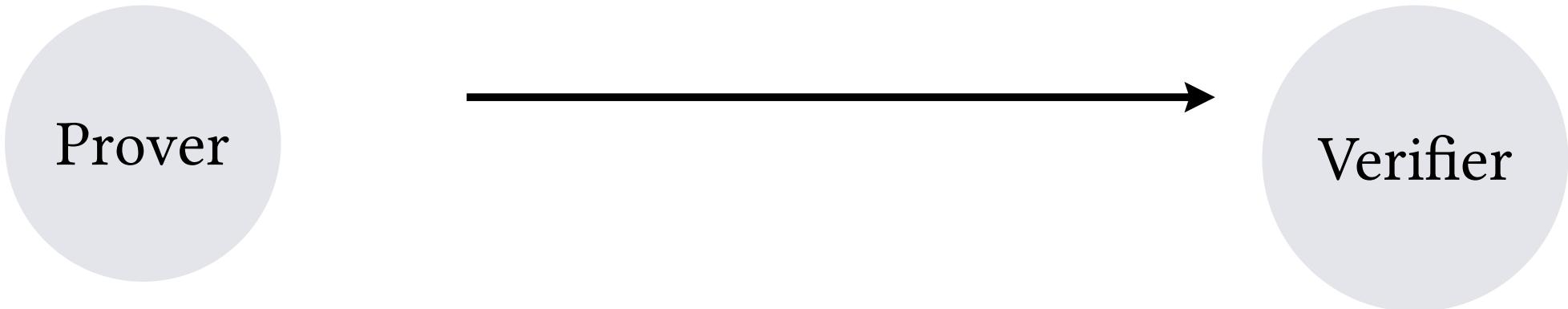
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proof verified using public data

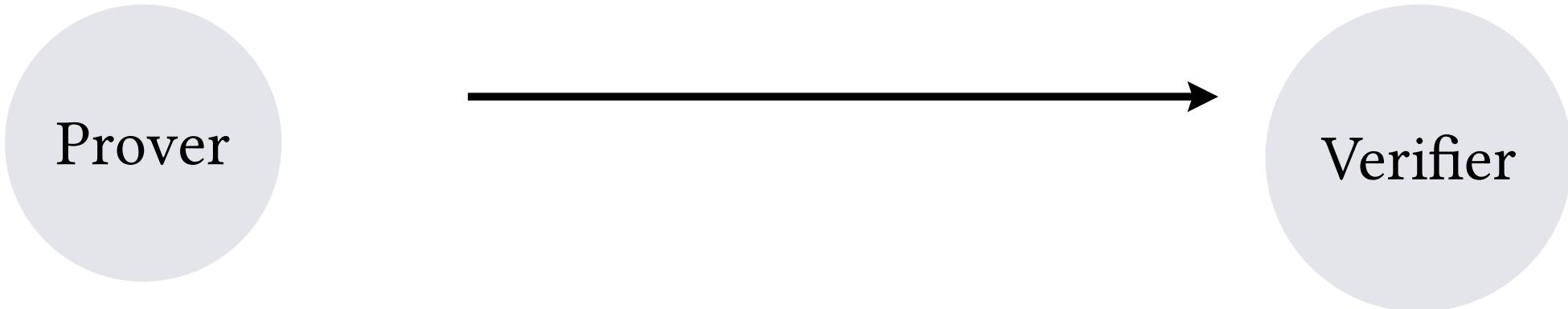
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There is **asymmetry** built into those systems. It is much quicker to verify than prove something.



# Verifiable computing vs ZKP (2)

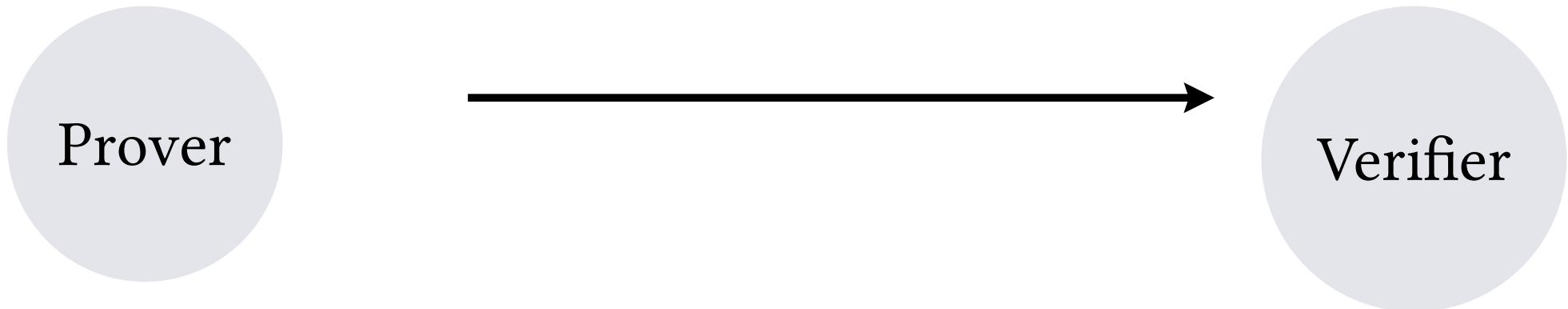
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$O(n)$  off-chain

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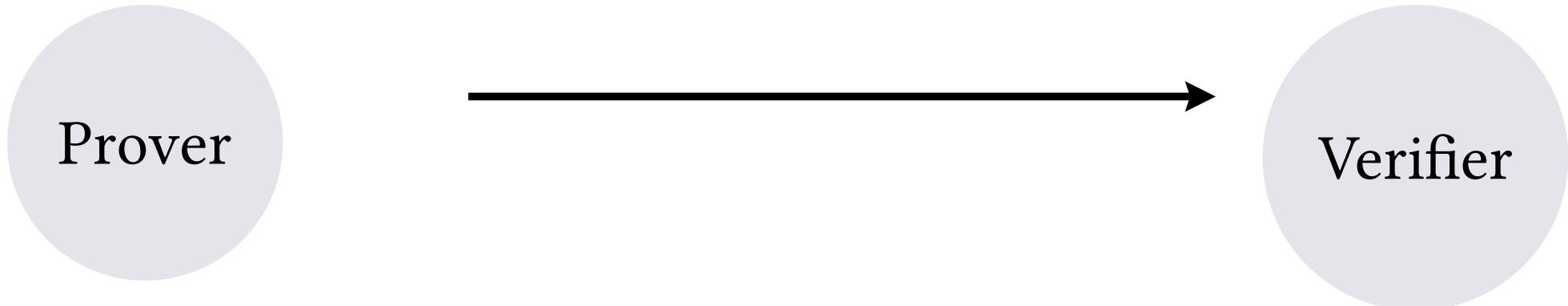


