# Your grades for **Winter 2016 CS 370 Midterm** Total Score: **30.0**

lmcase@uwaterloo.ca's assessments

**Summary**

## Class scores distribution

Total (/score/062db6c6-f7c3-4310-96d5-7fead9203972)

Q1 (/score/062db6c6-f7c3-4310-96d5-7fead9203972/Q1)

Q2 (/score/062db6c6-f7c3-4310-96d5-7fead9203972/Q2)

Q3 (/score/062db6c6-f7c3-4310-96d5-7fead9203972/Q3)

Q4 (/score/062db6c6-f7c3-4310-96d5-7fead9203972/Q4)

Q5 (/score/062db6c6-f7c3-4310-96d5-7fead9203972/Q5)

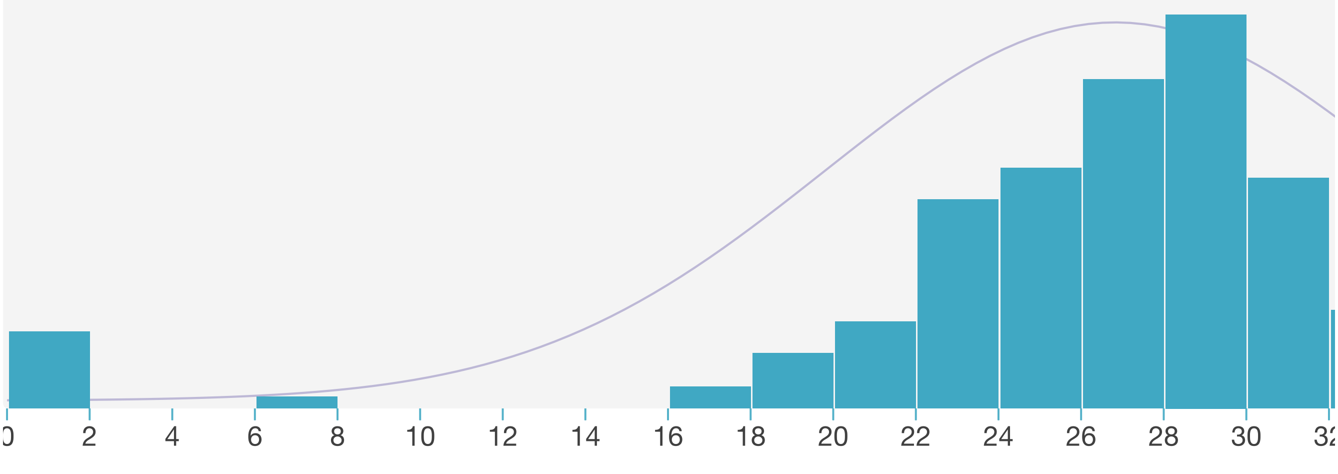
Q6 (/score/062db6c6-f7c3-4310-96d5-7fead9203972/Q6)

Q7 (/score/062db6c6-f7c3-4310-96d5-7fead9203972/Q7)

Q8 (/score/062db6c6-f7c3-4310-96d5-7fead9203972/Q8)

Q9 (/score/062db6c6-f7c3-4310-96d5-7fead9203972/Q9)

Students: 160 Median: 27 Mean: 25.25 Std. Dev: 6.625



Q1 Score: **4.0**



1. (4 marks) Consider the **normalized** floating-point number system $\mathcal{F}$ defined by $(t, \beta, L, U) = (4, 10, -3, 3)$ containing numbers of the form

$$[\pm 0.d_1 d_2 d_3 d_4]_\beta \times \beta^p \ ,$$

where $d_i \in \{0, 1, \ldots, 9\}$, $p \in \{-3, -2, -1, 0, 1, 2, 3\}$ and $d_1 \neq 0$ (except when all $d_i = 0$, representing a value of 0). Values not in $\mathcal{F}$, but within the exponent range, are rounded to the nearest floating-point number in $\mathcal{F}$.

(a) (1 mark) Give the size of machine epsilon.

