

# Fundamentos de análisis y diseño de algoritmos

Ecuaciones de recurrencia

# Recurrencias

Método de iteración

Método maestro\*

Método de sustitución

# Recurrencias

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Método de iteración

Expandir la recurrencia y expresarla como una suma de términos que dependen de  $n$  y de las condiciones iniciales

# Recurrencias

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$$T(n) = \hat{n} + 3T(n/4), T(1) = \Theta(1) \text{ y } n \text{ par}$$

Expandir la recurrencia 2 veces

# Recurrencias

$$T(n) = n + 3 T\left(\frac{n}{4}\right)$$

$$n + 3 \left( n/4 + 3 T\left(\frac{n}{16}\right) \right)$$

$$n + 3 \left( n/4 + 3 \left( n/16 + 3 T\left(\frac{n}{64}\right) \right) \right)$$

$$n + 3^1 n/4 + 3^2 n/4^2 + 3^3 T\left(\frac{n}{4^3}\right)$$

$$\downarrow \\ T(1)$$

$$T\left(\frac{n}{q}\right) = \frac{n}{q} + 3 T\left(\frac{n}{q^2}\right)$$

$$T\left(\frac{n}{q^2}\right) = \frac{n}{q^2} + 3 T\left(\frac{n}{q^3}\right)$$

# Recurrencias

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$$T(n) = n + 3T(n/4)$$

$$n + 3(n/4 + 3T(n/16))$$

$$n + 3(n/4 + 3(n/16 + 3T(n/64)))$$

$$n + 3^1 n/4 + 3^2 n/4^2 + 3^3 T(n/4^3)$$

¿Cuándo se detienen las iteraciones?

# Recurrencias

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$$T(n) = n + 3T(n/4)$$

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$$n + 3(n/4 + 3(n/16 + 3T(n/64)))$$

$$n + 3^1n/4 + 3^2n/4^2 + 3^3T(n/4^3)$$

¿Cuándo se detienen las iteraciones?

Cuando se llega a  $T(1)$

# Recurrencias

$$T(n) = n + 3T(n/4)$$

$$3^i T\left(\frac{n}{4^i}\right)$$

$$n + 3(n/4 + 3T(n/16))$$

$$n + 3(n/4 + 3(n/16 + 3T(n/64)))$$

$$\hookrightarrow n + 3^1 n/4 + 3^2 n/4^2 + 3^3 T(n/4^3)$$

$$n + 3 \frac{n}{4} + n \left(\frac{3}{4}\right)^2 + n \left(\frac{3}{4}\right)^3 + n \left(\frac{3}{4}\right)^4 + \dots + n \left(\frac{3}{4}\right)^{i-1} + 3^i T\left(\frac{n}{4^i}\right)$$

¿Cuándo se detienen las iteraciones?

Cuando se llega a  $T(1)$ , esto es, cuando  $(n/4^i) = 1$

$$4^i = n \quad i = \log_4(n)$$

$$\left(\frac{3}{4}\right)^0 n + \left(\frac{3}{4}\right)^1 n + \left(\frac{3}{4}\right)^2 n + \left(\frac{3}{4}\right)^3 n + \dots + \left(\frac{3}{4}\right)^{\log_4(n)-1} n + 3^{\log_4(n)} T(1)$$

# Recurrencias

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$$T(n) = n + 3T(n/4)$$

$$n + 3(n/4 + 3T(n/16))$$

$$n + 3(n/4 + 3(n/16 + 3T(n/64)))$$

$$n + 3^1 n/4 + 3^2 n/4^2 + 3^3 T(n/4^3)$$

$$n + 3^1 n/4 + 3^2 n/4^2 + 3^3 n/4^3 + \dots + 3^{\log_4 n} T(1)$$

¿Cuándo se detienen las iteraciones?

Cuando se llega a  $T(1)$ , esto es, cuando  $(n/4^i)=1$

# Recurrencias

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$$T(n) = n + 3T(n/4)$$

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$$n + 3^1 n/4 + 3^2 n/4^2 + 3^3 n/4^3 + \dots + 3^{\log_4 n} T(1)$$

Después de iterar, se debe tratar de expresar como una sumatoria con forma cerrada conocida

# Recurrencias

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$$T(n) = n + 3T(\lfloor n/4 \rfloor)$$

$$n + 3 (\lfloor n/4 \rfloor + 3T(\lfloor n/16 \rfloor))$$

$$n + 3 (\lfloor n/4 \rfloor + 3(\lfloor n/16 \rfloor + 3T(\lfloor n/64 \rfloor)))$$

$$n + 3^1 \lfloor n/4 \rfloor + 3^2 \lfloor n/4^2 \rfloor + 3^3 \lfloor n/4^3 \rfloor + \dots + 3^{\log_4 n} \Theta(1)$$

$$\leq n + 3n/4 + 3^2 n/4^2 + 3^3 n/4^3 + \dots + 3^{\log_4 n} \Theta(1)$$

# Recurrencias

$$T(n) = n + 3T(n/4)$$

$$n + 3(n/4 + 3T(n/16))$$

$$n + 3(n/4 + 3(n/16 + 3T(n/64)))$$

$$n + 3^1 n/4 + 3^2 n/4^2 + 3^3 n/4^3 + \dots + 3^{\log_4 n} \Theta(1)$$

$$\leq n + 3n/4 + 3^2 n/4^2 + 3^3 n/4^3 + \dots + 3^{\log_4 n} \Theta(1)$$

$$= \left( \sum_{i=0}^{\log_4 n} \left(\frac{3}{4}\right)^i n \right) + 3^{\log_4 n} \Theta(1) \quad n^{\log_4 \left(\frac{3}{4}\right)}$$

$$= n \left( \frac{(3/4)^{\log_4 n} - 1}{(3/4) - 1} \right) + n^{\log_4 3} = n * 4 \left( 1 - (3/4)^{\log_4 n} \right) + \Theta(n^{\log_4 3})$$

$$= O(n)$$

# Recurrencias

Resuelva por el método de iteración

$$T(n) = 2T\left(\frac{n}{2}\right) + 1, T(1) = \Theta(1)$$

$$\bar{T}(n) = 2 \left( 2 \bar{T}\left(\frac{n}{2^2}\right) + 1 \right) + 1$$

$$T(n) = 2 \left( 2 \left( 2 \bar{T}\left(\frac{n}{2^3}\right) + 1 \right) + 1 \right) + 1$$

$$T(n) = 2 \left( 2 \left( 2 \left( 2 \bar{T}\left(\frac{n}{2^4}\right) + 1 \right) + 1 \right) + 1 \right) + 1$$

$$T(n) = 2^4 T\left(\frac{n}{2^4}\right) + 2^3 + 2^2 + 2 + 2^0$$

$$T(n) = 2^i T\left(\frac{n}{2^i}\right) + \underbrace{2^{i-1} + 2^{i-2} + \dots + 2^2 + 2^1 + 2^0}$$

$$T(1) \quad 1 = \frac{n}{2^i} \quad i = \log_2(n)$$

$$\alpha^{\log_b(c)} = c^{\log_b(a)}$$

$$2^{\log_2(n)} T(1) + \sum_{i=0}^{\log_2(n)-1} 2^i$$

$$n^{\log_2(2)} \times c + \frac{2^{\log_2(n)-1+1} - 1}{2-1}$$

$$\cancel{\downarrow} n + n - 1 \underset{\approx}{=} O(n)$$

# Recurrencias

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Resuelva por el método de iteración

$$T(n) = 2T(n/2) + 1, T(1)= \Theta(1)$$

$$T(n) = 2T(n/2) + n, T(1)= \Theta(1)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$

$$T(n) = 2 \left( 2T\left(\frac{n}{2^1}\right) + 1 \right) + 1$$

$$T(n) = 2 \left( 2 \left( 2T\left(\frac{n}{2^2}\right) + 1 \right) + 1 \right) + 1$$

$$T(n) = 2 \left( 2 \left( 2 \left( 2T\left(\frac{n}{2^3}\right) + 1 \right) + 1 \right) + 1 \right) + 1$$

$$T(n) = 2^4 T\left(\frac{n}{2^4}\right) + 2^3 + 2^2 + 2^1 + 2^0$$

$$T(n) = 2^i T\left(\frac{n}{2^i}\right) + 2^{i-1} + 2^{i-2} + \dots + 2^1 + 2^0$$

$$T(1)$$

$$\frac{n}{2^i} = 1 \quad i = \log_2(n)$$

$$T(n) = 2^{\log_2(n)} T(1) + 2^{\log_2(n)-1} + 2^{\log_2(n)-2} + \dots + 2^1 + 2^0$$

$$T(n) = n T(1) + \sum_{i=0}^{\log_2(n)-1} 2^i$$

$$\sum_{k=0}^n ar^k = \frac{ar^{n+1} - a}{r-1}$$

$$T(n) = n \Theta(1) + \frac{2^{\log_2(n)-1+1} - 1}{2-1}$$

$$T(n) = n \Theta(1) + n - 1 \rightarrow \Theta(1)$$

$$T(n) = 2 \left[ T\left(\frac{n}{2}\right) + n \right] + n$$

$$= 2 \left( 2 \left[ T\left(\frac{n}{2^2}\right) + \frac{n}{2} \right] + \frac{n}{2} \right) + n$$

$$= 2^2 \left( 2 \left[ T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} \right] + \frac{n}{2^2} \right) + n$$

$$= 2^3 \left[ T\left(\frac{n}{2^3}\right) + \frac{n}{2^3} + n + n \right] + n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 3n$$

$$2^i + \left(\frac{n}{2^i}\right) + i n$$

$$\frac{n}{2^i} = 1 \quad i = \log_2(n)$$

$$2^{\log_2(n)} + (1) + \log_2(n) n$$

$$\cancel{n \cdot \Theta(1)} + \Theta(\log_2(n)) = \Theta(n \log(n))$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2 \quad \Theta(1) = T(1)$$

1)  $T(n) = 2 \left( 2T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2 \right) + n^2$

$$T(n) = 2^2 \left[ T\left(\frac{n}{2^2}\right) + 2\left(\frac{n}{2}\right)^2 + n^2 \right]$$

$$T(n) = 2^2 \left( 2T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2 \right) + 2\left(\frac{n}{2}\right)^2 + n^2$$

$$T(n) = 2^3 T\left(\frac{n}{2^3}\right) + 2^2 \left(\frac{n}{2^2}\right)^2 + 2\left(\frac{n}{2}\right)^2 + n^2$$

$$T(n) = 2^i T\left(\frac{n}{2^i}\right) + \left[ 2^{i-1} \left(\frac{n}{2^{i-1}}\right)^2 + 2^{i-2} \left(\frac{n}{2^{i-2}}\right)^2 + \dots + 2^1 \left(\frac{n}{2^1}\right)^2 + 2^0 \left(\frac{n}{2^0}\right)^2 \right]$$

$$i = \log_2(n)$$

$$T(n) = n T(1) + \sum_{i=0}^{\log_2(n)-1} 2^i \left(\frac{n}{2^i}\right)^2$$

$$T(n) = n \Theta(1) + \sum_{i=0}^{\log_2(n)-1} n^2 \left(\frac{1}{2^i}\right)$$

$$T(n) = n \Theta(1) + n^2 \sum_{i=0}^{\log_2(n)-1} \left(\frac{1}{2^i}\right)^i$$

$$T(n) = n \Theta(1) + n^2 \left( \frac{1}{2} \left( \frac{1}{2} \right)^{\log_2(n)} - 1 \right)$$

$$T(n) = n \Theta(1) + n^2 \left( \frac{1}{2} \left( \frac{1}{2} \right)^{\log_2(0.5)} - 1 \right)$$

$$T(n) = n \Theta(1) + n^2 \left( -2 \left( n^{-\frac{1}{2}} - 1 \right) \right)$$

$$T(n) = n \Theta(1) - 2n + 2n^2$$

$\Theta(n^2)$

# Recurrencias

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Resuelva por el método de iteración

$$T(n) = 2T(n/2) + 1, T(1)= \Theta(1)$$

$$T(n) = 2T(n/2) + n, T(1)= \Theta(1)$$

$$T(n) = T(n/2) + 1, T(1)= \Theta(1)$$

$$T(n) = 4T\left(\frac{n}{3}\right) + n + 1 \quad T(1) = 1$$

$$T(n) = 4\left(4T\left(\frac{n}{3^2}\right) + \frac{n}{3} + 1\right) + n + 1$$

$$T(n) = 4^2 + \left(\frac{n}{3^2}\right) + \left(\frac{4}{3}\right)n + 4 + n + 1$$

$$T(n) = 4^2 \left(4T\left(\frac{n}{3^3}\right) + \frac{n}{3^2} + 1\right) + \frac{4}{3}n + 4 + n + 1$$

$$T(n) = 4^3 T\left(\frac{n}{3^3}\right) + \frac{4^2}{3^2} n + 4^2 + \frac{4}{3} n + 4 + \left(\frac{4}{3}\right)^0 n + 4^0$$

$$T(n) = 4^i T\left(\frac{n}{3^i}\right) + \left(\frac{4}{3}\right)^{i-1} n + 4^{i-1} + \left(\frac{4}{3}\right)^{i-2} n + 4^{i-2} + \dots + \left(\frac{4}{3}\right)^0 n + 4^0$$

