Fundamentos de análisis y diseño de algoritmos

Ecuaciones de recurrencia

```
def factorial(n):
    if n==0:
        return 1
    else:
        return n*factorial(n-1)
```

$\sim n = 0 , F(n) = 1$

$F(0) = 1$

$F(n) = n \times F(n-1)$

$$F(n) = \begin{cases} 1 & n=0 \\ n \times F(n-1) & n > 0 \end{cases}$$

Recurrencias

Método de iteración

Método maestro*

Método de sustitución

Recurrencias

Método de iteración

Expandir la recurrencia y expresarla como una suma de términos que dependen de n y de las condiciones iniciales

Recurrencias

$T(n) = n + 3T(n/4)$, $T(1) = \Theta(1)$ y n par

Expandir la recurrencia 2 veces

0) $\qquad T(n) = 3\boxed{T\left(\dfrac{n}{4}\right)} + n$

1) $\quad T(n) = 3\left(3T\left(\dfrac{n}{4^2}\right) + \dfrac{n}{4}\right) + n$

$\qquad T(n) = 3^2\boxed{T\left(\dfrac{n}{4^2}\right)} + 3\dfrac{n}{4} + n$

2) $\quad T(n) = 3^2\left(3T\left(\dfrac{n}{4^3}\right) + \dfrac{n}{4^2}\right) + 3\dfrac{n}{4} + n$

$\qquad T(n) = 3^3 T\left(\dfrac{n}{4^3}\right) + \dfrac{3^2 n}{4^2} + \dfrac{3n}{4} + n$

Recurrencias

$T(n) = n + 3T(n/4)$

$n + 3(n/4 + 3T(n/16))$

$n + 3(n/4 + 3(n/16 + 3T(n/64)))$

$n + 3*n/4 + 3^2*n/4^2 + 3^3T(n/4^3)$

Recurrencias

$T(n) = n + 3T(n/4)$

$\qquad n + 3 (n/4 + 3T(n/16))$

$\qquad n + 3 (n/4 + 3(n/16 + 3T(n/64)))$

$\qquad n + 3*n/4 + 3^2*n/4^2 + 3^3 T(n/4^3)$

¿Cuándo se detienen las iteraciones?

$T(n/4^4)$

$T(n/4^5)$

Recurrencias

1) $T(n) = n + 3T(n/4)$

$\qquad n + 3 (n/4 + 3T(n/16))$

2) $\quad n + 3 (n/4 + 3(n/16 + 3T(n/64)))$

3) $\quad n + 3*n/4 + 3^2*n/4^2 + 3^3 T(n/4^3)$

$1^?$

¿Cuándo se detienen las iteraciones?

Cuando se llega a T(1)

$$T\left(\frac{n}{4i}\right) \rightarrow T(1)$$

$$\frac{n}{4^i} = 1$$

$$\boxed{i = \log_4(n)}$$

Recurrencias

$T(n) = n + 3T(n/4)$

$\quad\quad n + 3 (n/4 + 3T(n/16))$

$\quad n + 3 (n/4 + 3(n/16 + 3T(n/64)))$

$\quad n + 3*n/4 + 3^2*n/4^2 + 3^3T(n/4^3)$

¿Cuándo se detienen las iteraciones?

Cuando se llega a T(1), esto es, cuando $(n/4^i)=1$

Recurrencias

$T(n) = n + 3T(n/4)$

$n + 3 (n/4 + 3T(n/16))$

$n + 3 (n/4 + 3(n/16 + 3T(n/64)))$

$n + 3*n/4 + 3^2*n/4^2 + 3^3 T(n/4^3)$

$n + 3*n/4 + 3^2*n/4^2 + 3^3 n/4^3 + \ldots + 3^{\log 4n} T(1)$

¿Cuándo se detienen las iteraciones?

Cuando se llega a T(1), esto es, cuando $(n/4^i)=1$

Recurrencias

$T(n) = n + 3T(n/4)$

$\qquad n + 3 (n/4 + 3T(n/16))$

$\quad n + 3 (n/4 + 3(n/16 + 3T(n/64)))$

$\quad n + 3*n/4 + 3^2*n/4^2 + 3^3T(n/4^3)$

$\quad n + 3*n/4 + 3^2*n/4^2 + 3^3n/4^3 + \ldots + 3^{log4n}T(1)$

Después de iterar, se debe tratar de expresar como una sumatoria con forma cerrada conocida

Recurrencias

$T(n) = n + 3T(n/4\rfloor)$

$n + 3 (n/4\rfloor + 3T(n/16\rfloor))$

$n + 3 (n/4\rfloor + 3(n/16\rfloor + 3T(n/64\rfloor)))$

$n + 3*n/4\rfloor + 3^2*n/4^2\rfloor + 3^3(n/4^3\rfloor) + \ldots + 3^{\log 4n}\Theta(1)$

$\leq n + 3n/4 + 3^2n/4^2 + 3^3n/4^3 + \ldots + 3^{\log 4n}\Theta(1)$

Recurrencias

$T(n) = n + 3T(n/4)$

$n + 3 (n/4 + 3T(n/16))$

$n + 3 (n/4 + 3(n/16 + 3T(n/64)))$

$n + 3*n/4 + 3^2*n/4^2 + 3^3(n/4^3) + \ldots + 3^{log4n}\Theta(1)$

$\leq n + 3n/4 + 3^2n/4^2 + 3^3n/4^3 + \ldots + 3^{log4n}\Theta(1)$

$= \left(\sum_{i=0}^{\log_4 n} \left(\frac{3}{4}\right)^i n\right) + 3^{\log_4 n}\Theta(1)$

$= n\left(\frac{(3/4)^{(\log_4 n)} - 1}{(3/4) - 1}\right) + n^{\log_4 3} = n*4(1 - (3/4)^{(\log_4 n)}) + \Theta(n^{\log_4 3})$

$= O(n)$

$$T(n) = 3T\left(\frac{n}{4}\right) + n$$

$$T(n) = 3\left(3T\left(\frac{n}{4^2}\right) + \frac{n}{4}\right) + n = 3^2 T\left(\frac{n}{4^2}\right) + \frac{3}{4}n + n$$

$$T(n) = 3^2\left(3T\left(\frac{n}{4^3}\right) + \frac{n}{4^2}\right) + \frac{3}{4}n + n$$

2) $$T(n) = 3^3 T\left(\frac{n}{4^3}\right) + \frac{3^2}{4^2}n + \frac{3}{4}n + (1)n$$

i) $$T(n) = 3^i T\left(\frac{n}{4^i}\right) + \left(\frac{3}{4}\right)^{i-1}n + \left(\frac{3}{4}\right)^{i-2}n + \ldots + \left(\frac{3}{4}\right)^0 n$$

$$\frac{n}{4^i} = 1 \qquad i = \log_4(n)$$

$$T(n) = 3^{\log_4(n)} T(1) + \left(\frac{3}{4}\right)^{\log_4(n)-1}n + \left(\frac{3}{4}\right)^{\log_4(n)-2}n + \ldots + \left(\frac{3}{4}\right)^1 n + \left(\frac{3}{4}\right)^0 n$$

$$T(n) = n^{\log_4(3)} T(1) + \sum_{i=0}^{\log_4(n)-1}\left(\frac{3}{4}\right)^i n \qquad \sum_{i=0}^{n} ar^i = \frac{ar^{n+1}-a}{r-1}$$

$$a = n \qquad r = \frac{3}{4} \qquad n = \log_4(n)-1$$

$$T(n) = n^{\log_4(3)} T(1) + \frac{n\left(\frac{3}{4}\right)^{\log_4(n)} - n}{\frac{3}{4}-1}$$

$$\boxed{T(n) = n^{\log_4(3)} T(1) + \frac{n \times n^{\log_4\left(\frac{3}{4}\right)} - n}{-\frac{1}{4}}}$$

$$T(1) = \Theta(1)$$

$$\boxed{T(1) = C}$$

Recurrencias

Resuelva por el método de iteración

$T(n) = 2T(n/2) + 1, T(1) = \Theta(1)$

1) $T(n) = 2\left(2T\left(\dfrac{n}{2^2}\right) + 1\right) + 1 = 2^2 T\left(\dfrac{n}{2^2}\right) + 2 + 1$

2) $T(n) = 2^2\left(2T\left(\dfrac{n}{2^3}\right) + 1\right) + 2 + 1 = 2^3 T\left(\dfrac{n}{2^3}\right) + 2^2 + 2^1 + 2^0$

i) $T(n) = 2^i T\left(\dfrac{n}{2^i}\right) + 2^{i-1} + 2^{i-2} + \ldots + 2^1 + 2^0$

$T(1)$

$\dfrac{n}{2^i} = 1 \quad n = 2^i \quad i = \log_2(n)$

$$T(n) = 2^{\log_2(n)} T(1) + 2^{\log_2(n)-1} + 2^{\log_2(n)-2} + \dots + 2^1 + 2^0$$

$$T(n) = n \, T(1) + \sum_{i=0}^{\log_2(n)-1} 2^i$$

$$T(n) = c \, n + \frac{2^{\log_2(n)} - 1}{2 - 1}$$

$$T(n) = c \, n + n - 1 \qquad O(n)$$

Recurrencias

Resuelva por el método de iteración

$\times$ T(n) = 2T(n/2) + 1, T(1)= Θ(1)

T(n) = 2T(n/2) + n, T(1)= Θ(1)

$$T(n) = 2\boxed{T(n/2)} + n, \; T(1) = \Theta(1)$$

$$T(n) = 2\left(2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + \overset{n}{\boxed{n}} = 2^2 T\left(\frac{n}{2^2}\right) + n + n$$

$$T(n) = 2^2\left(2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + \overset{2n}{\boxed{n+n}}$$

$$T(n) = 2^3 T\left(\frac{n}{2^3}\right) + \overset{3n}{\boxed{n+n+n}}$$

$$T(n) = 2^i T\left(\frac{n}{2^i}\right) + i\,n$$

$$T(1) = T\left(\frac{n}{2^i}\right) \qquad \frac{n}{2^i} = 1 \qquad i = \log_2(n)$$

$$T(n) = 2^{\log_2(n)} T(1) + \log_2(n) \times n$$

$$T(n) = n \times T(1) + \log_2(n) \times n \qquad O(n \log_2(n))$$

Recurrencias

Resuelva por el método de iteración

$T(n) = 2T(n/2) + 1$, $T(1)= \Theta(1)$

$T(n) = 2T(n/2) + n$, $T(1)= \Theta(1)$

$T(n) = T(n/2) + 1$, $T(1)= \Theta(1)$

Recurrencias

Resuelva por el método de iteración

$T(n) = 2T(n/2) + 1$, $T(1) = \Theta(1)$

$T(n) = 2T(n/2) + n$, $T(1) = \Theta(1)$

$T(n) = T(n/2) + 1$, $T(1) = \Theta(1)$

Demuestre que $T(n) = T(n/2]) + n$, es $\Omega(n\log n)$

$$T(n) = 5T(n/4) + n,$$

$$T(n) = 5\left(5T\left(\frac{n}{4^2}\right) + \frac{n}{4}\right) + n = 5^2 T\left(\frac{n}{4^2}\right) + \frac{5}{4}n + n$$

$$T(n) = 5^2\left(5T\left(\frac{n}{4^3}\right) + \frac{n}{4^2}\right) + \frac{5}{4}n + n$$

$$\boxed{T(n) = 5^3 T\left(\frac{n}{4^3}\right) + \frac{5^2}{4^2}n + \frac{5}{4}n + n}$$

$$T(n) = 5^i T\left(\frac{n}{4^i}\right) + \left(\frac{5}{4}\right)^{i-1} n + \left(\frac{5}{4}\right)^{i-2} n + \dots + \left(\frac{5}{4}\right)^{1} n + \left(\frac{5}{4}\right)^{0} n$$

$$T(1) \qquad \frac{n}{4^i} = 1 \qquad i = \log_4(n)$$

$$T(n) = \boxed{5^{\log_4(n)} \, T(1) +} \sum_{i=0}^{\log_4(n)-1} \left(\frac{S}{4}\right)^i n$$

$$T(n) = n^{\log_4(S)} \, C \times n + n \left(\frac{\left(\frac{S}{4}\right)^{\log_4(n)} - 1}{\frac{S}{4} - 1} \right)$$

$$T(n) = n^{\log_4(S)} \times C \times n + 4n \left(n^{\log_4\left(\frac{S}{4}\right)} - 1 \right)$$
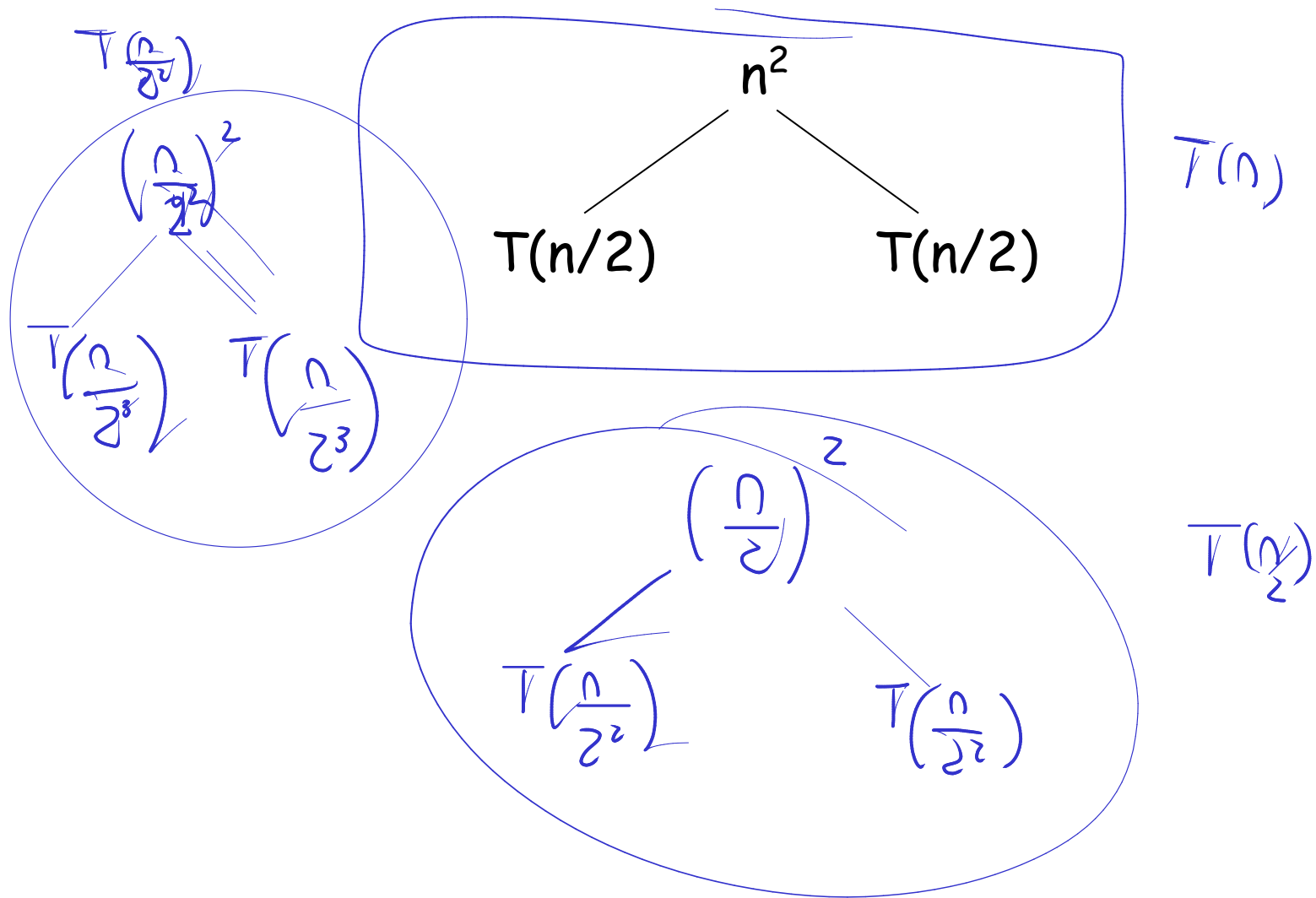
$$T(1) = C \times n$$

