Chapter 5

Elliptic Curves and Cryptography

The subject of elliptic curves encompasses a vast amount of mathematics.[1] Our aim in this section is to summarize just enough of the basic theory for cryptographic applications. For additional reading, there are a number of survey articles and books devoted to elliptic curve cryptography [14, 63, 72, 125], and many others that describe the number theoretic aspects of the theory of elliptic curves, including [25, 60, 68, 69, 123, 124, 127].

5.1 Elliptic curves

An *elliptic curve*[2] is the set of solutions to an equation of the form

$$Y^2 = X^3 + AX + B.$$

Equations of this type are called *Weierstrass equations* after the mathematician who studied them extensively during the 19^{th} century. Two examples of elliptic curves,

$$E_1 : Y^2 = X^3 - 3X + 3 \qquad \text{and} \qquad E_2 : Y^2 = X^3 - 6X + 5,$$

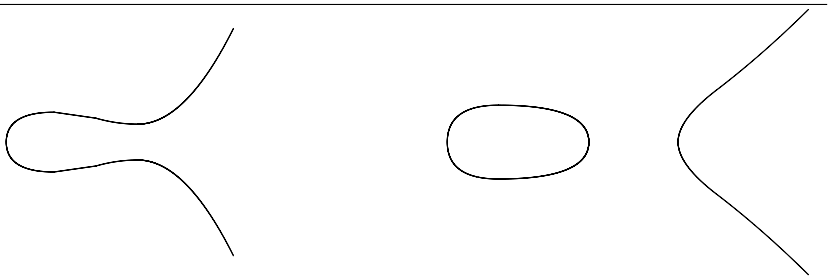
are illustrated in Figure 5.1.

An amazing feature of elliptic curves is that there is a natural way to take two points on an elliptic curve and "add" them to produce a third point. We

[1] Indeed, even before elliptic curves burst into cryptographic prominence, a well-known mathematician [68] opined that "it is possible to write endlessly on elliptic curves!"

[2] A word of warning. You may recall from high school geometry that an ellipse is a geometric object that looks like a squashed circle. Elliptic curves are *not* ellipses, and indeed, despite their somewhat unfortunate name, elliptic curves and ellipses have only the most tenuous connection with one another.

J. Hoffstein et al., *An Introduction to Mathematical Cryptography*,
DOI: 10.1007/978-0-387-77994-2_5,



$E_1 : Y^2 = X^3 - 3X + 3$ $E_2 : Y^2 = X^3 - 6X + 5$

Figure 5.1: Two examples of elliptic curves

put quotation marks around "add" because we are referring to an operation that combines two points in a manner analogous to addition in some respects (it is commutative and associative, and there is an identity), but very unlike addition in other ways. The most natural way to describe the "addition law" on elliptic curves is to use geometry.

Let P and Q be two points on an elliptic curve E, as illustrated in Figure 5.2. We start by drawing the line L through P and Q. This line L intersects E at three points, namely P, Q, and one other point R. We take that point R and reflect it across the x-axis (i.e., we multiply its Y-coordinate by -1) to get a new point R'. The point R' is called the "sum of P and Q," although as you can see, this process is nothing like ordinary addition. For now, we denote this strange addition law by the symbol $\oplus$. Thus we write[3]

$$P \oplus Q = R'.$$

Example 5.1. Let E be the elliptic curve

$$Y^2 = X^3 - 15X + 18. \tag{5.1}$$

The points $P = (7, 16)$ and $Q = (1, 2)$ are on the curve E. The line L connecting them is given by the equation[4]

$$L : Y = \frac{7}{3}X - \frac{1}{3}. \tag{5.2}$$

In order to find the points where E and L intersect, we substitute (5.2) into (5.1) and solve for X. Thus

[3]Not to be confused with the identical symbol $\oplus$ that we used to denote the XOR operation in a different context!

[4]Recall that the equation of the line through two points (x_1, y_1) and (x_2, y_2) is given by the point–slope formula $Y - y_1 = \lambda \cdot (X - x_1)$, where the slope λ is equal to $\frac{y_2 - y_1}{x_2 - x_1}$.

