PS6

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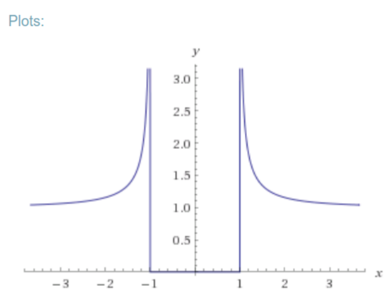
I'm running out of paper, that's why I wrote my homework in a pdf file. Otherwise I will never use it because it took a lot of time...

1 Electronic heat capacity of a supercondoctur

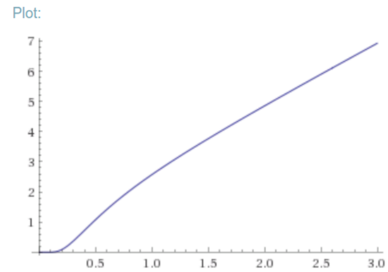
(a) DOS of the superconductor is
$\rho(E)dE = \int \frac{d^dk}{(2\pi)^d}\delta(E-E_k)dE = \int \frac{d^dk}{(2\pi)^d}\delta(\varepsilon-\varepsilon_k)d\varepsilon = \int \frac{d^dk}{(2\pi)^d}\delta(\varepsilon-\varepsilon_k)\frac{d\varepsilon}{dE}dE = \rho_0\frac{d\varepsilon}{dE}dE$
where $E = E_F \pm \sqrt{\Delta^2+(\varepsilon-E_F)^2}$
$\frac{dE}{d\varepsilon} = \pm\frac{\varepsilon-E_F}{\sqrt{\Delta^2+(\varepsilon-E_F)^2}} = \frac{|\varepsilon-E_F|}{\sqrt{\Delta^2+(\varepsilon-E_F)^2}} = \frac{\sqrt{(E-E_F)^2-\Delta^2}}{|E-E_F|},\ |E-E_F|\geq\Delta$
Therefore
$\rho(E) = \rho_0\frac{|E-E_F|}{\sqrt{(E-E_F)^2-\Delta^2}},\ |E-E_F|\geq\Delta$
otherwise $\rho(E)=0$

(b) The DOS looks like



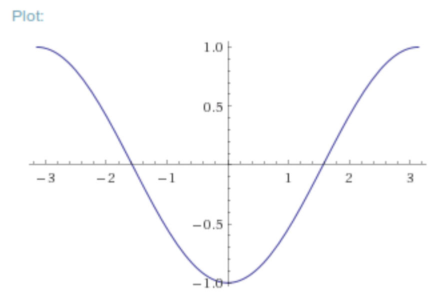
(c) Define $E-E_F = x$, total energy
$U = 2\int_\Delta^\infty (x+E_F)f(x)\rho(x)dx = 2\int_\Delta^\infty (x+E_F)\frac{1}{e^{x/T}+1}\rho_0\frac{x}{\sqrt{x^2-\Delta^2}}$
When $T \ll \Delta,\ f(x)\simeq e^{-x/T}$. Also assume $\rho_0\frac{x}{\sqrt{x^2-\Delta^2}}$ remains constant as long as $|x|\sim\Delta$ or $e^{-x/T}$ is extremely small.
Therefore
$U \simeq 2\int_\Delta^\infty (x+E_F)e^{-x/T}\rho_0$
Using integrals
$\int_\Delta^\infty e^{-x/T}dx = Te^{-\Delta/T}$
$\int_\Delta^\infty xe^{-x/T}dx = T(\Delta+T)e^{-\Delta/T}$
Then we have
$U \simeq 2T(\Delta+T+E_F)e^{-\Delta/T}\rho_0$

(d) The main contribution to electronic heat capacity comes from those density of states near the Fermi energy. We can then use the analytic result from (c)

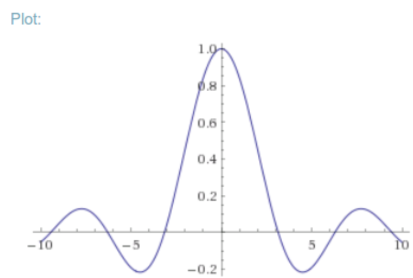


$C_e = (\frac{\partial U}{\partial T})_{N,V} \simeq 2(\frac{(\Delta+E_F)\Delta}{T} + 2\Delta + E_F + 2T)e^{-\Delta/T}\rho_0$

