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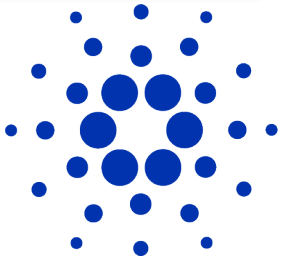

Zero-knowledge-proofs - part 1

Engineering workshop - Dec 2025 (Ireland)

Pawel Jakubas

Plan of the tutorial

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Let's get a little deeper than usual and understand what main building blocks of ZKP looks like

This tutorial will focus on

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1. sketching the landscape of what we want to understand during 3-4 parts
2.

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Let's get a little deeper than usual and understand what main building blocks of ZKP looks like

This tutorial will focus on

1. sketching the landscape of what we want to understand during 3-4 parts
2. cover the first part in some detail **elliptic curves**

Verifiable computing vs ZKP (1)

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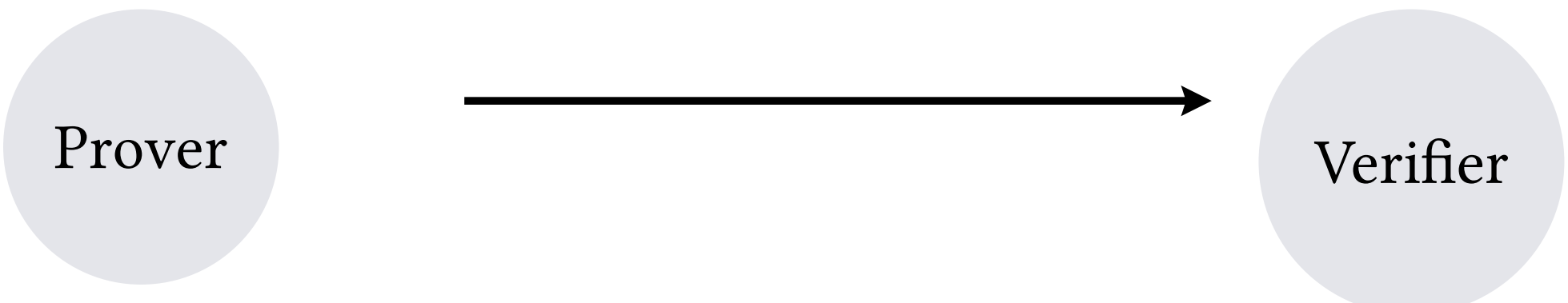
There is **asymmetry** built into those systems. It is much easy to get public key from secret. But not the other way



Verifiable computing vs ZKP (1)

There is **asymmetry** built into those systems. It is much easy to get public key from secret. But not the other way

secret -> (easy) -> public

Verifiable computing vs ZKP (1)

There is **asymmetry** built into those systems. It is much easy to get public key from secret. But not the other way

proof + public data shared

Verifiable computing vs ZKP (1)

There is **asymmetry** built into those systems. It is much easy to get public key from secret. But not the other way

public -> (very hard) -> secret

Verifiable computing vs ZKP (1)

There is **asymmetry** built into those systems. It is much easy to get public key from secret. But not the other way

proof verified using public data

Verifiable computing vs ZKP (2)

There is **asymmetry** built into those systems. It is much quicker to verify than prove something.

Verifiable computing vs ZKP (2)

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O(n) off-chain

Verifiable computing vs ZKP (2)

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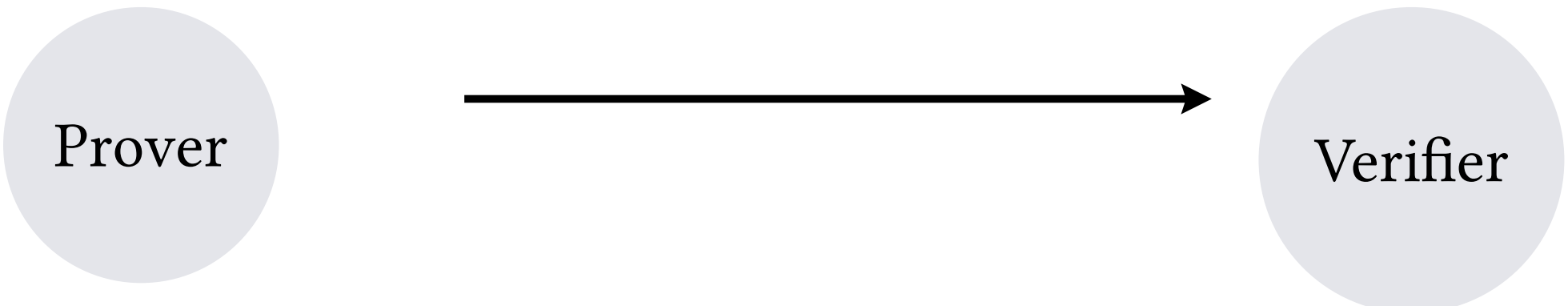
O(n) off-chain

O(log n) on-chain

ZKP (3)

Data sent to verifier is compressed, and can be hidden



ZKP (3)

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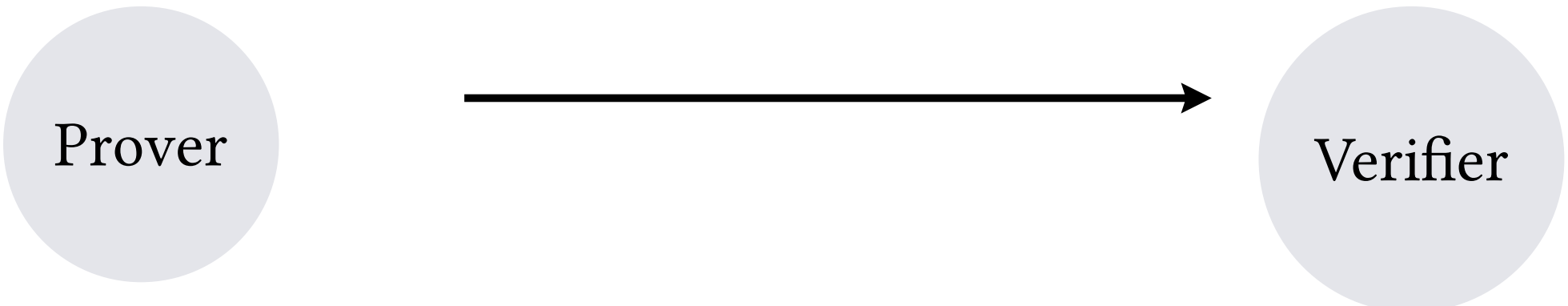

size: n

ZKP (3)

