6 This system of equations $Ax = b$ has *no solution* (they lead to $0 = 1$):

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9 \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector y in which subspace? Its dot product $y^{\mathrm{T}}b$ is 1, so no solution x.

8 In Figure 4.3, how do we know that Ax_r is equal to Ax? How do we know that this vector is in the column space? If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ what is x_r?

$x = \underbrace{x_r}_{\text{row space}} + \underbrace{x_n}_{\text{null space}} \Rightarrow \underbrace{Ax = Ax_r + Ax_n}_{\text{All } Ax \text{ are in } C(A).}$, $Ax_n = 0$

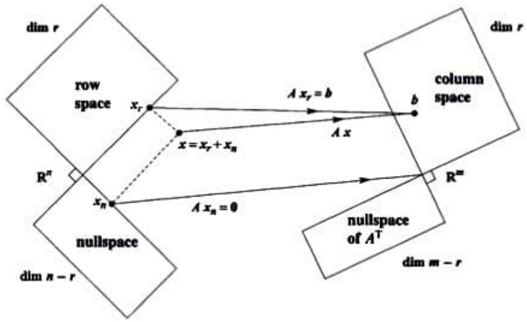

Figure 4.3: This update of Figure 4.2 shows the true action of A on $x = x_r + x_n$. Row space vector x_r to column space, nullspace vector x_n to zero.

14 The floor V and the wall W are not orthogonal subspaces, because they share a nonzero vector (along the line where they meet). No planes V and W in $\mathbf{R}^3$ can be orthogonal! Find a vector in the column spaces of both matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}$$

This will be a vector Ax and also $B\widehat{x}$. Think 3 by 4 with the matrix $[\,A \;\; B\,]$.

22 If $\boldsymbol{P}$ is the plane of vectors in $\mathbf{R}^4$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for $\boldsymbol{P}^{\perp}$. Construct a matrix that has $\boldsymbol{P}$ as its nullspace.

$(1,1,1,1)$ is a basis for $P^{\perp}$.

29 Find a matrix with $\boldsymbol{v} = (1, 2, 3)$ in the row space and column space. Find another matrix with $\boldsymbol{v}$ in the nullspace and column space. Which pairs of subspaces can $\boldsymbol{v}$ *not* be in?

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

row, colum space column, null space.

V can not be in null, row or left null, column.

11 Project b onto the column space of A by solving $A^TA\hat{x} = A^Tb$ and $p = A\hat{x}$:

(a) $A = \begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}$ and $b = \begin{bmatrix}2\\3\\4\end{bmatrix}$ (b) $A = \begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix}$ and $b = \begin{bmatrix}4\\4\\6\end{bmatrix}$.

Find $e = b - p$. It should be perpendicular to the columns of A.

$$A\hat{x} = \boxed{A(A^TA)^{-1}A^Tb} = Pb$$

Projection Matrix

12 Compute the projection matrices P_1 and P_2 onto the column spaces in Problem 11. Verify that P_1b gives the first projection p_1. Also verify $P_2^2 = P_2$.

(a)
$$\begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}\left(\begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}\begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}$$
$$=\begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1&1\\1&2\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}$$
$$=\begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}$$
$$=\begin{bmatrix}1&0\\-1&1\\0&0\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

(b)
$$\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix}\left(\begin{bmatrix}1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&0\\1&1&1\end{bmatrix}$$
$$=\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix}\left(\begin{bmatrix}2&2\\2&3\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&0\\1&1&1\end{bmatrix}$$
$$=\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix}\frac{1}{2}\begin{bmatrix}3&-2\\-2&2\end{bmatrix}\begin{bmatrix}1&1&0\\1&1&0\end{bmatrix}$$
$$=\frac{1}{2}\begin{bmatrix}1&0\\1&0\\-2&2\end{bmatrix}\begin{bmatrix}1&1&0\\1&1&0\end{bmatrix}$$
$$=\frac{1}{2}\begin{bmatrix}1&1&0\\1&1&0\\0&0&0\end{bmatrix}=\begin{bmatrix}0.5&0.5&0\\0.5&0.5&0\\0&0&0\end{bmatrix}$$

Questions 21–26 show that projection matrices satisfy $P^2 = P$ and $P^T = P$.