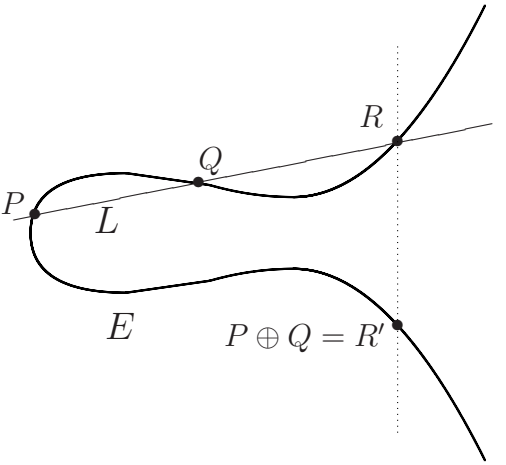


Figure 5.2: The addition law on an elliptic curve

$$\begin{aligned}\left(\frac{7}{3}X - \frac{1}{3}\right)^2 &= X^3 - 15X + 18,\\ \frac{49}{9}X^2 - \frac{14}{9}X + \frac{1}{9} &= X^3 - 15X + 18,\\ 0 &= X^3 - \frac{49}{9}X^2 - \frac{121}{9}X + \frac{161}{9}.\end{aligned}$$

We need to find the roots of this cubic polynomial. In general, finding the roots of a cubic is difficult. However, in this case we already know two of the roots, namely $X = 7$ and $X = 1$, since we know that P and Q are in the intersection $E \cap L$. It is then easy to find the other factor,

$$X^3 - \frac{49}{9}X^2 - \frac{121}{9}X + \frac{161}{9} = (X-7)\cdot(X-1)\cdot\left(X + \frac{23}{9}\right),$$

so the third point of intersection of L and E has X-coordinate equal to $-\frac{23}{9}$. Next we find the Y-coordinate by substituting $X = -\frac{23}{9}$ into equation (5.2). This gives $R = \left(-\frac{23}{9}, \frac{170}{27}\right)$. Finally, we reflect across the X-axis to obtain

$$P \oplus Q = \left(-\frac{23}{9}, -\frac{170}{27}\right).$$

There are a few subtleties to elliptic curve addition that need to be addressed. First, what happens if we want to add a point P to itself? Imagine what happens to the line L connecting P and Q if the point Q slides along the curve and gets closer and closer to P. In the limit, as Q approaches P, the line L becomes the tangent line to E at P. Thus in order to add P to

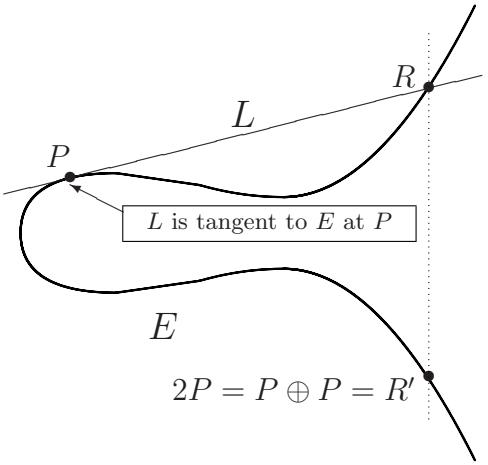


Figure 5.3: Adding a point P to itself

itself, we simply take L to be the tangent line to E at P, as illustrated in Figure 5.3. Then L intersects E at P and at one other point R, so we can proceed as before. In some sense, L still intersects E at three points, but P counts as two of them.

Example 5.2. Continuing with the curve E and point P from Example 5.1, we compute $P \oplus P$. The slope of E at P is computed by implicitly differentiating equation (5.1). Thus

$$2Y\frac{dY}{dX} = 3X^2 - 15, \qquad \text{so} \qquad \frac{dY}{dX} = \frac{3X^2 - 15}{2Y}.$$

Substituting the coordinates of $P = (7, 16)$ gives slope $\lambda = \frac{33}{8}$, so the tangent line to E at P is given by the equation

$$L : Y = \frac{33}{8}X - \frac{103}{8}. \tag{5.3}$$

Now we substitute (5.3) into the equation (5.1) for E, simplify, and factor:

$$\begin{aligned}\left(\frac{33}{8}X - \frac{103}{8}\right)^2 &= X^3 - 15X + 18,\\ X^3 - \frac{1089}{64}X^2 + \frac{2919}{32}X - \frac{9457}{64} &= 0,\\ (X-7)^2 \cdot \left(X - \frac{193}{64}\right) &= 0.\end{aligned}$$

