Machine Learning for Interest Rates: Using Auto-Encoders for the Risk-Neutral Modeling of Yield Curves

Andrei Lyashenko[1], Fabio Mercurio[2] and Alexander Sokol[3]

[1] *Quantitative Risk Management*
[2] *Bloomberg* *
[3] *CompatibL*

September 25, 2024

Abstract

In this paper, we use autoencoders (AE) to capture the historical dependence structure of interest rates, and introduce risk-neutral forward rate dynamics that are consistent with a given AE curve manifold. We first derive a general condition for the AE-based forward-rate curve to admit a no-arbitrage evolution. Then, by allowing a small convexity-driven deviation from the AE curve manifold, we derive a risk-neutral modeling framework that is arbitrage-free and incorporates the information built into the AE (low-dimensional) curve manifold. Finally, we showcase numerical results based on historical market swap data for multiple currencies, visualizing graphically some of our key concepts.

1 Introduction

The properties of an interest rate model are, to a large extent, determined by the choice of its state variables. Having too many state variables causes parameter estimation issues and negatively affects the performance of certain numerical methods such as American Monte Carlo. Having too few, or selecting them poorly, causes the model to misprice relevant market instruments or miss certain risks. Supplementing an optimal number of state variables with a realistic dependence structure allows the model to capture hedgeable sources of risk, while preserving

*The views and opinions expressed in this article are our own and do not represent the opinions of any firm or institution.

historical patterns of market movements. This is why achieving dimension reduction, i.e., decreasing the number of state variables with the least possible loss of accuracy, is of paramount importance.

The factor dependence structure in interest rate models is commonly defined through the instantaneous correlations of the modeled risk drivers. Such correlations, however, being local in nature, tend to provide a poor measure of curve interdependence over longer time horizons, causing the model to generate unrealistic co-movements of the state variables. In this paper, we follow a novel approach and introduce a Q-measure (risk-neutral) model construction procedure that captures the historical dependence structure of interest rates using autoencoders (AE), which are algorithms designed to optimally represent a high-dimensional dataset, in our case historical interest rates across all maturities, using a small number of latent variables.

When the AE latent variables are used as state variables of a Q-measure model, they make the model both consistent with historical interest rate patterns and capable of traditional calibration to market-implied prices. Such a model is, by construction, designed to exclude future curve shapes that are unlikely to occur in practice, thus leading to realistic, stable, and less costly hedges. The purpose of this paper is to introduce a more realistic interest rate modeling framework that incorporates empirical yield curve behavior. This is especially beneficial when pricing/hedging complex products that involve risks that cannot be hedged with liquid market instruments.

The use of AE for curve forecasting in the P-measure was introduced by Kondratyev (2018), who also used AE to predict the most likely shapes of a commodity forward curve. Sokol (2022) proposed using AE for finding the optimal set of state variables to represent a yield curve for the purposes of P- and Q-measure model construction. He demonstrated that training an AE on historical curve shapes leads to a more effective low-dimensional curve representation than parametric methods such as that of Nelson and Siegel (1987), or classic linear optimizations such as PCA. Andreasen (2023) confirmed the empirical findings of Kondratyev (2018) and Sokol (2022) and, by constructing a specific Q-measure short-rate model, showed that AE neural networks are very close to being arbitrage-free.

The AE training methods proposed by Sokol (2022) and Andreasen (2023) produce low dimensional representations for a finite set of swap rates with discrete maturities. While having the advantage of dealing with market observables directly, the rates produced by AE must then be used as an input to a curve builder to obtain instantaneous forward rates as a continuous function of the time to maturity. In this paper, we will instead build an AE for such continuous representation directly, the AE output being the entire interpolated curve rather than the term rates for a discrete set of maturities. This is important for our purposes because of the intricate set of no-arbitrage constraints that would be destroyed by applying bootstrapping after the fact.

2 Key assumptions and definitions

Most forward rate models in the Q-measure are specified using forward rates for the maturity times T that remain fixed as the observation time t advances. In contrast, historically estimated curve representations such as the Nelson-Siegel formula are specified using time to maturity $\tau = T - t$, defined relative to the observation time t, because the absolute maturity time T is meaningless in such a context. Since our approach is also based on historical estimation, we will use τ-parameterized variables in forward rate model construction, an approach proposed by Musiela (1993) and recently further developed by Lyashenko and Goncharov (2023). In particular, we will model the instantaneous forward rate observed at time t for time to maturity τ, which we denote by $f(t, \tau)$.

Introducing a column vector of N state variables (factors) $\boldsymbol{X} = (X_1, \ldots, X_N)^\intercal$, with the superscript $^\intercal$ denoting transposition, and a factor representation $f_X(\tau, \boldsymbol{X})$ for the forward rate curve, which depends on the time to maturity τ and factors $\boldsymbol{X}$,[1] we assume that the instantaneous forward rate curve can be represented in the following form:

$$f(t, \tau) = f_X(\tau, \boldsymbol{X}(t)) + \phi(\tau + t) + O(\sigma^2 t) \tag{1}$$

where $\boldsymbol{X}(t)$ is the stochastic process of model factors, $\phi(T)$ is the deviation of the initial curve at $t = 0$ from the factor representation for maturity T, and the last term is a convexity correction, which will be examined later in the paper.

An AE performs encoding (compression) of the higher-dimensional input $\boldsymbol{X}$ into a lower-dimensional set of latent variables $\boldsymbol{Z}$, followed by decoding, to obtain an approximation $\mathbb{X} : \mathbb{R}^K \to \mathbb{R}^N$ of the input.[2] With a slight abuse of notation, we write:

$$f_Z(\tau, \boldsymbol{Z}) = f_X(\tau, \mathbb{X}(\boldsymbol{Z})) \tag{2}$$

so the forward rate representation becomes:

$$f(t, \tau) = f_Z(\tau, \boldsymbol{Z}(t)) + \phi(\tau + t) + O(\sigma^2 t) \tag{3}$$

where $\boldsymbol{Z}(t)$ is the stochastic process of latent variables, and $f_Z(\tau, \boldsymbol{Z})$ is the AE representation (2) for the instantaneous forward rates, which depends on time to maturity τ and the latent variable vector. If the initial curve is located in close vicinity of the AE representation $f_Z(\tau, \boldsymbol{Z})$ like most historically observed curves, $\phi(T)$ will be small compared to the level of the modeled rates.

Inspired by the terminology introduced in Björk and Christensen (1999), we define the AE manifold $\mathcal{M}_Z^f$ as the family of all forward rate curves $f_Z(\tau, \boldsymbol{Z})$ parameterized by $\boldsymbol{Z} \in \mathbb{R}^K$: $\mathcal{M}_Z^f = \{f_Z(\tau, \boldsymbol{Z}) \,|\, \boldsymbol{Z} \in \mathbb{R}^K\}$. This manifold is embedded in

[1] In Section 5 below, we will consider a specific form and analyze conditions for it to be arbitrage-free.