$O(n)$  off-chain

$O(\log n)$  on-chain

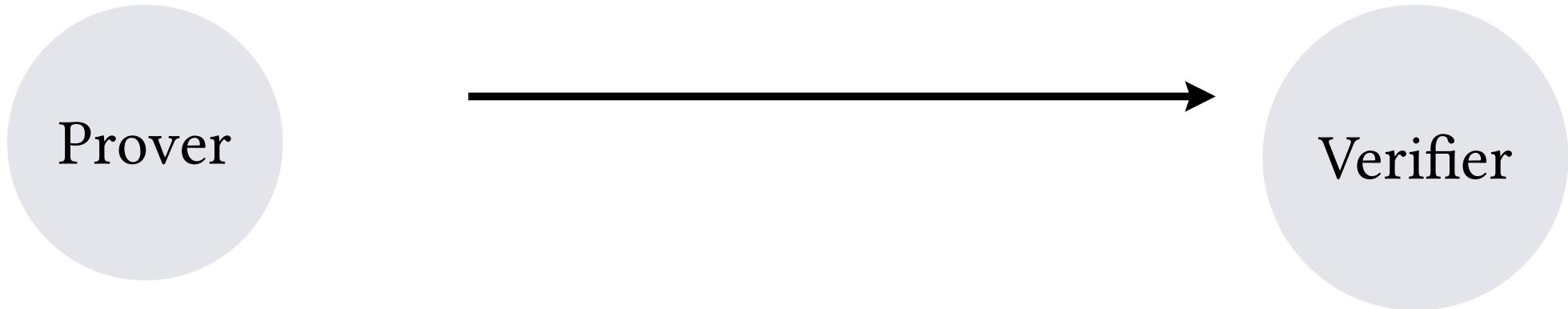
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Data sent to verifier is compressed, and can be hidden



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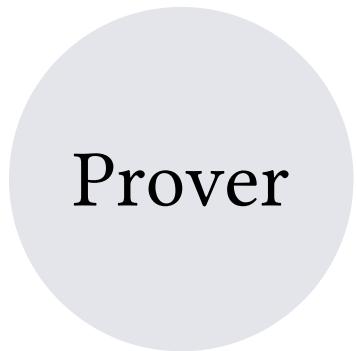
Data sent to verifier is compressed, and can be hidden



size: n

# ZKP (3)

Data sent to verifier is compressed, and can be hidden



Prover



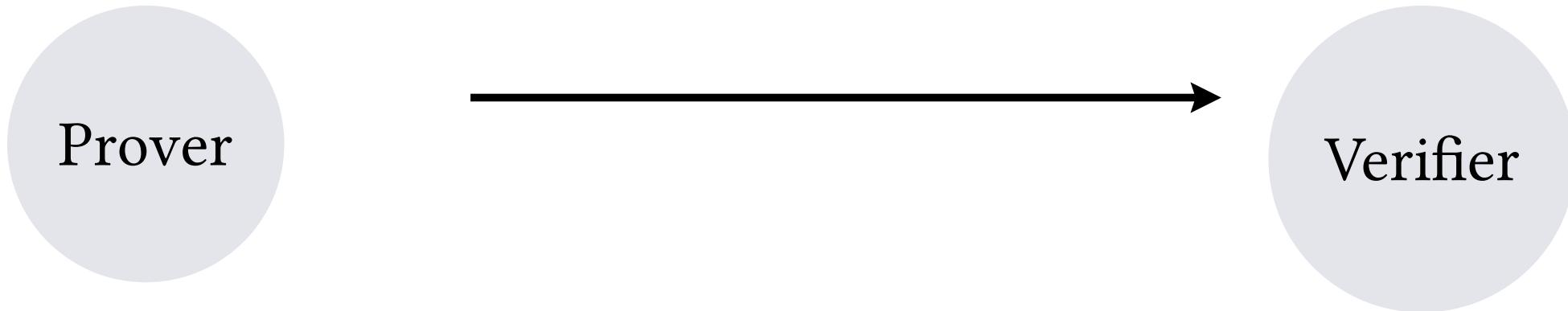
Verifier

size: n

size: at least  $\log n$

# ZKP (3)

Data sent to verifier is compressed, and can be hidden



size: n

size: at least  $\log n$

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# Modular arithmetics (1)

It is about integers.

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Let's assume we arithmetics **mod 8**. It means the possible values are  $0, 1, 2, 3, 4, 5, 6, 7$ . if we move below or above we need to wrap up.

# Modular arithmetics (1)

$$3 + 3 \bmod 8 = 6 \bmod 8$$

$$10 \bmod 8 = 2 \bmod 8$$

$$5 + 5 \bmod 8 = 2 \bmod 8$$

$$5 \cdot 5 \bmod 8 = 25 \bmod 8 = (3 \cdot 8 + 1) \bmod 8 = 1 \bmod 8$$

# Modular arithmetics (1)

$$3 + 3 \bmod 8 = 6 \bmod 8$$

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$$5 \cdot 5 \bmod 8 = 25 \bmod 8 = (3 \cdot 8 + 1) \bmod 8 = 1 \bmod 8$$

congruent groups

# Modular arithmetics (2)

addition mod 8      multiplication mod 8

0 1 2 3 4 5 6 7      1 2 3 4 5 6 7

1 2 3 4 5 6 7 0      2 4 6 0 2 4 6

2 3 4 5 6 7 0 1      3 6 1 4 7 2 5

3 4 5 6 7 0 1 2      4 0 4 0 4 0 4

4 5 6 7 0 1 2 3      5 2 7 4 1 6 3

5 6 7 0 1 2 3 4      6 4 2 0 6 4 2

6 7 0 1 2 3 4 5      7 6 5 4 3 2 1

7 0 1 2 3 4 5 6

# Modular arithmetics (2)

addition mod 8

0 1 2 3 4 5 6 7

1 2 3 4 5 6 7 0

2 3 4 5 6 7 0 1

3 4 5 6 7 0 1 2

4 5 6 7 0 1 2 3

5 6 7 0 1 2 3 4

6 7 0 1 2 3 4 5

7 0 1 2 3 4 5 6

multiplication mod 8

1 2 3 4 5 6 7

2 4 6 0 2 4 6

3 6 1 4 7 2 5

4 0 4 0 4 0 4

5 2 7 4 1 6 3

6 4 2 0 6 4 2

7 6 5 4 3 2 1

Let's solve the eq in mod 8:

