1.3 Funcion $\zeta(s)$ de Riemann

1) Integrar analiticamente la serie de fourier de $f(t) = t^2$ en el intervalo $-\pi \leq t \leq \pi$ y $\underbrace{f(t+2\pi) = f(t)}_{\text{Periodica.}}$ $T = 2\pi$

$A_{max} = \pi^2$

1) Calculamos la serie de fourier de $f(t) = t^2$

$$\Rightarrow f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \operatorname{sen}(n\omega_0 t)$$

Debemos encontrar los coeficientes a_0, a_n, b_n.

- $$\frac{1}{2} a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{2\pi}\left(\frac{t^3}{3}\right)\Bigg|_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\frac{\pi^3}{3} + \frac{\pi^3}{3}\right)$$

$$= \frac{\cancel{2}\pi^{\cancel{3}}}{3\,\cancel{2\pi}} = \frac{\pi^2}{3} = a_0$$

- $$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt = \frac{2}{2\pi}\int_{-\pi}^{\pi} t^2\cos(n\omega_0 t)\,dt$$, desarrollo por

partes: $\int u\,dv = uv - \int v\,du$ donde $u = t^2$, $dv = \cos(nt)\,dt$; $du = 2t$, $v = \frac{1}{n}\operatorname{sen}(nt)$

$$\Rightarrow \frac{1}{\pi}\left[t^2 \frac{1}{n}\operatorname{sen}(n\cdot t) - 2\frac{1}{n}\int_{-\pi}^{\pi}\operatorname{sen}(nt)\,t\,dt.\right.$$, nuevamente partes

$\int u'\,dv' = u'v' - \int v'\,du'$ donde $u' = t$, $dv' = \operatorname{sen}(nt)\,dt$; $du' = dt$, $v' = -\frac{1}{n}\cos(nt)$

$$\Rightarrow \frac{1}{n\pi}\left[t^2\operatorname{sen}(nt) - 2\int_{-\pi}^{\pi} t\operatorname{sen}(nt)\,dt\right.$$

$$\Rightarrow \frac{1}{n\pi}\left[t^2\operatorname{sen}(nt) - 2\left[-t\frac{1}{n}\cos(nt) - \left(-\frac{1}{n}\int_{-\pi}^{\pi}\cos(nt)\,dt\right)\right]\right]$$

$$= \frac{1}{n\pi}\left\{t^2\operatorname{sen}(nt) - 2\frac{1}{n}\left[-t\cos(nt) + \frac{1}{n}\operatorname{sen}(nt)\right]\right]\Bigg|_{-\pi}^{\pi}$$

$$= \frac{2}{n^2 \pi}\left[\pi \cos(n\pi) - (-\pi)\cos(n \cdot \pi)\right]$$

$$= \frac{2}{n^2 \pi}\left[2\pi \cos(n\pi)\right] = \frac{4}{n^2}\cos(n\pi) \quad ; \quad \cos(n\pi) = (-1)^n$$

$$\Rightarrow a_n = \frac{4}{n^2}(-1)^n.$$

• Ahora $b_n = \frac{2}{T}\int_0^T f(t)\, \text{Sen}(nw_0 t)\, dt$

considendarando $f(t)_{impar} * f(t)_{par} = f(t)_{impar}$

$$\int f(t)_{impar}\, dt = 0.$$

$$b_n = \frac{\cancel{2}}{\cancel{2}\pi}\int_{-\pi}^{\pi} t^2 \,\text{Sen}(nw_0 t)\, dt = 0.$$

$$\Rightarrow f(t) = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nt)$$

Ahora :

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos(nt)}{n^2}$$

$$t^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos(nt)}{n^2}$$

$$\frac{t^2}{4} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^2}{n^2}\cos(nt)$$

$$\frac{1}{12}t(t^2 - \pi^2) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\text{sen}(nt)$$

Considerando la identidad de Parseval.

$$\frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{4}\left(\frac{\pi^2}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\left(\frac{4}{n^2}(-1)^n\right)^2 + b_n^2\right]$$

$$\frac{1}{2\pi}\cdot\frac{t^3}{3}\Big|_{-\pi}^{\pi} = \frac{1}{4}\left(\frac{\pi^4}{9}\right) + \frac{1}{2}\cdot 4\sum_{n=1}^{\infty}\left(\frac{(-1)^{2n}}{n^2} + b_n^2\right]$$

Como t^3 es impar $b_n = \frac{(-1)^n}{n^3}$

Usando identidad de Parseval

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{12}t(t^2-\pi^2)\right)^2 dt = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^6}$$

Desarrollando la integral.

$$\frac{1}{\cancel{2}\pi}\int_{-\pi}^{\pi}\left[\frac{t^3}{12} - \frac{\pi^2 t}{12}\right]^2 dt = \frac{1}{\cancel{2}}\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^6}$$

$$\int_{-\pi}^{\pi}\frac{t^6}{12^2} - \frac{2\cdot t^3}{12}\cdot\frac{\pi^2 t}{12} + \frac{\pi^4 t^2}{12^2} = \pi\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^6}$$

$$\frac{t^7}{12^2\cdot 7} - \frac{2t^5\pi^2}{5\cdot 12^2} + \frac{\pi^4 t^3}{3\cdot 12^2}\Big|_{-\pi}^{\pi} = \pi\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^6}$$

$$\frac{\pi^7}{12^2\cdot 7} - \frac{2\pi^7}{5\cdot 12^2} + \frac{\pi^7}{3\cdot 12^2} + \frac{\pi^7}{12^2\cdot 7} - \frac{2\pi^7}{5\cdot 12^2} + \frac{\pi^7}{3\cdot 12^2} = \pi \sum_{n=1}^{\infty} \frac{1}{n^6}$$

Entonces

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$