[2] While AE algorithms usually rely on neural networks, we will use the term AE generically throughout the paper to refer to compression algorithms optimized for a specific dataset, whether or not neural networks are used for AE construction.

the larger manifold $\mathcal{M}_X^f$ of all forward rate curves parameterized by $\boldsymbol{X} \in \mathbb{R}^N$: $\mathcal{M}_X^f = \{f_X(\tau, \boldsymbol{X}) \,|\, \boldsymbol{X} \in \mathbb{R}^N\}$. Using a similar notation, we then also define the following X-manifold, i.e., the manifold for $\boldsymbol{X}$ generated by the variables in the latent space:

$$\mathcal{M}_Z^X = \left\{\boldsymbol{X} = \mathbb{X}(\boldsymbol{Z}) \,|\, \boldsymbol{Z} \in \mathbb{R}^K\right\} \tag{4}$$

Throughout the paper, we will continue to use the notation $\mathcal{M}_B^A$ to describe a manifold for A parameterized by B.

3 Time Shift Invariance and Static Arbitrage

Representations (1) and (3) are not necessarily arbitrage-free, and one has to impose specific conditions on the corresponding factor representations as well as on the factor (or latent variable) dynamics for no-arbitrage to hold. In particular, the absence of static arbitrage, which we are about to define, can be guaranteed if a time shift invariance constraint is imposed on our curve representation.

A model is said to be static arbitrage-free if it is free of arbitrage for zero volatility. This means that, under zero volatility, the forward rate for each given absolute maturity time T must be constant. All classic interest rate models in the Q-measure are free of static arbitrage by construction. However, this is generally not the case for the forward rate evolution given by (3). Indeed, requiring that the forward rate for a given absolute maturity time T is constant leads to:

$$f(t, T-t) = f(t, T-t)|_{t=0} = f(0, T) \tag{5}$$

that is, replacing $T - t$ by τ,

$$f(t, \tau) = f(0, \tau + t) \tag{6}$$

Substituting (3) into (6), we obtain the following static no-arbitrage constraint on AE curve representation: for every initial latent variable vector $\boldsymbol{Z}$ and time shift t, there must exist a latent variable vector $\boldsymbol{Z}'$ such that

$$f_Z(\tau + t, \boldsymbol{Z}) = f_Z(\tau, \boldsymbol{Z}') \tag{7}$$

i.e., the curve representation must be invariant to a time shift in τ. In what follows, we will use the terms "static no-arbitrage constraint" and "time-shift invariance" interchangeably.

Eq. (7) states that for any forward curve that lies on the AE manifold $\mathcal{M}_Z^f$, a curve obtained by removing the initial segment with time to maturity $0 < \tau < t$, and then shifting the rest of the curve back to the origin, must also lie on the same AE manifold.

4 Stationarity of the Long Forward Rate

Let $f_\infty(t)$ be the limit value of $f(t, \tau)$, as observed at time t for $\tau \to \infty$. Consistently with the typical assumptions used in economic theory, we assume that this limit exists and is finite. In fact, a well-known theorem by Dybvig, Ingersoll, and Ross (1996) states that in a frictionless no-arbitrage economy $f_\infty(t)$ can never fall, i.e., once it reaches a certain level for the observation time t, any lower level at a later time $t' > t$ will result in arbitrage: $f_\infty(t') \geq f_\infty(t)$ if $t' > t$.

If one further assumes that the economy is stationary over the long run, this rate can not rise either, as it would then be unable to come down to its prior level, and must therefore remain constant. We will adopt this stationarity requirement for our modeling framework as well, leading to the following constraint on the AE representation $f_Z(\tau, \boldsymbol{Z})$ for any $\boldsymbol{Z}$:

$$\lim_{\tau \to \infty} f_Z(\tau, \boldsymbol{Z}) = f_\infty \tag{8}$$

where the long rate f_∞ is a constant model parameter that does not depend on the observation time t. We notice that, in classic Q-measure modeling, this is equivalent to the common assumption of mean-reversion being strictly positive.

The long rate f_∞ cannot be observed in the market directly and must be estimated by a statistical method. Sokol (2014) proposed estimating f_∞ as the level where regressions of rates with different maturities near the long end of the curve intersect.

5 Arbitrage-Free Factor Representation

Given our assumption of a constant long-term forward rate f_∞, we are now in a position to postulate the following (classic) factor representation:

$$f_X(\tau, \boldsymbol{X}) = f_\infty + \boldsymbol{B}(\tau)\boldsymbol{X} \tag{9}$$

where $\boldsymbol{B}(\tau) = (B_1(\tau), \ldots, B_N(\tau))$ is a row vector of basis functions, i.e., the loadings of $\boldsymbol{X} = (X_1, \ldots, X_N)^\intercal$, which are such that each $B_i(\tau)$, $i = 1, \ldots, N$, has a zero limit for $\tau \to \infty$. If N is the same as the number of curve builder inputs, (9) will reproduce every curve builder input exactly. If N is less than the number of curve builder inputs, (9) will perform an approximate fit instead. Following a standard approach in Q-measure model construction, in the latter case we will absorb the error in fitting the initial market-implied forward curve into the correction term $\phi(\tau + t)$ in (1) to keep our models strictly arbitrage-free.

As we already pointed out, a factor representation expressed in terms of τ is not necessarily free of static arbitrage, and this is true for (9) as well. Lyashenko and Goncharov (2023) showed that for the forward curve (9) to satisfy the static no-arbitrage condition (7) the basis vector $\boldsymbol{B}(\tau)$ must be of the form:

$$\boldsymbol{B}(\tau) = \boldsymbol{B}_0 e^{-\tau \boldsymbol{D}} \tag{10}$$

where $\boldsymbol{D}$ is a square (generating) matrix and $\boldsymbol{B}_0$ is a row vector. This implies that the corresponding forward rate representation (9) can be written in the form

$$f_X(\tau, \boldsymbol{X}) = f_\infty + \sum_{i=1}^{L} \sum_{j=1}^{m_i} X_{s(i)+j}\, \tau^{j-1} e^{-\lambda_i \tau} \tag{11}$$

where $\lambda_1, \dots, \lambda_L$ are (distinct) eigenvalues of the generating matrix $\boldsymbol{D}$, $m_1, \dots, m_L$ are their (algebraic) multiplicities,[3] and the multiplicity counting function $s(i)$ is defined as follows

$$s(i) = \sum_{j=1}^{i-1} m_j, \quad i = 1, \dots, L \tag{12}$$

where we use the convention of empty sum being equal to zero, so $s(1) = 0$. Note that the power terms τ^{j-1} appear in (11) only in the case of a non-diagonalizable matrix $\boldsymbol{D}$, where some of the multiplicities m_i are larger than 1. All the eigenvalues λ_i of the generating matrix $\boldsymbol{D}$ must be strictly positive to satisfy the constant long rate requirement (8).

A well-known basis that satisfies (10) is the Nelson-Siegel basis, which was derived from the following form for the instantaneous forward rate:

$$f_X^{\mathrm{NS}}(\tau, \boldsymbol{X}) = X_0 + X_1 e^{-\lambda\tau} + X_2 \lambda\tau e^{-\lambda\tau} \tag{13}$$

However, the Nelson-Siegel representation, though being static arbitrage-free, is not a special case of our form (11) because it allows the maturity-independent term X_0 to vary from one curve observation to the next.