Notice that the X-coordinate of P, which is $X = 7$, appears as a double root of the cubic polynomial, so it was easy for us to factor the cubic. Finally, we

substitute $X = \frac{193}{64}$ into the equation (5.3) for L to get $Y = -\frac{223}{512}$, and then we switch the sign on Y to get

$$P \oplus P = \left(\frac{193}{64}, \frac{223}{512}\right).$$

A second potential problem with our "addition law" arises if we try to add a point $P = (a, b)$ to its reflection about the X-axis $P' = (a, -b)$. The line L through P and P' is the vertical line $x = a$, and this line intersects E in only the two points P and P'. (See Figure 5.4.) There is no third point of intersection, so it appears that we are stuck! But there is a way out. The solution is to create an extra point $\mathcal{O}$ that lives "at infinity." More precisely, the point $\mathcal{O}$ does not exist in the XY-plane, but we pretend that it lies on every vertical line. We then set

$$P \oplus P' = \mathcal{O}.$$

We also need to figure out how to add $\mathcal{O}$ to an ordinary point $P = (a, b)$ on E. The line L connecting P to $\mathcal{O}$ is the vertical line through P, since $\mathcal{O}$ lies on vertical lines, and that vertical line intersects E at the points P, $\mathcal{O}$, and $P' = (a, -b)$. To add P to $\mathcal{O}$, we reflect P' across the X-axis, which gets us back to P. In other words, $P \oplus \mathcal{O} = P$, so $\mathcal{O}$ acts like zero for elliptic curve addition.

Example 5.3. Continuing with the curve E from Example 5.1, notice that the point $T = (3, 0)$ is on the curve E and that the tangent line to E at T is the vertical line $X = 3$. Thus if we add T to itself, we get $T \oplus T = \mathcal{O}$.

Definition. An *elliptic curve* E is the set of solutions to a Weierstrass equation

$$E : Y^2 = X^3 + AX + B,$$

together with an extra point $\mathcal{O}$, where the constants A and B must satisfy

$$4A^3 + 27B^2 \neq 0.$$

The *addition law on* E is defined as follows. Let P and Q be two points on E. Let L be the line connecting P and Q, or the tangent line to E at P if $P = Q$. Then the intersection of E and L consists of three points P, Q, and R, counted with appropriate multiplicities and with the understanding that $\mathcal{O}$ lies on every vertical line. Writing $R = (a, b)$, the sum of P and Q is defined to be the reflection $R' = (a, -b)$ of R across the X-axis. This sum is denoted by $P \oplus Q$, or simply by $P + Q$.

Further, if $P = (a, b)$, we denote the reflected point by $\ominus P = (a, -b)$, or simply by $-P$; and we define $P \ominus Q$ (or $P - Q$) to be $P \oplus (\ominus Q)$. Similarly, repeated addition is represented as multiplication of a point by an integer,

$$nP = \underbrace{P + P + P + \cdots + P}_{n \text{ copies}}.$$

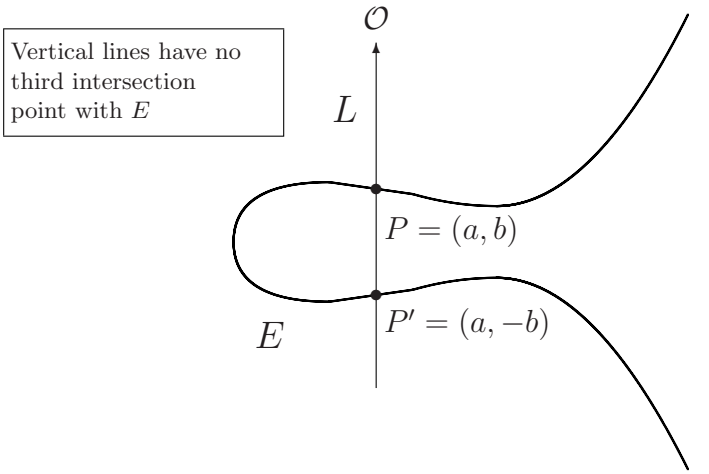


Figure 5.4: The vertical line L through $P = (a, b)$ and $P' = (a, -b)$

Remark 5.4. What is this extra condition $4A^3 + 27B^2 \neq 0$? The quantity $\Delta_E = 4A^3 + 27B^2$ is called the *discriminant of* E. The condition $\Delta_E \neq 0$ is equivalent to the condition that the cubic polynomial $X^3 + AX + B$ have no repeated roots, i.e., if we factor $X^3 + AX + B$ completely as

$$X^3 + AX + B = (X - e_1)(X - e_2)(X - e_3),$$

where e_1, e_2, e_3 are allowed to be complex numbers, then

$$4A^3 + 27B^2 \neq 0 \qquad \text{if and only if} \qquad e_1, e_2, e_3 \text{ are distinct.}$$

