

# Components of polarizability

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## 1 Two theorems

### 1.1 Wigner-Eckart theorem

$$\langle \alpha j m | T_q^{(k)} | \alpha' j' m' \rangle = (-1)^{2k} \langle \alpha j || \mathbf{T}^{(k)} || \alpha' j' \rangle \langle j m | j' m'; k q \rangle$$

contains Clebsh-Gordon coefficients

### 1.2 Decomposition rule

For a single subsystem,

$$\langle j || \mathbf{T}^{(k)} || j' \rangle = \delta_{j_2 j_2'} (-1)^{j' + j_1 + k + j_2} \sqrt{(2j' + 1)(2j_1 + 1)} \left\{ \begin{matrix} j_1 & j_1' & k \\ j' & j & j_2 \end{matrix} \right\} \langle j_1 || \mathbf{T}^{(k)} || j_1' \rangle$$

Wigner-6j symbol, recoupling of three angular momenta

## 2 DC Stark shifts

$$H_{int} = -\mathbf{d} \cdot \mathbf{E}$$

Second order time-independent perturbation theory gives,

$$\Delta E_\alpha = \sum_j \frac{|\langle \alpha | H_{int} | \beta_j \rangle|^2}{E_\alpha - E_{\beta_j}} = \langle \alpha | H_{stark} | \alpha \rangle$$

$$H_{stark} = \sum_j \frac{d_\mu |\beta_j\rangle \langle \beta_j| d_\nu}{E_\alpha - E_{\beta_j}} E_\mu E_\nu = S_{\mu\nu} E_\mu E_\nu$$

$$\text{where } S_{\mu\nu} = \sum_j \frac{d_\mu |\beta_j\rangle \langle \beta_j| d_\nu}{E_\alpha - E_{\beta_j}}$$

**A Cartesian tensor of rank 2 can be decomposed into irreducible spherical tensors of rank 0, 1, 2 as**

$$M_{\alpha\beta} = \frac{1}{3} M^{(0)} \delta_{\alpha\beta} + \frac{1}{4} M_\mu^{(1)} \epsilon_{\mu\alpha\beta} + M_{\alpha\beta}^{(2)}$$

where

$$M^{(0)} = M_{\mu\mu}$$

$$M_\mu^{(1)} = \epsilon_{\mu\sigma\tau} (M_{\sigma\tau} - M_{\tau\sigma})$$

$$M_{\alpha\beta}^{(2)} = M_{\alpha\beta} - \frac{1}{3} M_{\mu\mu} \delta_{\alpha\beta}$$

Same for  $S_{\mu\nu}$

$$S^{(0)} = S_{\mu\mu}$$

$$S_\mu^{(1)} = 0$$

$$S_{\mu\nu}^{(2)} = S_{\mu\nu} - \frac{1}{3} S_{\sigma\sigma} \delta_{\mu\nu}$$

$$\Delta E_\alpha = \frac{1}{3} \langle \alpha | S^{(0)} | \alpha \rangle E^2 + \langle \alpha | S_{\mu\nu}^{(2)} | \alpha \rangle E_\mu E_\nu$$

scalar and tensor shift

### 2.1 Scalar shift

Using Wigner-Eckart theorem, we have...

$$\alpha^{(0)}(J) = -\frac{2}{3} \sum_{J'} \frac{|\langle J || \mathbf{d} || J' \rangle|^2}{E_J - E_{J'}}$$

$$\Delta E_J^{(0)} = -\frac{1}{2} \alpha^{(0)}(J) E^2$$

independent of  $m_J$ : orientation-independent shift

## 2.2 Tensor shift

Switch to spherical basis (Roman indices: spherical components. Greek: Cartesian components)

$$\langle \alpha | S_{\mu\nu}^{(2)} | \alpha \rangle E_\mu E_\nu = \sum_q (-1)^q \langle J m_J | S_q^{(2)} | J m_J \rangle [\mathbf{E}\mathbf{E}]_{-q}^{(2)}$$

Applying Wigner-Eckart theorem one time, we find the only non-vanishing contribution comes from  $q = 0$ , and

$$[\mathbf{E}\mathbf{E}]_0^{(2)} = \frac{1}{\sqrt{6}}(3E_z^2 - E^2)$$

We define tensor polarizability as (in order to get the the correct normalization)

$$\alpha^{(2)}(J) = -\langle J || S_q^{(2)} || J \rangle \sqrt{\frac{8J(2J-1)}{3(J+1)(2J+3)}}$$

$$\Delta E_{J,m_J}^{(2)} = -\frac{1}{4}\alpha^{(2)}(J)(3E_z^2 - E^2) \frac{3m_J^2 - J(J+1)}{J(2J-1)}$$