6 Forward Rate Manifolds

Using the forward curve representation (9), we can build the forward rate curve manifold $\mathcal{M}_Z^f$ by first building an X-manifold $\mathcal{M}_Z^X$ with the AE transformation $\mathbb{X} : \mathbb{R}^K \to \mathbb{R}^N$ trained to the historical observations of factor vectors $\boldsymbol{X}$, which are obtained by bootstrapping historical rate data using representation (9), and then setting

$$\mathcal{M}_Z^f = \left\{ f_Z(\tau, \boldsymbol{Z}) = f_\infty + \boldsymbol{B}(\tau)\mathbb{X}(\boldsymbol{Z}) \,|\, \boldsymbol{Z} \in \mathbb{R}^K \right\} \tag{14}$$

Note that the K-dimensional manifold $\mathcal{M}_Z^f$ is embedded in the full N-dimensional linear (affine) manifold of forward rate curves

$$\mathcal{M}_X^f = \left\{ f_X(\tau, \boldsymbol{X}) = f_\infty + \boldsymbol{B}(\tau)\boldsymbol{X} \,|\, \boldsymbol{X} \in \mathbb{R}^N \right\} \tag{15}$$

While the full linear (affine) manifold $\mathcal{M}_X^f$ satisfies the static no-arbitrage property (7) because of (10), the (nonlinear) manifold $\mathcal{M}_Z^f$ generally does not.

It follows from (10) that the (nonlinear) manifold $\mathcal{M}_Z^f$ satisfies the static no-arbitrage property (7) if and only if the corresponding X-manifold $\mathcal{M}_Z^X$ defined by (4) is invariant with respect to multiplication by $e^{-t\boldsymbol{D}}$ for any $t > 0$, in the sense that if $\boldsymbol{X} \in \mathcal{M}_Z^X$ then $e^{-t\boldsymbol{D}}\boldsymbol{X} \in \mathcal{M}_Z^X$. We represent this invariance property by using the following shorthand notation:

$$e^{-t\boldsymbol{D}}\mathcal{M}_Z^X \equiv \mathcal{M}_Z^X, \quad t > 0 \tag{16}$$

In combination with our assumption that all eigenvalues of the generating matrix $\boldsymbol{D}$ are positive, the invariance condition (16) implies that the zero point $\boldsymbol{X} = \boldsymbol{0}$ must belong to the manifold $\mathcal{M}_Z^X$ at least asymptotically.

[3] Geometric multiplicities should be always equal to 1 to ensure that the basis functions are linearly independent.

7 The Linear Case

Consider the case of a linear transformation $\mathbb{X} : \mathbb{R}^K \to \mathbb{R}^N$

$$\mathbb{X}(\boldsymbol{Z}) = \boldsymbol{G}\boldsymbol{Z} \tag{17}$$

where $\boldsymbol{G}$ is an $N \times K$ matrix of full rank. This case is very instructive since (17) has the same form as a PCA representation, a very important benchmark widely used in financial analysis and modeling.

Based on (17), we want to see when the linear forward curve representation

$$f_Z(\tau, \boldsymbol{Z}) = f_\infty + \boldsymbol{B}(\tau)\boldsymbol{G}\boldsymbol{Z} \tag{18}$$

satisfies the static no-arbitrage condition (7). Defining the new basis $\boldsymbol{B}_G(\tau) = \boldsymbol{B}(\tau)\boldsymbol{G}$ we can write the above equation in the classic factor form

$$f_Z(\tau, \boldsymbol{Z}) = f_\infty + \boldsymbol{B}_G(\tau)\boldsymbol{Z} \tag{19}$$

As discussed in Section 5, the forward curve representation (19) satisfies the static no-arbitrage condition (7) if and only if the basis $\boldsymbol{B}_G(\tau)$ is of the exponential matrix form (10) and $f_Z(\tau, \boldsymbol{Z})$ can be written in the form (11):

$$f_Z(\tau, \boldsymbol{Z}) = f_\infty + \sum_{i=1}^{L'} \sum_{j=1}^{m'_i} Z_{s'(i)+j}\, \tau^{j-1} e^{-\lambda'_i \tau} \tag{20}$$

where $\lambda'_1, \ldots, \lambda'_{L'}$ and $m'_1, \ldots, m'_{L'}$ are, respectively, the eigenvalues and their multiplicities for the generating matrix of the basis $\boldsymbol{B}_G(\tau)$, and $s'(i)$ is the multiplicity counting function of the form (12):

$$s'(i) = \sum_{j=1}^{i-1} m'_j, \quad i = 1, \ldots, L' \tag{21}$$

Since

$$f_Z(\tau, \boldsymbol{Z}) = f_X(\tau, \boldsymbol{G}\boldsymbol{Z}), \tag{22}$$

comparing the right hand sides of equations (11) and (20), we immediately see that (22) can hold true for any $\boldsymbol{Z} \in \mathbb{R}^K$ and $\tau > 0$ only if eigenvalues $\lambda'_1, \ldots, \lambda'_{L'}$ are a subset of eigenvalues $\lambda_1, \ldots, \lambda_L$, and multiplicities $m'_1, \ldots, m'_{L'}$ are less or equal than the corresponding multiplicities $m_1, \ldots, m_L$.

We thus have the following result: *The only linear dimensionality reductions that preserve the static no-arbitrage invariance property* (16) *consist of reducing multiplicities of some of the eigenvalues of the generating matrix* $\boldsymbol{D}$*, which includes dropping some eigenvalues by reducing their multiplicities to zero.*

This result is very important because it shows that an empirically estimated principal component basis would not generally be suitable for no-arbitrage modeling of a yield curve as it is unlikely to satisfy the static no-arbitrage condition. An alternative derivation can be found in Lyashenko, Mercurio, and Sokol (2024), along with additional descriptions, motivations, and modeling details.

8 No-arbitrage Condition

In this section, we derive a general condition for the forward rate manifold $\mathcal{M}_Z^f$ to admit a no-arbitrage evolution. Throughout this section, we assume that the AE mapping $\mathbb{X}(\boldsymbol{Z})$ is twice continuously differentiable.

For any adapted volatility matrix $\boldsymbol{\sigma}^Z(t)$ of size $K \times M$, $M \leq K$, we say that the forward rate manifold $\mathcal{M}_Z^f$ admits an arbitrage-free evolution with latent variable volatility $\boldsymbol{\sigma}^Z(t)$ if, for any $\boldsymbol{Z}_0 \in \mathbb{R}^K$ there exists an adapted vector drift process $\boldsymbol{\mu}^Z(t)$ of size K, such that the manifold $\mathcal{M}_Z^f$ bound forward curve evolution given by

$$f^{\mathcal{M}}(t,\tau) = f_Z(\tau, \boldsymbol{Z}(t)) = f_\infty + \boldsymbol{B}(\tau)\mathbb{X}(\boldsymbol{Z}(t)) \tag{23}$$

is arbitrage-free, where process $\boldsymbol{Z}(t)$ evolves according to

$$\mathrm{d}\boldsymbol{Z}(t) = \boldsymbol{\mu}^Z(t)\,\mathrm{d}t + \boldsymbol{\sigma}^Z(t)\,\mathrm{d}\boldsymbol{W}(t), \quad \boldsymbol{Z}(0) = \boldsymbol{Z}_0 \tag{24}$$

and $\boldsymbol{W}(t)$ is a standard M-dimensional Q-Brownian motion.

