5 Find the eigenvalues of A and B (easy for triangular matrices) and $A+B$:

$$A=\begin{bmatrix}3&0\\1&1\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}1&1\\0&3\end{bmatrix} \quad\text{and}\quad A+B=\begin{bmatrix}4&1\\1&4\end{bmatrix}.$$

A) $\lambda=1,3$ / B) $\lambda=1,3$

Eigenvalues of $A+B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

A+B) $\lambda=3,5$ ∴ not equal.

10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain from those answers why A^{100} is close to A^∞:

A has $\lambda=1, 0.4$ (1,2) (1,-1).

$$A=\begin{bmatrix}.6&.2\\.4&.8\end{bmatrix} \quad\text{and}\quad A^\infty=\begin{bmatrix}1/3&1/3\\2/3&2/3\end{bmatrix}.$$

↳ $\lambda=1,0$ / $A^{100} \to \lambda_1=1,\ \lambda_2=(0.4)^{100}$ ⇒ near zero.

∴ A^{100} is near A^∞.

19 A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these (give the answers where possible):

(a) the rank of B

(b) the determinant of B^TB

(c) the eigenvalues of B^TB

(d) the eigenvalues of $(B^2+I)^{-1}$.

a) $B=3-1=2$.

b) $\det(B)=0$ (Singular) $\to \det(B^TB)=0$.

c) ?

d) $B^2x_1=0\cdot x_1,\ B^2x_2=1\cdot x_2,\ B^2x_3=4x_3$

$B^2x_1+Ix_1=x_1,\ B^2x_2+Ix_2=2x_2,\ B^2x_3+Ix_3=5x_3$.

∴ $1, \frac{1}{2}, \frac{1}{5}$

27 Find the rank and the four eigenvalues of A and C:

$$A=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} \quad\text{and}\quad C=\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}.$$

A) $\text{rank}(A)=1 \longrightarrow 1+1+1+1=\lambda+0+0+0=4 \to (4,0,0,0)$.

c) $\text{rank}(C)=2 \longrightarrow 1+1+1+1=\lambda_1+\lambda_2+0+0=4$.

34 Find the eigenvalues of this permutation matrix P from $\det(P-\lambda I)=0$. Which vectors are not changed by the permutation? They are eigenvectors for $\lambda=1$. Can you find three more eigenvectors?

$\lambda^4=1 \to \lambda=\pm1, \pm i$

$$P=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}.$$

5 Find the eigenvalues of A and B (easy for triangular matrices) and $A+B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A+B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A+B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

A) $\lambda = 1, 3$ / B) $\lambda = 1, 3$

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10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain from those answers why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

A has $\lambda = 1, 0.4$ $(1,2)$ $(1,-1)$.

↳ $\lambda = 1, 0$ / $A^{100} \to \lambda_1 = 1$ $\lambda_2 = (0.4)^{100}$ ⇒ near zero.

∴ A^{100} is near A^∞.

19 A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2. This information is enough to find three of these (give the answers where possible):

(a) the rank of B

(b) the determinant of $B^{\mathrm{T}}B$

(c) the eigenvalues of $B^{\mathrm{T}}B$

(d) the eigenvalues of $(B^2+I)^{-1}$.

a) $B = 3 - 1 = 2$.

b) $\det(B) = 0$ (Singular) → $\det(B^{T}B) = 0$.

c) ?

d) $B^2x_1 = 0 \cdot x_1$, $B^2x_2 = 1 \cdot x_2$, $B^2x_3 = 4x_3$

$B^2x_1 + Ix_1 = x_1$, $B^2x_2 + Ix_2 = 2x_2$, $B^2x_3 + Ix_3 = 5x_3$.

∴ $1, \frac{1}{2}, \frac{1}{5}$

27 Find the rank and the four eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

A) $\operatorname{rank}(A) = 1 \longrightarrow 1+1+1+1 = \lambda + 0 + 0 + 0 = 4 \to (4, 0, 0, 0)$.

C) $\operatorname{rank}(C) = 2 \longrightarrow 1+1+1+1 = \lambda_1 + \lambda_2 + 0 + 0 = 4$.

34 Find the eigenvalues of this permutation matrix P from $\det(P - \lambda I) = 0$. Which vectors are not changed by the permutation? They are eigenvectors for $\lambda = 1$. Can you find three more eigenvectors?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\lambda^4 = 1 \to \lambda = \pm 1, \pm i$

$AS = S\lambda$.

