Exercises 3

1. A one-dimensional classification problem for three classes, A, B and C and pdf-s $f_A(x)$, $f_B(x)$, $f_C(x)$ is defined as:

   $p_A = 1/2, f_A(x) = 1/8$ for $x \in [-4, 4]$

   $p_B = 1/3, f_B(x) = 3(1-x^2)/4$ for $x \in [-1, 1]$

   $p_C = 1/6, f_C(x) = x/8$ for $x \in [0, 4]$

   Estimate optimal classification boundaries and decision rules for the system. Explain the general estimation steps and draw a figure that describes the estimation process and the results.

   Answer: In general in a classification problem with a number of classes C_k, with *a priori* probabilities p_{C_k} and pdf's $f_{C_k}(x)$, the optimal decision rule for a given outcome x is to choose the class C_{opt} that maximizes the expression:

   $$C_{opt} = \arg\max_{C_k} p_{C_k} \cdot f_{C_k}(x)$$

   Here we set:

   $g_A(x) = p_A \cdot f_A(x) = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$ when $x \in [-4, 4]$

   $g_B(x) = p_B \cdot f_B(x) = \frac{1}{3} \cdot \frac{3(1-x^2)}{4} = \frac{1-x^2}{4}$ when $x \in [-1, 1]$

   $g_C(x) = p_C \cdot f_C(x) = \frac{1}{6} \cdot \frac{x}{8} = \frac{x}{48}$ when $x \in [0, 4]$

   In order to find the classification borders we need to solve the following system of inequalities:

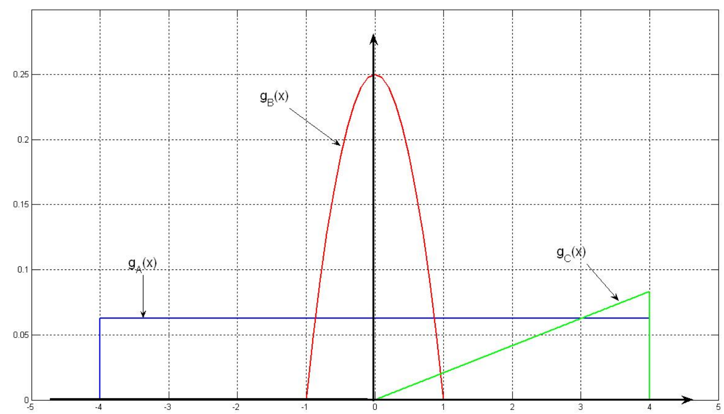
   $$\begin{cases} g_A(x) > g_B(x) & (1) \\ g_A(x) > g_C(x) & (2) \\ g_B(x) > g_C(x) & (3) \end{cases}$$

   so that $x \in C_k$, where

   $$C_k = \begin{cases} A & if & (1) \wedge (2) \\ B & if & (3) \wedge \neg(1) \\ C & if & \neg(2) \wedge \neg(3) \end{cases}$$

   Now, let us solve this system. Equation (1) yields:

   $$\frac{1}{16} > \frac{1-x^2}{4} \Leftrightarrow x^2 > \frac{3}{4} \Rightarrow \left\{ x > \frac{\sqrt{3}}{2} \vee x < -\frac{\sqrt{3}}{2} \right.$$



Figur 1: Illustration of the classification problem

Likewise, equation (2) yields:

$$\frac{1}{16} > \frac{x}{48} \Leftrightarrow x < 3$$

Before we calculate the solution to equation (3), let us plot the three functions $g_{C_k}(x)$.See Figure **??**

We see that the solution to equation (3) is obsolete, since it's equivalent with the negated solution to equation (1) and the solution to equation (2). (Reason is that $3 > \sqrt{3}/2$). In other words, the classification boundaries are:

$$C_k = \begin{cases} A & if \quad -4 < x < -\frac{\sqrt{3}}{2} \quad \vee \quad \frac{\sqrt{3}}{2} < x < 3 \\ B & if \quad -\frac{\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2} \\ C & if \quad \quad 3 < x < 4 \end{cases}$$

2. For a classification problem with two classes C_A and C_B, are the a priori probabilities $p_A = 3/4$ and $p_B = 1/4$. Assume the following pdf-s

$$p(\bar{z}|C_k) = \frac{1}{2\pi|det\Sigma_k|^{1/2}} e^{-(\bar{z}-m_k)^T \Sigma_k^{-1}(\bar{z}-m_k)/2}$$

with

$$m_A = m_B = 0,$$

and

$$\Sigma_A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \Sigma_B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Estimate decision boudaries for the problem!

