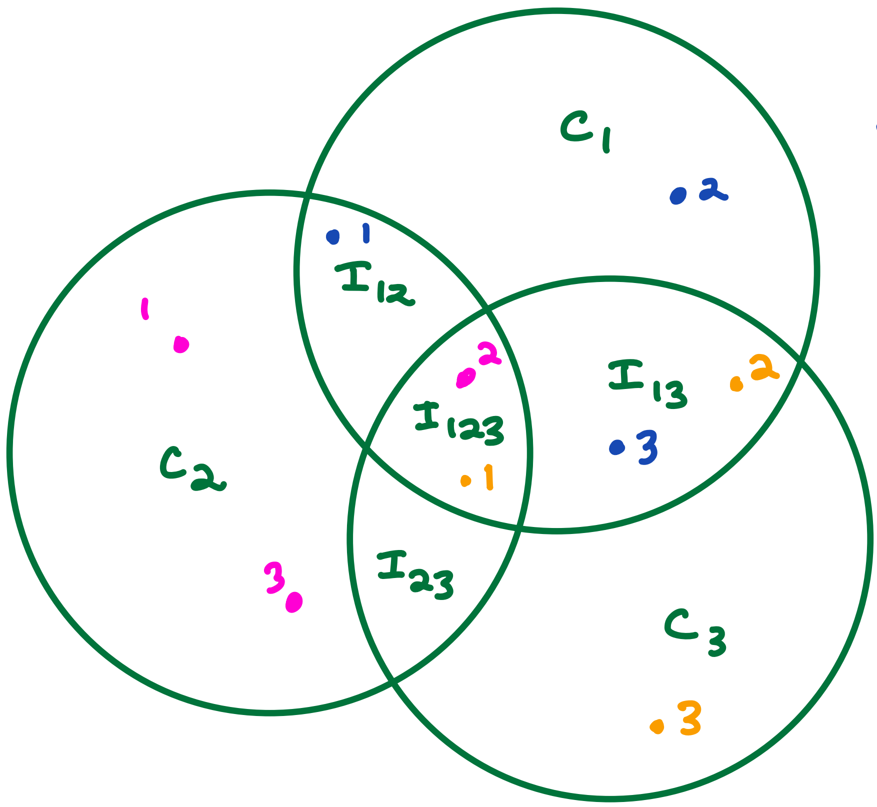
Multiple Plant Growth Model

To scale from a single plant to multiple plants we must consider competing canopies

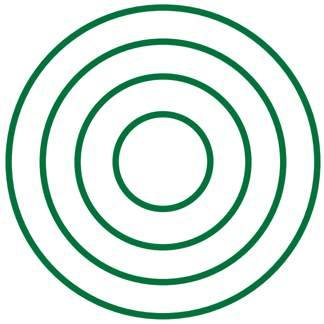
→ can be described by proportion of a lower canopy covered by a higher canopy

Consider 3 overlapping canopies



must have same # of test points per canopy

Finding I_{123} analytically can be difficult. Thus, a numerical method is used.



Let each canopy be composed of 5000 points arranged in concentric circles.

Say canopy c_i has center coordinates (c_{ix}, c_{iy}) with radius r_i.

Say canopy c_j has center coordinates (c_{jx}, c_{jy}) with radius r_j.

Consider point k, known to lie within canopy c_i, with coordinates (x_k, y_k).

We say point k is also within canopy c_j iff

$$\sqrt{(x_k - c_{jx})^2 + (y_k - c_{jy})^2} < r_j$$

We can then construct an $n \times m$ matrix A, where n is the # of sampling points in a canopy and m is the number of plants in the canopy, such that

$$B_{kj} = \begin{cases} 1 & \text{if } \sqrt{(x_k - c_{jx})^2 + (y_k - c_{jy})^2} < r_j \\ 0 & \text{otherwise} \end{cases}$$

"belonging matrix"

where $k = 1, \dots, n$ and $j = 1, \dots, m$.