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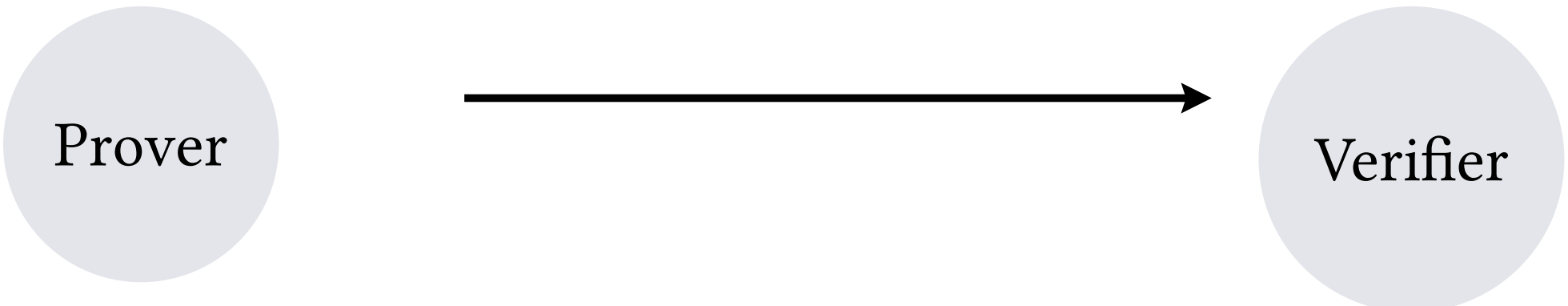

size: n

size: at least log n

ZKP (3)

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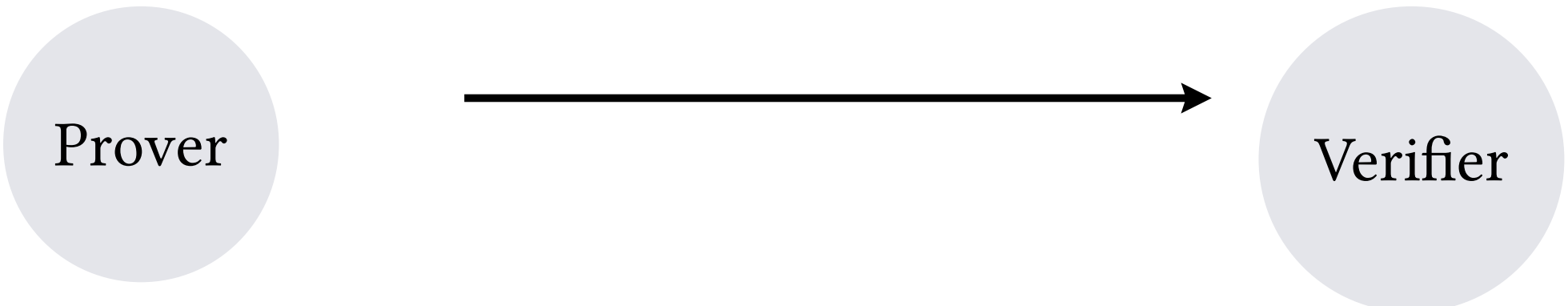

size: n

size: at least log n

size: at least log n

Modular arithmetics (1)

It is about integers.

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Let's assume we arithmetics **mod 8**. It means the possible values are 0,1,2,3,4,5,6,7. if we move below or above we need to wrap up.

Modular arithmetics (1)

$$3 + 3 \bmod 8 = 6 \bmod 8$$

$$10 \bmod 8 = 2 \bmod 8$$

$$5 + 5 \bmod 8 = 2 \bmod 8$$

$$5 \cdot 5 \bmod 8 = 25 \bmod 8 = (3 \cdot 8 + 1) \bmod 8 = 1 \bmod 8$$

Modular arithmetics (1)

$$3 + 3 \bmod 8 = 6 \bmod 8$$

$$10 \bmod 8 = 2 \bmod 8$$

$$5 + 5 \bmod 8 = 2 \bmod 8$$

$$5 \cdot 5 \bmod 8 = 25 \bmod 8 = (3 \cdot 8 + 1) \bmod 8 = 1 \bmod 8$$

congruent groups

Modular arithmetics (2)

addition mod 8								multiplication mod 8							
0	1	2	3	4	5	6	7	1	2	3	4	5	6	7	
1	2	3	4	5	6	7	0	2	4	6	0	2	4	6	
2	3	4	5	6	7	0	1	3	6	1	4	7	2	5	
3	4	5	6	7	0	1	2	4	0	4	0	4	0	4	
4	5	6	7	0	1	2	3	5	2	7	4	1	6	3	
5	6	7	0	1	2	3	4	6	4	2	0	6	4	2	
6	7	0	1	2	3	4	5	7	6	5	4	3	2	1	
7	0	1	2	3	4	5	6								

Modular arithmetics (2)

addition mod 8

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

multiplication mod 8

1	2	3	4	5	6	7	
2	4	6	0	2	4	6	
3	6	1	4	7	2	5	
4	0	4	0	4	0	4	
5	2	7	4	1	6	3	
6	4	2	0	6	4	2	
7	6	5	4	3	2	1	

Let's solve the eq in mod 8:

$$9(2x + 7) - 6 \equiv 2x + 6$$

Modular arithmetics (2)

addition mod 8

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

multiplication mod 8

1	2	3	4	5	6	7	
2	4	6	0	2	4	6	
3	6	1	4	7	2	5	
4	0	4	0	4	0	4	
5	2	7	4	1	6	3	
6	4	2	0	6	4	2	
7	6	5	4	3	2	1	

Let's solve the eq in mod 8:

$9(2x + 7) - 6 \equiv 2x + 6$

$19^*2x + 19^*7 - 6 \equiv 2x + 6$

Modular arithmetics (2)

addition mod 8

```
0 1 2 3 4 5 6 7
1 2 3 4 5 6 7 0
2 3 4 5 6 7 0 1
3 4 5 6 7 0 1 2
4 5 6 7 0 1 2 3
5 6 7 0 1 2 3 4
6 7 0 1 2 3 4 5
7 0 1 2 3 4 5 6
```

multiplication mod 8

```
1 2 3 4 5 6 7
2 4 6 0 2 4 6
3 6 1 4 7 2 5
4 0 4 0 4 0 4
5 2 7 4 1 6 3
6 4 2 0 6 4 2
7 6 5 4 3 2 1
```

Let's solve the eq in mod 8:

$9(2x + 7) - 6 \equiv 2x + 6$

$19^*2x + 19^*7 - 6 \equiv 2x + 6$

$38x + 133 - 6 \equiv 2x + 6$ # 133 mod 8 = 5

Modular arithmetics (2)

addition mod 8

```
0 1 2 3 4 5 6 7
1 2 3 4 5 6 7 0
2 3 4 5 6 7 0 1
3 4 5 6 7 0 1 2
4 5 6 7 0 1 2 3
5 6 7 0 1 2 3 4
6 7 0 1 2 3 4 5
7 0 1 2 3 4 5 6
```

multiplication mod 8

```
1 2 3 4 5 6 7
2 4 6 0 2 4 6
3 6 1 4 7 2 5
4 0 4 0 4 0 4
5 2 7 4 1 6 3
6 4 2 0 6 4 2
7 6 5 4 3 2 1
```