Answer: The decision boundary is given by the "equal probability solution": $Pr(A|\bar{x}) = Pr(B|\bar{x})$. Now multiply each side with $Pr(\bar{x})$, giving $Pr(\bar{x}) \cdot Pr(A|\bar{x}) = Pr(\bar{x}) \cdot Pr(B|\bar{x})$. However, according to Bayes' rule, $Pr(\bar{x}) \cdot Pr(A|\bar{x}) = Pr(A) \cdot Pr(\bar{x}|A)$ which gives us the decision boundary is given by $Pr(A) \cdot Pr(\bar{x}|A) = Pr(B) \cdot Pr(\bar{x}|B)$. Thus we need to solve the equation

$$p_A \cdot p(\bar{x}|A) = p_B \cdot p(\bar{x}|B)$$

for $\bar{x} = [xy]$. Since we have the expression for $p(\bar{x}|C_k)$ and p_{C_k}, this is trivial. We will solve the equation as an inequality for the case $\bar{x} \in A$, i.e. $p_A \cdot p(\bar{x}|A) > p_B \cdot p(\bar{x}|B)$ (this is to have correct boundary conditions):

$$\frac{p_A}{2\pi\left|\det \Sigma_A\right|^{1/2}} e^{\frac{-(\bar{x}-m_A)^T \Sigma_A^{-1} (\bar{x}-m_A)}{2}} > \frac{p_B}{2\pi\left|\det \Sigma_B\right|^{1/2}} e^{\frac{-(\bar{x}-m_B)^T \Sigma_B^{-1} (\bar{x}-m_B)}{2}} \Leftrightarrow$$

$$\frac{3}{4}\frac{1}{2\pi\sqrt{4}} e^{\frac{-\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}{2}} > \frac{1}{4}\frac{1}{2\pi\sqrt{4}} e^{\frac{-\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}{2}} \Leftrightarrow$$

$$\ln 3 - \begin{bmatrix} x/8 & y/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} > -\begin{bmatrix} x/2 & y/8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow$$

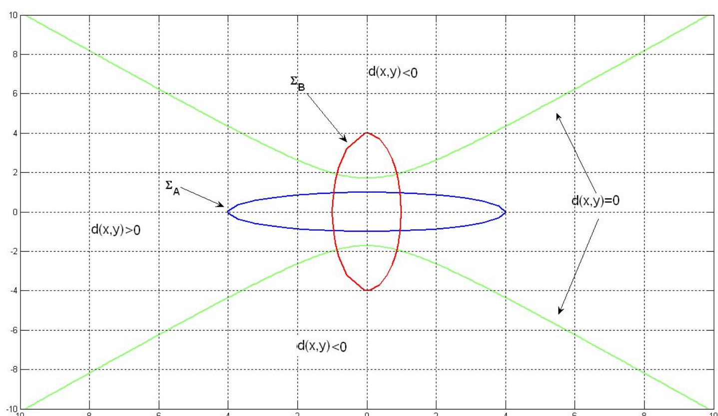
$$\ln 3 - \left(\frac{x^2}{8} + \frac{y^2}{2}\right) > -\left(\frac{x^2}{2} + \frac{y^2}{8}\right) \Leftrightarrow$$

$$x^2 - y^2 + \frac{8}{3}\ln 3 > 0$$

Now define $d(\bar{x}) \equiv d(x, y) \equiv x^2 - y^2 + \frac{8}{3}\ln 3$, then our classification rule becomes:

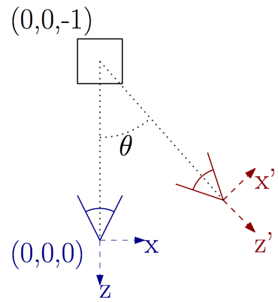
$$C_k = \begin{cases} A & if \quad d(\bar{x}) > 0 \\ B & if \quad d(\bar{x}) < 0 \end{cases}$$

Note that $d(x, y) = 0$ defines a hyperbolic curve, see Figure **??**



Figur 2: Illustration of the 2D classification problem

3. A robot is trying to gather 3D information from an object. Since it has only a single camera, it rotates around the object to obtain multiple views from it, as seen in the figure below.



a) What is the relation between points P in 3D space and their image projections p on the image camera at $(0,0)$? Consider unit focal length, $f=1$, and centered image origin, with x increasing to the right and y increasing up.

b) What is the relation between 3D points $P=(x,y,z)$ in the original coordinate frame and $P'=(x',y',z')$ in the new coordinate frame after rotating θ radians?

c) What is the relation between image points p and p'?

d) At some point the encoders of the motors fail and the robot doesn't know how large θ is. Estimate θ given point correspondences between the two images. How many point do you need?

