Exercises 2

1. Describe how to estimate the equation of a line that minimizes the sum of squared orthogonal distances for a set of points. Apply it on a following set of points:

$$p_1 = (-6, -2), p_2 = (-3, -1), p_3 = (0, 0), p_4 = (1, 1), p_5 = (3, 2)$$

Answer:

First we normalize the coordinates with respect to the mean position.

$$\bar{p} = \frac{1}{N} \sum p_i = \frac{1}{5}(-6 - 3 + 0 + 1 + 3, -2 - 1 + 0 + 1 + 2) = (-1, 0)$$

$$p'_i = p_i - \bar{p} = (x'_i, y'_i)$$

$$p_1 = (-5, -2),\ p_2 = (-2, -1),\ p_3 = (1, 0),\ p_4 = (2, 1),\ p_5 = (4, 2)$$

Then we compute the elements of the covariance matrix.

$$C_{xx} = \frac{1}{N} \sum x_i'^2 = \frac{1}{5}(5^2 + 2^2 + 1^2 + 2^2 + 4^2) = \frac{1}{5}50$$

$$C_{xy} = \frac{1}{N} \sum x'_i y'_i = \frac{1}{5}(5 \cdot 2 + 2 \cdot 1 + 1 \cdot 0 + 2 \cdot 1 + 4 \cdot 2) = \frac{1}{5}22$$

$$C_{yy} = \frac{1}{N} \sum y_i'^2 = \frac{1}{5}(2^2 + 1^2 + 0^2 + 1^2 + 2^2) = \frac{1}{5}10$$

We can ignore the $N = 5$, since we are only interested in the normal direction given by the eigenvector corresponding to the smallest eigenvalue of

$$C = \begin{pmatrix} C_{xx} & C_{xy} \\ C_{xy} & C_{yy} \end{pmatrix}$$

$$|\lambda I - 5C| = \begin{vmatrix} \lambda - 50 & -22 \\ -22 & \lambda - 10 \end{vmatrix} = \lambda^2 - 60\lambda + 16 = 0$$

The smallest eigenvector is thus

$$\lambda = 30 - \sqrt{30^2 - 16} = 30 - \sqrt{884} \approx 0.26786$$

and the corresponding eigenvector

$$u = (30 - \sqrt{884} - 10, 22) = (20 - \sqrt{884}, 22)$$

The mean $\bar{p} = (-1, 0)$ has to be on this line. This the line is given by

$$(20 - \sqrt{884}) \cdot x + 22 \cdot y + (20 - \sqrt{884}) = 0.$$

2. Instead of minimizing the sum orthogonal errors, fit the points in previous exercise to a line $y = kx + L$ by minimizing the sum of squared errors in y.

   Answer: We set up a system of equations

$$A \begin{pmatrix} k \\ L \end{pmatrix} = b$$

   where

$$A = \begin{pmatrix} -6 & 1 \\ -3 & 1 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

   and solve for k and L.

$$\begin{pmatrix} k \\ L \end{pmatrix} = (A^T A)^{-1}(A^T b) = \begin{pmatrix} 55 & -5 \\ -5 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 22 \\ 0 \end{pmatrix} = \frac{1}{250} \begin{pmatrix} 5 & 5 \\ 5 & 55 \end{pmatrix} \begin{pmatrix} 22 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.44 \\ 0.44 \end{pmatrix}$$

3. An image has been smoothed with the following kernel:

$$h = k \cdot [1, 5, 10, 10, 5, 1]$$

   Can repeated convolutions of an image with the kernel

$$g = \frac{1}{2}[1, 1]$$

   be used to obtain the same result as with the first kernel? If yes, how many convolutions are needed? If no, explain the reasons why.

   What should the constant k be so that the filter gain is equal to 1?

   Answer: We see that $g * g = \frac{1}{4}[1, 2, 1]$, thus $g * g * g * g = \frac{1}{16}[1, 4, 6, 4, 1]$ and $g_*^5 = \frac{1}{32}[1, 5, 10, 10, 5, 1]$. Therefore, if $k = \frac{1}{32}$ we have $h = g_*^5$, i.e. five convolutions with the g kernel yields the h kernel for the given k-value.

4. (a) Show that the differential kernel $d_x^1 = \frac{1}{2}[1, 0, -1]$ is preferable from $d_x^2 = [1, -1]$ as an approximation of the first order derivative.