$T(1)$

$$1 = \frac{n}{3^i}$$

$$i = \log_3(n)$$

$$T(n) = 4^{\log_3(n)} \times T(1) + \sum_{i=0}^{\log_3(n)-1} \left( \left(\frac{4}{3}\right)^i n + 4^i \right)$$

$$T(n) = \underbrace{n^{\log_3(4)}}_{\text{base case}} + n \left( \frac{\frac{4}{3}^{\log_3(4)} - 1}{4/3 - 1} \right) + \left( \frac{4^{\log_3(n)} - 1}{4 - 1} \right)$$

$$T(n) = n^{\log_3(4)} + 3n \times n^{\log_3(\frac{4}{3})} = 3n^{\log_3(\frac{4}{3})} + \underbrace{n^{\log_3(4)}}_{\frac{4}{3}}$$

$$6(n^{\log_3(4)})$$

$$\frac{\log_3(\frac{4}{3}) + 1}{\log_3(4) - \log_3(3) + 1}$$

# Recurrencias

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Resuelva por el método de iteración

$$T(n) = 2T(n/2) + 1, T(1)= \Theta(1)$$

$$T(n) = 2T(n/2) + n, T(1)= \Theta(1)$$

$$T(n) = T(n/2) + 1, T(1)= \Theta(1)$$

Demuestre que  $T(n) = T(n/2\lfloor) + n$ , es  $\Omega(n\log n)$

# Recurrencias

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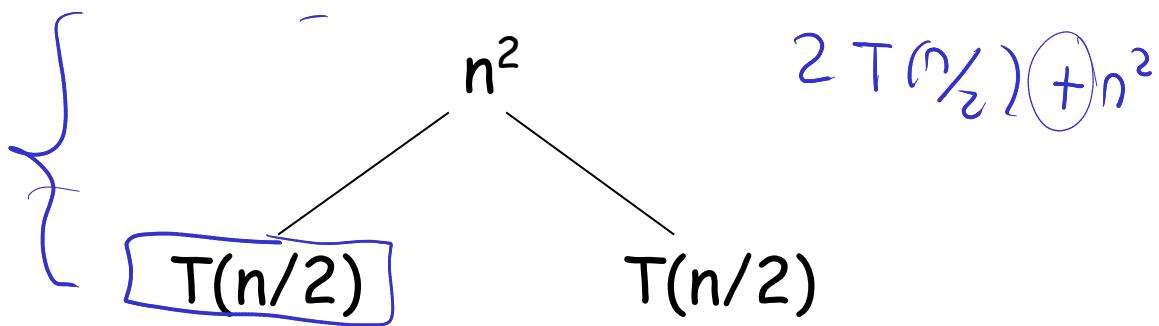
Iteración con árboles de recursión

$$T(1) = 1$$

$$T(n) = 2T(n/2) + n^2$$

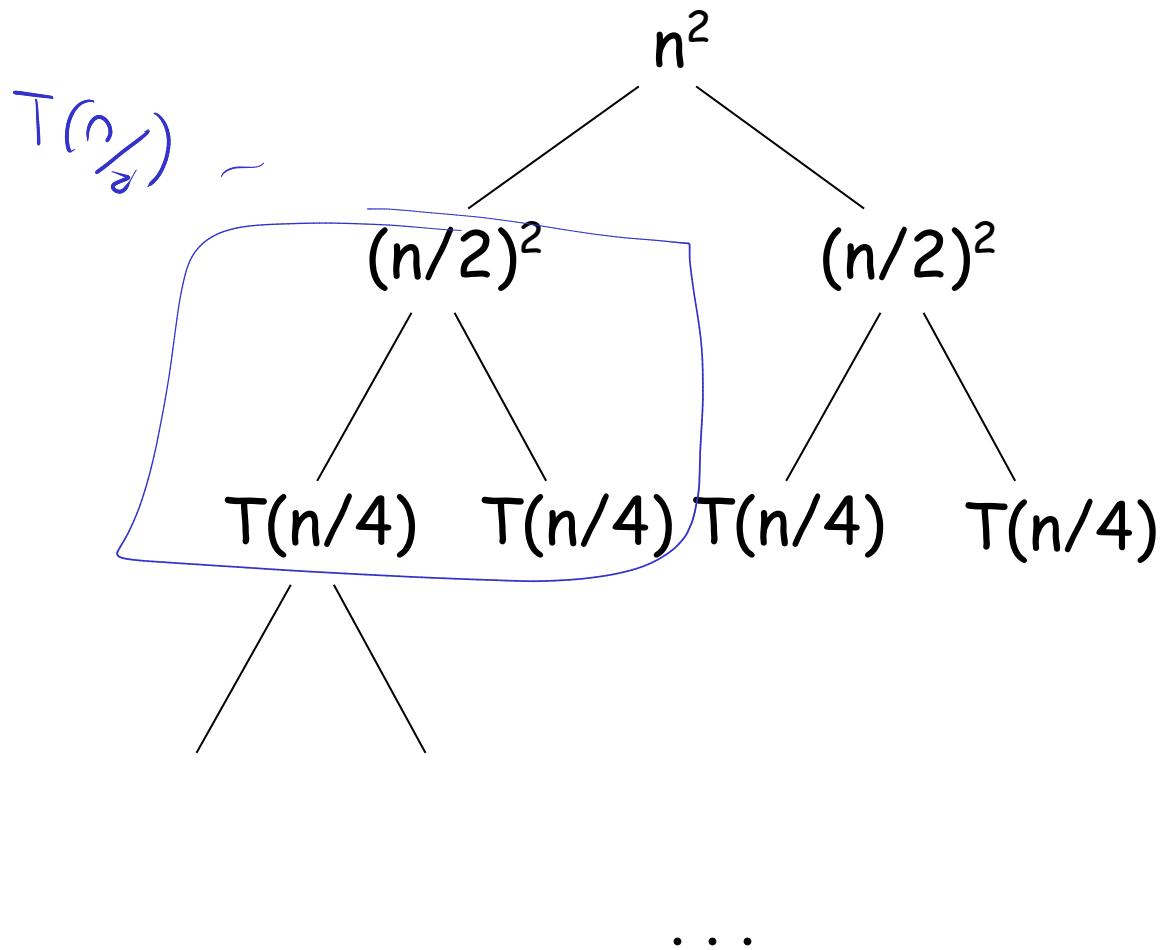
# Recurrencias

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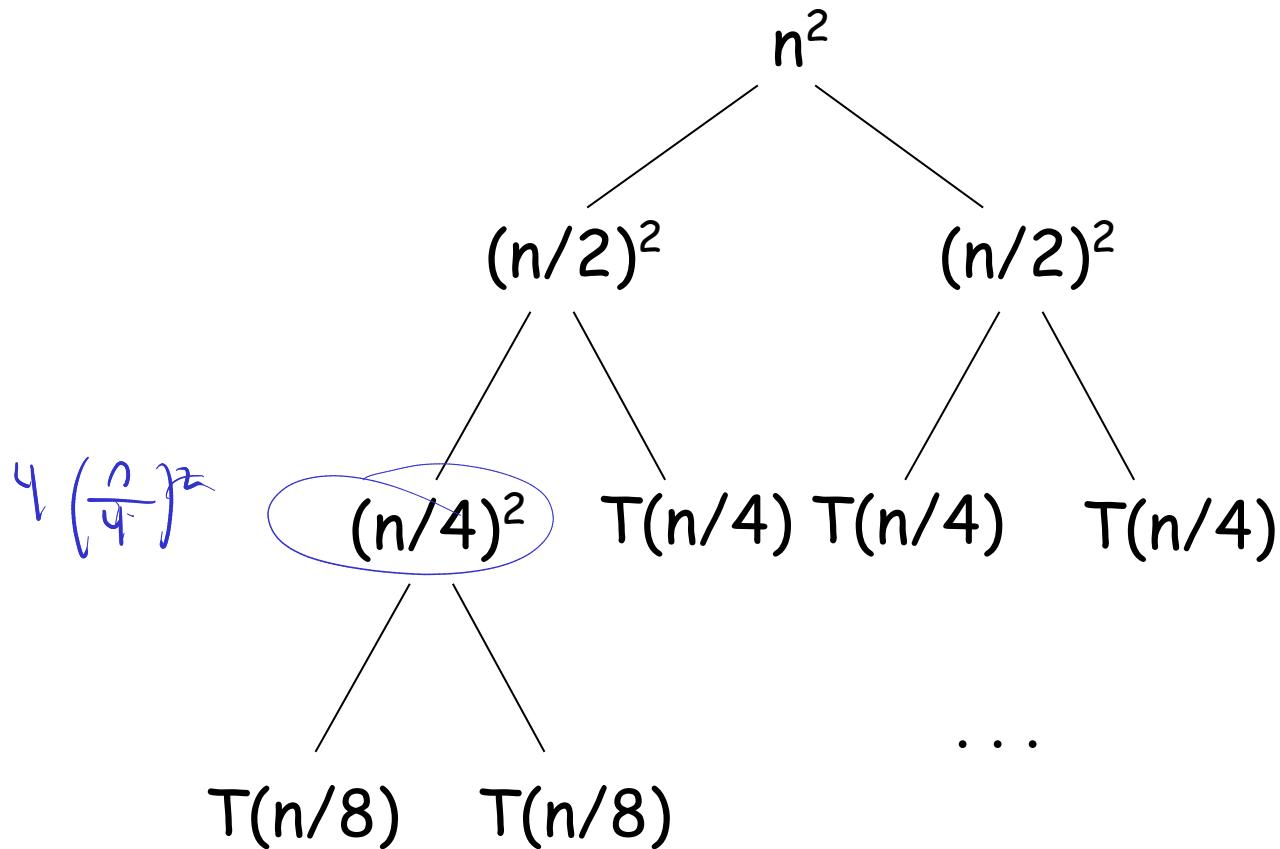