$$9(2x + 7) - 6 \equiv 2x + 6$$

# Modular arithmetics (2)

addition mod 8

0 1 2 3 4 5 6 7

1 2 3 4 5 6 7 0

2 3 4 5 6 7 0 1

3 4 5 6 7 0 1 2

4 5 6 7 0 1 2 3

5 6 7 0 1 2 3 4

6 7 0 1 2 3 4 5

7 0 1 2 3 4 5 6

multiplication mod 8

1 2 3 4 5 6 7

2 4 6 0 2 4 6

3 6 1 4 7 2 5

4 0 4 0 4 0 4

5 2 7 4 1 6 3

6 4 2 0 6 4 2

7 6 5 4 3 2 1

Let's solve the eq in mod 8:

$$9(2x + 7) - 6 \equiv 2x + 6$$

$$19^*2x + 19^*7 - 6 \equiv 2x + 6$$

# Modular arithmetics (2)

addition mod 8

0 1 2 3 4 5 6 7

1 2 3 4 5 6 7 0

2 3 4 5 6 7 0 1

3 4 5 6 7 0 1 2

4 5 6 7 0 1 2 3

5 6 7 0 1 2 3 4

6 7 0 1 2 3 4 5

7 0 1 2 3 4 5 6

multiplication mod 8

1 2 3 4 5 6 7

2 4 6 0 2 4 6

3 6 1 4 7 2 5

4 0 4 0 4 0 4

5 2 7 4 1 6 3

6 4 2 0 6 4 2

7 6 5 4 3 2 1

Let's solve the eq in mod 8:

$$9(2x + 7) - 6 \equiv 2x + 6$$

$$19^*2x + 19^*7 - 6 \equiv 2x + 6$$

$$38x + 133 - 6 \equiv 2x + 6 \quad \# \quad 133 \text{ mod } 8 = 5$$

# Modular arithmetics (2)

addition mod 8

0 1 2 3 4 5 6 7

1 2 3 4 5 6 7 0

2 3 4 5 6 7 0 1

3 4 5 6 7 0 1 2

4 5 6 7 0 1 2 3

5 6 7 0 1 2 3 4

6 7 0 1 2 3 4 5

7 0 1 2 3 4 5 6

multiplication mod 8

1 2 3 4 5 6 7

2 4 6 0 2 4 6

3 6 1 4 7 2 5

4 0 4 0 4 0 4

5 2 7 4 1 6 3

6 4 2 0 6 4 2

7 6 5 4 3 2 1

Let's solve the eq in mod 8:

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$$19^*2x + 19^*7 - 6 \equiv 2x + 6$$

$$38x + 133 - 6 \equiv 2x + 6 \quad \# \text{ } 133 \text{ mod } 8 = 5$$

$$6x + 5 - 6 \equiv 2x + 6$$

# Modular arithmetics (2)

addition mod 8

0 1 2 3 4 5 6 7

1 2 3 4 5 6 7 0

2 3 4 5 6 7 0 1

3 4 5 6 7 0 1 2

4 5 6 7 0 1 2 3

5 6 7 0 1 2 3 4

6 7 0 1 2 3 4 5

7 0 1 2 3 4 5 6

multiplication mod 8

1 2 3 4 5 6 7

2 4 6 0 2 4 6

3 6 1 4 7 2 5

4 0 4 0 4 0 4

5 2 7 4 1 6 3

6 4 2 0 6 4 2

7 6 5 4 3 2 1

Let's solve the eq in mod 8:

$$9(2x + 7) - 6 \equiv 2x + 6$$

$$19^*2x + 19^*7 - 6 \equiv 2x + 6$$

$$38x + 133 - 6 \equiv 2x + 6 \quad \# \text{ } 133 \text{ mod } 8 = 5$$

$$6x + 5 - 6 \equiv 2x + 6$$

$$6x + 5 - 6 + 6 \equiv 2x + 6 + 6$$

# Modular arithmetics (2)

addition mod 8

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

multiplication mod 8

1	2	3	4	5	6	7
2	4	6	0	2	4	6
3	6	1	4	7	2	5
4	0	4	0	4	0	4
5	2	7	4	1	6	3
6	4	2	0	6	4	2
7	6	5	4	3	2	1

Let's solve the eq in mod 8:

$$\begin{aligned} 9(2x + 7) - 6 &\equiv 2x + 6 \\ 19^*2x + 19^*7 - 6 &\equiv 2x + 6 \\ 38x + 133 - 6 &\equiv 2x + 6 \quad \# 133 \text{ mod } 8 = 5 \\ 6x + 5 - 6 &\equiv 2x + 6 \\ 6x + 5 - 6 + 6 &\equiv 2x + 6 + 6 \\ 6x + 5 &\equiv 2x + 4 \quad \# 12 \text{ mod } 8 = 4 \\ 6x - 2x + 5 - 5 &\equiv 2x - 2x + 4 - 5 \end{aligned}$$

# Modular arithmetics (2)

addition mod 8

0 1 2 3 4 5 6 7

1 2 3 4 5 6 7 0

2 3 4 5 6 7 0 1

3 4 5 6 7 0 1 2

4 5 6 7 0 1 2 3

5 6 7 0 1 2 3 4

6 7 0 1 2 3 4 5

7 0 1 2 3 4 5 6

multiplication mod 8

1 2 3 4 5 6 7

2 4 6 0 2 4 6

3 6 1 4 7 2 5

4 0 4 0 4 0 4

5 2 7 4 1 6 3

6 4 2 0 6 4 2

7 6 5 4 3 2 1

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$$19^*2x + 19^*7 - 6 \equiv 2x + 6$$

$$38x + 133 - 6 \equiv 2x + 6 \ # 133 \text{ mod } 8 = 5$$

$$6x + 5 - 6 \equiv 2x + 6$$

$$6x + 5 - 6 + 6 \equiv 2x + 6 + 6$$

$$6x + 5 \equiv 2x + 4 \ # 12 \text{ mod } 8 = 4$$

$$6x - 2x + 5 - 5 \equiv 2x - 2x + 4 - 5$$

$$4x \equiv 7 \ # -1 \text{ mod } 8 = 7$$

Now we do **NOT have multiplication inverse** for 4, ie. we cannot divide by 4 in modulo 8, ie. solve this equation We have only multiplication inverse for 1 which is 1; 3 which is 3; 5 which is 5, and 7 which is 7.

# Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

# Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$$19(2x + 7) - 6 \equiv 2x + 6$$

# Modular arithmetics (3)

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0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

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1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

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$$5x + 1 - 6 \equiv 2x + 6$$

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0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$$\begin{aligned}19(2x + 7) - 6 &\equiv 2x + 6 \\19^*2x + 19^*7 - 6 &\equiv 2x + 6 \\5x + 1 - 6 &\equiv 2x + 6 \\5x + 1 - 6 + 6 &\equiv 2x + 6 + 6\end{aligned}$$

# Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

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# Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$$\begin{aligned}19(2x + 7) - 6 &\equiv 2x + 6 \\19^*2x + 19^*7 - 6 &\equiv 2x + 6 \\5x + 1 - 6 &\equiv 2x + 6 \\5x + 1 - 6 + 6 &\equiv 2x + 6 + 6 \\5x + 1 &\equiv 2x + 1 \\x &\equiv 0\end{aligned}$$

We have solution:  $\{.., -22, -11, 0, 11, 22, \dots\}$  As in each row of mult table there is 1 we have inverse for each congruence group!

# Modular arithmetics (4)

addition mod 13

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
3	4	5	6	7	8	9	10	11	12	0	1	2
4	5	6	7	8	9	10	11	12	0	1	2	3
5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5
7	8	9	10	11	12	0	1	2	3	4	5	6
8	9	10	11	12	0	1	2	3	4	5	6	7
9	10	11	12	0	1	2	3	4	5	6	7	8
10	11	12	0	1	2	3	4	5	6	7	8	9
11	12	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

As in each row of mult table there is 1 we have inverse for each congruence group! => we want to work with mod PRIME as we want inverses!

# Modular arithmetics (4)

addition mod 13

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
3	4	5	6	7	8	9	10	11	12	0	1	2
4	5	6	7	8	9	10	11	12	0	1	2	3
5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5
7	8	9	10	11	12	0	1	2	3	4	5	6
8	9	10	11	12	0	1	2	3	4	5	6	7
9	10	11	12	0	1	2	3	4	5	6	7	8
10	11	12	0	1	2	3	4	5	6	7	8	9
11	12	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

diagonal: 1 2 3 4 5 6 7 8 9 10 11 12

val: 1 4 9 3 12 10 10 12 3 9 4 1

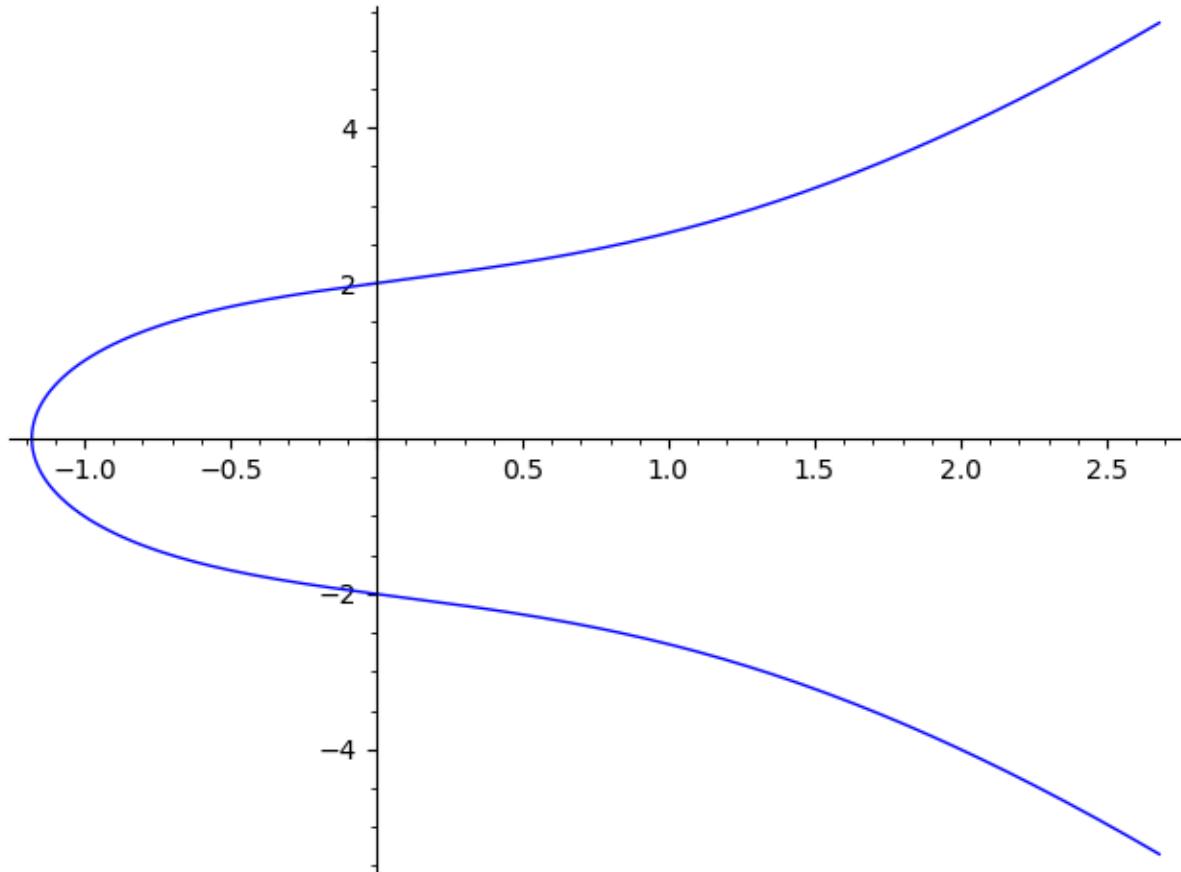
- not always square  
is possible within  
modulus
- 1, 3, 4, 9, 10, 12

are quadratic  
residues

# Elliptic curves (1)

```
sage: # Let's plot the following  
elliptic curve in R  
sage: #  $y^2 == x^3 + 2x + 4$  in R
```

```
sage: E = EllipticCurve([2,4]);  
sage: P = E.plot()  
sage: P.save("ellipticR.png")
```