$\uparrow$ # of points per canopy

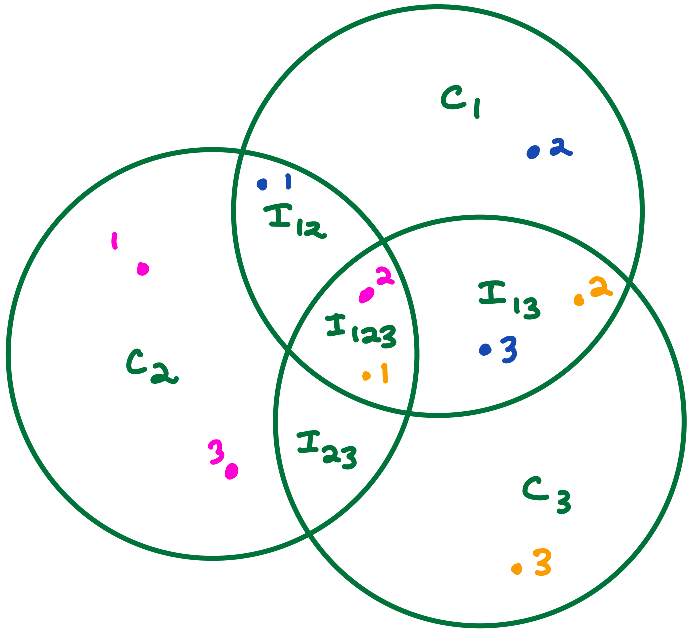
$\uparrow$ # of canopies

For the example canopy above with 3 canopies and 9 test points, $n=3$ and $m=3$

$$B^{(1)} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

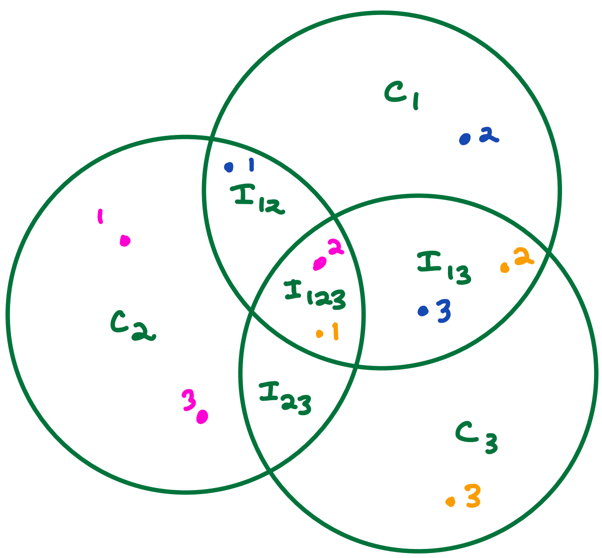
$$B^{(3)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

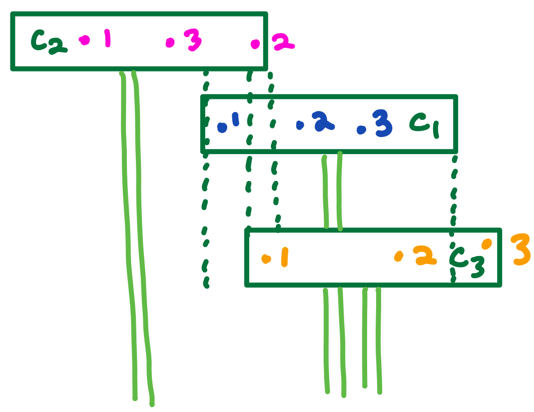


Let the heights of the 3 canopies be

$$h_1 = 0.7,\ h_2 = 0.8,\ h_3 = 0.5.$$



$\Longrightarrow$



To account for plant heights define an $m \times 1$ vector

$$y^{(k)} = H(h_k - h_i)$$

where h_i is the height of plant i, h_k is the height of plant k and H is the Heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow y^{(i)}_k = \begin{cases} 1 & \text{if } h_i \leq h_k \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow y^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad y^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad y^{(3)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $C^{(k)}$ be an $m \times 1$ vector such that

$$C^{(k)} = B^{(k)} y^{(k)} - 1$$

Then the C vectors represent counts of the canopies above a particular point.

$$\Rightarrow \quad c^{(1)} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c^{(3)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - 1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

We can use the c vectors to modify R_0 for each test point in the canopy growth rate equation:

$$R(t) = R_0 (1 - \exp\{-\kappa \gamma(t)\})$$

In fact, we will say

$$R_0^{(i)} = R_0 \left[1 - \frac{1}{n} \sum_{k=1}^{n} c_k^{(i)} \alpha \right]$$

where α is the absorption factor for a single leaf of the crop.