(See Exercise 5.3.) Curves with $\Delta_E = 0$ have singular points (see Exercise 5.4). The addition law does not work well on these curves. That is why we include the requirement that $\Delta_E \neq 0$ in our definition of an elliptic curve.

Theorem 5.5. *Let E be an elliptic curve. Then the addition law on E has the following properties:*

(a)	$P + \mathcal{O} = \mathcal{O} + P = P$	*for all* $P \in E$.	[Identity]
(b)	$P + (-P) = \mathcal{O}$	*for all* $P \in E$.	[Inverse]
(c)	$(P + Q) + R = P + (Q + R)$	*for all* $P, Q, R \in E$.	[Associative]
(d)	$P + Q = Q + P$	*for all* $P, Q \in E$.	[Commutative]

In other words, the addition law makes the points of E into an abelian group. (*See Section 2.5 for a general discussion of groups and their axioms.*)

Proof. As we explained earlier, the identity law (a) and inverse law (b) are true because $\mathcal{O}$ lies on all vertical lines. The commutative law (d) is easy to

verify, since the line that goes through P and Q is the same as the line that goes through Q and P, so the order of the points does not matter.

The remaining piece of Theorem 5.5 is the associative law (c). One might not think that this would be hard to prove, but if you draw a picture and start to put in all of the lines needed to verify (c), you will see that it is quite complicated. There are many ways to prove the associative law, but none of the proofs are easy. After we develop explicit formulas for the addition law on E (Theorem 5.6), you can use those formulas to check the associative law by a direct (but painful) calculation. More perspicacious, but less elementary, proofs may be found in [69, 123, 127] and other books on elliptic curves. □

Our next task is to find explicit formulas to enable us to easily add and subtract points on an elliptic curve. The derivation of these formulas uses elementary analytic geometry, a little bit of differential calculus to find a tangent line, and a certain amount of algebraic manipulation. We state the results in the form of an algorithm, and then briefly indicate the proof.

Theorem 5.6 (Elliptic Curve Addition Algorithm)**.** *Let*

$$E : Y^2 = X^3 + AX + B$$

be an elliptic curve and let P_1 and P_2 be points on E.
(a) *If $P_1 = \mathcal{O}$, then $P_1 + P_2 = P_2$.*
(b) *Otherwise, if $P_2 = \mathcal{O}$, then $P_1 + P_2 = P_1$.*
(c) *Otherwise, write $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$.*
(d) *If $x_1 = x_2$ and $y_1 = -y_2$, then $P_1 + P_2 = \mathcal{O}$.*
(e) *Otherwise, define λ by*

$$\lambda = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} & \text{if } P_1 \neq P_2, \\[2ex] \dfrac{3x_1^2 + A}{2y_1} & \text{if } P_1 = P_2, \end{cases}$$

and let

$$x_3 = \lambda^2 - x_1 - x_2 \qquad \text{and} \qquad y_3 = \lambda(x_1 - x_3) - y_1.$$

Then $P_1 + P_2 = (x_3, y_3)$.

Proof. Parts (a) and (b) are clear, and (d) is the case that the line through P_1 and P_2 is vertical, so $P_1 + P_2 = \mathcal{O}$. (Note that if $y_1 = y_2 = 0$, then the tangent line is vertical, so that case works, too.) For (e), we note that if $P_1 \neq P_2$, then λ is the slope of the line through P_1 and P_2, and if $P_1 = P_2$, then λ is the slope of the tangent line at $P_1 = P_2$. In either case the line L is given by the equation $Y = \lambda X + \nu$ with $\nu = y_1 - \lambda x_1$. Substituting the equation for L into the equation for E gives

$$(\lambda X + \nu)^2 = X^3 + AX + B,$$

so

$$X^3 - \lambda^2 X^2 + (A - 2\lambda\nu)X + (B - \nu^2) = 0.$$

We know that this cubic has x_1 and x_2 as two of its roots. If we call the third root x_3, then it factors as

$$X^3 - \lambda^2 X^2 + (A - 2\lambda\nu)X + (B - \nu^2) = (X - x_1)(X - x_2)(X - x_3).$$

