Chapter 5.1

4 Which row exchanges show that these "reverse identity matrices" J_3 and J_4 have $|J_3| = -1$ but $|J_4| = +1$?

$$\det\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} = -1 \quad \text{but} \quad \det\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} = +1.$$

Exchange row 1 and 3 to show $|J_3| = -1$.

Exchange row 1 and 4, then 2 and 3 to show $|J_4| = +1$.

16 Find the determinants of a rank one matrix and a skew-symmetric matrix:

$$A = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & -4 & 5\end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix}0&1&3\\-1&0&4\\-3&-4&0\end{bmatrix}.$$

$$A = \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1 & -4 & 5\end{bmatrix} = \begin{bmatrix}1&-4&5\\2&-8&10\\3&-12&15\end{bmatrix}$$ $\times -4$, $\times 5$: linearly dependent $\Rightarrow \det(A) = 0$.

$n \times n$ matrix

$\det(cA) = c^n \det(A)$

$$K = \begin{bmatrix}0&1&3\\-1&0&4\\-3&-4&0\end{bmatrix} \quad K^T = -K.$$

$\det(K^T) = \det(K) = \det(-K) = (-1)^3\det(K)$

$\det(K) + \det(K) = 0 \quad \therefore \det(K) = 0$

$-(-(-12)) + 3(4)$

$= 0$

$\therefore \det(K) = 0.$

22 From $ad - bc$, find the determinants of A and A^{-1} and $A - \lambda I$:

$$A = \begin{bmatrix}2&1\\1&2\end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{3}\begin{bmatrix}2&-1\\-1&2\end{bmatrix} \quad \text{and} \quad A - \lambda I = \begin{bmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{bmatrix}.$$

Which two numbers λ lead to $\det(A - \lambda I) = 0$? Write down the matrix $A - \lambda I$ for each of those numbers λ—it should not be invertible.

$\det(A^{-1}) = 1/\det(A)$

$\det(A) = 3$. $\det(A^{-1}) = \begin{bmatrix}\frac{2}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3}\end{bmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$

$\det(A - \lambda I) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3. \Rightarrow \lambda = 1$ or 3.

$$\begin{bmatrix}1&1\\1&1\end{bmatrix} \quad \begin{bmatrix}-1&1\\1&-1\end{bmatrix}$$

24 Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix}3&3&4\\6&8&7\\-3&5&-9\end{bmatrix} = \begin{bmatrix}1&0&0\\2&1&0\\-1&4&1\end{bmatrix}\begin{bmatrix}3&3&4\\0&2&-1\\0&0&-1\end{bmatrix} = LU.$$

Find the determinants of L, U, A, $U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

$\det(A) = \underbrace{\det(L)}_{=1}\det(U) = \det(U) = -6.$

$\det(U^{-1}L^{-1}) = \det((LU)^{-1}) = -\frac{1}{6}$, $\det((LU)^{-1}A) = 1.$

Chapter 5.2

4 Find two ways to choose nonzeros from four different rows and columns:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix} \quad (B \text{ has the same zeros as } A).$$

Is det A equal to $1+1$ or $1-1$ or $-1-1$? What is det B?

$a_{11}a_{23}a_{32}a_{41} \Rightarrow -1. \qquad a_{14}a_{23}a_{32}a_{41} \rightarrow 1. \Rightarrow \det A = 1-1=0.$

$\det(B) = 64-16 = 48.$

12 Find the cofactor matrix C and multiply A times C^T. Compare AC^T with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

$C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad AC^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{4}C^T = \frac{C^T}{\det(A)}$

24 With block multiplication, $A = LU$ has $A_k = L_k U_k$ in the top left corner:

$$A = \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} = \begin{bmatrix} L_k & 0 \\ * & * \end{bmatrix}\begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}.$$

(a) Suppose the first three pivots of A are $2, 3, -1$. What are the determinants of L_1, L_2, L_3 (with diagonal 1's) and U_1, U_2, U_3 and A_1, A_2, A_3?

(b) If A_1, A_2, A_3 have determinants 5, 6, 7 find the three pivots from equation (3).

(a) $\det(L_1) = \det(L_2) = \det(L_3) = 1.$ $\det(U_1) = \det(A_1) = 2$, $\det(U_2) = \det(A_2) = 6$ $\det(U_3) = \det(A_3) = -6$

(b) $\det A_{k-1} \times d_k = \det A_k. \rightarrow 5, \frac{6}{5}, \frac{7}{6}$

Chapter 5.3

8 Find the cofactors of A and multiply AC^{T} to find det A:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and } AC^{\mathrm{T}} = \_\_\_\_.$$

If you change that 4 to 100, why is det A unchanged?

$$C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix} \qquad AC^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\det(A) = 3$, The (1,3) cofactor of A is 0. $\Rightarrow$ no change.

14 L is lower triangular and S is symmetric. Assume they are invertible:

To invert triangular L symmetric S

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \qquad S = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

(a) Which three cofactors of L are zero? Then L^{-1} is also lower triangular.

(b) Which three pairs of cofactors of S are equal? Then S^{-1} is also symmetric.

(c) The cofactor matrix C of an orthogonal Q will be ____. *Why*?

(a) C_{21}, C_{31}, C_{32} $\qquad AA^T = A^TA = I \rightarrow A^{-1} = A^T.$

(b) $C_{12} = C_{21}, \; C_{31} = C_{13}, \; C_{32} = C_{23}$

(c)