Lab 10

Conditioning number of a linear system. Direct methods for solving linear systems

Gauss method with partial pivoting

Consider the linear system $Ax = b$, with $A = (a(i,j))_{i,j=\overline{1,n}}$ and $b = (b(1), ..., b(n))'$.

Algorithm:

Input: n-order of the system; A-matrix of coefficients; b-vector of free terms;

Output: x-vector of the solutions or a message in case of incompatibility of the system

1. For $p = 1, ..., n-1$

   Let $abs(a(q,p)) = \max(abs(a(p:n,p)))$.

   If $a(q,p) = 0$ then "Message"; Exit

   If $q \neq p$ interchange the lines p and q from A and b.

   Perform the necessary operations for obtaining zeros on the column p, below $a(p,p)$.

   Apply the transformations also to the vector b.

2. If $a(n,n) = 0$ then "Message"; Exit
3. For $i = n : -1 : 1$ do

   Compute $x(i)$.

4. Display x.

Problems:

1. Consider the system:

$$Ax = b,$$

with

$$A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 32 \\ 23 \\ 33 \\ 31 \end{bmatrix}.$$

a) Solve this system and find its conditioning number. (Use Matlab function *cond*.)

b) Solve the system

$$A\tilde{x} = \tilde{b},$$

where

$$\tilde{b} = \begin{bmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{bmatrix}.$$

Compute *the input relative error* $\frac{\|b-\tilde{b}\|}{\|b\|}$ and *the output relative error* $\frac{\|x-\tilde{x}\|}{\|x\|}$. Compute $\frac{\|x-\tilde{x}\|}{\|x\|} \Big/ \frac{\|b-\tilde{b}\|}{\|b\|}$

c) Solve the system

$$\bar{A}\bar{x} = b,$$

with

$$\bar{A} = \begin{bmatrix} 10 & 7 & 8.1 & 7.2 \\ 7.08 & 5.04 & 6 & 5 \\ 8 & 5.98 & 9.89 & 9 \\ 6.99 & 4.99 & 9 & 9.98 \end{bmatrix}.$$

Compute *the input relative error* $\frac{\|A-\bar{A}\|}{\|A\|}$ and *the output relative error* $\frac{\|x-\bar{x}\|}{\|x\|}$. Compute $\frac{\|x-\bar{x}\|}{\|x\|} \Big/ \frac{\|A-\bar{A}\|}{\|A\|}$.

2. Find the conditioning numbers of the Vandermonde matrices $V(t_k)$ for the points $t_k = \frac{1}{k},\ k = \overline{1,n}$, for $n = \overline{10, 15}$.

3. Implement Gauss method for solving linear systems, using partial elimination. Solve the following system of equations:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 5 \\ -1 & 1 & -5 & 3 \\ 3 & 1 & 7 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 31 \\ -2 \\ 18 \end{bmatrix}.$$

(Facultative) **4.** Find the conditioning numbers of the Vandermonde matrices $V(t_k)$ for the points $t_k = -1 + \frac{2}{n}k \in [-1, 1]$, with $k = \overline{1, n}$, for $n = \overline{10, 15}$.

5. Find the conditioning numbers of the Hilbert matrices $H_n = (h_{ij})_{\substack{1\le i\le n,\\ 1\le j\le n}}$ with $h_{ij} = \frac{1}{i+j-1}$, for $n = 10 : 15$.