   Answer: One way to show this is by means of Taylor expansion

$$f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + R_3(x),$$

   where the rest $R_3(x) = \frac{1}{3!}f^{(3)}(\psi)h^3$, $\exists \psi \in [x, x + h]$ depends on the third order derivative. With the first kernel we get

$$d_x^1 * f(x) = \frac{1}{2}(f(x + 1) - f(x - 1)) =$$

$$= \frac{1}{2}((f(x) + f'(x) + \frac{f''(x)}{2!}) - (f(x) - f'(x) + \frac{f''(x)}{2!}) + R_3(x) =$$

$$= f'(x) + R_3(x),$$

while first the second kernel we get

$$d_x^2 * f(x) = f(x+1) - f(x) = f(x) + f'(x) + \frac{f''(x)}{2!} + R_3(x) - f(x) =$$

$$= f'(x) + \frac{f''(x)}{2!} + R_3(x).$$

Thus in the first case the error depends only on the third order derivative, while the second case also depends on the second order derivative.

5. A ball is moving with constant velocity straight towards a camera along the optical axis. At time $t_0 = 0$ it covers 500 pixels, and at time $t_1 = 3$ it covers 750 pixels. At what time does it cover 1000 pixels? (The camera is assumed to be of pinhole type.)

   Answer: A ball moving towards a pinhole camera will be projected as a circle on the image plane. We make the assumption that the entire intersection is visible in the image (see figure). At t_0 the ball covers 500 pixels, why the radius $r_0 = \sqrt{500/\pi}$. Similarly, at t_1, $r_1 = \sqrt{750/\pi} = \sqrt{1.5}r_0$ and at $t_2 = \sqrt{1000/\pi} = \sqrt{2}r_0$. Furthermore at t_i the ball is at distance Z_i. Let the ball have radius R. From the figure we see that

$$\frac{r_i}{f} = \frac{R}{Z_i} \Leftrightarrow r_i Z_i = Rf$$

   We can now write the equalities

$$r_0 Z_0 = r_1 Z_1 = \sqrt{1.5} r_0 Z_1 \Leftrightarrow Z_0 = \sqrt{1.5} Z_1 \Leftrightarrow Z_1 = \frac{Z_0}{\sqrt{1.5}}$$
$$r_0 Z_0 = r_2 Z_2 = \sqrt{2} r_0 Z_2 \Leftrightarrow Z_0 = \sqrt{2} Z_2 \Leftrightarrow Z_2 = \frac{Z_0}{\sqrt{2}}$$

   We know that the ball moves the distance $Z_0 - \frac{Z_0}{\sqrt{1.5}}$ in three time steps, so the speed $v = \frac{\sqrt{1.5}Z_0 - Z_0}{3\sqrt{1.5}}$
   Finally we can compute time t_2

$$t_2 = \frac{Z_0 - Z_2}{v} = \frac{Z_0 - \frac{Z_0}{\sqrt{2}}}{\frac{\sqrt{1.5}Z_0 - Z_0}{3\sqrt{1.5}}} = \frac{1 - \frac{1}{\sqrt{2}}}{\frac{\sqrt{1.5}-1}{3\sqrt{1.5}}} = 3\frac{\sqrt{3} - \sqrt{1.5}}{\sqrt{3} - \sqrt{2}}$$

6. You are given the following binary image:

   Assuming that $x = 1$ for the first column, compute the following:

   - Moments: $m_{00},\ m_{10},\ m_{01}$ and m_{20}
   - Centers of gravity: x_0 and y_0
   - Central moments: $\mu_{00},\ \mu_{01}$ and μ_{02}

$$\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1
\end{array}$$

Answer: Image moments for an image I are defined as:

$$m_{ij} = \sum_y \sum_x I(x,y) x^i y^j$$

Hence we get:

$$m_{00} = 17,\ m_{10} = 58,\ m_{01} = 53,\ m_{20} = 248$$

Centers of gravity, x_0 and y_0 are means in x- and y-directions:

$$x_0 = \frac{m_{10}}{m_{00}} \approx 3.41$$
$$y_0 = \frac{m_{01}}{m_{00}} \approx 3.12$$

Finally central moments are defined as

$$\mu_{ij} = \sum_y \sum_x I(x,y)(x - x_0)^i (y - y_0)^j$$

So we get:

$$\mu_{00} = 17,\ \mu_{01} = 0,\ \mu_{02} \approx 63.76$$