# Recurrencias

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# Recurrencias



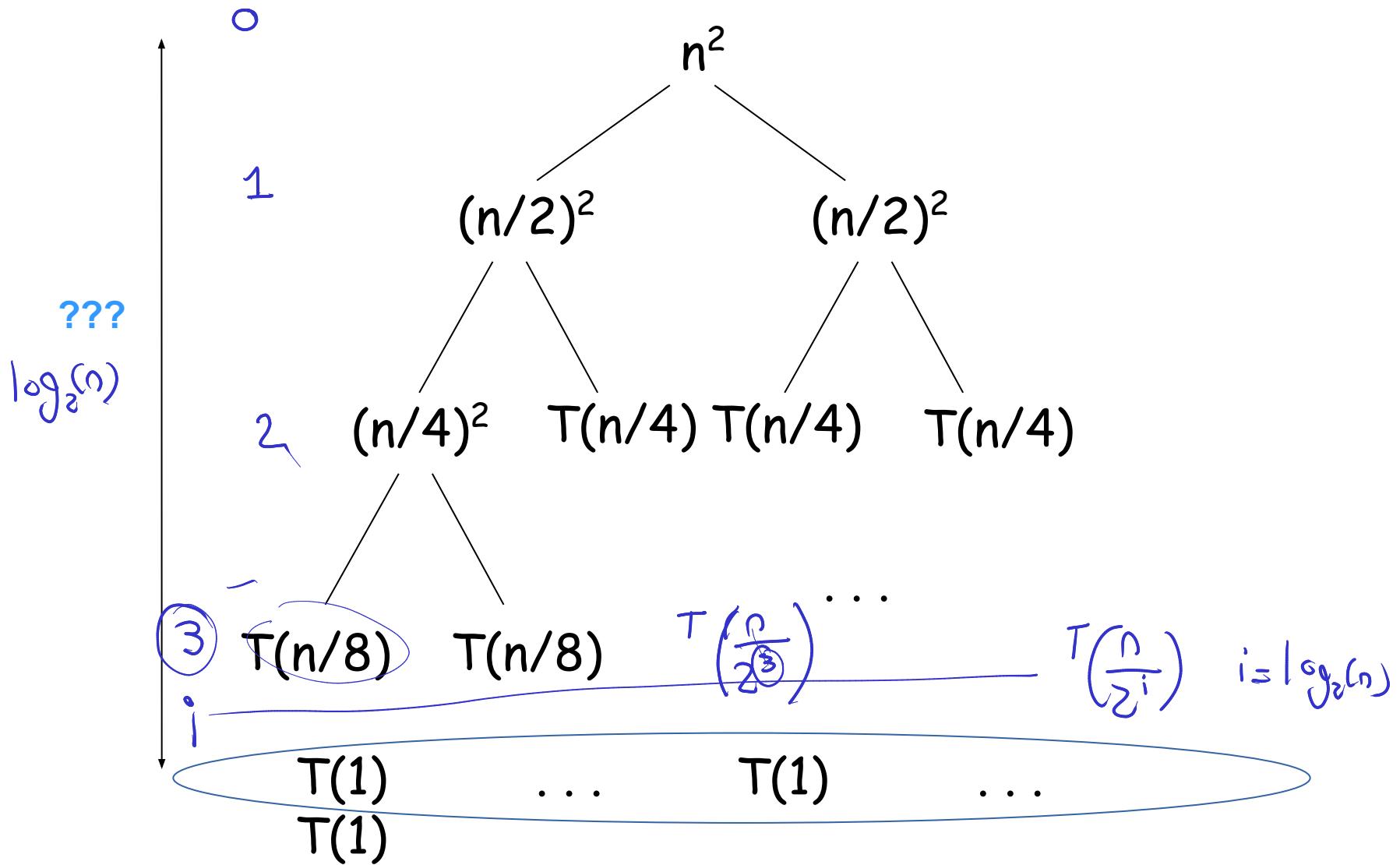
$$n^4 = \frac{n^2}{2^0}$$

$$\frac{n^2}{2} = \frac{n^2}{2^1}$$

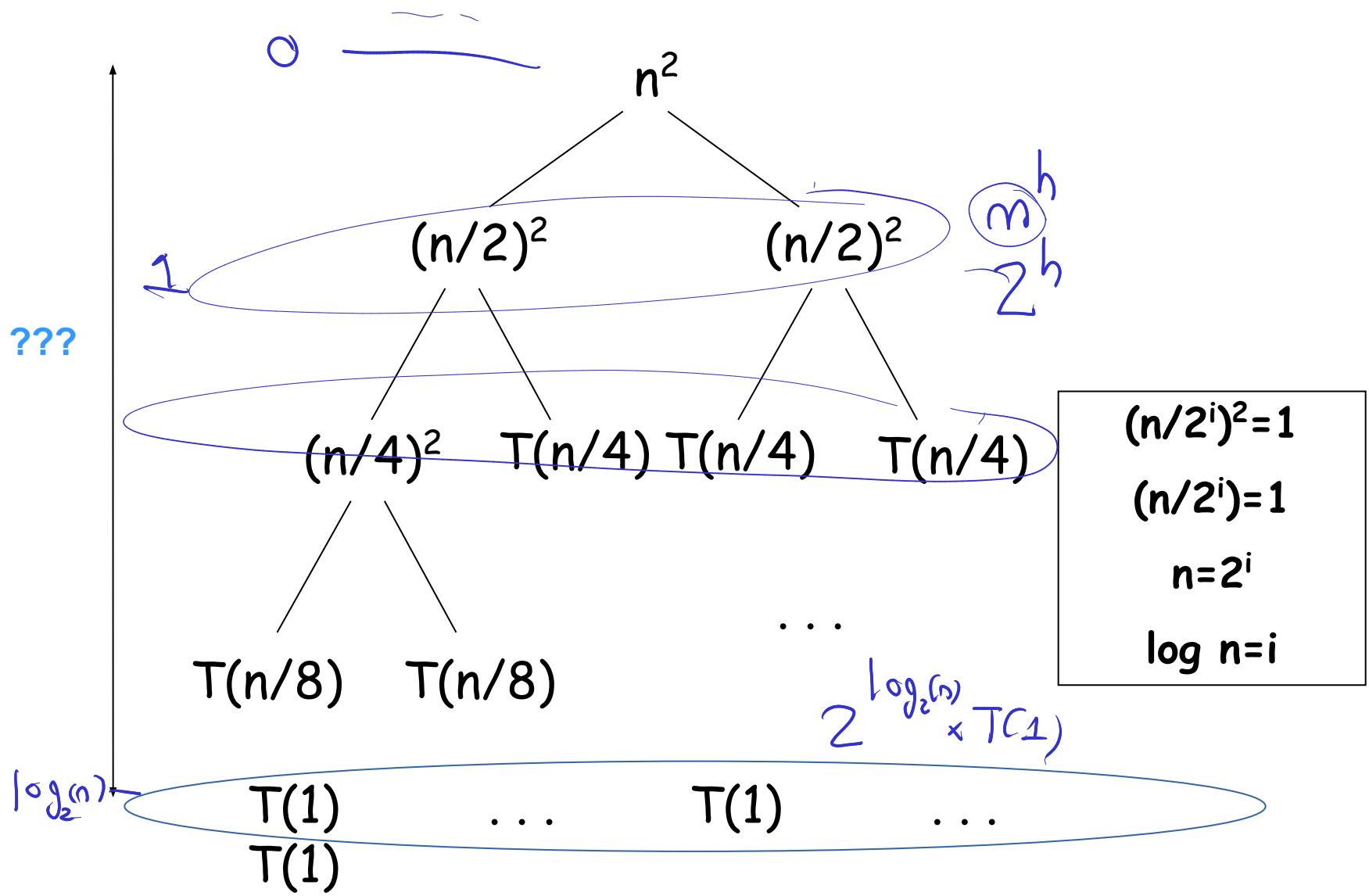
$$\frac{n^2}{4} = \frac{n^2}{2^2}$$

$$\frac{n^2}{8} = \frac{n^2}{2^3}$$

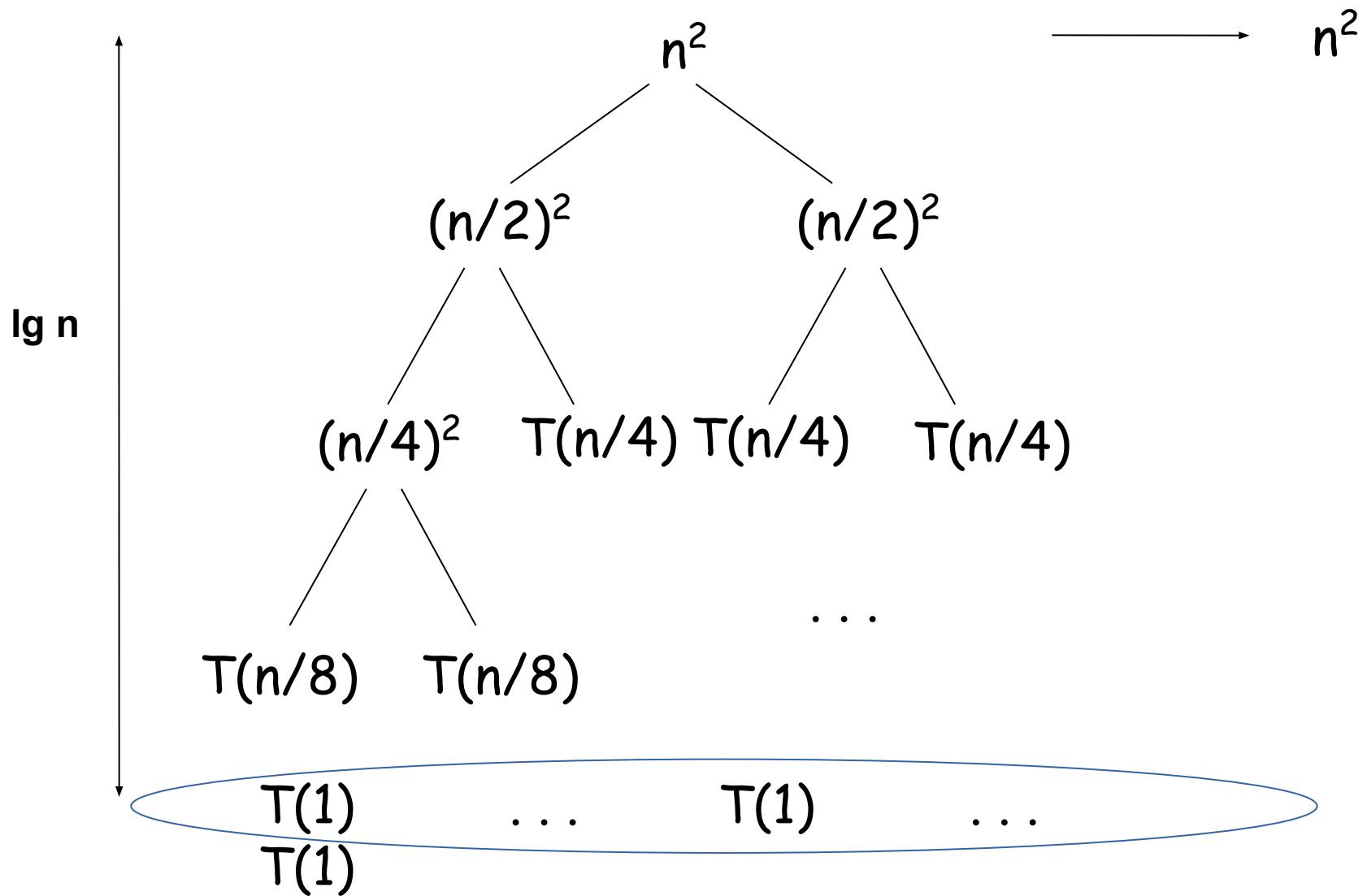
# Recurrencias



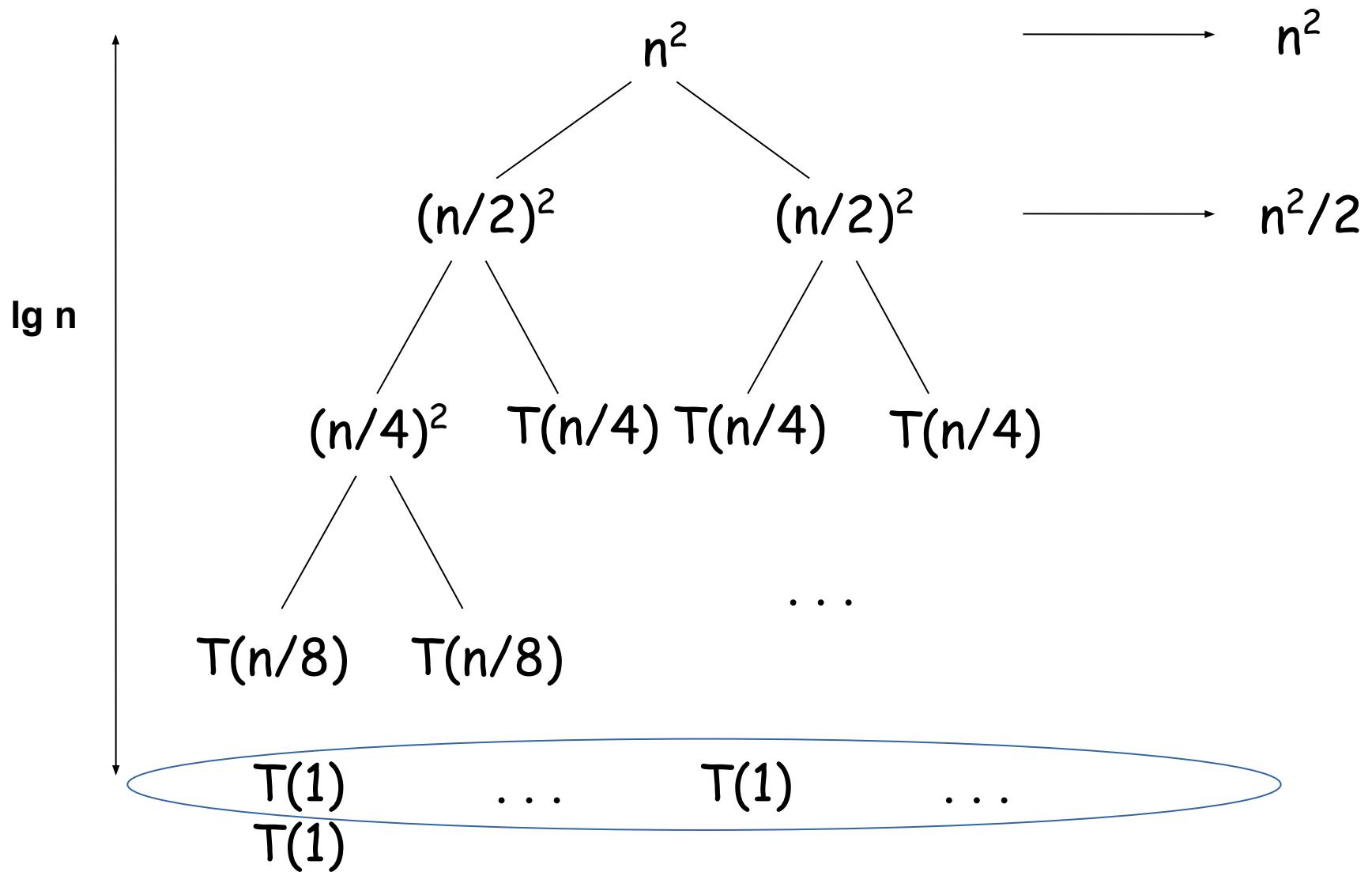
# Recurrencias



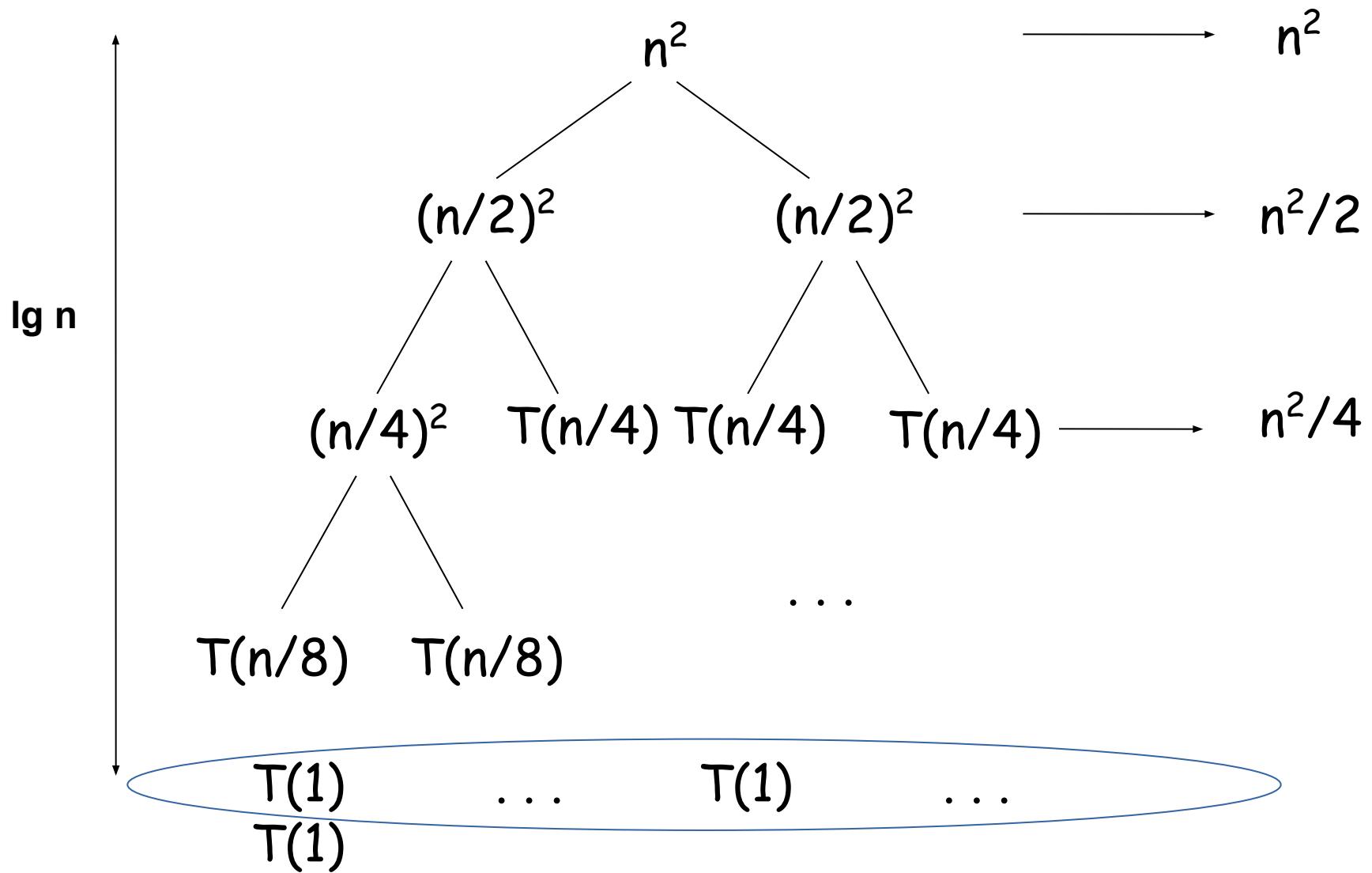
# Recurrencias



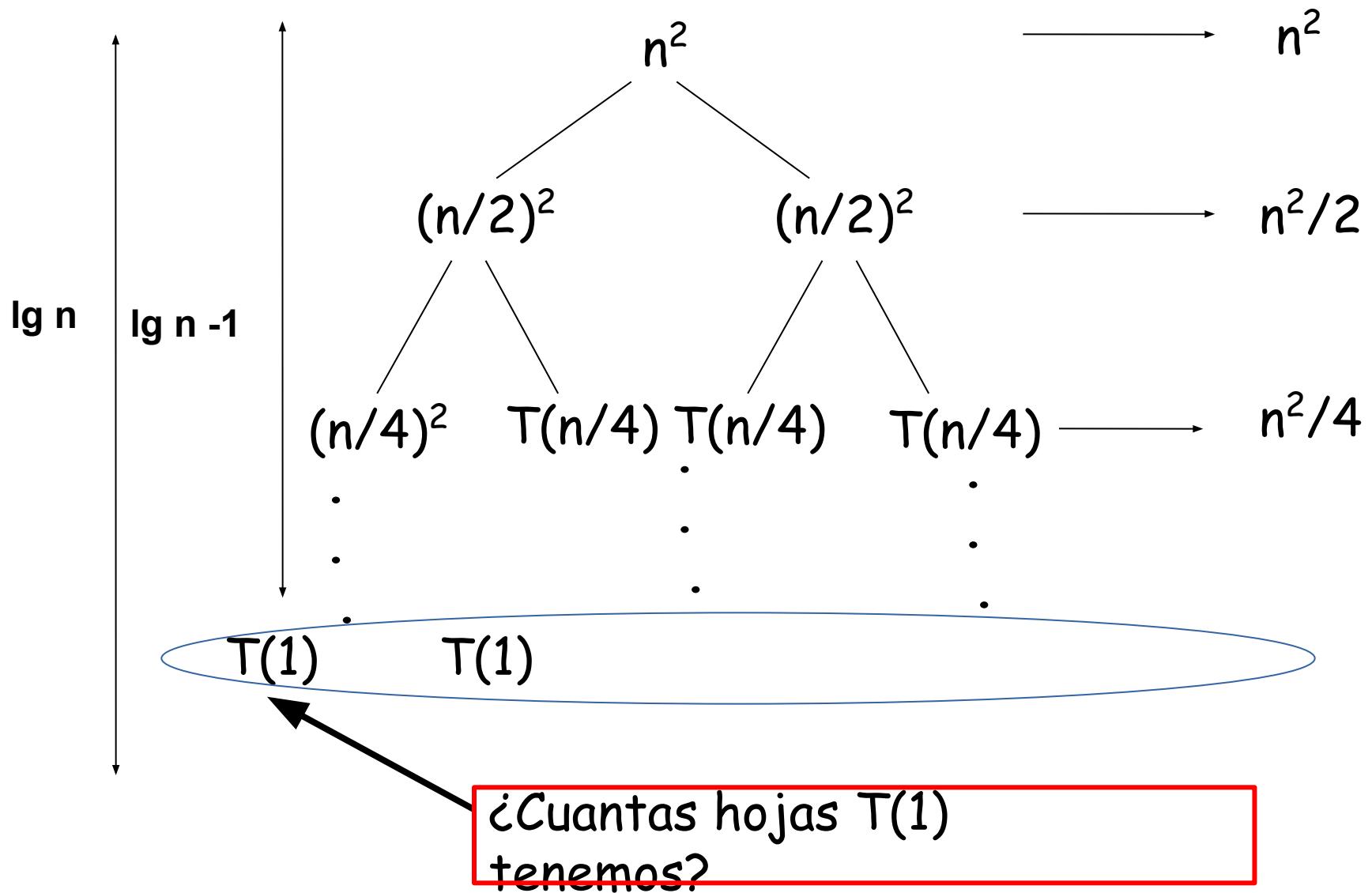
