2. Let S be the set of all integers $0, \pm 1, \pm 2, \ldots, \pm n, \ldots$. For a, b in S define $*$ by $a * b = a - b$. Verify the following:
   (a) $a * b \neq b * a$ unless $a = b$.
   (b) $(a * b) * c \neq a * (b * c)$ in general. Under what conditions on a, b, c is $(a * b) * c = a * (b * c)$?
   (c) The integer 0 has the property that $a * 0 = a$ for every a in S.
   (d) For a in S, $a * a = 0$.
3. Let S consist of the two objects $\square$ and $\triangle$. We define the operation $*$ on S by subjecting $\square$ and $\triangle$ to the following conditions:
   1. $\square * \triangle = \triangle = \triangle * \square$.
   2. $\square * \square = \square$.
   3. $\triangle * \triangle = \square$.

   Verify by explicit calculation that if a, b, c are any elements of S (i.e., a, b and c can be any of $\square$ or $\triangle$), then:
   (a) $a * b$ is in S.
   (b) $(a * b) * c = a * (b * c)$.
   (c) $a * b = b * a$.
   (d) There is a particular a in S such that $a * b = b * a = b$ for *all* b in S.
   (e) Given b in S, then $b * b = a$, where a is the particular element in Part (d).

2. SET THEORY

With the changes in the mathematics curriculum in the schools in the United States, many college students have had some exposure to set theory. This introduction to set theory in the schools usually includes the elementary notions and operations with sets. Going on the assumption that many readers will have some acquaintance with set theory, we shall give a rapid survey of those parts of set theory that we shall need in what follows.

First, however, we need some notation. To avoid the endless repetition of certain phrases, we introduce a shorthand for these phrases. Let S be a collection of objects; the objects of S we call the *elements* of S. To denote that a given element, a, is an element of S, we write $a \in S$—this is read "a is an element of S." To denote the contrary, namely that an object a is *not* an element of S, we write $a \notin S$. So, for instance, if S denotes the set of all positive integers $1, 2, 3, \ldots, n, \ldots$, then $165 \in S$, whereas $-13 \notin S$.

We often want to know or prove that given two sets S and T, one of these is a part of the other. We say that S is a *subset* of T, which we write $S \subset T$ (read "S is contained in T") if every element of S is an element of T.

In terms of the notation we now have: $S \subset T$ if $s \in S$ implies that $s \in T$. We can also denote this by writing $T \supset S$, read "T contains S." (This does not exclude the possibility that $S = T$, that is, that S and T have exactly the same elements.) Thus, if T is the set of all positive integers and S is the set of all positive even integers, then $S \subset T$, and S is a subset of T. In the definition given above, $S \supset S$ for any set S; that is, S is always a subset of itself.

We shall frequently need to show that two sets S and T, defined perhaps in distinct ways, are equal, that is, they consist of the same set of elements. The usual strategy for proving this is to show that both $S \subset T$ and $T \subset S$. For instance, if S is the set of all positive integers having 6 as a factor and T is the set of all positive integers having both 2 and 3 as factors, then $S = T$. (Prove!)

The need also arises for a very peculiar set, namely one having no elements. This set is called the *null* or *empty* set and is denoted by $\varnothing$. It has the property that it is a subset of *any* set S.

Let A, B be subsets of a given set S. We now introduce methods of constructing other subsets of S from A and B. The first of these is the *union* of A and B, written $A \cup B$, which is defined: $A \cup B$ is that subset of S consisting of those elements of S that are elements of A *or* are elements of B. The "or" we have just used is somewhat different in meaning from the ordinary usage of the word. Here we mean that an element c is in $A \cup B$ if it is in A, or is in B, or is in *both*. The "or" is not meant to exclude the possibility that both things are true. Consequently, for instance, $A \cup A = A$.

If $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 10\}$, then $A \cup B = \{1, 2, 3, 4, 6, 10\}$.