**Normalization.** Let  $\mathbf{E} = E_z \hat{z}$ ,  $\Delta E_{J,\pm J}^{(2)} = -\frac{1}{2}\alpha^{(2)}(J)E_z^2$  depends on  $m_J$

Use one more time Wigner-Eckart theorem, we have

$$\alpha^{(2)}(J) = \sum_{J'} (-1)^{J+J'+1} \sqrt{\frac{40J(2J+1)(2J-1)}{3(J+1)(2J+3)}} \left\{ \begin{matrix} 1 & 1 & 2 \\ J & J & J' \end{matrix} \right\} \frac{|\langle J || \mathbf{d} || J' \rangle|^2}{E_J - E_{J'}}$$

## 2.3 Note

$J = 0$  or  $J = \frac{1}{2}$ , no tensor shift

## 3 AC Stark shifts (light shifts)

$$\mathbf{E}(\mathbf{r}) = \hat{\varepsilon} E_0^{(+)}(\mathbf{r}) e^{-i\omega t} + c.c.$$

Second order time-dependent perturbation theory

$$\Delta E_\alpha = -\sum_\beta \frac{2\omega_{\beta\alpha} |\langle \alpha | \hat{\varepsilon} \cdot \mathbf{d} | \beta \rangle|^2 |E_0^{(+)}|^2}{\hbar(\omega_{\beta\alpha}^2 - \omega^2)}$$

$$\Delta E_\alpha = -\frac{1}{2} \mathbf{d}^{(+)} \cdot \mathbf{E}^{(-)} - \frac{1}{2} \mathbf{d}^{(-)} \cdot \mathbf{E}^{(+)} = -Re[\alpha(\omega)] |E_0^{(+)}|^2$$

$$\alpha(\omega) = -\sum_\beta \frac{2\omega_{\beta\alpha} |\langle \alpha | \hat{\varepsilon} \cdot \mathbf{d} | \beta \rangle|^2}{\hbar(\omega_{\beta\alpha}^2 - \omega^2)}$$

Define

$$\alpha_{\mu\nu}(\omega) = \sum_\beta \frac{2\omega_{\beta\alpha} \langle \alpha | d_\mu | \beta \rangle \langle \beta | d_\nu | \alpha \rangle}{\hbar(\omega_{\beta\alpha}^2 - \omega^2)}$$

$$\Delta E_\alpha = -Re[\alpha_{\mu\nu}(\omega)] (E_0^{(-)})_\mu (E_0^{(+)})_\nu$$

$$\alpha_{\mu\nu}(J, m_J, \omega) = \sum_{J'} \frac{2\omega_{J'J} T_{\mu\nu}}{\hbar(\omega_{J'J}^2 - \omega^2)}$$

$$T_{\mu\nu} = \sum_{m'_J} \langle J m_J | d_\mu | J' m'_J \rangle \langle J' m'_J | d_\nu | J m_J \rangle$$

### 3.1 Scalar shift

$$T^{(0)} = T_{\mu\mu}$$

Using Wigner-Eckart theorem,

$$T^{(0)} = |\langle J || \mathbf{d} || J' \rangle|^2$$

$$\alpha^{(0)}(J, \omega) = \sum_{J'} \frac{2\omega_{J'J} |\langle J || \mathbf{d} || J' \rangle|^2}{3\hbar(\omega_{J'J}^2 - \omega^2)}$$

$m_J$  independent

### 3.2 Vector shift

$$T_\mu^{(1)} = \epsilon_{\mu\sigma\tau} (T_{\sigma\tau} - T_{\tau\sigma}) = 2\epsilon_{\mu\sigma\tau} 2T_{\sigma\tau}$$

$$T_q^{(1)} = (-1)^{J+J'} (-i) \sqrt{\frac{24(2J+1)}{J(J+1)}} \left\{ \begin{matrix} 1 & 1 & 1 \\ J & J & J' \end{matrix} \right\} |\langle J || \mathbf{d} || J' \rangle|^2 m_J \delta_{q0}$$

$$\alpha^{(1)}(J, \omega) = \sum_{J'} (-1)^{J+J'+1} \sqrt{\frac{6J(2J+1)}{J+1}} \left\{ \begin{matrix} 1 & 1 & 1 \\ J & J & J' \end{matrix} \right\} \frac{\omega_{J'J} |\langle J || \mathbf{d} || J' \rangle|^2}{\hbar(\omega_{J'J}^2 - \omega^2)}$$