Let's solve the eq in mod 8:

$9(2x + 7) - 6 \equiv 2x + 6$

$19^*2x + 19^*7 - 6 \equiv 2x + 6$

$38x + 133 - 6 \equiv 2x + 6$ # 133 mod 8 = 5

$6x + 5 - 6 \equiv 2x + 6$

Modular arithmetics (2)

addition mod 8

```
0 1 2 3 4 5 6 7
1 2 3 4 5 6 7 0
2 3 4 5 6 7 0 1
3 4 5 6 7 0 1 2
4 5 6 7 0 1 2 3
5 6 7 0 1 2 3 4
6 7 0 1 2 3 4 5
7 0 1 2 3 4 5 6
```

multiplication mod 8

```
1 2 3 4 5 6 7
2 4 6 0 2 4 6
3 6 1 4 7 2 5
4 0 4 0 4 0 4
5 2 7 4 1 6 3
6 4 2 0 6 4 2
7 6 5 4 3 2 1
```

Let's solve the eq in mod 8:

$$9(2x + 7) - 6 \equiv 2x + 6$$

$$19^*2x + 19^*7 - 6 \equiv 2x + 6$$

$$38x + 133 - 6 \equiv 2x + 6 \quad \# 133 \bmod 8 = 5$$

$$6x + 5 - 6 \equiv 2x + 6$$

$$6x + 5 - 6 + 6 \equiv 2x + 6 + 6$$

Modular arithmetics (2)

addition mod 8

```
0 1 2 3 4 5 6 7
1 2 3 4 5 6 7 0
2 3 4 5 6 7 0 1
3 4 5 6 7 0 1 2
4 5 6 7 0 1 2 3
5 6 7 0 1 2 3 4
6 7 0 1 2 3 4 5
7 0 1 2 3 4 5 6
```

multiplication mod 8

```
1 2 3 4 5 6 7
2 4 6 0 2 4 6
3 6 1 4 7 2 5
4 0 4 0 4 0 4
5 2 7 4 1 6 3
6 4 2 0 6 4 2
7 6 5 4 3 2 1
```

Let's solve the eq in mod 8:

$$9(2x + 7) - 6 \equiv 2x + 6$$

$$19^*2x + 19^*7 - 6 \equiv 2x + 6$$

$$38x + 133 - 6 \equiv 2x + 6 \quad \# \ 133 \bmod 8 = 5$$

$$6x + 5 - 6 \equiv 2x + 6$$

$$6x + 5 - 6 + 6 \equiv 2x + 6 + 6$$

$$6x + 5 \equiv 2x + 4 \quad \# \ 12 \bmod 8 = 4$$

$$6x - 2x + 5 - 5 \equiv 2x - 2x + 4 - 5$$

Modular arithmetics (2)

addition mod 8

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

multiplication mod 8

1	2	3	4	5	6	7	
2	4	6	0	2	4	6	
3	6	1	4	7	2	5	
4	0	4	0	4	0	4	
5	2	7	4	1	6	3	
6	4	2	0	6	4	2	
7	6	5	4	3	2	1	

Let's solve the eq in mod 8:

$$9(2x + 7) - 6 \equiv 2x + 6$$
$$19^*2x + 19^*7 - 6 \equiv 2x + 6$$
$$38x + 133 - 6 \equiv 2x + 6 \quad \# 133 \bmod 8 = 5$$
$$6x + 5 - 6 \equiv 2x + 6$$
$$6x + 5 - 6 + 6 \equiv 2x + 6 + 6$$
$$6x + 5 \equiv 2x + 4 \quad \# 12 \bmod 8 = 4$$
$$6x - 2x + 5 - 5 \equiv 2x - 2x + 4 - 5$$
$$4x \equiv 7 \quad \# -1 \bmod 8 = 7$$

Now we do **NOT have multiplication inverse** for 4, ie. we cannot divide by 4 in modulo 8, ie. solve this equation We have only multiplication inverse for 1 which is 1; 3 which is 3; 5 which is 5, and 7 which is 7.

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$19(2x + 7) - 6 \equiv 2x + 6$

Modular arithmetics (3)

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0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

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$19(2x + 7) - 6 \equiv 2x + 6$

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Modular arithmetics (3)

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0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$19(2x + 7) - 6 \equiv 2x + 6$

$19^*2x + 19^*7 - 6 \equiv 2x + 6$

$5x + 1 - 6 \equiv 2x + 6$

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$19(2x + 7) - 6 \equiv 2x + 6$

$19^*2x + 19^*7 - 6 \equiv 2x + 6$

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$5x + 1 - 6 + 6 \equiv 2x + 6 + 6$

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

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$5x + 1 - 6 \equiv 2x + 6$

$5x + 1 - 6 + 6 \equiv 2x + 6 + 6$

$5x + 1 \equiv 2x + 1$

Modular arithmetics (3)

addition mod 11

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	0
2	3	4	5	6	7	8	9	10	0	1
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
6	7	8	9	10	0	1	2	3	4	5
7	8	9	10	0	1	2	3	4	5	6
8	9	10	0	1	2	3	4	5	6	7
9	10	0	1	2	3	4	5	6	7	8
10	0	1	2	3	4	5	6	7	8	9

multiplication mod 11

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

Let's solve in mod 11:

$19(2x + 7) - 6 \equiv 2x + 6$

$19^*2x + 19^*7 - 6 \equiv 2x + 6$

$5x + 1 - 6 \equiv 2x + 6$

$5x + 1 - 6 + 6 \equiv 2x + 6 + 6$

$5x + 1 \equiv 2x + 1$

$x \equiv 0$

We have solution: **{.., −22, −11, 0, 11, 22, ...}** As in each row of mult table there is 1 we have inverse for each congruence group!

Modular arithmetics (4)

addition mod 13

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
3	4	5	6	7	8	9	10	11	12	0	1	2
4	5	6	7	8	9	10	11	12	0	1	2	3
5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5
7	8	9	10	11	12	0	1	2	3	4	5	6
8	9	10	11	12	0	1	2	3	4	5	6	7
9	10	11	12	0	1	2	3	4	5	6	7	8
10	11	12	0	1	2	3	4	5	6	7	8	9
11	12	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

As in each row of mult table there is 1 we have inverse for each congruence group! => we want to work with mod **PRIME** as we want inverses!

