# Analysis of a Curve Fitting Technique for an Aerospace Model Using Numerical Approximations

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C++ | Honors | Spring 2014 |

The purpose of the parameter study is two-fold. The first is to identify the relationships between parameters of the model as a single parameter changes. The second is to investigate the reliability of curve fitting numerical landing solutions from the model so a general equation can be used to estimate additional solutions.

An aerospace problem from *Solving Ordinary Differential Equations and Boundary Value Problems: Computing and Modeling* by C. Henry Edwards and David E. Penney is used to validate the RK4 integrator. The case study also provides a base solution for the parameter study. Validating the code ensures our numerical integrator is providing accurate solutions.

**Case Study:**

A lunar lander is at an altitude of 53,000 m above the moon's surface, and is moving directly toward the surface with a velocity of -410.28 m/s. At a point between the current altitude and the lunar surface, the lander's retrorockets will fire and decelerate the lander so it will make a soft landing. The net acceleration of the lander can be described by the following non-linear differential equation.

$$a = \frac{d^2r}{dt^2} = T - \frac{GM}{r^2}$$

G is a gravitational constant and M is the moon's mass. The variable T is the acceleration from the retrorocket which fires directly upward at 4 $m/s^2$. The second term is the lander's acceleration due to the gravity from the moon. This term is derived by equating Newton's second law with Newton's law of gravitation. The 2nd order ODE can be split into a system of two 1st order differential equations which describe the altitude, velocity, and acceleration of the lander throughout it's decent. The system of equations is written as.

Position: $r(t)$

Velocity: $\frac{dr}{dt} = v$

Acceleration: $\frac{dv}{dt} = 4 - \frac{4.9 \times 10^{12}}{r^2}$

The initial conditions for position and velocity are required to solve the equation numerically. For this model, both bodies are considered to be point masses; therefore the lander's altitude and the lunar surface are measured from the center of the moon. The radius of the moon is, R = 1,740,000 m.

Solving for the initial altitude and initial velocity is done analytically using the separation of variables method and is shown below.

Assume

$$v = \frac{dr}{dt}$$

Therefore

$$\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\left(\frac{dr}{dt}\right) = v\left(\frac{dv}{dr}\right)$$

Using the chain rule, the 2nd notation is written as two 1$^{st}$ order ODEs.

$$v\left(\frac{dv}{dr}\right) = T - \frac{GM}{r^2}$$

The notation is then substituted into the acceleration equation.

$$\frac{v^2}{2} = Tr + \frac{GM}{r} + C$$

Separating the variables and integrating yields the above general equation.

The lander's retrorocket has two distinct states. Therefore, two specific solutions exist. Before the retrorocket ignites the acceleration from thrust is 0 m/s$^2$ and after ignition the acceleration is 4 m/s$^2$. The first specific equation is assigned a constant, $C_1$, which satisfies the equation for the first state when T = 0. The second equation is assigned a different constant, $C_2$, which satisfies the equation for the second retrorocket state when T = 4. The point where the retrorocket should ignite is found at the intersection of the two equations.

Solve for $C_1$ when T = 0, v = -410.28, and r = 1,793,000 = 1,740,000 + 53,000 (the initial point at the current altitude). This yields the specific equation for the first retrorocket state.

$$\frac{v^2}{2} = \frac{GM}{r} - 2648800$$

Then, solve for $C_2$ when T = 4, v = 0, and r = R = 1,740,000 (the landing point on the surface). This yields the specific equation for the second retrorocket state. In the second state, the lander is stationary on the surface of the moon.

$$\frac{v^2}{2} = 4r + \frac{GM}{r} - 9776092$$

Set the equations equal to each other and solve for r to determine the initial altitude for retrorocket ignition.

