Fundamentos de análisis y diseño de algoritmos

Ecuaciones de recurrencia

Recurrencias

Método de iteración

Método maestro*

Método de sustitución

Recurrencias

Método de iteración

Expandir la recurrencia y expresarla como una suma de términos que dependen de n y de las condiciones iniciales

$$T(n) = 4\boxed{T\left(\frac{n}{3}\right)} + 3n^2$$

$$\left(\frac{n}{3^2}\right)^2 = \frac{n^2}{9^2} \quad n \in \mathbb{Z}$$

1) $$T(n) = 4\left(4T\left(\frac{n}{3^2}\right) + 3\left(\frac{n}{3}\right)^2\right) + 3n^2$$

$$T(n) = 4^2\boxed{T\left(\frac{n}{3^2}\right)} + 4 \times 3\left(\frac{n}{3}\right)^2 + 3n^2$$

$$T(n) = 4^2\left(4T\left(\frac{n}{3^3}\right) + 3\left(\frac{n}{3^2}\right)^2\right) + 4 \times 3\left(\frac{n}{3}\right)^2 + 3n^2$$

$$4^3T\left(\frac{n}{3^3}\right) + 4^2 \times 3\left(\frac{n}{3^2}\right)^2 + 4 \times 3\left(\frac{n}{3}\right)^2 + 3n^2$$

$$T(n) = 2\boxed{T\left(\frac{n}{4}\right)} + 2n \qquad T(1) = \Theta(1)$$

$$T(n) = 2\left(2T\left(\frac{n}{4^2}\right) + 2\frac{n}{4}\right) + 2n$$

$$T(n) = 2^2\boxed{T\left(\frac{n}{4^2}\right)} + 2 \times \frac{2n}{4} + 2n$$

$$T(n) = 2^2\left(2T\left(\frac{n}{4^3}\right) + \frac{2n}{4^2}\right) + 2 \times \frac{2n}{4} + 2n$$

$$T(n) = 2^3 T\left(\frac{n}{4^3}\right) + 2^2 \times \frac{2n}{4^2} + \frac{2 \times 2n}{4} + 2n$$

$$T(n) = 2^i T\left(\frac{n}{4^i}\right) + 2^{i-1} \times \frac{2n}{4^{i-1}} + 2^{i-2} \times \frac{2n}{4^{i-2}} + \cdots + 2^2 \times \frac{2n}{4^2} + 2^1 \times \frac{2n}{4^1} + \frac{2^0}{4^0} 2n$$

$$\frac{n}{4^i} = 1 \qquad i = \log_4(n)$$

$$T(n) = 2^{\log_4(n)} T\left(\frac{n}{4^i}\right) + 2^{\log_4(n)-1} \times \frac{2n}{4^{\log_4(n)-1}} + 2^{\log_4(n)-2} \times \frac{2n}{4^{\log_4(n)-2}} + \cdots + 2^2 \times \frac{2n}{4^2} + 2^1 \times \frac{2n}{4^1} + \frac{2^0}{4^0} 2n$$

$$2^{\log_4(n)} T(1) + \sum_{i=0}^{\log_4(n)-1} \left(\frac{2}{4}\right)^i \times 2n \qquad \sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

$$n^{\log_4(2)} (1) + 2n \sum_{r=0}^{\log(n)-1} \left(\frac{1}{2}\right)^i$$

$$n^{0.5} + 2n \left(\frac{\left(\frac{1}{2}\right)^{\log(n)} - 1}{1/2 - 1}\right) = \sqrt{n} + 2n \left(\frac{n^{\log\left(\frac{1}{2}\right)} - 1}{-1/2}\right)$$

$$\sqrt{n} - 4n \left(n^{\log\left(\frac{1}{2}\right)} - 1\right) = O(n)$$

Recurrencias

T(n) = n + 3$\boxed{T(n/4)}$, T(1)= Θ(1) y n par

Expandir la recurrencia 2 veces

1) $T(n) = n + 3\left(\frac{n}{4} + 3T\left(\frac{n}{4^2}\right)\right)$

$T(n) = n + \frac{3n}{4} + 3^2 T\left(\frac{n}{4^2}\right)$

2) $T(n) = n + \frac{3n}{4} + 3^2\left(\frac{n}{4^2} + 3T\left(\frac{n}{4^3}\right)\right)$

$T(n) = n + \frac{3n}{4} + \left(\frac{3}{4}\right)^2 n + 3^3 T\left(\frac{n}{4^3}\right)$

Recurrencias

$T(n) = n + 3T(n/4)$

$n + 3\ (\ n/4 + 3T(n/16))$

$n + 3\ (\ n/4 + 3(n/16\ \ + 3T(n/64)\ ))$

$n + 3*n/4 + 3^2*n/4^2\ \ + 3^3T(n/4^3)$

Recurrencias

$T(n) = n + 3T(n/4)$ $\quad$ $T(1) \in \Theta(1)$

$n + 3\,(\,n/4 + 3T(n/16))$

$n + 3\,(\,n/4 + 3(n/16 \;+ 3T(n/64)\,))$

$n + 3 * n/4 + 3^2 * n/4^2 \;+ 3^3 T(n/4^3)$

¿Cuándo se detienen las iteraciones?

Recurrencias

$T(n) = n + 3T(n/4)$

$n + 3\ (\ n/4 + 3T(n/16))$

$n + 3\ (\ n/4 + 3(n/16\ \ + 3T(n/64)\ ))$

$n + 3*n/4 + 3^2*n/4^2\ \ + 3^3T(n/4^3)$

¿Cuándo se detienen las iteraciones?

Cuando se llega a $T(1)$

Recurrencias

$T(n) = n + 3T(n/4)$

$n + 3\ (\ n/4 + 3T(n/16))$

$n + 3\ (\ n/4 + 3(n/16\ \ + 3T(n/64)\ ))$

$n + 3{*}n/4 + 3^2{*}n/4^2\ \ + 3^3T(n/4^3)$

$T\left(\frac{n}{4^i}\right) = T(1)$

¿Cuándo se detienen las iteraciones?

Cuando se llega a T(1), esto es, cuando $(n/4^i)=1$

$\frac{n}{4^i} = 1$ $\quad n = 4^i \quad i = \log_4(n)$

Recurrencias

$T(n) = n + 3T(n/4)$

$n + 3\,(\,n/4 + 3T(n/16))$

$n + 3\,(\,n/4 + 3(n/16\ + 3T(n/64)\,))$

$n + 3*n/4 + 3^2*n/4^2\ + 3^3T(n/4^3)$

$\log_4(n) = 1$

$n + 3*n/4 + 3^2*n/4^2 + 3^3n/4^3 + \ldots + 3^{\log 4n}T(1)$

$\frac{3^i}{4^i}\, n\ 4$

¿Cuándo se detienen las iteraciones?

Cuando se llega a T(1), esto es, cuando $(n/4^i)=1$

Recurrencias

$T(n) = n + 3T(n/4)$

$n + 3 (n/4 + 3T(n/16))$

$n + 3 (n/4 + 3(n/16 + 3T(n/64)))$

$n + 3*n/4 + 3^2*n/4^2 + 3^3T(n/4^3)$

$n + 3*n/4 + 3^2*n/4^2 + 3^3n/4^3 + \ldots + 3^{\log 4n}T(1)$

Después de iterar, se debe tratar de expresar como una sumatoria con forma cerrada conocida

Recurrencias

$T(n) = n + 3T(n/4\rfloor)$

$n + 3\ (\ n/4\rfloor + 3T(n/16\rfloor))$

$n + 3\ (\ n/4\rfloor + 3(n/16\rfloor\ + 3T(n/64\rfloor)\ ))$

$n + 3{*}n/4\rfloor + 3^2{*}n/4^2\rfloor\ + 3^3(n/4^3\rfloor) + \ldots + 3^{\log 4n}\Theta(1)$

$\leq\ n + 3n/4 + 3^2n/4^2 + 3^3n/4^3 + \ldots + 3^{\log 4n}\Theta(1)$

Recurrencias

$T(n) = n + 3T(n/4)$

$n + 3\ (\ n/4 + 3T(n/16))$

$n + 3\ (\ n/4 + 3(n/16\ + 3T(n/64)\ ))$

$n + 3*n/4 + 3^2*n/4^2\ + 3^3(n/4^3) + \ldots + 3^{\log 4n}\Theta(1)$

$\leq\ n + 3n/4 + 3^2n/4^2 + 3^3n/4^3 + \ldots + 3^{\log 4n}\Theta(1)$

$$= \left(\sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{4}\right)^i n\right) + 3^{\log_4 n}\Theta(1)$$

$$= n\left(\frac{(3/4)^{(\log_4 n)} - 1}{(3/4) - 1}\right) + n^{\log_4 3} = n * 4(1 - (3/4)^{(\log_4 n)}) + \Theta(n^{\log_4 3})$$

$= O(n)$

Recurrencias

Resuelva por el método de iteración

$T(n) = 2T(n/2) + 1, \; T(1) = \Theta(1)$

$$T(n) = 2\left(2T\left(\frac{n}{2^2}\right) + 1\right) + 1 = 2^2 \boxed{T\left(\frac{n}{2^2}\right)} + 2 + 1$$

$$T(n) = 2^2\left(2T\left(\frac{n}{2^3}\right) + 1\right) + 2 + 1 = 2^3 T\left(\frac{n}{2^3}\right) + 2^2 + 2^1 + 2^0$$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + 2^{k-1} + 2^{k-2} + \dots + 2^1 + 2^0$$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} 2^i$$

k ésima expansión

$\underline{T(1)}$ $\qquad \frac{n}{2^k} = 1 \qquad k = \log_2(n)$

$$T(n) = 2^{\log_2(n)} T(1) + \sum_{i=0}^{\log_2(n)-1} 2^i$$

$$T(n) = n\,T(1) + \frac{2^{\log_2(n)} - 1}{2-1} = cn + n - 1$$

$c = 3$ $\qquad \boxed{T(n) = 4n - 1}$

Recurrencias

Resuelva por el método de iteración

$T(n) = 2T(n/2) + 1,\ T(1) = \Theta(1)$

- $T(n) = 2T(n/2) + n,\ T(1) = \Theta(1)$

1

$\frac{n}{2^k} = 1 \qquad k = \log_2(n)$

$2\left(2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + n = 2^2 T\left(\frac{n}{2^2}\right) + \frac{2}{2}n + n$

$2^2\left(2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + \frac{2}{2}n + n = 2^3 T\left(\frac{n}{2^3}\right) + \underbrace{\cancelto{1}{\frac{2^2}{2^2}}n + \cancelto{1}{\frac{2}{2}}n + n}_{3n}$

$2^k T\left(\frac{n}{2^k}\right) + kn \qquad 2^{\log_2(n)} T(1) + \log_2(n) \times n = \boxed{cn + \log_2(n) \times n}$