2 Calculating the Sommerfeld expansion for a 1-d metal

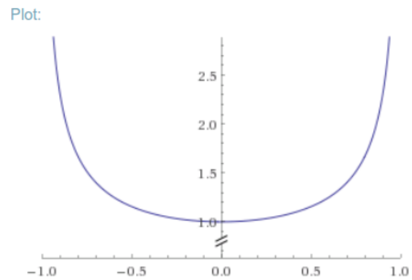


(a) If the band is half fill, the electronic density in k space is $g\frac{a}{2\pi},\ k \in (-\frac{\pi}{2a}, \frac{\pi}{2a})$, where g is the spin degeneracy, $g = 2$ for electrons
Therefore, the electronic density in x space is
$\int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} g\frac{a}{2\pi}e^{-ikx}dk = \frac{g}{2}\text{sinc}(\frac{\pi x}{2a}),\ x \in (-\frac{a}{2}, \frac{a}{2})$, where $\text{sinc}(y) = \frac{\sin(y)}{y}$



(b) Fermi energy of system corresponds to the highest ε_k. Suppose the number density of the electron per unit cell is $n = \frac{N}{a}$, the band will fill up to $k_F = \pm\frac{1}{g}\frac{2\pi}{a}n$ (Note n is also the filling factor of the band).
Therefore,
$E_F = \mu = -t\cos(ka) \mid_{k=\pm\frac{1}{g}\frac{2\pi}{a}n} = -t\cos(\frac{1}{g}2\pi n)$

(c) Density of states
$\rho(\varepsilon)d\varepsilon = \int \frac{dk}{2\pi}\delta(\varepsilon - \varepsilon_k)d\varepsilon = \int \frac{dk}{2\pi}\delta(k - k_\varepsilon)\frac{dk}{d\varepsilon}d\varepsilon = g\frac{a}{2\pi}\frac{1}{at\sin(ka)}d\varepsilon = \frac{g}{2\pi}\frac{1}{\sqrt{t^2-\varepsilon^2}}d\varepsilon,\ \varepsilon \in (-t, t)$



(d) Total energy in the band (in a unit cell)
$U = \int_{-t}^{t} \varepsilon f(\varepsilon)\rho(\varepsilon) d\varepsilon = \int_{-t}^{t} \frac{g}{2\pi}\varepsilon \frac{1}{\sqrt{t^2-\varepsilon^2}} \frac{1}{e^{(\varepsilon-\mu)/T}+1} d\varepsilon$
Using Sommerfeld expansion
$\int_{-t}^{t} h(\varepsilon)\frac{1}{e^{\varepsilon/T}+1} d\varepsilon = \int_{-t}^{\mu} h(\varepsilon)d\varepsilon + \int_{-t}^{\mu} h(\varepsilon)(\frac{1}{e^{(\varepsilon-\mu)/T}+1} - 1)d\varepsilon + \int_{\mu}^{t} h(\varepsilon)\frac{1}{e^{(\varepsilon-\mu)/T}+1} d\varepsilon = \int_{-t}^{\mu} h(\varepsilon)d\varepsilon - \int_{-t}^{\mu} h(\varepsilon)\frac{1}{e^{-(\varepsilon-\mu)/T}+1} d\varepsilon + \int_{\mu}^{t} h(\varepsilon)\frac{1}{e^{(\varepsilon-\mu)/T}+1} d\varepsilon$
Substitute $(\varepsilon - \mu)$ to $-(\varepsilon - \mu)$ in the second integration, and let the upper limits of the integration $t + \mu, t - \mu$ be ∞
$\int_{-t}^{\mu} h(\varepsilon)d\varepsilon + \int_0^{\infty}(h(\mu+\varepsilon) - h(\mu-\varepsilon))\frac{1}{e^{\varepsilon/T}+1} d\varepsilon$
where
$h(\mu+\varepsilon) - h(\mu-\varepsilon) = 2\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} h^{(2n+1)}(\mu)\varepsilon^{2n+1}$
Therefore
$\int_{-t}^{\mu} h(\varepsilon)d\varepsilon + 2\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} h^{(2n+1)}(\mu)\int_0^{\infty} \frac{x^{2n+1}}{e^x+1} dx T^{2n+2}$
Up to T^4, we have
$U = \frac{g}{2\pi}\{-\sqrt{t^2-\mu^2} + \frac{\pi^2}{6}\frac{t^2}{\sqrt{t^2-\mu^2}^3}T^2 + \frac{7\pi^4}{360}\frac{3t^2\mu}{\sqrt{t^2-\mu^2}^5}T^4\}$