# Recurrencias



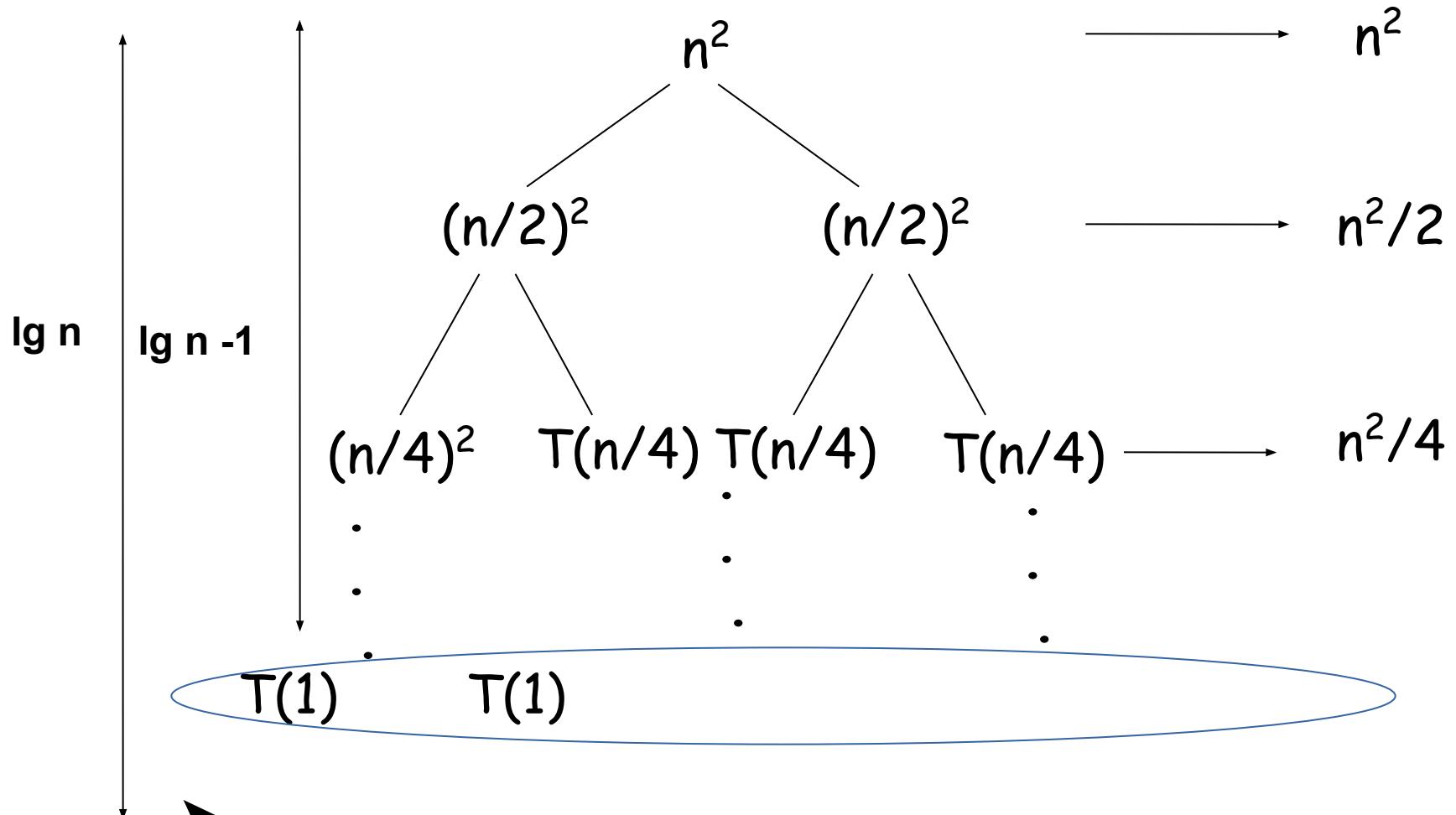
# Recurrencias



# Recurrencias

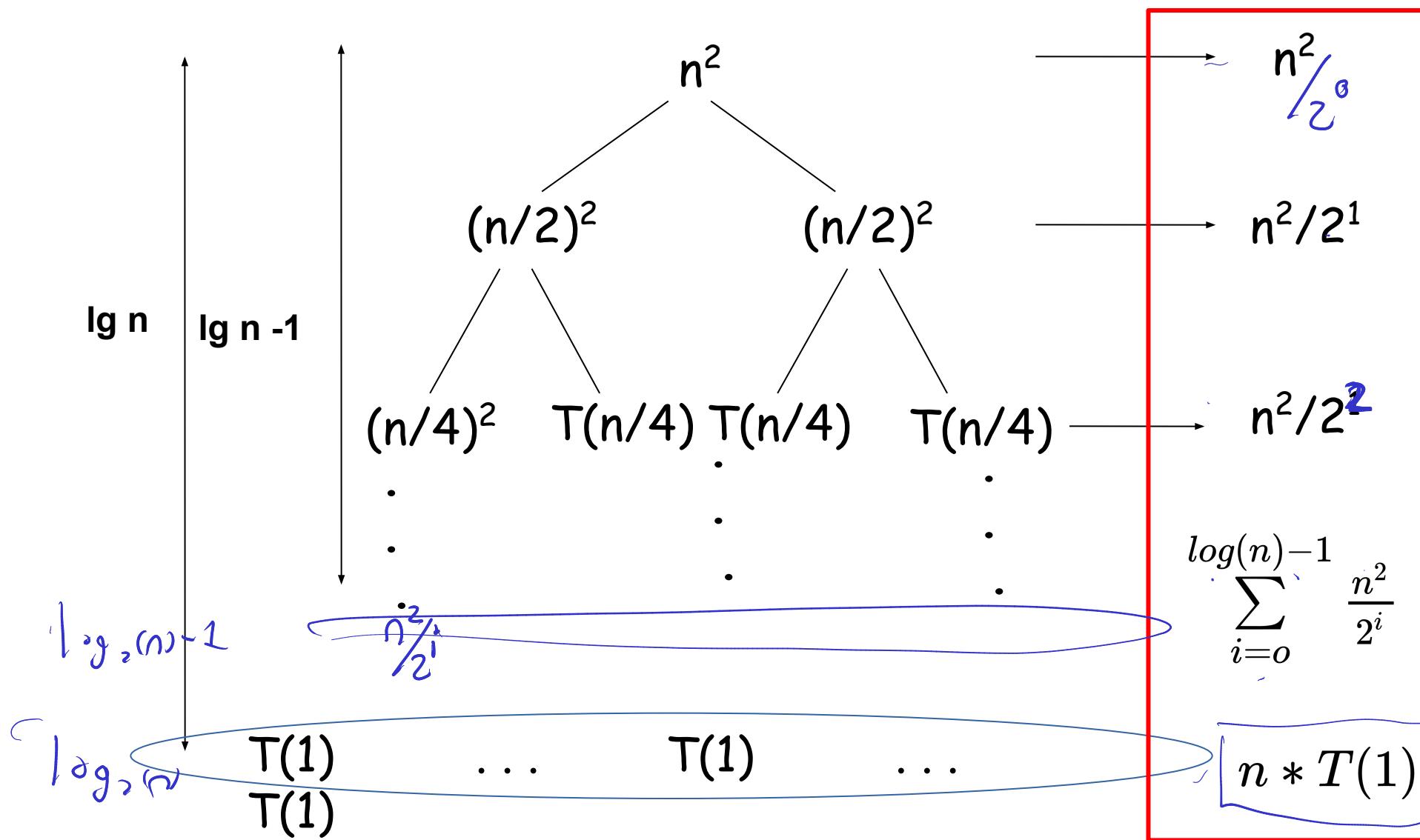


# Recurrencias



Si recuerda en un árbol m-ario se tienen máximo  $m^h$ . En este caso al ser arbol binario  $m=2$ , tenemos  $2^{\log(n)}$  hojas. Por lo tanto se

# Recurrencias



# Recurrencias

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$$T(n) = n * T(1) + \sum_{i=0}^{\Theta(\log(n))} \frac{n^2}{2^i}$$

$$T(n) = n * c + n^2 \frac{0.5^{\log(n)} - 1}{0.5 - 1}$$

$$T(n) = n * c + n^2 \frac{n^{\log(0.5)} - 1}{-0.5}$$