Now multiply out the right-hand side and look at the coefficient of X^2 on each side. The coefficient of X^2 on the right-hand side is $-x_1 - x_2 - x_3$, which must equal $-\lambda^2$, the coefficient of X^2 on the left-hand side. This allows us to solve for $x_3 = \lambda^2 - x_1 - x_2$, and then the Y-coordinate of the third intersection point of E and L is given by $\lambda x_3 + \nu$. Finally, in order to get $P_1 + P_2$, we must reflect across the X-axis, which means replacing the Y-coordinate with its negative. □

5.2 Elliptic curves over finite fields

In the previous section we developed the theory of elliptic curves geometrically. For example, the sum of two distinct points P and Q on an elliptic curve E is defined by drawing the line L connecting P to Q and then finding the third point where L and E intersect, as illustrated in Figure 5.2. However, in order to apply the theory of elliptic curves to cryptography, we need to look at elliptic curves whose points have coordinates in a finite field $\mathbb{F}_p$. This is easy to do. We simply define an *elliptic curve over* $\mathbb{F}_p$ to be an equation of the form

$$E : Y^2 = X^3 + AX + B \qquad \text{with } A, B \in \mathbb{F}_p \text{ satisfying } 4A^3 + 27B^2 \neq 0,$$

and then we look at the points on E with coordinates in $\mathbb{F}_p$, which we denote by

$$E(\mathbb{F}_p) = \big\{(x, y) : x, y \in \mathbb{F}_p \text{ satisfy } y^2 = x^3 + Ax + B\big\} \cup \{\mathcal{O}\}.$$

Remark 5.7. For reasons that are explained later, we also require that $p \geq 3$. Elliptic curves over $\mathbb{F}_2$ are actually quite important in cryptography, but they are somewhat more complicated, so we delay our discussion of them until Section 5.7.

Example 5.8. Consider the elliptic curve

$$E : Y^2 = X^3 + 3X + 8 \quad \text{over the field } \mathbb{F}_{13}.$$

We can find the points of $E(\mathbb{F}_{13})$ by substituting in all possible values $X = 0, 1, 2, \ldots, 12$ and checking for which X values the quantity $X^3 + 3X + 8$ is a square modulo 13. For example, putting $X = 0$ gives 8, and 8 is not a square

modulo 13. Next we try $X = 1$, which gives $1+3+8 = 12$. It turns out that 12 is a square modulo 13; in fact, it has two square roots,

$$5^2 \equiv 12 \pmod{13} \qquad \text{and} \qquad 8^2 \equiv 12 \pmod{13}.$$

This gives two points $(1, 5)$ and $(1, 8)$ in $E(\mathbb{F}_{13})$. Continuing in this fashion, we end up with a complete list,

$$E(\mathbb{F}_{13}) = \{\mathcal{O}, (1,5), (1,8), (2,3), (2,10), (9,6), (9,7), (12,2), (12,11)\}.$$

Thus $E(\mathbb{F}_{13})$ consists of nine points.

Suppose now that P and Q are two points in $E(\mathbb{F}_p)$ and that we want to "add" the points P and Q. One possibility is to develop a theory of geometry using the field $\mathbb{F}_p$ instead of $\mathbb{R}$. Then we could mimic our earlier constructions to define $P + Q$. This can be done, and it leads to a fascinating field of mathematics called algebraic geometry. However, in the interests of brevity of exposition, we instead use the explicit formulas given in Theorem 5.6 to add points in $E(\mathbb{F}_p)$. But we note that if one wants to gain a deeper understanding of the theory of elliptic curves, then it is necessary to use some of the machinery and some of the formalism of algebraic geometry.

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points in $E(\mathbb{F}_p)$. We define the sum $P_1 + P_2$ to be the point (x_3, y_3) obtained by applying the elliptic curve addition algorithm (Theorem 5.6). Notice that in this algorithm, the only operations used are addition, subtraction, multiplication, and division involving the coefficients of E and the coordinates of P and Q. Since those coefficients and coordinates are in the field $\mathbb{F}_p$, we end up with a point (x_3, y_3) whose coordinates are in $\mathbb{F}_p$. Of course, it is not completely clear that (x_3, y_3) is a point in $E(\mathbb{F}_p)$.