The no-arbitrage condition for the instantaneous forward rate $f^{\mathcal{M}}(t,\tau)$ is given by the Musiela (1993) parameterization of the HJM equation (Heath, Jarrow, and Morton 1992):

$$\mathrm{d}f^{\mathcal{M}}(t,\tau) = \left(\frac{\partial f^{\mathcal{M}}(t,\tau)}{\partial \tau} + \boldsymbol{\sigma}^f(t,\tau)^\intercal \int_0^\tau \boldsymbol{\sigma}^f(t,u)\,\mathrm{d}u\right)\mathrm{d}t + \boldsymbol{\sigma}^f(t,\tau)^\intercal\,\mathrm{d}\boldsymbol{W}(t) \tag{25}$$

where $\boldsymbol{\sigma}^f(t,\tau)$ is the adapted column volatility vector of the forward rate. The first drift term simply captures the effect of advancing the observation time t on the forward rate parameterized as a function of $\tau = T - t$. It appears only as a result of the changing frame of reference, and is absent when the HJM model is written down in its canonical form for the absolute maturity time T rather than $\tau = T - t$.

Substituting the forward rate representation (23) into the Musiela HJM equation (25), and using (10) and the dynamics of $\mathbb{X}(\boldsymbol{Z}(t))$, which follows from (24) and Ito's lemma, we get, dropping dependence on t for readability,

$$\boldsymbol{\sigma}^f(\tau)^\intercal = \boldsymbol{B}(\tau)\frac{\partial \mathbb{X}(\boldsymbol{Z})}{\partial \boldsymbol{Z}}\boldsymbol{\sigma}^Z \tag{26}$$

and

$$\boldsymbol{B}(\tau)\left(\frac{\partial \mathbb{X}(\boldsymbol{Z})}{\partial \boldsymbol{Z}}\boldsymbol{\mu}^Z + \boldsymbol{\Phi}\right) = -\boldsymbol{B}(\tau)\boldsymbol{D}\mathbb{X}(\boldsymbol{Z}) + \boldsymbol{B}(\tau)\boldsymbol{v}^X \int_0^\tau \boldsymbol{B}(s)^\intercal\,\mathrm{d}s \tag{27}$$

where $\boldsymbol{v}^X$ is the covariance matrix of $\mathbb{X}(\boldsymbol{Z})$:

$$\boldsymbol{v}^X = \frac{\partial \mathbb{X}(\boldsymbol{Z})}{\partial \boldsymbol{Z}}\boldsymbol{v}^Z\left(\frac{\partial \mathbb{X}(\boldsymbol{Z})}{\partial \boldsymbol{Z}}\right)^\intercal \tag{28}$$

and $\boldsymbol{\Phi}$ is the convexity term:

$$\boldsymbol{\Phi} = \frac{1}{2}\sum_{i,j=1}^{K} \frac{\partial^2 \mathbb{X}(\boldsymbol{Z})}{\partial Z_i \partial Z_j} \left[\boldsymbol{v}^Z\right]_{i,j} \tag{29}$$

with $\boldsymbol{v}^Z = \boldsymbol{\sigma}^Z \left(\boldsymbol{\sigma}^Z\right)^\intercal$.

Lyashenko and Goncharov (2023) showed that the last term in Eq. (27) can be written in the form

$$\boldsymbol{B}(\tau)\boldsymbol{v}^X \int_0^\tau \boldsymbol{B}(s)^\intercal \, \mathrm{d}s = \tilde{\boldsymbol{B}}(\tau)\boldsymbol{\Omega} \tag{30}$$

where $\tilde{\boldsymbol{B}}(\tau)$ is a basis of the form (10) that includes (extends) basis $\boldsymbol{B}(\tau)$, and vector $\boldsymbol{\Omega}$ has entries that are linear combinations of entries of matrix $\boldsymbol{v}^X$.

Thus, for the forward curve evolution given by (23) to be arbitrage-free, the drift $\boldsymbol{\mu}^Z$ of the latent vector process $\boldsymbol{Z}(t)$ defined by (24) must satisfy the following condition:

$$\boldsymbol{B}(\tau)\frac{\partial \mathbb{X}(\boldsymbol{Z})}{\partial \boldsymbol{Z}}\boldsymbol{\mu}^Z = -\boldsymbol{B}(\tau)\boldsymbol{D}\mathbb{X}(\boldsymbol{Z}) - \boldsymbol{B}(\tau)\boldsymbol{\Phi} + \tilde{\boldsymbol{B}}(\tau)\boldsymbol{\Omega} \tag{31}$$

The no-arbitrage condition (31) represents a constraint on the mapping $\mathbb{X}(\boldsymbol{Z})$ since it generally cannot be solved for $\boldsymbol{\mu}^Z$ because the dimensionality of $\boldsymbol{\mu}^Z$ is K while the dimensionality of the space spanned by the basis $\boldsymbol{B}(\tau)$ is $N > K$ and the dimensionality of the extended basis $\tilde{\boldsymbol{B}}(\tau)$ is greater than N.

The term $\boldsymbol{B}(\tau)\boldsymbol{\Phi}$ is a convexity adjustment that is zero in the linear case while the term $\tilde{\boldsymbol{B}}(\tau)\boldsymbol{\Omega}$ is the HJM arbitrage adjustment term. The vectors $\boldsymbol{\Phi}$ and $\boldsymbol{\Omega}$ are linear with respect to the entries of the covariance matrix $\boldsymbol{v}^Z$, and thus are quadratic with respect to the entries of the volatility vector $\boldsymbol{\sigma}^Z$.

In the zero volatility case $\boldsymbol{\sigma}^Z = 0$, the no-arbitrage condition (31) becomes:

$$\frac{\partial \mathbb{X}(\boldsymbol{Z})}{\partial \boldsymbol{Z}}\boldsymbol{\mu}^Z = -\boldsymbol{D}\mathbb{X}(\boldsymbol{Z}) \tag{32}$$

Reintroducing back the time dependence, we note that condition (32) is equivalent to: $\mathrm{d}\mathbb{X}(\boldsymbol{Z}(t)) = -\boldsymbol{D}\mathbb{X}(\boldsymbol{Z}(t))\,\mathrm{d}t$, so $\mathbb{X}(\boldsymbol{Z}(t)) = e^{-t\boldsymbol{D}}\mathbb{X}(\boldsymbol{Z}_0)$, which is in turn equivalent to (16). This means that we can always solve (32) for $\boldsymbol{\mu}^Z$ if the X-manifold $\mathcal{M}_Z^X$ satisfies the invariance condition (16). Thus, we have obtained the following important result: *the no-arbitrage condition* (31) *is satisfied to the approximation order of* $O(\sigma^2)$ *if the invariance condition* (16) *holds.*

The zero-volatility condition (32) has the geometric interpretation of vector $\boldsymbol{D}\mathbb{X}(\boldsymbol{Z})$ being tangential to the manifold $\mathcal{M}_Z^X$ at the point $\mathbb{X}(\boldsymbol{Z})$. Likewise, the no-arbitrage condition (31) is equivalent to its right-hand side being tangential to the forward rate manifold $\mathcal{M}_Z^f$. This is fully consistent with the results of Björk and Christensen (1999).