$$T(n) = n * c + n^2 \frac{n^{-1} - 1}{-0.5}$$

$$T(n) = \cancel{n * c} - \frac{n}{0.5} + \frac{n^2}{0.5} = O(n^2)$$

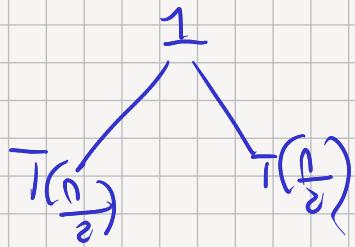
# Recurrencias

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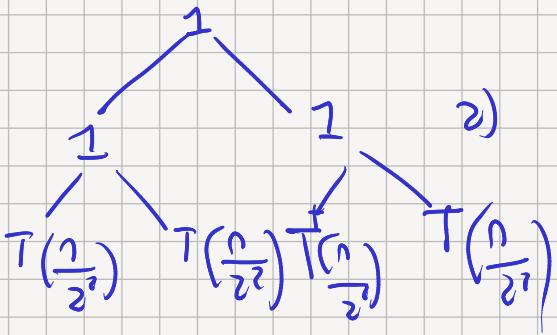
Resuelva construyendo el árbol

$$T(n) = 2T(n/2) + 1, T(1)= \Theta(1)$$

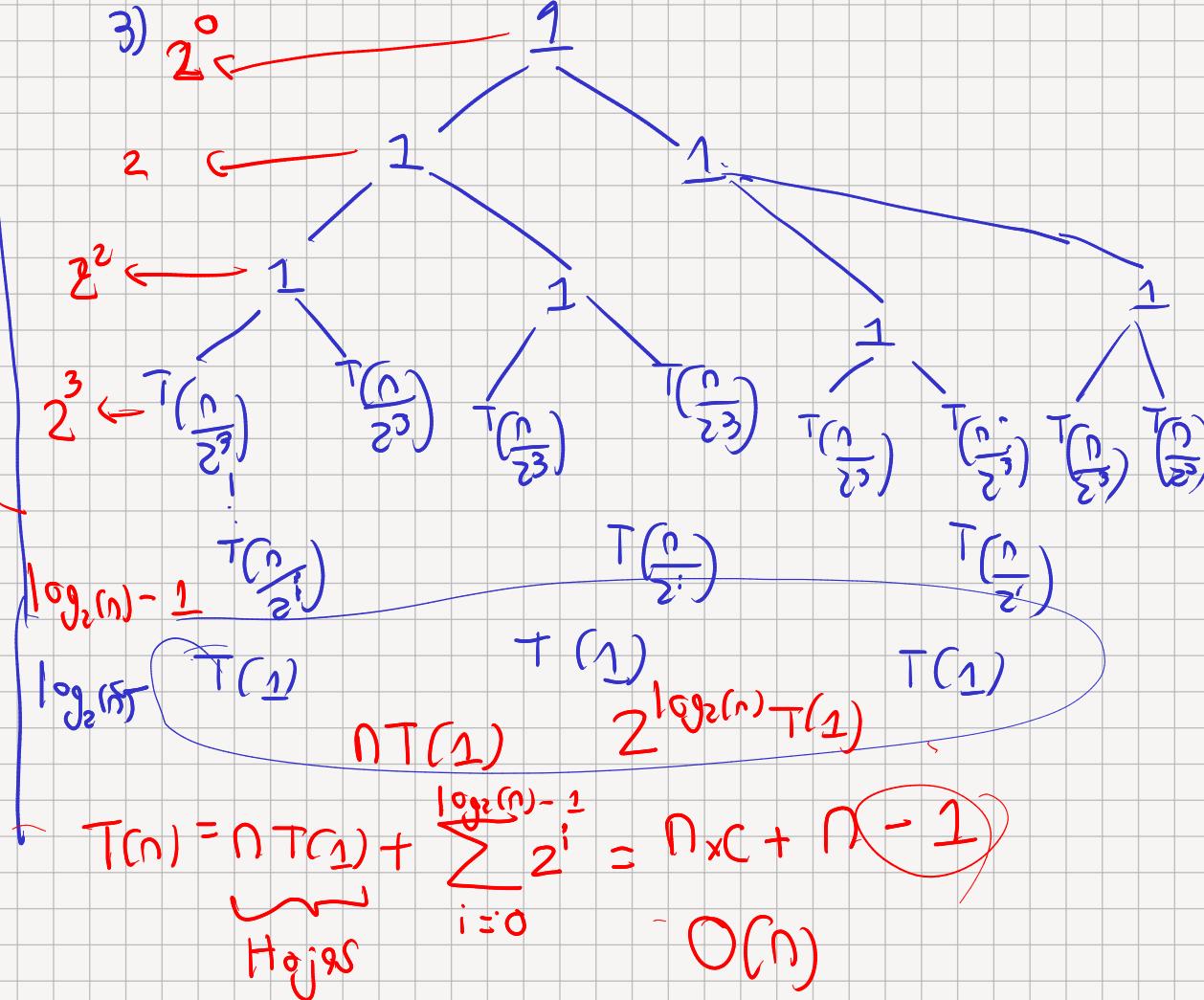
$$T(n) = 2T(n/2) + n, T(1)= \Theta(1)$$



$$2^T \left(\frac{c}{2}\right) + 1$$



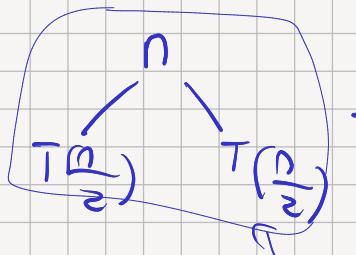
1)