We now proceed to our second way of constructing new sets from old. Again let A and B be subsets of a set S; by the *intersection* of A and B, written $A \cap B$, we shall mean the subset of S consisting of those elements that are both in A *and* in B. Thus, in the example above, $A \cap B = \{2\}$. It should be clear from the definitions involved that $A \cap B \subset A$ and $A \cap B \subset B$. Particular examples of intersections that hold universally are: $A \cap A = A$, $A \cap S = A$, $A \cap \varnothing = \varnothing$.

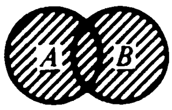
This is an opportune moment to introduce a notational device that will be used time after time. Given a set S, we shall often be called on to describe the subset A of S, whose elements satisfy a certain property P. We shall write this as $A = \{s \in S \mid s \text{ satisfies } P\}$. For instance, if A, B are subsets of S, then $A \cup B = \{s \in S \mid s \in A \text{ or } s \in B\}$ while $A \cap B = \{s \in S \mid s \in A \text{ and } s \in B\}$.

Although the notions of union and intersection of subsets of S have been defined for two subsets, it is clear how one can define the union and intersection of any number of subsets.

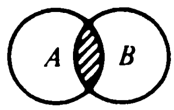
We now introduce a third operation we can perform on sets, the *difference* of two sets. If A, B are subsets of S, we define $A - B = \{a \in A \mid a \notin B\}$.

So if A is the set of all positive integers and B is the set of all even integers, then $A - B$ is the set of all positive odd integers. In the particular case when A is a subset of S, the difference $S - A$ is called the *complement* of A in S and is written A'.

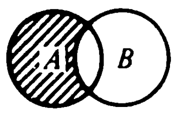
We represent these three operations pictorially. If A is (A) and B is (B), then

1. $A \cup B =$ 

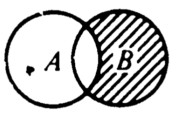
is the shaded area.

2. $A \cap B =$ 

is the shaded area.

3. $A - B =$ 

is the shaded area.

4. $B - A =$ 

is the shaded area.

Note the relation among the three operations, namely the equality $A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$. As an illustration of how one goes about proving the equality of sets constructed by such set-theoretic constructions, we prove this latter alleged equality. We first show that $(A \cap B) \cup (A - B) \cup (B - A) \subset A \cup B$; this part is easy for, by definition, $A \cap B \subset A$, $A - B \subset A$, and $B - A \subset B$, hence

$$(A \cap B) \cup (A - B) \cup (B - A) \subset A \cup A \cup B = A \cup B.$$

Now for the other direction, namely that $A \cup B \subset (A \cap B) \cup (A - B) \cup (B - A)$. Given $u \in A \cup B$, if $u \in A$ and $u \in B$, then $u \in A \cap B$, so it is certainly in $(A \cap B) \cup (A - B) \cup (B - A)$. On the other hand, if $u \in A$ but $u \notin B$, then, by the very definition of $A - B$, $u \in A - B$, so again it is certainly in $(A \cap B) \cup (A - B) \cup (B - A)$. Finally, if $u \in B$ but $u \notin A$, then $u \in B - A$, so again it is in $(A \cap B) \cup (A - B) \cup (B - A)$. We have thus covered all the possibilities and have shown that $A \cup B \subset (A \cap B) \cup (A - B) \cup (B - A)$. Having the two opposite containing relations of $A \cup B$ and $(A \cap B) \cup (A - B) \cup (B - A)$, we obtain the desired equality of these two sets.

We close this brief review of set theory with yet another construction we can carry out on sets. This is the *Cartesian product* defined for the two sets A, B by $A \times B = \{(a, b) \mid a \in A, b \in B\}$, where we *declare* the ordered pair (a, b) to be equal to the ordered pair (a_1, b_1) if and only if $a = a_1$ and $b = b_1$. Here, too, we need not restrict ourselves to two sets; for instance, we can define, for sets A, B, C, their Cartesian product as the set of ordered triples (a, b, c), where $a \in A$, $b \in B$, $c \in C$ and where equality of two ordered triples is defined component-wise.

