Ex 1 • Construct a natural cubic spline that passes through $(1,2), (2,3), (3,5)$

$x_0=1, \ f_0=2$

$x_1=2, \ f_1=3$

$x_2=3, \ f_2=5$

(a) $S(x)=\begin{cases} S_0(x), & x\in[1,2] \\ S_1(x), & x\in[2,3] \end{cases}$

$S_0(x)=a_0+b_0(x-\underbrace{1}_{=x_0})+c_0(x-1)^2+d_0(x-1)^3$

$S_0'(x)=b_0+2c_0(x-1)+3d_0(x-1)^2$

$S_0''(x)=2c_0+6d_0(x-1)$

$S_1(x)=a_1+b_1(x-\underbrace{2}_{=x_1})+c_1(x-2)^2+d_1(x-2)^3$

$S_1'(x)=b_1+2c_1(x-2)+3d_1(x-2)^2$

$S_1''(x)=2c_1+6d_1(x-2)$

See Def. 10, Course 5 (slide 14):

(b) $\begin{cases} S_0(x_0)=f(x_0) \\ S_0(x_1)=f(x_1) \end{cases} \Rightarrow \begin{cases} \underline{\underline{a_0}}=f(1)=\underline{\underline{2}} \\ a_0+b_0+c_0+d_0=f(2)=3 \end{cases}$

$\begin{cases} S_1(x_1)=f(x_1) \\ S_1(x_2)=f(x_2) \end{cases} \Rightarrow \begin{cases} \underline{\underline{a_1}}=f(2)=\underline{\underline{3}} \\ a_1+b_1+c_1+d_1=f(3)=5 \end{cases}$

(c) $S_0(x_1)=S_1(x_1) \Rightarrow a_0+b_0+c_0+d_0=a_1=3$

(d) $S_0'(x_1)=S_1'(x_1) \Rightarrow b_0+2c_0+3d_0=b_1$

(e) $S_0''(x_1)=S_1''(x_1) \Rightarrow 2c_0+6d_0=2c_1$

(f) $S_0''(x_0)=0, \ S_1''(x_2)=0 \Rightarrow$ $2c_0=0 \Rightarrow \underline{c_0=0}$

$2c_1+6d_1=0 \Rightarrow c_1+3d_1=0$

$$\begin{cases} a_0 = 2 \\ a_1 = 3 \\ c_0 = 0 \\ a_0 + b_0 + c_0 + d_0 = 3 \\ a_1 + b_1 + c_1 + d_1 = 5 \\ b_0 + 2c_0 + 3d_0 = b_1 \\ c_0 + 3d_0 = c_1 \end{cases}$$

$c_1 + 3d_1 = 0 \Rightarrow d_1 = -\frac{c_1}{3}$

$\Rightarrow \begin{cases} b_0 + d_0 = 1 \Rightarrow b_0 = 1 - d_0 \\ 3d_0 = c_1 \Rightarrow d_0 = \frac{c_1}{3} \Rightarrow b_0 = 1 - \frac{c_1}{3} \end{cases}$

$\begin{cases} b_1 + c_1 - \frac{c_1}{3} = 2 \quad | \cdot 3 \\ 1 - \frac{c_1}{3} + c_1 = b_1 \quad | \cdot 3 \end{cases} \overset{*}{\Rightarrow}$

$\overset{*}{\Rightarrow}$ $3b_1 + 2c_1 = 6$

$3b_1 - 2c_1 = +3$ $\quad -$

$4c_1 = 3 \Rightarrow \boxed{c_1 = \frac{3}{4}}$

$b_1 = \frac{6 - 2c_1}{3} = \frac{6 - \frac{3}{2}}{3} = \frac{9}{6}$

$\boxed{b_1 = \frac{3}{2}}$

$d_1 = -\frac{c_1}{3} = -\frac{3}{4} \cdot \frac{1}{3} = -\frac{1}{4}$ $\quad \boxed{d_1 = -\frac{1}{4}}$

$d_0 = \frac{c_1}{3} = \frac{1}{4}$ $\quad \boxed{d_0 = \frac{1}{4}}$

$b_0 = 1 - d_0 = 1 - \frac{1}{4} = \frac{3}{4}$

$\boxed{b_0 = \frac{3}{4}}$

$$S(x) = \begin{cases} S_0(x) = 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3 & x \in [1,2] \\ S_1(x) = 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3 & x \in [2,3] \end{cases}$$

(Ex 2) * Construct a clamped spline S that passes through the points $(1,2)$, $(2,3)$, $(3,5)$ and has $S'(1)=2$, $S'(3)=1$.