9 Risk-Neutral Modeling Framework

In this section, we derive a risk-neutral modeling framework that incorporates information built into the forward rate manifold $\mathcal{M}_Z^f$. We continue to follow the notation introduced in the previous section and assume that the X-manifold $\mathcal{M}_Z^X$ satisfies the invariance condition (16), which implies that there exists $\boldsymbol{\mu}^Z(\boldsymbol{Z})$ that satisfies (32) for any $\boldsymbol{Z} \in \mathbb{R}^K$. Using this drift specification in dynamics (24), process $\boldsymbol{Z}(t)$ then follows:

$$\mathrm{d}\boldsymbol{Z}(t) = \boldsymbol{\mu}^Z(\boldsymbol{Z}(t))\,\mathrm{d}t + \boldsymbol{\sigma}^Z(t)\,\mathrm{d}\boldsymbol{W}(t), \quad \boldsymbol{Z}(0) = \boldsymbol{Z}_0 \tag{33}$$

This latent vector process $\boldsymbol{Z}(t)$, combined with Eq. (23), defines the corresponding forward rate evolution, $f^{\mathcal{M}}(t,\tau) = f_\infty + \boldsymbol{B}(\tau)\mathbb{X}(\boldsymbol{Z}(t))$, which lies on the forward rate manifold $\mathcal{M}_Z^f$ by construction. The forward rate evolution $f^{\mathcal{M}}(t,\tau)$ is generally not arbitrage-free since the drift $\boldsymbol{\mu}^Z$ satisfies (32) by definition and thus it satisfies the no-arbitrage condition (31) only approximately up to order $O(\sigma^2)$. To construct a non-trivial arbitrage-free forward rate model, we must relax the requirement that its evolution is bound to the manifold $\mathcal{M}_Z^f$.

In general, an arbitrage-free forward rate evolution that is close to $f^{\mathcal{M}}(t,\tau)$ can be built as follows. Using the approach of Lyashenko and Goncharov (2023), we choose a forward rate evolution of the form

$$f^{AF}(t,\tau) = f_\infty + \boldsymbol{B}(\tau)\boldsymbol{X}(t) + \tilde{\boldsymbol{B}}(\tau)\tilde{\boldsymbol{X}}(t) \tag{34}$$

and make it arbitrage-free by assuming that the factor vectors $\boldsymbol{X}(t)$ and $\tilde{\boldsymbol{X}}(t)$ satisfy:

$$\mathrm{d}\boldsymbol{X}(t) = -\boldsymbol{D}\boldsymbol{X}(t)\,\mathrm{d}t + \boldsymbol{\sigma}^X(t)\,\mathrm{d}\boldsymbol{W}(t), \quad \boldsymbol{X}(0) = \mathbb{X}(\boldsymbol{Z}_0) \tag{35}$$

$$\mathrm{d}\tilde{\boldsymbol{X}}(t) = \left(-\tilde{\boldsymbol{D}}\tilde{\boldsymbol{X}}(t) + \boldsymbol{\Omega}(t)\right)\mathrm{d}t, \quad \tilde{\boldsymbol{X}}(0) = 0 \tag{36}$$

where $\tilde{\boldsymbol{D}}$ is the generating matrix of basis $\tilde{\boldsymbol{B}}(\tau)$, which also has positive eigenvalues, and vector $\boldsymbol{\Omega}$ is defined by (30). We then set the volatility process $\boldsymbol{\sigma}^X$ to be equal to the volatility process of $\mathbb{X}(\boldsymbol{Z}(t))$:

$$\boldsymbol{\sigma}^X(t) = \frac{\partial \mathbb{X}(\boldsymbol{Z}(t))}{\partial \boldsymbol{Z}}\boldsymbol{\sigma}^Z(t) \tag{37}$$

The auxiliary factor $\tilde{\boldsymbol{X}}(t)$ in (34) serves as a no-arbitrage adjustment and it can be written as

$$\tilde{\boldsymbol{X}}(t) = \int_0^t e^{-(t-s)\tilde{\boldsymbol{D}}}\boldsymbol{\Omega}(s)\,\mathrm{d}s \tag{38}$$

Since $\mathbb{X}(\boldsymbol{Z}(0)) = \boldsymbol{X}(0)$ and

$$\mathrm{d}\left(\mathbb{X}(\boldsymbol{Z}(t)) - \boldsymbol{X}(t)\right) = \left[-\boldsymbol{D}\left(\mathbb{X}(\boldsymbol{Z}(t)) - \boldsymbol{X}(t)\right) + \boldsymbol{\Phi}(t)\right]\mathrm{d}t \tag{39}$$

we have:

$$\mathbb{X}(\boldsymbol{Z}(t)) - \boldsymbol{X}(t) = \int_0^t e^{-(t-s)\boldsymbol{D}}\boldsymbol{\Phi}(s)\,\mathrm{d}s \tag{40}$$

Using (10), (23), (34), (38) and (40) we then get:

$$f^{\mathcal{M}}(t,\tau) - f^{AF}(t,\tau) = \boldsymbol{B}(\tau)\left(\mathbb{X}(\boldsymbol{Z}(t)) - \boldsymbol{X}(t)\right) - \tilde{\boldsymbol{B}}(\tau)\tilde{\boldsymbol{X}}(t)$$
$$= \int_0^t \left(\boldsymbol{B}_0 e^{-(t+\tau-s)\boldsymbol{D}}\boldsymbol{\Phi}(s) - \tilde{\boldsymbol{B}}_0 e^{-(t+\tau-s)\tilde{\boldsymbol{D}}}\boldsymbol{\Omega}(s)\right) \mathrm{d}s \tag{41}$$

Since both $\boldsymbol{\Phi}$ and $\boldsymbol{\Omega}$ are of the order of magnitude $O(\sigma^2)$ and the generating matrices $\boldsymbol{D}$ and $\tilde{\boldsymbol{D}}$ have positive eigenvalues, the deviation of $f^{AF}(t,\tau)$ from $f^{\mathcal{M}}(t,\tau)$ is of the order of magnitude $O(\sigma^2 t)$. Therefore, the forward rate curve $f^{AF}(t,\tau)$ lies in a $O(\sigma^2 t)$ neighborhood of the manifold $\mathcal{M}_Z^f$.

The forward curve evolution $f^{\mathcal{M}}(t,\tau)$ can be regarded as a "shadow" forward curve evolution on the forward rate manifold $\mathcal{M}_Z^f$ that corresponds to the arbitrage-free evolution $f^{AF}(t,\tau)$. Both the arbitrage-free curve $f^{AF}(t,\tau)$ and its corresponding shadow curve $f^{\mathcal{M}}(t,\tau)$ start from the same initial curve, $f^{AF}(0,\tau) = f^{\mathcal{M}}(0,\tau)$, and they share the same volatility process

$$\sigma^f(t,\tau) = \boldsymbol{B}(\tau)\boldsymbol{\sigma}^X(t) = \boldsymbol{B}(\tau)\frac{\partial \mathbb{X}(\boldsymbol{Z}(t))}{\partial \boldsymbol{Z}}\boldsymbol{\sigma}^Z(t) \tag{42}$$

where $\boldsymbol{\sigma}^Z(t)$ is the volatility process one is free to specify and calibrate.