PROBLEMS

Easier Problems

1. Describe the following sets verbally.
(a) $S = \{$Mercury, Venus, Earth, . . . , Pluto$\}$.
(b) $S = \{$Alabama, Alaska, . . . , Wyoming$\}$.

2. Describe the following sets verbally.
(a) $S = \{2, 4, 6, 8, \ldots\}$.
(b) $S = \{2, 4, 8, 16, 32, \ldots\}$.
(c) $S = \{1, 4, 9, 16, 25, 36, \ldots\}$.

3. If A is the set of all residents of the United States, B the set of all Canadian citizens, and C the set of all women in the world, describe the sets $A \cap B \cap C$, $A - B$, $A - C$, $C - A$ verbally.

4. If $A = \{1, 4, 7, a\}$ and $B = \{3, 4, 9, 11\}$ and you have been told that $A \cap B = \{4, 9\}$, what must a be?

5. If $A \subset B$ and $B \subset C$, prove that $A \subset C$.

6. If $A \subset B$, prove that $A \cup C \subset B \cup C$ for any set C.

7. Show that $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

8. Prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. What does this look like pictorially?

9. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

10. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

11. Write down all the subsets of $S = \{1, 2, 3, 4\}$.

Middle-Level Problems

***12.** If C is a subset of S, let C' denote the complement of C in S. Prove the *De Morgan Rules* for subsets A, B of S, namely:
(a) $(A \cap B)' = A' \cup B'$.
(b) $(A \cup B)' = A' \cap B'$.

***13.** Let S be a set. For any two subsets of S we define

$$A + B = (A - B) \cup (B - A) \quad \text{and} \quad A \cdot B = A \cap B.$$

Prove that:

(a) $A + B = B + A$.
(b) $A + \varnothing = A$.
(c) $A \cdot A = A$.
(d) $A + A = \varnothing$.
(e) $A + (B + C) = (A + B) + C$.
(f) If $A + B = A + C$, then $B = C$.
(g) $A \cdot (B + C) = A \cdot B + A \cdot C$.

***14.** If C is a finite set, let $m(C)$ denote the number of elements in C. If A, B are finite sets, prove that

$$m(A \cup B) = m(A) + m(B) - m(A \cap B).$$

15. For three finite sets A, B, C find a formula for $m(A \cup B \cup C)$. (**Hint:** First consider $D = B \cup C$ and use the result of Problem 14.)

16. Take a shot at finding $m(A_1 \cup A_2 \cup \cdots \cup A_n)$ for n finite sets $A_1, A_2, \ldots, A_n$.

17. Use the result of Problem 14 to show that if 80% of all Americans have gone to high school and 70% of all Americans read a daily newspaper, then *at least* 50% of Americans have both gone to high school and read a daily newspaper.

18. A public opinion poll shows that 93% of the population agreed with the government on the first decision, 84% on the second, and 74% on the third, for three decisions made by the government. At least what percentage of the population agreed with the government on all three decisions? (**Hint:** Use the results of Problem 15.)

19. In his book *A Tangled Tale*, Lewis Carroll proposed the following riddle about a group of disabled veterans: "Say that 70% have lost an eye, 75% an ear, 80% an arm, 85% a leg. What percentage, *at least*, must have lost all four?" Solve Lewis Carroll's problem.

***20.** Show, for finite sets A, B, that $m(A \times B) = m(A)m(B)$.

21. If S is a set having five elements:

(a) How many subsets does S have?
(b) How many subsets having four elements does S have?
(c) How many subsets having two elements does S have?

Harder Problems

22. (a) Show that a set having n elements has 2^n subsets.
(b) If $0 < m < n$, how many subsets are there that have exactly m elements?