Recurrencias

Resuelva por el método de iteración

$T(n) = 2T(n/2) + 1$, $T(1)= \Theta(1)$

$T(n) = 2T(n/2) + n$, $T(1)= \Theta(1)$

$T(n) = T(n/2) + 1$, $T(1)= \Theta(1)$

Recurrencias

Resuelva por el método de iteración

$T(n) = 2T(n/2) + 1$, $T(1)= \Theta(1)$

$T(n) = 2T(n/2) + n$, $T(1)= \Theta(1)$

$T(n) = T(n/2) + 1$, $T(1)= \Theta(1)$

Demuestre que $T(n) = T(n/2\rfloor) + n$, es $\Omega(nlogn)$

Recurrencias

Iteración con árboles de recursión

$T(1) = 1$

$T(n) = 2T(n/2) + n^2$

$$T(n) = 2\left(2T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right) + n^2$$

$$T(n) = 2^2 T\left(\frac{n}{2^2}\right) + 2\frac{n^2}{2^2} + n^2$$

$$T(n) = 2^2\left(2T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right) + 2\frac{n^2}{2^2} + n^2$$

$$T(n) = 2^3 T\left(\frac{n}{2^3}\right) + 2^2\left(\frac{n}{2^2}\right)^2 + \frac{2n^2}{2^2} + n^2$$

$$T(n) = 2^3 T\left(\frac{n}{2^3}\right) + 2^2 \times \frac{n^2}{2^4} + 2\frac{n^2}{2^2} + n^2$$

$$T(n) = 2^3\left(2T\left(\frac{n}{2^4}\right) + \left(\frac{n}{2^3}\right)^2\right) + 2^2 \times \frac{n^2}{2^4} + 2\frac{n^2}{2^2} + n^2$$

$$T(n) = 2^4 T\left(\frac{n}{2^4}\right) + 2^3 \frac{n^2}{2^6} + 2^2 \frac{n^2}{2^4} + \frac{2n^2}{2^2} + n^2$$

$$T(n) = 2^4 T\left(\frac{n}{2^4}\right) + \frac{n^2}{2^3} + \frac{n^2}{2^2} + \frac{n^2}{2^1} + \frac{n^2}{2^0}$$

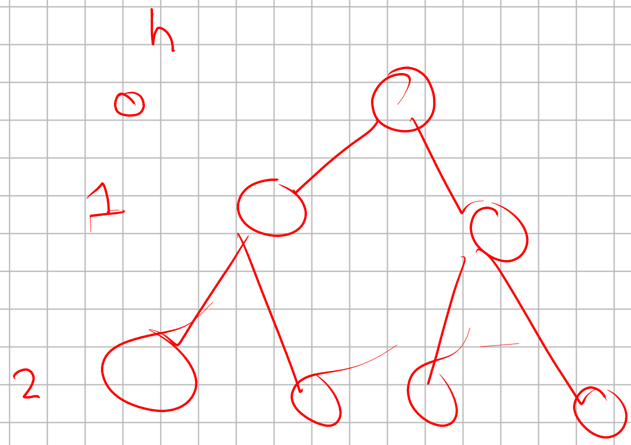
$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + \frac{n^2}{2^{k-1}} + \frac{n^2}{2^{k-2}} + \cdots + \frac{n^2}{2^1} + \frac{n^2}{2^0} \qquad k = \log_2(n)$$

$$T(n) = 2^{\log_2(n)} T(1) + \sum_{i=0}^{\log_2(n)-1} n^2 \frac{1}{2^i} \qquad 1 = 1^i \qquad \frac{1^i}{2^i} = \left(\frac{1}{2}\right)^i$$

$$T(n) = n + n^2 \sum_{i=0}^{\log_2(n)-1} \left(\frac{1}{2}\right)^i = n + n^2\left(\frac{\left(\frac{1}{2}\right)^{\log_2(n)} - 1}{\frac{1}{2} - 1}\right)$$

$$T(n) = n + n^2\left(\frac{n^{\log_2(0.5)} - 1}{-\frac{1}{2}}\right) = n - 2n^2\left(n^{-1} - 1\right)$$