$$\frac{GM}{r} - 2648800 = 4r + \frac{GM}{r} - 9776092$$

$$r_0 = r = 1781870 \text{ m}$$

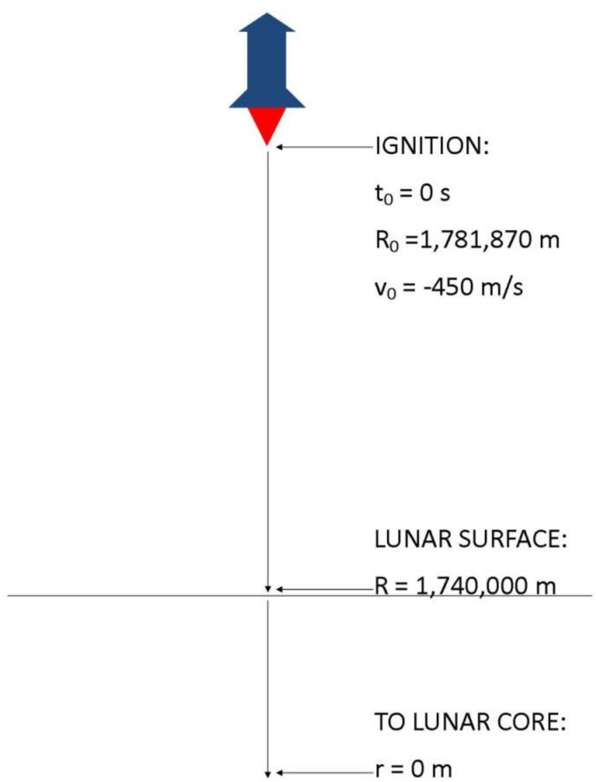
Substituting this r value into the equation when acceleration from thrust is 0 m/s$^2$ gives the initial velocity. Note the lander is moving in the negative r direction and the sign of the velocity should be changed accordingly.

$$\frac{v^2}{2} = \frac{GM}{1781870} - 2648800$$

$$v = 450 \text{ m/s}$$

$$\therefore v_0 = -450 \ \text{m/s}$$

Figure#06 depicts the lander at the retrorocket's ignition time and altitude. Initial values for the model are given. The landing point on the lunar surface is also shown measured from the center of the moon.




Figure#06: The diagram depicts the lander's flight path from the altitude at ignition to the lunar surface. The initial conditions are given at retrorocket ignition. Also, the altitude at the lunar surface measured from center of the moon is given.

The case study provided two solution tables. The first table contains data points from the time at ignition (t=0) until 200 seconds after ignition (t=200). These values are the lower limit and upper limit of integration, and are given by the case study. In a real-world scenario when the upper limit is not given, an estimate should be made and then possibly adjusted based on whether a point determined to be a landing solution is found. Table#04 lists the input values for the first numerical simulation.

| | |
|---:|---:|
| **# Equations** | 2 |
| **Initial Time ($t_0$)** | 0 |
| **Initial Altitude ($r_0$)** | 1,781,870 |
| **Initial Velocity ($v_0$)** | -450 |
| **Upper Limit ($t_n$)** | 200 |
| **Number of Steps (n)** | 200 |
| **Step size (h, calculated value)** | 1 |

Table #04: Inputs for the first case study simulation.

Table#05 shows the data from the first simulation compared to the case study's solutions. No graphs are shown because the simulation's domain is not refined enough to identify an accurate solution. On the following two pages, Figure#07 and Figure#08 show graphs of the data with a reduced domain. The graphs (not shown) for Table#05 look similar to those of Figure#07 and Figure#08, but show the entire time domain.

| Time | Case Study Altitude | Simulation Altitude | Case Study Velocity | Simulation Velocity |
|---|---|---|---|---|
| 0 | 1781870 | 1781870 | -450 | -450 |
| 20 | 1773360 | 1773360 | -401.04 | -401.04 |
| 40 | 1765826 | 1765826 | -352.37 | -352.37 |
| 60 | 1759264 | 1759264 | -303.95 | -303.95 |
| 80 | 1753667 | 1753667 | -255.74 | -255.74 |
| 100 | 1749033 | 1749033 | -207.73 | -207.73 |
| 120 | 1745357 | 1745357 | -159.86 | -159.86 |
| 140 | 1742637 | 1742637 | -112.11 | -112.11 |
| 160 | 1740872 | 1740872 | -64.45 | -64.45 |
| 180 | 1740059 | 1740059 | -16.83 | -16.83 |
| 200 | 1740199 | 1740198 | 30.77 | 30.77 |

Table #05: The table contains the data set for the first numerical simulation along with the expected case study values. The highlighted rows (t = 180 and t = 200) mark the time interval in which the lander makes a soft touchdown.