$E = \frac{\beta}{2} \times \beta^{-t} = \frac{10}{2} \times 10^{-4} = 5 \times 10^{-4}$

(b) (1 mark) What is the smallest positive number in $\mathcal{F}$?

$0.1000 \times 10^{-3}$

(c) (1 mark) What number in $\mathcal{F}$ is $fl(0.00328173003)$?

$fl(0.00328173003) = 0.003282 = 0.3282 \times 10^{-2}$

(d) (1 mark) What would be the result of evaluating $(800 + 950)$ in $\mathcal{F}$?

$fl(800 + 950) = fl(1750) =$ overflow since largest element is $0.9999 \times 10^3$ which is less than result.

Q2 Score: **4.0**



2. (4 marks) Suppose you are a computer that performs arithmetic in the floating-point number system $\mathcal{F}(t, \beta, L, U) = \mathcal{F}(3, 10, -5, 5)$. Consider the expression,

$$(1320 + 23) - 1350$$

(a) (1 mark) Evaluate the expression using the floating-point number system $\mathcal{F}(3, 10, -5, 5)$. Values within the exponent range but not in the number system should be rounded to the nearest element in $\mathcal{F}(3, 10, -5, 5)$.

$fl(1320 + 23) = fl(1343) = 0.134 \times 10^4 = 1340$

$fl(1340 - 1350) = fl(-10) = -0.1 \times 10^2 = -10$

(b) (1 mark) Calculate the relative error in the above computation.

$(1320 + 23) - 1350 = -7$

$\frac{|x - \bar{x}|}{|x|} = \frac{|-7 - (-10)|}{|-7|} = \frac{3}{7} = 0.429$

(c) (2 marks) How could you compute the expression in (a) differently to get a lower relative error? Evaluate the relative error for your method to show that it is smaller.

Take $(1320 - 1350)$ first then add 23 to have all operations to be operands of similar magnitude

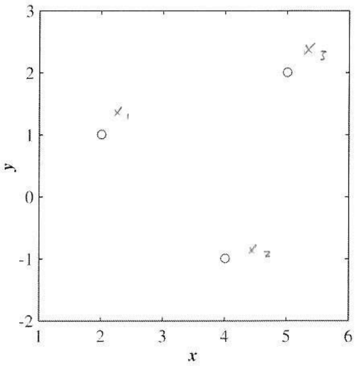
$fl(1320 - 1350) = fl(-30) = -30$

$fl(-30 + 23) = fl(-7) = -7$

$\frac{|x - \bar{x}|}{|x|} = \frac{|-7 - (-7)|}{|-7|} = 0 < \frac{3}{7}$

3. (6 marks) Suppose you are interested in fitting a quadratic polynomial to three points.

(a) (3 marks) The figure shows three interpolation points. Write down the Lagrange form of the polynomial that interpolates those points. You do not need to simplify the polynomial.




$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-4)(x-5)}{(2-4)(2-5)}$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-2)(x-5)}{(4-2)(4-5)}$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-2)(x-4)}{(5-2)(5-4)}$$

$$p(x) = L_1(x) + (-1)L_2(x) + (2)L_3(x)$$

$$L_1 = \frac{(x-4)(x-5)}{6} \quad L_2(x) = \frac{(x-2)(x-5)}{-2} \quad L_3(x) = \frac{(x-2)(x-4)}{3}$$

(b) (3 marks) Suppose you want to fit the points from part (a) with a single polynomial of the form $p(x) = A + Bx + Cx^2$. Write down the matrix equation (involving the Vandermonde matrix) that you would solve to find values for $A$, $B$ and $C$. You do not need to solve the system.