$$n - 2n + 2n^2 = \boxed{2n^2 - n} \qquad O(n^2)$$



$m = 2$

2^0

2^1

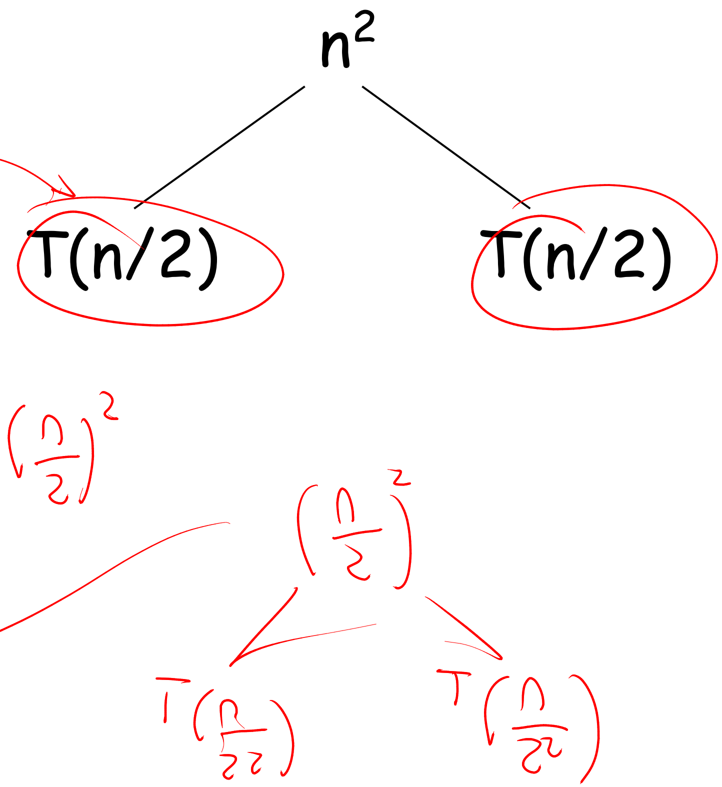
2^2

2^3

$\vdots$

Recurrencias

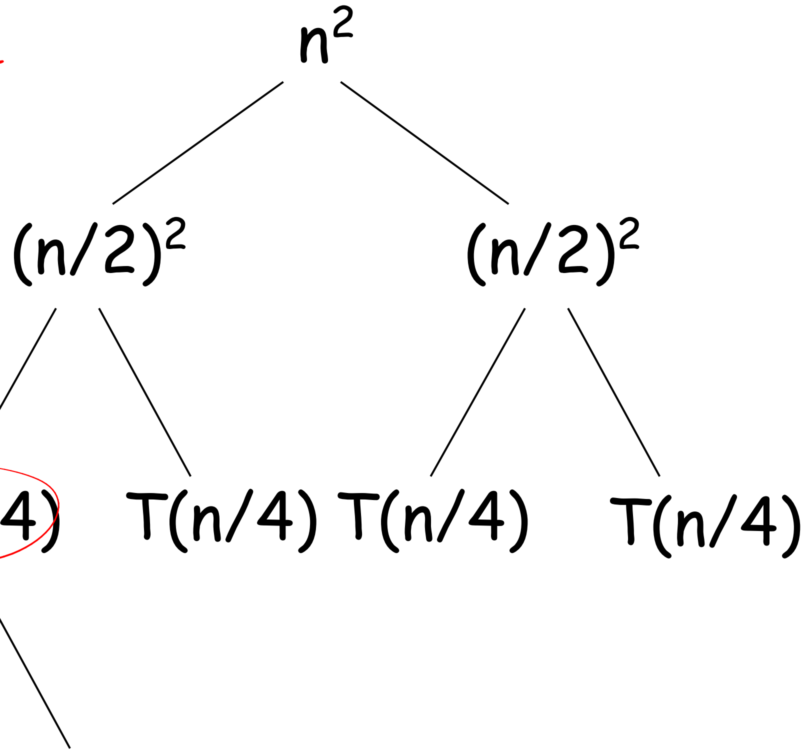
$T(n) = 2T(n/2) + n^2$



$T(n/2) = 2T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2$

Recurrencias

$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2$



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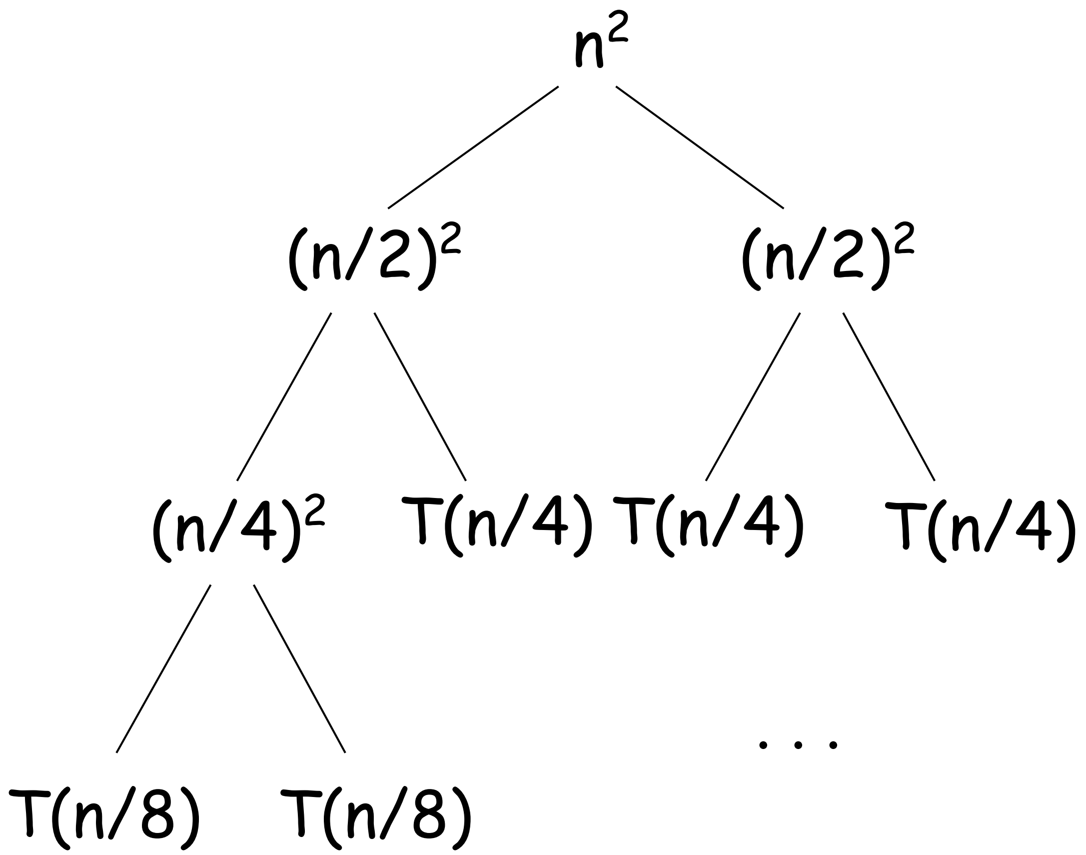
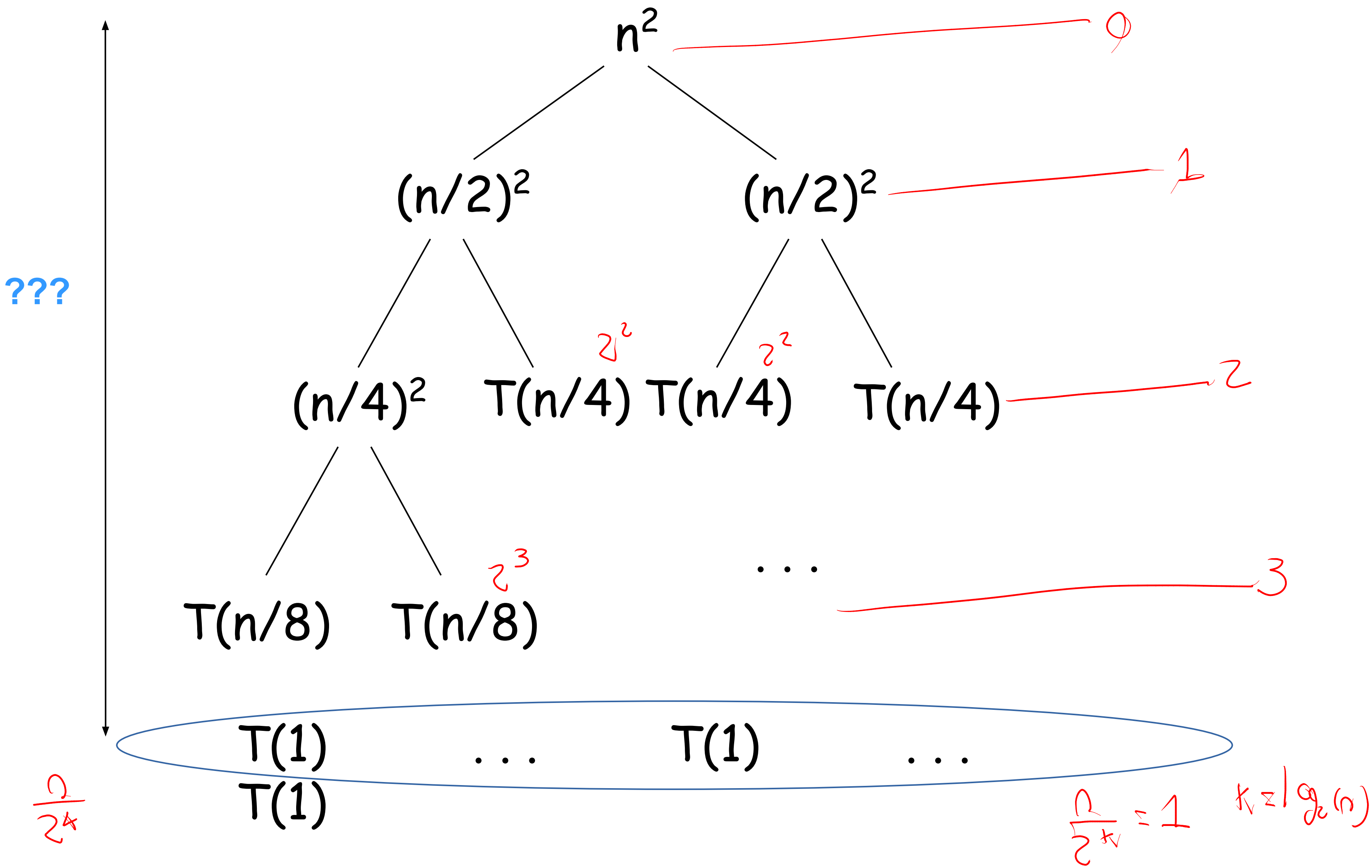
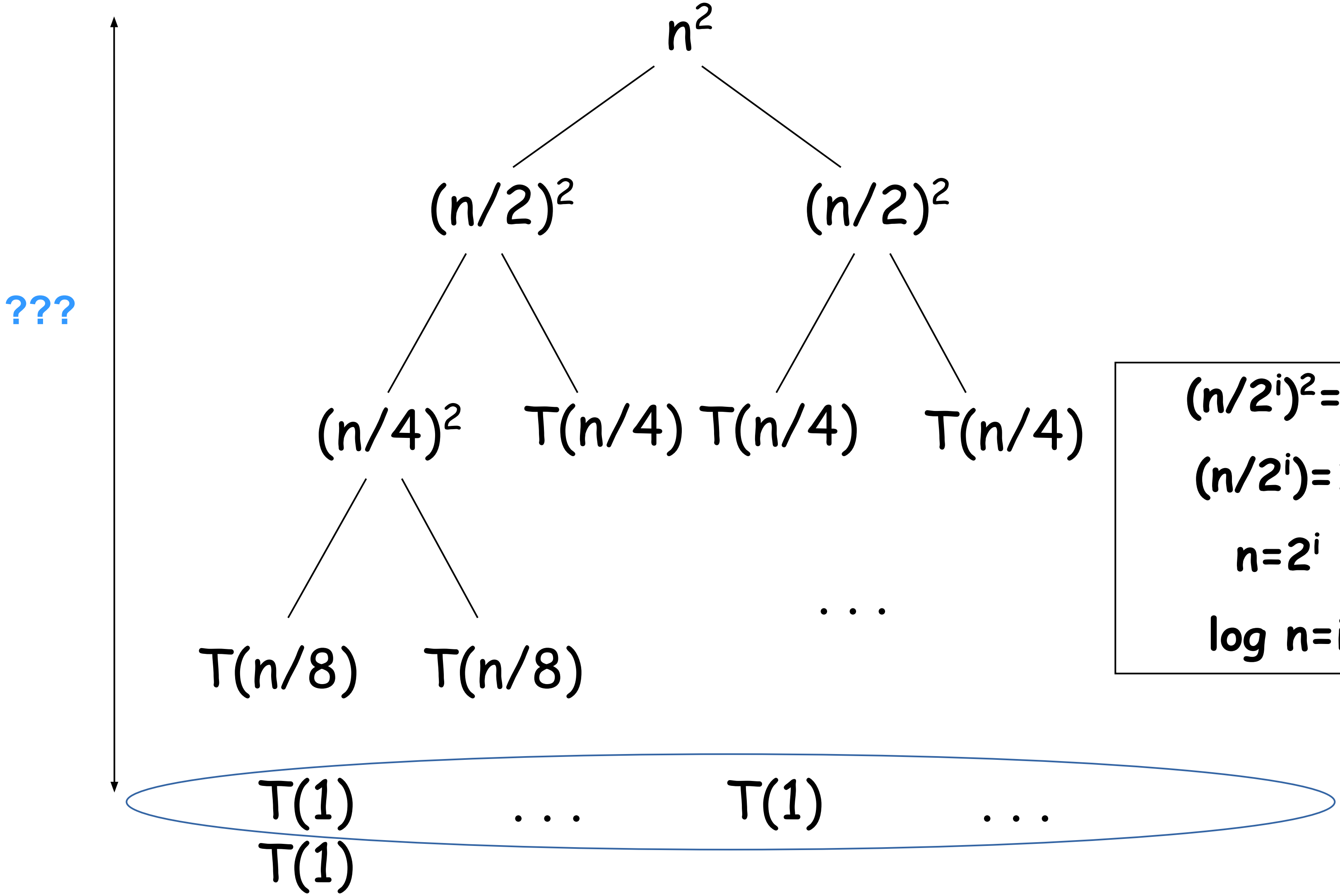
Recurrencias



