GBE2031-41: Linear Algebra for Biomedical Engineer

Final Examination

Exam duration: 1 hour and 15 minutes

Problem 1:

a) If a, b, c are orthonormal vectors in $\mathcal{R}^3$, what are the possible determinants of the matrix A with columns $2a, 3b, and\ 5c$? Explain why.

$$A = \begin{bmatrix} 2a & 3b & 5c \end{bmatrix}$$

b) For a matrix A, suppose cofactor C_{11} of the first entry a_{11} is **zero.** What information does that give about A^{-1}? Can this inverse exist? Why?

c) Find the 3 eigenvalues of the matrix A below and find all of its eigenvectors. Is diagonalization $S^{-1}AS = \Lambda$ possible? Why?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 2:

a) If C is any symmetric matrix, show that e^C is the positive definite matrix. We can see that e^C is symmetric – which test will you use to show that e^C is positive definite?

b) A is a 3 by 3 matrix. Let's suppose v_1, v_2, v_3 are orthonormal eigenvectors (with eigenvalues 1,2,3) of the symmetric matrix A^TA. Show that Av_1, Av_2, Av_3 are orthogonal by rewriting and simplifying $(Av_i)^T(Av_j)$.

c) For the matrix A in subproblem (b), find the three matrices U, Σ, V that go into the singular value decomposition (SVD) $A = U\Sigma\mathrm{V}^T$

Problem 3:

QR factorization of the matrix A (e.g., via Gram-Schmidt) yields $A = QR$

where,

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \quad , \quad R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

a) Which columns of A were **orthogonal** to begin with, if any?

b) What is the orthogonal **projection** $\boldsymbol{p}$ of the vector $b = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ onto $C(A)$?

c) If we are minimizing $||Ax - b||$ (i.e., solving the least square problem) for $b = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, we should be able to quickly get an upper-triangular system of equations $U\hat{x} = c$ for the least-square solution $\hat{x}$. So, what are the upper-triangular matrix $\boldsymbol{U}$ and the right-hand side $\boldsymbol{c}$?