Ex. 5. Calculem la curvatura de $g = \rho(x,y)^2 (dx^2 + dy^2)$

escrivim $\boxed{\rho(x,y) = e^{u(x,y)}}$ pq $\rho > 0$

$$g = e^{2u}(dx^2 + dy^2)$$

Calculem símbols de Christoffel fent servir que $g_{11} = g_{22} = e^{2u}$ i $g_{12} = g_{21} = 0$

Per l'exercici 13 de la llista 2, $\Gamma^i_{ik} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^k}$ $\forall i,k$ i $\Gamma^i_{kk} = \frac{-1}{2g_{ii}} \frac{\partial g_{kk}}{\partial x^i}$ $\forall i \neq k$

$$\Gamma^1_{11} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial x} = \frac{1}{2e^{2u}} e^{2u} 2u_x = \underline{u_x} \qquad \Gamma^2_{11} = -\frac{1}{2g_{22}} \frac{\partial g_{11}}{\partial y} = \underline{-u_y}$$

i per simetria $\Gamma^2_{22} = \underline{u_y}$ i $\Gamma^1_{22} = -u_x$

Finalment $\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2e^{2u}} \frac{\partial e^{2u}}{\partial y} = u_y$ i per simetria $\Gamma^2_{12} = \Gamma^2_{21} = u_x$

Per tant:

$$\nabla_{\partial_x} \partial_x = u_x \partial_x - u_y \partial_y \qquad \nabla_{\partial_y} \partial_y = -u_x \partial_x + u_y \partial_y \quad \text{i} \quad \nabla_{\partial_x} \partial_y = u_y \partial_x + u_x \partial_y$$

Calculem $g(R(\partial_x, \partial_y)\partial_y, \partial_x)$

$$R(\partial_x, \partial_y)\partial_y = \nabla_{\partial_x} \nabla_{\partial_y} \partial_y - \nabla_{\partial_y} \nabla_{\partial_x} \partial_y = \nabla_{\partial_x}(-u_x \partial_x + u_y \partial_y) - \nabla_{\partial_y}(u_y \partial_x + u_x \partial_y)$$

$$= -u_{xx}\partial_x - u_x \nabla_{\partial_x}\partial_x + u_{xy}\partial_y + u_y \nabla_{\partial_x}\partial_y - \left(u_{yy}\partial_x + u_y \nabla_{\partial_y}\partial_x + u_{xy}\partial_y + u_x \nabla_{\partial_y}\partial_y\right)$$

$$= (-u_{xx} - u_{yy})\partial_x + u_x \nabla_{\partial_x}\partial_x - u_x \nabla_{\partial_y}\partial_y$$

$$= (-u_{xx} - u_{yy})\partial_x - u_x(u_x\partial_x - u_y\partial_y) - u_x(-u_x\partial_x + u_y\partial_y) = (-u_{xx} - u_{yy})\partial_x = -\Delta u\, \partial_x$$

$$g(R(\partial_x, \partial_y)\partial_y, \partial_x) = -\Delta u\, e^{2u}$$

La curvatura seccional és

$$\sec = \frac{g(R(\partial_x, \partial_y)\partial_y, \partial_x)}{g(\partial_x, \partial_x)\, g(\partial_y, \partial_y) - g(\partial_x, \partial_y)^2} = \frac{-\Delta u\, e^{2u}}{e^{4u}} = -e^{-2u} \Delta u$$ //