Theorem 5.9. *Let E be an elliptic curve over $\mathbb{F}_p$ and let P and Q be points in $E(\mathbb{F}_p)$.*

(a) *The elliptic curve addition algorithm (Theorem 5.6) applied to P and Q yields a point in $E(\mathbb{F}_p)$. We denote this point by $P + Q$.*

(b) *This addition law on $E(\mathbb{F}_p)$ satisfies all of the properties listed in Theorem 5.5. In other words, this addition law makes $E(\mathbb{F}_p)$ into a finite group.*

Proof. The formulas in Theorem 5.6(e) are derived by substituting the equation of a line into the equation for E and solving for X, so the resulting point is automatically a point on E, i.e., it is a solution to the equation defining E. This shows why (a) is true, although when $P = Q$, a small additional argument is needed to indicate why the resulting cubic polynomial has a double root. For (b), the identity law follows from the addition algorithm steps (a) and (b), the inverse law is clear from the addition algorithm Step (d), and the commutative law is easy, since a brief examination of the addition algorithm shows that switching the two points leads to the same result. Unfortunately, the associative law is not so clear. It is possible to verify the associative law directly

using the addition algorithm formulas, although there are many special cases to consider. The alternative is to develop more of the general theory of elliptic curves, as is done in the references cited in the proof of Theorem 5.5. □

Example 5.10. We continue with the elliptic curve

$$E : Y^2 = X^3 + 3X + 8 \qquad \text{over } \mathbb{F}_{13}$$

from Example 5.8, and we use the addition algorithm (Theorem 5.6) to add the points $P = (9, 7)$ and $Q = (1, 8)$ in $E(\mathbb{F}_{13})$. Step (e) of that algorithm tells us to first compute

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 7}{1 - 9} = \frac{1}{-8} = \frac{1}{5} = 8,$$

where recall that all computations[5] are being performed in the field $\mathbb{F}_{13}$, so $-8 = 5$ and $\frac{1}{5} = 5^{-1} = 8$. Next we compute

$$\nu = y_1 - \lambda x_1 = 7 - 8 \cdot 9 = -65 = 0.$$

Finally, the addition algorithm tells us to compute

$$\begin{aligned} x_3 &= \lambda^2 - x_1 - x_2 = 64 - 9 - 1 = 54 = 2, \\ y_3 &= -(\lambda x_3 + \nu) = -8 \cdot 2 = -16 = 10. \end{aligned}$$

This completes the computation of

$$P + Q = (1, 8) + (9, 7) = (2, 10) \qquad \text{in } E(\mathbb{F}_{13}).$$

Similarly, we can use the addition algorithm to add $P = (9, 7)$ to itself. Keeping in mind that all calculations are in $\mathbb{F}_{13}$, we find that

$$\lambda = \frac{3x_1^2 + A}{2y_1} = \frac{3 \cdot 9^2 + 3}{2 \cdot 7} = \frac{246}{14} = 1 \quad \text{and} \quad \nu = y_1 - \lambda x_1 = 7 - 1 \cdot 9 = 11.$$

Then

$$x_3 = \lambda^2 - x_1 - x_2 = 1 - 9 - 9 = 9 \quad \text{and} \quad y_3 = -(\lambda x_3 + \nu) = -1 \cdot 9 - 11 = 6,$$

so $P + P = (9, 7) + (9, 7) = (9, 6)$ in $E(\mathbb{F}_{13})$. In a similar fashion, we can compute the sum of every pair of points in $E(\mathbb{F}_{13})$. The results are listed in Table 5.1.

It is clear that the set of points $E(\mathbb{F}_p)$ is a finite set, since there are only finitely many possibilities for the X-and Y-coordinates. More precisely, there are p possibilities for X, and then for each X, the equation

[5]This is a good time to learn that $\frac{1}{5}$ is a *symbol* for a solution to the equation $5x = 1$. In order to assign a value to the symbol $\frac{1}{5}$, you must know where that value lives. In $\mathbb{Q}$, the value of $\frac{1}{5}$ is the usual number with which you are familiar, but in $\mathbb{F}_{13}$ the value of $\frac{1}{5}$ is 8, while in $\mathbb{F}_{11}$ the value of $\frac{1}{5}$ is 9. And in $\mathbb{F}_5$ the symbol $\frac{1}{5}$ is not assigned a value.

