LLISTA 3 Exercici 1: Calculeu el tensor de curvatura de S^3.

- Primer donem la mètrica. A partir de la parametrització

$$\begin{cases} x = \sin\alpha \\ y = \cos\alpha \sin\beta \\ z = \cos\alpha \cos\beta \sin\gamma \\ t = \cos\alpha \cos\beta \cos\gamma \end{cases} \qquad \alpha, \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \gamma \in (0, 2\pi)$$

i un càlcul ens dóna la mètrica

$$\boxed{ds^2 = d\alpha^2 + \cos^2\alpha\, d\beta^2 + \cos^2\alpha \cos^2\beta\, d\gamma^2}$$

- Calculem els SÍMBOLS de CHRISTOFFEL on $x_1 = \alpha$, $x_2 = \beta$, $x_3 = \gamma$ utilitzem l'exercici 13 de la llista 2 per mètriques diagonals on $g_{11} = 1$, $g_{22} = \cos^2\alpha$ i $g_{33} = \cos^2\alpha \cos^2\beta$

- $\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{11}^3 = 0$ $\qquad \nabla_{\partial_\alpha}\partial_\alpha = 0$

- $\Gamma_{12}^1 = 0 \quad \Gamma_{12}^2 = \frac{1}{2\cos^2\alpha}\frac{\partial \cos^2\alpha}{\partial\alpha} = -\frac{\sin\alpha}{\cos\alpha} \quad \Gamma_{12}^3 = 0$ $\qquad \nabla_{\partial_\alpha}\partial_\beta = \nabla_{\partial_\beta}\partial_\alpha = -\frac{\sin\alpha}{\cos\alpha}\partial_\beta$

- $\Gamma_{22}^1 = -\frac{1}{2}\frac{\partial\cos^2\alpha}{\partial\alpha} = \cos\alpha\sin\alpha, \quad \Gamma_{22}^2 = 0, \quad \Gamma_{22}^3 = 0$ $\qquad \nabla_{\partial_\beta}\partial_\beta = \cos\alpha\sin\alpha\,\partial_\alpha$

- $\Gamma_{13}^1 = \Gamma_{13}^2 = 0 \quad \Gamma_{13}^3 = \frac{1}{2\cos^2\alpha\cos^2\beta}\frac{\partial\cos^2\alpha\cos^2\beta}{\partial\alpha} = -\frac{\sin\alpha}{\cos\alpha}$ $\qquad \nabla_{\partial_\alpha}\partial_\gamma = \nabla_{\partial_\gamma}\partial_\alpha = -\frac{\sin\alpha}{\cos\alpha}\partial_\gamma$

- $\Gamma_{23}^1 = \Gamma_{23}^2 = 0, \quad \Gamma_{23}^3 = \frac{1}{2\cos^2\alpha\cos^2\beta}\frac{\partial\cos^2\alpha\cos^2\beta}{\partial\beta} = -\frac{\sin\beta}{\cos\beta}$ $\qquad \nabla_{\partial_\beta}\partial_\gamma = \nabla_{\partial_\gamma}\partial_\beta = -\frac{\sin\beta}{\cos\beta}\partial_\gamma$

- $\Gamma_{33}^1 = -\frac{1}{2}\frac{\partial\cos^2\alpha\cos^2\beta}{\partial\alpha} = \cos\alpha\sin\alpha\cos^2\beta$

  $\Gamma_{33}^2 = -\frac{1}{2\cos^2\alpha}\frac{\partial\cos^2\alpha\cos^2\beta}{\partial\beta} = \cos\beta\sin\beta \qquad \Gamma_{33}^3 = 0$

$$\nabla_{\partial_\gamma}\partial_\gamma = \cos\alpha\sin\alpha\cos^2\beta\,\partial_\alpha + \cos\beta\sin\beta\,\partial_\beta$$

- Càlcul del tensor:

$$R(\partial_\alpha,\partial_\beta)\partial_\alpha = \nabla_{\partial_\alpha}\nabla_{\partial_\beta}\partial_\alpha - \nabla_{\partial_\beta}\nabla_{\partial_\alpha}\partial_\alpha = \nabla_{\partial_\alpha}\left(-\frac{\sin\alpha}{\cos\alpha}\partial_\beta\right) = \frac{-1}{\cos^2\alpha}\partial_\beta + \frac{\sin^2\alpha}{\cos^2\alpha}\partial_\beta$$
$$= -\partial_\beta$$

$$R(\partial_\alpha,\partial_\beta)\partial_\beta = \nabla_{\partial_\alpha}\nabla_{\partial_\beta}\partial_\beta - \nabla_{\partial_\beta}\nabla_{\partial_\alpha}\partial_\beta = \nabla_{\partial_\alpha}\left(\cos\alpha\sin\alpha\,\partial_\alpha\right) - \nabla_{\partial_\beta}\left(-\frac{\sin\alpha}{\cos\alpha}\partial_\beta\right)$$
$$= \left(-\sin^2\alpha + \cos^2\alpha\right)\partial_\alpha + \frac{\sin\alpha}{\cos\alpha}\cos\alpha\sin\alpha\,\partial_\alpha = \cos^2\alpha\,\partial_\alpha$$