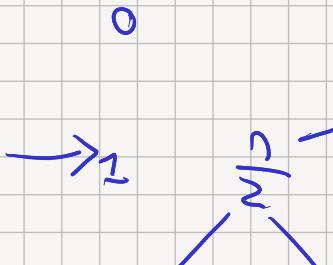
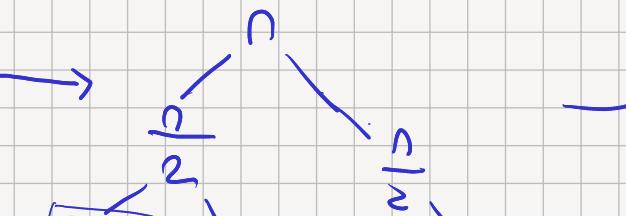
$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T(1) = \Theta(1)$$

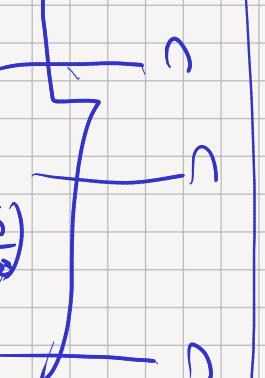
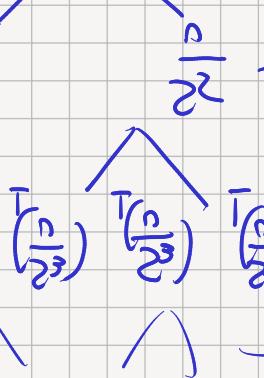
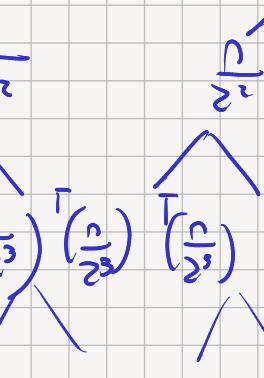
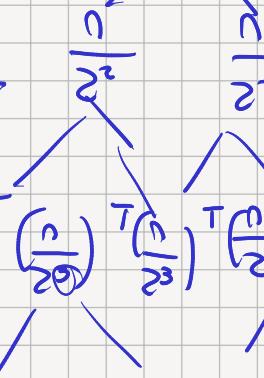
$$T(n) = O(n \log(n))$$



$$T\left(\frac{n}{2^2}\right)$$



③



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$\log_2(n)-1$

$$2^{\log_2(n)} \times T(1) + \sum_{i=0}^{\log_2(n)-1} n =$$

$$cn + n + \sum_{i=2}^{\log_2(n)-1} n = cn + n + n \times (\log_2(n) - 1)$$

$$cn + n \log_2(n) = O(n \log_2(n))$$

# Recurrencias

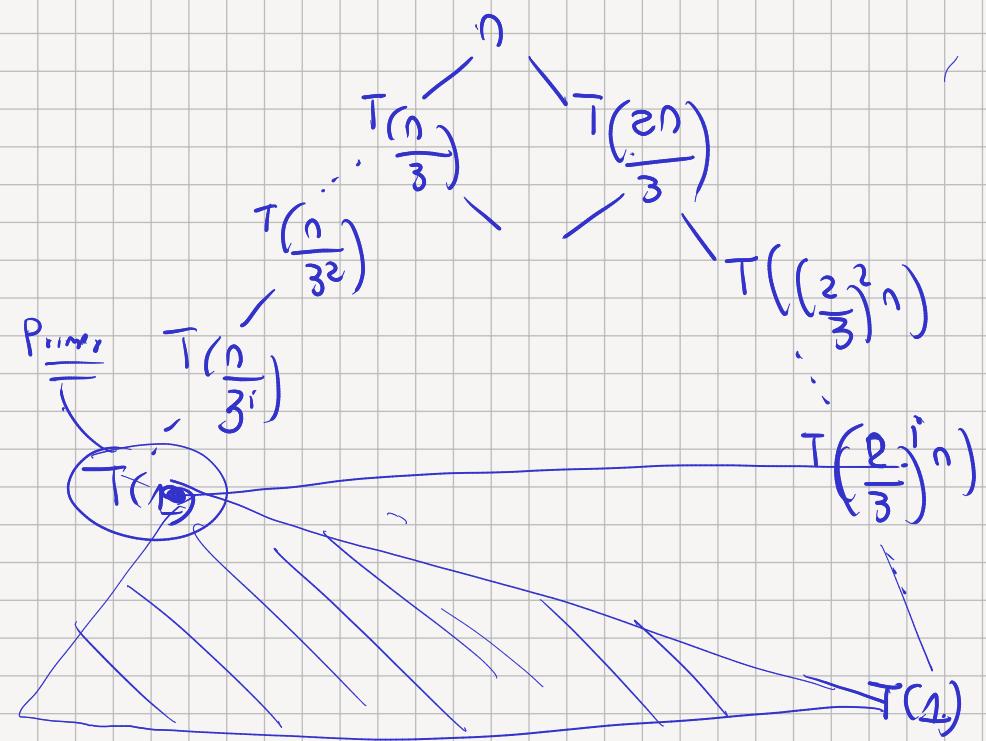
Resuelva la recurrencia  $T(n) = T(n/3) + T(2n/3) + n$

Indique una cota superior y una inferior

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$

$$T(n) \approx T\left(\frac{n}{3^2}\right) + T\left(\frac{2n}{3^2}\right) + \frac{n}{3} + T\left(\frac{2n}{3^2}\right) + T\left(\frac{2^2n}{3^2}\right) + \frac{2n}{3} + 2$$

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + D$$

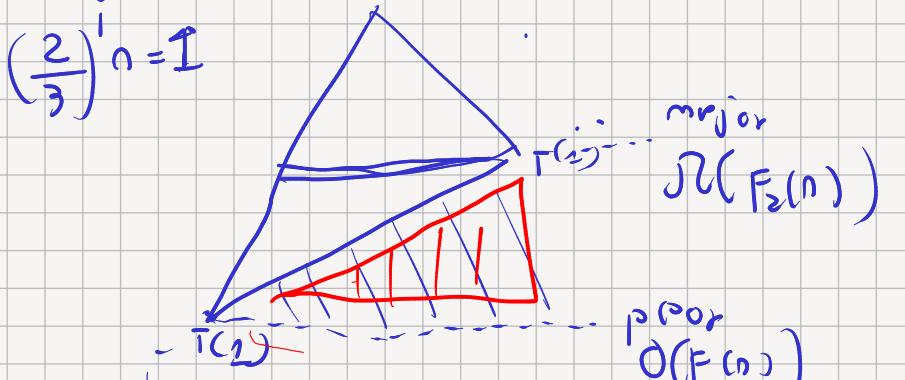


$$\sum T(n) = n + 2T\left(\frac{n}{3}\right)$$

$$O(T(n)) = n + 2T\left(\frac{2}{3}n\right)$$

$$\frac{n}{3^i} = i = \log_3(n)$$

$$\left(\frac{2}{3}\right)^i n = 1 = \log_{\frac{2}{3}}\left(\frac{n}{2}\right)$$

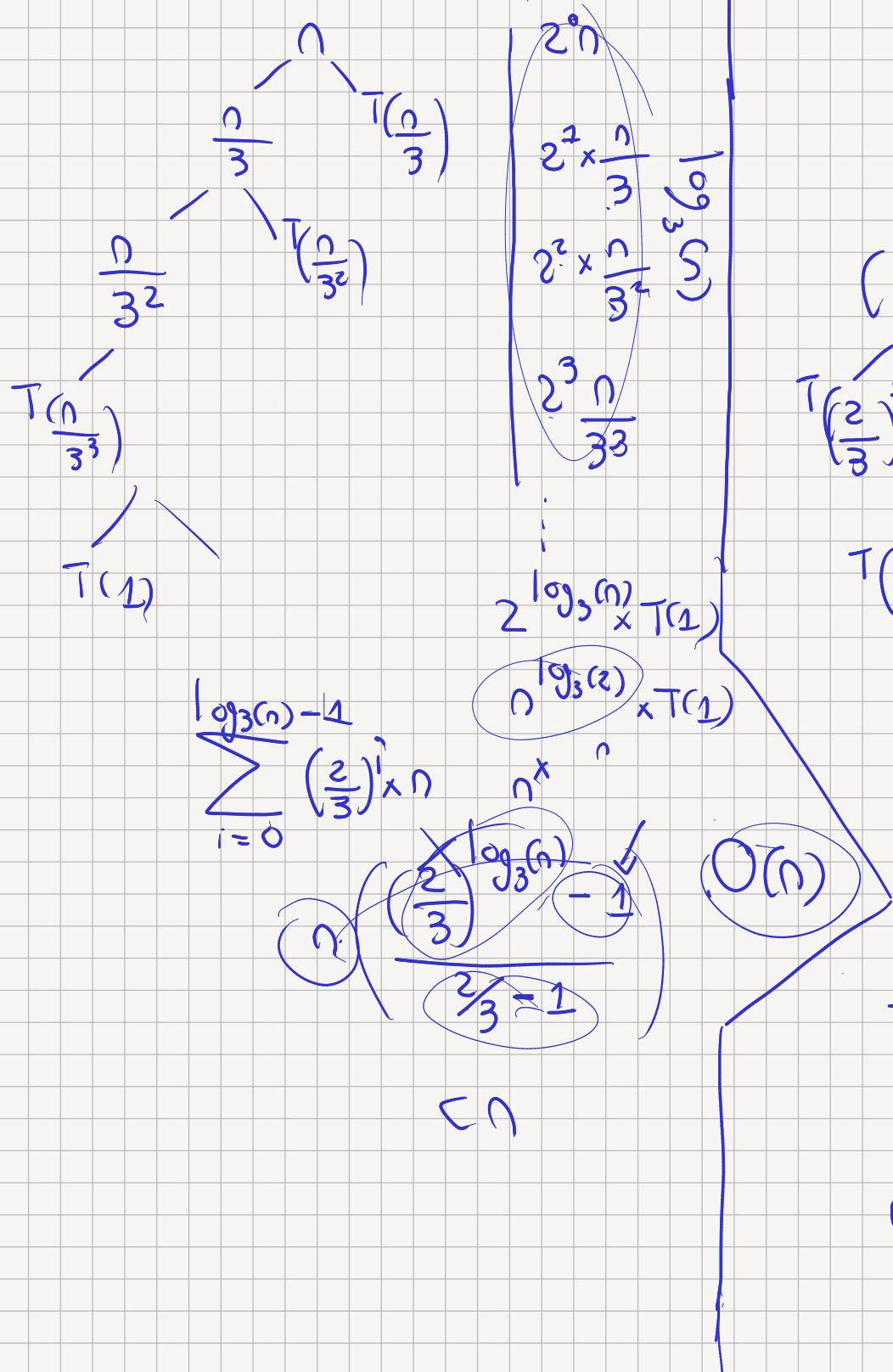


$$\log_{\frac{2}{3}}(n) = \frac{\log_3(n)}{\log_3(\frac{2}{3})}$$

$$\log_{\frac{2}{3}}(n) = \frac{\log_3(n)}{\log_3(2) - \log_3(3)} \approx -0.3n$$

$$\log_{\frac{2}{3}}(n) = -2.70 \times \log_3(n)$$

$$T(n) = n + 2T\left(\frac{n}{3}\right)$$

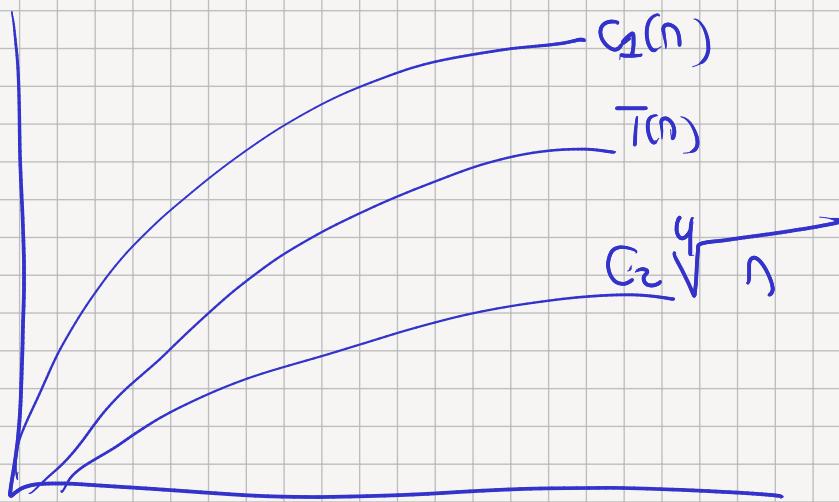


$$T(n) = n + 2T\left(\frac{2}{3}n\right)$$

$$T(n) = n + 2T\left(\frac{2}{3}n\right)$$

Another recurrence tree diagram for  $T(n) = n + 2T\left(\frac{2}{3}n\right)$ . The root node is labeled  $n$ . It branches into two children, each labeled  $T\left(\frac{2}{3}n\right)$ . These further branch into four children, each labeled  $T\left(\frac{2}{3}^2 n\right)$ . This pattern continues, with the  $i$ -th level having  $2^i$  nodes, each labeled  $T\left(\frac{2}{3}^i n\right)$ . The tree has  $\log_2(n)$  levels. The cost at each node is  $n$ , and there are additional costs of  $2^i \times \left(\frac{2}{3}\right)^i n$  for the edges between levels. The total cost is  $n \log_2(n) + 2^i n$ .