(e) Up to T^3, the heat capacity of the electron gas is
$C_e = (\frac{\partial U}{\partial T})_{V,N} = \frac{g}{2\pi}\{\frac{\pi^2}{3}\frac{t^2}{\sqrt{t^2-\mu^2}^3}T + \frac{7\pi^4}{30}\frac{t^2\mu}{\sqrt{t^2-\mu^2}^5}T^3\}$

(f) If chemical potential μ is held constant at E_F, the number of particles in a unit cell is
$N = \int_{-t}^{t} f(\varepsilon)\rho(\varepsilon)d\varepsilon = \int_{-t}^{t} \frac{g}{2\pi}\frac{1}{\sqrt{t^2-\varepsilon^2}}\frac{1}{e^{(\varepsilon-\mu)/T}+1} d\varepsilon$
Using Sommerfeld expansion, we have
$N = \frac{g}{2\pi}\{\arctan(\frac{3t^2\mu}{\sqrt{t^2-\mu^2}}) + \frac{\pi}{2}\} + 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\rho^{(2n+1)}(\mu)\int_0^{\infty}\frac{x^{2n+1}}{e^x+1} dx T^{2n+2}$
For example, up to T^4
$N = \frac{g}{2\pi}\{\arctan(\frac{3t^2\mu}{\sqrt{t^2-\mu^2}}) + \frac{\pi}{2} + \frac{\pi^2}{6}\frac{\mu}{\sqrt{t^2-\mu^2}^3}T^2 + \frac{7\pi^4}{360}\frac{t^2+2\mu^2}{\sqrt{t^2-\mu^2}^5}T^4\}$
Consistency check: if $\mu = 0$, ie, half filled, then to the 0th order, $N = \frac{g}{4}$, number density $n = g\frac{a}{4}$, Fermi vector $k_F = \pm\frac{1}{g}\frac{2\pi}{a}n = \pm\frac{\pi}{2}$, $\mu = 0$

3 Adiabatic expansion

(a) Proof.
$\text{RHS} = (\frac{\partial P}{\partial V})_{S,N} = (\frac{\partial P}{\partial V})_{T,N} + (\frac{\partial P}{\partial T})_{V,N}(\frac{\partial T}{\partial V})_{S,N} = (\frac{\partial P}{\partial V})_{T,N}\{1 + \frac{(\frac{\partial P}{\partial T})_{V,N}(\frac{\partial T}{\partial V})_{S,N}}{(\frac{\partial P}{\partial V})_{T,N}}\}$
where
$\frac{(\frac{\partial P}{\partial T})_{V,N}(\frac{\partial T}{\partial V})_{S,N}}{(\frac{\partial P}{\partial V})_{T,N}} = -(\frac{\partial V}{\partial T})_{P,N}(\frac{\partial T}{\partial V})_{S,N} = (\frac{\partial V}{\partial T})_{P,N}\frac{(\frac{\partial S}{\partial V})_{T,N}}{(\frac{\partial S}{\partial T})_{V,N}}$
Therefore
$\text{RHS} = (\frac{\partial P}{\partial V})_{T,N}\{1 + (\frac{\partial V}{\partial T})_{P,N}\frac{(\frac{\partial S}{\partial V})_{T,N}}{(\frac{\partial S}{\partial T})_{V,N}}\} = \frac{(\frac{\partial P}{\partial V})_{T,N}}{(\frac{\partial S}{\partial T})_{V,N}}\{(\frac{\partial S}{\partial T})_{V,N} + (\frac{\partial V}{\partial T})_{P,N}(\frac{\partial S}{\partial V})_{T,N}\} = \frac{(\frac{\partial P}{\partial V})_{T,N}}{(\frac{\partial S}{\partial T})_{V,N}}(\frac{\partial S}{\partial T})_{P,N} = \frac{C_P}{C_V}(\frac{\partial P}{\partial V})_{T,N} = \frac{C_P}{C_V(\frac{\partial V}{\partial P})_{T,N}} = -\frac{C_P}{C_V V\kappa_T} = \text{LHS}$