Recurrencias



Recurrencias



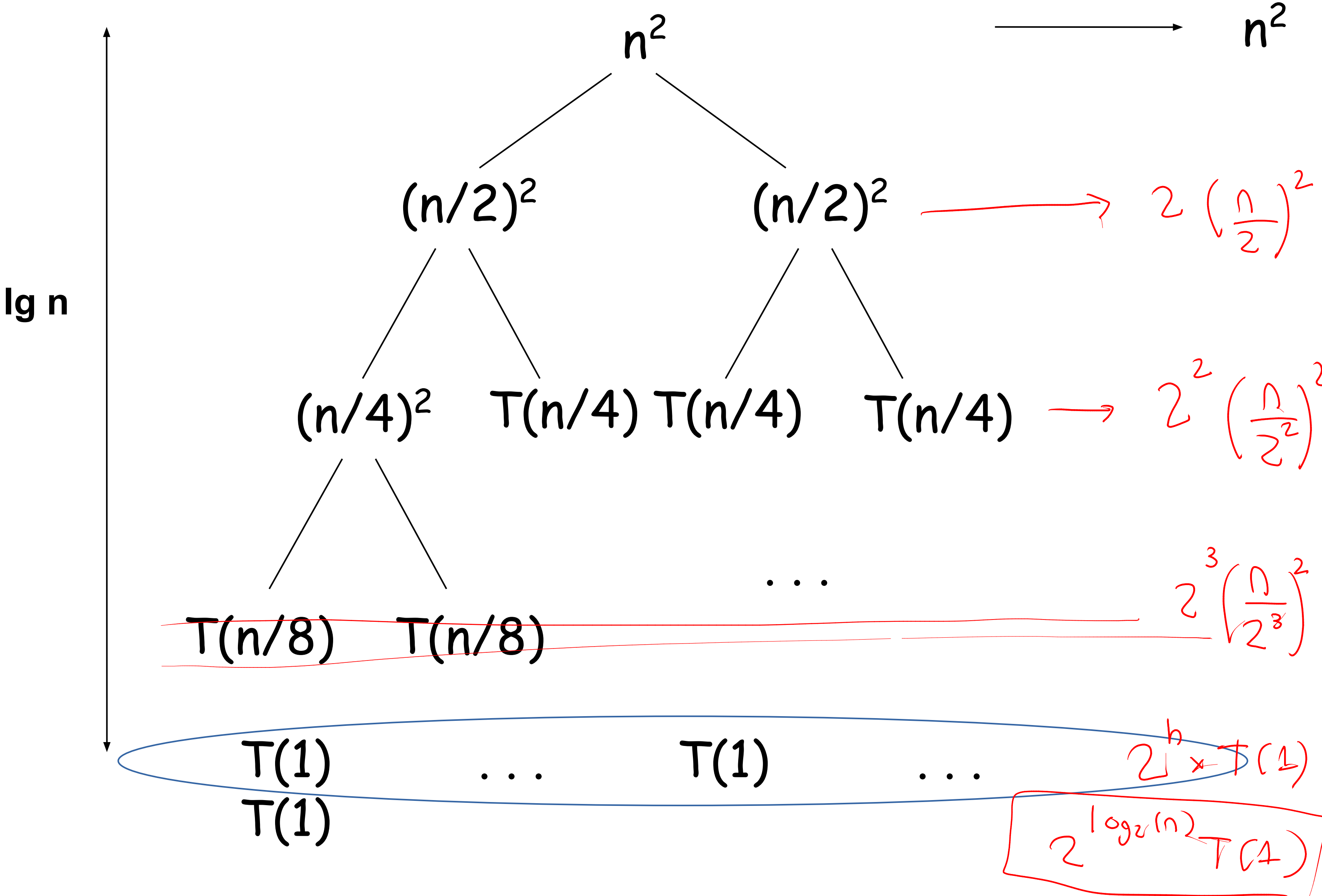
$(n/2^i)^2=1$

$(n/2^i)=1$

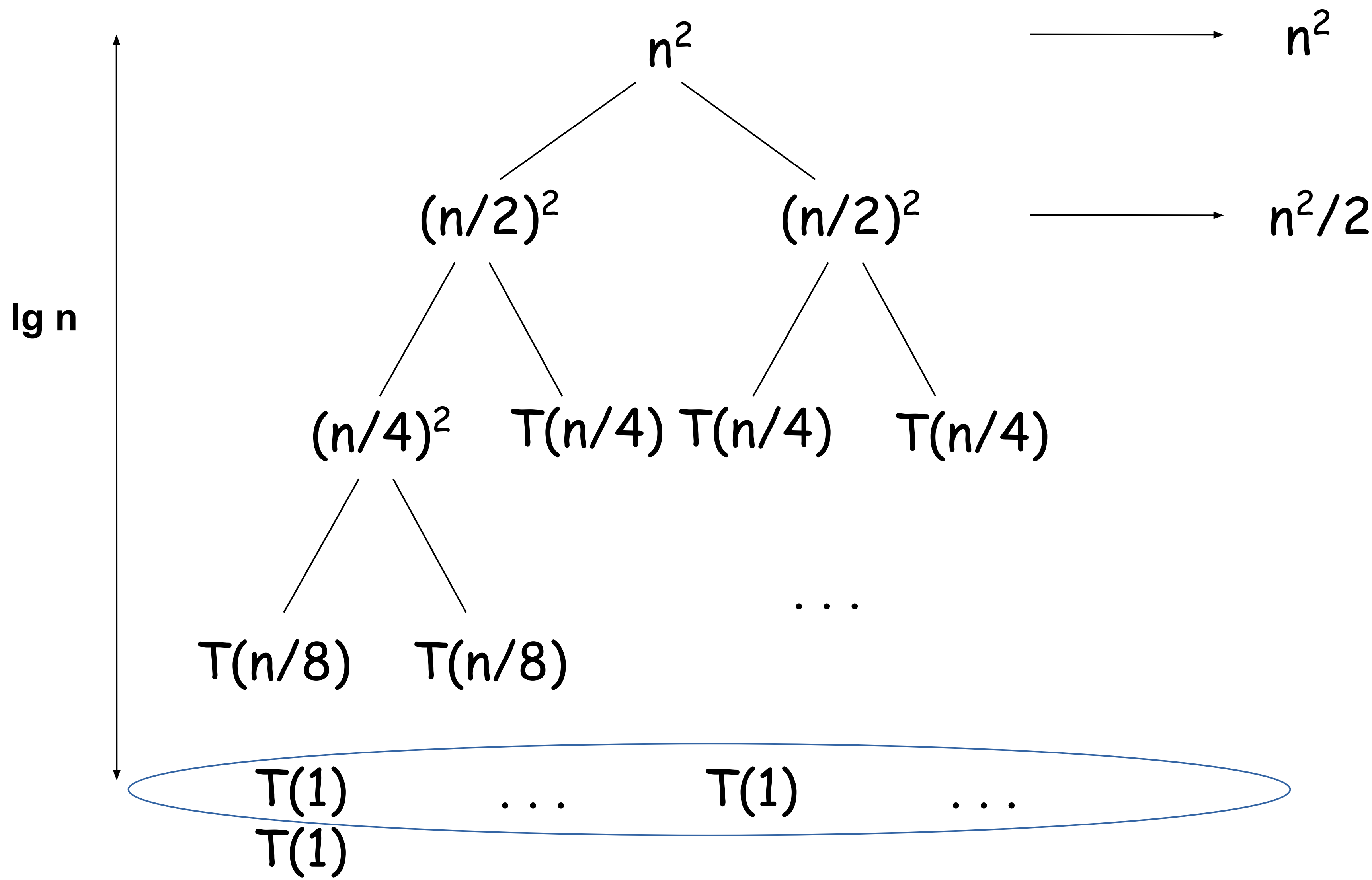
$n=2^i$

$\log n=i$

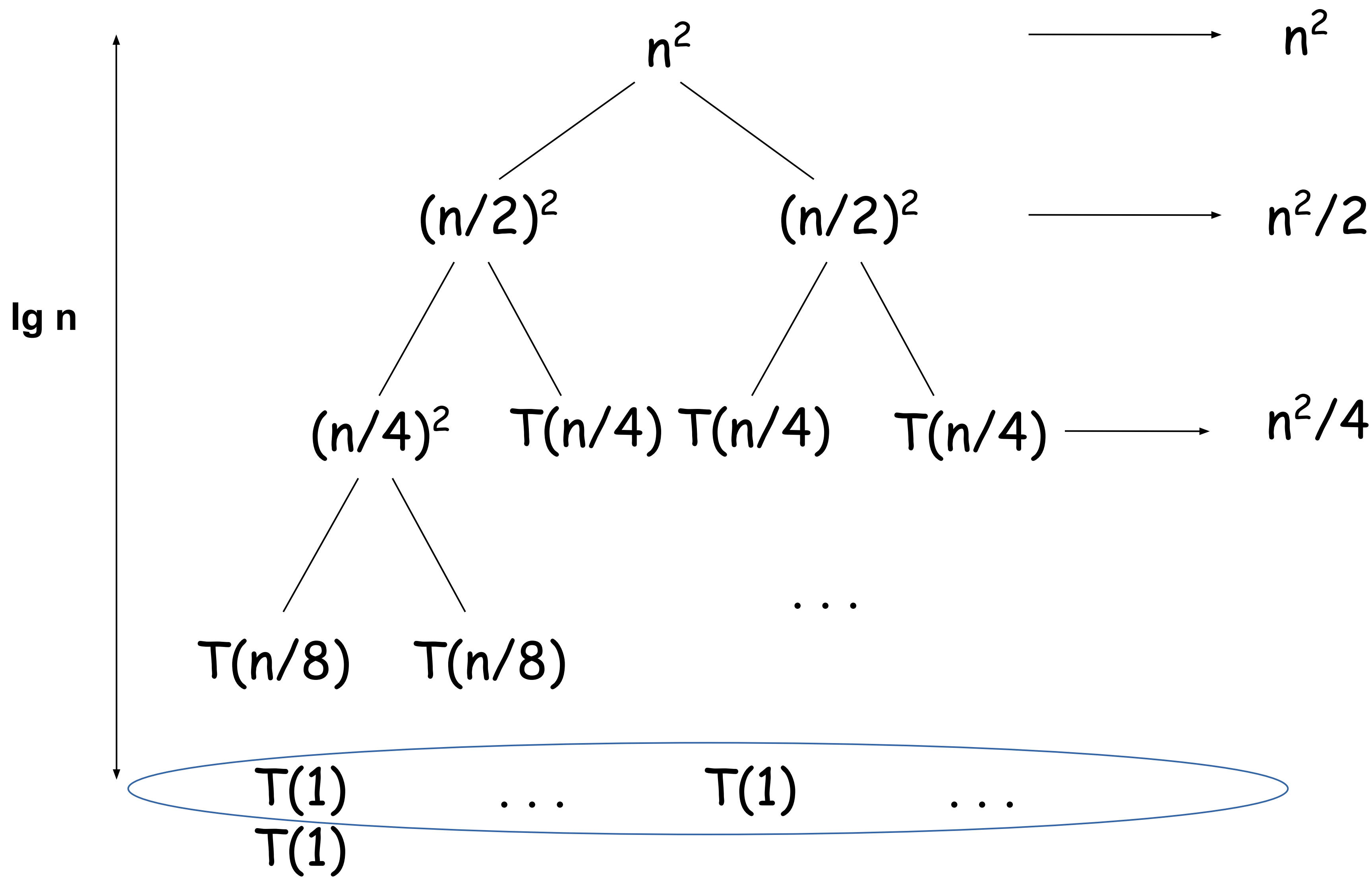
Recurrencias



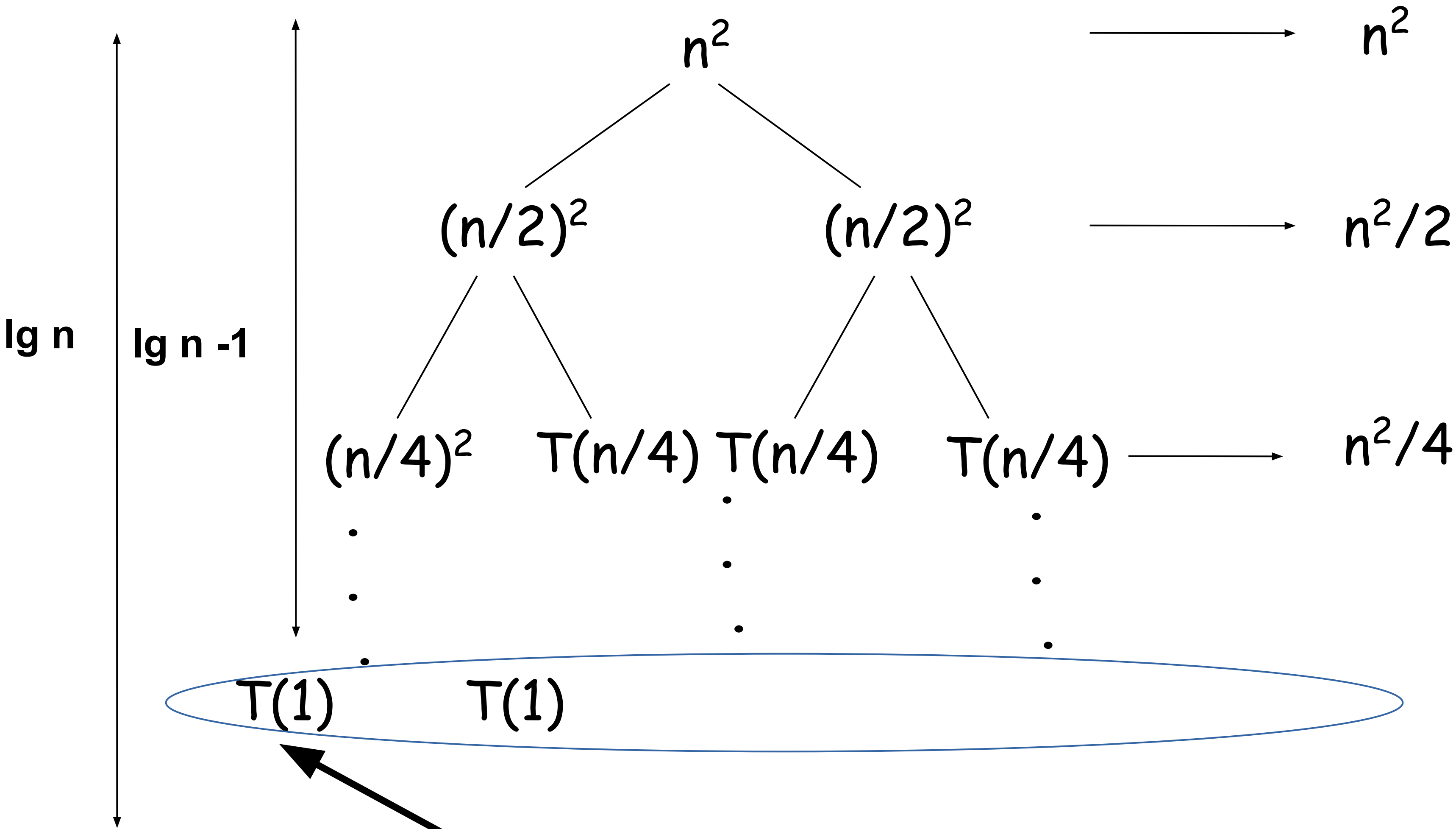
Recurrencias



Recurrencias

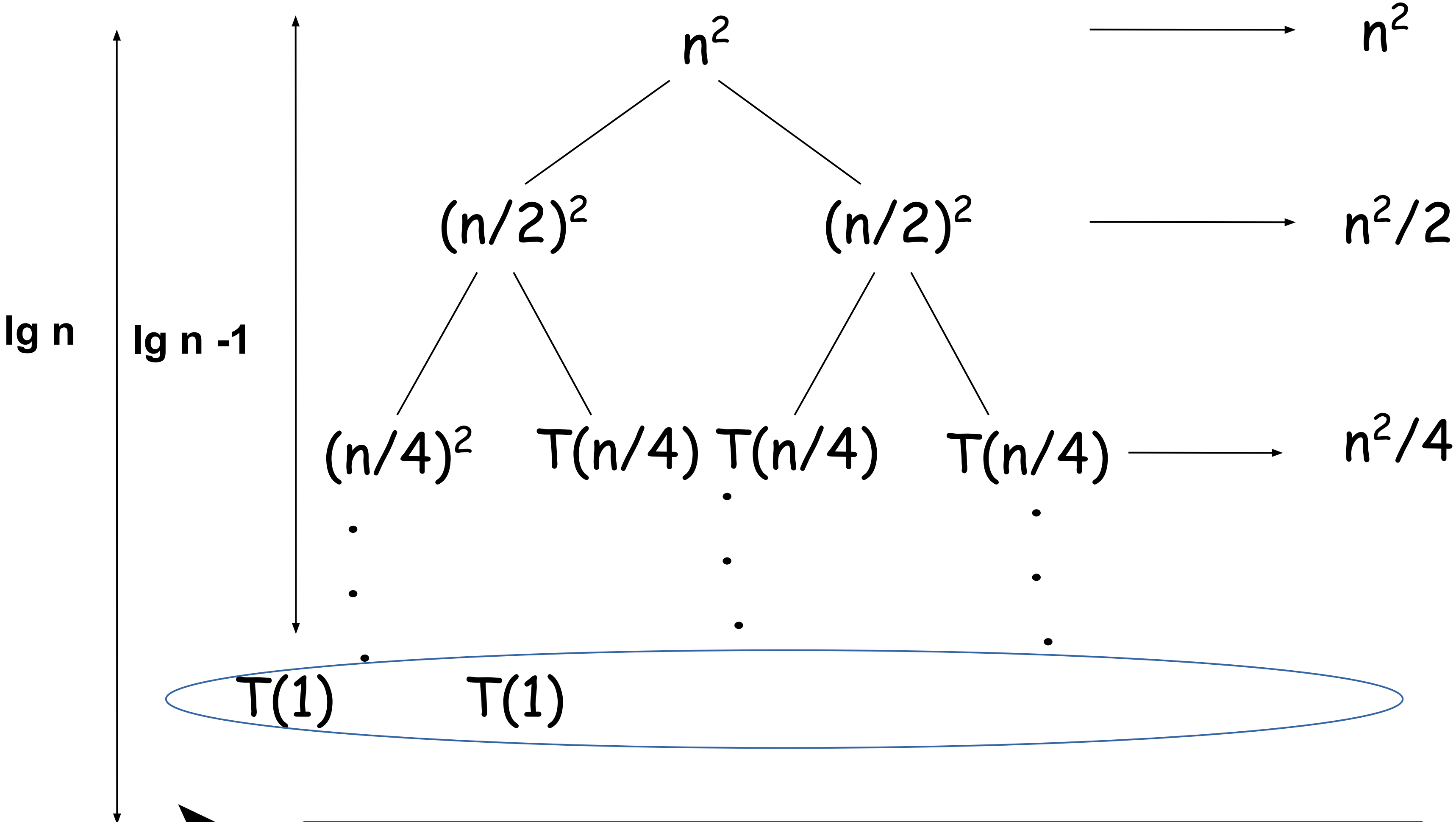


Recurrencias



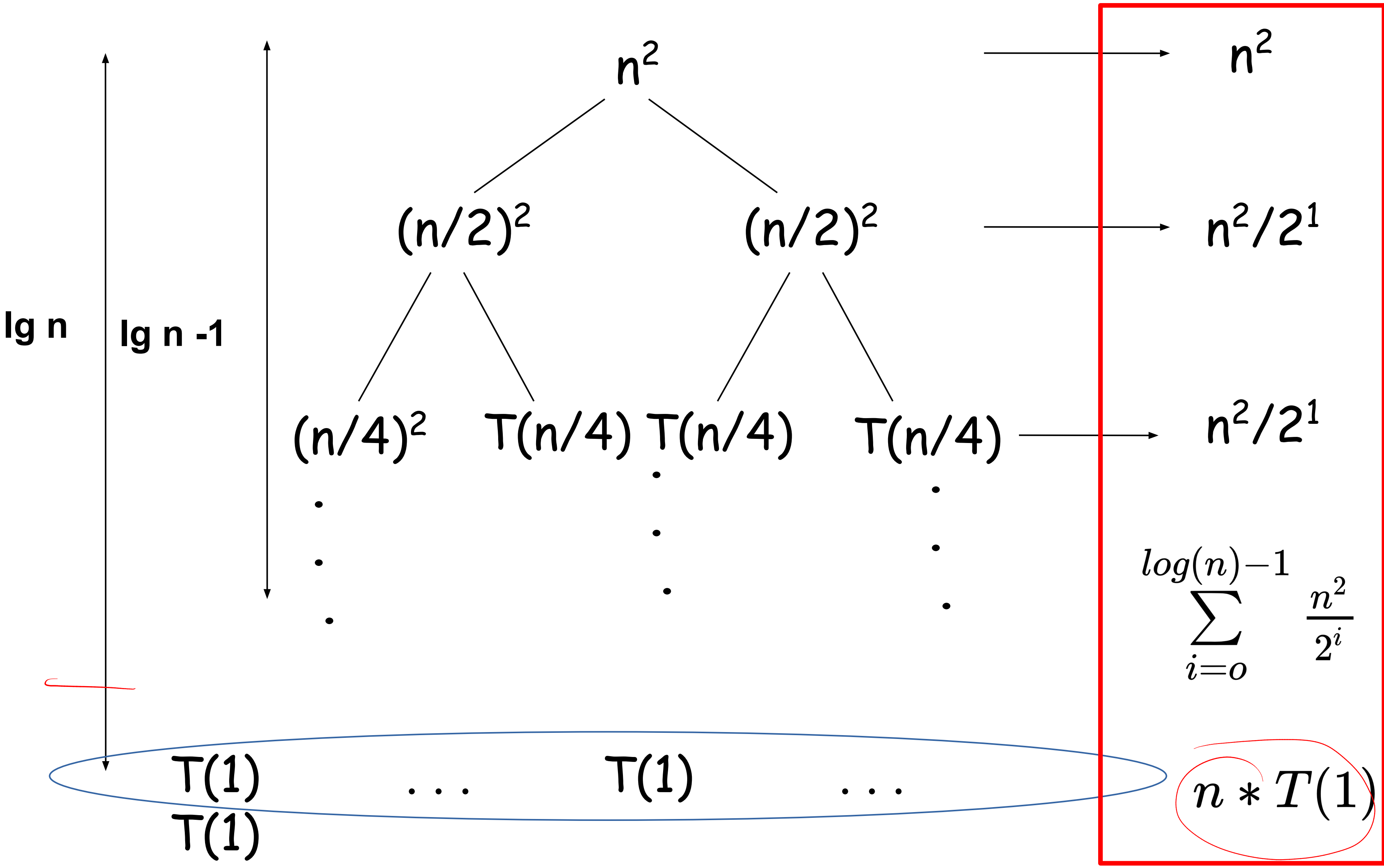
¿Cuantas hojas T(1) tenemos?

Recurrencias



Si recuerda en un árbol m-ario se tienen máximo m^h. En este caso al ser arbol binario m=2, tenemos $2^{\log(n)}$ hojas. Por lo tanto se

Recurrencias



Recurrencias

$$T(n) = n * T(1) + \sum_{i=o}^{log(n)-1} \frac{n^2}{2^i}$$

