APh190a Final HW

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1 Definition and Explanation

1. Rotating wave approximation: terms in the Hamiltonian which oscillate at high frequencies are neglected. In that sense, the sin() function of the electric field is approximated by a rotating one. That's why it's called rotating wave approximation.

   Dipole approximation: assume that the electric field is a function of time only without spatial dependence. Higher orders of the eletric field-atom interactions, like the gradient or the curvation of the field, or the recoil motion of the atom, are neglected.

2. Now since the electric field is quantized, the Rabi frequency depends not on the classical field strength (amplitude), rather than the number of photons in the mode. Moreover, due to quantization, the volume of the system, which determines the miminum available electric field per photon, comes into play. Also, because of zero-point energy which is truly a quantum effect, the atom will still exhibit oscillation when there is no electric field.

   As for the eigenvectors, now the atom becomes dressed, which means the eigenvectors are a combined eigenvectors of the atom and the electric field.

3. Rate equation approximation: the polartization dephasing time is much shorter than the variation of electric field amplitude. Which means one can assume a constant eletric field amplitude at a certain time, solve the corresponding susceptibility and then go ahead to another time. Since the dephasing of polarization is assumed as quickly as it's generated, it's called rate equation approximation.

4. (a) The population of the two levels (diagonal terms in the density matrix), as well as dipole terms (off-digaonal elements in the density matrix), will show decay behavior due to interactions with the bath. Especially, one of the two decay terms has population difference dependence and the other only depends on the population of the upper level. The first is the stimulated term and the second accounts for spontaneous decay.

   (b) The two-level system will thermalize to the same temperature as the bath. More explicitly, due to the finite transition rate, the system will show linewidth corresponding to thermal broadening.

5. On the one hand, becasue of coupling to the bath or some other kind of interactions, the population difference of the two levels will show a decay behavior. On the other hand, the gain of the system depends on the population difference. Therefore, the steady state solution of the population inversion, thus the gain, will show a saturation behavior.

   In the operation of a laser, this means that one can't increase the gain to infinity by increasing the laser input power, there will be saturation at some point.

2 Brillouin Amplification

1. Heisenberg equations of motion

   $\frac{db^+}{dt} = \frac{i}{\hbar}[H, b^+] = i\Omega[b^+b, b^+] + i\kappa a A^* e^{i\omega_p t}[b, b^+] = i\Omega b^+ + i\kappa a A^* e^{i\omega_p t}$ (phonon equation)

   $\frac{da}{dt} = \frac{i}{\hbar}[H, a] = i\omega_s[a^+a, a] + i\kappa b^+ A e^{-i\omega_p t}[a^+, a] = -i\omega_s a - i\kappa b^+ A e^{-i\omega_p t}$ (Stokes equation)

Add phenomenlogical damping terms

$\frac{db^+}{dt} = (i\Omega - \Gamma)b^+ + i\kappa a A^* e^{i\omega_p t}$ (phonon equation)

$\frac{da}{dt} = (-i\omega_s - \gamma)a - i\kappa b^+ A e^{-i\omega_p t}$ (Stokes equation)

2. Since $\Gamma \gg \gamma$, we can solve the phonon equation assuming a is a constant first

   $b^+(t) = \int_{-\infty}^{t} e^{(i\Omega-\Gamma)(t-z)} i\kappa a(z) A^* e^{i\omega_p z} dz \simeq i\kappa A^* a(t) \int_{-\infty}^{t} e^{(i\Omega-\Gamma)(t-z)} e^{i\omega_p z} dz$

   Plug into the Stokes equation

   $\frac{da}{dt} = (-i\omega_s - \gamma)a + \kappa^2|A|^2 a \int_{-\infty}^{t} e^{(i\Omega-\Gamma)(t-z)} e^{-i\omega_p (t-z)} dz = (-i\omega_s - \gamma)a + \frac{\kappa^2|A|^2}{i(\omega_p-\Omega)+\Gamma} a = \{i(-\omega_s - \frac{\kappa^2|A|^2(\omega_p-\Omega)}{(\omega_p-\Omega)^2+\Gamma^2}) + (\frac{\kappa^2|A|^2\Gamma}{(\omega_p-\Omega)^2+\Gamma^2} - \gamma)\}a$