(b) For an ideal gas, $PV = NT$, $(\frac{\partial V}{\partial P})_{T,N} = -\frac{NT}{P^2}$, $\frac{C_P}{C_V} = \gamma$ is a constant
Then we have

$(\frac{\partial P}{\partial V})_{S,N} = -\frac{\gamma P^2}{NT} = -\frac{\gamma P}{V}$
$(\frac{\partial P}{P})_{S,N} = -\gamma(\frac{\partial V}{V})_{S,N}$
$(PV^\gamma)_{S,N} = constant$

(c) For monoatomic ideal gas in d dimensions
$U = U_{\text{kinetic}} = \frac{1}{2}dNT$ (equipartition theorem)
$C_V = (\frac{\partial U}{\partial T})_{V,N} = \frac{1}{2}dN$
$C_P = C_V + N = (\frac{1}{2}d + 1)N$ (ideal gas law)
Then $\gamma_{\text{mono}} = 1 + \frac{2}{d}$
For diatomic ideal gas in d dimensions (with no vibrational excitations)
$U = U_{\text{kinetic}} + U_{\text{rotational}}$
Assume the rotation is confined to $3d$ and due to two rotation axes. The partition function for a single molecule is given in PS5, which is
$Z_1 = \frac{8\pi^2 IT}{h^2}$
where I is the moment of inertia about the center axis perpendicular to the line connecting the two atoms
Free energy $F = -T\ln(Z_N) = -NT\{\ln(\frac{8\pi^2 IT}{h^2 N}) + 1\}$
Entropy $S = -(\frac{\partial F}{\partial T})_{N,V} = N\{\ln(\frac{8\pi^2 IT}{h^2 N}) + 2\}$
$U_{\text{rotational}} = F + TS = NT$ (not suprising because of the equipartition theorem)
$C_V = (\frac{\partial U}{\partial T})_{V,N} = (\frac{1}{2}d + 1)N$
$C_P = C_V + N = (\frac{1}{2}d + 2)N$
Then $\gamma_{\text{di}} = 1 + \frac{2}{d+2}$

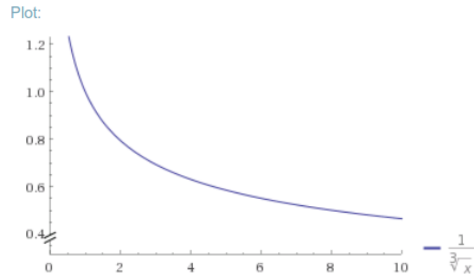
4 White dwarves

(a) For non-reletivistic free electrons, we have (derived in class)
$E_F = \frac{\hbar^2}{2m}(3\pi^2\frac{N}{V})^{\frac{2}{3}}$ (include $g = 2$ for electrons)
where $N = \frac{2M}{m_{He}}$, $V = \frac{4}{3}\pi R^3$, m_{He} is the atomic mass of helium (ignore the mass contribution from the electrons and all electrons are not bound to the atom, thus each nuclei will give two electrons)
Therefore $E_F = \frac{\hbar^2}{2m}(\frac{9}{4}\frac{\pi}{R^3}\frac{2M}{m_{He}})^{\frac{2}{3}}$

(b) Total energy of the electron gas is (derived in class)
$U_e = \frac{3}{5}E_F N = \frac{3}{5}\frac{\hbar^2}{2m}(\frac{3\pi^2}{V})^{\frac{2}{3}}N^{\frac{5}{3}} = \frac{3}{5}\frac{\hbar^2}{2m}(\frac{9}{4}\frac{\pi}{R^3})^{\frac{2}{3}}(\frac{2M}{m_{He}})^{\frac{5}{3}}$