Answer:

a) Since z-axis points backwards, z-values are inverted in the projection.

$$P=(x,y,z) \Rightarrow p=\left(\frac{x}{-z},\frac{y}{-z}\right)$$

b)

$$t=\begin{pmatrix}0\\0\\-1\end{pmatrix},\quad R=\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}$$

Transforming the 3D points look the same as transfoming the camera, but in opposite direction. We first move the camera forward towards the object (-t), then rotate (R) and finally move back backwards (t).

$$P'=R(P-t)+t=R(P-(t-R^{-1}t))=R(P-e)$$

Here e corresponds to the final translation of the camera.

$$e=t-R^{-1}t=\begin{pmatrix}0\\0\\-1\end{pmatrix}-\begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}0\\0\\-1\end{pmatrix}=\begin{pmatrix}\sin\theta\\0\\-1+\cos\theta\end{pmatrix}$$

c) There is an essential matrix $E=RT_e$ for which $P'^TEP=0$ holds.

$$T_e=\begin{pmatrix}0 & 1-\cos\theta & 0\\ -(1-\cos\theta) & 0 & -\sin\theta\\ 0 & \sin\theta & 0\end{pmatrix}$$

$$E = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1-\cos\theta & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & -(1-\cos\theta) & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}$$

Then the projections can be related by

$$\begin{pmatrix} p'_x & p'_y & -1 \end{pmatrix} E \begin{pmatrix} p_x \\ p_y \\ -1 \end{pmatrix} = 0$$

d)

$$\begin{pmatrix} p'_x & p'_y & -1 \end{pmatrix} \begin{pmatrix} 0 & -(1-\cos\theta) & 0 \\ -(1-\cos\theta) & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ -1 \end{pmatrix} = 0$$

$$\begin{pmatrix} p'_x & p'_y & -1 \end{pmatrix} \begin{pmatrix} -(1-\cos\theta)p_y \\ -(1-\cos\theta)p_x + \sin\theta \\ \sin\theta p_y \end{pmatrix} = 0$$

$$-(1-\cos\theta)p_y p'_x - (1-\cos\theta)p_x p'_y + \sin\theta p'_y - \sin\theta p_y = 0$$

$$(p_y p'_x + p_x p'_y)\cos\theta + (p'_y - p_y)\sin\theta = (p_y p'_x + p_x p'_y)$$

Finally, solve for θ by removing the cosine and then using arcsin.

4. Assume that you have a robot translating in space. For every point in time a camera (with focal length f), mounted on the robot, observes two points A and B by measuring the image coordinates (x_A, y_A) and (x_B, y_B), as well as the motion field (x'_A, y'_A) and (x'_B, y'_B). Set up a linear system of equations from which is possible to determine the translational direction of the robot (T_X, T_Y, T_Z) up to an unknown scale factor, including the depths of the two points Z_A and Z_B. Explain how it is possible to determine the direction, even if the scale factor is unknown.

   Answer: By derivating the equation for the perspective mapping

$$\frac{x}{f} = \frac{X}{Z}$$
$$\frac{y}{f} = \frac{Y}{Z}$$

we get

$$\frac{\dot{x}}{f} = \frac{\dot{X}}{Z} - \frac{X}{Z}\frac{\dot{Z}}{Z}$$
$$\frac{\dot{y}}{f} = \frac{\dot{Y}}{Z} - \frac{Y}{Z}\frac{\dot{Z}}{Z}$$

With $(\dot{X}, \dot{Y}, \dot{Z})^T = -(T_X, T_Y, T_Z)^T$, corresponding to the a pure translation, you get

$$\frac{\dot{x}}{f} = -\frac{T_X}{Z} + \frac{x}{f}\frac{T_Z}{Z}$$
$$\frac{\dot{y}}{f} = -\frac{T_Y}{Z} + \frac{y}{f}\frac{T_Z}{Z}$$

and if you multiply with fZ you get

$$\dot{x}Z + fT_X = xT_Z$$
$$\dot{y}Z + fT_Y = yT_Z$$

Suppose we have the measurements (x, y) and $(\dot{x}, \dot{y})$ for two points A och B. We can then write the equations above as a system of equations

$$\begin{pmatrix} f & 0 & \dot{x}_A & 0 \\ 0 & f & \dot{y}_A & 0 \\ f & 0 & 0 & \dot{x}_B \\ 0 & f & 0 & \dot{y}_B \end{pmatrix} \begin{pmatrix} T_X \\ T_Y \\ Z_A \\ Z_B \end{pmatrix} = T_Z \begin{pmatrix} x_A \\ y_A \\ x_B \\ y_B \end{pmatrix}$$

to determine Z, T_X and T_Y as a function of T_Z, which is not known. This means that we can only determine the motion (T_X, T_Y, T_Z) and depths (Z_A and Z_B) up to some unknown scale factor. The translational direction though is determined by

$$\left(\frac{T_X}{T_Z}, \frac{T_Y}{T_Z}, 1\right)^T,$$

which can be determined, but in order to know the speed we need more information, such as e.g. Z_A or Z_B.