   The optical gain of the Stokes field is given by

   $g = \frac{\kappa^2|A|^2\Gamma}{(\omega_p-\Omega)^2+\Gamma^2} - \gamma$

   The gain has its maximum when

   $\omega_p = \Omega$

   Or the energy per photon of the pump field is equal to the energy of the phonon

3. The maximum gain is

   $g_m = \frac{\kappa^2|A|^2}{\Gamma} - \gamma$

   Therefore the threshold pumping level is

   $|A|^2_{thres} = \frac{\gamma\Gamma}{\kappa^2}$

3 Purcell Effect

1. The Hamiltonian is

   $H = E_1 n_1 + E_2 n_2 + \hbar\omega a^+ a - \frac{\hbar\Omega_0}{2}(\sigma_+ a + \sigma_- a^+)$

   Then the Heisenberg equation of motion for n_2 is

   $\frac{dn_2}{dt} = \frac{i}{\hbar}[H, n_2] = -i\frac{\Omega_0}{2}[\sigma_+, n_2]a - i\frac{\Omega_0}{2}[\sigma_-, n_2]a^+ = i\frac{\Omega_0}{2}\sigma_+ a - i\frac{\Omega_0}{2}\sigma_- a^+$

   where

   $\frac{d\sigma_- a^+}{dt} = \frac{i}{\hbar}[H, \sigma_- a^+] = \frac{i}{\hbar}E_1[n_1, \sigma_-]a^+ + \frac{i}{\hbar}E_2[n_2, \sigma_-]a^+ + i\omega\sigma_-[a^+ a, a^+] - i\frac{\Omega_0}{2}[\sigma_+ a, \sigma_- a^+] = \frac{i}{\hbar}E_1\sigma_- a^+ - \frac{i}{\hbar}E_2\sigma_- a^+ + i\omega\sigma_- a^+ - i\frac{\Omega_0}{2}\{(n_2 - n_1)a^+ a + n_2\} = -i(\Omega-\omega)\sigma_- a^+ - i\frac{\Omega_0}{2}\{(n_2-n_1)a^+ a + n_2\}$

   For anti-nodes of the resonator

   $\langle a^+ a \rangle = n_p$

   which is hardly affected by the interaction with the atom

   Also

   $n_1 + n_2 = 1$ (identity matrix)

   Therefore

   $\frac{d\sigma_- a^+}{dt} = -i(\Omega-\omega)\sigma_- a^+ - i\frac{\Omega_0}{2}\{(2n_2 - 1)n_p + n_2\}$

   Add damping

   $\frac{d\sigma_- a^+}{dt} = (-i(\Omega-\omega) - \frac{1}{2\tau_p})\sigma_- a^+ - i\frac{\Omega_0}{2}\{(2n_2 - 1)n_p + n_2\}$

   Since τ_p is the fastest rate, we can solve $\sigma_- a^+$ first

   $\sigma_- a^+ = -i\frac{\Omega_0}{2}\{(2n_2 - 1)n_p + n_2\}\int_{-\infty}^{t} e^{(-i(\Omega-\omega)-\frac{1}{2\tau_p})(t-z)} dz = -i\frac{\Omega_0}{2}\frac{(2n_2-1)n_p + n_2}{i(\Omega-\omega)+\frac{1}{2\tau_p}}$

   Therefore

$\frac{dn_2}{dt} = -\frac{\Omega_0^2}{4}\frac{(2n_2-1)n_p+n_2}{-i(\Omega-\omega)+\frac{1}{2\tau_p}} - \frac{\Omega_0^2}{4}\frac{(2n_2-1)n_p+n_2}{i(\Omega-\omega)+\frac{1}{2\tau_p}} = -\frac{\Omega_0^2}{4\tau_p}\frac{(2n_2-1)n_p+n_2}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2} = \frac{\Omega_0^2}{4\tau_p}\frac{2n_2-1}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}n_p + \frac{\Omega_0^2}{4\tau_p}\frac{n_2}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}$

2. The decay rate of the electron is

   $\gamma_e = \frac{\Omega_0^2}{4\tau_p}\frac{2n_2-1}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}n_p + \frac{\Omega_0^2}{4\tau_p}\frac{n_2}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2} = \frac{\Omega_0^2}{4}\frac{\frac{\omega}{Q}(2n_2-1)}{(\Omega-\omega)^2+(\frac{\omega}{2Q})^2}n_p + \frac{\Omega_0^2}{4}\frac{\frac{\omega}{Q}}{(\Omega-\omega)^2+(\frac{\omega}{2Q})^2}$

   where the first term in the stimulated term and the second one is the spontaneous decay

