Exercises 1

1. With homogeneous point and line coordinates in the plane, $x = (x_1, x_2, x_3)^T$ and $l = (l_1, l_2, l_3)^T$, the equation of a straight line can be written $l^T x = 0$. Show using the same notation that

   (a) the intersection x of two straight lines l and l' is given by $x = l \times l'$,

   (b) the line l between two points x and x' can be written as $l = x \times x'$,

   where "$\times$" denotes the cross product.

   Solution:

   (a) Set up the equation system that defines a common point for two lines, and compare the result to that of computing a cross product of $l' \times l''$ in terms of explicit coordinates.

   (b) From the qualities of a cross product follows directly that x and x' satisfies $l^T x = (x \times x')^T x = 0$ and $l^T x' = (x \times x')^T x' = 0$ respectively, which shows that both these points are on the given line.

2. Projective transformations between two planar surfaces (which includes the perspective projection between a plane in the world and the image plane) can be expressed as transformations of the type

   $$y = Ax$$

   where $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ are homogeneous coordinates in each respective plane and A is a non-singular 3×3-matrix. Determine the following geometric objects transformed using this projection:

   (a) a line $l^T x = 0$, where l is a vector of length 3,

   (b) an ellipse $x^T C x = 0$, where C is a positive definite 3×3-matrix.

   Solution:

   (a) Since A is non-singular it can be inverted and $x = A^{-1} y$. Thus $l^T x = l^T A^{-1} y = 0$ and $l_y^T y = 0$, where $l_y = A^{-T} l$.

   (b) The same can be done for the ellipse $x^T C x = y^T A^{-T} C A^{-1} y = 0$, such that $y^T C_y y = 0$, where $C_y = A^{-T} C A^{-1}$.

3. Assume you have four points in a plane with coordinates $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ respectively. Illustrate the mappings of these points, computed through:

(i) the similarity transformation

$$\begin{pmatrix} 2 & 2 & 1 \\ -2 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) the affine transformation

$$\begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii) den projective transformation

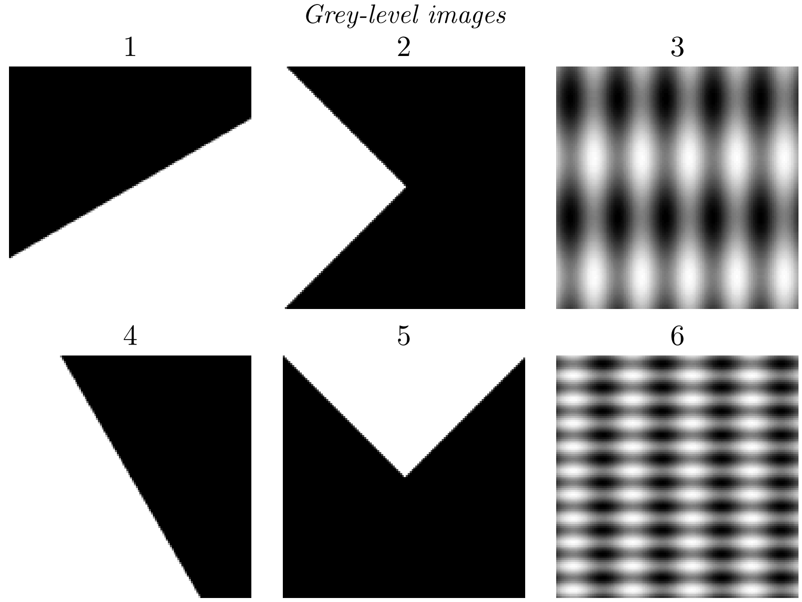
$$\begin{pmatrix} 2 & 2 & 1 \\ -2 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

Interpret the results geometrically.

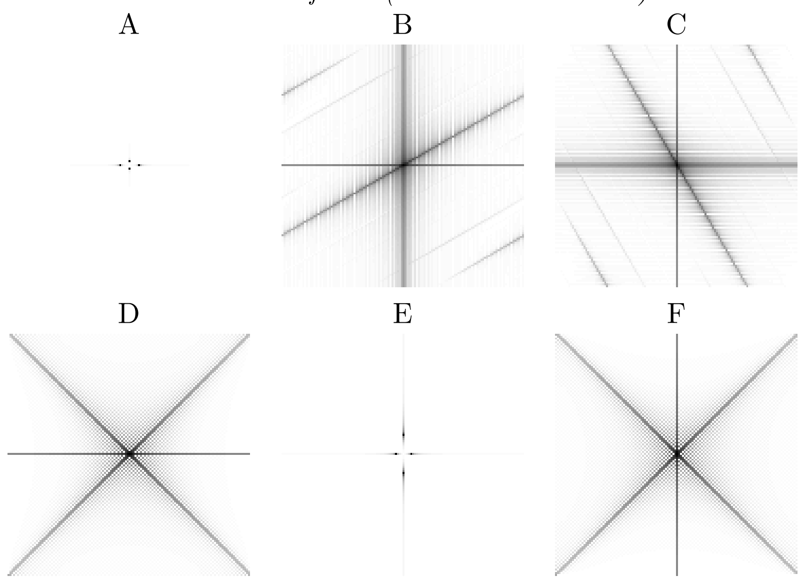
Solution: Perform the matrix multiplications using the homogeneous coordinates of the unit square with each of the transformations, as well as a final projection to 2D coordinates. Observe that the similarity transformation keeps all angles and parallel lines, the affine transformation only the parallel lines, whereas the projective transformation does not keep any of these characteristics.

4. The figures below shows six images and their centered and inverted Fourier transforms (The dark areas in the Fourier transforms correspond to high values). Match the correct images to the correct Fourier transforms. Motivate your answers well.