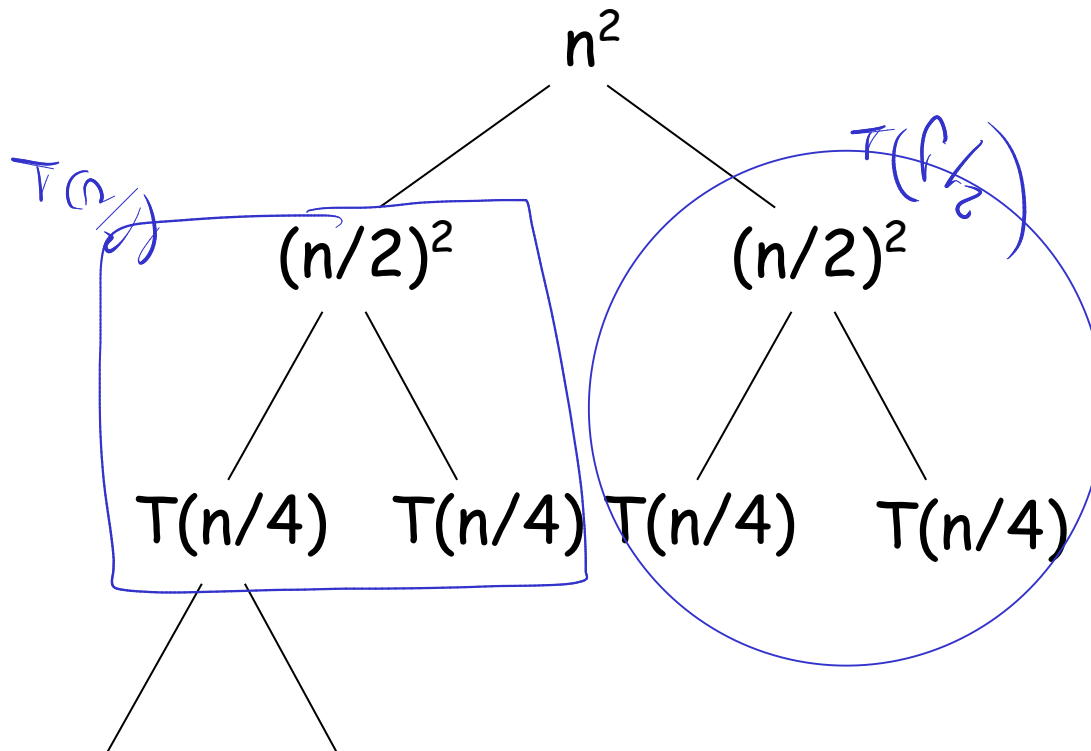$$\boxed{T(1) = C}$$

Recurrencias

Iteración con árboles de recursión
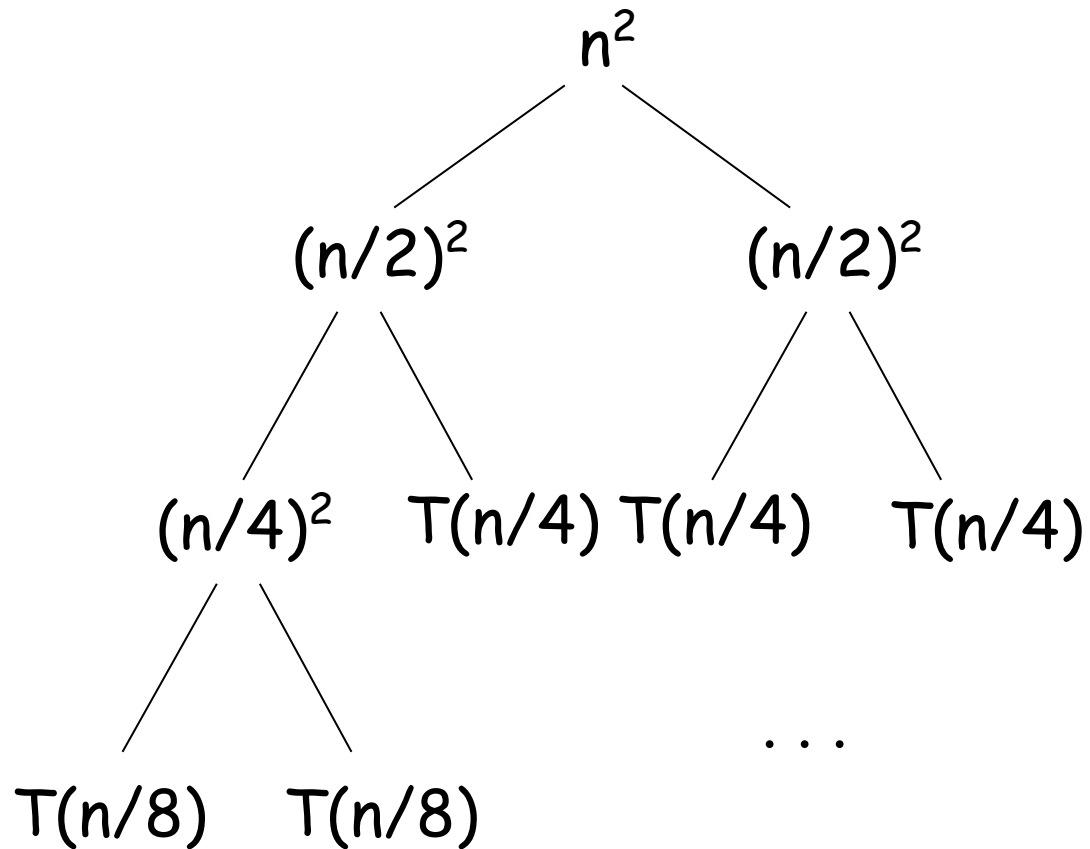
$T(1) = 1$

$T(n) = 2T(n/2) + n^2$

Recurrencias

$T\left(\frac{n}{2^4}\right)$

$\left(\frac{n}{2^4}\right)^2$

$T\left(\frac{n}{2^3}\right)$ $T\left(\frac{n}{2^3}\right)$

n^2

$T(n/2)$ $T(n/2)$

$T(n)$

$\left(\frac{n}{2}\right)^2$

$T\left(\frac{n}{2^2}\right)$ $T\left(\frac{n}{2^2}\right)$

$T\left(\frac{n}{2}\right)$

Recurrencias

$$n^2$$

$$(n/2)^2 \qquad (n/2)^2$$

$$T(n/4) \quad T(n/4) \quad T(n/4) \quad T(n/4)$$

. . .

Recurrencias

Recurrencias

0

1

???

$\log_2(n)$

2

3

n^2

$(n/2)^2$ $(n/2)^2$

$(n/4)^2$ $T(n/4)$ $T(n/4)$ $T(n/4)$

$i \Rightarrow T\left(\dfrac{n}{2^i}\right)$

$T(n/8)$ $T(n/8)$ . . .

$\dfrac{n}{2^i} = 1$

$T(1)$ . . . $T(1)$ . . . $i = \log_2(n)$
$T(1)$

Recurrencias


```
                          n²

            (n/2)²                    (n/2)²

      (n/4)²      T(n/4)   T(n/4)        T(n/4)

   T(n/8)  T(n/8)              . . .

     T(1)        . . .      T(1)        . . .
       T(1)
```

???

$(n/2^i)^2=1$

$(n/2^i)=1$

$n=2^i$

$\log n=i$

Recurrencias

$2^0 \Rightarrow$ n^2 $\longrightarrow n^2$

$2^1 \longrightarrow$ $(n/2)^2$ $(n/2)^2$ $\rightarrow \dfrac{n^2}{8^2} + \dfrac{n^2}{2^2} = \dfrac{2n^2}{2^2}$

lg n

2^2 $(n/4)^2$ T(n/4)   T(n/4)   T(n/4) $= 4\left(\dfrac{n}{2^2}\right)^2$

T(n/8)   T(n/8)   . . . $8\left(\dfrac{n}{2^3}\right)^2$

T(1)   . . .   T(1)   . . .
T(1)

Recurrencias

lg n

n^2 → n^2

$(n/2)^2$ $(n/2)^2$ → $n^2/2$

$(n/4)^2$ $T(n/4)$ $T(n/4)$ $T(n/4)$

$T(n/8)$ $T(n/8)$ . . .

$T(1)$ . . . $T(1)$ . . .