   $\gamma_{e,sp} = \frac{\Omega_0^2}{4}\frac{\frac{\omega}{Q}}{(\Omega-\omega)^2+(\frac{\omega}{2Q})^2}$

   For small detuning compared to optical decay rate

   $\gamma_{e,sp} \simeq \frac{\Omega_0^2 Q}{\omega} \propto \frac{Q}{V_{mode}}$

3. For two-level system in vacuum, the corresponding spontaneous decay rate is exactly

   $\gamma_{e,sp}^{(0)} = \frac{\Omega_0^2 Q}{\omega}$

   Then

   $\gamma_e = \frac{\Omega_0^2}{4}\frac{\frac{\omega}{Q}(2n_2-1)}{(\Omega-\omega)^2+(\frac{\omega}{2Q})^2}n_p + \frac{\Omega_0^2}{4}\frac{\frac{\omega}{Q}}{(\Omega-\omega)^2+(\frac{\omega}{2Q})^2} = \gamma_{e,sp}^{(0)}\frac{(\frac{\omega}{2Q})^2(2n_2-1)}{(\Omega-\omega)^2+(\frac{\omega}{2Q})^2}n_p + \gamma_{e,sp}^{(0)}\frac{(\frac{\omega}{2Q})^2}{(\Omega-\omega)^2+(\frac{\omega}{2Q})^2}$

   For small detuning compared to optical decay rate

   $\gamma_e \simeq \gamma_{e,sp}^{(0)}(2n_2-1)n_p + \gamma_{e,sp}^{(0)} \propto \frac{Q}{V_{mode}}\{(2n_2-1)n_p+1\}$

   The decay rate can be greatly enhanced if $\frac{Q}{V_{mode}}$ is very large

4. We can then write the Heisenberg equation of motion for a^+a

   $\frac{da^+a}{dt} = \frac{i}{\hbar}[H, a^+a] = -i\frac{\Omega_0}{2}\sigma_+[a, a^+a] - i\frac{\Omega_0}{2}\sigma_-[a^+, a^+a] = -i\frac{\Omega_0}{2}\sigma_+ a + i\frac{\Omega_0}{2}\sigma_- a^+ = -\frac{dn_2}{dt} = \frac{\Omega_0^2}{4\tau_p}\frac{2n_2-1}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}a^+a + \frac{\Omega_0^2}{4\tau_p}\frac{n_2}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}$

   Add damping term

   $\frac{da^+a}{dt} = (\frac{\Omega_0^2}{4\tau_p}\frac{2n_2-1}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2} - \frac{1}{\tau})a^+a + \frac{\Omega_0^2}{4\tau_p}\frac{n_2}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}$

   Since the damping of $\sigma_- a^+$ comes from the damping of the electron and the damping of the photon, then

   $\frac{1}{\tau_p} \simeq \frac{1}{\tau} > \frac{\Omega_0^2}{4\tau_p}\frac{2n_2-1}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}$

   Therefore

   $a^+a(t) \simeq a^+a(t=-\infty)e^{(\frac{\Omega_0^2}{4\tau_p}\frac{2n_2-1}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}-\frac{1}{\tau})t} + \frac{\Omega_0^2}{4\tau_p}\frac{n_2(t)}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}\int_{-\infty}^{t} e^{(\frac{\Omega_0^2}{4\tau_p}\frac{2n_2-1}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}-\frac{1}{\tau})(t-z)}dz \simeq a^+a(t=-\infty)e^{-\frac{1}{\tau}t} + \frac{\Omega_0^2}{4\tau_p}\frac{n_2(t)}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}\int_{-\infty}^{t} e^{-\frac{1}{\tau}(t-z)}dz \simeq \frac{\Omega_0^2}{4\tau_p}\frac{n_2(t)}{(\Omega-\omega)^2+(\frac{1}{2\tau_p})^2}$

   Since a^+a has the same time dependence as n_2, if we assume the optical decay rate is faster than the decay rate of the electron, the approximation still holds for the decay rate of the photon number in this mode.