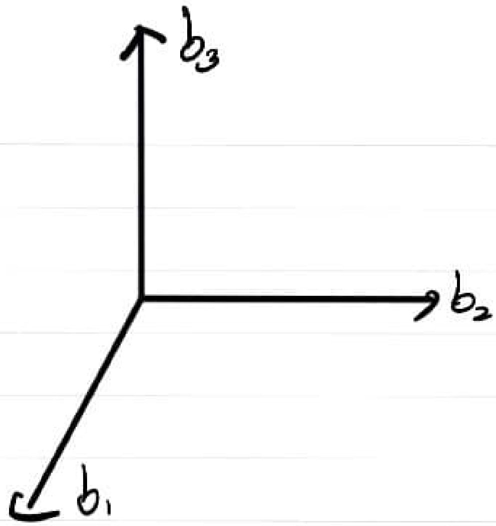
# Chapter 3.1




**10** Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors $(b_1, b_2, b_3)$ with $b_1 = b_2$.
(b) The plane of vectors with $b_1 = 1$.
(c) The vectors with $b_1 b_2 b_3 = 0$.
(d) All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.
(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.
(f) All vectors with $b_1 \le b_2 \le b_3$.

$\vec{x}, \vec{y} \rightarrow c\vec{x} + d\vec{y}$
$\subset$sub. $\quad \subset$sub.

(a) $(x_1, x_1, x_3) = \vec{x} \quad (y_1, y_1, y_3) = \vec{y}$
$c\vec{x} + d\vec{y} = (cx_1 + dy_1, cx_1 + dy_1, cx_3 + dy_3)$

**20** For which right sides (find a condition on $b_1, b_2, b_3$) are these systems solvable?

(a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

a) $\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - b_1 \\ 0 & 0 & 0 & b_1 + b_3 \end{array}\right]$ $\begin{array}{l} = 0 \\ = 0 \end{array}$ b) $\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_1 + b_3 \end{array}\right] = 0$

$\therefore b_2 = b_1$ and $b_3 = -b_1$. $\qquad \therefore b_3 = -b_1$

**25** Suppose $Ax = b$ and $Ay = b^*$ are both solvable. Then $Az = b + b^*$ is solvable. What is $z$? This translates into: If $b$ and $b^*$ are in the column space $C(A)$, then $b + b^*$ is in $C(A)$.

$Az = b + b^* \longrightarrow z = x + y$

# Chapter 3.2

$A = m \times n$ matrix
$L = m \times m$ matrix (m represents # of rows in A)
$U = m \times n$ matrix

**5** By row operations reduce each matrix to its echelon form $U$. Write down a 2 by 2 lower triangular $L$ such that $B = LU$.

(a) $A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}$ (b) $B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 6 & 2 & 6 \\ 3 & 9 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}}_{U} \qquad B = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}}_{U}$$

* Construct A whose column space contains $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and null space $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$r = 2$ $\quad$ $n - r = 1, n = 3$

$$A = \begin{array}{c} \\ 3 \end{array} \begin{array}{c} 3 \\ \begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix} \end{array}$$

$1 + 2a = 0, \ 4 + 2b = 0, \ 6 + 2c = 0$

$\therefore a = -\frac{1}{2}, b = -2, c = -3.$

**19** Prove that $U$ and $A = LU$ have the same nullspace when $L$ is invertible:

If $Ux = 0$ then $LUx = 0$. If $LUx = 0$, how do you know $Ux = 0$?

if $LUx = 0$, $Ux = L^{-1} \cdot 0 = 0$.
$\Rightarrow$ U and LU have the same nullspace.

* Construct a Matrix A whose column space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and whose nullspace contains $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

↳ row: 3 $\quad$ ↳ $n - r = 2, n = 3, r = 1.$

$\Rightarrow r = 1$, but $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent.
no solution.

$R: E_A$

**32** If the special solutions to $Rx = 0$ are in the columns of these $N$, go backward to find the nonzero rows of the reduced matrices $R$:

$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $N = \begin{bmatrix} \ \end{bmatrix}$ (empty 3 by 1).