Modular arithmetics (4)

addition mod 13

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
3	4	5	6	7	8	9	10	11	12	0	1	2
4	5	6	7	8	9	10	11	12	0	1	2	3
5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5
7	8	9	10	11	12	0	1	2	3	4	5	6
8	9	10	11	12	0	1	2	3	4	5	6	7
9	10	11	12	0	1	2	3	4	5	6	7	8
10	11	12	0	1	2	3	4	5	6	7	8	9
11	12	0	1	2	3	4	5	6	7	8	9	10
12	0	1	2	3	4	5	6	7	8	9	10	11

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

diagonal:	1	2	3	4	5	6	7	8	9	10	11	12
val:	1	4	9	3	12	10	10	12	3	9	4	1

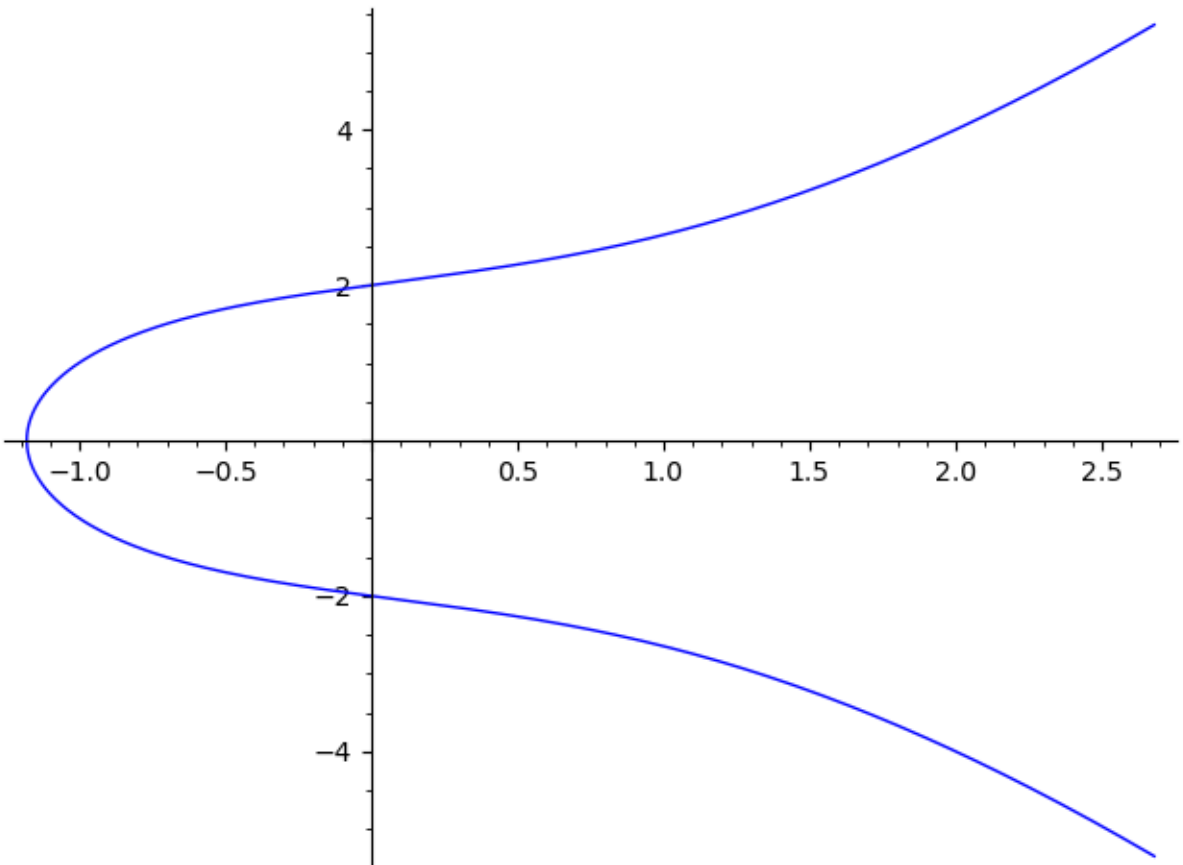
- not always square is possible within modulus
- 1, 3, 4, 9, 10, 12

  are **quadratic residues**

Elliptic curves (1)

```
sage: # Let's plot the following elliptic curve in R
sage: # y^2 == x^3 + 2x + 4 in R

sage: E = EllipticCurve([2,4]);
sage: P = E.plot()
sage: P.save("ellipticR.png")
```



Elliptic curves (2)

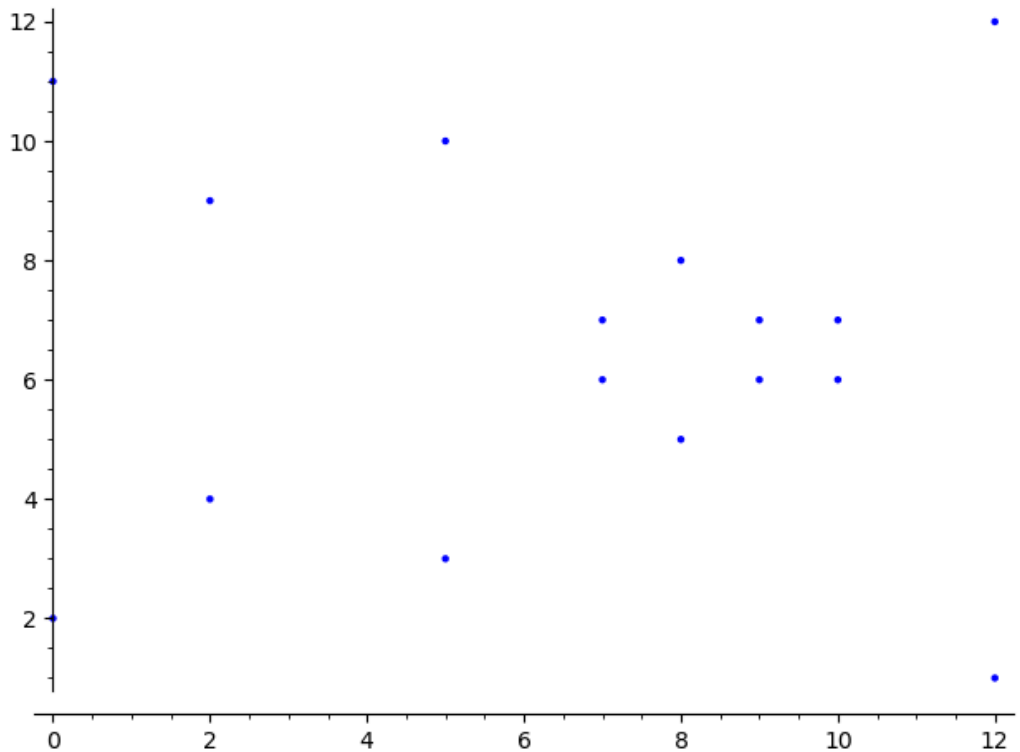
```
y^2 == x^3 + 2x + 4 in mod 13

x=0, y^2= 4 => (0,2) and (0,11) -> two points
x=1, y^2= 1 + 2 + 4 = 7 -> no point
x=2, y^2= 8 + 4 + 4 = 16 mod 13 = 3 -> (2,4) and (2,9)
x=3, y^2= 1 + 6 + 4 = 11 -> no point
x=4, y^2= 12 + 8 + 4 = 24 mod 13 = 11 -> no point
x=5, y^2= 8 + 10 + 4 = 9 -> (5,3) and (5,10)
x=6, y^2= 8 + 12 + 4 = 24 mod 13 = 11 -> no point
x=7, y^2= 5 + 14 + 4 = 23 mod 13 = 10 -> (7,6) and (7,7)
x=8, y^2= 5 + 16 + 4 = 25 mod 13 = 12 -> (8,5) and (8,8)
x=9, y^2= 1 + 18 + 4 = 23 mod 13 = 10 -> (9,6) and (9,7)
x=10, y^2= 12 + 20 + 4 = 36 mod 13 = 10 -> (10,6) and (10,7)
x=11, y^2= 5 + 22 + 4 = 31 mod 13 = 5 -> no point
x=12, y^2= 12 + 24 + 4 = 40 mod 13 = 1 -> (12,1) and (12,12)
```

For some x there are no solutions when we have mod 13, for the rest we have two!