$T(1)$

Recurrencias

Recurrencias

lg n

lg n -1

0

1

2

n^2 → n^2

$(n/2)^2$ $(n/2)^2$ → $n^2/2$

$(n/4)^2$ T(n/4) T(n/4) T(n/4) → $n^2/4$

$2^{\log_2(n)}$

T(1) T(1)
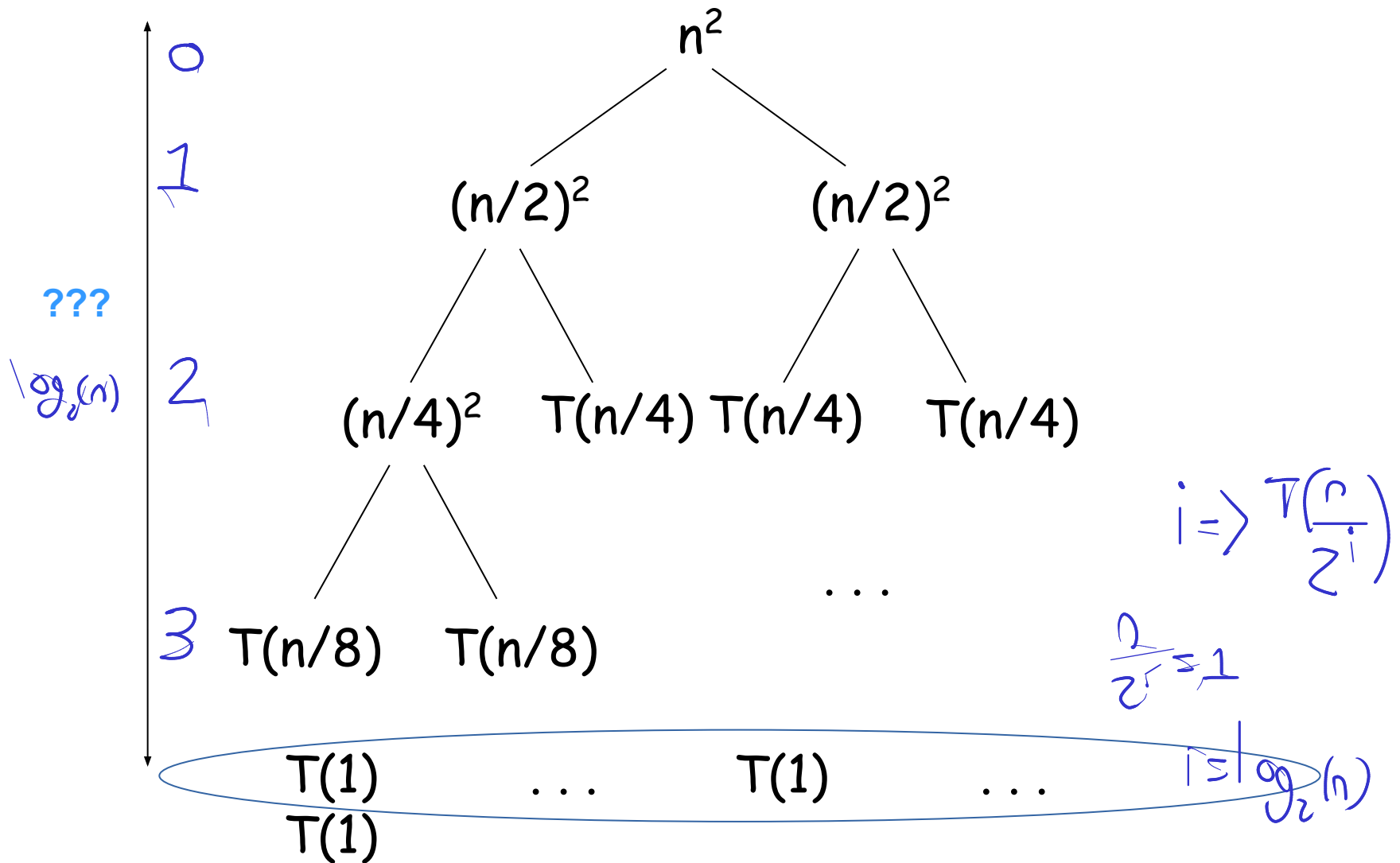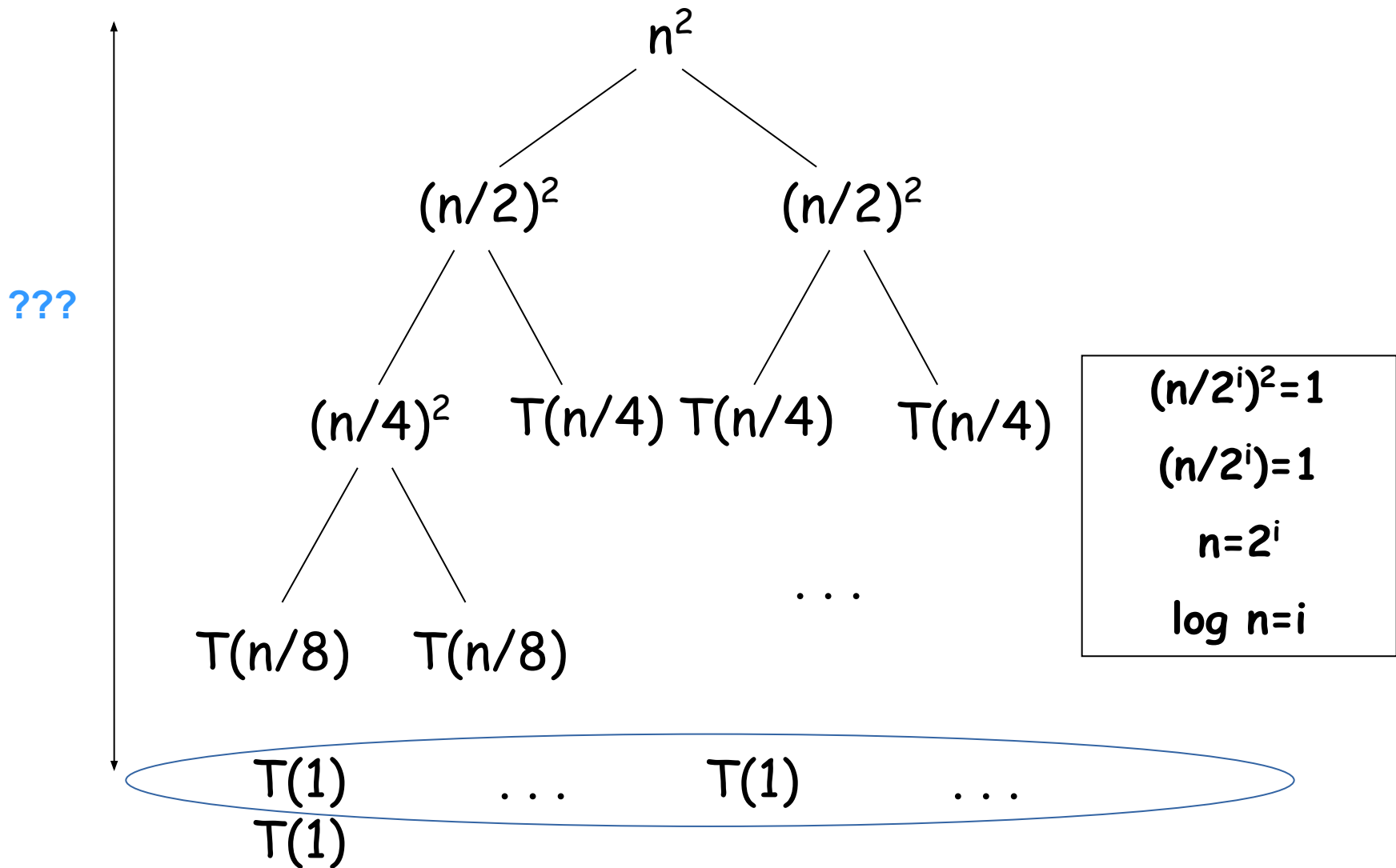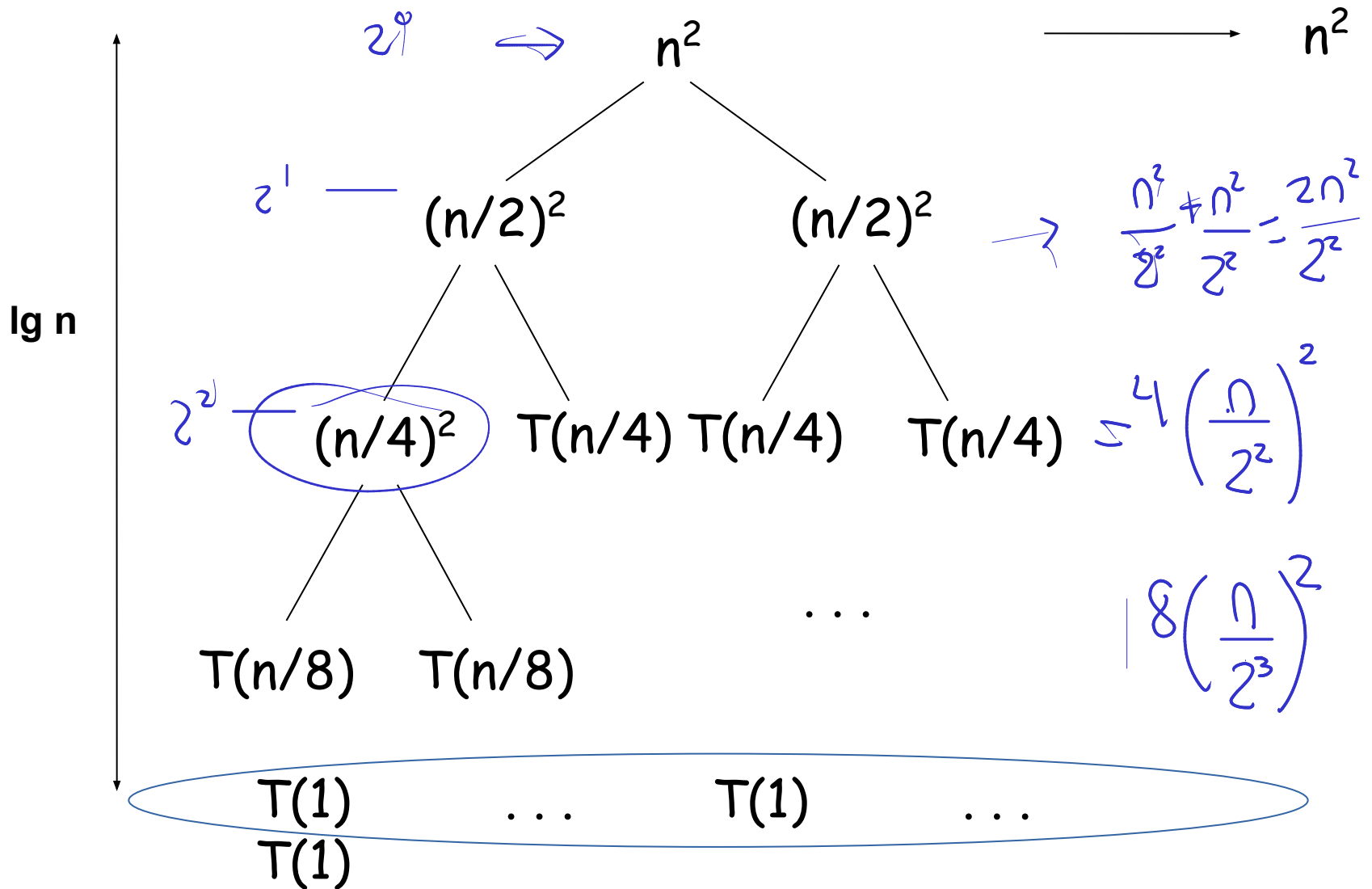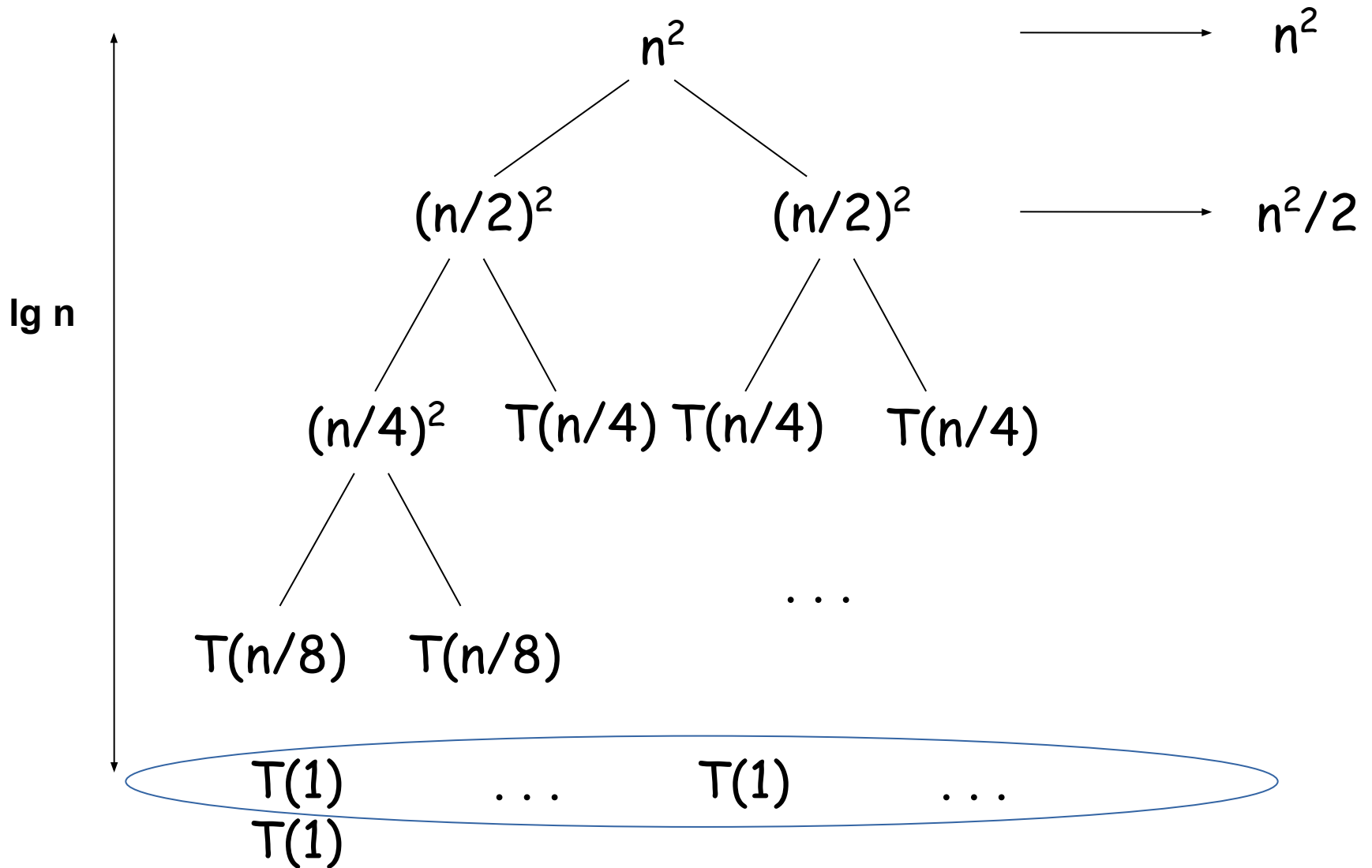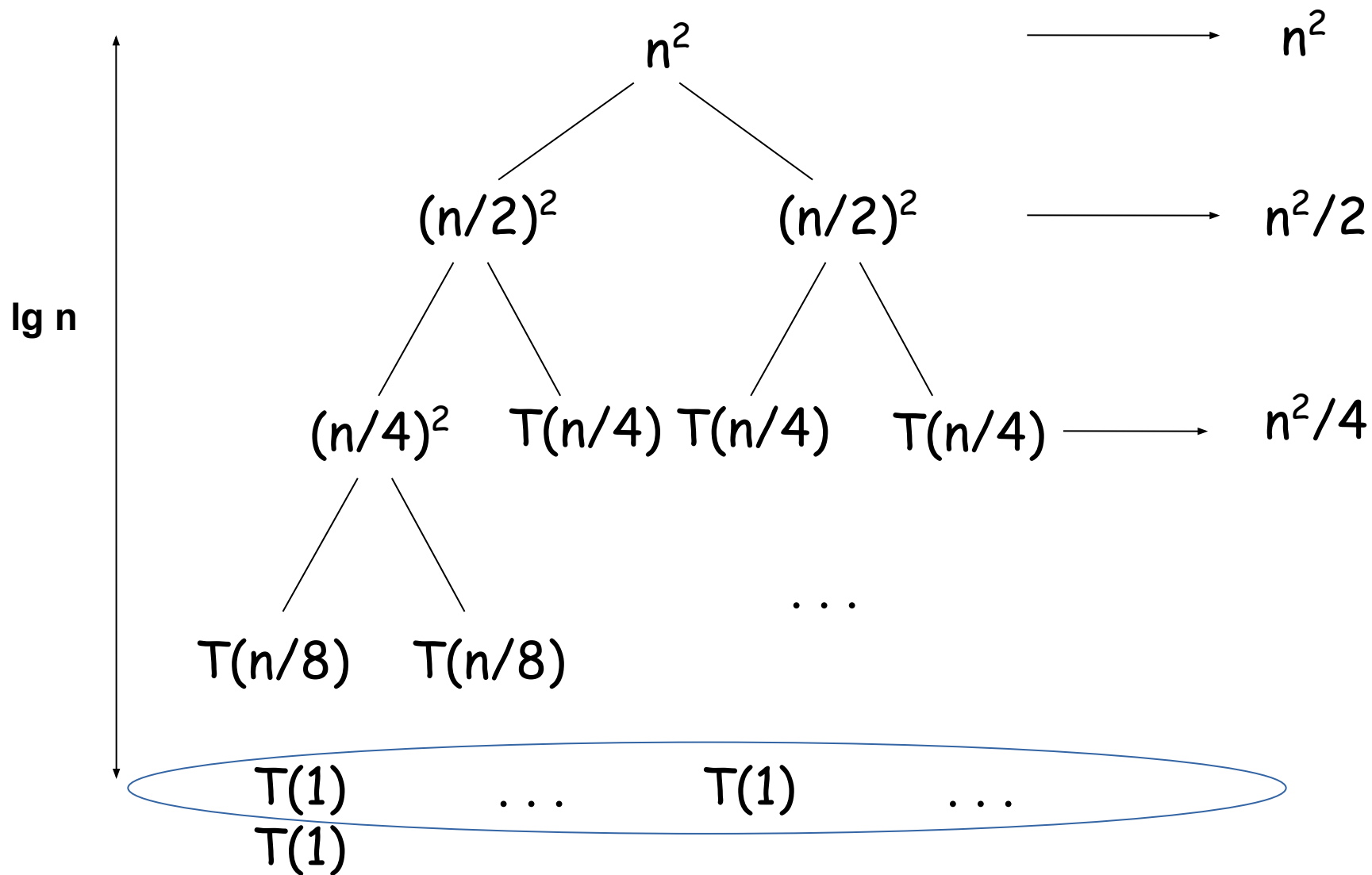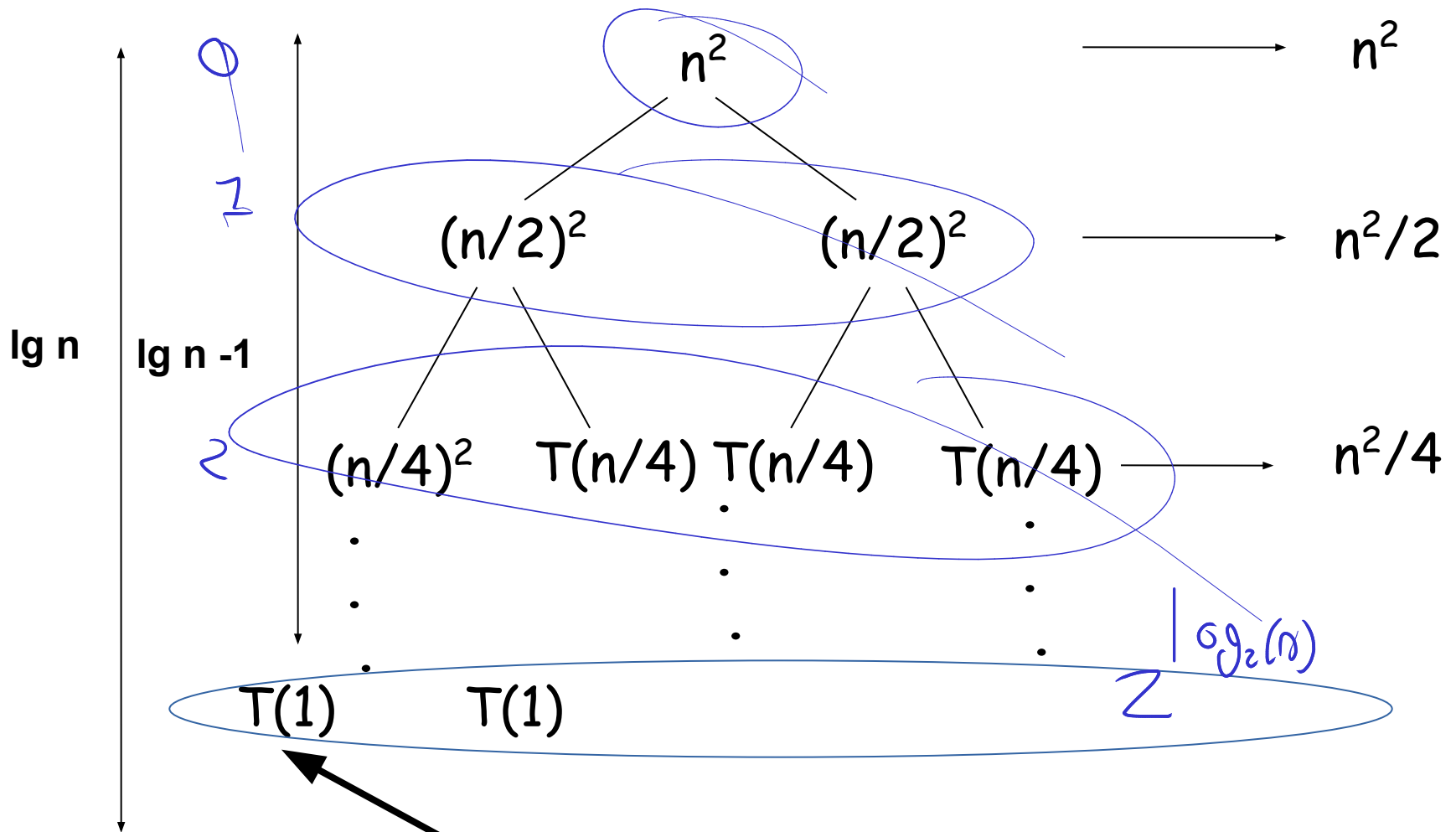
¿Cuantas hojas T(1)
tenemos?