①

$$m \begin{array}{c} 3 \\ \begin{bmatrix} & & \end{bmatrix} \end{array} \rightarrow r \begin{array}{c} 3 \\ \begin{bmatrix} 1 & r_2 & r_3 \end{bmatrix} \end{array}$$

$\Rightarrow n - r = 2, r = 1$

$2 + r_2 = 0, \ 3 + r_3 = 0$

$\therefore R = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$

②

$n - r = 1, r = 2$

$$\begin{array}{c} 3 \\ \begin{bmatrix} 1 & 0 & r_1 \\ 0 & 1 & r_2 \end{bmatrix} \end{array}$$

$r_1 = 0, r_2 = 0$

$\therefore R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

③

$n - r = 0.$

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$\therefore R = I.$

* Construct a Matrix A with $N(A) = \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\quad N(A) = \left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

$n - r = 2, n = 4, r = 2$ $\quad$ $n - r = 1, n = 4, r = 3$

$$A = \begin{array}{c} \\ 2 \end{array} \begin{array}{c} 4 \\ \begin{bmatrix} 1 & 1 & -4 & -4 \\ 0 & 1 & -2 & -1 \end{bmatrix} \end{array} \rightarrow \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$A = \begin{array}{c} 4 \\ \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \end{array}$$

# Chapter 3.3

**14** Transpose $P$ in problem 13. Then find the $r$ pivot columns of $P^T$. Transposing back, **this produces an $r$ by $r$ invertible submatrix $S$ inside $P$ and $A$:**

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 7 \end{bmatrix}$ find $P$ (3 by 2) and then the invertible $S$ (2 by 2).

**12** *If $A$ has rank $r$, then it has an $r$ by $r$ submatrix $S$ that is invertible.* Remove $m - r$ rows and $n - r$ columns to find an invertible submatrix $S$ inside $A$, $B$, and $C$. You could keep the pivot rows and pivot columns:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**13** Suppose $P$ contains only the $r$ pivot columns of an $m$ by $n$ matrix. Explain why this $m$ by $r$ submatrix $P$ has rank $r$.

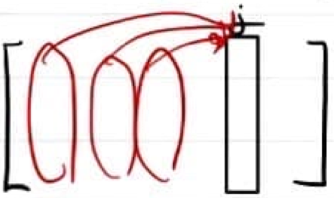
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{P} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix}, \quad P^T = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \qquad S^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

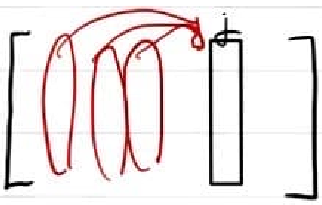
**17** (a) Suppose column $j$ of $B$ is a combination of previous columns of $B$. Show that column $j$ of $AB$ is the same combination of previous columns of $AB$. Then $AB$ cannot have new pivot columns, so **rank($AB$) $\le$ rank($B$)**.

(b) Find $A_1$ and $A_2$ so that rank$(A_1B) = 1$ and rank$(A_2B) = 0$ for $B = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$.

a) B




AB




b) $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$

# Chapter 3.4

**4** Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

$$\begin{array}{cccc|c} & y & & t & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & \boxed{3} & 1 & \boxed{2} & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x+3y+z+2t=1.$

$2z+4t=1.$

$x=-3y-z-2t+1=-3y-\frac{1}{2}+2t-2t+1=-3y+\frac{1}{2}.$

$z=\frac{1}{2}-2t$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

**19** Find the rank of $A$ and also of $A^TA$ and also of $AA^T$:

① $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix}$ and ② $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$.

①

$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -4 \end{bmatrix}$ rank=2

$A^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 27 & 6 \\ -21 & -4 \end{bmatrix}$ rank=2

$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 5 \\ 5 & 4 & 21 \end{bmatrix}$ rank=2

②

$A = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$ rank=2

$A^T = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$AA^T = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$ rank=2

$A^TA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix}$ rank=2