3. MAPPINGS

One of the truly universal concepts that runs through almost every phase of mathematics is that of a *function* or *mapping* from one set to another. One could safely say that there is no part of mathematics where the notion does not arise or play a central role. The definition of a function from one set to another can be given in a formal way in terms of a subset of the Cartesian product of these sets. Instead, here, we shall give an informal and admittedly nonrigorous definition of a mapping (function) from one set to another.

Let S, T be sets; a *function* or *mapping* f from S to T is a *rule* that assigns to *each* element $s \in S$ a *unique* element $t \in T$. Let's explain a little more thoroughly what this means. If s is a given element of S, then there is *only one* element t in T that is associated to s by the mapping. As s varies over S, t varies over T (in a manner depending on s). Note that by the definition given, the following is *not* a mapping. Let S be the set of all people in the world and T the set of all countries in the world. Let f be the rule that assigns to every person his or her country of citizenship. Then f is not a mapping from S to T. Why not? Because there are people in the world that enjoy a dual citizenship; for such people there would not be a *unique* country of citizenship. Thus, if Mary Jones is both an English and French citizen, f would not make sense, as a mapping, when applied to Mary Jones. On the other hand, the rule $f: \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, defined by $f(a) = a^2$ for $a \in \mathbb{R}$, is a perfectly good function from $\mathbb{R}$ to $\mathbb{R}$. It should be noted that $f(-2) = (-2)^2 = 4 = f(2)$, and $f(-a) = f(a)$ for all $a \in \mathbb{R}$.

We denote that f is a mapping from S to T by $f: S \to T$ and for the $t \in T$ mentioned above we write $t = f(s)$; we call t the *image* of s under f.

The concept is hardly a new one for any of us. Since grade school we have constantly encountered mappings and functions, often in the form of formulas. But mappings need not be restricted to sets of numbers. As we see below, they can occur in any area.

Examples

1. Let $S = \{$all men who have ever lived$\}$ and $T = \{$all women who have ever lived$\}$. Define $f: S \to T$ by $f(s) =$ mother of s. Therefore, f(John F. Kennedy) $=$ Rose Kennedy, and according to our definition, Rose Kennedy is the image under f of John F. Kennedy.

2. Let $S = \{$all legally employed citizens of the United States$\}$ and $T = \{$positive integers$\}$. Define, for $s \in S$, $f(s)$ by $f(s) =$ Social Security Number of s. (For the purpose of this text, let us assume that all legally employed citizens of the United States have a Social Security Number.) Then f defines a mapping from S to T.

3. Let S be the set of all objects for sale in a grocery store and let $T = \{\text{all real numbers}\}$. Define $f: S \to T$ by $f(s) =$ price of s. This defines a mapping from S to T.

4. Let S be the set of all integers and let $T = S$. Define $f: S \to T$ by $f(m) = 2m$ for any integer m. Thus the image of 6 under this mapping, $f(6)$, is given by $f(6) = 2 \cdot 6 = 12$, while that of -3, $f(-3)$, is given by $f(-3) = 2(-3) = -6$. If $s_1, s_2 \in S$ are in S and $f(s_1) = f(s_2)$, what can you say about s_1 and s_2?

5. Let $S = T$ be the set of all real numbers; define $f: S \to T$ by $f(s) = s^2$. Does every element of T come up as an image of some $s \in S$? If not, how would you describe the set of all images $\{f(s) \mid s \in S\}$? When is $f(s_1) = f(s_2)$?

6. Let $S = T$ be the set of all real numbers; define $f: S \to T$ by $f(s) = s^3$. This is a function from S to T. What can you say about $\{f(s) \mid s \in S\}$? When is $f(s_1) = f(s_2)$?