Recurrencias

$$n^2 \longrightarrow n^2$$

$$(n/2)^2 \qquad (n/2)^2 \longrightarrow n^2/2$$

lg n | lg n -1

$$(n/4)^2 \quad T(n/4) \; T(n/4) \quad T(n/4) \longrightarrow n^2/4$$

$$T(1) \qquad T(1)$$
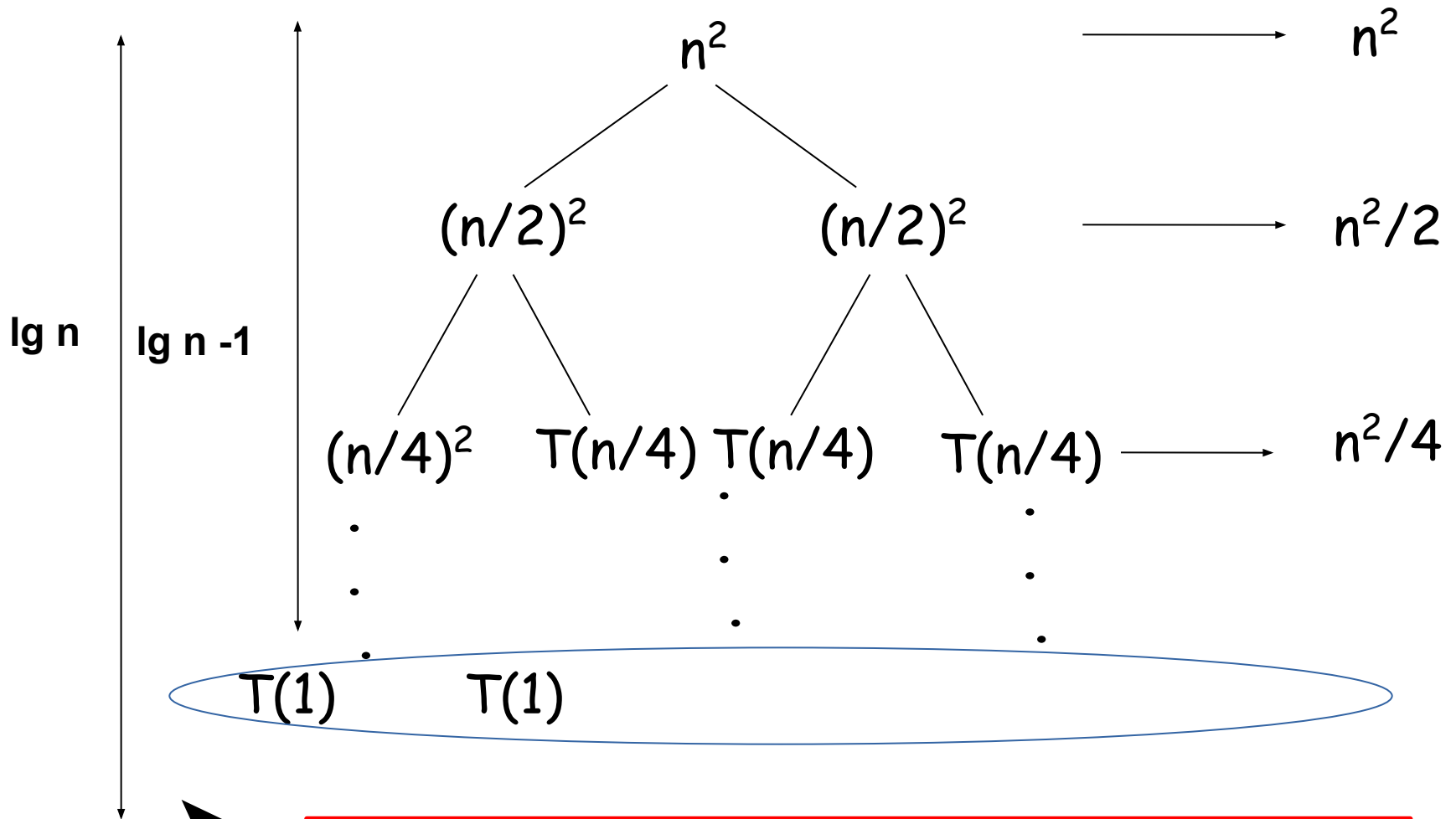
Si recuerda en un árbol m-ario se tienen máximo m^h. En este caso al ser arbol binario m=2, tenemos $2^{\log(n)}$ hojas. Por lo tanto se

