Problem 1

(a)

$$Au = b$$

$$A\hat{u} = \hat{b}$$

$$A(u + \Delta u) = b + \Delta b$$

$$Au + A\Delta u = b + \Delta b$$

$$b + A\Delta u = b + \Delta b$$

$$A\Delta u = \Delta b$$

$$\Delta u = A^{-1}\Delta b$$

$$||\Delta u|| = ||A^{-1}\Delta b|| \leq ||A^{-1}|| \cdot ||\Delta b||$$

$$||u - \hat{u}|| \leq ||A^{-1}|| \cdot ||b - \hat{b}||$$

$$\frac{||u - \hat{u}||}{||u||} \leq \frac{||A^{-1}|| \cdot ||b - \hat{b}||}{||u||}$$

Since $||b|| = ||Au|| \leq ||A|| \cdot ||u||$, $||u|| \geq \frac{||b||}{||A||}$

$$\frac{||u - \hat{u}||}{||u||} \leq \frac{||A|| \cdot ||A^{-1}|| \cdot ||b - \hat{b}||}{||b||}$$

$$\boxed{\frac{||u - \hat{u}||}{||u||} \leq \kappa(A) \cdot \frac{||b - \hat{b}||}{||b||}}$$

(b)

$$A^{-1}r = A^{-1}Ae$$

$$e = A^{-1}r$$

$$||e|| = ||A^{-1}r|| \leq ||A^{-1}|| \cdot ||r||$$

$$\frac{||e||}{||u||} \leq \frac{||A^{-1}|| \cdot ||r||}{||u||}$$

Since $||b|| = ||Au|| \leq ||A|| \cdot ||u||$ so $\frac{1}{||u||} \leq \frac{||A||}{||b||}$:

$$\frac{||e||}{||u||} \leq ||A^{-1}|| \cdot ||r|| \cdot \frac{||A||}{||b||}$$

$$\frac{||e||}{||u||} \leq ||A||\,||A^{-1}|| \cdot \frac{||r||}{||b||}$$

$$\frac{||e||}{||u||} \leq \kappa(A) \cdot \frac{||r||}{||b||}$$

From $r = Ae$:

$$||r|| = ||Ae|| \leq ||A|| \cdot ||e||$$

$$||e|| \geq \frac{||r||}{||A||}$$

$$\frac{||e||}{||u||} \geq \frac{||r||}{||A||} \cdot \frac{1}{||u||}$$

Since $u = A^{-1}b$, $||u|| = ||A^{-1}b|| \leq ||A^{-1}|| \cdot ||b||$ so $\frac{1}{||u||} \geq \frac{1}{||A^{-1}|| \cdot ||b||}$:

$$\frac{||e||}{||u||} \geq \frac{||r||}{||A||} \cdot \frac{1}{||A^{-1}|| \cdot ||b||}$$

$$\frac{||e||}{||u||} \geq \frac{1}{||A||\,||A^{-1}||} \cdot \frac{||r||}{||b||}$$

$$\frac{||e||}{||u||} \geq \frac{1}{\kappa(A)} \cdot \frac{||r||}{||b||}$$

Thus,

$$\boxed{\frac{1}{\kappa(A)} \frac{||r||}{||b||} \leq \frac{||e||}{||u||} \leq \kappa(A) \frac{||r||}{||b||}}$$

Interpretation: a smaller $k(A)$ (i.e. well conditioned matrix) means the relationship between error e and residual r is tight. For a higher $k(A)$, the relationship is less specific and a small r can mean a large e.

Problem 2

(a)

Let
$(v_j)_i = \sin(j\pi x_i) = \sin\left(\frac{j\pi i}{m}\right), \qquad i = 1, \ldots, m-1.$

$$(Av_j)_i = \frac{1}{h^2}(-(v_j)_{i-1} + 2(v_j)_i - (v_j)_{i+1})$$

$$(Av_j)_i = \frac{1}{h^2}(-\sin(\frac{j\pi(i-1)}{m}) + 2\sin(\frac{j\pi i}{m}) - \sin(\frac{j\pi(i+1)}{m})).$$