7. Let T be any nonempty set and let $S = T \times T$, the Cartesian product of T with itself. Define $f: T \times T \to T$ by $f(t_1, t_2) = t_1$. This mapping from $T \times T$ to T is called the *projection* of $T \times T$ onto its first component.

8. Let S be the set of all positive integers and let T be the set of all positive rational numbers. Define $f: S \times S \to T$ by $f(m, n) = m/n$. This defines a mapping from $S \times S$ to T. Note that $f(1, 2) = \frac{1}{2}$ while $f(3, 6) = \frac{3}{6} = \frac{1}{2} = f(1, 2)$, although $(1, 2) \neq (3, 6)$. Describe the subset of $S \times S$ consisting of those (a, b) such that $f(a, b) = \frac{1}{2}$.

The mappings to be defined in Examples 9 and 10 are mappings that occur for any nonempty sets and play a special role.

9. Let S, T be nonempty sets, and let t_0 be a fixed element of T. Define $f: S \to T$ by $f(s) = t_0$ for every $s \in S$; f is called a *constant* function from S to T.

10. Let S be any nonempty set and define $i: S \to S$ by $i(s) = s$ for every $s \in S$. We call this function of S to itself the *identity function* (or *identity mapping)* on S. We may, at times, denote it by i_S (and later in the book, by e).

Now that we have the notion of a mapping we need some way of identifying when two mappings from one set to another are equal. This is not God given; it is for us to decide how to declare $f = g$ where $f: S \to T$ and $g: S \to T$. What is more natural than to define this equality via the actions of f and g on the elements of S? More precisely, we declare that $f = g$ if and only if $f(s) = g(s)$ for *every* $s \in S$. If S is the set of all real numbers and f is defined on S by $f(s) = s^2 + 2s + 1$, while g is defined on S by $g(s) = (s + 1)^2$, our definition of the equality of f and g is merely a statement of the familiar identity $(s + 1)^2 = s^2 + 2s + 1$.

Having made the definition of equality of two mappings, we now want to single out certain types of mappings by the way they behave.

Definition. The mapping $f: S \to T$ is *onto* or *surjective* if every $t \in T$ is the image under f of some $s \in S$; that is, if and only if, given $t \in T$, there exists an $s \in S$ such that $t = f(s)$.

In the examples we gave earlier, in Example 1 the mapping is not onto, since not every woman that ever lived was the mother of a male child. Similarly, in Example 2 the mapping is not onto, for not every positive integer is the Social Security Number of some U.S. citizen. The mapping in Example 4 fails to be onto because not every integer is even; and in Example 5, again, the mapping is not onto, for the number -1, for instance, is not the square of any real number. However, the mapping in Example 6 is onto because every real number has a unique real cube root. The reader can decide whether or not the given mappings are onto in the other examples.

If we define $f(S) = \{f(s) \in T \mid s \in S\}$, another way of saying that the mapping $f: S \to T$ is onto is by saying that $f(S) = T$.

Another specific type of mapping plays an important and particular role in what follows.

Definition. A mapping $f: S \to T$ is said to be *one-to-one* (written 1-1) or *injective* if for $s_1 \neq s_2$ in S, $f(s_1) \neq f(s_2)$ in T. Equivalently, f is 1-1 if $f(s_1) = f(s_2)$ implies that $s_1 = s_2$.

In other words, a mapping is 1-1 if it takes distinct objects into distinct images. In the examples of mappings we gave earlier, the mapping of Example 1 is not 1-1, since two brothers would have the same mother. However in Example 2 the mapping is 1-1 because distinct U.S. citizens have distinct Social Security numbers (provided that there is no goof-up in Washington, which is unlikely). The reader should check if the various other examples of mappings are 1-1.

