Week 2 / Exercise 5

Show $\mathcal{E}_1$ and $\mathcal{E}_2$ are independent iff $P(\mathcal{E}_1 \cap \mathcal{E}_2) = P(\mathcal{E}_1) \cdot P(\mathcal{E}_2)$ ⓪

Our definition of independence is:

$$Pr[\mathcal{E}_1 \mid \mathcal{E}_2] = Pr[\mathcal{E}_1] \quad ①$$

Starting with our definition of conditional probability:

$$Pr[\mathcal{E}_1 \mid \mathcal{E}_2] = Pr[\mathcal{E}_1 \cap \mathcal{E}_2] \,/\, Pr[\mathcal{E}_2]$$

$$Pr[\mathcal{E}_1 \mid \mathcal{E}_2] \cdot Pr[\mathcal{E}_2] = Pr[\mathcal{E}_1 \cap \mathcal{E}_2] \quad ②$$

We know from our definition of independence that $Pr[\mathcal{E}_1 \mid \mathcal{E}_2] = Pr[\mathcal{E}_1]$

Substitute ② into ①:

$$Pr[\mathcal{E}_1] \cdot Pr[\mathcal{E}_2] = Pr[\mathcal{E}_1 \cap \mathcal{E}_2]$$

We have proven one direction:
$\mathcal{E}_1$, $\mathcal{E}_2$ are independent $\rightarrow$ ⓪ holds

Now we want to prove:
⓪ holds $\rightarrow$ $\mathcal{E}_1$, $\mathcal{E}_2$ are independent

Start with: $P(\mathcal{E}_1 \cap \mathcal{E}_2) = P(\mathcal{E}_1) \cdot P(\mathcal{E}_2)$

$$P(\mathcal{E}_1 \cap \mathcal{E}_2) \,/\, P(\mathcal{E}_2) = P(\mathcal{E}_1) \quad (\text{LHS} = \text{cond. def})$$

$$Pr[\mathcal{E}_1 \mid \mathcal{E}_2] = P[\mathcal{E}_1]$$

This meets our definition of independence.