$$R(\partial_\alpha,\partial_\beta)\partial_\gamma = \nabla_{\partial_\alpha}\nabla_{\partial_\beta}\partial_\gamma - \nabla_{\partial_\beta}\nabla_{\partial_\alpha}\partial_\gamma = \nabla_{\partial_\alpha}\left(-\frac{\sin\beta}{\cos\beta}\partial_\gamma\right) - \nabla_{\partial_\beta}\left(-\frac{\sin\alpha}{\cos\alpha}\partial_\gamma\right)$$
$$= -\frac{\sin\beta}{\cos\beta}\,\frac{-\sin\alpha}{\cos\alpha}\partial_\gamma + \frac{\sin\alpha}{\cos\alpha}\,\frac{-\sin\beta}{\cos\beta}\partial_\gamma = 0$$

$$R(\partial_\alpha,\partial_\gamma)\partial_\alpha = \nabla_{\partial_\alpha}\nabla_{\partial_\gamma}\partial_\alpha - \nabla_{\partial_\gamma}\nabla_{\partial_\alpha}\partial_\alpha = \nabla_{\partial_\alpha}\left(-\frac{\sin\alpha}{\cos\alpha}\partial_\gamma\right) =$$
$$= \frac{-\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha}\partial_\gamma + \frac{\sin^2\alpha}{\cos^2\alpha}\partial_\gamma = -\partial_\gamma$$

$$R(\partial_\alpha,\partial_\gamma)\partial_\beta = \nabla_{\partial_\alpha}\nabla_{\partial_\gamma}\partial_\beta - \nabla_{\partial_\gamma}\nabla_{\partial_\alpha}\partial_\beta = \nabla_{\partial_\alpha}\left(-\frac{\sin\beta}{\cos\beta}\partial_\gamma\right) - \nabla_{\partial_\gamma}\left(-\frac{\sin\alpha}{\cos\alpha}\partial_\beta\right)$$
$$= \frac{\sin\beta}{\cos\beta}\,\frac{\sin\alpha}{\cos\alpha}\partial_\gamma - \frac{\sin\alpha}{\cos\alpha}\,\frac{\sin\beta}{\cos\beta}\partial_\gamma = 0$$

Per calcular $R(\partial_\alpha,\partial_\gamma)\partial_\gamma$ fem servir les simetries:

$$g\left(R(\partial_\alpha,\partial_\gamma)\partial_\gamma,\partial_\alpha\right) = -g\left(R(\partial_\alpha,\partial_\gamma)\partial_\alpha,\partial_\gamma\right) = -g(\partial_\gamma,\partial_\gamma) = -\cos^2\alpha\cos^2\beta$$

$$g\left(R(\partial_\alpha,\partial_\gamma)\partial_\gamma,\partial_\beta\right) = g\left(R(\partial_\alpha,\partial_\gamma)\partial_\gamma,\partial_\beta\right) = 0\,,\quad g\left(R(\partial_\alpha,\partial_\gamma)\partial_\gamma,\partial_\gamma\right) = 0$$

Per tant $\quad R(\partial_\alpha,\partial_\gamma)\partial_\gamma = -\cos^2\alpha\cos^2\beta\,\partial_\alpha$

Per calcular $R(\partial_\beta, \partial_\gamma)\partial_\alpha$ apliquem la simetria

$$R(u,v)w + R(v,w)u + R(w,u)v = 0$$

i tenim $R(\partial_\beta, \partial_\gamma)\partial_\alpha = 0$

$$R(\partial_\beta, \partial_\gamma)\partial_\beta = \nabla_{\partial_\beta}\nabla_{\partial_\gamma}\partial_\beta - \nabla_{\partial_\gamma}\nabla_{\partial_\beta}\partial_\beta = \nabla_{\partial_\beta}\left(-\frac{\sin\beta}{\cos\beta}\partial_\alpha\right) - \nabla_{\partial_\gamma}\left(\cos\alpha\sin\alpha\,\partial_\alpha\right)$$

$$= \frac{-\cos^2\beta - \sin^2\beta}{\cos^2\beta}\partial_\gamma + \left(\frac{\sin\beta}{\cos\beta}\right)^2\partial_\gamma - \cos\alpha\sin\alpha\left(-\frac{\sin\alpha}{\cos\alpha}\right)\partial_\gamma$$

$$= (-1 + \sin^2\alpha)\partial_\gamma = -\cos^2\alpha\,\partial_\gamma$$

Per calcular $R(\partial_\beta, \partial_\gamma)\partial_\gamma$ tornem a fer servir simetries del tensor de Riemann

$$g(R(\partial_\beta, \partial_\gamma)\partial_\gamma, \partial_\alpha) = -g(R(\partial_\beta, \partial_\gamma)\partial_\alpha, \partial_\gamma) = 0$$

$$g(R(\partial_\beta, \partial_\gamma)\partial_\gamma, \partial_\beta) = -g(R(\partial_\beta, \partial_\gamma)\partial_\beta, \partial_\gamma) = -g(-\cos^2\alpha\,\partial_\gamma, \partial_\gamma) = -\cos^4\alpha\cos^2\beta$$

$$g(R(\partial_\beta, \partial_\gamma)\partial_\gamma, \partial_\gamma) = 0$$

Per tant

$$R(\partial_\beta, \partial_\gamma)\partial_\gamma = \cos^2\alpha\cos^2\beta\,\partial_\beta$$

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