Recurrencias
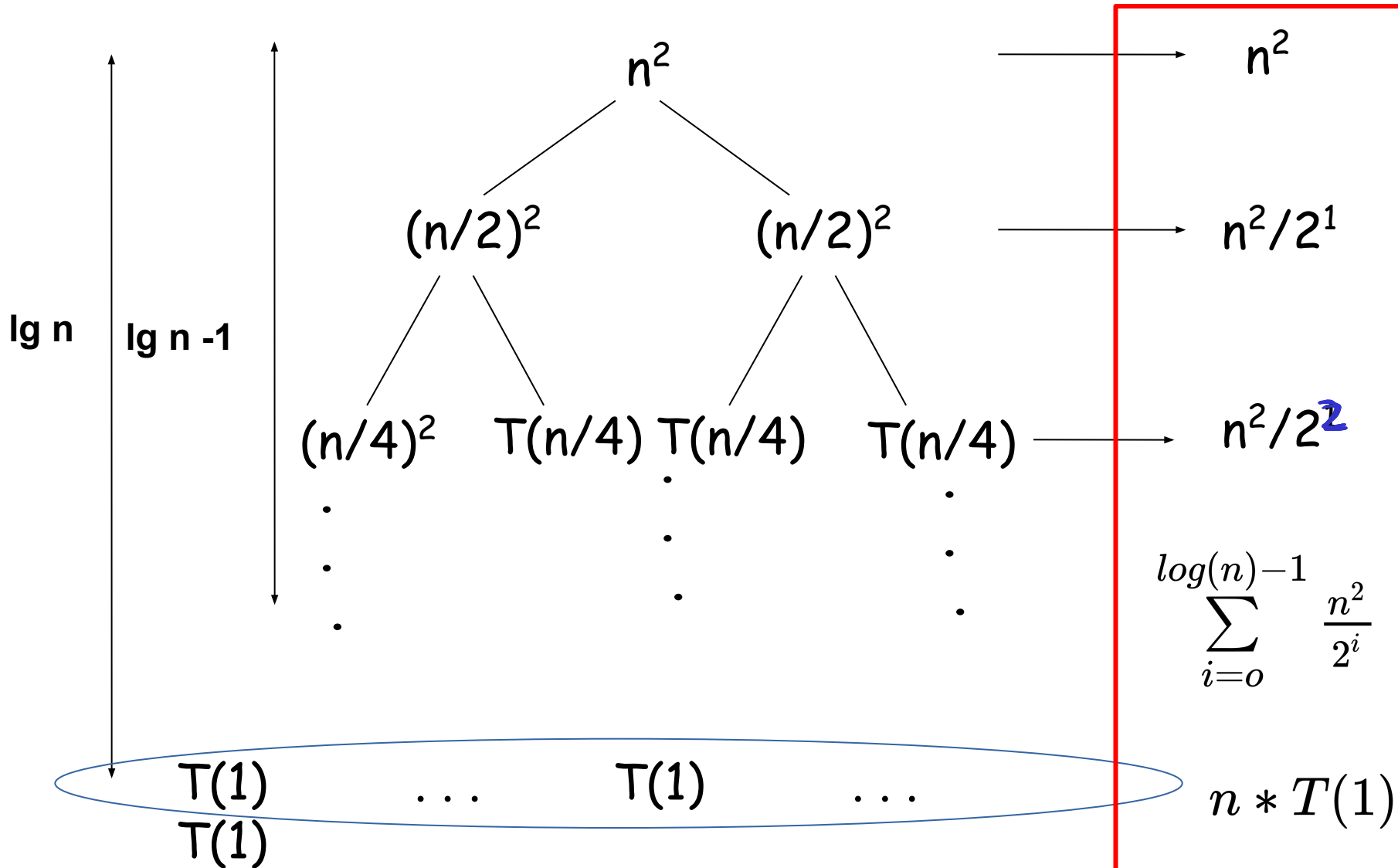
Recurrencias

$$T(n) = n * T(1) + \sum_{i=o}^{log(n)-1} \frac{n^2}{2^i}$$

$$T(n) = n * c + n^2 \frac{0.5^{log(n)} - 1}{0.5 - 1}$$

$$T(n) = n * c + n^2 \frac{n^{log(0.5)} - 1}{-0.5}$$

$$T(n) = n * c + n^2 \frac{n^{-1} - 1}{-0.5}$$

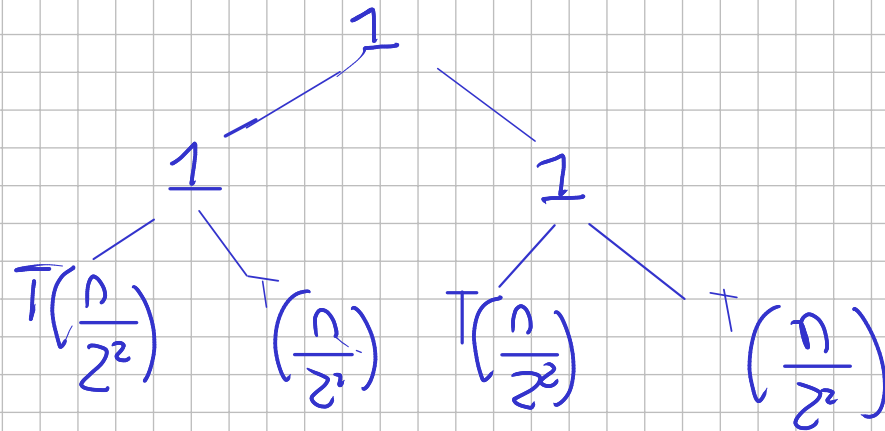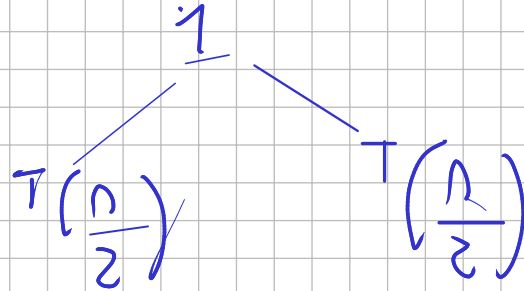$$T(n) = n * c - \frac{n}{0.5} + \frac{n^2}{0.5} = O(n^2)$$
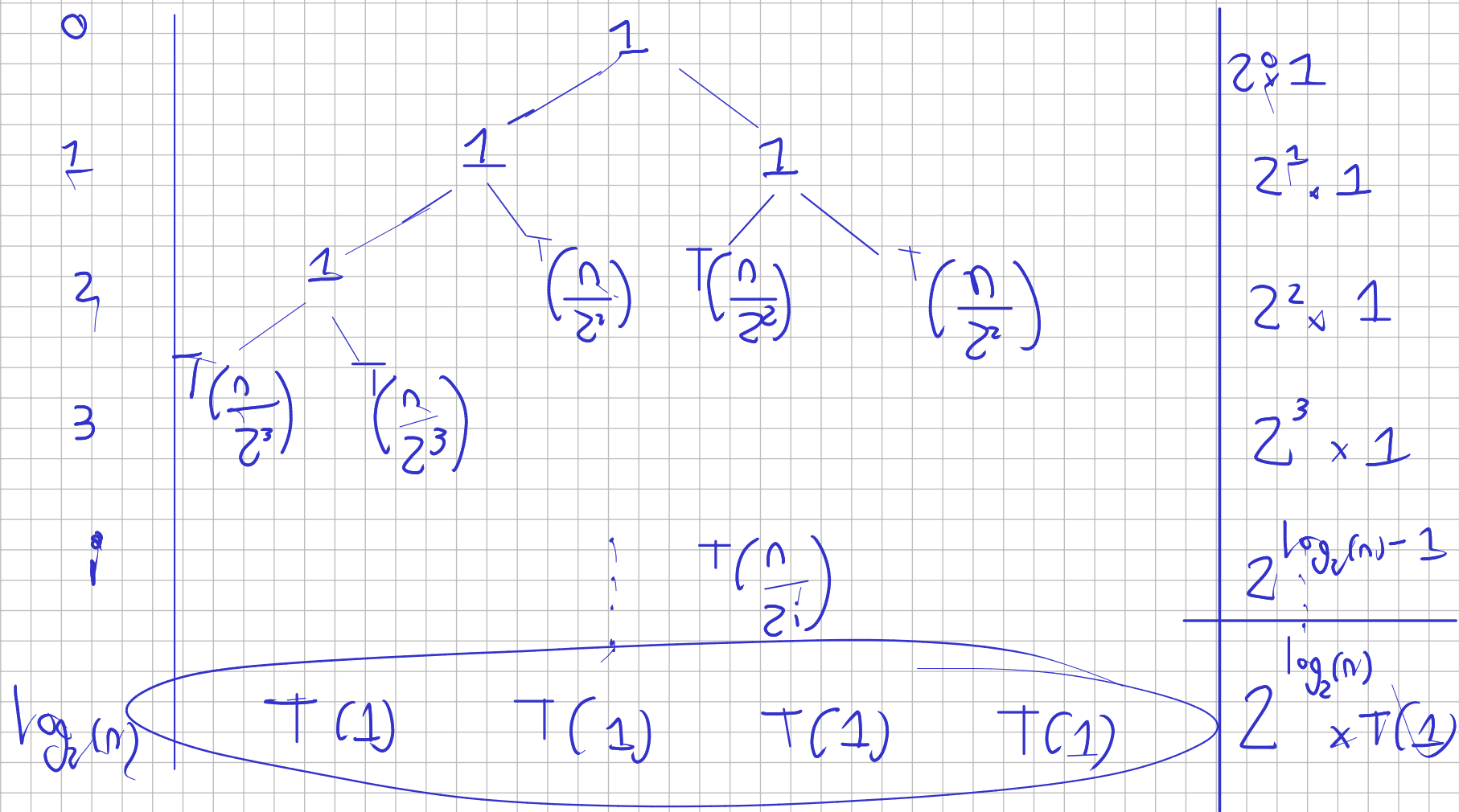
Recurrencias

Resuelva construyendo el árbol

$T(n) = 2T(n/2) + 1$, $T(1) = \Theta(1)$

$T(n) = 2T(n/2) + n$, $T(1) = \Theta(1)$

$$\Rightarrow T(n) = 2T(n/2) + 1, \quad T(1) = \Theta(1)$$

0

1

2

3

i

$\log_2(n)$

1

1 1

1 $T\left(\dfrac{n}{2^2}\right)$ $T\left(\dfrac{n}{2^2}\right)$ $T\left(\dfrac{n}{2^2}\right)$

$T\left(\dfrac{n}{2^3}\right)$ $T\left(\dfrac{n}{2^3}\right)$

$T\left(\dfrac{n}{2^i}\right)$

$T(1) \qquad T(1) \qquad T(1) \qquad T(1)$

$2^0 \cdot 1$

$2^1 \cdot 1$

$2^2 \times 1$

$2^3 \times 1$

$2^{\log_2(n)-1}$

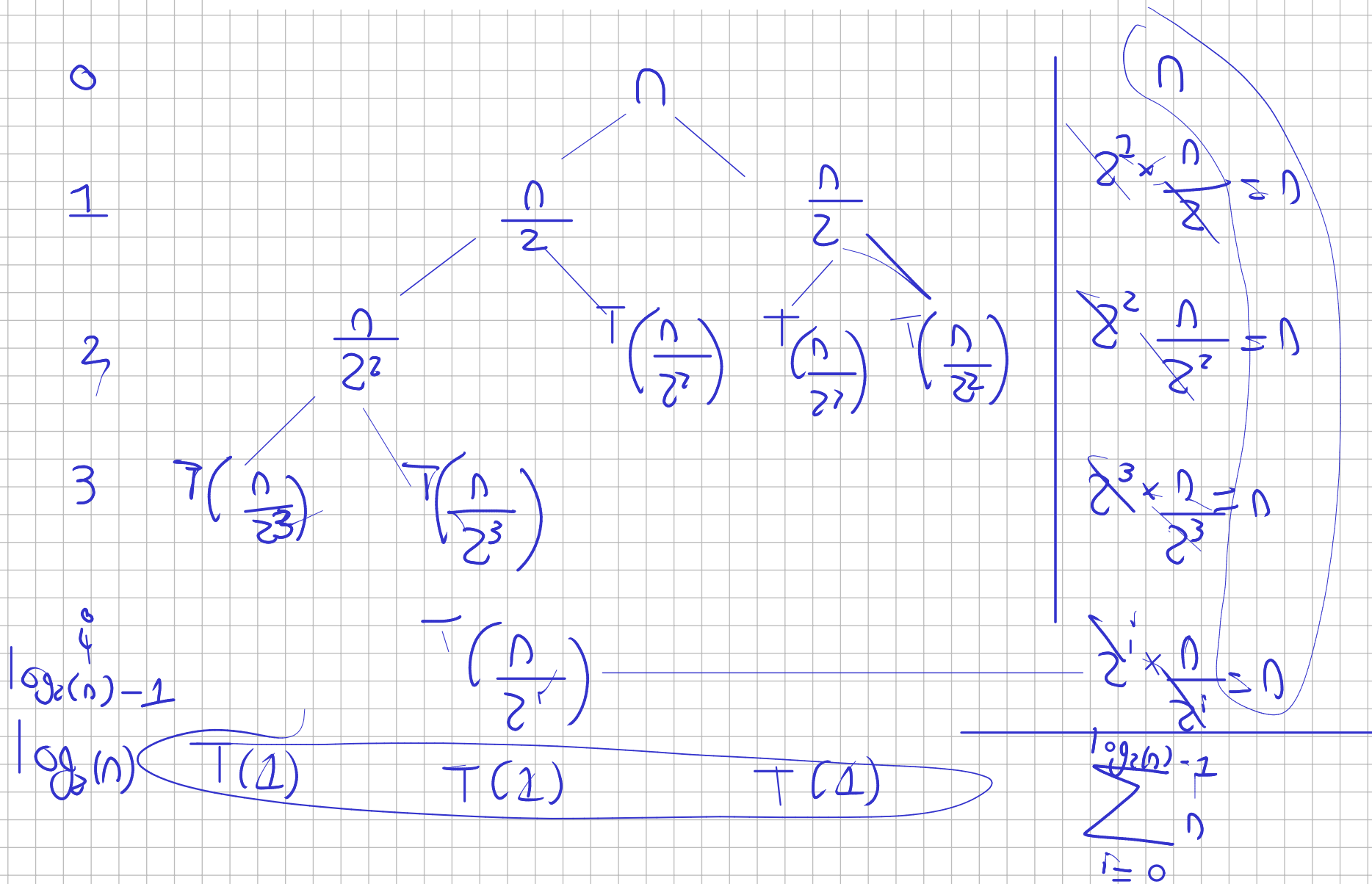$2^{\log_2(n)} \times T(1)$

$\dfrac{n}{2^i} \leq 1 \qquad\qquad i \equiv \log_2(n)$

$2^{\log_2(n)} T(1) + \sum_{i=0}^{\log_2(n)-1} 2^i$

$$T(n) = 2^{\log_2(n)} T(1) + \sum_{i=0}^{\log_2(n)-1} 2^i$$

$$T(n) = nc + 2 \frac{2^{\log_2(n)} - 1}{2 - 1} \boxed{= nc + n - 1}$$

$$T(n) = 2T(n/2) + n, \quad T(1) = \Theta(1)$$

0

n

1

$\dfrac{n}{2}$ $\dfrac{n}{2}$

$T\left(\dfrac{n}{2}\right)$ $T\left(\dfrac{n}{2^2}\right)$ $T\left(\dfrac{n}{2^2}\right)$

2

$\dfrac{n}{2^2}$

3

$T\left(\dfrac{n}{2^3}\right)$ $T\left(\dfrac{n}{2^3}\right)$

$\log_2(n) - 1$

$T\left(\dfrac{n}{2^i}\right)$

$\log_2(n)$

$T(1) \quad T(2) \quad T(1)$

$2^1 \times \dfrac{n}{2} = n$

$2^2 \dfrac{n}{2^2} = n$

$2^3 \times \dfrac{n}{2^3} = n$

$2^i \times \dfrac{n}{2^i} = n$

$\displaystyle\sum_{i=0}^{\log_2(n) - 1} n$

$$T(n) = n\,T(1) + \sum_{i=0}^{\log_2(n)-1} n$$

$$\sum_{i=1}^{n} c = c \times n$$
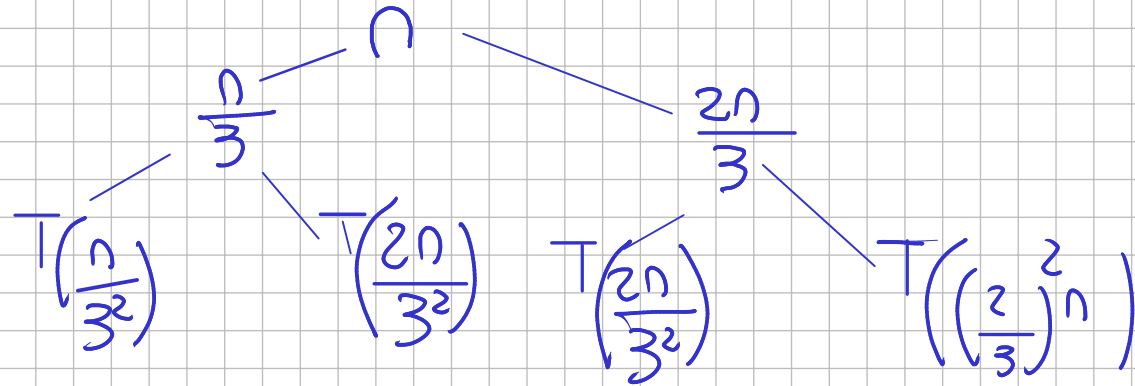
$$T(n) = c\,n + \log_2(n) \times n$$