	$\mathcal{O}$	$(1,5)$	$(1,8)$	$(2,3)$	$(2,10)$	$(9,6)$	$(9,7)$	$(12,2)$	$(12,11)$
$\mathcal{O}$	$\mathcal{O}$	$(1,5)$	$(1,8)$	$(2,3)$	$(2,10)$	$(9,6)$	$(9,7)$	$(12,2)$	$(12,11)$
$(1,5)$	$(1,5)$	$(2,10)$	$\mathcal{O}$	$(1,8)$	$(9,7)$	$(2,3)$	$(12,2)$	$(12,11)$	$(9,6)$
$(1,8)$	$(1,8)$	$\mathcal{O}$	$(2,3)$	$(9,6)$	$(1,5)$	$(12,11)$	$(2,10)$	$(9,7)$	$(12,2)$
$(2,3)$	$(2,3)$	$(1,8)$	$(9,6)$	$(12,11)$	$\mathcal{O}$	$(12,2)$	$(1,5)$	$(2,10)$	$(9,7)$
$(2,10)$	$(2,10)$	$(9,7)$	$(1,5)$	$\mathcal{O}$	$(12,2)$	$(1,8)$	$(12,11)$	$(9,6)$	$(2,3)$
$(9,6)$	$(9,6)$	$(2,3)$	$(12,11)$	$(12,2)$	$(1,8)$	$(9,7)$	$\mathcal{O}$	$(1,5)$	$(2,10)$
$(9,7)$	$(9,7)$	$(12,2)$	$(2,10)$	$(1,5)$	$(12,11)$	$\mathcal{O}$	$(9,6)$	$(2,3)$	$(1,8)$
$(12,2)$	$(12,2)$	$(12,11)$	$(9,7)$	$(2,10)$	$(9,6)$	$(1,5)$	$(2,3)$	$(1,8)$	$\mathcal{O}$
$(12,11)$	$(12,11)$	$(9,6)$	$(12,2)$	$(9,7)$	$(2,3)$	$(2,10)$	$(1,8)$	$\mathcal{O}$	$(1,5)$

Table 5.1: Addition table for $E : Y^2 = X^3 + 3X + 8$ over $\mathbb{F}_{13}$

$$Y^2 = X^3 + AX + B$$

shows that there are at most two possibilities for Y. (See Exercise 1.34.) Adding in the extra point $\mathcal{O}$, this shows that $\#E(\mathbb{F}_p)$ has at most $2p+1$ points. However, this estimate is considerably larger than the true size.

When we plug in a value for X, there are three possibilities for the value of the quantity

$$X^3 + AX + B.$$

First, it may be a quadratic residue modulo p, in which case it has two square roots and we get two points in $E(\mathbb{F}_p)$. This happens about 50% of the time. Second, it may be a nonresidue modulo p, in which case we discard X. This also happens about 50% of the time. Third, it might equal 0, in which case we get one point in $E(\mathbb{F}_p)$, but this case happens very rarely.[6] Thus we might expect that the number of points in $E(\mathbb{F}_p)$ is approximately

$$\#E(\mathbb{F}_p) \approx 50\% \cdot 2 \cdot p + 1 = p + 1.$$

A famous theorem of Hasse, later vastly generalized by Weil and Deligne, says that this is true up to random fluctuations.

Theorem 5.11 (Hasse)**.** *Let E be an elliptic curve over $\mathbb{F}_p$. Then*

$$\#E(\mathbb{F}_p) = p + 1 - t_p \qquad \textit{with } t_p \textit{ satisfying} \quad |t_p| \le 2\sqrt{p}.$$

Definition. The quantity

$$t_p = p + 1 - \#E(\mathbb{F}_p)$$

appearing in Theorem 5.11 is called the *trace of Frobenius* for $E/\mathbb{F}_p$. We will not explain the somewhat technical reasons for this name, other than to say that t_p appears as the trace of a certain 2-by-2 matrix that acts as a linear transformation on a certain two-dimensional vector space associated to $E/\mathbb{F}_p$.

[6]The congruence $X^3 + AX + B \equiv 0 \pmod{p}$ has at most three solutions, and if p is large, the chance of randomly choosing one of them is very small.