# Elliptic curves (2)

$$y^2 == x^3 + 2x + 4 \text{ in mod } 13$$

$x=0, y^2 = 4 \Rightarrow (0,2) \text{ and } (0,11) \rightarrow$  two points

$x=1, y^2 = 1 + 2 + 4 = 7 \rightarrow$  no point

$x=2, y^2 = 8 + 4 + 4 = 16 \text{ mod } 13 = 3 \rightarrow (2,4) \text{ and } (2,9)$

$x=3, y^2 = 1 + 6 + 4 = 11 \rightarrow$  no point

$x=4, y^2 = 12 + 8 + 4 = 24 \text{ mod } 13 = 11 \rightarrow$  no point

$x=5, y^2 = 8 + 10 + 4 = 9 \rightarrow (5,3) \text{ and } (5,10)$

$x=6, y^2 = 8 + 12 + 4 = 24 \text{ mod } 13 = 11 \rightarrow$  no point

$x=7, y^2 = 5 + 14 + 4 = 23 \text{ mod } 13 = 10 \rightarrow (7,6) \text{ and } (7,7)$

$x=8, y^2 = 5 + 16 + 4 = 25 \text{ mod } 13 = 12 \rightarrow (8,5) \text{ and } (8,8)$

$x=9, y^2 = 1 + 18 + 4 = 23 \text{ mod } 13 = 10 \rightarrow (9,6) \text{ and } (9,7)$

$x=10, y^2 = 12 + 20 + 4 = 36 \text{ mod } 13 = 10 \rightarrow (10,6) \text{ and } (10,7)$

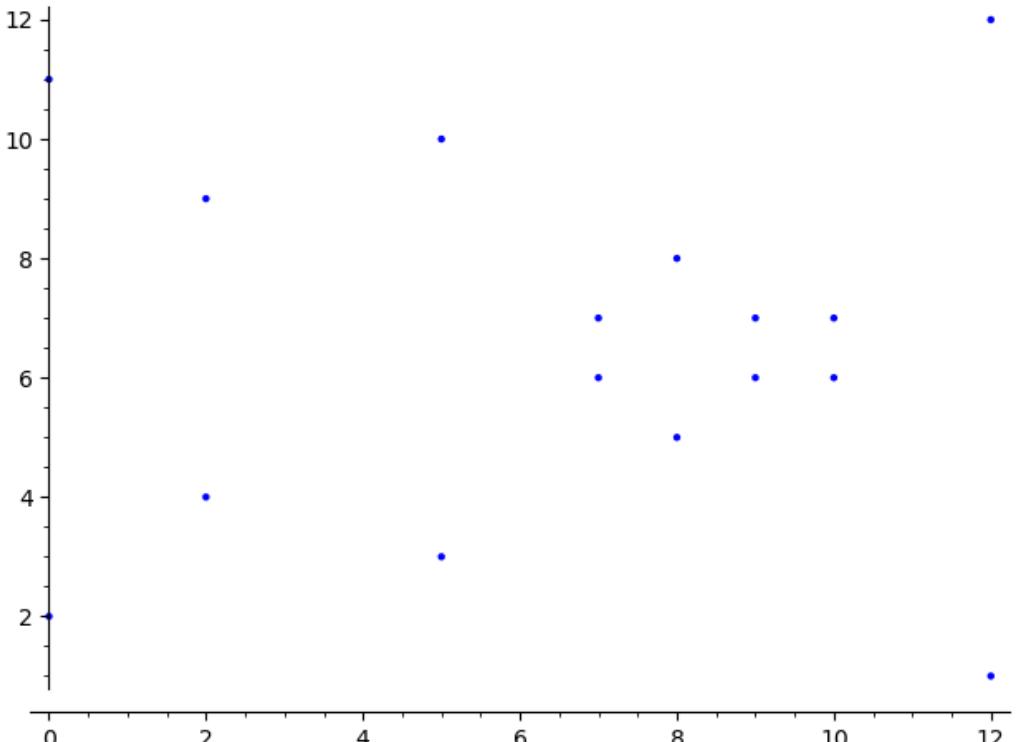
$x=11, y^2 = 5 + 22 + 4 = 31 \text{ mod } 13 = 5 \rightarrow$  no point

$x=12, y^2 = 12 + 24 + 4 = 40 \text{ mod } 13 = 1 \rightarrow (12,1) \text{ and } (12,12)$

For some  $x$  there are no solutions when we have mod 13, for the rest we have two!

# Elliptic curves (3)

```
sage: F13=GF(13)
sage: a = F13(2)
sage: b = F13(4)
sage: # discriminant obeys condition
sage: F13(6)*(F13(4)^3+F13(27)*b^2) != F13(0)
True
sage: E = EllipticCurve(F13,[a,b]) #  $y^2 == x^3 + 2x + 4$ 
sage: P = E(0,2) #  $2^2 == 0^3 + 2 \cdot 0 + 4 \pmod{13}$ 
sage: P.xy()
(0, 2)
sage: INF=E(0)
sage: try:
....:     INF.xy()
....: except ZeroDivisionError:
....:     pass
....:
sage: P = E.plot()
sage: P.save("elliptic13.png")
```



# Modular arithmetics (5)

addition mod 13

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
3	4	5	6	7	8	9	10	11	12	0	1	2
4	5	6	7	8	9	10	11	12	0	1	2	3
5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5
7	8	9	10	11	12	0	1	2	3	4	5	6
8	9	10	11	12	0	1	2	3	4	5	6	7
9	10	11	12	0	1	2	3	4	5	6	7	8
10	11	12	0	1	2	3	4	5	6	7	8	9
11	12	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11

Addition forms a **group** as

- 0 is identity element
- addition is associative op
- addition is closed op
- each element has the inverse