$$T(n) = n * c + n^2 \frac{0.5^{log(n)} - 1}{0.5 - 1}$$

$$T(n) = n * c + n^2 \frac{n^{log(0.5)} - 1}{-0.5}$$

$$T(n) = n * c + n^2 \frac{n^{-1} - 1}{-0.5}$$

$$T(n) = n * \overset{1}{c} - \frac{n}{0.5} + \frac{n^2}{0.5} = O(n^2)$$

c=1

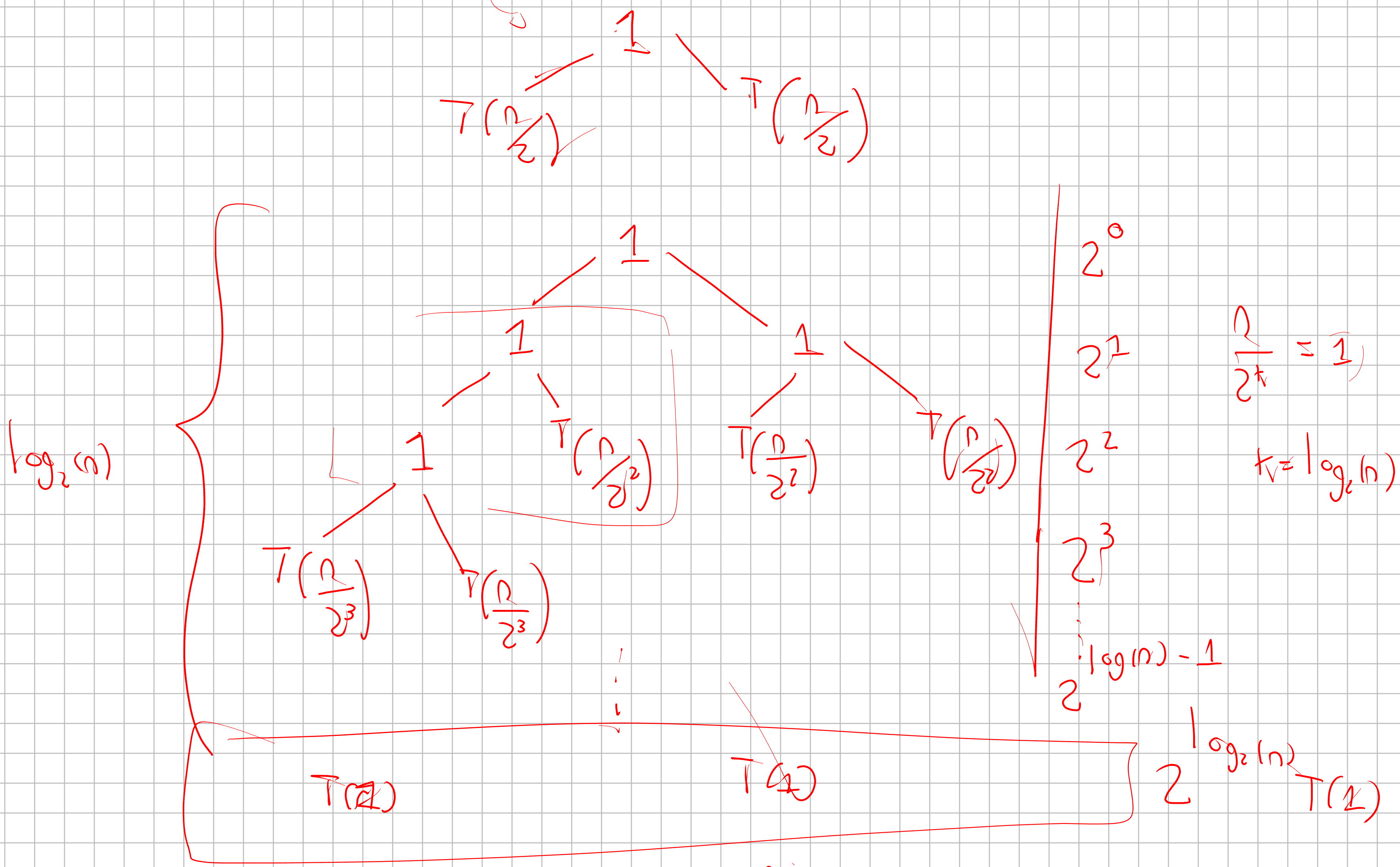
Recurrencias

Resuelva construyendo el árbol

$T(n) = 2T(n/2) + 1, T(1) = \Theta(1)$

$T(n) = 2T(n/2) + n, T(1) = \Theta(1)$

$T(n) = 2T\left(\frac{n}{2}\right) + 1$, $T(1) = \Theta(1)$

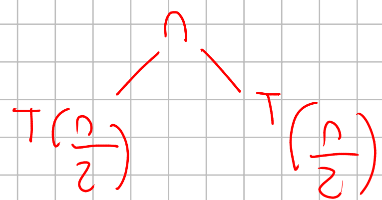


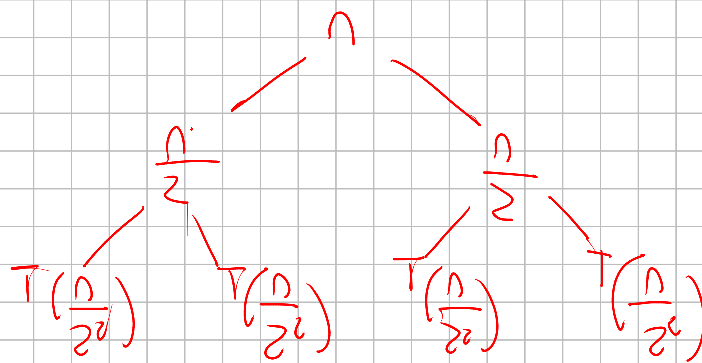
$$T(n) = nc + \sum_{k=0}^{\log_2(n)-1} 2^k = cn + \frac{\overset{n}{\cancel{2^{\log_2(n)}}} - 1}{2-1} = cn + n - 1$$