Elliptic curves (3)

```
sage: F13=GF(13)
sage: a = F13(2)
sage: b = F13(4)
sage: # discriminant obeys condition
sage: F13(6)*(F13(4)*a^3+F13(27)*b^2) != F13(0)
True
sage: E = EllipticCurve(F13,[a,b]) # y^2 == x^3 + 2x + 4
sage: P = E(0,2) # 2^2 == 0^3 + 2*0 + 4 mod 13
sage: P.xy()
(0, 2)
sage: INF=E(0)
sage: try:
....:     INF.xy()
....: except ZeroDivisionError:
....:     pass
....:
sage: P = E.plot()
sage: P.save("elliptic13.png")
```



Modular arithmetics (5)

addition mod 13

0	1	2	3	4	5	6	7	8	9	10	11	12	
1	2	3	4	5	6	7	8	9	10	11	12	0	
2	3	4	5	6	7	8	9	10	11	12	0	1	
3	4	5	6	7	8	9	10	11	12	0	1	2	
4	5	6	7	8	9	10	11	12	0	1	2	3	
5	6	7	8	9	10	11	12	0	1	2	3	4	
6	7	8	9	10	11	12	0	1	2	3	4	5	
7	8	9	10	11	12	0	1	2	3	4	5	6	
8	9	10	11	12	0	1	2	3	4	5	6	7	
9	10	11	12	0	1	2	3	4	5	6	7	8	
10	11	12	0	1	2	3	4	5	6	7	8	9	
11	12	0	1	2	3	4	5	6	7	8	9	10	
12	0	1	2	3	4	5	6	7	8	9	10	11	

Addition forms a **group** as

- 0 is identity element
- addition is associative op
- addition is closed op
- each element has the inverse

Modular arithmetics (5)

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

Mult forms a **group** as

- 1 is identity element
- mult is associative op
- mult is closed op
- each element has the inverse (thanks to p being prime)

Modular arithmetics (5)

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

- $3 \cdot 3 = 9$
- $3 \cdot 3 \cdot 3 = 1$
- $3 \cdot 3 \cdot 3 \cdot 3 = 3$

we can NOT generate EACH element -> 3 is not generator

Modular arithmetics (5)

multiplication mod 13

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

- $8 \cdot 8 = 12$
- $8 \cdot 8 \cdot 8 = 5$
- $8 \cdot 8 \cdot 8 \cdot 8 = 1$

we can NOT generate EACH element -> 8 is not generator

Modular arithmetics (5)

multiplication mod 13

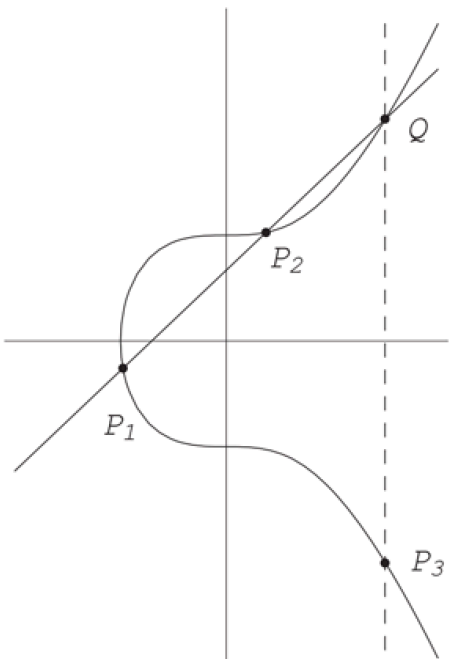
1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

- $2 \cdot 2 = 4$
- $2 \cdot 2 \cdot 2 = 8$
- $2 \cdot 2 \cdot 2 \cdot 2 = 3$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 6$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 12$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 11$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 9$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 5$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 10$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 7$
- $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 1$

we can generate EVERY element -> 2 is **generator**, group is **cyclic**

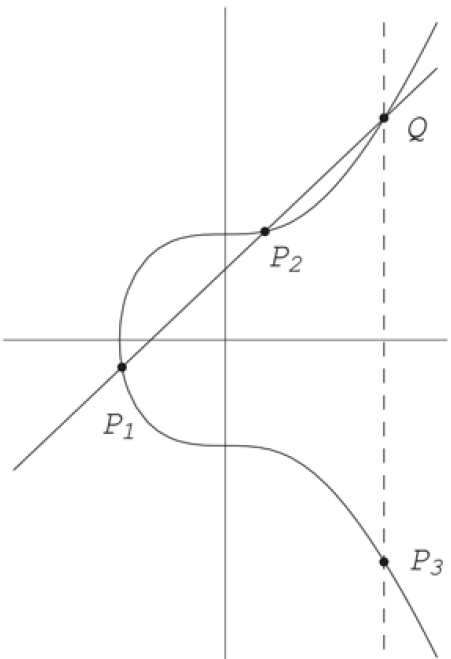
Elliptic curves (4)

Visually addition looks like here for elliptic curves with one remark: it is modulo PRIME



Elliptic curves (4)