21 Multiply the matrix $P = A(A^TA)^{-1}A^T$ by itself. Cancel to prove that $P^2 = P$. Explain why $P(Pb)$ always equals Pb: The vector Pb is in the column space so its projection is ____.

22 Prove that $P = A(A^TA)^{-1}A^T$ is symmetric by computing P^T. Remember that the inverse of a symmetric matrix is symmetric.

$$P^T = (A(A^TA)^{-1}A^T)^T = A((A^TA)^{-1})^TA^T = A(A^TA)^{-1}A^T.$$

28 Use $P^T = P$ and $P^2 = P$ to prove that the length squared of column 2 always equals the diagonal entry P_{22}. This number is $\frac{2}{6} = \frac{4}{36} + \frac{4}{36} + \frac{4}{36}$ for

$$P = \frac{1}{6}\begin{bmatrix}5&\boxed{2}&-1\\2&\boxed{2}&2\\-1&\boxed{2}&5\end{bmatrix}.$$

$$P^2 = P = P^TP = \frac{1}{36}\begin{bmatrix}5&2&-1\\ \boxed{2\;\;2\;\;2}\\-1&2&5\end{bmatrix}\begin{bmatrix}5&\boxed{2}&-1\\2&\boxed{2}&2\\-1&\boxed{2}&5\end{bmatrix}$$
$$=\frac{1}{36}\begin{bmatrix}9&9&9\\8&8&8\end{bmatrix}$$

$2\times2+2\times2+2\times2 \Rightarrow \frac{4}{36}+\frac{4}{36}+\frac{4}{36}=\frac{2}{6}$

$\therefore$ length squared of column 2 always equals the diagonal entry P_{22}.

9 For the closest parabola $b = C + Dt + Et^2$ to the same four points, write down the unsolvable equations $Ax = b$ in three unknowns $x = (C, D, E)$. Set up the three normal equations $A^TA\hat{x} = A^Tb$ (solution not required). In Figure 4.9a you are now fitting a parabola to 4 points—what is happening in Figure 4.9b?

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}, \quad A^TA\hat{x} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

11 The average of the four times is $\hat{t} = \frac{1}{4}(0 + 1 + 3 + 4) = 2$. The average of the four b's is $\hat{b} = \frac{1}{4}(0 + 8 + 8 + 20) = 9$.

(a) Verify that the best line goes through the center point $(\hat{t}, \hat{b}) = (2, 9)$.

(b) Explain why $C + D\hat{t} = \hat{b}$ comes from the first equation in $A^TA\hat{x} = A^Tb$.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$$

$C=1,\ D=4 \Rightarrow 1+4t.$

a) $b = 9$ when $t=2$.

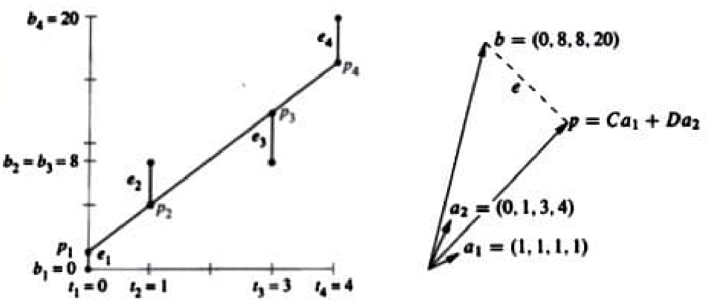

Figure 4.9: **Problems 1–11:** The closest line $C + Dt$ matches $Ca_1 + Da_2$ in $\mathbf{R}^4$.

b) A line goes through the m points when we exactly solve $Ax = b$. Generally we can't do it. Two unknowns C and D determine a line, so A has only $n = 2$ columns. To fit the m points, we are trying to solve m equations (and we only want two!):

$$Ax = b \quad \text{is} \quad \begin{matrix} C + Dt_1 = b_1 \\ C + Dt_2 = b_2 \\ \vdots \\ C + Dt_m = b_m \end{matrix} \quad \text{with} \quad A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}. \qquad (5)$$

$\rightarrow$

Fitting points by a straight line is so important that we give the two equations $A^TA\hat{x} = A^Tb$, once and for all. The two columns of A are independent (unless all times t_i are the same). So we turn to least squares and solve $A^TA\hat{x} = A^Tb$.

$$\textbf{Dot-product matrix} \quad A^TA = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix}\begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} = \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix}. \qquad (6)$$

$\rightarrow$

On the right side of the normal equation is the 2 by 1 vector A^Tb:

$$A^Tb = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_ib_i \end{bmatrix}.$$

17 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\hat{x} = (C, D)$ and draw the closest line.