$$T(n) \rightarrow O(n) \text{ y } \Omega(n^{0.25}) = \underline{\sqrt{n}}$$



# Recurrencias

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Método maestro

Permite resolver recurrencias de la forma:

$$T(n) = aT(n/b) + f(n), \text{ donde } a \geq 1, b > 1$$

# Recurrencias

---

Dado  $T(n) = aT(n/b) + f(n)$ , donde  $a \geq 1$ ,  $b > 1$ , se puede acotar asintóticamente como sigue:

1.  $T(n) = \Theta(n^{\log_b a})$

Si  $f(n) = O(n^{\log_b a - \varepsilon})$  para algún  $\varepsilon > 0$

2.  $T(n) = \Theta(n^{\log_b a} \lg n)$

Si  $f(n) = \Theta(n^{\log_b a})$  para algún  $\varepsilon > 0$

3.  $T(n) = \Theta(f(n))$

Si  $f(n) = \Omega(n^{\log_b a + \varepsilon})$  para algún  $\varepsilon > 0$  si  $a * f(n/b) \leq c * f(n)$

para algún  $c < 1$

# Recurrencias

Dado  $T(n) = 9T(n/3) + n$

$$a = 9$$

$$b = 3$$

$$f(n) = n$$

$$n^{\log_3 9} = n^2 \quad \text{vs} \quad f(n) = n$$

Es  $f(n) = O(n^{\log_b a - \varepsilon})$  ?

Es  $n = O(n^{2-\varepsilon})$  ?

$$\begin{aligned} & n^{\log_b a} \\ & n^{\log_3 9} = n^2 \end{aligned}$$

$$\begin{aligned} \Theta(n^{\log_b a}) & n \text{ vs } O(n^{2-\varepsilon}) \\ \cancel{\Theta(n^2)} & n \text{ vs } O(n) \checkmark \end{aligned}$$

# Recurrencias

---

Dado  $T(n) = 9T(n/3) + n$

$$n^{\log_3 9} = n^2 \quad \text{vs} \quad f(n) = n$$

Es  $f(n) = O(n^{\log_b a - \varepsilon})$  ?

Es  $n = O(n^{2-\varepsilon})$  ?

Si  $\varepsilon = 1$  se cumple que  $O(n) = O(n)$ , por lo tanto, se cumple que:

$$T(n) = \Theta(n^2)$$

# $\alpha T\left(\frac{n}{b}\right) + F(n)$ Recurrencias

$$\frac{2}{3} = \frac{1}{b}$$

$$T(n) = T(2n/3) + 1$$

$$a = 1$$

$$n^{\log_{3/2} 1} = n^0 = 1 \quad \text{vs} \quad f(n) = 1$$

$$b = \frac{3}{2}$$

$$\text{Es } f(n) = O(n^{\log_b a - \varepsilon}) \quad ?$$

$$\begin{aligned} & \cap \\ & n^{\log_b a} \\ & n^{\log_{3/2} 1} = n^0 \end{aligned}$$

$$\text{Es } 1 = O(n^{0-\varepsilon}) \quad ?$$

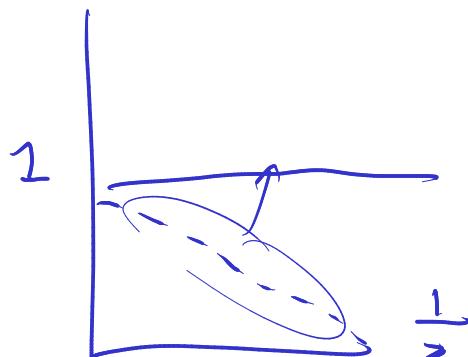
$$\begin{aligned} 1) \quad f(n) &\in \\ & O(n^{\log_b a - \varepsilon}) \end{aligned}$$

No existe  $\varepsilon > 0$

$$1 \in O(n^{0-\varepsilon})$$

$$1 \in O(n^{-1})$$

$$1 \in O\left(\frac{1}{n}\right)$$



# Recurrencias

$$T(n) = T(2n/3) + 1$$

$$n^{\log_{3/2} 1} = n^0 = 1 \quad \text{vs} \quad f(n) = 1$$

Es  $f(n) = \Theta(n^{\log_b a})$  ?  $1 \in \Theta(n^{\log_1 1})$

Es  $1 = \Theta(1)$  ?  $1 \in \Theta(n^0)$

Si, por lo tanto, se cumple que:  $1 \in \Theta(1)$

$$T(n) = \Theta(1 * \lg n) = \Theta(\lg n)$$

$$T(n) = \Theta\left(n^{\frac{\log_{1/3} 2}{\log_{1/3} 3}} \lg(n)\right)$$

# Recurrencias

$$T(n) = 3 T(n/4) + n \lg n$$

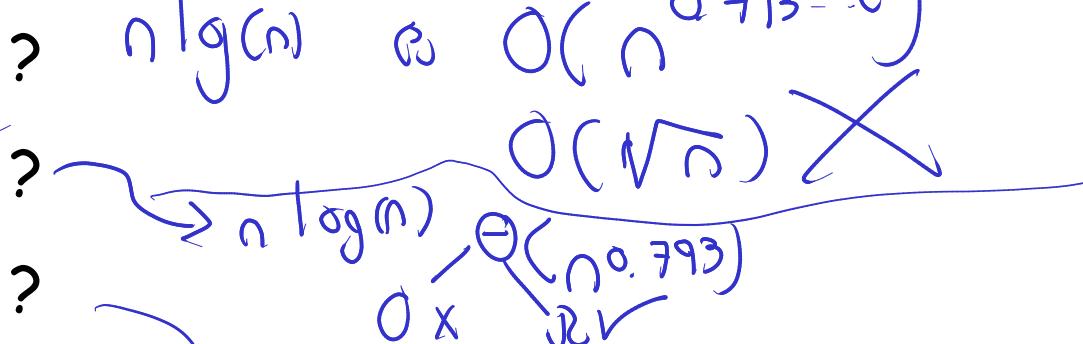
$$\alpha = 3 \quad f = 4$$

$$n^{\log_4 3} = n^{0.793}$$

vs  $f(n) = n \lg n$

$$n^{\log_b a} = n^{\log_3 4}$$

1) Es  $f(n) = O(n^{\log_b a - \varepsilon})$



Si, y además,  $af(n/b) \leq cf(n)$

$$3(n/4) \lg(n/4) \leq cn \lg n$$

$\Omega(n)$   $\Sigma$

$$3(n/4) \lg n - 3(n/4) * 2 \leq cn \lg n$$

$$(3/4)n \lg n \leq cn \lg n \rightarrow c = 3/4 \text{ y se concluye } T(n) = \Theta(n \lg n)$$

$c < 1$

# Recurrencias

---

$$T(n) = 2T(n/2) + nlgn$$

Muestre que no se puede resolver por el método maestro

# Recurrencias

Resuelva usando método del maestro

$$T(n) = 4T(n/2) + n$$

$$T(n) = 4T(n/2) + n^2$$

$$T(n) = 4T(n/2) + n^3$$

Dado  $T(n) = aT(n/b) + f(n)$ , donde  $a \geq 1$ ,  $b > 1$ , se puede acotar asintóticamente como sigue:

1.  $T(n) = \Theta(n^{\log_b a})$

Si  $f(n) = O(n^{\log_b a - \varepsilon})$  para algún  $\varepsilon > 0$

2.  $T(n) = \Theta(n^{\log_b a} \lg n)$

Si  $f(n) = \Theta(n^{\log_b a})$  para algún  $\varepsilon > 0$

3.  $T(n) = \Theta(f(n))$

Si  $f(n) = \Omega(n^{\log_b a + \varepsilon})$  para algún  $\varepsilon > 0$  si  $a * f(n/b) \leq c * f(n)$

$c < 1$

$$T(n) = 4 T\left(\frac{n}{2}\right) + n$$

$$n^{\log_b 4} = n^{\log_2 4} = n^2$$

4)  $n$  es  $\Theta(n^{2-\epsilon})$

$$n \in \Omega(n)$$

$$\Theta(n^2)$$

Dado  $T(n) = aT(n/b) + f(n)$ , donde  $a \geq 1$ ,  $b > 1$ , se puede acotar asintóticamente como sigue:

1.  $T(n) = \Theta(n^{\log_b a})$

Si  $f(n) = O(n^{\log_b a - \epsilon})$  para algún  $\epsilon > 0$

2.  $T(n) = \Theta(n^{\log_b a} \lg n)$

Si  $f(n) = \Theta(n^{\log_b a})$  para algún  $\epsilon > 0$

3.  $T(n) = \Theta(f(n))$

Si  $f(n) = \Omega(n^{\log_b a + \epsilon})$  para algún  $\epsilon > 0$  si  $a * f(n/b) \leq c * f(n)$

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2$$

1)  $n^2 \in O(n^{2-\varepsilon})$   $\times$

2)  $n^2 \in \Theta(n^2)$   $\checkmark$

$\Theta(n^2 \log(n))$

Dado  $T(n) = aT(n/b) + f(n)$ , donde  $a \geq 1$ ,  $b > 1$ , se puede acotar asintóticamente como sigue:

1.  $T(n) = \Theta(n^{\log_b a})$

Si  $f(n) = O(n^{\log_b a - \varepsilon})$  para algún  $\varepsilon > 0$

2.  $T(n) = \Theta(n^{\log_b a} \lg n)$

Si  $f(n) = \Theta(n^{\log_b a})$  para algún  $\varepsilon > 0$

3.  $T(n) = \Theta(f(n))$

Si  $f(n) = \Omega(n^{\log_b a + \varepsilon})$  para algún  $\varepsilon > 0$  si  $a * f(n/b) \leq c * f(n)$

$$3) T(n) = 4T\left(\frac{n}{2}\right) + n^3$$

1)  $n^3 \rightsquigarrow O(n^2 - \epsilon) \times$

2)  $n^3 \rightsquigarrow \Theta(n^2) \times$

3)  $n^3 \rightsquigarrow \Omega(n^{2+\epsilon})$   
 $\Omega(n^3)$

Dado  $T(n) = aT(n/b) + f(n)$ , donde  $a \geq 1$ ,  $b > 1$ , se puede acotar asintóticamente como sigue:

$$1. T(n) = \Theta(n^{\log_b a})$$

$$\text{Si } f(n) = O(n^{\log_b a - \epsilon}) \text{ para algún } \epsilon > 0$$

$$2. T(n) = \Theta(n^{\log_b a} \lg n)$$

$$\text{Si } f(n) = \Theta(n^{\log_b a}) \text{ para algún } \epsilon > 0$$

$$3. T(n) = \Theta(f(n))$$

$$\text{Si } f(n) = \Omega(n^{\log_b a + \epsilon}) \text{ para algún } \epsilon < 0 \quad \text{si } a * f(n/b) \leq c * f(n)$$