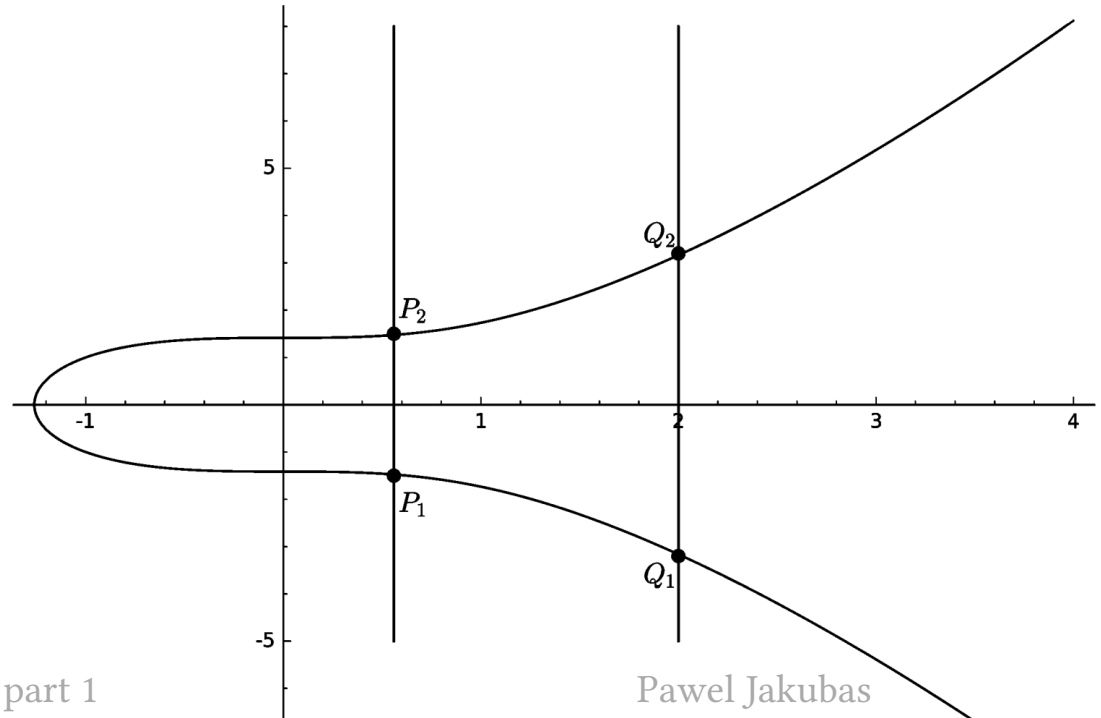
It is called **chord-and-tangent** rule and visually is looks like below $P_1 + P_2 = P_3$



Elliptic curves (4)

y=INF is identity element, and because of that $P_1^{-1} = P_2$

$Q_1^{-1} = Q_2$



Elliptic curves (4)

chord-and-tangent rule algebraically is following

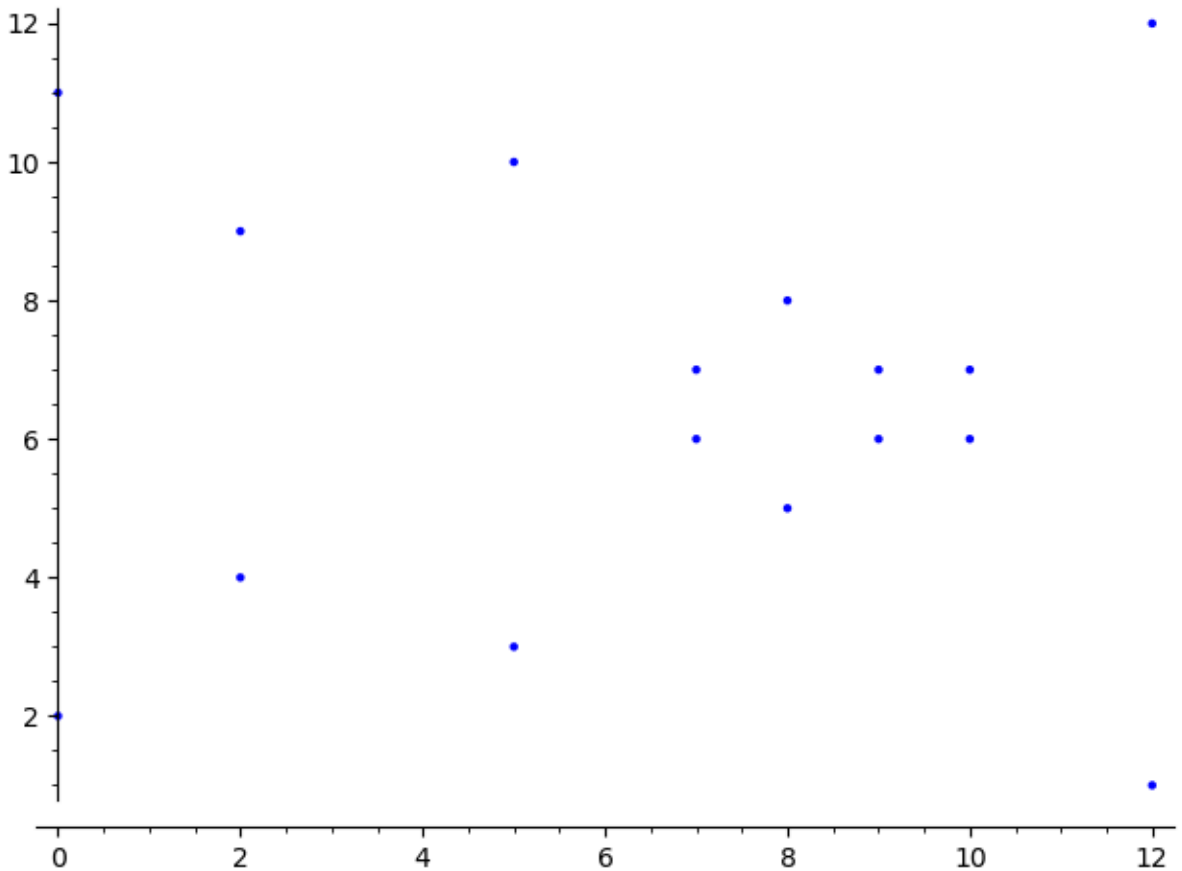
(Tangent Rule) If $P = (x, y)$ with $y \neq 0$, the group law $P \oplus P = (x', y')$ is defined as follows:

$$x' = \left(\frac{3x^2+a}{2y}\right)^2 - 2x \quad , \quad y' = \left(\frac{3x^2+a}{2y}\right)(x - x') - y$$

