(b) Repeat (a) for sum instead of product.

(c) Verify in (a) and (b) that the sum of the eigenvalues of $A + B$ equals the sum of the individual eigenvalues of A and B. Then verify the same for products.

(d) Explain why (c) is true for any $n \times n$ matrices A and B.

36. In $(x - a_1)(x - a_2) \cdots (x - a_n)$, show that the coefficient of x^{n-1} is $-(a_1 + a_2 + \cdots + a_n)$.

37. Go through all the details of the proof of Theorem (5.17b). In particular:

(a) Verify Equation (5.19) by expanding $\det(\lambda I - A)$ by minors along the first row, then expanding the first determinant again, and so on, and

(b) Substitute Equation (5.19) into Equation (5.18) and use Exercise 36.

5.3 DIAGONALIZATION

We begin with a fact that is essential to both theory and applications, and it will be used repeatedly throughout the chapter. The computation that verifies this fact is surprisingly simple.

(5.20)

> ***Theorem*** Suppose that the $n \times n$ matrix A has *n linearly independent* eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Form an $n \times n$ matrix S whose ith column is $\mathbf{v}_i$. Then S is invertible and $S^{-1}AS$ is a diagonal matrix Λ:
>
> $$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$
>
> where λ_i is the eigenvalue of A associated with $\mathbf{v}_i$.

Proof Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent eigenvectors of A and $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$, $1 \le i \le n$. Construct

$$S = \left[\mathbf{v}_1 \vdots \mathbf{v}_2 \vdots \cdots \vdots \mathbf{v}_n\right]$$

Now $rk(S) = n$, since its columns are linearly independent, so S is invertible.

Compute the product AS one column at a time,

$$AS = A\left[\mathbf{v}_1 \vdots \cdots \vdots \mathbf{v}_n\right] = \left[A\mathbf{v}_1 \vdots \cdots \vdots A\mathbf{v}_n\right] = \left[\lambda_1\mathbf{v}_1 \vdots \cdots \vdots \lambda_n\mathbf{v}_n\right]$$

Next factor the last matrix as indicated.

$$\left[\lambda_1\mathbf{v}_1 \vdots \cdots \vdots \lambda_n\mathbf{v}_n\right] = \left[\mathbf{v}_1 \vdots \cdots \vdots \mathbf{v}_n\right]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

IMPORTANT You should check this last multiplication to see why we get $S\Lambda$ and *not* ΛS (i.e., see ΛS multiples the ith *row* by λ_i and $S\Lambda$ multiples the ith *column* by λ_i). Therefore, we have shown

(5.21) $$AS = S\Lambda \quad \text{which implies} \quad S^{-1}AS = \Lambda$$

(by multiplying on the left by S^{-1}), so we are done. ■

Note that if we had multiplied by S^{-1} on the right in (5.21), we would have obtained the following corollary.

(5.22)

> ***Corollary*** If A, Λ, and S are as in Theorem (5.20), then
>
> $$A = S\Lambda S^{-1}$$

Example 1 Let A be the matrix of Example 7 in Section 5.2, and let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the eigenvectors of A computed there. Thus

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \frac{3}{4} \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and $A\mathbf{v}_1 = -\mathbf{v}_1$, $A\mathbf{v}_2 = 6\mathbf{v}_2$, $A\mathbf{v}_3 = 6\mathbf{v}_3$. If we let

$$S = \begin{bmatrix} 1 & \frac{3}{4} & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and compute} \quad S^{-1} = \begin{bmatrix} \frac{4}{7} & -\frac{3}{7} & 0 \\ \frac{4}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then we can check directly that

$$S^{-1}AS = \begin{bmatrix} \frac{4}{7} & -\frac{3}{7} & 0 \\ \frac{4}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & \frac{3}{4} & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & & \\ & 6 & \\ & & 6 \end{bmatrix} = \Lambda$$ ■

NOTE Given an A, then S and Λ are *not* unique. If we use different eigenvectors or change their order, we will get a different S and possibly a different Λ.

Example 2 For the A and S in Example 1, let $S_1 = [\mathbf{v}_3 \vdots \mathbf{v}_2 \vdots \mathbf{v}_1]$. Then $S_1 \ne S$ and

$$S_1^{-1}AS_1 = \Lambda_1 \quad \text{where} \quad \Lambda_1 = \begin{bmatrix} 6 & & \\ & 6 & \\ & & -1 \end{bmatrix}$$ ■

-1, 6, 6

$\det(\lambda I - A) = 0$

$\begin{vmatrix} \lambda-2 & -3 & 0 \\ -4 & \lambda-3 & 0 \\ 0 & 0 & \lambda-6 \end{vmatrix} = 0$

$(\lambda-6)\left[(\lambda-2)(\lambda-3) - 12\right] = 0$

$\lambda^2 - 5\lambda + 6 - 12$

$-6 = 0$

-6

1 1 1