Recurrencias

Resuelva la recurrencia $T(n) = T(n/3) + T(2n/3) + n$
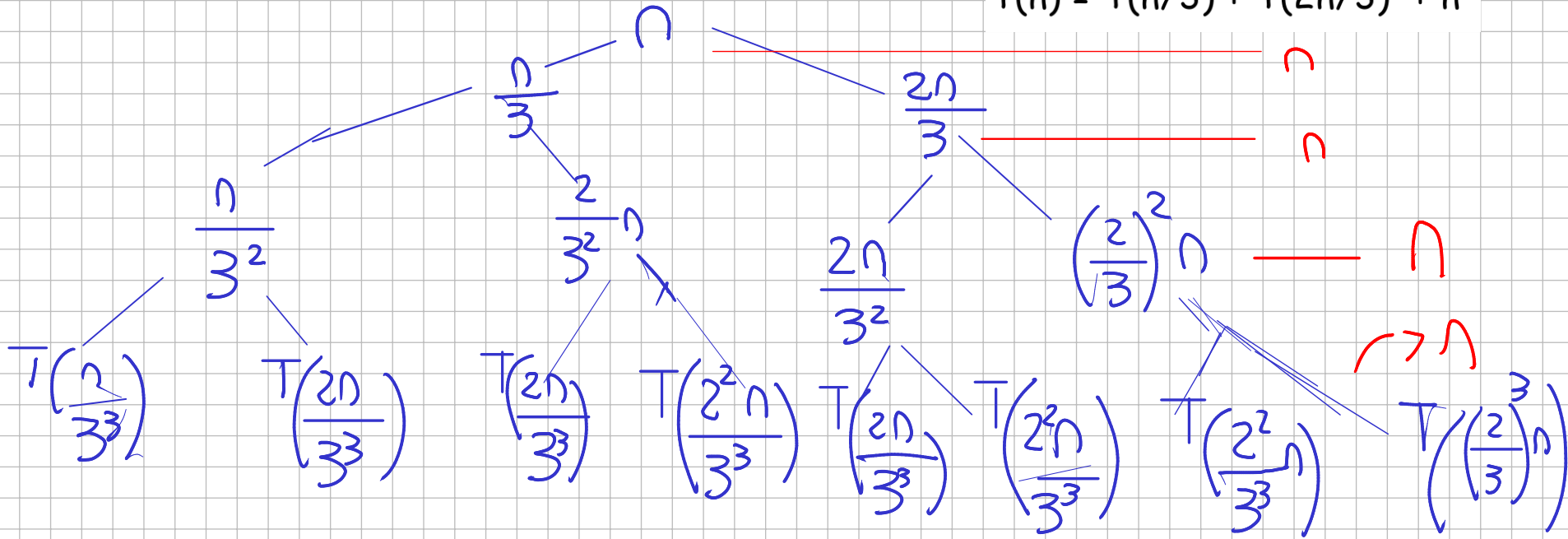
Indique una cota superior y una inferior
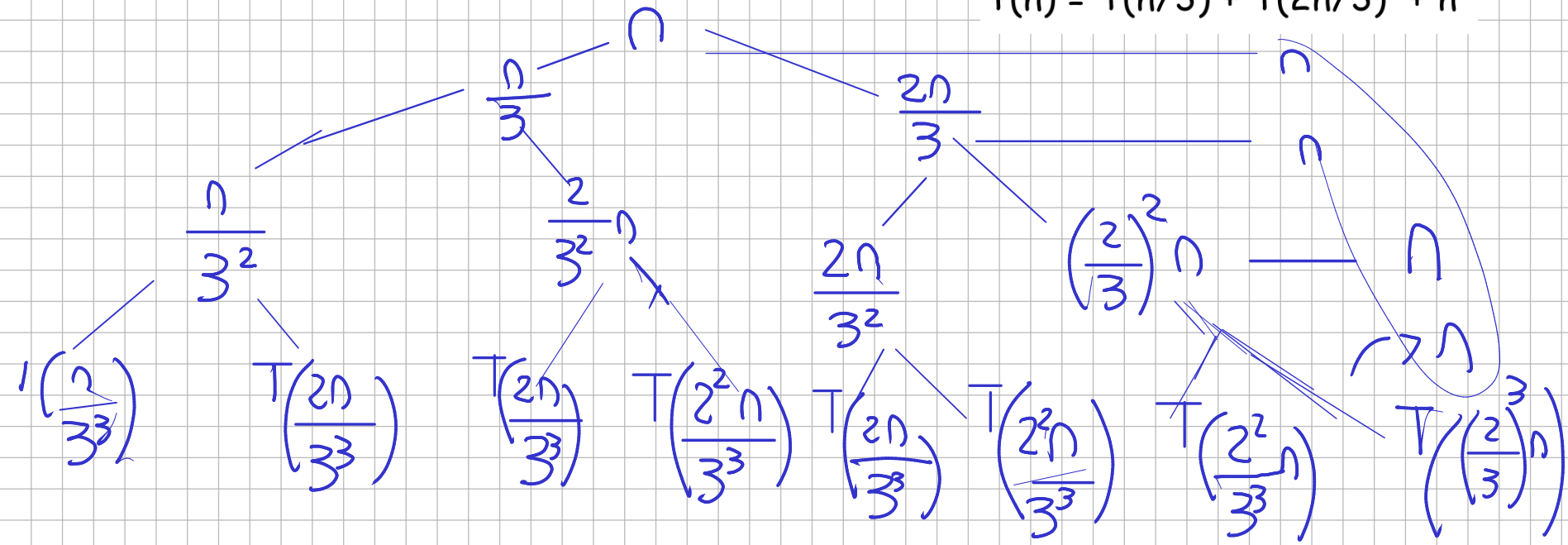
$$T(n) = T(n/3) + T(2n/3) + n$$

$$n = \frac{2}{3}n$$

$$n$$

$$T\left(\frac{n}{3}\right) \qquad T\left(\frac{2n}{3}\right)$$

$$n$$

$$\frac{n}{3} \qquad \frac{2n}{3}$$

$$T\left(\frac{n}{3^2}\right) \qquad T\left(\frac{2n}{3^2}\right) \qquad T\left(\frac{2n}{3^2}\right) \qquad T\left(\left(\frac{2}{3}\right)^2 n\right)$$

$$T(n) = T(n/3) + T(2n/3) + n$$

Recursion tree for $T(n) = T(n/3) + T(2n/3) + n$.

Top level: n with n to the right.

Second level: $\frac{n}{3}$ and $\frac{2n}{3}$, summing to n.

Third level: $\frac{n}{3^2}$, $\frac{2n}{3^2}$, $\frac{2n}{3^2}$, $\left(\frac{2}{3}\right)^2 n$, summing to n.
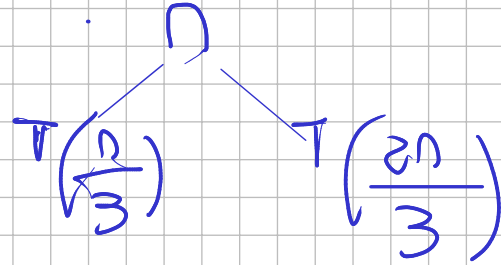
Leaves:
$T\left(\frac{n}{3^3}\right)$, $T\left(\frac{2n}{3^3}\right)$, $T\left(\frac{2n}{3^3}\right)$, $T\left(\frac{2^2 n}{3^3}\right)$, $T\left(\frac{2n}{3^3}\right)$, $T\left(\frac{2^2 n}{3^3}\right)$, $T\left(\frac{2^2 n}{3^3}\right)$, $T\left(\left(\frac{2}{3}\right)^3 n\right)$

$$\frac{1}{3^2} + \frac{2}{3^2} + \frac{2}{3^2} + \frac{4}{3^2} = \frac{9}{3^2}$$

$$\frac{1}{3^3} + \frac{2}{3^3} + \frac{2}{3^3} + \frac{4}{3^3} + \frac{2}{3^3} + \frac{4}{3^3} + \frac{4}{3^3} + \frac{8}{3^3} = \frac{27}{3^3} = 1$$
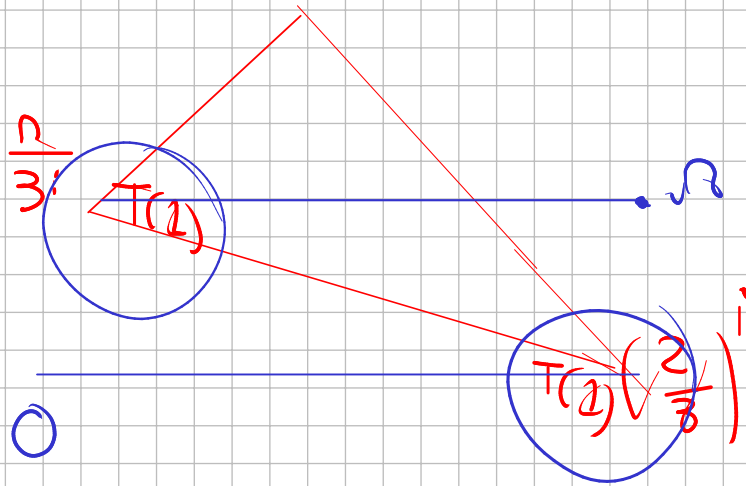
$$T(n) = T(n/3) + T(2n/3) + n$$

n

$\dfrac{n}{3}$ $\dfrac{2n}{3}$ n

$\dfrac{n}{3^2}$ $\dfrac{2}{3^2}n$ $\dfrac{2n}{3^2}$ $\left(\dfrac{2}{3}\right)^2 n$ n n

$T\left(\dfrac{n}{3^3}\right)$ $T\left(\dfrac{2n}{3^3}\right)$ $T\left(\dfrac{2n}{3^3}\right)$ $T\left(\dfrac{2^2 n}{3^3}\right)$ $T\left(\dfrac{2n}{3^3}\right)$ $T\left(\dfrac{2^2 n}{3^3}\right)$ $T\left(\dfrac{2^2}{3^3}n\right)$ $T\left(\left(\dfrac{2}{3}\right)^3 n\right)$

$\dfrac{n}{3^i}$

$i_1 \le \log_3(n)$

$\left(\dfrac{2}{3}\right)^i \le n$

$\left(\dfrac{2}{3}\right)^i n$

$\left(\dfrac{2}{3}\right)^i \le n$

$i \log_{\frac{2}{3}}\left(\dfrac{2}{3}\right) \le \log_{\frac{2}{3}}(n)$

$\dfrac{n}{3^i}$ $T(1)$ n

0 $T(1)\left(\dfrac{2}{3}\right)^i$

$i_2 = \log_{\frac{2}{3}}(n)$

$$2^{\log_3(n)} T(1) + \sum_{i=0}^{\log_3(n)-1} n$$

$$\boxed{n^{\log_3(2)} C + \left(+ \log_3(n) \times n \right)}$$

Cota
inferior

$$\log_3(n) = \frac{\log_2(n)}{\log_3(2)}$$

$$2^{\log_{\frac{2}{3}}(n)} + \sum_{i=0}^{\log_{\frac{2}{3}}(n)-1} n$$
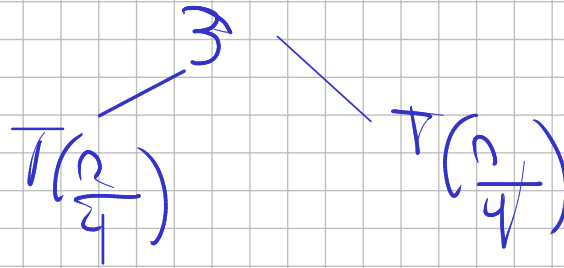
$$C \times \log(n) \times n$$

$$\boxed{n^{\log_{\frac{2}{3}}(2)} \times C + \left(\log_{\frac{2}{3}}(n) \times n \right)}$$

Cota
supro

$$\log_{\frac{2}{3}}(n) = \frac{\log_2(n)}{\log_2\left(\frac{2}{3}\right)}$$

$$\Theta\left(n \log(n) \right)$$

$$T(n) = 2T(n/4) + 3$$
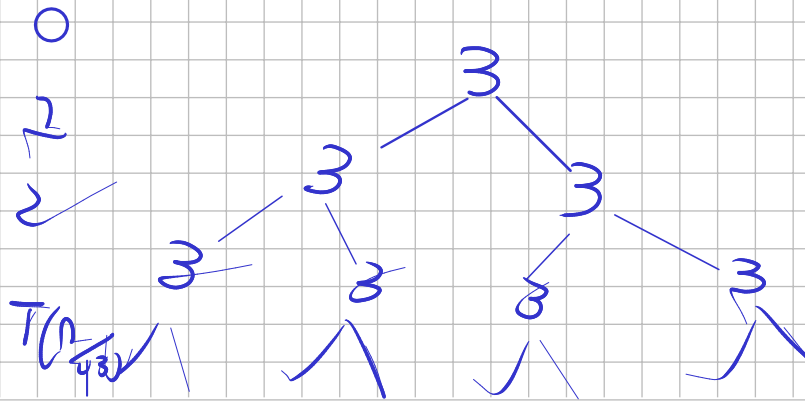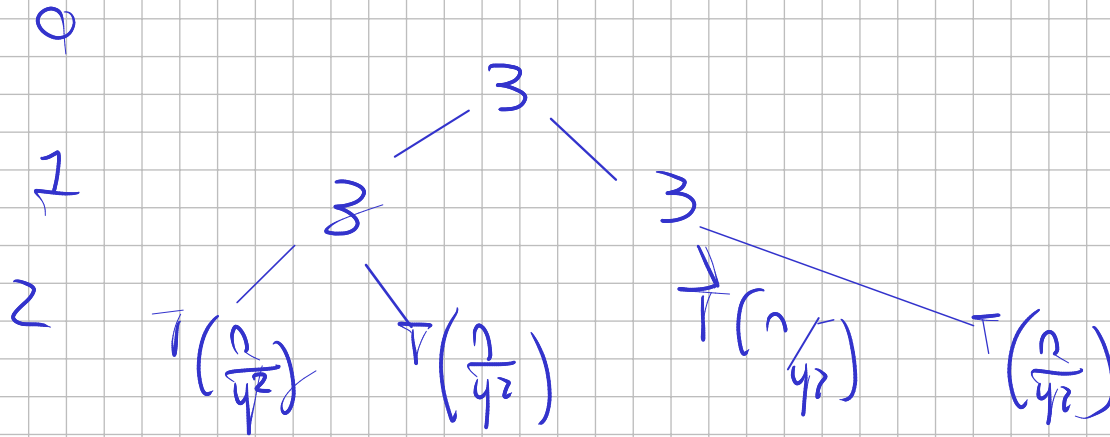
0

1

3

3   3

$T\left(\dfrac{n}{4^2}\right)$ $T\left(\dfrac{n}{4^2}\right)$ $T\left(n/4^2\right)$ $T\left(\dfrac{n}{4^2}\right)$