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Q4 Score: **5.0**



4. (5 marks) We wish to interpolate the 3 points $(-1, 2)$, $(0, 1)$ , and $(2, 2)$ with a piecewise quadratic polynomial of the form,

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0 x + c_0 x^2 & \text{for } -1 \le x < 0 \\ S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 & \text{for } \quad 0 \le x \le 2 \end{cases}$$

(a) (4 marks) Write down the equations that enforce that $S(x)$ is an interpolator of the points $(-1, 2)$, $(0, 1)$, and $(2, 2)$, and that $S(x)$ is continuous and smooth (ie. continuous first derivative). The **only unknowns** in your equations should be $a_0$, $b_0$, $c_0$, $a_1$, $b_1$, and $c_1$.

$S_0(-1) = 2$ $\quad S_0(0) = 1$ $\quad S_1(0) = 1$

$\boxed{a_0 - b_0 + c_0 = 2}$ $\quad \boxed{a_0 = 1}$ $\quad \boxed{a_1 - 2b_1 + 4c_1 = 1}$

$S_1(2) = 2$

$\boxed{a_1 = 2}$

$$S'(x) = \begin{cases} S_0'(x) = b_0 + 2c_0 x & -1 \le x < 0 \\ S_1'(x) = b_1 + 2c_1(x-2) & 0 \le x \le 2 \end{cases}$$

$S_0'(0) = S_1'(0)$

$\boxed{b_0 = b_1 - 4c_1}$

(b) (1 mark) Suppose you want to make $S(x)$ periodic and smooth (so that the periodic function is differentiable at all knot points). What additional constraint would accomplish this? Again, state the constraint as an equation involving only $a_0$, $b_0$, $c_0$, $a_1$, $b_1$, and $c_1$.

$S_0'(-1) = S_1'(2)$

$\boxed{b_0 - 2c_0 = b_1}$

Q5 Score: **2.0**

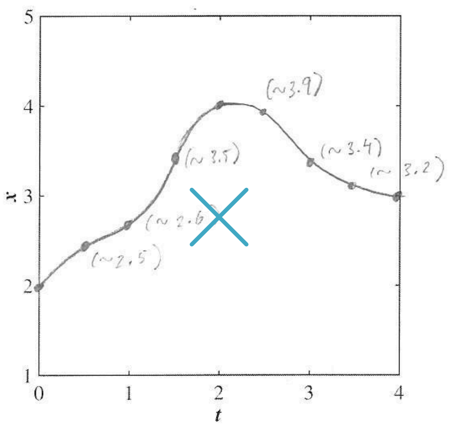


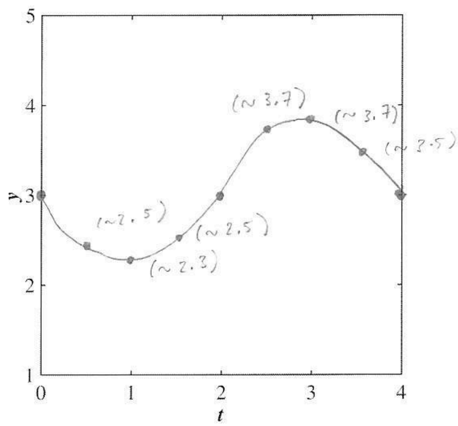
5. (4 marks) Consider the following table of data.

| | | | |
|---|---|---|---|
| $k$ | 1 | 2 | 3 |
| $t$ | 0 | 2 | 4 |
| $x$ | 2 | 4 | 3 |
| $y$ | 3 | 3 | 3 |

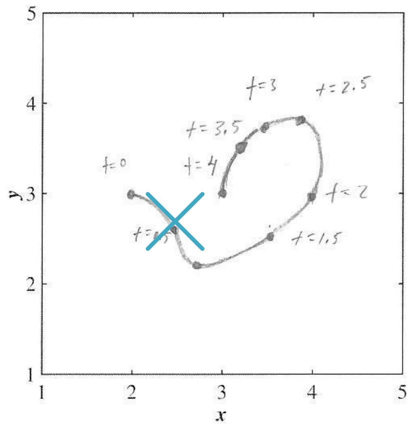
(a) (3 marks) On the axes below, sketch the cubic spline interpolating curves for $(t, x)$ and $(t, y)$ using your best guess at the shape. Assume clamped boundary conditions, where

$$\frac{dx}{dt}(0) = 1 \qquad \frac{dx}{dt}(4) = 0 \qquad \frac{dy}{dt}(0) = -1 \qquad \frac{dy}{dt}(4) = -1$$




(b) (1 mark) On the axis below, sketch the cubic-spline parametric curve $(x(t), y(t))$ using the interpolating splines from part (a).