There are a few anomalous approximations. For example, the altitude at 200 seconds is off by 1 meter. The vast majority of the approximations exactly match the case study's solutions. The differences between the simulation and expected values are never greater than 1 meter. The approximations following each irregularity return to the expected solution. Because the simulation solutions never diverges from the case study solutions, it's reasonable to assume these errors are caused by rounding errors, not an error in the model or RK4 integrator.

The step size for the first simulation is 1second, so it's possible to find a more accurate value for landing time value from the first simulation. However, examining the data points along the entire domain ($0 \leq t \leq 200$) in increments of 20 seconds shows the simulation's solutions never diverged from the case study solution at any point. The inputs for the simulation can be refined based on the values in Table#05, to identify a more accurate landing solution.

Table#05 shows the lander possibly touched down on the surface of the moon somewhere between 180 and 200 seconds. The altitude at 180 seconds is closer to the radius of the moon than the altitude at 200 seconds. The same is true for the lander's velocity. New limits of integration ($180 \leq t \leq 190$) are used to identify a more accurate landing solution. The simulation is run again using the following inputs in Table#06.

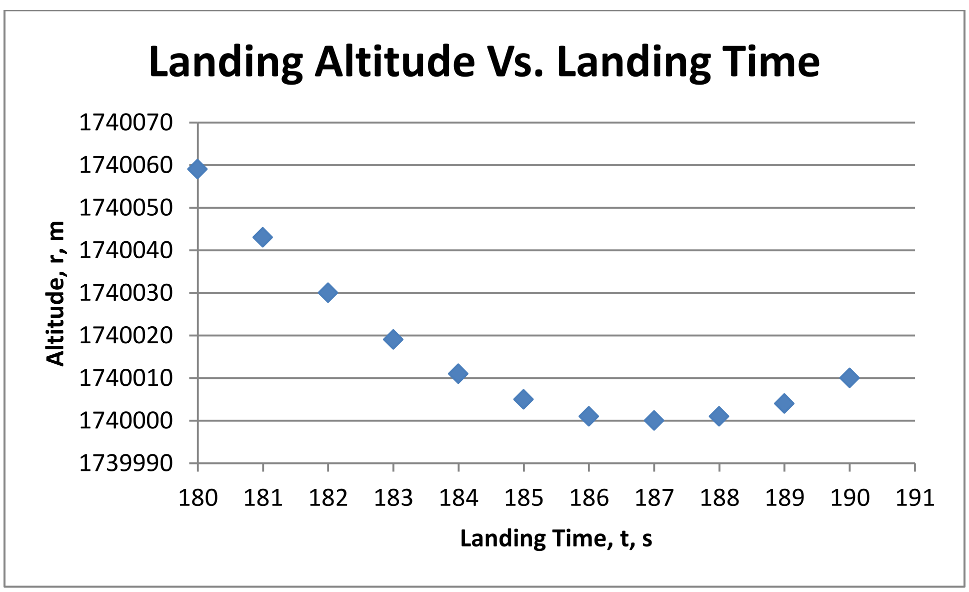
| | |
|---:|---:|
| **# Equations** | 2 |
| **Initial Time ($t_0$)** | 180 |
| **Initial Altitude ($r_0$)** | 1,740,059 |
| **Initial Velocity ($v_0$)** | -16.83 |
| **Upper Limit ($t_n$)** | 190 |
| **Number of Steps (n)** | 10 |
| **Step size (h, calculated value)** | 1 |

Table #06: Inputs for the second case study simulation.