(c) The gravitational potential between two objects with mass m_1 and m_2 is
$U(m_1, m_2) = -\frac{Gm_1m_2}{r}$
Then consider piling up a sphere with radius R and density ρ, the work done on this process is
$W = -\int_0^R \frac{Gm(r)dm}{r} = -\int_0^R \frac{G\frac{4}{3}\pi r^3\rho}{r}4\pi r^2\rho dr = -\int_0^R G\frac{4}{3}\pi^2\rho^2 r^4 dr = -G\frac{16}{15}\pi^2\rho^2 R^5$
Using $M = \frac{4}{3}\pi R^3\rho$, $W = -\frac{3}{5}\frac{GM^2}{R}$
Therefore the gravitational energy of the star is
$U_g = -\frac{3}{5}\frac{GM^2}{R}$

(d) Total energy of the star
$U(R) = U_e + U_g = \frac{3}{5}\frac{\hbar^2}{2m}(\frac{9}{4}\pi)^{\frac{2}{3}}(\frac{2M}{m_{He}})^{\frac{5}{3}}\frac{1}{R^2} - \frac{3}{5}\frac{GM^2}{R} = \frac{A}{R^2} - \frac{B}{R}$
In equilibrium
$\frac{dU}{dR} = 0$, $\frac{2A}{R^3} = \frac{B}{R^2}$, $R = \frac{2A}{B}$
Plug in the detailed numbers, we have
$A = 1.941 \times 10^{56}\text{J}\cdot\text{m}^2(\frac{M}{M_S})^{\frac{5}{3}}$
$B = 4.004 \times 10^{49}\text{J}\cdot\text{m}\ (\frac{M}{M_S})^2$
where M_S is the mass of the sun
Then $R = 9.695 \times 10^6\text{m}(\frac{M}{M_S})^{-\frac{1}{3}}$



The Fermi energy is
$E_F = \frac{5}{3}\frac{U_e}{N} = \frac{5}{3}\frac{A}{R^2 N} = 68.67\text{keV}(\frac{M}{M_S})^{\frac{4}{3}}$
whereas the rest energy of the electron is 511keV, E_F is 13.4% os the rest energy when $M = M_S$
The electrons can't be atreated non-reletivistically

(e) For ultra-relativistic electrons, the Fermi momentum is the same as the non-relativistic case
$p_F = \hbar(3\pi^2\frac{N}{V})^{\frac{1}{3}}$ (include $g = 2$ for electrons)
In ultra-relativistic limit
$E_F = p_F c = \hbar c(3\pi^2\frac{N}{V})^{\frac{1}{3}}$
Total energy of the electron gas is
$U_e = \frac{3}{4}E_F N = \frac{3}{4}\hbar c(\frac{3\pi^2}{V})^{\frac{1}{3}}N^{\frac{4}{3}} = \frac{3}{4}\hbar c(\frac{9}{4}\frac{\pi}{R^3})^{\frac{1}{3}}(\frac{2M}{m_{He}})^{\frac{4}{3}}$
(Sorry I didn't derive it... Just copied from wiki. It's too late and the derivation is trivial...)
Total energy of the star is
$U(R) = U_e + U_g = \frac{3}{4}\hbar c(\frac{9}{4}\pi)^{\frac{1}{3}}(\frac{2M}{m_{He}})^{\frac{4}{3}}\frac{1}{R} - \frac{3}{5}\frac{GM^2}{R} = \frac{C}{R} - \frac{B}{R}$
where
$C = 9.262 \times 10^{49}\text{J}\cdot\text{m }(\frac{M}{M_S})^{\frac{4}{3}}$
$B = 4.004 \times 10^{49}\text{J}\cdot\text{m }(\frac{M}{M_S})^2$ is the same as before

(f) For the start to be stable, $\frac{dU}{dR} = -\frac{C-B}{R^2} < 0$, i.e.,
$9.262(\frac{M}{M_S})^{\frac{4}{3}} > 4.004(\frac{M}{M_S})^2$
$\frac{M}{M_S} < (\frac{9.262}{4.004})^{\frac{3}{2}} \simeq 3.52$
This is a strange number... which should be around 1.4