PS9

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Where is PS8?

1 BEC in d-dimensions

(a) For ideal Bose gas
$\frac{N}{V} = \frac{1}{V}\frac{\zeta}{1-\zeta} + \int \frac{d^dk}{(2\pi)^d}\frac{1}{e^{\epsilon_k\beta}/\zeta-1} = \frac{1}{V}\frac{\zeta}{1-\zeta} + \int \frac{d^dk}{(2\pi)^d}\zeta e^{-\epsilon_k\beta}(1+\zeta e^{-\epsilon_k\beta} + (\zeta e^{-\epsilon_k\beta})^2 + ...) = \frac{1}{V}\frac{\zeta}{1-\zeta} + \sum_{n=1}^{\infty}\int \frac{d^dk}{(2\pi)^d}e^{-n\epsilon_k\beta}\zeta^n$
Define $I_1 = \int \frac{d^dk}{(2\pi)^d}e^{-\epsilon_k\beta} = \int \frac{d^dk}{(2\pi)^d}e^{-\frac{1}{2\alpha}k^4\beta}$, then
$I_n = \int \frac{d^dk}{(2\pi)^d}e^{-n\epsilon_k\beta} = \int \frac{d^dk}{(2\pi)^d}e^{-\frac{n}{2\alpha}k^4\beta} = \frac{1}{n^{d/4}}\int \frac{d^d(kn^{1/4})}{(2\pi)^d}e^{-\frac{1}{2\alpha}(kn^{1/4})^4\beta}$
Therefore
$\frac{N}{V} = \frac{1}{V}\frac{\zeta}{1-\zeta} + I_1\sum_{n=1}^{\infty}\frac{\zeta^n}{n^{d/4}}$
Note that $I_1 \le \int_{k\le 1}\frac{d^dk}{(2\pi)^d}e^{-\frac{1}{2\alpha}k^4\beta} + \int_{k>1}\frac{d^dk}{(2\pi)^d}e^{-\frac{1}{2\alpha}k^2\beta} \le \text{const} + \int \frac{d^dk}{(2\pi)^d}e^{-\frac{1}{2\alpha}k^2\beta} = \text{const} + \frac{1}{\lambda_T^d} < \infty$
When $d < 5$, the sum will diverge if $\zeta \to 1$, which means ζ can't reach 1. Therefore, $d = 5$ is the minimum dimension in which BEC can occur

(b) If BEC can occur, at $T = T_{BEC}$
$\frac{N}{V} = I_1\sum_{n=1}^{\infty}\frac{1}{n^{d/4}}$
By again looking at I_1
$I_1 = \int \frac{d^dk}{(2\pi)^d}e^{-\frac{1}{2\alpha}k^4\beta} = \frac{1}{\beta^{d/4}}\int \frac{d^d(k\beta^{1/4})}{(2\pi)^d}e^{-\frac{1}{2\alpha}(k\beta^{1/4})^4} = \frac{a}{\beta^{d/4}}$, where a is a proportionality constant.
Then
$\frac{N}{V} = bT_{BEC}^{d/4}$
$T_{BEC} \propto (\frac{N}{V})^{4/d}$
when $d = 5$, $T_{BEC} \propto (\frac{N}{V})^{4/5}$

(c) If the dispersion relation changes to $\epsilon_k = \frac{1}{2\alpha}|k^2 - k_0^2|^2$
$\frac{N}{V} = \frac{1}{V}\frac{\zeta}{1-\zeta} + \sum_{n=1}^{\infty}\int \frac{d^dk}{(2\pi)^d}e^{-n\epsilon_k\beta}\zeta^n$
where I_1' is now
$I_1' = \int \frac{d^dk}{(2\pi)^d}e^{-\epsilon_k\beta} = \int \frac{d^dk}{(2\pi)^d}e^{-\frac{1}{2\alpha}|k^2-k_0^2|^2\beta}$
$I_n' = \int \frac{d^dk}{(2\pi)^d}e^{-n\epsilon_k\beta} = \int \frac{d^dk}{(2\pi)^d}e^{-\frac{n}{2\alpha}|k^2-k_0^2|^2\beta}$
where
$\int \frac{d^dk}{(2\pi)^d}e^{-\frac{n}{2\alpha}|k^2-k_0^2|^2\beta} \ge \int_{k^2>k_0^2}\frac{d^dk}{(2\pi)^d}e^{-\frac{n}{2\alpha}(k^4+k_0^4-2k_0^2k^2)\beta} \ge \int_{k^2>k_0^2}\frac{d^dk}{(2\pi)^d}e^{-\frac{n}{2\alpha}(2k^4-2k_0^4)\beta} = e^{\frac{n}{\alpha}k_0^4\beta}\int_{k^2>k_0^2}\frac{d^dk}{(2\pi)^d}e^{-\frac{n}{\alpha}k^4\beta} = \frac{1}{n^{d/4}}e^{\frac{n}{\alpha}k_0^4\beta}\int_{(k(n)^{1/4})^2>(k_0(n)^{1/4})^2}\frac{d^d(k(n)^{1/4})}{(2\pi)^d}e^{-\frac{1}{\alpha}(k(n)^{1/4})^4\beta} \ge \frac{1}{n^{d/4}}e^{\frac{n}{\alpha}k_0^4\beta}\int_{(k(n)^{1/4})^2>k_0^2}\frac{d^d(k(n)^{1/4})}{(2\pi)^d}e^{-\frac{1}{\alpha}(k(n)^{1/4})^4\beta} = C\frac{1}{n^{d/4}}e^{\frac{n}{\alpha}k_0^4\beta}$
where
$C = \int_{k^2>k_0^2}\frac{d^dk}{(2\pi)^d}e^{-\frac{1}{\alpha}k^4\beta} < \infty$
Therefore
$\frac{N}{V} \ge \frac{1}{V}\frac{\zeta}{1-\zeta} + C\sum_{n=1}^{\infty}\frac{\zeta^n}{n^{d/4}}e^{\frac{n}{\alpha}k_0^4\beta}$
The sum will always diverge as $\zeta \to 1$, BEC can never occur