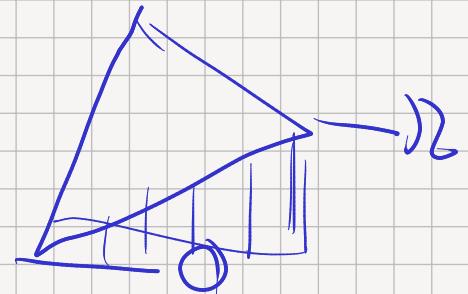
$$4 F\left(\frac{n}{2}\right) \leq C * f(n)$$

$$4 \left(\frac{n}{2}\right)^3 \leq C n^3 \rightarrow \frac{4n^3}{8} \leq C n^3 \quad \frac{1}{2} \leq C$$

$$C \geq \frac{1}{2}$$

$$\Theta(n^3)$$

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$$



$$T(n) = 2T\left(\frac{n}{3}\right) + n$$

$$n^{\log_3 2} = n^{\log_3 2}$$

1)  $n \in O(n^{0.63})$   ~~$\times$~~

2)  $n \in \Theta(n^{0.63})$   ~~$\times$~~

3)  $n \in \Omega(n^{0.63+\epsilon})$

$$T(n)$$

$$O(f(c/n)) \leq c \times f(n)$$

$$2\left(\frac{n}{3}\right) \leq c \times n$$

$$c \geq \frac{2}{3}$$

$$T(n) = 2T\left(\frac{2n}{3}\right) + n$$

$$n^{\log_3 2}$$

$$n^{\frac{\log_2 2}{\log_2 \frac{2}{3}}} = n^{\frac{1}{1}}$$

1)  $n \in O(n^{1.70-\epsilon})$

$$\Omega(n)$$

$$T(n) = \Theta(n^{1.70})$$

$$\Theta(n n^{0.7})$$

$$\Theta(n \sqrt{n})$$

$$O \geq \frac{1}{x}$$

$$x = \frac{1}{0.7}$$

# Recurrencias

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Método de sustitución

Suponer la forma de la solución y probar por inducción matemática

# Recurrencias

---

$$T(n) = 2T(\lfloor n/2 \rfloor) + n, T(1) = 1$$

Suponer que la solución es de la forma  $T(n)=O(n\lg n)$

Probar que  $T(n) \leq cn\lg n$ .

Se supone que se cumple para  $\underline{n/2}$  y se prueba para  $n$

Hipótesis inductiva:  $T(n/2) \leq c(n/2)\lg(n/2)$

# Recurrencias

---

$$T(n) = 2T(\lfloor n/2 \rfloor) + n, T(1) = 1$$

Probar que  $T(n) \leq cn \lg n$ .

Hipótesis inductiva:  $T(n/2) \leq cn/2 \lg (n/2)$

Paso inductivo:

$$T(n) \leq 2(c\lfloor n/2 \rfloor \lg (n/2)) + n$$

$$(n \lg n) - n \cancel{+ n}$$

$$\leq cn \lg (n/2) + n$$

$$= cn \lg (n) - cn + n, \text{ para } c \geq 1, \text{ haga } c=1$$

$$\leq cn \lg n$$

# Recurrencias

---

$$T(n) = 2T(\lfloor n/2 \rfloor) + n, \quad T(1) = 1$$

Probar que  $T(n) \leq cn\lg n$ .

Paso base: si  $c=1$ , probar que  $T(1)=1$  se cumple

$$T(1) \leq 1 * 1 \lg 1 ?$$

$$1 \leq 0 ?$$

No, se debe escoger otro valor para  $c$

# Recurrencias

---

$$T(n) = 2T(\lfloor n/2 \rfloor) + n, T(1) = 1$$

Probar que  $T(n) \leq cn\lg n$ .

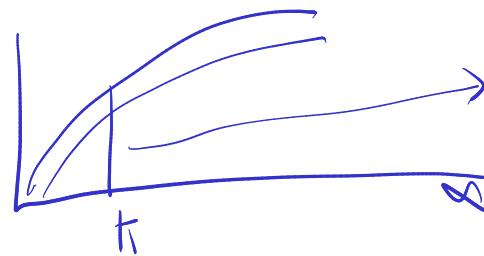
Paso base: si  $c=2$ , probar que  $T(1)=1$  se cumple

$$T(1) \leq 2 * 1 \lg 1 ?$$

$$1 \leq 0 ?$$

No, se puede variar  $k$ .

Para esto, se calcula  $T(2)$  y se toma como valor inicial



# Recurrencias

---

Probar que  $T(n) \leq cn\lg n$ .

$$T(2) = 2T(0) + 2 = 4$$

Paso base: si  $c=1$ , probar que  $T(2)=4$  se cumple

$$T(2) \leq 1 * 2 \lg 2 ?$$

$$4 \leq 2 ?$$

No, se puede variar  $c$ .

# Recurrencias

---

Probar que  $T(n) \leq cn\lg n$ .

$$T(2) = 2T(0) + 2 = 4$$

Paso base: si  $c=3$ , probar que  $T(2)=4$  se cumple

$$T(2) \leq 3 \cdot 2 \lg 2 ?$$

$$4 \leq 6 ?$$

Si, se termina la demostración

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$

$$T(1) = 1$$

$$\log_2 n = \log_2 2 = n^{\frac{1}{2}}$$

$$T(n) \leq C \times n$$

$$n \quad T(n) \geq C \times n$$

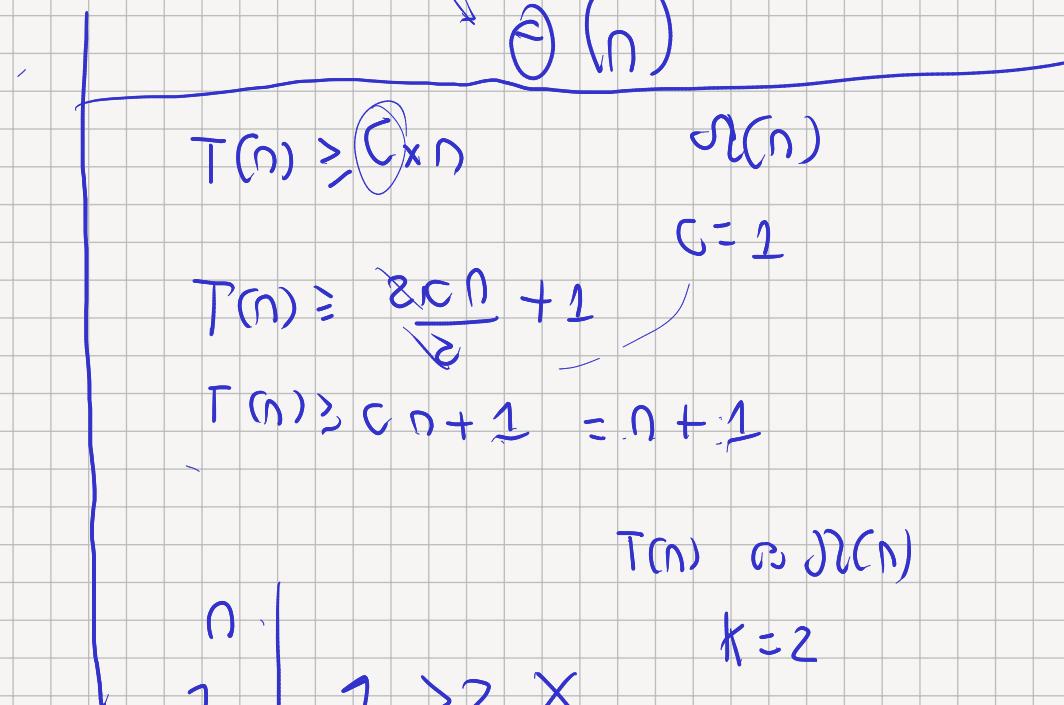
$\Theta(n)$

$$T(n) \leq \frac{2Cn}{2} + 1$$

$$T(n) \leq Cn + 1 \quad C=4$$

$$T(n) \leq 4n + 1$$

n	
1	
2	
4	
	$1 \leq 5 \checkmark$ $3 \leq 13 \checkmark$ $7 \leq 17 \checkmark$



n	
1	
2	
4	$1 \geq 2 \times \cancel{X}$ $3 \geq 3 \checkmark$ $7 \geq 5 \checkmark$

$$T(n) \approx \Omega(n)$$

$k=2$

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2 \quad T(1) = 1$$

$$n^{\log_2 9} = n^{\log_2 2} = n$$

$$T(n) \in \Theta(n^2)$$

$$T(n) \in O(n^2)$$

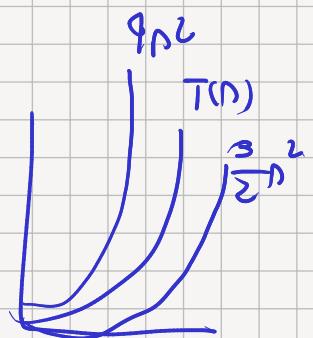
$$T(n) \leq \underline{C}n^2$$

$$T(n) \leq 2\left(C\left(\frac{n}{2}\right)^2\right) + n^2$$

$$C = 16$$

$$T(n) \leq 2\frac{Cn^2}{4} + n^2 = 8n^2 + n^2 = 9n^2$$

	$T(n)$	$9n^2$
1	1	9 ✓
2	6	36 ✓
4	28	144 ✓
8	120	576 ✓



1)  $n^2 \in O(n^{-\epsilon}) \times$

2)  $n^2 \in \Theta(n) \times$

3)  $n^2 \in \Omega(n^{2+\epsilon})$

$$\Omega(n^2)$$

$$aF\left(\frac{n}{b}\right) \leq C \times n^2 \quad c < 1$$

$$2\left(\frac{n}{2}\right)^2 \leq C \times n^2 \Rightarrow \frac{1}{2} \leq C \checkmark$$

$$\Theta(F(n)) = \Theta(n^2)$$

$$T(n) \in \Omega(n^2)$$

$$T(n) \geq 2\frac{Cn^2}{4} + n^2 = \frac{1}{2}n^2 + n^2 = \frac{3}{2}n^2$$

	$T(n)$	$\frac{3}{2}n^2$	$k \geq 3$
1	1	$\frac{3}{2}$ ✓	X
2	6	6 ✓	
4	28	24 ✓	
8	120	96 ✓	

# Recurrencias

---

$$T(n) = T(n-1) + T(n-2) + 1, \quad T(1) = O(1), \quad T(2) = O(1)$$

Suponer que la solución es de la forma  $T(n)=O(2^n)$

Probar que  $T(n) \leq c2^n$ .

Se supone que se cumple para  $n-1$  y se prueba para  $n$

Hipótesis inductiva:  $T(n-1) \leq c2^{(n-1)}$  y  $T(n-2) \leq c2^{(n-2)}$

$$T(n) \leq C 2^{n-1} + C 2^{n-2} + 1$$

$$\begin{aligned} T(n) &\leq C \left( \frac{2^n}{2} + \frac{2^n}{4} \right) + 1 \leq C \times \frac{3}{4} 2^n + 1 \\ &\leq 3 \times 2^n + 1 \end{aligned}$$

$C=4$

$$T(1) = 10$$

$$T(2) = 20$$

$O(2^n)$

$n$	$T(n)$	$3 \times 2^n + 1$
1	10	7 X
2	20	13 X
3	30	25 X
4	80	49 X
5	80	97 ✓

$T(n) \in O(2^n)$

ANS

$n$	$-$	$3 \times 2^n + 1$
6	130	193
7	210	385
8	340	769
9	550	1837

# Recurrencias

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$$T(n) = T(n-1) + T(n-2) + 1, \quad T(1) = O(1), \quad T(2) = O(1)$$

Ahora se debe probar que:  $T(n) \leq c2^n$

$$T(1) \leq c2^1 \rightarrow 1 \leq 2*c$$

$$T(2) \leq c2^2 \rightarrow 1 \leq 4*c$$

$$T(3) \leq c2^3 \rightarrow 2 \leq 8*c$$

$$T(4) \leq c2^4 \rightarrow 3 \leq 16*c$$

$$T(5) \leq c2^5 \rightarrow 5 \leq 32*c$$

$$T(6) \leq c2^6 \rightarrow 8 \leq 64*c$$

$$T(7) \leq c2^7 \rightarrow 13 \leq 128*c$$

$$T(8) \leq c2^8 \rightarrow 21 \leq 256*c$$

Con  $c=1$ , se cumple.

# Referencias

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Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. *Introduction to Algorithms*, Third Edition (3rd ed.). The MIT Press. Chapter 4

# Gracias

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Próximo tema:

Divide y vencerás