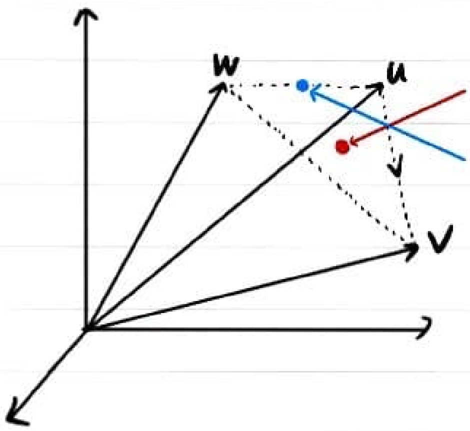
Chapter 1.1

20)

20 Locate $\frac{1}{3}u+\frac{1}{3}v+\frac{1}{3}w$ and $\frac{1}{2}u+\frac{1}{2}w$ in Figure 1.5b. Challenge problem: Under what restrictions on c, d, e, will the combinations $cu+dv+ew$ fill in the dashed triangle? To stay in the triangle, one requirement is $c \geq 0, d \geq 0, e \geq 0$.



1) $\frac{1}{3}u+\frac{1}{3}v+\frac{1}{3}w$: the center of the triangle between u,v,w

$\frac{1}{2}u+\frac{1}{2}w$ lies between u and w.

2) To fill the triangle, $c\geq 0, d\geq 0, e\geq 0$ and $c+d+e=1$.

31)

31 Write down three equations for c, d, e so that $cu+dv+ew=b$. Can you somehow find c, d, and e?

$$u=\begin{bmatrix}2\\-1\\0\end{bmatrix}\quad v=\begin{bmatrix}-1\\2\\-1\end{bmatrix}\quad w=\begin{bmatrix}0\\-1\\2\end{bmatrix}\quad b=\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

$$2c-d=1.$$
$$-c+2d-e=0.$$
$$-d+2e=0. \qquad \therefore\ c=\frac{3}{4},\ d=\frac{1}{2},\ e=\frac{1}{4}$$

Chapter 1.2

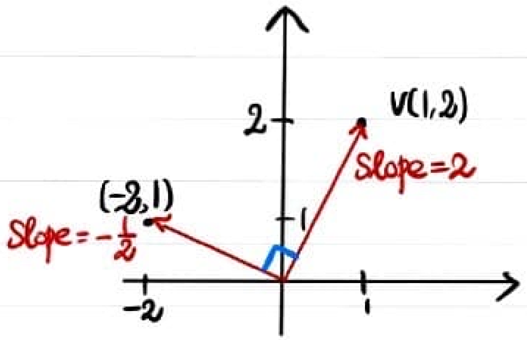
5)

5 Find unit vectors u_1 and u_2 in the directions of $v=(3,1)$ and $w=(2,1,2)$. Find unit vectors U_1 and U_2 that are perpendicular to u_1 and u_2.

$$u_1=\frac{u_1}{\|u_1\|}=\frac{(3,1)}{\sqrt{10}} \qquad u_2=\frac{u_2}{\|u_2\|}=\frac{(2,1,2)}{\sqrt{10}}$$

$$U_1=\frac{(-1,3)}{\sqrt{10}}\ \text{or}\ \frac{(1,-3)}{\sqrt{10}} \qquad U_2=\frac{(1,-2,0)}{\sqrt{10}}$$

10) **10** Draw arrows from $(0,0)$ to the points $\boldsymbol{v} = (1,2)$ and $\boldsymbol{w} = (-2,1)$. Multiply their slopes. That answer is a signal that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the arrows are _____.



$V \cdot W = 0$

$\Rightarrow$ the vectors are perpendicular.

20) **20** The "Law of Cosines" comes from $(\boldsymbol{v}-\boldsymbol{w}) \cdot (\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$:

Cosine Law $\quad \|\boldsymbol{v}-\boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|\cos\theta + \|\boldsymbol{w}\|^2.$

If $\theta < 90°$ show that $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2$ is larger than $\|\boldsymbol{v}-\boldsymbol{w}\|^2$ (the third side).

When $\theta < 90°$, $V \cdot W$ is positive.
thus, $\|V\|^2 + \|W\|^2$ is larger than $\|V-W\|^2$.

22) **22** The Schwarz inequality $|\boldsymbol{v} \cdot \boldsymbol{w}| \le \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ by algebra instead of trigonometry:

(a) Multiply out both sides of $(v_1w_1 + v_2w_2)^2 \le (v_1^2 + v_2^2)(w_1^2 + w_2^2)$.

(b) Show that the difference between those two sides equals $(v_1w_2 - v_2w_1)^2$. This cannot be negative since it is a square—so the inequality is true.

$\cancel{v_1^2w_1^2} + 2v_1w_1v_2w_2 + \cancel{v_2^2w_2^2} \le \cancel{v_1^2w_1^2} + v_1^2w_2^2 + v_2^2w_1^2 + \cancel{v_2^2w_2^2}$

$v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2 = (v_1w_2 - v_2w_1)^2 \ge 0.$

Chapter 1.3

3) **3** Solve these three equations for y_1, y_2, y_3 in terms of B_1, B_2, B_3:

$$Sy = B \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

Write the solution y as a matrix $A = S^{-1}$ times the vector B. Are the columns of S independent or dependent?

$$\begin{aligned} y_1 &= B_1 \\ y_1 + y_2 &= B_2 \\ y_1 + y_2 + y_3 &= B_3 \end{aligned} \qquad \begin{aligned} y_1 &= B_1 \\ y_2 &= -B_1 + B_2 \\ y_3 &= -B_2 + B_3 \end{aligned} \qquad = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

10) **10** A *forward* difference matrix Δ is *upper* triangular:

$$\Delta z = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 - z_1 \\ z_3 - z_2 \\ 0 - z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b.$$

Find z_1, z_2, z_3 from b_1, b_2, b_3. What is the inverse matrix in $z = \Delta^{-1} b$?

$$\begin{aligned} z_1 &= -b_1 - b_2 - b_3 \\ z_2 &= -b_2 - b_3 \\ z_3 &= -b_3 \end{aligned} \qquad = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1} b$$

14) **14** If (a, b) is a multiple of (c, d) with $abcd \neq 0$, *show that* (a, c) *is a multiple of* (b, d). This is surprisingly important; two columns are falling on one line. You could use numbers first to see how a, b, c, d are related. The question will lead to:

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent columns when it has dependent rows.

$(a,b) = (3,6)$

$(c,d) = (1,2) \Rightarrow \frac{a}{c} = \frac{b}{d} = 3 \Rightarrow ad = bc. \Rightarrow \frac{a}{b} = \frac{c}{d}$

dependent rows $\Rightarrow$ dependent columns.