4 True or false: If the columns of S (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) S is invertible (d) S is diagonalizable.

a) / b) true

c) true / d) false

8 Diagonalize the Fibonacci matrix by completing S^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $S\Lambda^k S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2)$.

27 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = S\sqrt{\Lambda}\,S^{-1}$. Why is there no real matrix square root of B?

A) $R = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

B) $\sqrt{B} \rightarrow \sqrt{9}$ and $\sqrt{-1} \Rightarrow$ not real.

34 If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $AB = BA$, show that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also a diagonal matrix. B has the same eigen_____ as A but different eigen_____ . These diagonal matrices B form a two-dimensional subspace of matrix space. $AB - BA = 0$ gives four equations for the unknowns a, b, c, d—find the rank of the 4 by 4 matrix.

7 (a) Find a symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.
(b) How do you know it must have a negative pivot?
(c) How do you know it can't have two negative eigenvalues?

a) $(\lambda-1)^2-b^2=0$. if $b=2 \longrightarrow \lambda=-1$ or 3.

b) pivots have same sign as λ.

c) trace $=\lambda_1+\lambda_2=2 \rightarrow A$ can't have two negative eigenvalues.

13 What are the eigenvalues of $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$? Create a 4 by 4 skew-symmetric matrix ($A^T = -A$) and verify that all its eigenvalues are imaginary.

$\lambda^2+b^2=0$. $\lambda=\pm ib$.

19 Find the eigenvector matrix S for A and for B. Show that S doesn't collapse at $d=1$, even though $\lambda=1$ is repeated. Are the eigenvectors perpendicular?

$$A = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -d & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix} \quad \text{have} \quad \lambda = 1, d, -d.$$

A) $\begin{matrix} -\lambda & d & 0 \\ d & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{matrix}$ $\quad (1-\lambda)(\lambda^2-d^2)=0.$ $\Rightarrow \lambda=1$ or $\pm d$.

$\begin{bmatrix} -1 & d & 0 \\ d & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}=0$ / $\begin{bmatrix} -d & d & 0 \\ d & -d & 0 \\ 0 & 0 & 1-d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}=0$ / $\begin{bmatrix} d & d & 0 \\ d & d & 0 \\ 0 & 0 & 1+d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}=0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2d \end{bmatrix}$

$\Rightarrow$ perpendicular for A and not for B.
$(A^T=A)$ $\quad (B^T \neq B)$.

A.

24 What number b in $\begin{bmatrix} 2 & b \\ 1 & 0 \end{bmatrix}$ makes $A = Q\Lambda Q^T$ possible? What number makes $A = S\Lambda S^{-1}$ impossible? What number makes A^{-1} impossible?

1) $A=Q\Lambda Q^T \longrightarrow$ symmetry $\therefore b=1$.

2) repeated λ if $b=-1$.

3) $b=0 \longrightarrow$ singular.

9 Find the 3 by 3 matrix A and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

$4x_1^2 + 4x_2^2 + 16x_3^2 + 2(-4x_1x_2 - 8x_2x_3 + 8x_3x_1)$

$$\begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot = 4

rank = 1

det(S) = 0

Since A is square, symmetric, → # of eigenvalues = pivots = 1.

trace $A = 4+4+16 = \lambda_1 + \underbrace{\lambda_2 + \lambda_3}_{=0} = 24$ ∴ eigenvalues = 24, 0, 0.

12 For what numbers c and d are A and B positive definite? Test the 3 determinants:

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}.$$

A) $(c>0)$ and $(c^2-1>0)$ and $(c(c^2-1) - (c-1) + (1-c)) > 0$ ∴ $c>1$

B) impossible.

26 In the Cholesky factorization $A = C^{\mathrm{T}}C$, with $C^{\mathrm{T}} = L\sqrt{D}$, the square roots of the pivots are on the diagonal of C. Find C (upper triangular) for

1) $A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}$ and 2) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}$.

$C = \sqrt{D}L^T$

28 Without multiplying $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find

(a) the determinant of A (b) the eigenvalues of A
(c) the eigenvectors of A (d) a reason why A is symmetric positive definite.

a) $|1\times 10\times 1| = 10$.

b) $\lambda = 2, 5$

c) $A = LDL^T$. ⇒ $(\cos\theta, \sin\theta)$, $(-\sin\theta, \cos\theta)$

d) $\lambda > 0$.