2

0

1

2

3

3   3

3   3   3   3

$T\left(\dfrac{n}{4^3}\right)$

$2^0 \times 3$

$2^1 \times 3'$

$2^2 \times 3$

$$3$$

$$3 \qquad 3$$

$$3 \qquad T\left(\frac{n}{4^2}\right) \qquad T\left(\frac{n}{4^2}\right) \qquad T\left(\frac{n}{4^2}\right)$$

$$T\left(\frac{n}{4^3}\right) \qquad T\left(\frac{n}{4^3}\right)$$

$$T\left(\frac{n}{4^i}\right)$$

$$T(1) \qquad\qquad T(1)$$

$$\frac{n}{4^i} \leq 1 \qquad i = \log_4(n)$$

1) T(n) = 5T(n/4) + $\boxed{5}$

$8^0 \times S$

$8^1 \times S = S^3$

$S^2 \times S = S^3$

2) T(n) = 8T(n/6) + $15n$

$8 \times 15\dfrac{n}{6}$

1) $15\dfrac{n}{6}$

$15 \times \dfrac{n}{6^2}$

$T(n-1)$ $\boxed{n-1}$

$T(n-2)$
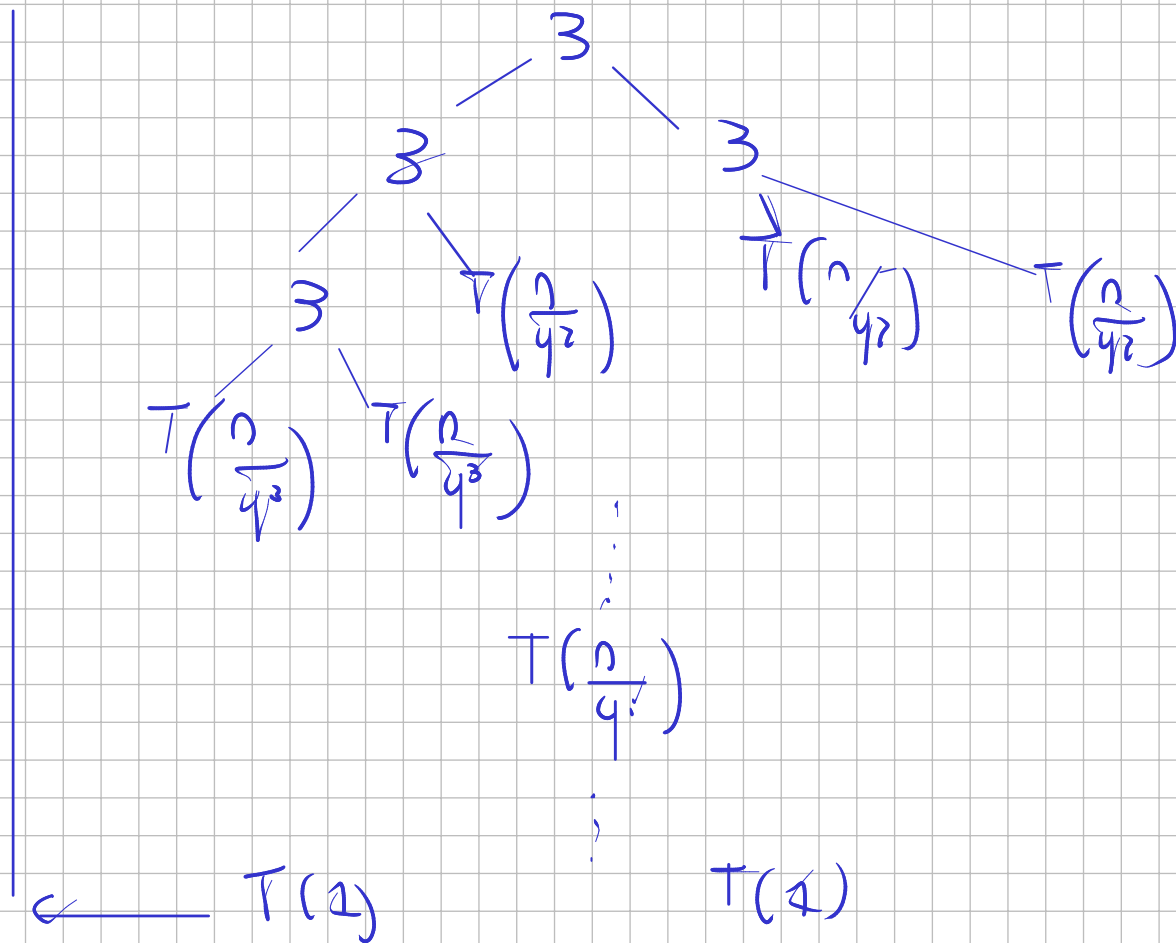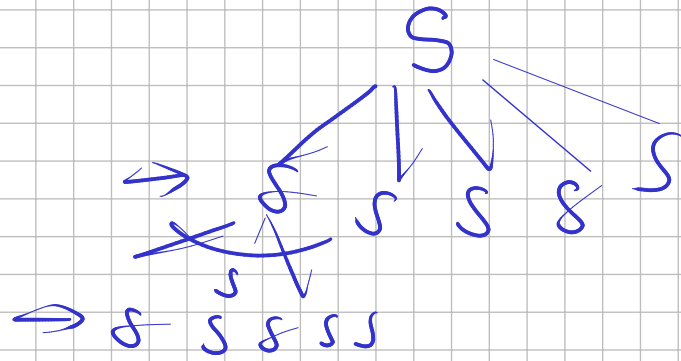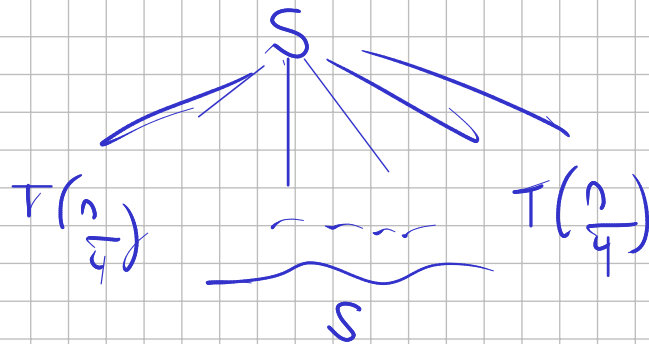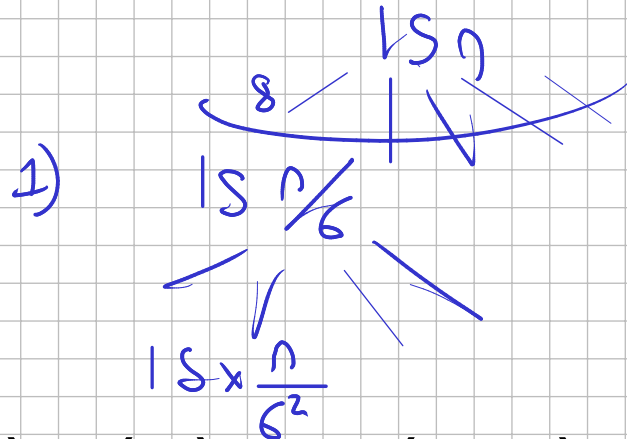
3) T(n) = T(n-1) + n

Recurrencias

Método maestro

Permite resolver recurrencias de la forma:

$T(n) = aT(n/b) + f(n)$, donde $a \geq 1$, $b > 1$

$$T(n) = \boxed{a}\,T\!\left(\frac{n}{b}\right) + \boxed{F(n)}$$

*#prob

Conquista

Divide y venceras

↳ Tamaño

5 3 8 10 | 1 4 2 3

Dividir

5 3 8 10   1 4 2 3

5 3   8 10   1 4   2 3

5   3   8   10   1   4   2   3

O(n)   3 5   8 10   1 4   2 3

$\boxed{a = 2}$

$\boxed{b = 2}$

$F(n) = n$

$$\boxed{2T\!\left(\frac{n}{2}\right) + n}$$

Conquista

3 5 8 10   1 2 3 4

1 2 3 3 4 5 8 10
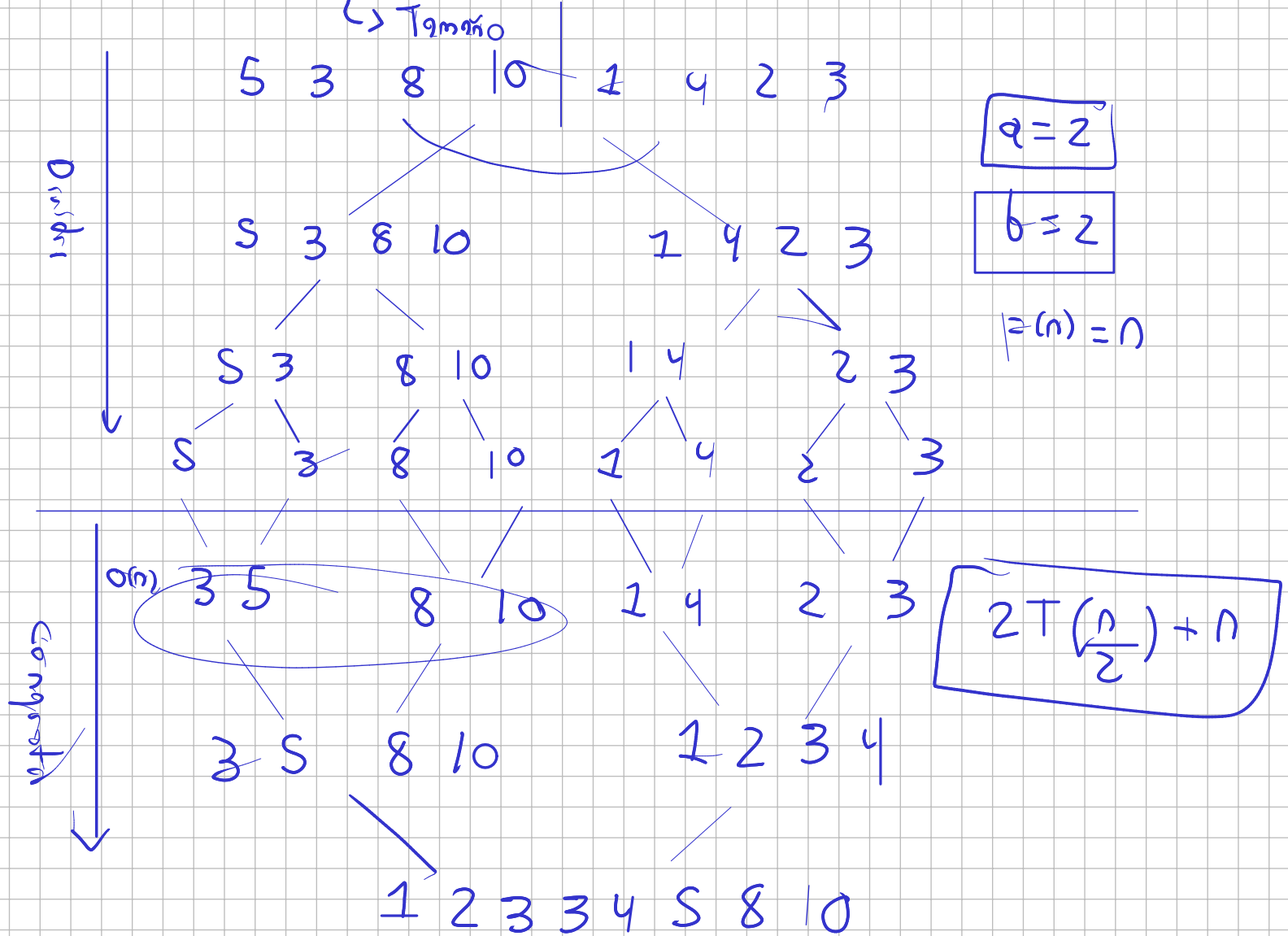
Recurrencias

Dado T(n) = aT(n/b) + f(n), donde a≥1, b>1, se puede acotar asintóticamente como sigue:

1. $T(n)=\Theta(n^{\log_b a})$

Si $f(n)=O(n^{\log_b a - \varepsilon})$ para algún $\varepsilon > 0$

2. $T(n)=\Theta(n^{\log_b a} \lg n)$

Si $f(n)=\Theta(n^{\log_b a})$ para algún $\varepsilon > 0$

3. $T(n)=\Theta(f(n))$

Si $f(n)=\Omega(n^{\log_b a + \varepsilon})$ para algún $\varepsilon > 0$ si a*f(n/b) ≤c*f(n)

para algun c<1

Recurrencias

Dado $T(n) = 9T(n/3) + n$

$$n^{\log_3 9} = n^2 \quad \textbf{Vs} \quad f(n) = n$$

Es $f(n) = O(n^{\log_b a - \varepsilon})$?

Es $n = O(n^{2 - \varepsilon})$?

Recurrencias

Dado $T(n) = 9T(n/3) + n$

$$n^{\log_3 9} = n^2 \quad \textbf{Vs} \quad f(n) = n$$

Es $f(n) = O(n^{\log_b a - \varepsilon})$?

Es $n = O(n^{2-\varepsilon})$?

Si $\varepsilon = 1$ se cumple que $n = O(n)$, por lo tanto, se cumple que:

$$T(n) = \Theta(n^2)$$

Dado T(n) = 9T(n/3) + n

$a = 9$ $b = 3$ $F(n) = n$ $T(1) = \boxed{10}$

1)

n es $O\left(n^{\log_3 9 - \varepsilon}\right)$

n es $O\left(n^{2 - \varepsilon}\right)$ $\varepsilon = 0.5$

n es $O(n^{1.5})$ ✓

$$T(n) = \Theta\left(n^{\log_b 9}\right)$$

$$T(n) = \Theta(n^2) \leftarrow$$

$$T(n) = 9T\left(\frac{n}{3}\right) + n$$

$$T(n) = 9^2 T\left(\frac{n}{3^2}\right) + 9\frac{n}{3} + n$$

$$T(n) = 9^3 T\left(\frac{n}{3^3}\right) + \frac{9^2}{3^2}n + \frac{9}{3}n + n$$

$$T(n) = 9^i T\left(\frac{n}{3^i}\right) + \left[\left(\frac{9}{3}\right)^{i-1} n + \left(\frac{9}{3}\right)^{i-2} + \cdots \left(\frac{9}{3}\right)^0 n\right]$$

$$i = \log_3(n)$$

$$T(n) = \boxed{9^{\log_3 n}} + (1) + \sum_{r=0}^{\log_2(n) - 1} 3^i$$

$$T(n) = n^2 T(1) + \frac{3^{\log_3(n)} - 1}{3 - 1}$$

$$T(n) = C\,n^2 + \frac{n - 1}{2}$$

Recurrencias

$T(n) = T(2n/3) + 1$

$$n^{\log_{3/2}1}=n^0=1 \quad \textbf{Vs} \quad f(n)=1$$

Es $f(n)=O(n^{\log_b a-\varepsilon})$?