Since $\sin(\alpha+\theta) + \sin(\alpha-\theta) = 2\sin(\alpha)\cos(\theta)$, with $\alpha = \frac{j\pi i}{m}$ and $\theta = \frac{j\pi}{m}$:

$$\sin\left(\frac{j\pi(i+1)}{m}\right) + \sin\left(\frac{j\pi(i-1)}{m}\right) = 2\sin\left(\frac{j\pi i}{m}\right)\cos\left(\frac{j\pi}{m}\right).$$

$$(Av_j)_i = \frac{1}{h^2}\left(2 - 2\cos\left(\frac{j\pi}{m}\right)\right)\sin\left(\frac{j\pi i}{m}\right).$$

Thus

$$Av_j = \lambda_j v_j,$$

so vectors $\boxed{(v_j)_i = \sin(j\pi x_i)}$ are eigenvectors of A.

(b)

From (a) we know that $\lambda_j = \frac{1}{h^2}\left(2 - 2\cos\left(\frac{j\pi}{m}\right)\right)$.

Using $2 - 2\cos\theta = 4\sin^2(\theta/2)$,

$$\boxed{\lambda_j = \frac{4}{h^2}\sin^2\left(\frac{j\pi}{2m}\right), \qquad j = 1, \ldots, m-1.}$$

(c)

Since A is symmetric positive definite,

$$\kappa(A) = \frac{\lambda_{\max}}{\lambda_{\min}}.$$

$$\lambda_{\min} = \lambda_1 = \frac{4}{h^2}\sin^2\left(\frac{\pi}{2m}\right).$$

$$\lambda_{\max} = \lambda_{m-1} = \frac{4}{h^2}\sin^2\left(\frac{(m-1)\pi}{2m}\right) = \frac{4}{h^2}\cos^2\left(\frac{\pi}{2m}\right).$$

So:

$$\kappa(A) = \frac{\cos^2\left(\frac{\pi}{2m}\right)}{\sin^2\left(\frac{\pi}{2m}\right)} = \cot^2\left(\frac{\pi}{2m}\right).$$

As $m \to \infty$, $\frac{\pi}{2m} \to 0$ and $\sin\frac{\pi}{2m} \sim \frac{\pi}{2m} \sim 0$, $\cos\frac{\pi}{2m} \sim 1$:

$$\kappa(A) = \cot^2(\frac{\pi}{2m}) \sim \left(\frac{1}{\frac{\pi}{2m}}\right)^2 = \left(\frac{2m}{\pi}\right)^2.$$

Thus,

$$\boxed{\kappa(A) = O(m^2) \quad \text{as } m \to \infty.}$$

Problem 3

$$u(x) = sin(k\pi x) + c(x - \frac{1}{2})^3$$

$$u'(x) = k\pi \cdot cos(k\pi x) + 3c(x - \frac{1}{2})^2$$

$$u''(x) = -k^2\pi^2 \cdot sin(k\pi x) + 6c(x - \frac{1}{2})$$

$$-u''(x) + \gamma u(x) = f(x)$$

$$f(x) = -\left(-k^2\pi^2 \sin(k\pi x) + 6c\left(x - \frac{1}{2}\right)\right) + \gamma\left(\sin(k\pi x) + c\left(x - \frac{1}{2}\right)^3\right)$$

$$f(x) = k^2\pi^2 \sin(k\pi x) - 6c\left(x - \frac{1}{2}\right) + \gamma \sin(k\pi x) + \gamma c\left(x - \frac{1}{2}\right)^3$$

$$f(x) = \left(k^2\pi^2 + \gamma\right)\sin(k\pi x) - 6c\left(x - \frac{1}{2}\right) + \gamma c\left(x - \frac{1}{2}\right)^3$$

$$\boxed{f(x) = \left(k^2\pi^2 + \gamma\right)\sin(k\pi x) - 6c\left(x - \frac{1}{2}\right) + \gamma c\left(x - \frac{1}{2}\right)^3}$$

Problem 4

Note: in the results below, iteration 0 is not counted.
(a) 1000+. Did not meet required tolerances after 1000 iterations.
(b) 102 iterations.
(c) 1 iteration. This makes sense because without the cubic factor, the residual is enough for CG to descent in one iteration, because the RHS lies entirely in the span of one eigenvector of A.

(d) 1 iteration.

(e) 1 iteration.

(f) 1 iteration for both 1 and 2 processors.