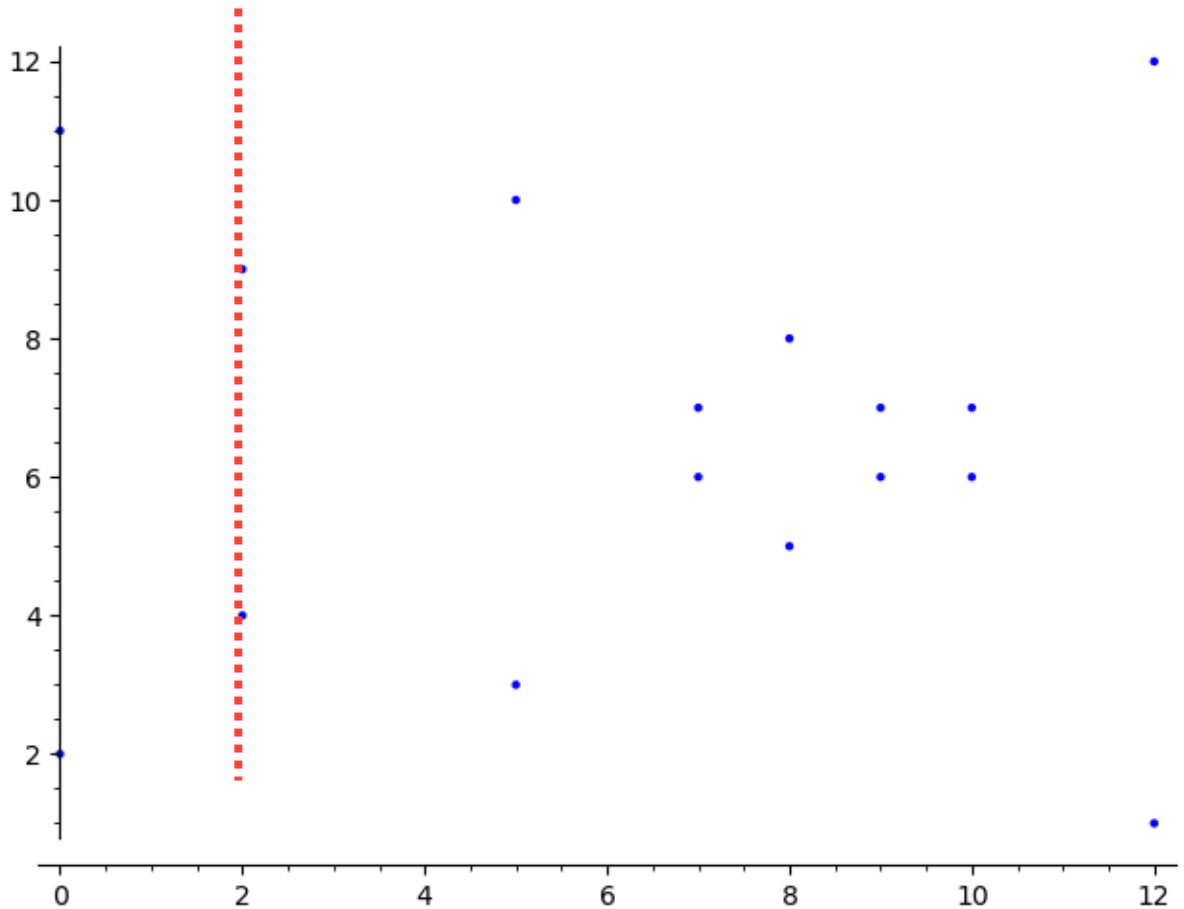
(Chord Rule) If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ such that $x_1 \neq x_2$, the group law $R = P \oplus Q$ with $R = (x_3, y_3)$ is defined as follows:

$$x_3 = \left(\frac{y_2-y_1}{x_2-x_1}\right)^2 - x_1 - x_2 \quad , \quad y_3 = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x_1 - x_3) - y_1$$

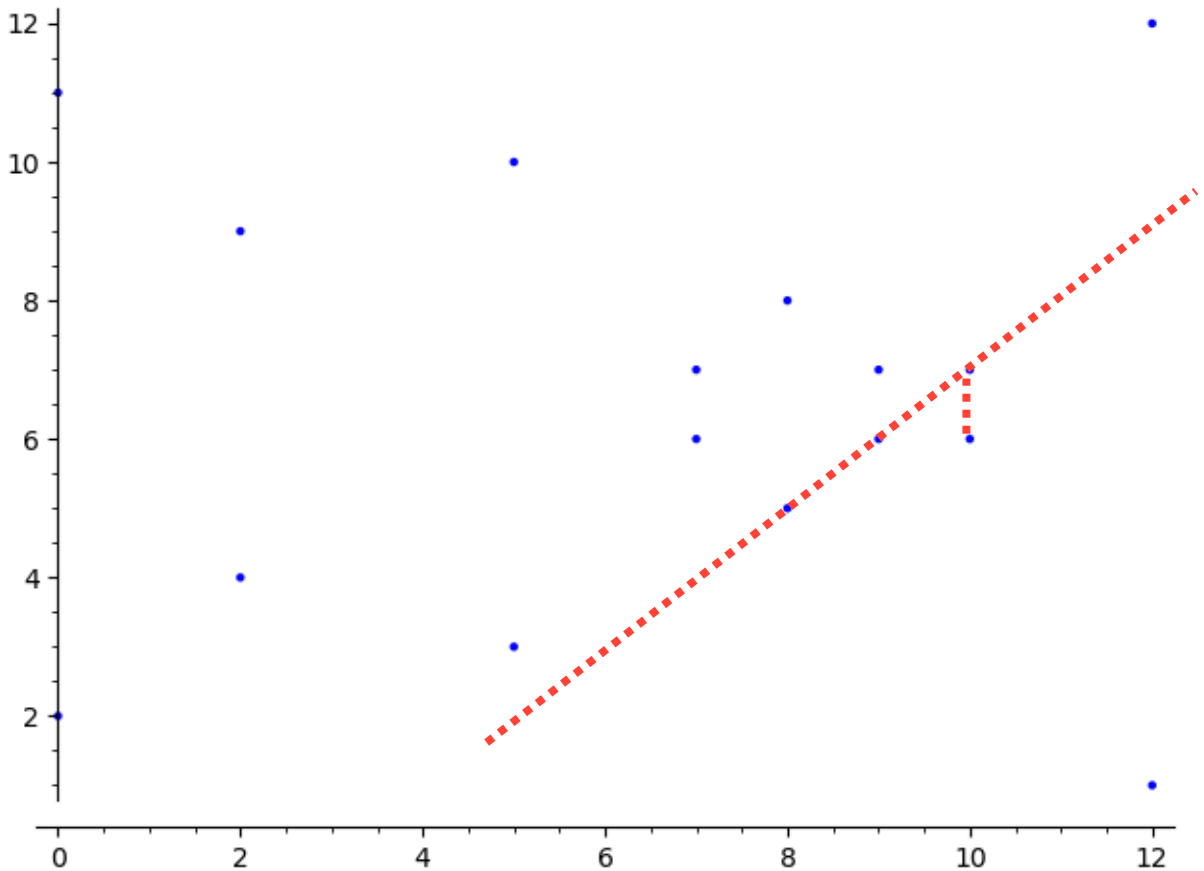
Elliptic curves (5) (mod 13)



Elliptic curves (5) (mod 13)



Elliptic curves (5) (mod 13)



Elliptic curves (6) (mod 13)

```
sage: F13 = GF(13)
sage: a = F13(2)
sage: b = F13(4)
sage: E = EllipticCurve(F13,[a,b]) # y^2 == x^3 + 2x + 4
sage: INF=E(0)
sage: E(2,4) + E(2,9) == INF
True
sage: E(8,5) + E(9,6) == E(10,7)
False
sage: E(8,5) + E(9,6) == E(10,6)
True
```

Elliptic curves (7)