2 Virial expansion for interacting gases

(a) Grand canonical partition function

$\mathcal{Z} = \sum_N \sum_i e^{-\beta(E_i-\mu N)} = \sum_N \zeta^N e^{-\beta\{\sum_{i\in\{1,...N\}}\frac{P_i^2}{2m}+\sum_{i<j\in\{1,...N\}}V(r_{ij})\}} = \sum_N \zeta^N \Pi_{i\in\{1,...N\}} e^{-\beta\frac{P_i^2}{2m}} \Pi_{i<j\in\{1,...N\}} e^{-\beta V(r_{ij})}$

where

$\Pi_{i\in\{1,...N\}} e^{-\beta\frac{P_i^2}{2m}} = (\frac{1}{\lambda_T^3})^N/N!$

$\Pi_{i<j\in\{1,...N\}} e^{-\beta V(r_{ij})} = \Pi_{i<j\in\{1,...N\}}(1+e^{-\beta V(r_{ij})}-1) \simeq \sum_{i<j\in\{1,...N\}}(1+e^{-\beta V(r_{ij})}-1) = V^N + V^{N-2}\frac{N(N-1)}{2}\int d^3r_1 \int d^3r_2(e^{-\beta V(r_{12})}-1) = V^N + V^{N-1}\frac{N(N-1)}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1)$

Thus

$\mathcal{Z} = \sum_N \zeta^N(\frac{1}{\lambda_T^3})^N/N!\{V^N + V^{N-1}\frac{N(N-1)}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1)\} = \sum_N\{\zeta^N(\frac{V}{\lambda_T^3})^N/N! + \zeta^{N-2}(\frac{V}{\lambda_T^3})^{N-2}/(N-2)!(\zeta\frac{1}{\lambda_T^3})^2\frac{V}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1)\} = e^{\zeta\frac{V}{\lambda_T^3}}(1+(\zeta\frac{1}{\lambda_T^3})^2\frac{V}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1))$

Define $\frac{1}{\lambda_T^3} = Z_1$

$\mathcal{Z} \simeq e^{\zeta Z_1 V}\{1+\zeta^2 Z_1^2\frac{V}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1)\}$

Or

$\mathcal{Z} \simeq e^{\zeta Z_1 V}\{1+\zeta^2 Z_1^2\frac{1}{2}\int d^3r_1\int d^3r_2(e^{-\beta V(r_{12})}-1)\}$

(b) Grand canonical potential

$w = -T\ln\mathcal{Z} \simeq -T\zeta Z_1 V - T\ln\{1+\zeta^2 Z_1^2\frac{V}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1)\} = -PV$

$N = -(\frac{\partial w}{\partial\mu})_{T,V} = -\beta\zeta(\frac{\partial w}{\partial\zeta})_{T,V} \simeq \zeta Z_1 V + \frac{\zeta^2 Z_1^2 V\int d^3r_{12}(e^{-\beta V(r_{12})}-1)}{1+\zeta^2 Z_1^2\frac{V}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1)} \simeq \zeta Z_1 V(1+\zeta Z_1\frac{1}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1))$

Therefore

$P \simeq T\zeta Z_1\{1+\zeta Z_1\frac{1}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1)\}$

where

$\zeta Z_1 \simeq \frac{N}{V}\{1-\frac{N}{V}\int d^3r_{12}(e^{-\beta V(r_{12})}-1)\}$

Then

$P \simeq T\frac{N}{V}\{1-\frac{N}{V}\frac{1}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1)\}$

For the given interaction

$\frac{1}{2}\int d^3r_{12}(e^{-\beta V(r_{12})}-1) = -2\pi\int_0^b r^2dr + 2\pi\int_b^\infty drr^2(e^{\beta\frac{a}{r^6}}-1) \sim -2\pi\frac{1}{3}b^3 + 2\pi\int_b^\infty dr\beta\frac{a}{r^4} = 2\pi(\beta\frac{a}{3b^3}-\frac{1}{3}b^3)$

Therefore

$P \simeq T\frac{N}{V}\{1-\frac{N}{V}2\pi(\beta\frac{a}{3b^3}-\frac{1}{3}b^3)\} = T\{\frac{N}{V}-2\pi\frac{N^2}{V^2}(\beta\frac{a}{3b^3}-\frac{1}{3}b^3)\}$

$P + 2\pi\frac{N^2}{V^2}\frac{a}{3b^3} = T\frac{N}{V}(1+\frac{2\pi N}{3V}b^3) \simeq \frac{NT}{V-\frac{2\pi N}{3}b^3}$

And

$B_1 = -2\pi(\beta\frac{a}{3b^3}-\frac{1}{3}b^3)$