We can now add back the initial fit error term

$$\phi(\tau) = f(0,\tau) - f^{AF}(0,\tau) = f(0,\tau) - (f_\infty + \boldsymbol{B}(\tau)\mathbb{X}(\boldsymbol{Z}_0)) \tag{43}$$

to define the following arbitrage-free forward curve evolution

$$f(t,\tau) = f^{AF}(t,\tau) + \phi(t+\tau) \tag{44}$$

that is fully consistent with the initial term structure $f(0,\tau)$. The initial latent vector $\boldsymbol{Z}_0$ can be chosen to minimize the fit error term $\phi(\tau)$.

Using (41), (44) can then be expressed in terms of the forward rate manifold $\mathcal{M}_Z^f$ as follows:

$$f(t,\tau) = f^{\mathcal{M}}(t,\tau) + \phi(t+\tau) - \int_0^t \left(\boldsymbol{B}(t+\tau-s)\boldsymbol{\Phi}(s) - \tilde{\boldsymbol{B}}(t+\tau-s)\boldsymbol{\Omega}(s)\right) \mathrm{d}s \tag{45}$$

which finally makes representation (3) explicit.

We next discuss some practical aspects of using the modeling framework introduced in this section. In particular, we will show that the above formulas can be considerably simplified if we impose a certain structure on the mapping $\mathbb{X}(\boldsymbol{Z})$.

10 A Generating Manifold

Eq. (45) defines a class of arbitrage-free forward rate models whose evolution, modulo $\phi(t+\tau)$, lies in the manifold $\mathcal{M}_Z^f$ for zero volatility. For non-zero volatility, the departure from $\mathcal{M}_Z^f$ is caused by the "convexity" integral term in (45), which is of the order of $O(\sigma^2 t)$. These results follow from the assumption that the manifold $\mathcal{M}_Z^X$ satisfies the invariance condition (16). In practice, however, this condition can be quite restrictive. In fact, training $\mathbb{X}(\boldsymbol{Z})$ to historical data does not necessarily lead to an X-manifold $\mathcal{M}_Z^X$ that is invariant with respect to multiplication by the exponential matrix $e^{-t\boldsymbol{D}}$ for any $t > 0$.

To construct a manifold $\mathcal{M}_Z^X$ that automatically satisfies (16), we introduce the following simple recipe. We start from a new AE transformation $\mathbb{U} : \mathbb{R}^{K-1} \to \mathbb{R}^N$ to define a generating manifold with one less dimension $\boldsymbol{Y} = (Z_2, \ldots, Z_K)^\intercal \in \mathbb{R}^{K-1}$:

$$\mathcal{M}_Y^X = \{\boldsymbol{X} = \mathbb{U}(\boldsymbol{Y}) \,|\, \boldsymbol{Y} \in \mathbb{R}^{K-1}\} \tag{46}$$

Then, we define the full K-dimensional manifold $\mathcal{M}_Z^X$ as the result of applying the time shift to this generating manifold:

$$\mathcal{M}_Z^X = \{\boldsymbol{X} = \mathbb{X}(\boldsymbol{Z}) = e^{-Z_1\boldsymbol{D}}\mathbb{U}(\boldsymbol{Y}) \,|\, \boldsymbol{Z} = (Z_1, \boldsymbol{Y}^\intercal)^\intercal \in \mathbb{R}^K\} \tag{47}$$

As a result, the time-shift invariance condition (16) is satisfied by $\mathcal{M}_Z^X$ by construction. Using representation (47), we can train the $\mathbb{U}(\boldsymbol{Y})$ autoencoder with $K-1$ latent variables $\boldsymbol{Y}$, instead of K latent variables $\boldsymbol{Z}$, to the historical data.

The manifold representation (47) does much more than just ensuring that condition (16) is satisfied. It provides a local system of coordinates given by the decomposition of the latent variable vector $\boldsymbol{Z}$ into two components: i) the time shift Z_1 corresponding to the implied-forward (zero-volatility) movements of the curve and ii) the generating manifold state vector $\boldsymbol{Y}$ tracking deviations from the implied-forward trajectory. Most importantly, the representation (47) leads to significant simplifications in the formulas derived in the previous section. Indeed, since

$$\frac{\partial \mathbb{X}(\boldsymbol{Z}(t))}{\partial \boldsymbol{Z}} = \left(-\boldsymbol{D}\mathbb{X}(\boldsymbol{Z}(t)), e^{-Z_1\boldsymbol{D}}\frac{\partial \mathbb{U}(\boldsymbol{Y}(t))}{\partial \boldsymbol{Y}}\right) \tag{48}$$

then solution $\boldsymbol{\mu}^Z(\boldsymbol{Z})$ to Eq. (32) has a very simple form:

$$\boldsymbol{\mu}^Z(\boldsymbol{Z}) = (1, 0, \ldots, 0)^\intercal \tag{49}$$

Therefore, the manifold driver process $\boldsymbol{Z}(t)$, given by (33), can be expressed as

$$\boldsymbol{Z}(t) = (t_0 + t, \boldsymbol{Y}(t)^\intercal)^\intercal \tag{50}$$

where $\boldsymbol{Y}(t)$ is a driftless process defined by

$$\mathrm{d}\boldsymbol{Y}(t) = \boldsymbol{\sigma}^Y(t)\,\mathrm{d}\boldsymbol{W}(t), \quad \boldsymbol{Y}(0) = \boldsymbol{Y}_0 \tag{51}$$

Here $\boldsymbol{\sigma}^Y(t)$ is an adapted volatility matrix to be specified and calibrated, and initial values t_0 and $\boldsymbol{Y}_0 \in \mathbb{R}^{K-1}$ can be determined by fitting $f^{\mathcal{M}}(0,\tau) = f_\infty + \boldsymbol{B}(\tau)e^{-t_0\boldsymbol{D}}\mathbb{U}(\boldsymbol{Y}_0)$ to the initial forward curve.

Because of (50) and (51), we have:

$$\boldsymbol{\sigma}^Z(t) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{\sigma}^Y(t) \end{pmatrix} \tag{52}$$

and Eq. (37), which defines the volatility process $\boldsymbol{\sigma}^X(t)$, becomes:

$$\boldsymbol{\sigma}^X(t) = e^{-(t_0+t)\boldsymbol{D}} \frac{\partial \mathbb{U}(\boldsymbol{Y}(t))}{\partial \boldsymbol{Y}} \boldsymbol{\sigma}^Y(t) \tag{53}$$

These equations allow us to work directly with the generating manifold $\mathcal{M}_Y^X$. This considerably simplifies the task of generating a risk-neutral curve evolution consistent with an empirically estimated manifold using the approach described in the previous section. In fact, after these transformations, Eq. (45) becomes:

$$\begin{aligned} f(t,\tau) = f_\infty + \boldsymbol{B}(T) e^{-t_0 \boldsymbol{D}} \mathbb{U}(\boldsymbol{Y}(t)) + \phi(T) \\ - \int_0^t \Big(\boldsymbol{B}(T-s)\boldsymbol{\Phi}(s) - \tilde{\boldsymbol{B}}(T-s)\boldsymbol{\Omega}(s) \Big)\, \mathrm{d}s \end{aligned} \tag{54}$$

where we replaced $t+\tau$ with the maturity T for readability.