Given a mapping $f: S \to T$ and a subset $A \subset T$, we may want to look at $B = \{s \in S \mid f(s) \in A\}$; we use the notation $f^{-1}(A)$ for this set B, and call $F^{-1}(A)$ the *inverse image of A under f*. Of particular interest is $f^{-1}(t)$, the inverse image of the subset $\{t\}$ of T consisting of the element $t \in T$ alone. If the inverse image of $\{t\}$ consists of only one element, say $s \in S$, we could try to define $f^{-1}(t)$ by defining $f^{-1}(t) = s$. As we note below, this need not be a mapping from T to S, but is so if f is 1-1 and onto. We shall use the same notation f^{-1} in cases of both subsets and elements. This f^{-1} does *not* in general define a mapping from T to S for several reasons. First, if f is not onto, then there is some t in T which is not the image of any element s, so $f^{-1}(t) = \varnothing$. Second, if f is not 1-1, then for some $t \in T$ there are at least two distinct $s_1 \neq s_2$ in S such that $f(s_1) = t = f(s_2)$. So $f^{-1}(t)$ is *not* a unique element of S—something we require in our definition of mapping. However, if f is both 1-1 and onto T, then f^{-1} indeed defines a mapping of T onto S. (Verify!) This brings us to a very important class of mappings.

Definition. The mapping $f: S \to T$ is said to be a 1-1 *correspondence* or *bijection* if f is both 1-1 and onto.

Now that we have the notion of a mapping and have singled out various types of mappings, we might very well ask: "Good and well, but what can we do with them?" As we shall see in a moment, we can introduce an operation of combining mappings in certain circumstances.

Consider the situation $g : S \to T$ and $f: T \to U$. Given an element $s \in S$, then g sends it into the element $g(s)$ in T; so $g(s)$ is ripe for being acted on by f. Thus we get an element $f(g(s)) \in U$. We claim that this procedure provides us with a mapping from S to U. (Verify!) We define this more formally in the

Definition. If $g : S \to T$ and $f: T \to U$, then the *composition* (or *product*), denoted by $f \circ g$, is the mapping $f \circ g : S \to U$ defined by $(f \circ g)(s) = f(g(s))$ for every $s \in S$.

Note that to compose the two mappings f and g—that is, for $f \circ g$ to have any sense—the *terminal set*, T, for the mapping g *must be* the *initial set* for the mapping f. *One special time when we can always compose any two mappings is when $S = T = U$, that is, when we map S into itself.* Although special, this case is of the utmost importance.

We verify a few properties of this composition of mappings.

Lemma 1.3.1. If $h : S \to T$, $g: T \to U$, and $f: U \to V$, then $f \circ (g \circ h) = (f \circ g) \circ h$.

Proof. How shall we go about proving this lemma? To verify that two mappings are equal, we merely must check that they do the same thing to every element. Note first of all that both $f \circ (g \circ h)$ and $(f \circ g) \circ h$ define mappings from S to V, so it makes sense to speak about their possible equality.

Our task, then, is to show that for every $s \in S$, $(f \circ (g \circ h))(s) = ((f \circ g) \circ h)(s)$. We apply the definition of composition to see that

$$(f \circ (g \circ h))(s) = f((g \circ h)(s)) = f(g(h(s))).$$

Unraveling

$$((f \circ g) \circ h)(s) = (f \circ g)(h(s)) = f(g(h(s))),$$

we do indeed see that

$$(f \circ (g \circ h))(s) = ((f \circ g) \circ h)(s)$$

for every $s \in S$. Consequently, by definition, $f \circ (g \circ h) = (f \circ g) \circ h$. □

(The symbol □ will always indicate that the proof has been completed.)

This equality is described by saying that mappings, under composition, satisfy the *associative law*. Because of the equality involved there is really no need for parentheses, so we write $f \circ (g \circ h)$ as $f \circ g \circ h$.

Lemma 1.3.2. If $g : S \to T$ and $f : T \to U$ are both 1-1, then $f \circ g : S \to U$ is also 1-1.