Grey-level images



Fourier transforms (centered och inverted)



Solution: The images and their Fourier transforms are matched as follows:

(a) 1–C
(b) 2–D
(c) 3–A
(d) 4–B
(e) 5–F
(f) 6–E

5. A grey-level image $f\colon \Omega \to [0, z_{max}]$ has a histogram

$$p(z) = \frac{\pi}{2z_{max}} \sin\left(\frac{\pi}{2}\frac{z}{z_{max}}\right).$$

Determine the grey-level transformation $T\colon [0, z_{max}] \to [-\frac{z_{max}}{2}, \frac{z_{max}}{2}]$ so that the grey-levels in the transformed image $g(x, y) = T(f(x, y))$ become uniformly distributed in the interval $[-\frac{z_{max}}{2}, \frac{z_{max}}{2}]$. For which $z \in [0, z_{max}]$ does this transformation result in a stretching of grey-levels?

Solution: An image $f\colon \Omega \to [0, 1]$ with a normalized histogram $p\colon [0, 1] \to \mathbb{R}$ satisfying

$$\int_{z=0}^{1} p(z)\, dz = 1$$

is histogram equalized by a transformation $T'\colon [0, 1] \to [0, 1]$

$$T'(z) = \int_{\xi=0}^{z} p(\xi)\, d\xi.$$

For the transformation $T\colon [0, z_{max}] \to [-\frac{z_{max}}{2}, \frac{z_{max}}{2}]$ searched for in this exercise, the corresponding function is given by

$$T(z) = -\frac{z_{max}}{2} + z_{max} \int_{\xi=0}^{z} p(\xi)\, d\xi.$$

Since the integral can be evaluated as

$$\int_{\xi=0}^{z} p(\xi)\,d\xi = \int_{\xi=0}^{z} \frac{\pi}{2z_{max}} \sin\left(\frac{\pi}{2}\frac{\xi}{z_{max}}\right)\,d\xi = \left(1 - \cos\left(\frac{\pi}{2}\frac{z}{z_{max}}\right)\right),$$

the requested transformation becomes

$$T(z) = -\frac{z_{max}}{2} + z_{max}\left(1 - \cos\left(\frac{\pi}{2}\frac{z}{z_{max}}\right)\right).$$

To determine for which values of z the transformation is stretched, we search for those z for which

$$\frac{\partial T}{\partial z} \geq 1.$$

The derivation becomes

$$\frac{\partial T}{\partial z} = \frac{\pi}{2} \sin\left(\frac{\pi}{2}\frac{z}{z_{max}}\right),$$

which results in the interval

$$z \geq z_{max}\,\frac{2}{\pi} \arcsin \frac{2}{\pi}.$$

6. An image f normalized to the interval $[0,1]$ has a normalized distribution function $p_f(z) = 2(1-z)$. Compute the monotonically increasing grey-level transformation $z' = T(z)$ that transforms this image into an image g with distribution function $p_g(z') = 2z'$.

   Solution: Given a (normalized) histogram with the distribution

$$p(z) = 2(1-z)$$

we want to transform it to a (normalized) histogram of the form

$$p'(z') = 2z'$$

using a transformation

$$z' = T(z)$$

where T is monotonically increasing. Assume an infinitesimally small area element that is preserved, it follow that

$$p(z)\,dz = p'(z')\,dz'$$

or with the values inserted (note: $T'(z) = dz'/dz \rightarrow dz' = T'(z)dz$)

$$2(1-z)\,dz = 2T(z)T'(z)dz.$$

After integration this leads to

$$2\left(z - \frac{z^2}{2} + C\right) = (T(z))^2$$

which corresponds to a monotonically increasing solution

$$T(z) = \sqrt{2z - z^2 + C}.$$

To determine the integration constant C we can use $T(0) = 0$, which leads to $C = 0$. The result is thus

$$T(z) = \sqrt{2z - z^2}$$

which satisfies $T(0) = 0$ as well as $T(1) = 1$.

7. A filter consists of the following coefficients

$$[1, 2, \underset{\cdot}{0}, -2, -1]$$

   (a) Compute and draw the transfer function in the Fourier domain.

   (b) Determine the differential expression that this filter corresponds to.

   (c) Can you divide this filter into a collection of simpler components?

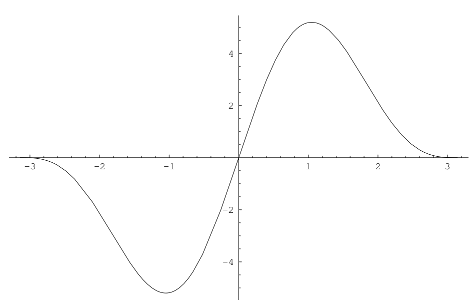
Solution: Define

$$\varphi(\omega) = \sum_{x=-\infty}^{\infty} c_x \, e^{i\omega x}$$

where $\omega \in [-\pi, \pi]$. Assume that $x = 0$ is the center of the filter (denoted by a small point). Then it follows that

$$\varphi(\omega) = -e^{-i2\omega} - 2e^{-i\omega} + 2e^{i\omega} + e^{i2\omega} = i\,(4\sin\omega + 2\sin 2\omega)$$

The Fourier transform is purely imaginary since the filter is anti-symmetric around zero. Graphically, the imaginary part looks as follows:



From the Taylor series $\sin(x) = x - \frac{x^3}{3!} + \mathcal{O}(x^5)$ is can be concluded that for small ω

$$\varphi(\omega) = 8i\omega + \mathcal{O}(\omega^3).$$

The linear behaviour around the origin suggests that the filter acts like a derivative for low frequencies, but since it goes towards zero when $\omega \to \pm\pi$, the filter suppresses frequencies close to the sampling rate $\omega = \pm\pi$. The filter thus acts like a combination of a derivative and a low-pass filter.