### 3.3 Tensor shift

$$T_{\alpha\beta}^{(2)} = T_{\alpha\beta} - \frac{1}{3}T_{\mu\mu}\delta_{\alpha\beta}$$

$$T_q^{(2)} = (-1)^{J+J'} \sqrt{\frac{5(2J+1)}{J(J+1)(2J-1)(2J+3)}} \begin{Bmatrix} 1 & 1 & 2 \\ J & J & J' \end{Bmatrix} |\langle J || \mathbf{d} || J' \rangle|^2 [m_J^2 - J(J+1)]\delta_{q0}$$

$$\alpha^{(2)}(J, \omega) = \sum_{J'} (-1)^{J+J'} \sqrt{\frac{40J(2J+1)(2J-1)}{3(J+1)(2J+3)}} \begin{Bmatrix} 1 & 1 & 2 \\ J & J & J' \end{Bmatrix} \frac{\omega_{J',J} |\langle J || \mathbf{d} || J' \rangle|^2}{\hbar(\omega_{J',J}^2 - \omega^2)}$$

### 3.4 Total light shift

$$\Delta E(J, m_J, \omega) = -\alpha^{(0)}(J, \omega) |E_0^{(+)}|^2 - \alpha^{(1)}(J, \omega) (i\mathbf{E}_0^{(-)} \times \mathbf{E}_0^{(+)})_z \frac{m_J}{J} - \alpha^{(2)}(J, \omega) \frac{3|E_{0z}^{(+)}|^2 - |E_0^{(+)}|^2}{2} \frac{3m_J^2 - J(J+1)}{J(2J-1)}$$

**Normalization.** Note  $(i\mathbf{E}_0^{(-)} \times \mathbf{E}_0^{(+)})_z = |\mathbf{E}_{0,-1}^{(+)}|^2 - |\mathbf{E}_{0,1}^{(-)}|^2$ . Then  $\sigma^+(\mathbf{E}_0^{(+)} = E_{0,-1}^{(+)})$  has light shift  $\Delta E_{J,\pm J}^{(1)} = \mp \alpha^{(1)}(J) |E_{0,-1}^{(+)}|^2$

### 3.5 Note

1. Linear polarization drives the scalar and tensor light shifts, thus acts as an effective dc electric field
2. Circular polarization drives the vector light shift, acts as an effective magnetic field
3.  $J = 0$  or  $J = \frac{1}{2}$ , no tensor light shift
4.  $J = 0$ , no vector light shift

## 4 Decay rate and calculation of reduced matrix elements

Wigner-Eckart theorem

$$\Gamma_{n'J' \rightarrow nJ} = \frac{\omega_{n'J' \rightarrow nJ}^3}{3\pi\epsilon_0\hbar c^3} \frac{2J+1}{2J'+1} |\langle nJ || \mathbf{d} || n'J' \rangle|^2$$

Using the decomposition rule

$$\langle nJ || \mathbf{d} || n'J' \rangle = (-1)^{L'+L+1+S} \sqrt{(2J'+1)(2L+1)} \begin{Bmatrix} L & L' & 1 \\ J' & J & S \end{Bmatrix} \langle nL || \mathbf{d} || nL' \rangle$$

$$A_T(n'J') = \sum_J \Gamma_{n'J' \rightarrow nJ} = \frac{|\langle nL || \mathbf{d} || nL' \rangle|^2}{3\pi\epsilon_0\hbar c^3} \frac{2L+1}{2L'+1} \sum_J \omega_{n'J' \rightarrow nJ}^3 (2J+1)(2L'+1) \begin{Bmatrix} L & L' & 1 \\ J' & J & S \end{Bmatrix}^2$$

The branching ratio is then

$$\frac{\Gamma_{n'J' \rightarrow nJ}}{A_T(n'J')} = \frac{\omega_{n'J' \rightarrow nJ}^3 (2J+1) \begin{Bmatrix} L & L' & 1 \\ J' & J & S \end{Bmatrix}^2}{\sum_J \omega_{n'J' \rightarrow nJ}^3 (2J+1) \begin{Bmatrix} L & L' & 1 \\ J' & J & S \end{Bmatrix}^2}$$

Then we can compute the corresponding  $|\langle nJ || \mathbf{d} || n'J' \rangle|^2$

It's the same as Boyd thesis when considering different  $m_J, m_{J'}$  states using Clebsch-Gordon coefficients and taking  $\omega^3$  term into  $\zeta$  correction. (Eq 3.9-3.10)

## 5 References

1. Steck, 'Quantum and Atomic Optics', chapter 7.
2. Boyd thesis
3. Martin thesis