$R_0^{(1)} = R_0 (1 - \alpha/3)$ $\quad$ $R_0^{(2)} = R_0$ $\quad$ $R_0^{(3)} = R_0 (1 - \alpha)$

Using the single plant growth model as a starting point, and adding some terms for competition between plants, we obtain the following for crop growth:

Plant height $h(t)$

$$\frac{dh(t)}{dt} = a_h R(t) h(t) \left(1 - \frac{h(t)}{K_h}\right) - d_h h(t)$$

Leaf Area $A(t)$

$$\frac{dA(t)}{dt} = L_A a_L(T_{avg}, t) \exp\left\{-\left(\frac{T_{cum}(t) - b_L(T_{avg}, t)}{c_L(T_{avg}, t)}\right)^2\right\} - d_L T_{SL}(T_{avg}, t) A(t)$$

Ground Cover $G(t)$

$$\frac{dG(t)}{dt} = a_g A(t) \exp\left\{-\left(\frac{t - b_g}{c_g}\right)^2\right\}$$

Leaf Area Index $\gamma(t)$

$$\gamma(t) = A(t) / G(t)$$

Radiation $R(t)$

$$R(t) = R_0 \left(1 - \exp\{-K\gamma(t)\}\right)$$

Canopy Biomass $c(t)$

$$\frac{dc(t)}{dt} = c_e R(t) G(t) \left(1 - \frac{c(t)}{K_c}\right) - d_c c(t)$$

Fruit Biomass $F(t)$

$$\frac{dF(t)}{dt} = a_f T_{sf}(T_{avg}) \frac{dc(t)}{dt} F(t) \left\{1 - \frac{F(t)}{c(t)}\right\} - d_f F(t)$$

Variable Glossary

a_h = plant height growth rate

a_L = leaf area growth rate

a_g = ground cover growth rate

a_f = fruit biomass growth rate

m = # of plants

h_0 = initial plant height

A_0 = initial leaf area

G_0 = initial ground cover

C_0 = initial canopy cover

F_0 = initial fruit mass

b_L = determines time of peak growth of leaf area

b_g = determines time of peak growth of ground cover

c_L = determines duration of peak growth time (leaves)

c_g = determines duration of peak growth time (canopy)

d_h = plant height decay rate

d_L = leaf decay rate

d_c = canopy decay rate

d_f = fruit decay rate

K_h = maximum plant height (height carrying capacity)

K_c = maximum canopy area (canopy carrying capacity)

L_A = leaf area per leaf

$\theta_R(t)$ = incident angle of solar radiation, measured from horizontal

c_e = photosynthesis efficiency coefficient

k = extinction coefficient for photosynthesis

θ_L = leaf angle

ω = tuning parameter for pod temperature sensitivity

R_0 = available photosynthetically active at canopy

DAS = days after sowing

t_{final} = total sim time

α = absorption factor for a single leaf

ρ = density of leaves in a single canopy

ρ_{comp} = density of test points for canopy competition

τ = thickness of canopy

T_{opt} = optimal temp for pod growth

T_{ceil} = temp above which pods will not grow

T_{crit} = temp below which pods will not grow

$$T_{avg}(t) = \frac{1}{24} \sum_{j=1}^{24} T_j(t)$$ daily average temp

$$T_{eff}(T_{avg}, t) = \frac{1}{24} \sum_{j=1}^{24} \left(T_j(t) - T_{crit}\right)$$ daily effective temp

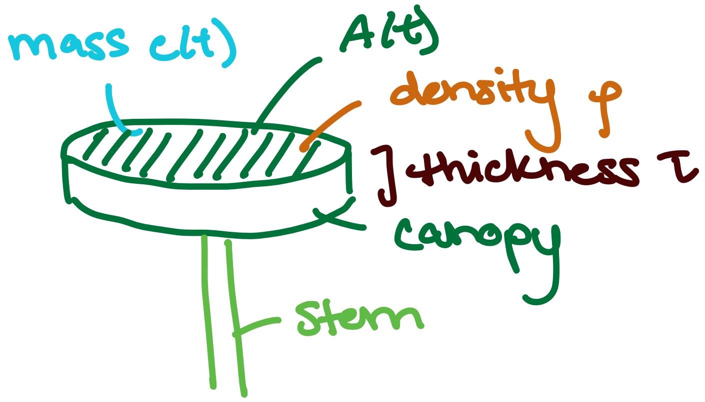
$$T_{cum}(T_{avg}, t) = \int_0^{DAS} T_{eff}(T_{avg}, t)\, dt$$ cumulative thermal time

