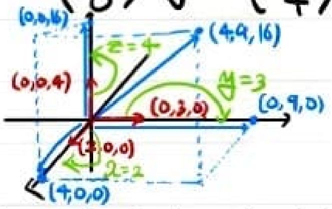
Chapter 2.1

2 If the equations in Problem 1 are multiplied by 2, 3, 4 they become $DX = B$:

$$\begin{aligned}2x + 0y + 0z &= 4\\ 0x + 3y + 0z &= 9\\ 0x + 0y + 4z &= 16\end{aligned} \quad \text{or} \quad DX = \begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}4\\9\\16\end{bmatrix} = B$$

Why is the row picture the same? Is the solution X the same as x? What is changed in the column picture—the columns or the right combination to give B?

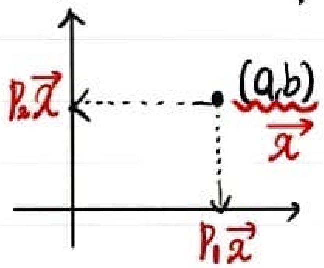
$$\begin{pmatrix}2\\0\\0\end{pmatrix}\overset{2}{\overset{\shortparallel}{x}} + \begin{pmatrix}0\\3\\0\end{pmatrix}\overset{3}{\overset{\shortparallel}{y}} + \begin{pmatrix}0\\0\\4\end{pmatrix}\overset{4}{\overset{\shortparallel}{z}} = \begin{bmatrix}4\\9\\16\end{bmatrix}$$



* Projection

20 What 2 by 2 matrix P_1 projects the vector (x, y) onto the x axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2, you get (①) and (②).

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\0\end{pmatrix}, \quad \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\y\end{pmatrix}$$



①: (5,0)

②: (0,7)

5 The first of these equations plus the second equals the third:

$$\begin{aligned}x + y + z &= 2\\ x + 2y + z &= 3\\ 2x + 3y + 2z &= 5.\end{aligned}$$

교선이 $2x+3y+2z=5$를 포함하고 있다.

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also ____. The equations have infinitely many solutions (the whole line L). Find three solutions on L.

Satisfy the third equation = sum of the first two.

$$\begin{array}{rl} & x+2y+z=3\\ - & x+y+z=2\\ \hline \end{array}$$

$(y=1)$ and $(x+z=3)$.

$\therefore (0,1,1), (1,1,0), (-1,1,2)$

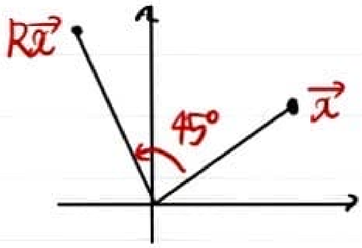
* Rotation Matrix

21 What 2 by 2 matrix R rotates every vector through 45°? The vector $(1,0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0,1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$. Those determine the matrix. Draw these particular vectors in the xy plane and find R.

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix} \quad \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}$$

$$\therefore \begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\end{pmatrix} = \begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}$$

$\theta = \frac{\pi}{4}$



17 Find the matrix P that multiplies (x, y, z) to give (y, z, x). Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z).

$$P: \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} \quad \begin{pmatrix}y\\z\\x\end{pmatrix}$$

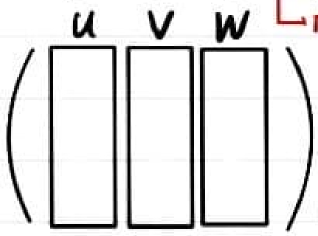
$$Q: \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}y\\z\\x\end{pmatrix} \quad \begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$Q \cdot P\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow QP = I \Rightarrow Q = P^{-1}$$

32 Suppose u and v are the first two columns of a 3 by 3 matrix A. Which third columns w would make this matrix singular? Describe a typical column picture of $Ax = b$ in that singular case, and a typical row picture (for a random b).

→ $n \times n$ matrix, rank $< n$: Singular.

⇒ Linearly dependent.



A is Singular when w is a combination $cu + dv$.

$b \neq 0 \rightarrow$ no solution

- A typical column picture has b outside the plane u, v, w.
- A typical row picture has the intersection line of two planes parallel to the third plane.