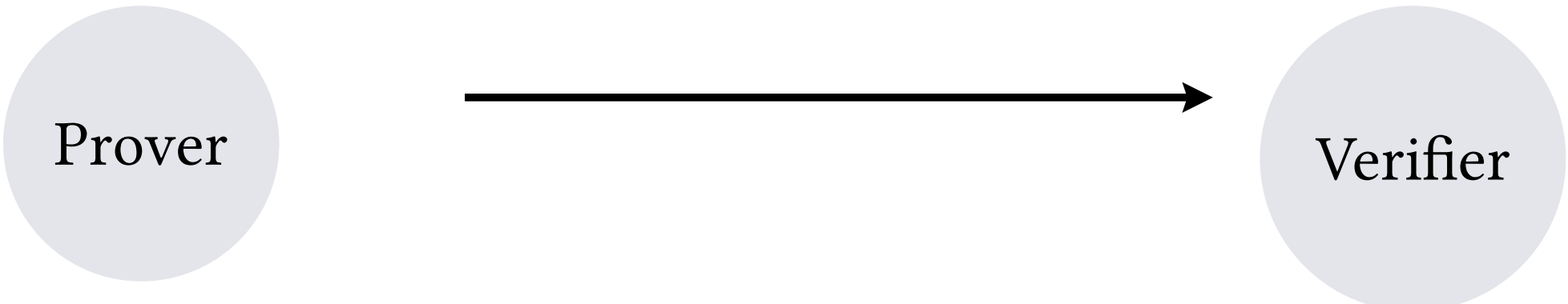
```
sage: # Bitcoin's secp256k1 curve
sage: # p = 2^256-2^32-977
sage: p =115792089237316195423570985008687907853269984665640564039457584007908834671663
sage: p.is_prime()
True
sage: p.nbits()
256
sage: Fp = GF(p)
sage: secp256k1 = EllipticCurve(Fp,[0,7])
sage: # Base point
sage: gx= 55066263022277343669578718895168534326250603453777594175500187360389116729240L
sage: gy= 32670510020758816978083085130507043184471273380659243275938904335757337482424L
sage: G = secp256k1(Fp(gx), Fp(gy))
```

Elliptic curves (8)

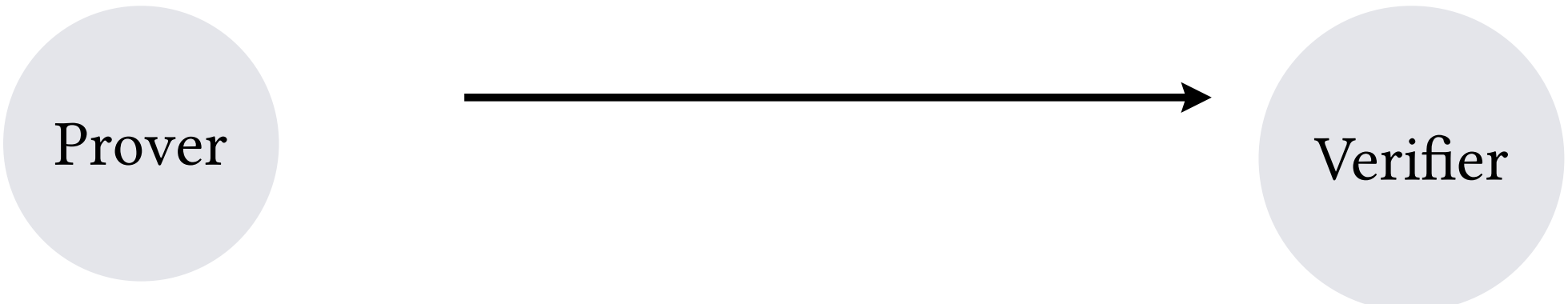
```
sage: # we have x + y = 9 to solve
sage: # PROVER provided a solution (x=2, y=7) and has the proof for it
sage: #
sage: xHidden = 2*G
sage: yHidden = 7*G
sage:
sage: # VERIFIER knows 9 which is public knowledge and gets solution hidden in POINTS
sage: rhsPoint = 9*G
sage: rhsPoint == xHidden + yHidden
True

sage: xHidden
(89565891926547004231252920425935692360644145829622209833684329913297188986597 :
1215839929969383032296780861271339863615536788704162817679887195478837165393O : 1)
```

Elliptic curves (8)



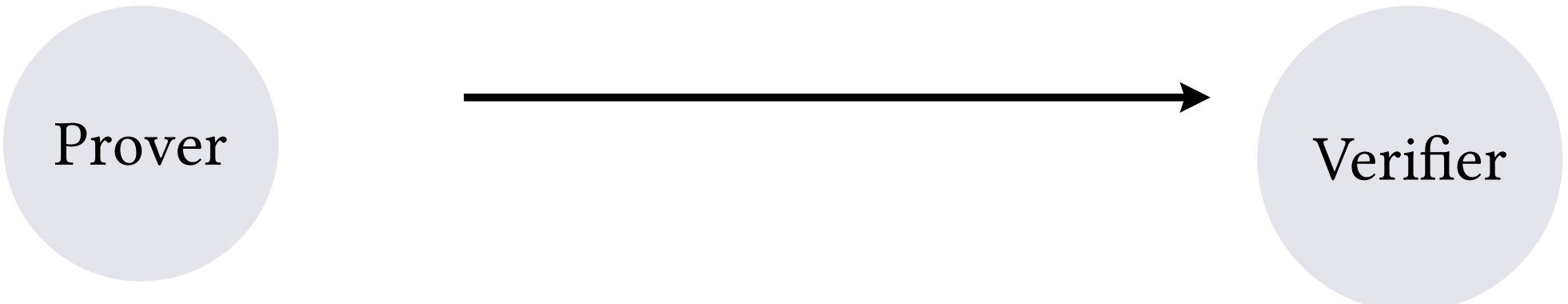
Elliptic curves (8)



x=2, y=7 and sends xG and yG

can check **xG + yG = 9G**, but cannot retrieve x and y

Elliptic curves (8)



x=2, y=7 and sends xG and yG

can check **xG + yG = 9G**, but cannot retrieve x and y

holomorphic encryption

Elliptic curves (9)

At this moment we can solve problems that are linear, meaning can be expressed as set of linear expressions:

$$a_{11} * x_1 + a_{12} * x_2 + ... = b_1$$

$$a_{21} * x_1 + a_{22} * x_2 + ... = b_2$$

$$...$$

$$a_{n1} * x_1 + a_{n2} * x_2 + ... = b_n$$

but we cannot solve:

$$\mathbf{xy = 9}$$

Elliptic curves (9)

At this moment we can solve problems that are linear, meaning can be expressed as set of linear expressions:

$$a_{11} * x_1 + a_{12} * x_2 + ... = b_1$$

$$a_{21} * x_1 + a_{22} * x_2 + ... = b_2$$

$$...$$

$$a_{n1} * x_1 + a_{n2} * x_2 + ... = b_n$$

but we cannot solve:

xy = 9

=> pairings

That's it! More to come in the future

⇧ My notes on GitHub

Get in touch 👋