$(c+1)n - 1$ $\qquad c = 3$

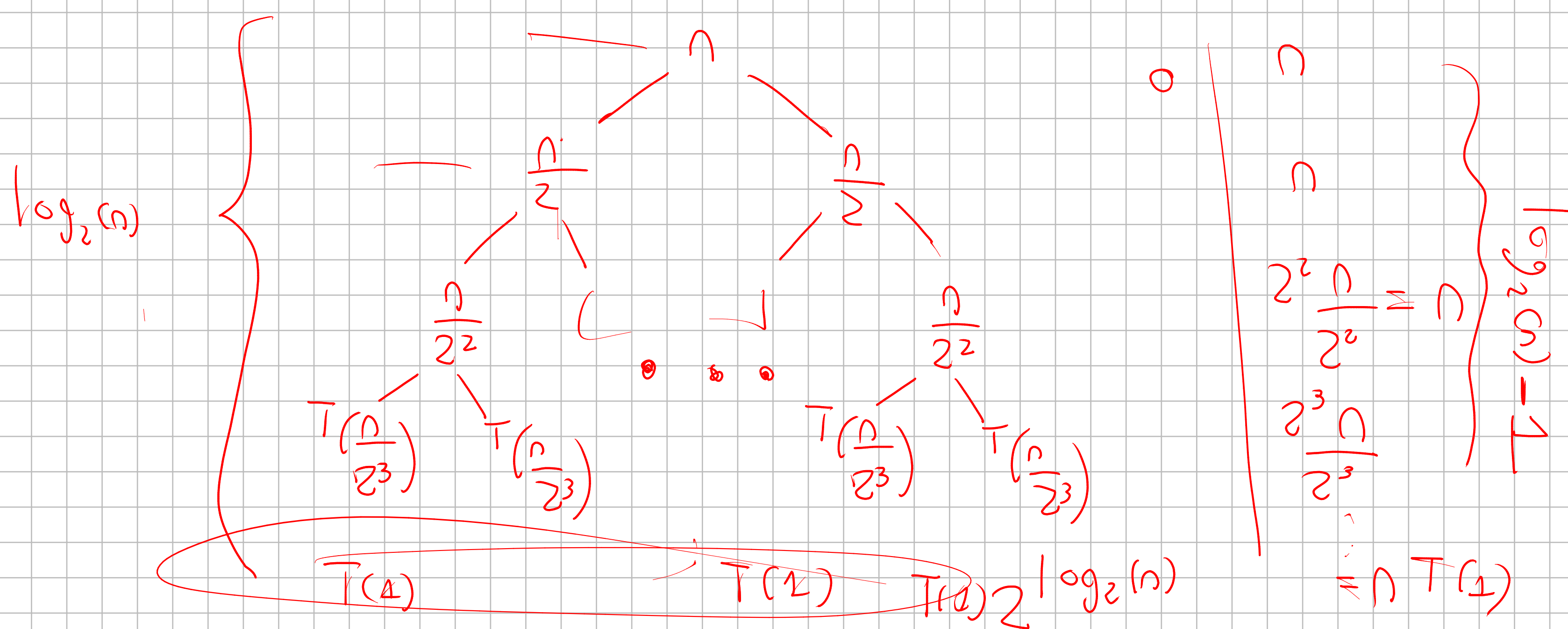
$\underline{4n - 1}$

$2T\left(\frac{n}{2}\right) + n \qquad T(1) = 1$





$0, 1, 2, \dots \log_2(n) - 1$



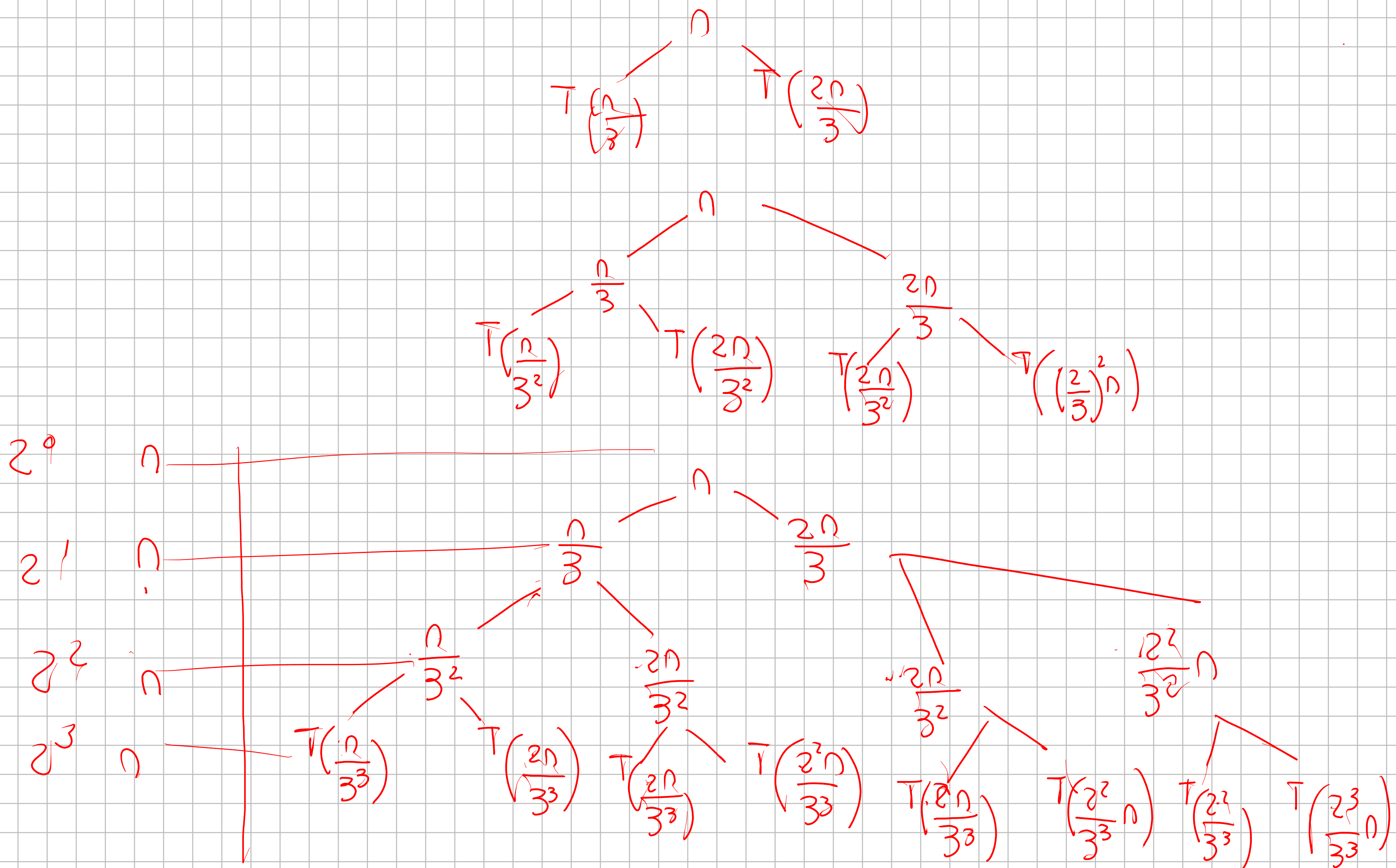
$$\boxed{T(n) = nT(1) + \log_2(n) \times n} \qquad \boxed{T(n) = n + n\log_2(n)}$$

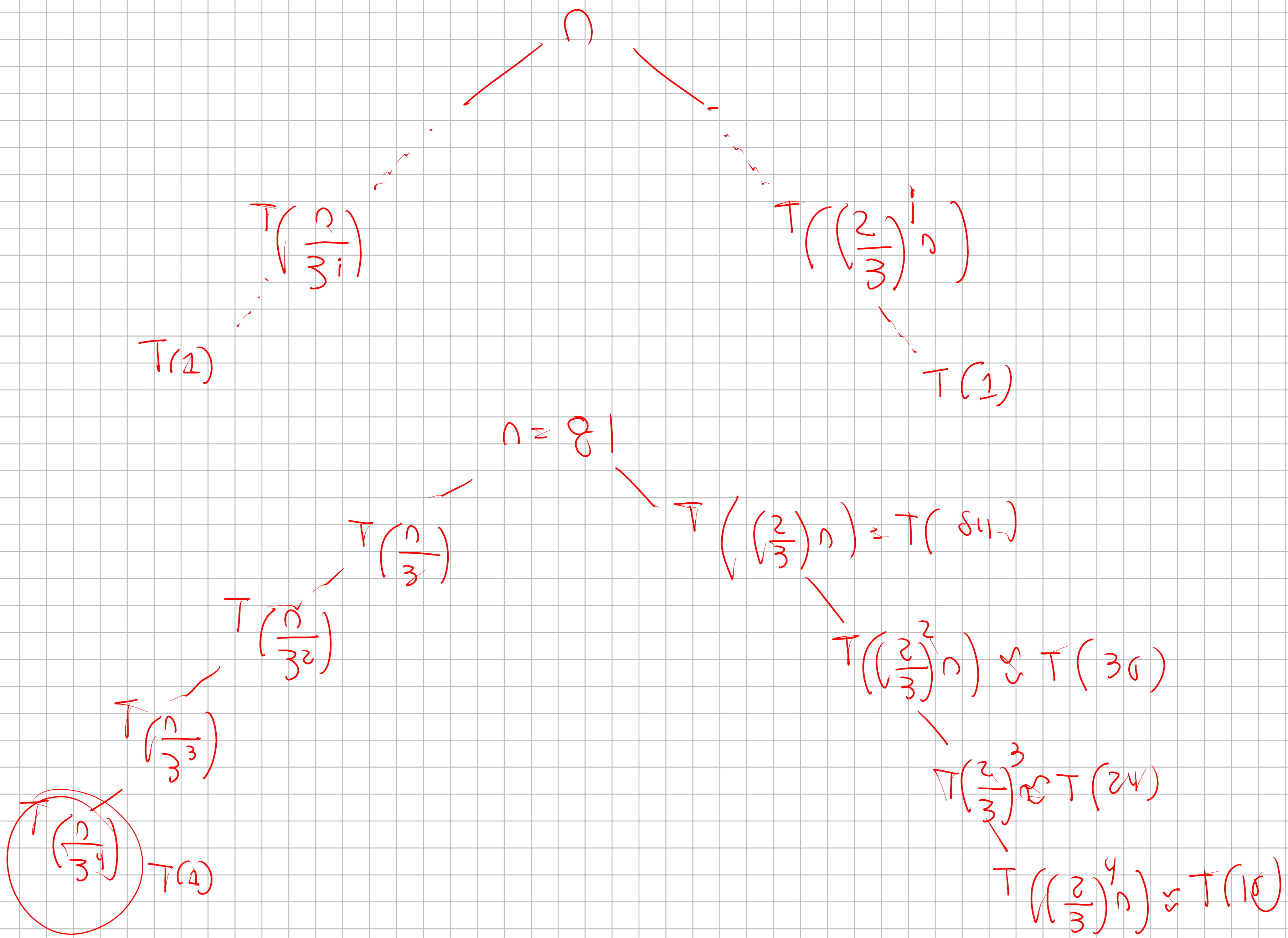
Recurrencias

Resuelva la recurrencia $T(n) = T(n/3) + T(2n/3) + n$

Indique una cota superior y una inferior

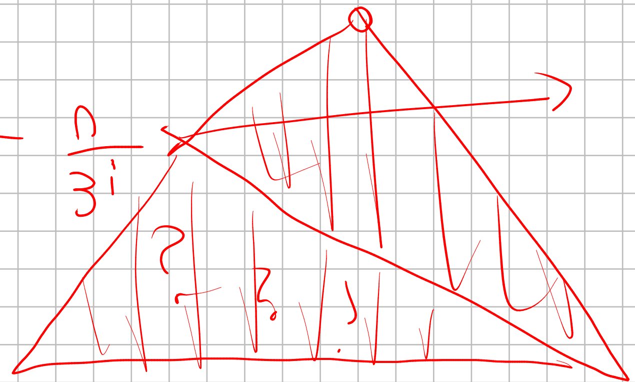
$$T(n) = T(n/3) + T\left(\frac{2n}{3}\right) + n$$