Proof. Let us suppose that $(f \circ g)(s_1) = (f \circ g)(s_2)$; thus, by definition, $f(g(s_1)) = f(g(s_2))$. Since f is 1-1, we get from this that $g(s_1) = g(s_2)$; however, g is also 1-1, thus $s_1 = s_2$ follows. Since $(f \circ g)(s_1) = (f \circ g)(s_2)$ forces $s_1 = s_2$, the mapping $f \circ g$ is 1-1. □

We leave the proof of the next Remark to the reader.

Remark. If $g : S \to T$ and $f : T \to U$ are both onto, then $f \circ g : S \to U$ is also onto.

An immediate consequence of combining the Remark and Lemma 1.3.2 is to obtain

Lemma 1.3.3. If $g : S \to T$ and $f : T \to U$ are both bijections, then $f \circ g : S \to U$ is also a bijection.

If f is a 1-1 correspondence of S onto T, then the "object" $f^{-1} : T \to S$ defined earlier can easily be shown to be a 1-1 mapping of T onto S. In this case it is called the *inverse* of f. In this situation we have

Lemma 1.3.4. If $f : S \to T$ is a bijection, then $f \circ f^{-1} = i_T$ and $f^{-1} \circ f = i_S$, where i_S and i_T are the identity mappings of S and T, respectively.

Proof. We verify one of these. If $t \in T$, then $(f \circ f^{-1})(t) = f(f^{-1}(t))$. But what is $f^{-1}(t)$? By definition, $f^{-1}(t)$ is that element $s_0 \in S$ such that

$t = f(s_0)$. So $f(f^{-1}(t)) = f(s_0) = t$. In other words, $(f \circ f^{-1})(t) = t$ for every $t \in T$; hence $f \circ f^{-1} = i_T$, the identity mapping on T. $\square$

We leave the last result of this section for the reader to prove.

Lemma 1.3.5. If $f: S \to T$ and i_T is the identity mapping of T onto itself and i_S is that of S onto itself, then $i_T \circ f = f$ and $f \circ i_S = f$.

PROBLEMS

Easier Problems

1. For the given sets S, T determine if a mapping $f: S \to T$ is clearly and unambiguously defined; if not, say why not.
- **(a)** S = set of all women, T = set of all men, $f(s)$ = husband of s.
- **(b)** S = set of positive integers, $T = S$, $f(s) = s - 1$.
- **(c)** S = set of positive integers, T = set of nonnegative integers, $f(s) = s - 1$.
- **(d)** S = set of nonnegative integers, $T = S$, $f(s) = s - 1$.
- **(e)** S = set of all integers, $T = S$, $f(s) = s - 1$.
- **(f)** S = set of all real numbers, $T = S$, $f(s) = \sqrt{s}$.
- **(g)** S = set of all positive real numbers, $T = S$, $f(s) = \sqrt{s}$.

2. In those parts of Problem 1 where f does define a function, determine if it is 1-1, onto, or both.

***3.** If f is a 1-1 mapping of S onto T, prove that f^{-1} is a 1-1 mapping of T onto S.

***4.** If f is a 1-1 mapping of S onto T, prove that $f^{-1} \circ f = i_S$.

5. Give a proof of the Remark after Lemma 1.3.2.

***6.** If $f: S \to T$ is onto and $g: T \to U$ and $h: T \to U$ are such that $g \circ f = h \circ f$, prove that $g = h$.

***7.** If $g: S \to T$, $h: S \to T$, and if $f: T \to U$ is 1-1, show that if $f \circ g = f \circ h$, then $g = h$.

8. Let S be the set of all integers and $T = \{1, -1\}$; $f: S \to T$ is defined by $f(s) = 1$ if s is even, $f(s) = -1$ if s is odd.
- **(a)** Does this define a function from S to T?
- **(b)** Show that $f(s_1 + s_2) = f(s_1)f(s_2)$. What does this say about the integers?
- **(c)** Is $f(s_1 s_2) = f(s_1)f(s_2)$ also true?