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$$

$3C + 2D = 35$

$2C + 6D = 42$

$7C = 63,\ C = 9,\ D = 4$

$\therefore b = 9 + 4t.$

23 Find q_1, q_2, q_3 (orthonormal) as combinations of a, b, c (independent columns). Then write A as QR:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

$\lfloor \qquad \rfloor \lfloor x_3 \rfloor$

(a) Suppose the a's are orthonormal. Show that $x_1 = a_1^T b$.

(b) Suppose the a's are orthogonal. Show that $x_1 = a_1^T b / a_1^T a_1$.

(c) If the a's are independent, x_1 is the first component of _____ times b.

a) $b = x_1 a_1 + x_2 a_2 + x_3 a_3$, $a_1^T b = x_1 a_1^T a_1 + x_2 a_1^T a_2 + x_3 a_1^T a_3$
$= x_1(1) + x_2(0) + x_3(0) = x_1$

b) $a_1^T b = x_1 a_1^T a_1 + x_2 a_1^T a_2 + x_3 a_1^T a_3 = x_1 a_1^T a_1 \quad \therefore \quad x_1 = \dfrac{a_1^T b}{a_1^T a_1}$

c) Let $[a_1\ a_2\ a_3]$ be A. $\rightarrow A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$.

Since a's are independant and A is square (3×3), A is invertible.

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} b. \quad \therefore A^{-1}$.

15 (a) Find orthonormal vectors q_1, q_2, q_3 such that q_1, q_2 span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

(b) Which of the four fundamental subspaces contains q_3?

(c) Solve $Ax = (1, 2, 7)$ by least squares.

a) $A = (1, 2, -2)$

$q_1 = \frac{1}{3}(1, 2, -2)$

$B = b - \dfrac{A^T b}{A^T A} A = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \dfrac{-9}{9} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$q_2 = \frac{1}{3}(2, 1, 2)$

$q_3 = (\alpha, \beta, \gamma) \rightarrow q_3^T q_1 = 0$ and $q_3^T q_2 = 0$.

$\frac{1}{3}(\alpha + 2\beta - 2\gamma) = 0$ and $\frac{1}{3}(2\alpha + \beta + 2\gamma) = 0$

$3\alpha + 3\beta = 0 \rightarrow \beta = -\alpha, \ \gamma = -\frac{1}{2}\alpha.$

$q_3 = (\alpha, -\alpha, -\frac{1}{2}\alpha)$. $q_3^T q_3 = \alpha^2 + \alpha^2 + \frac{1}{4}\alpha^2 = 1 \quad \therefore \alpha = \pm\frac{2}{3} \quad \therefore q_3 = \frac{1}{3}(2, -2, -1).$

(b) $q_1 \in C(A)$, $q_2 \in C(A)$, q_1 and q_2 are linearly independant.

rank$(A) = 2 \rightarrow$ Therefore, q_1 and q_2 are bases of $C(A)$.

Since q_3 is perpendicular to both q_1 and q_2, which are the bases of $C(A)$,

$N(A^T)$ contains q_3.

$A^T A \hat{x} = A^T b$

$\downarrow$

$R^T Q^T Q R \hat{x} = R^T Q^T b$

$R^T R \hat{x} = R^T Q^T b$, R^T is square

$\Rightarrow \boxed{R\hat{x} = Q^T b}$

(c) $A = QR = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} R$, $R = Q^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$

$R\hat{x} = Q^T \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$

$\therefore \hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

23 Find q_1, q_2, q_3 (orthonormal) as combinations of a, b, c (independent columns). Then write A as QR:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

$A = (1,0,0)$, $q_1 = (1,0,0)$

$B = b - \frac{A^Tb}{A^TA}A = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$, $q_2 = (0,0,1)$

$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B$

$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{18}{9}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$ $\therefore$ $q_3 = (0,1,0)$

$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $A = QR$, $R = Q^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

$\therefore$ $A = QR$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$