Problema 1. *Demostreu utilitzant camps de Jacobi que la curvatura seccional del pla hiperbòlic* $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$*, amb la mètrica* $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$*, és constant igual a* -1*.*

Demostració. Considerem la variació de geodèsiques $f(t,s) := (x+s, e^t)$. Considerem el camp de Jacobi $J(t) := \mathrm{d}f(\partial_s)|_{(t,0)}(t) = (1,0)$ (que escrit en una referència ortonormal és $J(t) = e^{-t}(e^t,0)$) ortogonal a $\gamma(t) := f(t,0)$. Fixem-nos que $(e^t,0)$ és paral·lel al llarg de γ. També, fixem-nos que $\langle J(t), \gamma(t)\rangle = \mathbb{H}^2$. Per tant, és suficient veure que $J'' - J = 0$ per a comprovar que $\mathbb{H}^2$ té curvatura constant -1. Tenim

$$\begin{aligned}
\nabla_{\gamma'}\nabla_{\gamma'}J(t) &= \nabla_{\gamma'}\nabla_{\gamma'}e^{-t}(e^t,0)\\
&= \nabla_{\gamma'}(-e^{-t}(e^t,0) + e^{-t}\nabla_{\gamma'}(e^t,0))\\
&= \nabla_{\gamma'}(-e^{-t}(e^t,0))\\
&= e^{-t}(e^t,0) - e^{-t}\nabla_{\gamma'}(e^t,0)\\
&= e^{-t}(e^t,0)\\
&= J(t)
\end{aligned}$$

d'on deduïm que $J'' - J = 0$, com volíem veure. $\square$

Problema 2. *Donats dos camps de Jacobi* $J_1(t), J_2(t)$ *definits al llarg d'una geodèsica* $\gamma(t)$ *i tals que* $J_1(0) = J_2(0) = 0$ *demostreu*

$$g(J_1(t), J_2(t)) = g(J_1(t), J_2'(t))t^2 - \frac{1}{3}(\gamma'(0), J_1'(0), J_2'(0), \gamma'(0))t^4 + O(t^5)$$

Demostració. Escrivim $\nabla_{\gamma'}^{(k)} := \nabla_{\gamma'} \overset{k)}{\cdots} \nabla_{\gamma'}$. Com $J_i(t)$ de Jacobi i $J_i(0) = 0$, $\nabla_{\gamma'}^{(2)} J_i(0) = R(\gamma'(0), J(0))\gamma'(0) = 0$ i $\nabla_{\gamma'}^{(3)} J_i(0) = R(\gamma'(0), \nabla_{\gamma'} J_i(0))\gamma'(0)$. Per inducció, trobem que

$$\nabla_{\gamma'}^{(k)} g(J_1(t), J_2(t)) = \sum_{j=0}^{k} \binom{k}{j} g(\nabla_{\gamma'}^{(k-j)} J_1(t), \nabla_{\gamma'}^{(j)} J_2(t))$$

D'aquestes observacions i que $J_i(0) = 0$, deduïm que

$$\begin{aligned}
g(J_1(0), J_2(0)) &= 0\\
\nabla_{\gamma'} g(J_1(0), J_2(0)) &= 0\\
\nabla_{\gamma'}^{(2)} g(J_1(0), J_2(0)) &= 2g(\nabla_{\gamma'}J_1(0), \nabla_{\gamma'}J_2(0))\\
\nabla_{\gamma'}^{(3)} g(J_1(0), J_2(0)) &= \sum_{j\in\{0,3\}} g(\nabla_{\gamma'}^{(3-j)} J_1(0), \nabla_{\gamma'}^{(j)} J_2(0)) + \sum_{j\in\{1,2\}} g(\nabla_{\gamma'}^{(k-j)} J_1(0), \nabla_{\gamma'}^{(j)} J_2(0))\\
&= \sum_{\sigma\in S_2} g(R(\gamma'(0), J_{\sigma(1)}(0))\gamma'(0), \nabla_{\gamma'}J_{\sigma(2)}(0))\\
&= 0\\
\nabla_{\gamma'}^{(4)} g(J_1(0), J_2(0)) &= 4g(\nabla_{\gamma'}^{(3)} J_1(0), \nabla_{\gamma'}J_2(0)) + 4g(\nabla_{\gamma'}J_1(0), \nabla_{\gamma'}^{(3)} J_2(0))\\
&= 4\big\{g(R(\gamma'(0), \nabla_{\gamma'}J_1(0))\gamma'(0), \nabla_{\gamma'}J_2(0)) + g(\nabla_{\gamma'}J_1(0), R(\gamma'(0), \nabla_{\gamma'}J_2(0))\gamma'(0))\big\}\\
&= 4\big\{(\gamma'(0), \nabla_{\gamma'}J_1(0), \gamma'(0), \nabla_{\gamma'}J_2(0)) + (\gamma'(0), \nabla_{\gamma'}J_2(0), \gamma'(0), \nabla_{\gamma'}J_1(0))\big\}\\
&= -8(\gamma'(0), \nabla_{\gamma'}J_1(0), \nabla_{\gamma'}J_2(0), \gamma'(0))
\end{aligned}$$