Chapter 2.2

6 ① Choose a coefficient b that makes this system singular. ② Then choose a right side g that makes it solvable. Find two solutions in that singular case.

too many solutions.

$$2x + by = 16$$
$$4x + 8y = g.$$

$4x + 8y = g.$

$4x + 2by = 32.$

$\underbrace{(8-2b)}_{=0} y = g - 32.$

$\therefore b = 4$ (Singular case)

$\therefore g = 32.$

$\therefore \quad 2x + 4y = 16.$

$4x + 8y = 32.$

19 Which number q makes this system singular and which right side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$x + 4y - 2z = 1$$
$$x + 7y - 6z = 6$$
$$3y + qz = t.$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & \boxed{q+4} & t-5 \end{array}\right)$$

if $q = -4$, the system is singular (zero pivot)

if $t = 5$, infinitely many solutions.

$z = 1, \ y = 3, \ x = -9.$

12 Reduce this system to upper triangular form by two row operations:

$$2x + 3y + z = 8$$
$$4x + 7y + 5z = 20$$
$$-2y + 2z = 0.$$

Circle the pivots. Solve by back substitution for z, y, x.

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 0 & -2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

$2x + 3y + z = 8$

$y + 3z = 4$

$8z = 8$

$\therefore x = 2, y = 1, z = 1$

25 For which three numbers a will elimination fail to give three pivots?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \text{ is singular for three values of } a.$$

$a = 2$: equal columns

$a = 4$: equal rows … 질문하기

$a = 0$: zero column

Chapter 2.3

6 If every column of A is a multiple of $(1,1,1)$, then Ax is always a multiple of $(1,1,1)$. Do a 3 by 3 example. How many pivots are produced by elimination?

$$\underset{A}{\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} \alpha x+\beta y+\gamma z \\ \alpha x+\beta y+\gamma z \\ \alpha x+\beta y+\gamma z \end{pmatrix}$$

$$(\alpha x+\beta y+\gamma z)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$: Ax is always a multiple of $(1,1,1)$.

elimination

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$: only one pivot.

8 The ***determinant*** of $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det M = ad - bc$. Subtract ℓ times row 1 from row 2 to produce a new M^*. Show that $\det M^* = \det M$ for every ℓ. When $\ell = c/a$, *the product of pivots equals the determinant*: $(a)(d - \ell b)$ equals $ad - bc$.

$$M^* = \begin{pmatrix} a & b \\ c-a\ell & d-b\ell \end{pmatrix}$$

$$\det(M^*) = a(d-b\ell) - b(c-a\ell)$$
$$= ad - bc.$$
$$= \det(M)$$

* 의미) elimination을 해도 det에는 영향을 주지 않는다. ($E_{32}E_{31}E_{21}A$)

$$M^* = \begin{pmatrix} a & b \\ 0 & d-\frac{bc}{a} \end{pmatrix}$$

$$a\left(d-\frac{bc}{a}\right) = ad - bc = \det(M)$$

⇒ Matrix의 det, 즉 solution에 대한 특징이 바뀌지 않는다.

31 Find elimination matrices E_{21} then E_{32} then E_{43} to change K into U:

$$E_{43}\,E_{32}\,E_{21} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}.$$

Apply those three steps to the identity matrix I, to multiply $E_{43}E_{32}E_{21}$.

$$E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & b & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_{43} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & c & 1 \end{bmatrix}$$

$$\therefore E_{43}E_{32}E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ ab & b & 1 & \\ abc & bc & c & 1 \end{bmatrix}$$

Chapter 2.4

29 Which matrices E_{21} and E_{31} produce zeros in the (2, 1) and (3, 1) positions of $E_{21}A$ and $E_{31}A$?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}.$$

Find the single matrix $E = E_{31}E_{21}$ that produces both zeros at once. Multiply EA.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

34 Find all matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that satisfy $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}A$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b, & a+b \\ c+d, & c+d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c, & b+d \\ a+c, & b+d \end{bmatrix}$$

$$\therefore b = c, \ a = d. \Rightarrow A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$