# Modular arithmetics (5)

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

Mult forms a **group** as

- 1 is identity element
- mult is associative op
- mult is closed op
- each element has the inverse  
(thanks to p being prime)

# Modular arithmetics (5)

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

- $3 \cdot 3 = 9$
- $3 \cdot 3 \cdot 3 = 1$
- $3 \cdot 3 \cdot 3 \cdot 3 = 3$

we can NOT generate EACH element -> 3 is not generator

# Modular arithmetics (5)

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

- $8 \cdot 8 = 12$
- $8 \cdot 8 \cdot 8 = 5$
- $8 \cdot 8 \cdot 8 \cdot 8 = 1$

we can NOT generate EACH element -> 8 is not generator

# Modular arithmetics (5)

multiplication mod 13

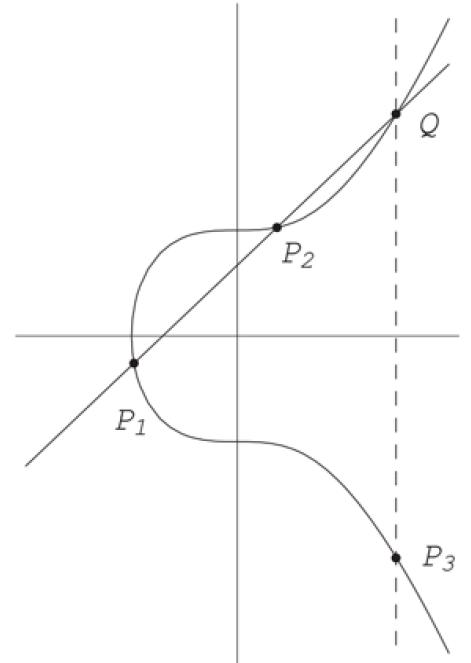
1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

- $2 \cdot 2 = 4$
- $2 \cdot 2 \cdot 2 = 8$
- $2 \cdot 2 \cdot 2 \cdot 2 = 3$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 6$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 12$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 11$
- $2 \cdot 2 = 9$
- $2 \cdot 2 = 5$
- $2 \cdot 2 = 10$
- $2 \cdot 2 = 7$
- $2 \cdot 2 = 1$

we can generate EVERY element -> 2 is **generator**, group is **cyclic**

# Elliptic curves (4)

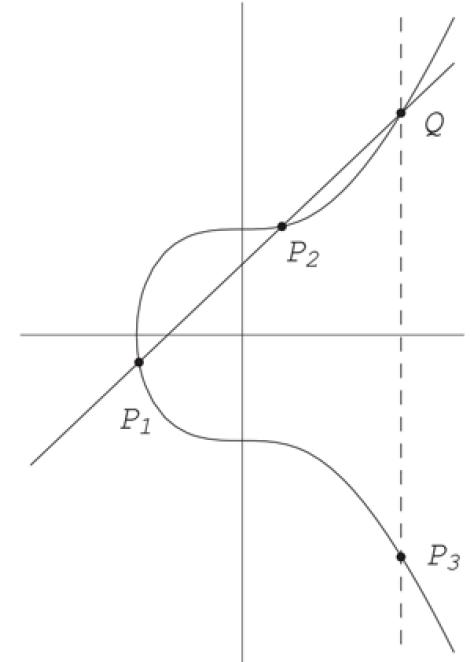
Visually addition looks like here for elliptic curves with one remark: it is modulo PRIME



# Elliptic curves (4)

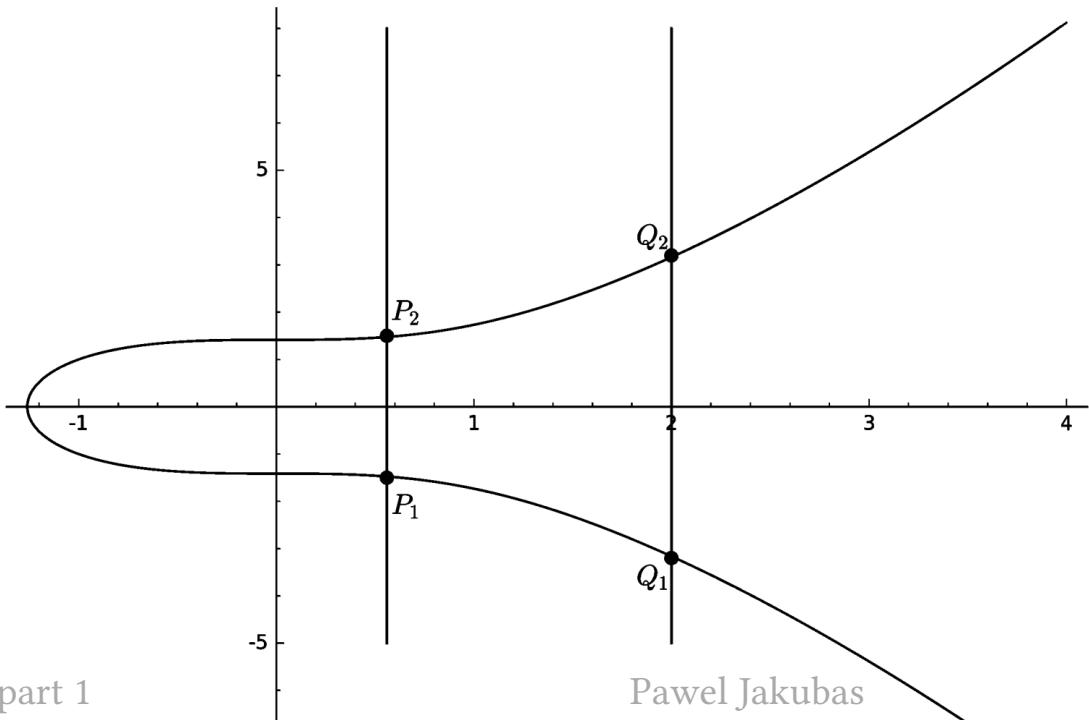
It is called **Chord-and-tangent rule**

visually  $P_1 + P_2 = P_3$



# Elliptic curves (4)

$y=INF$  is identity element, and because of that  $P_1^{-1} = P_2$   
 $Q_1^{-1} = Q_2$