$$T_{SL}(T_{avg}, t) = \frac{T_{opt} - T_{crit}}{T_{avg}(t) - T_{crit}}$$ leaf sensitivity to temp

$$T_{sf}(T_{avg}) = \begin{cases} 1, & T_{avg} \leq T_{opt} \\ 1 - \left|1 - \omega \dfrac{T_{avg} - T_{crit}}{T_{opt} - T_{crit}}\right|, & T_{opt} < T_{avg} < T_{ceil} \\ 0 & T_{avg} \geq T_{ceil} \end{cases}$$

fruit sensitivity to temp

For canopy $c(t)$, it will have density φ and thickness τ because we are treating it as a disk of constant thickness and varying radius:



Leaf area $A(t)$ is then given by

$$\varphi = \frac{c}{A\tau} \Rightarrow \boxed{A = \frac{c}{\varphi\tau}}$$

At time t, the canopy area A is then represented as n test points where

$$n = \text{floor}(\varphi A)$$

From which we can construct the m matrices

$$A^{(1)}, \dots, A^{(m)} \in \mathbb{R}^{n \times m}$$

the m vectors

$$\eta^{(1)}, \dots, \eta^{(m)} \in \mathbb{R}^{n \times 1}$$

and the m vectors

$$c^{(1)}, \dots, c^{(m)} \in \mathbb{R}^{n \times 1}$$

then use the c vectors to create m scalars

$$R_0^{(1)}, \dots, R_0^{(m)} \in \mathbb{R}^{1\times 1}$$

These scalars can then be used in the next time step for the height and canopy models.

Below are the discretized model equations

Plant height

need $h(0) = h_0 \neq 0$

$$h(t+\Delta t) = \Delta t \left[a_h^{R(t)} h(t) \left(1 - \frac{h(t)}{K_h}\right) - d_h h(t) \right] + h(t)$$

Leaf Area

$A(0) = A_0 = 0$

$$A(t+\Delta t) = \left[L_A a_L(T(t), t) \exp\left\{ -\left(\frac{T_c(t) - b_L(T(t), t)}{c_L(T(t), t)} \right)^2 \right\} - d_L T_{SL}(T(t), t) A(t) \right] \Delta t + A(t)$$

Ground Cover

$G(0) = G_0 = 0$

$$G(t+\Delta t) = a_g A(t) \exp\left\{ -\left(\frac{t - b_g}{c_g} \right)^2 \right\} \Delta t + G(t)$$

Leaf Area Index $\gamma(t)$

$$\gamma(t) = A(t) / G(t)$$

Radiation

$R(0) = R_0$ ← changes based on time of day/year

$$R(t) = R_0\left(1 - \exp\{-K\gamma(t)\}\right)$$

Canopy Biomass

$c(0) = c_0 = 0$

$$c(t+\Delta t) = \left[c_e R(t)\, G(t)\left(1 - \frac{c(t)}{K_c}\right) - d_c\, c(t)\right]\Delta t + c(t)$$

Fruit Biomass

$F(0) = F_0 \neq 0$

$$F(t+\Delta t) = \left[a_f\, T_{sp}\{T(t)\}\, \frac{dc(t)}{dt}\, F(t)\left\{1 - \frac{F(t)}{c(t)}\right\} - d_f F(t)\right]\Delta t + F(t)$$

Algorithm

① Set all variables in the variable glossary.

② Set coordinates for all m plants.

③ If 24 hours have passed since the last update of T_{avg}, T_{eff}, T_{cum}, T_{SL}, and T_{sf}, then update them. Otherwise continue to next step.

④ For each of the m plants, discretize the canopies C according to p_{comp} and find R_0 for each plant.

5. Use solar model to determine $\theta_R(t)$, then calculate $\theta = \theta_R - \theta_L$ and then K, the extinction coefficient for photosynthesis.

6. Evaluate $A(t+\Delta t)$, $G(t+\Delta t)$, $\delta(t+\Delta t)$.

7. Evaluate $R(t)$.

8. Evaluate $h(t+\Delta t)$ and $c(t)$.

9. Evaluate $F(t)$.

10. Repeat steps 3-9 until t_{final} is reached.