Aleshores,

$$\begin{aligned}
g(J_1(t), J_2(t)) &= \sum_{k=0}^{4} \frac{g(J_1(0), J_2(0))t^k}{k!} + O(t^5)\\
&= \frac{2}{2!}g(\nabla_{\gamma'}J_1(0), \nabla_{\gamma'}J_2(0))t^2 - \frac{8}{4!}(\gamma'(0), \nabla_{\gamma'}J_1(0), \nabla_{\gamma'}J_2(0), \gamma'(0)) + O(t^5)\\
&= g(\nabla_{\gamma'}J_1(0), \nabla_{\gamma'}J_2(0))t^2 - \frac{1}{3}(\gamma'(0), \nabla_{\gamma'}J_1(0), \nabla_{\gamma'}J_2(0), \gamma'(0)) + O(t^5)
\end{aligned}$$

com volíem. $\square$

Deduïu que respecte un sistema de coordenades geodèsiques $(x_1, \ldots, x_n)$ *al voltant d'un punt tenim*

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{r,s} R_{irsj}(0)x_r x_s + O(|x|^3)$$

on $R_{irsj} = g(R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_r})\frac{\partial}{\partial x_s}, \frac{\partial}{\partial x_j})$.

Demostració. Sigui $(U, (x_1, \ldots, x_n))$ carta amb coordenades geodèsiques d'una varietat riemanniana (M, g) i $p \in U$. Suposem el sistema de coordenades centrat en p. Aleshores, $g_{ij}(0) = \delta_{ij}$ i la parametrització en coordenades de les geodèsiques que passen per p són de la forma $t \mapsto at$, on $a \in \mathbb{R}^{\dim M}$. Sigui $\gamma(t) = (x_1 t, \ldots, x_n t)$ i les variacions en geodèsiques $\gamma_i(t, s) := (tx_1, \ldots, t(x_i + s), \ldots, tx_n)$. Aleshores, $J_i(t) := d\gamma_i(\partial_s)|_{(t,0)}(t)$ és un camp de Jacobi i $J_i(t) = t\partial_{x_i}$ (per com hem definit γ_i), d'on deduïm que tal que $J_i(0) = 0$ i $\nabla_{\gamma'} J_i(t) = \partial_{x_i}$. Com $g(V_i(t), V_j(t)) = g(t\partial_{x_i}, t\partial_{x_j}) = t^2 g(\gamma(t))$,

$$\begin{aligned}
g(\gamma(t)) &= g(x_1 t, \ldots, x_n t) \\
&= \frac{1}{t^2} g(J_i(t), J_j(t)) \\
&= \frac{1}{t^2}\Big(g(\nabla_{\gamma'} J_i(0), \nabla_{\gamma'} J_j(0))t^2 - \frac{1}{3}(\gamma'(0), \nabla_{\gamma'} J_i(0), \nabla_{\gamma'} J_j(0), \gamma'(0))t^4 + O(t^5)\Big) \\
&= \frac{1}{t^2}\Big(g(\partial_{x_i}(0), \partial_{x_j}(0))t^2 - \frac{1}{3}\Big(\sum_r x_r \partial_{x_r}(0), \partial_{x_i}(0), \partial_{x_j}(0), \sum_r x_s \partial_{x_s}(0)\Big)t^4 + O(t^5)\Big) \\
&= g(\partial_{x_i}(0), \partial_{x_j}(0)) - \frac{1}{3}\sum_{r,s}(\partial_{x_r}(0), \partial_{x_i}(0), \partial_{x_j}(0), \partial_{x_s}(0))(x_r t)(x_s t) + O(t^3) \\
&= g_{ij}(0) - \frac{1}{3}\sum_{r,s} R_{rijs}(0)(x_r t)(x_s t) + O(t^3) \\
&= \delta_{ij} - \frac{1}{3}\sum_{r,s} R_{irsj}(0)(x_r t)(x_s t) + O(t^3)
\end{aligned}$$

d'on resulta

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{r,s} R_{irsj}(0)x_r x_s + O(\|x\|^3)$$

que és el que volíem veure. $\square$