11 Construction of the Generating Manifold

Based on the definition (47) of the extended manifold $\mathcal{M}_Z^X$, it is easy to realize that the generating manifold mapping function $\mathbb{U}(\boldsymbol{Y})$ can only be defined up to an exponential scaling $e^{\gamma \boldsymbol{D}}$, where γ is any arbitrary real number. To fix this scaling issue, we need to add a normalization condition. For instance, we can normalize $\mathbb{U}(\boldsymbol{Y})$ by imposing the condition

$$\|\mathbb{U}(\boldsymbol{Y})\| = \delta, \quad \boldsymbol{Y} \in \mathbb{R}^{K-1} \tag{55}$$

where δ is a positive number. Since matrix $\boldsymbol{D}$ has positive eigenvalues, this condition, combined with the representation $\mathbb{X}(\boldsymbol{Z}) = e^{-Z_1 \boldsymbol{D}} \mathbb{U}(\boldsymbol{Y})$, uniquely defines $\mathbb{U}(\boldsymbol{Y})$ if $\mathbb{X}(\boldsymbol{Z}) \neq 0$.

To train the generating manifold mapping function $\mathbb{U}(\boldsymbol{Y})$ normalized by (55), we first map all the historical observations of state vector $\boldsymbol{X}$ to the sphere of radius δ by applying the static arbitrage-free time shift to satisfy the normalization condition (55). That is, for any historical observation $\boldsymbol{X}^h$, we find the corresponding values Z_1^h and $\boldsymbol{U}^h \in \mathbb{R}^N$, $\|\boldsymbol{U}^h\| = \delta$ such that

$$\boldsymbol{X}^h = e^{-Z_1^h \boldsymbol{D}} \boldsymbol{U}^h \tag{56}$$

Once all the historical data $\boldsymbol{X}^h$ are converted to the normalized data $\boldsymbol{U}^h$, we can train the AE mapping function $\mathbb{U}(\boldsymbol{Y})$ directly to $\boldsymbol{U}^h$. A graphical illustration of the previous concepts is provided in Figures 1, 2 and 3, which have been generated using historical market swap data for multiple currencies.[4] In Figure 1, we show a

[4] We used a combined LIBOR and OIS swap-rate dataset with swap maturities 2Y, 3Y, 5Y, 10Y, 15Y, 20Y, 30Y, observed from May 1994 to June 2024, for the following currencies: AUD, CAD, CHF, CZK, DKK, EUR, GBP, JPY, NOK, NZD, SEK, USD. To convert the historical swap rates to $\boldsymbol{X}$, we used (11) with the following parameters: $\lambda_1 = 4\%$, $\lambda_2 = 8\%$ and $\lambda_3 = 12\%$ per year, algebraic multiplicities $m_{1,2,3} = 1$, and $f_\infty = 6\%$.

set of zero-volatility (static arbitrage-free) trajectories (blue lines), each one beginning from its historical observation $\boldsymbol{X}^h$ and ending at $\boldsymbol{X} = \mathbf{0}$. These trajectories intersect the spheres of varying sizes $\delta = 0.1, 0.2, 0.3$, forming a scatter pattern of points $\boldsymbol{U}^h$ on the surface of each sphere, as shown in greater detail in Figure 2. These scatter patterns are well approximated by a closed curve, which can then serve as the generating manifold $\mathcal{M}_Y^X$. Note that Figures 1, 2 show three possible choices for the parameter δ but only one is needed for model construction.

Figure 3 compares AE and PCA results in the $\delta = 0.1$ case. We found that the generating manifold $\mathcal{M}_Y^X$ (blue line) fits the time-shifted historical data well (green markers). We also note significant nonlinearity of the manifold and its deviation from the intersection of the PCA plane with the sphere (red line), indicating that PCA is not a suitable representation for a curve evolving under the Q-measure even when it provides a good fit to the initial data.

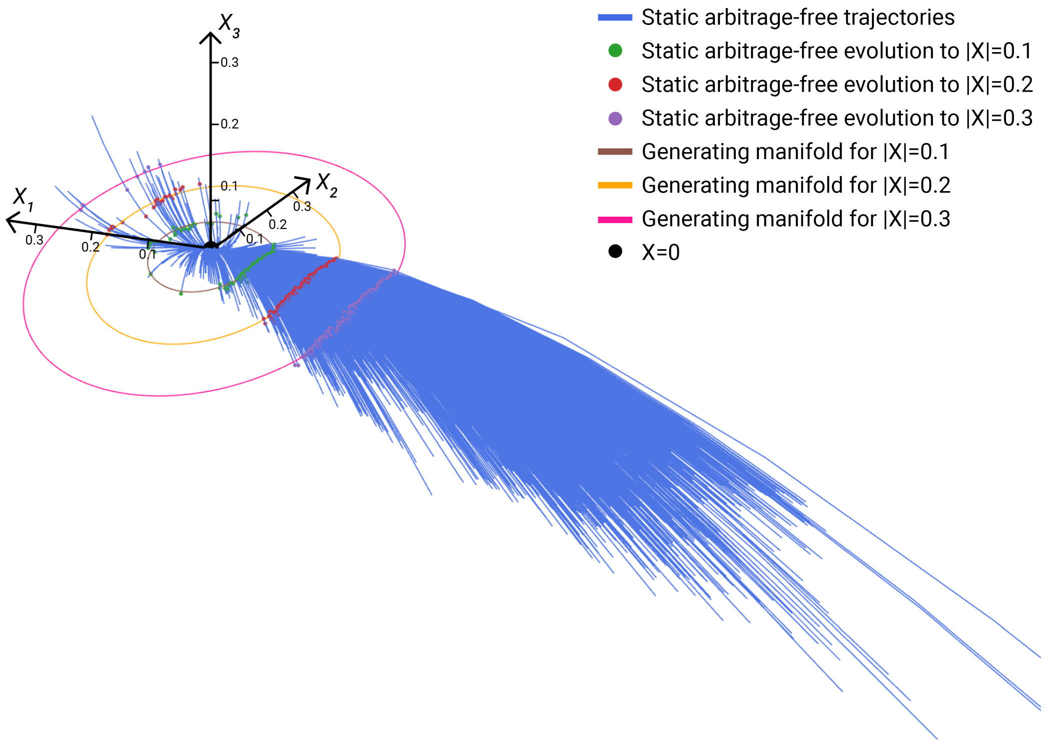

Figure 1: Static arbitrage-free trajectories (in blue) and generating manifold rings $\mathcal{M}_Y^X$ for three possible choices of $\delta = 0.1, 0.2, 0.3$.