n
$T\left(\frac{n}{3^i}\right)$
$T\left(\left(\frac{2}{3}\right)^i n\right)$
$T(1)$
$T(1)$
$n = 81$
$T\left(\frac{n}{3}\right)$
$T\left(\left(\frac{2}{3}\right)n\right) = T(54)$
$T\left(\frac{n}{3^2}\right)$
$T\left(\left(\frac{2}{3}\right)^2 n\right) = T(36)$
$T\left(\frac{n}{3^3}\right)$
$T\left(\frac{2}{3}\right)^3 = T(24)$
$T\left(\frac{n}{3^4}\right)$
$T(1)$
$T\left(\left(\frac{2}{3}\right)^4 n\right) = T(16)$

$\log_3(n) \left\{ \Omega(f(n)) \leftarrow \frac{n}{3^i} \right.$



n

n

n

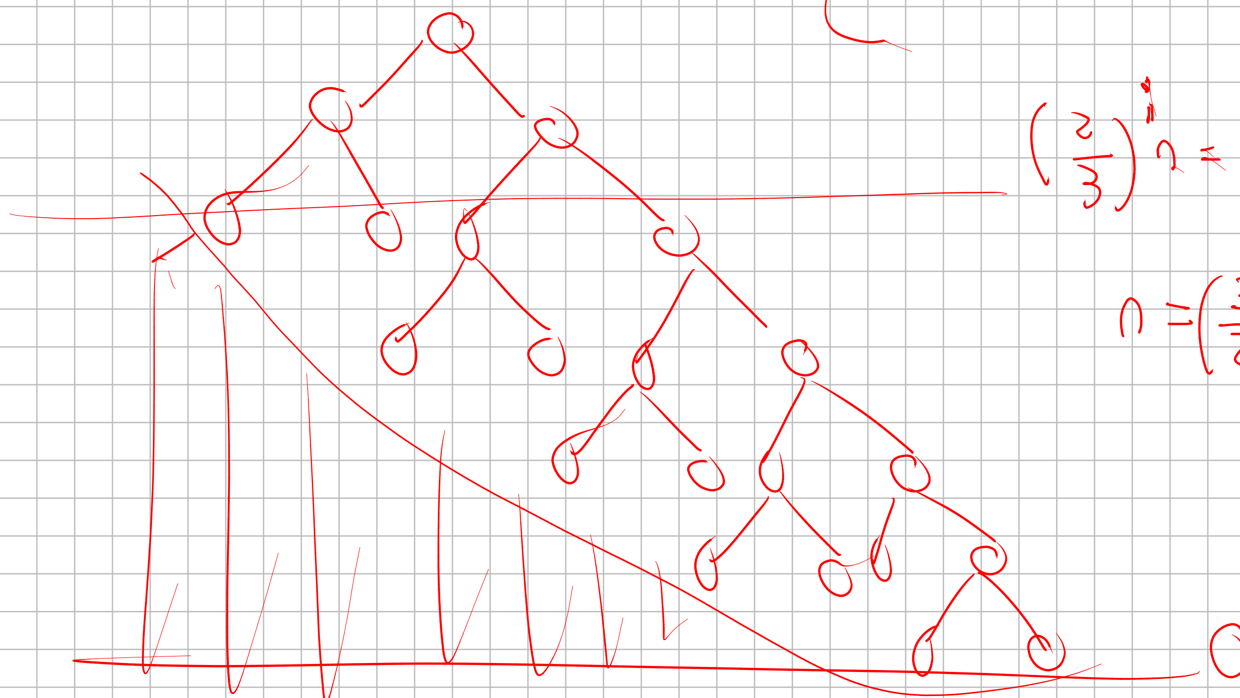
n

$\log_{\frac{3}{2}}(n)$

$\left(\frac{2}{3}\right)^i n$

$O(f(n))$

Ω

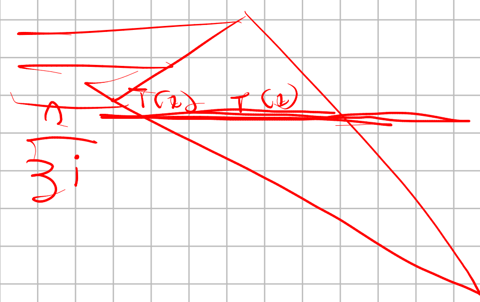


$\left(\frac{2}{3}\right)^i n = 1$

$n = \left(\frac{3}{2}\right)^i$

Cota inf

$$\log_3(n) \begin{cases} 0 \\ n \\ -1 \end{cases}$$



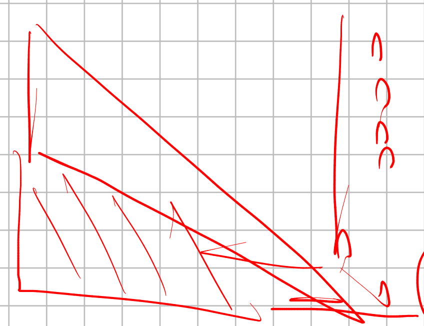
$m = 2$

$$2^{\log_3(n)} \times T(1) + \log_3(n) \times n$$

< 1

$$n^{\log_3(2)} T(1) + n \log_3(n)$$

$$\Omega(n \log_3(n))$$



$$\left(\frac{2}{3}\right)^i n = 1 \qquad i = \log_{\frac{3}{2}}(n)$$

1.70

$$T(n) = n^{\log_{\frac{3}{2}}(2)} T(1) + n \log_{\frac{3}{2}}(n)$$

$$O(n^2)$$

Recurrencias

Método maestro

Permite resolver recurrencias de la forma:

$T(n) = aT(n/b) + f(n)$, donde $a \geq 1$, $b > 1$

Recurrencias

Dado T(n) = aT(n/b) + f(n), donde a≥1, b>1, se puede acotar asintóticamente como sigue:

1. $T(n)=\Theta(n^{\log_b a})$

Si $f(n)=O(n^{\log_b a-\varepsilon})$ para algún $\varepsilon>0$

2. $T(n)=\Theta(n^{\log_b a}\lg n)$

Si $f(n)=\Theta(n^{\log_b a})$ para algún $\varepsilon>0$

3. $T(n)=\Theta(f(n))$

Si $f(n)=\Omega(n^{\log_b a+\varepsilon})$ para algún $\varepsilon>0$ si a*f(n/b) ≤c*f(n)

para algun c<1

Recurrencias

Dado T(n) = 9T(n/3) + n

$a = 9$

$b = 3$

$$n^{\log_3 9} = n^2 \quad \textbf{Vs} \quad f(n) = n$$

Es $f(n) = O(n^{\log_b a - \varepsilon})$?

Es $n = O(n^{2-\varepsilon})$?

$$n^{\log_b a} = n^{\log_3(9)} = n^2$$

n es $O(n^{2-\epsilon})$

$\Theta(n^2)$

n es $O(n^{1.9})$

Recurrencias

Dado T(n) = 9T(n/3) + n

$$n^{\log_3 9} = n^2 \quad \textbf{Vs} \quad f(n) = n$$

Es $f(n) = O(n^{\log_b a - \varepsilon})$?

Es $n = O(n^{2-\varepsilon})$?

Si $\varepsilon = 1$ se cumple que $n = O(n)$, por lo tanto, se cumple que:

$$T(n) = \Theta(n^2)$$

Recurrencias

$T(n) = T(2n/3) + 1$

$$n^{\log_{3/2} 1} = n^0 = 1 \quad \textbf{Vs} \quad f(n) = 1$$

$a = 1 \quad b = \frac{3}{2}$

Es $f(n) = O(n^{\log_b a - \varepsilon})$?

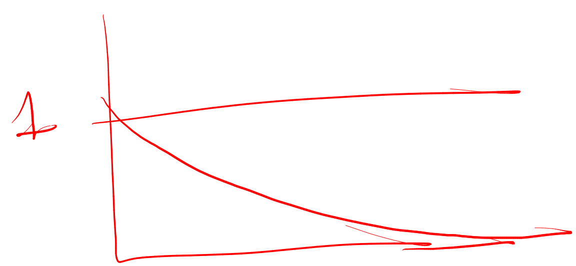
Es $1 = O(n^{0-\varepsilon})$?

$n^{\log_b a} = n^{\log_{3/2} 1} = n^0$

No existe $\varepsilon > 0$

$\frac{1}{n}$

1) 1 es $O(n^{0-\epsilon})$



Recurrencias

T(n) = T(2n/3) + 1

$$n^{\log_{3/2} 1} = n^0 = 1 \quad \text{vs} \quad f(n) = 1$$

Es $f(n) = \Theta(n^{\log_b a})$?

1 es $\Theta(n^0)$

Es $1 = \Theta(1)$?

1 es $\Theta(1)$??

Si, por lo tanto, se cumple que:

$$T(n) = \Theta(1 * \lg n) = \Theta(\lg n)$$

Recurrencias

T(n) =3 T(n/4) + nlgn

$$n^{\log_4 3} = n^{0.793} \quad \text{vs} \quad f(n) = n\lg n$$

Es $f(n) = O(n^{\log_b a - \varepsilon})$?

Es $f(n) = \Theta(n^{\log_b a})$?

Es $f(n) = \Omega(n^{\log_b a + \varepsilon})$?