Es $1=O(n^{0-\varepsilon})$?

No existe $\varepsilon>0$

Recurrencias

T(n) = T(2n/3) + 1

$$n^{\log_{3/2} 1} = n^0 = 1 \quad \text{vs} \quad f(n) = 1$$

Es $\quad f(n) = \Theta(n^{\log_b a}) \quad$?

Es $\quad 1 = \Theta(1) \quad$?

Si, por lo tanto, se cumple que:

$$T(n) = \Theta(1 * \lg n) = \Theta(\lg n)$$

.

$$T(n) = T\left(\sqrt{\frac{2n}{3}}\right) + 1 \qquad a = 1$$

$$b = \frac{3}{2}$$

$$f(n) = 1$$

1. $T(n) = \Theta(n^{\log_b a})$

Si $f(n) = O(n^{\log_b a - \varepsilon})$ para algún $\varepsilon > 0$

2. $T(n) = \Theta(n^{\log_b a} \lg n)$

Si $f(n) = \Theta(n^{\log_b a})$ para algún $\varepsilon > 0$

3. $T(n) = \Theta(f(n))$

Si $f(n) = \Omega(n^{\log_b a + \varepsilon})$ para algún $\varepsilon > 0$ si $a \ast f(n/b)$
$\leq c \ast f(n)$ $\quad c < 1$

1) 1 es $O\left(n^{\log_{\frac{3}{2}} 1 - \varepsilon}\right)$

~~1 es $O(n^{0-\varepsilon})$~~

1 es $O\left(\dfrac{1}{n^{\varepsilon}}\right)$

2) es 1 es $\Theta\left(n^{\log_{\frac{3}{2}} 1}\right)$

1 es $\Theta(1)$

$T(n) = \Theta\left(n^{\log_b a} \log(n)\right)$

$T(n) = \Theta\left(n^0 \log(n)\right)$

$$T(n) = \Theta(\log(n))$$

Recurrencias

$T(n) = 3\,T(n/4) + n\lg n$

$$n^{\log_4 3} = n^{0.793} \quad \text{vs} \quad f(n) = n\lg n$$

Es $f(n) = O(n^{\log_b a - \varepsilon})$?

Es $f(n) = \Theta(n^{\log_b a})$?

Es $f(n) = \Omega(n^{\log_b a + \varepsilon})$?

Si, y además, af(n/b)≤cf(n)

3(n/4)lg(n/4) ≤ cnlgn

3(n/4)lgn - 3(n/4)*2 ≤ cnlgn

(3/4)nlgn ≤ cnlgn → c=3/4 y se concluye que $T(n) = \Theta(n\lg n)$

$$T(n) = 3T\left(\frac{n}{4}\right) + \log(n)$$

$$a = 3 \qquad b = 4 \qquad f(n) = n\log(n)$$

1) $n\log(n) \Longrightarrow O\left(n^{\log_4 3 - \varepsilon}\right)$

$\cancel{n\log(n)}$ es $O\left(n^{0.793 - \varepsilon}\right)$ $\qquad \varepsilon = 0.093$

2) $n\log(n) \Longrightarrow \Theta\left(n^{0.793}\right)$ ✗

3) $n\log(n) \Longrightarrow \Omega\left(n^{0.793 + \varepsilon}\right)$ $\qquad \varepsilon = 0.207$

$n\log(n) \Longrightarrow \Omega(n)$ ✓ $\qquad\qquad n > 0$

$3\left(\frac{n}{4}\log\left(\frac{n}{4}\right)\right) \leq c * n\log(n)$ $\qquad$ $\frac{3}{4}\cancel{n}\log\left(\frac{n}{4}\right) \leq c * \cancel{n}\log(n)$

$\frac{3}{4}\log\left(\frac{n}{4}\right) \leq c\log(n)$ $\qquad$ $\frac{3}{4}\log(n) - \frac{3}{4}\log(4) \leq c\log(n)$

$\frac{3}{4}\cancel{\log(n)} \leq c\log(n)$ $\qquad\qquad$ $\frac{3}{4}\log(n) - \frac{3}{\cancel{2}} \leq c\log(n)$ $\qquad n \ggg$

$\boxed{c \geq \frac{3}{4}}$ $\qquad\qquad \left[\frac{3}{4}, 1\right)$

$$T(n) = \Theta(n\log(n))$$

Recurrencias

$T(n) = 2T(n/2) + n\lg n$

$$\log_b a = \log_2 2 = 1$$

Muestre que no se puede resolver p...

1. $T(n) = \Theta(n^{\log_b a})$
   Si $f(n) = O(n^{\log_b a - \varepsilon})$ para algún $\varepsilon > 0$

2. $T(n) = \Theta(n^{\log_b a} \lg n)$
   Si $f(n) = \Theta(n^{\log_b a})$ para algún $\varepsilon > 0$

3. $T(n) = \Theta(f(n))$
   Si $f(n) = \Omega(n^{\log_b a + \varepsilon})$ para algún $\varepsilon > 0$ si $a*f(n/b)$
   $\leq c*f(n)$

   $c < 1$

1) $n\log(n)$ es $O(n^{1-\epsilon})$ ✗

2) $n\log(n)$ es $\Theta(n^1)$ ✗

3) $n\lg(n)$ es $\Omega(n^{1+\epsilon})$

   $\Omega(n^{1\cdot 1})$

$2\frac{n}{2}\log\left(\frac{n}{2}\right) \leq C_* n\log(n) \Rightarrow n\log\left(\frac{n}{2}\right) \leq C_* n\log(n)$

$\boxed{C \geq 1}$ $\boxed{C < 1}$ ✗

Recurrencias

Resuelva usando método del maestro

$a=4$ $b=2$

$T(n) = 4T(n/2) + n$

$T(n) = 4T(n/2) + n^2$

$T(n) = 4T(n/2) + n^3$

1. $T(n) = \Theta(n^{\log_b a})$
   Si $f(n) = O(n^{\log_b a - \varepsilon})$ para algún $\varepsilon > 0$

2. $T(n) = \Theta(n^{\log_b a} \lg n)$
   Si $f(n) = \Theta(n^{\log_b a})$ para algún $\varepsilon > 0$

3. $T(n) = \Theta(f(n))$
   Si $f(n) = \Omega(n^{\log_b a + \varepsilon})$ para algún $\varepsilon > 0$ si $a*f(n/b)$
   $\leq c*f(n)$ $c < 1$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$a = 4 \quad b = 2 \qquad f(n) = n \qquad \boxed{T(n) = \Theta(n^2)}$$

$$n \text{ es } O(n^{2-\varepsilon})$$

$$n \text{ es } O(n) \qquad \varepsilon = 1$$

1. $T(n) = \Theta(n^{\log_b a})$

   Si $f(n) = O(n^{\log_b a - \varepsilon})$ para algún $\varepsilon > 0$

2. $T(n) = \Theta(n^{\log_b a} \lg n)$

   Si $f(n) = \Theta(n^{\log_b a})$ para algún $\varepsilon > 0$

3. $T(n) = \Theta(f(n))$

   Si $f(n) = \Omega(n^{\log_b a + \varepsilon})$ para algún $\varepsilon > 0$ si $a*f(n/b) \leq c*f(n)$

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2$$

1) n^2 es $O(n^{2-\varepsilon})$ ✗

2) n^2 es $\Theta(n^2)$ ✓

$$T(n) = \Theta(n^2 \log(n))$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n^3$$

1) n^3 es $O(n^{2-\varepsilon})$ ✗

2) n^3 es $\Theta(n^2)$ ✗

3) n^3 es $\Omega(n^{2+\varepsilon})$ ✓

$$4 \times \left(\frac{n}{2}\right)^3 \leq C \times n^3 \qquad \frac{1}{2} n^3 \leq C n^3$$

$$C \geq \frac{1}{2} \quad \left[\frac{1}{2}, 1\right) ✓$$

$$\boxed{\Theta(n^3)}$$

Recurrencias

Método de sustitución

Suponer la forma de la solución y probar por inducción matemática

Recurrencias

$T(n)=2T(\lfloor n/2\rfloor)+n$, $T(1)=1$

Suponer que la solución es de la forma $T(n)=O(n\lg n)$

Probar que $T(n)\leq cn\lg n$.

Se supone que se cumple para $n/2$ y se prueba para n

Hipotesis inductiva: $T(n/2) \leq cn/2\lg(n/2)$

Recurrencias

$T(n)=2T(\lfloor n/2 \rfloor)+n$, $T(1)=1$

Probar que $T(n) \leq cn\lg n$.

Hipótesis inductiva: $T(n/2) \leq cn/2 \lg(n/2)$

Paso inductivo:

$T(n) \leq 2(cn/2 \lg(n/2)) + n$

$\qquad \leq cn\lg(n/2) + n$

$\qquad = cn\lg(n) - cn + n$, para $c \geq 1$, haga $c=1$

$\qquad \leq cn\lg n$

Recurrencias

$T(n)=2T(\lfloor n/2 \rfloor)+n, \; T(1)=1$

Probar que $T(n) \leq cn \lg n$.

Paso base: si c=1, probar que $T(1)=1$ se cumple

$T(1) \leq 1*1\lg 1$?

$1 \leq 0$?

No, se debe escoger otro valor para c

Recurrencias

$T(n)=2T(\lfloor n/2 \rfloor)+n$, $T(1)=1$

Probar que $T(n) \leq cn \lg n$.

Paso base: si c=2, probar que $T(1)=1$ se cumple

$T(1) \leq 2*1 \lg 1$?

$1 \leq 0$?

No, se puede variar k.

Para esto, se calcula T(2) y se toma como valor inicial

Recurrencias

Probar que T(n)≤cnlgn.

T(2)=2T(0)+2=4

T(2) ≤ 1*2lg 2 ?

  4 ≤ 2 ?

No, se puede variar c.

Recurrencias

Probar que $T(n) \leq cn \lg n$.

$T(2) = 2T(0) + 2 = 4$

$T(2) \leq 3*2 \lg 2$?

$4 \leq 6$?

Si, se termina la demostración

$\rightarrow T(n) = 2T\left(\frac{n}{2}\right) + 1$ $T(n) = O(n)$

$T(1) = 3$

Paso inductivo $T(n) \le C \times n$

$T(n) \le C \cdot \frac{n}{2} + 1$

$C = 1$ $C = 10$

$3 \le \frac{1}{2} + 1 \;\times$ $3 \le \frac{10n}{2} + 1$ $3 \le 4 \;\checkmark$

n	$T(n)$	$\le \frac{n}{2} + 1$
1	3	6
2	7	11
4	15	21
8	31	41
16	63	81
32	127	161

$$\boxed{T(n) = 4T\left(\frac{n}{2}\right) + n^2} \qquad T(1) \le 10$$

$$a = 4 \qquad b = 2 \qquad \log_2 4 = 2$$

$$n^2 \text{ es } \Theta(n^2)$$

$$T(n) = \Theta\left(n^2 \log(n)\right)$$

$$T(n) \le C_* n^2 \cdot \log(n) \qquad C = 100$$

$$T(n) \le 4C\left(\frac{n}{2}\right)^2 \log\left(\frac{n}{2}\right) + n^2$$

$$T(n) \le 100\, n^2 \log\left(\frac{n}{2}\right) + n^2$$

n	$T^1(n)$
1	
2	
4	
8	
16	
32	

$$T(\;) \ge C n^2 \log(n)$$

$$T(n) \le n^2 \log\left(\frac{n}{2}\right) + n^2$$

$T(n)$	n
10	1
4 4	2
176	4
768	8
3328	16
14336	32

1. $T(n) = \Theta(n^{\log_b a})$

Si $f(n) = O(n^{\log_b a - \varepsilon})$ para algún $\varepsilon > 0$

2. $T(n) = \Theta(n^{\log_b a} \lg n)$

Si $f(n) = \Theta(n^{\log_b a})$ para algún $\varepsilon > 0$

3. $T(n) = \Theta(f(n))$

Si $f(n) = \Omega(n^{\log_b a + \varepsilon})$ para algún $\varepsilon > 0$ si $a*f(n/b) \le c*f(n)$

Recurrencias

$T(n)=T(n-1)+T(n-2)+1$, $T(1)=O(1)$, $T(2) = O(1)$

Suponer que la solución es de la forma $T(n)=O(2^n)$

Probar que $T(n) \leq c2^n$.

Se supone que se cumple para n-1 y se n-2 prueba para n

Hipotesis inductiva: $T(n-1) \leq c2^{(n-1)}$ y $T(n-2) \leq c2^{(n-2)}$

Recurrencias

$T(n)=T(n-1)+T(n-2)+1$, $T(1)=O(1)$, $T(2)=O(1)$

Ahora se debe probar que: $T(n) \leq c2^n$

$$T(1) \leq c2^1 \rightarrow 1 \leq 2*c$$

$$T(2) \leq c2^2 \rightarrow 1 \leq 4*c$$

$$T(3) \leq c2^3 \rightarrow 2 \leq 8*c$$

$$T(4) \leq c2^4 \rightarrow 3 \leq 16*c$$

$$T(5) \leq c2^5 \rightarrow 5 \leq 32*c$$

$$T(6) \leq c2^6 \rightarrow 8 \leq 64*c$$

$$T(7) \leq c2^7 \rightarrow 13 \leq 128*c$$

$$T(8) \leq c2^8 \rightarrow 21 \leq 256*c$$

Con $c = 1$, se cumple.

Referencias

Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd ed.). The MIT Press. Chapter 4

Gracias

Próximo tema:

Divide y vencerás