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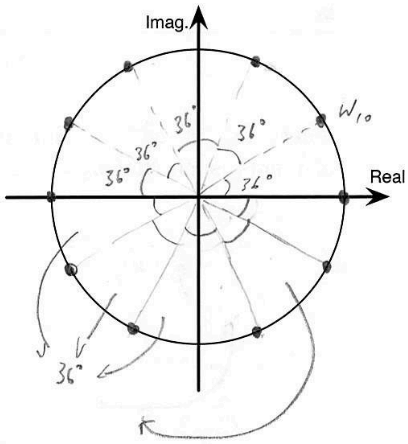
Q6 Score: **5.0**



6. (5 marks) The following questions deal with complex numbers as used in Fourier analysis.

(a) (2 marks) The figure shows a unit circle in the complex plane. Place a dot on each complex number that is a $10^{\text{th}}$-root of one, and **label** $W_{10}$.




(b) (3 marks) Given that $W_{10} = e^{i2\pi/10}$, prove that

$$\sum_{n=0}^{9} W_{10}^n = 0 \,.$$

$$\sum_{n=0}^{9} W_{10}^n = \frac{1-(W_{10})^{10}}{1-(W_{10})} = \frac{1-e^{\frac{i2\pi}{10}\cancel{10}}}{1-W_{10}} = \frac{1-1}{1-W_{10}} = 0$$

$W_{10} \neq 1$

$$S = \sum_{n=0}^{9} x^n = 1 + x + x^2 \ldots + x^9$$

$$Sx = x + x^2 + x^3 \ldots + x^{10}$$

$$S - Sx = S(1-x) = 1 - x^{10}$$

$$\Rightarrow S = \frac{1-x^{10}}{1-x}$$

$$\Rightarrow \sum_{n=0}^{9} x^n = \frac{1-x^{10}}{1-x}$$

Q7 Score: **4.0**



7. (4 marks) Let $f$ be an infinite periodic sequence such that $f_n = f_{n+N}$. Note that the DFT of $f$ is

$$F_k = \sum_{n=0}^{N-1} f_n \overline{W}_N^{nk} \quad \text{for} \quad k = 0, 1, \ldots, N-1$$

where $W_N = e^{2\pi i/N}$. Suppose we create a new sequence $g$ by shifting the sequence $f$ by $s$ elements, such that $g_n = f_{n-s}$. Let $G_k$ be the DFT of $g$. Show that

$$G_k = \overline{W}_N^{sk} F_k \quad \text{for} \quad k = 0, 1, \ldots, N-1 \, .$$

$$G_k = \sum_{n=0}^{N-1} g_n \overline{W}_N^{nk}$$

$$= \sum_{n=0}^{N-1} f_{n-s} \overline{W}_N^{nk}$$

$m = n - s \implies n = m + s$ $\quad$ $n = N-1$, $m+s = N-1$, $m = N-1-s$

$$= \sum_{m=-s}^{N-1-s} f_m \overline{W}_N^{(m+s)k}$$

$$= \sum_{m=-s}^{N-1-s} \left( f_m \overline{W}_N^{mk} \left( \overline{W}_N^{ks} \right) \right)$$

$$= \overline{W}_N^{ks} \sum_{m=-s}^{N-1-s} f_m \overline{W}_N^{mk}$$

$$= \overline{W}_N^{ks} \sum_{m=0}^{N-1} f_m \overline{W}_N^{mk}$$ — since $f_m$ is periodic we can rearrange the sum.

$$= \overline{W}_N^{sk} F_k$$

Q8 Not graded



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Clearly state the question number being answered and refer the marker to this page

Q9 Not graded

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