Si, y además, af(n/b)≤cf(n)

$3(n/4)\lg(n/4) \leq cn\lg n$

$3(n/4)\lg n - 3(n/4)*2 \leq cn\lg n$

$(3/4)n\lg n \leq cn\lg n \rightarrow$ c=3/4 y se concluye que $T(n) = \Theta(n\lg n)$

$n^{\log_b a}$

$n^{\log_4 3} = n^{0.793}$

1) $n\log(n)$ $O(n^{0.793-\epsilon})$

2) $n\log(n)$ $\Theta(n^{0.793})$

$\Omega(n^{0.793})$ ✓ $\cancel{O}(n^{0.793})$

3) $n\log(n)$ $\Omega(n^{0.793+\epsilon})$

$\Omega(n)$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Sol: $T(n) = cn + \log_2(n) \times n$

$\sigma(\log_2(n))$

$a = 2 \quad b = 2$

$n^{\log_2 2} = n^1$

1) n es $O(n^{1-\epsilon})$ X

2) n es $\Theta(n)$

$$\Theta(n \log(n))$$

Dado $T(n) = aT(n/b) + f(n)$, donde $a \geq 1$, $b > 1$, se puede acc
asintóticamente como sigue:

1. $T(n) = \Theta(n^{\log_b a})$

Si $f(n) = O(n^{\log_b a - \varepsilon})$ para algún $\varepsilon > 0$

2. $T(n) = \Theta(n^{\log_b a} \lg n)$

Si $f(n) = \Theta(n^{\log_b a})$ para algún $\varepsilon > 0$

3. $T(n) = \Theta(f(n))$

Si $f(n) = \Omega(n^{\log_b a + \varepsilon})$ para algún $\varepsilon > 0$ si $a*f(n/b) \leq c*f(n)$

$T(n) = 2T(\frac{n}{2}) + 1$

$a=2 \quad b=2 \quad n^{\log_b a} = n^1$

$1 \in O(n^{1-\epsilon})$

$1 \in O(n^{0-7})$ ✓

$\Theta(n)$

$T(n) = (c+1)n - 1$

Dado $T(n) = aT(n/b) + f(n)$, donde $a \geq 1$, $b > 1$, se puede acc
asintóticamente como sigue:

1. $T(n)=\Theta(n^{\log_b a})$

Si $f(n)=O(n^{\log_b a-\varepsilon})$ para algún $\varepsilon>0$

2. $T(n)=\Theta(n^{\log_b a} \lg n)$

Si $f(n)=\Theta(n^{\log_b a})$ para algún $\varepsilon>0$

3. $T(n)=\Theta(f(n))$

Si $f(n)=\Omega(n^{\log_b a+\varepsilon})$ para algún $\varepsilon>0$ si $a*f(n/b) \leq c*f(n)$

$T(n) = 2T\left(\frac{n}{2}\right) + n^2$

$a = 2 \qquad b = 2 \qquad n^{\log_b a} = n$

1) $n^2 \in O(n^{1-\epsilon})$ X

2) $n^2 \in \Theta(n)$ X

3) $n^2 \in \Omega(n^{1+\epsilon})$ ✓

$a\,f\left(\frac{n}{b}\right) \leq c * f(n) \qquad c < 1$

$2\left(\frac{n}{2}\right)^2 \leq c * n^2$

$\frac{1}{2}\cancel{n^2} \leq c * \cancel{n^2} \qquad n > 0$

$c \geq \frac{1}{2} \quad \wedge \quad c < 1$ ✓

$T(n) = \Theta(n^2)$

Dado $T(n) = aT(n/b) + f(n)$, donde $a \geq 1$, $b > 1$, se puede acc
asintóticamente como sigue:

1. $T(n) = \Theta(n^{\log_b a})$

Si $f(n) = O(n^{\log_b a - \varepsilon})$ para algún $\varepsilon > 0$

2. $T(n) = \Theta(n^{\log_b a} \lg n)$

Si $f(n) = \Theta(n^{\log_b a})$ para algún $\varepsilon > 0$

3. $T(n) = \Theta(f(n))$

Si $f(n) = \Omega(n^{\log_b a + \varepsilon})$ para algún $\varepsilon > 0$ si $a * f(n/b) \leq c * f(n)$

Recurrencias

$T(n) = 2T(n/2) + n\lg n$

Muestre que no se puede resolver por el método maestro

Recurrencias

Resuelva usando método del maestro

$T(n) = 4T(n/2) + n$

$T(n) = 4T(n/2) + n^2$

$T(n) = 4T(n/2) + n^3$

Recurrencias

Método de sustitución

Suponer la forma de la solución y probar por inducción matemática

Recurrencias

$T(n)=2T(\lfloor n/2 \rfloor)+n,\ T(1)=1$

$$\log_a(n) = \frac{\log_2(n)}{\log_a(2)} = c\lg_2(n)$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Suponer que la solución es de la forma $T(n)=O(n\lg n)$

Probar que $\underline{T(n)\leq \boxed{c}\,n\lg n}$.

Se supone que se cumple para $n/2$ y se prueba para n

Hipotesis inductiva: $T(n/2) \leq cn/2\lg(n/2)$ Verdadera

Recurrencias

$T(n)=2T(\lfloor n/2 \rfloor)+n$, $T(1)=1$

Probar que $T(n) \leq cn\lg n$.

Hipótesis inductiva: $T(n/2) \leq cn/2 \lg (n/2)$

Paso inductivo:

$T(n) \leq 2(cn/2 \lg (n/2)) + n$

$\leq cn \lg (n/2) + n$

$= cn \lg (n) - cn + n$, para $c \geq 1$, haga $c=1$

$\leq cn \lg n$

Recurrencias

$T(n)=2T(\lfloor n/2 \rfloor)+n$, $T(1)=1$

Probar que $T(n) \leq cn\lg n$.

Paso base: si $c=1$, probar que $T(1)=1$ se cumple

$T(1) \leq 1*1\lg 1$?

$1 \leq 0$?

No, se debe escoger otro valor para c

Recurrencias

$T(n)=2T(\lfloor n/2 \rfloor)+n$, $T(1)=1$

Probar que $T(n) \leq cn\lg n$.

Paso base: si $c=2$, probar que $T(1)=1$ se cumple

$T(1) \leq 2*1\lg 1$?

$1 \leq 0$?

No, se puede variar k.

Para esto, se calcula $T(2)$ y se toma como valor inicial

Recurrencias

Probar que $T(n) \leq cn\lg n$.

$T(2)=2T(0)+2=4$

Paso base: si $c=1$, probar que $T(2)=4$ se cumple

$T(2) \leq 1*2\lg 2$?

$4 \leq 2$?

No, se puede variar c.

Recurrencias

Probar que $T(n) \leq cnlgn$.

$T(2)=2T(0)+2=4$

Paso base: si $c=3$, probar que $T(2)=4$ se cumple

$T(2) \leq 3*2lg\ 2$?

$4 \leq 6$?

Si, se termina la demostración

$T(n) = 2T\left(\frac{n}{2}\right) + 1$ $\qquad O(n)$ $\qquad T(1) = 1$

$T(n) \leq C_1 \times n \qquad n \geq K$

$T\left(\frac{n}{2}\right) = C_2 \times \frac{n}{2}$

$2T\left(\frac{n}{2}\right) + 1 \leq C_1 \times n$

hipotesis inductiva

$P\left(\frac{n}{2}\right) \rightarrow P(n)$

$2 \times \boxed{C_2 \frac{n}{2}} + 1 \leq C_1 \times n$

$C_2 n + 1 \leq C_1 \times n$

$2n + 1 \leq 3n$

$\lim_{n \to \infty} 2n + 1 \leq 3n$

$2 \leq 3$ ✓

$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad T(n) = O(n\log(n))$$

$$T\left(\frac{n}{2}\right) \leq \underline{c_1 \times \frac{n}{2}} \log\left(\frac{n}{2}\right)$$

$$2\boxed{T\left(\frac{n}{2}\right)} + n \leq c_2 \times n\log(n)$$

$$2\left(c_1 \times \frac{n}{2}\log\left(\frac{n}{2}\right)\right) \leq c_2 \times n\log(n)$$

$$c_1 \times n\log\left(\frac{n}{2}\right) \leq c_2 \times n\log(n)$$

$$c_1 \times n\left(\log(n) - \log(2)\right) \leq c_2 \times n\log(n)$$

$$c_1 n\log(n) - c_1 n\log(2) \leq c_2 \times n\log(n)$$

$$c_1 n\log(n) - c_1 n \leq c_2 \times n\log(n)$$

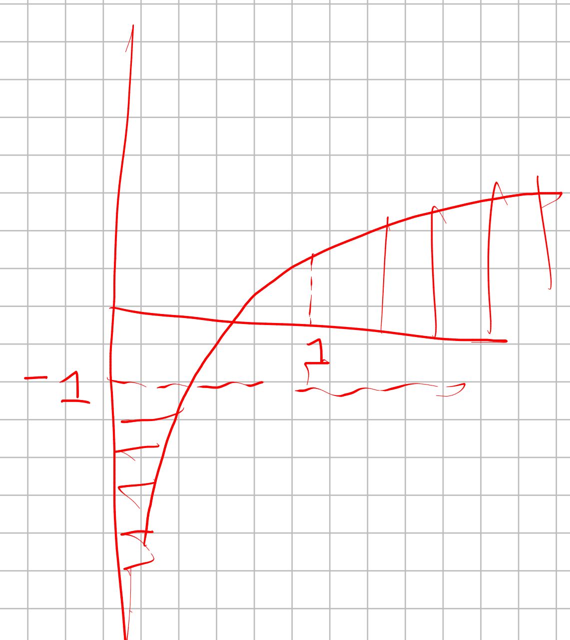
$$5n\log(n) - 5n \leq 10 \times n\log(n)$$

$$-5n\log(n) - 5n \leq 0$$

$$5n\log(n) + 5n \geq 0$$

$$\cancel{5n}\log(n) \geq -\cancel{5n} \qquad n > 0$$

$$\log(n) \geq -1 \qquad n > 0$$



Recurrencias

$T(n)=T(n-1)+T(n-2)+1,\ T(1)=O(1),\ T(2) = O(1)$

Suponer que la solución es de la forma $T(n)=O(2^n)$

Probar que $T(n)\leq c2^n$.

Se supone que se cumple para $n-1$ y se $n-2$ prueba para n

Hipotesis inductiva: $T(n-1) \leq c2^{(n-1)}$ y $T(n-2) \leq c2^{(n-2)}$

Recurrencias

$T(n)=T(n-1)+T(n-2)+1,\ T(1)=O(1),\ T(2) = O(1)$

Ahora se debe probar que: $T(n) \le c2^n$

$T(n) \le c_1 2^{n-1} + c_2 2^{n-2}$

$T(n) \le c_1 2^n + c_2 2^n$

$T(n) \le c\, 2^n$

$T(1) \le c2^1 \rightarrow 1 \le 2*c$

$T(2) \le c2^2 \rightarrow 1 \le 4*c$

$T(3) \le c2^3 \rightarrow 2 \le 8*c$

$T(4) \le c2^4 \rightarrow 3 \le 16*c$

$T(5) \le c2^5 \rightarrow 5 \le 32*c$

$T(6) \le c2^6 \rightarrow 8 \le 64*c$

$T(7) \le c2^7 \rightarrow 13 \le 128*c$

$T(8) \le c2^8 \rightarrow 21 \le 256*c$

Con $c = 1$, se cumple.

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2 \qquad \Theta(n^2)$$

$$T(n) \in O(n^2)$$

$$T\left(\frac{n}{2}\right) \le c \times \left(\frac{n}{2}\right)^2$$

$$T(n) \le c \times n^2$$

$$2\boxed{T\left(\frac{n}{2}\right)} + n^2 \le c_1 \times n^2$$

$$2c_2 \times \frac{n^2}{4} + n^2 \le c_1 \times n^2$$

$$cn^2 + n^2 \le c_1 \times n^2$$

$$(c+1)\, n^2 \le c_1 \times n^2$$

$$cn^2 \le c_1 n^2$$

$$3n^2 \le 4n^2 \qquad 3 \le 4$$

$$T\left(\frac{n}{2}\right) \checkmark$$

$$T(n) \in \Omega(n^2)$$

$$T\left(\frac{n}{2}\right) \ge c \times \left(\frac{n}{2}\right)^2$$

$$2T\left(\frac{n}{2}\right) + n^2 \ge cn^2$$

$$2c_2\left(\frac{n}{2}\right)^2 + n^2 \ge cn^2$$

$$c_3 n^2 + n^2 \ge cn^2$$

$$c_4 n^2 \ge cn^2$$

$$5n^2 \ge 4n^2$$

$$5 \ge 4$$

Referencias

Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd ed.). The MIT Press. Chapter 4

Gracias

Próximo tema:

Divide y vencerás