# Elliptic curves (4)

## Chord-and-tangent rule algebraically

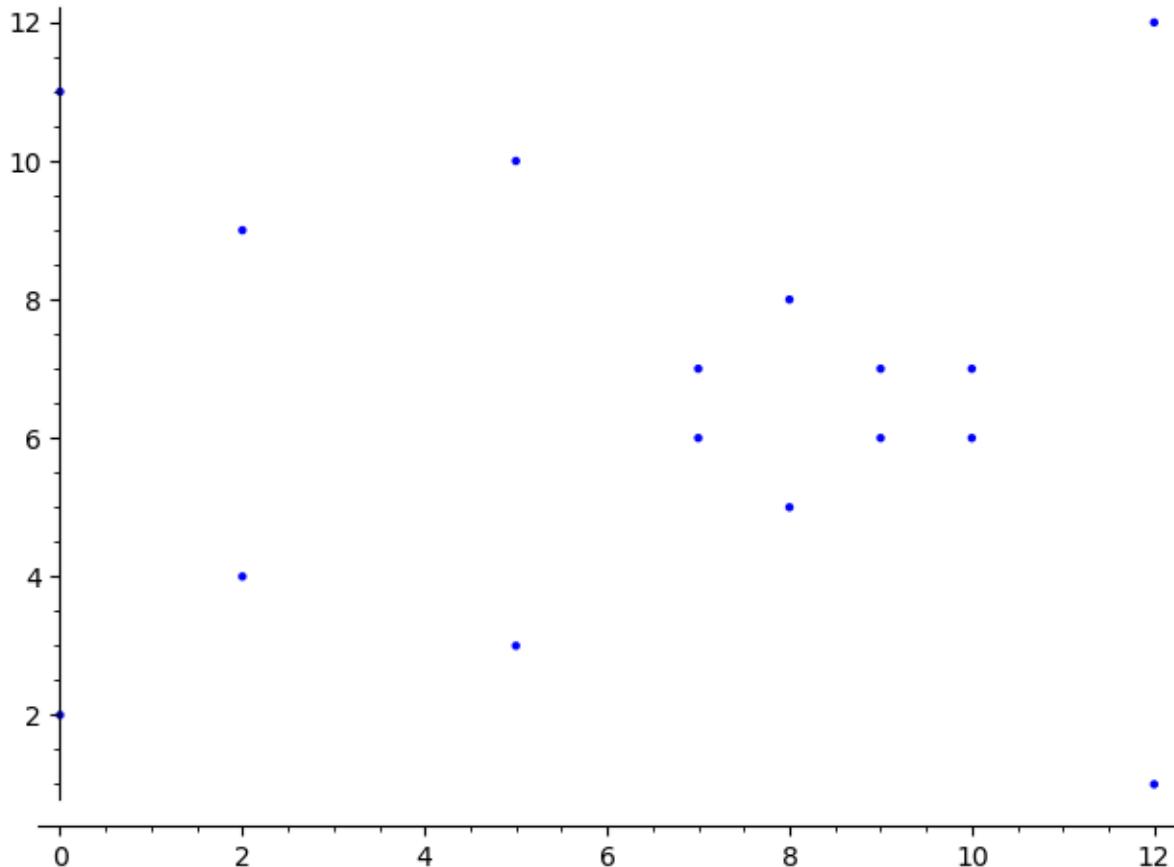
(Tangent Rule) If  $P = (x, y)$  with  $y \neq 0$ , the group law  $P \oplus P = (x', y')$  is defined as follows:

$$x' = \left( \frac{3x^2 + a}{2y} \right)^2 - 2x \quad , \quad y' = \left( \frac{3x^2 + a}{2y} \right) (x - x') - y$$

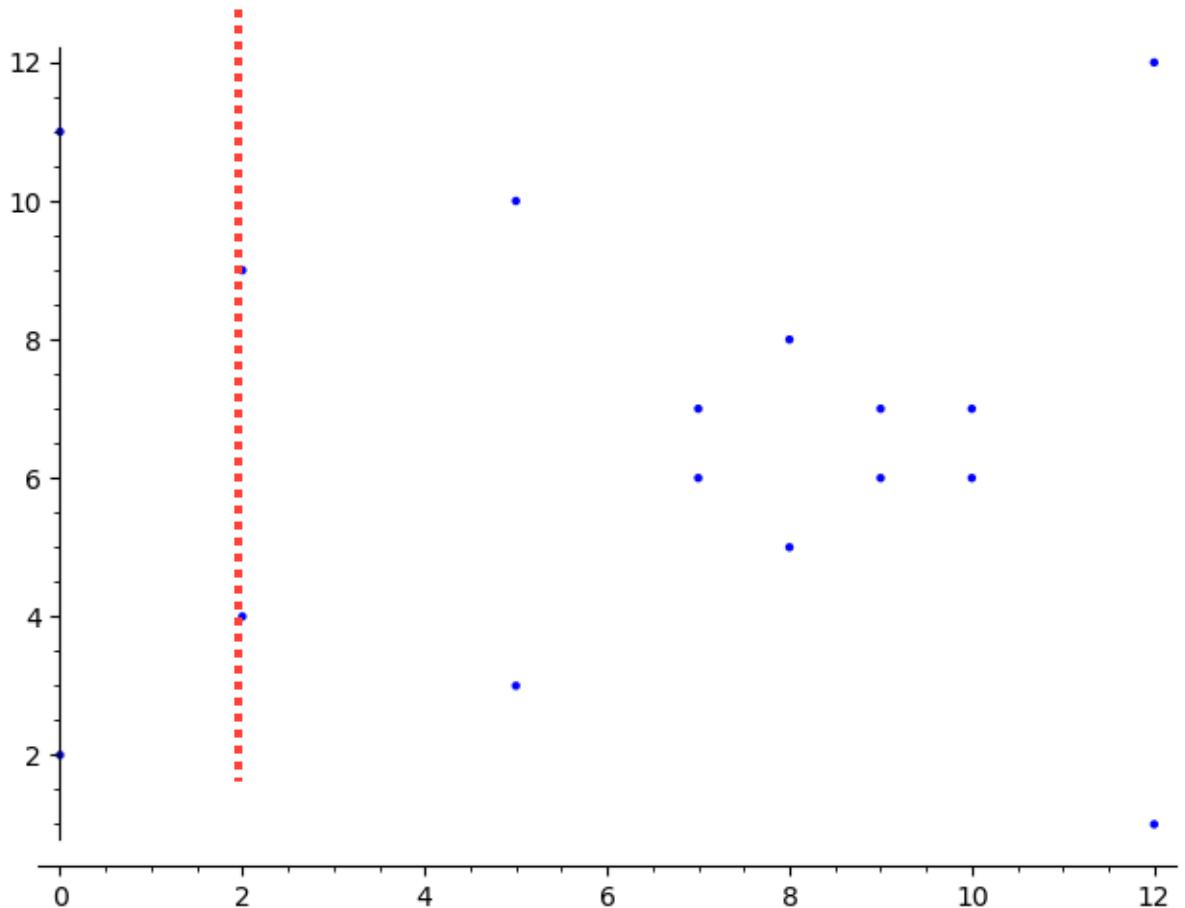
(Chord Rule) If  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  such that  $x_1 \neq x_2$ , the group law  $R = P \oplus Q$  with  $R = (x_3, y_3)$  is defined as follows:

$$x_3 = \left( \frac{y_2 - y_1}{x_2 - x_1} \right)^2 - x_1 - x_2 \quad , \quad y_3 = \left( \frac{y_2 - y_1}{x_2 - x_1} \right) (x_1 - x_3) - y_1$$

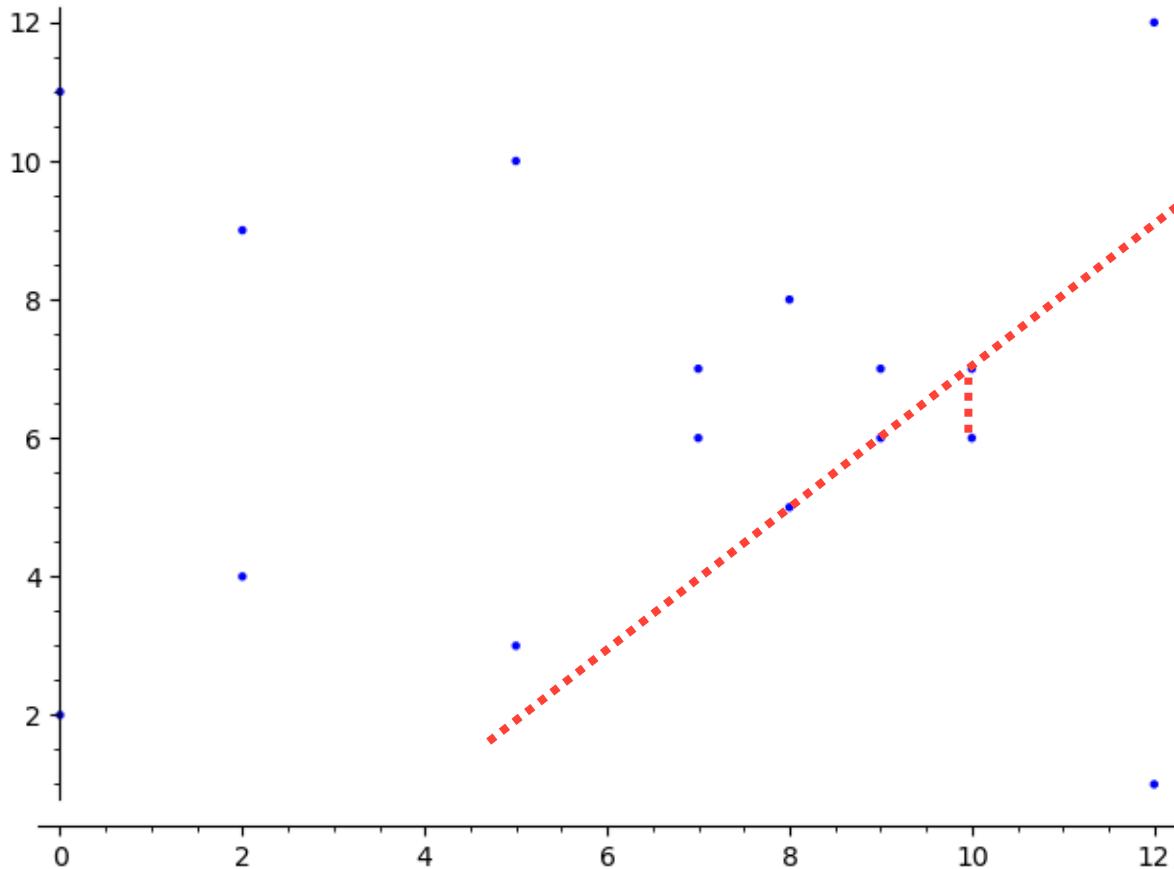
# Elliptic curves (5) (mod 13)



# Elliptic curves (5) (mod 13)



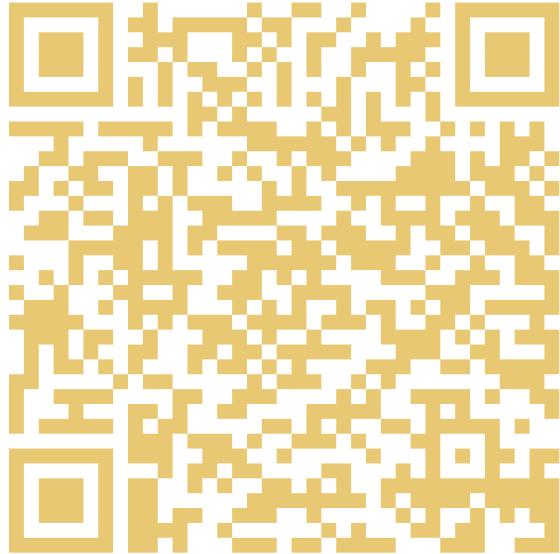
# Elliptic curves (5) (mod 13)



# Elliptic curves (6) (mod 13)

```
sage: F13 = GF(13)
sage: a = F13(2)
sage: b = F13(4)
sage: E = EllipticCurve(F13,[a,b]) # y^2 == x^3 + 2x + 4
sage: INF=E(0)
sage: E(2,4) + E(2,9) == INF
True
sage: E(8,5) + E(9,6) == E(10,7)
False
sage: E(8,5) + E(9,6) == E(10,6)
True
```

# That's it! More to come in the future



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