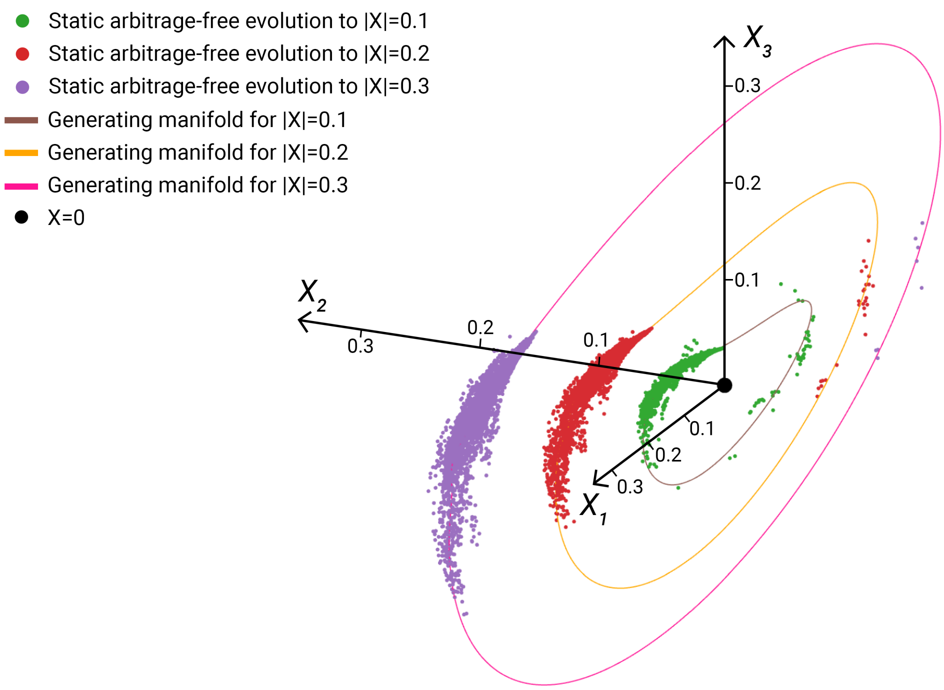

Figure 2: Detail of the same generating manifold rings $\mathcal{M}_Y^X$ for $\delta = 0.1, 0.2, 0.3$ from a different vantage point.

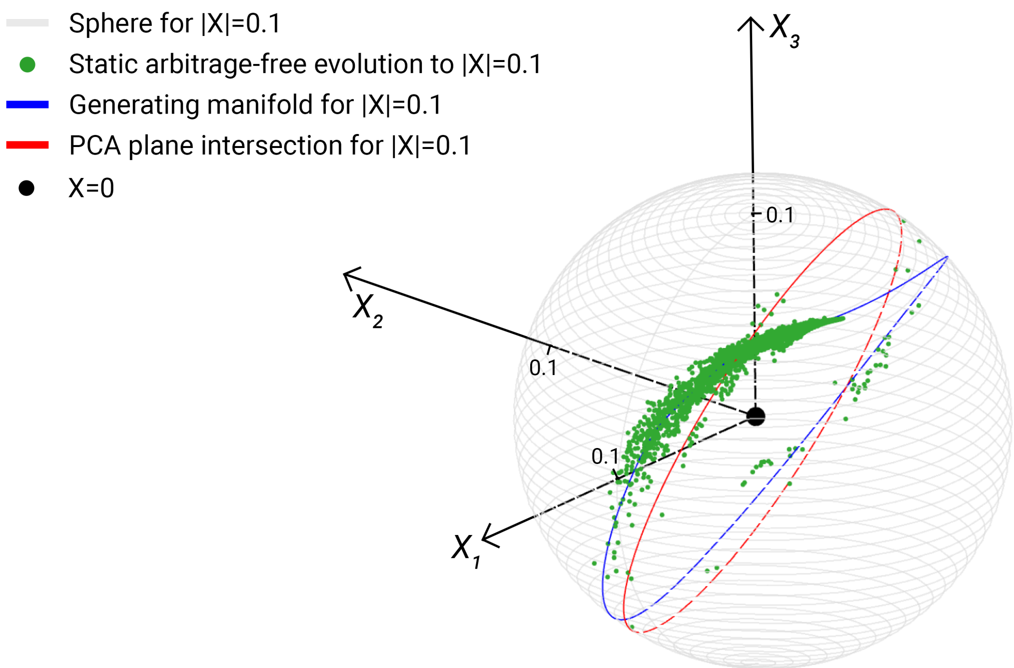

Figure 3: The generating manifold $\mathcal{M}_Y^X$ compared to the intersection of PCA plane with the sphere (55) for $\delta = 1$.

12 Conclusion

In this paper, we presented a new Q-measure interest rate modeling approach that is based on a (low-dimensional) nonlinear AE representation of the yield curve. We recognized that, for no-arbitrage to hold, future yield curves cannot lie on the AE curve manifold exactly without introducing highly restrictive constraints on volatility that would render a model unsuitable for practical use. Contrary to Björk and Christensen (1999), who imposed the constraint that a Q-model's future curves must lie exactly on the manifold defined by a given parametric form, we permitted a small departure from the curve manifold of the order of $O(\sigma^2 t)$. This way, we were able to make our modeling framework consistent with no-arbitrage for a general form of volatility, even stochastic, while still accurately reproducing historical yield curve shapes.

In a follow-up paper, we plan to discuss in more detail the practical aspects of using our methodology, and to analyze specific examples of Q-measure models in our framework that can be calibrated to cross-sectional data and implemented for pricing or risk management purposes.

Bibliography

[1] Sokol, A. Autoencoder Market Models for Interest Rates. SSRN https://ssrn.com/abstract=4300756, 2022.

[2] Nelson, C. R. and Siegel, A. F. Parsimonious Modeling of Yield Curves. *The Journal of Business*, 60(4):473, 1987. ISSN 0021-9398. doi: https://doi.org/10.1086/296409.

[3] Andreasen, J. Decoding the Autoencoder. *Wilmott*, September, 2023.

[4] Musiela, M. Stochastic pdes and term structure models. *Journées Internationales de Finance, IGR-AFFI*, June, 1993.

[5] Lyashenko, A. and Goncharov, Y. Introducing the Factor Heath-Jarrow-Morton Term Structure Modelling Framework. *Risk*, June 2023.

[6] Björk, T. and Christensen, B. Interest rate dynamics and consistent forward rate curves. *Mathematical Finance*, 9, 1999.

[7] Dybvig, P. H., Ingersoll, J. E., and Ross, S. A. Long Forward and Zero-Coupon Rates Can Never Fall. *The Journal of Business*, 69(1):1, 1996. ISSN 0021-9398. doi: https://doi.org/10.1086/209677.

[8] Sokol, A. *Long-Term Portfolio Simulation: For XVA, Limits, Liquidity and Regulatory Capital*. Risk Books, 2014. ISBN 9781782720959.

[9] Lyashenko, A., Mercurio, F., and Sokol, A. Autoencoder-Based Risk-Neutral Model for Interest Rates. SSRN https://www.ssrn.com/abstract=4836728, 2024.

[10] Heath, D., Jarrow, R., and Morton, A. Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica*, 60(1):77–105, 1992. ISSN 00129682. doi: https://doi.org/10.2307/2951677.