$x_0=1,\ f_0=2$

$x_1=2,\ f_1=3$

$x_2=3,\ f_2=5$

$$S(x)=\begin{cases} S_0(x), & x\in[1,2] \\ S_1(x), & x\in[2,3] \end{cases}$$

$$S_0(x)=a_0+b_0(x-\underset{}{\overset{=x_0}{1}})+c_0(x-1)^2+d_0(x-1)^3$$

$$S_1(x)=a_1+b_1(x-\underbrace{2}_{=x_1})+c_1(x-2)^2+d_1(x-2)^3$$

$S_0'(x)=b_0+2c_0(x-1)+3d_0(x-1)^2$ $\qquad S_0''(x)=2c_0+6d_0(x-1)$

$S_1'(x)=b_1+2c_1(x-2)+3d_1(x-2)^2$ $\qquad S_1''(x)=2c_1+6d_1(x-2)$

$$\begin{cases} S_0(x_0)=f(x_0) \\ S_1(x_1)=f(x_1) \\ S_0(x_1)=f(x_1) \\ S_1(x_2)=f(x_2) \\ S_0(x_1)=S_1(x_1) \\ S_0'(x_1)=S_1'(x_1) \\ S_0''(x_1)=S_1''(x_1) \\ S_0'(x_0)=f'(x_0) \\ S_1'(x_2)=f'(x_2) \end{cases}$$

$S_0(1)=a_0=f(1)=2$

$S_1(2)=a_1=f(2)=3$

$S_0(2)=a_0+b_0+c_0+d_0=3$

$S_1(3)=a_1+b_1+c_1+d_1=5$

$S_0(2)=a_0+b_0+c_0+d_0=a_1=3$

$S_0'(2)=b_0+2c_0+3d_0=b_1\ (=S_1'(2))$

$S_0''(2)=2c_0+6d_0=2c_1\ (=S_1''(2))$

$\underbrace{S_0'(1)}_{=S'(1)}=b_0=2$ (from hypothesis)

$S_1'(3)=1=b_1+2c_1+3d_1$ (from hyphothesis)

For a_0, b_0, c_0, d_0 we have:

$$\begin{cases} \boxed{a_0 = 2} \\ \frac{a_0}{2} + \frac{b_0}{2} + c_0 + d_0 = 3 \\ \boxed{b_0 = 2} \\ b_0 + 2c_0 + 3d_0 = b_1 \\ 2c_0 + 6d_0 = 2c_1 \quad |:2 \end{cases} \Rightarrow \begin{cases} c_0 + d_0 = -1 \\ c_0 + 3d_0 = c_1 \\ b_0 + 2c_0 + 3d_0 = b_1 \\ b_1 + c_1 + d_1 = 2 \\ b_1 + 2c_1 + 3d_1 = 1 \quad \ominus \\ \hline -c_1 - 2d_1 = 1 \Rightarrow \end{cases}$$

$$c_1 + 2d_1 = -1$$

For a_1, b_1, c_1, d_1 we have:

$$\begin{cases} \boxed{a_1 = 3} \\ a_1 + b_1 + c_1 + d_1 = 5 \\ b_1 + 2c_1 + 3d_1 = 1 \end{cases}$$

We need to solve the system:

$$\begin{cases} c_0 + d_0 = -1 \Rightarrow c_0 = -1 - d_0 \\ c_0 + 3d_0 - c_1 = 0 \Rightarrow c_1 = c_0 + 3d_0 = -1 - d_0 + 3d_0 = 2d_0 - 1 \\ 2c_0 + 3d_0 - b_1 = -2 \Rightarrow 2d_0 = c_1 + 1 \Rightarrow d_0 = \frac{c_1 + 1}{2} \\ b_1 + c_1 + d_1 = 2 \quad |\cdot 3 \\ b_1 + 2c_1 + 3d_1 = 1 \quad \ominus \end{cases}$$

$$c_0 = -1 - \frac{1 + c_1}{2} = \frac{-3 - c_1}{2}$$

$$2b_1 + c_1 = 5$$

$$\begin{cases} 2c_0 + 3d_0 - b_1 = -2 \\ c_1 + 2b_1 = 5 \end{cases} \Leftrightarrow \begin{cases} -3 - c_1 + \frac{3c_1 + 3}{2} - b_1 = -2 \quad |\cdot 2 \\ c_1 + 2b_1 = 5 \quad \oplus \end{cases}$$

$$-6 - 2c_1 + 3c_1 + 3 + c_1 = 1$$

$$2c_1 = 4 \Rightarrow \boxed{c_1 = 2} \quad \boxed{b_1 = \frac{3}{2}}$$

$$S(x) = \begin{cases} 2 + 2(x-1) - \frac{5}{2}(x-1)^2 + \frac{3}{2}(x-1)^3 \quad (S_0(x)) \quad x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + 2(x-2)^2 - \frac{3}{2}(x-2)^3 \quad (S_1(x)) \quad x \in [2,3] \end{cases}$$

$$\boxed{d_0 = \frac{3}{2}} \quad \boxed{c_0 = -\frac{5}{2}} \quad \boxed{d_1 = -\frac{3}{2}}$$