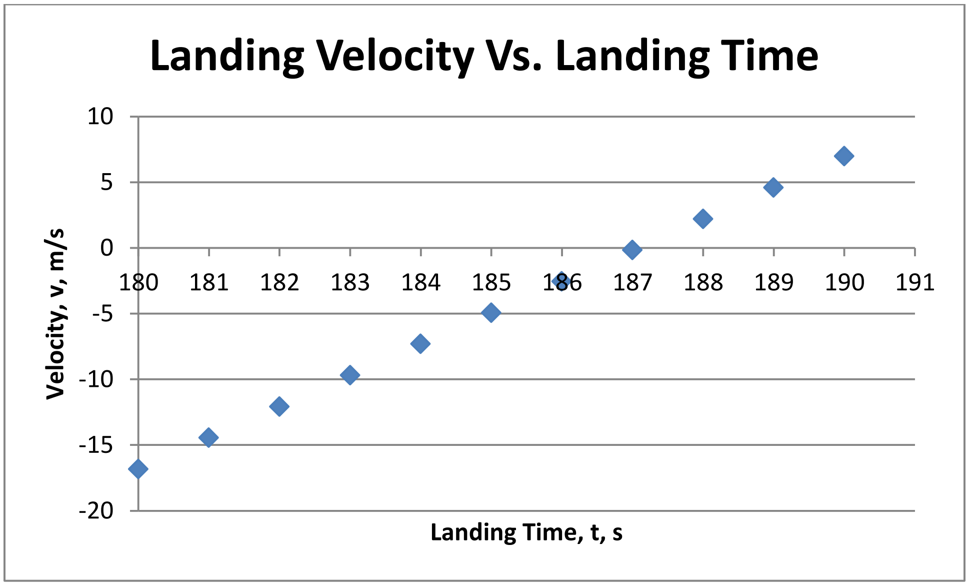
Table#07 lists the data points from the second simulation compared to the expected case study solutions. The altitude and velocity are plotted against time in Figure#07 and Figure#08.

| Time | Case Study Altitude | Simulation Altitude | Case Study Velocity | Simulation Velocity |
|---:|---:|---:|---:|---:|
| 180 | 1740059 | 1740059 | -16.83 | -16.83 |
| 181 | 1740044 | 1740043 | -14.45 | -14.45 |
| 182 | 1740030 | 1740030 | -12.07 | -12.07 |
| 183 | 1740019 | 1740019 | -9.69 | -9.69 |
| 184 | 1740011 | 1740011 | -7.31 | -7.31 |
| 185 | 1740005 | 1740005 | -4.93 | -4.93 |
| 186 | 1740001 | 1740001 | -2.55 | -2.55 |
| 187 | 1740000 | 1740000 | -0.17 | -0.17 |
| 188 | 1740001 | 1740001 | 2.21 | 2.21 |
| 189 | 1740004 | 1740004 | 4.59 | 4.59 |
| 190 | 1740010 | 1740010 | 6.97 | 6.97 |

Table#07: The approximations for the second case study simulation (column 3 and column 5) compared to the expected case study values (column 2 and column 4). The highlighted row (t = 187) contains the landing time, altitude, and velocity for the lander when it makes its soft touchdown.




Figure#07: The figure shows the relationship between altitude and time across the domain ($180 \le t \le 190$).




Figure#08: The figure shows the relationship between velocity and time across the domain (180≤t≤190).

The highlighted row in Table#07 highlights the case study's accepted landing solution compared to the results from the simulation. The results show the lander touched down at 187 seconds after retrorocket ignition. The time, altitude, and velocity values all agree with the expected solution. The lander's altitude at 187 seconds is exactly at the moon's surface. However, the velocity is -0.17 m/s. While this value is not exactly zero, it's within a tolerance deemed reasonable by the authors. Even if the velocity was exactly 0 m/s, it's still an approximation. Even the altitude which appears to show a perfect touchdown is an approximation.

The accuracy of the RK4 method can be shown by increasing the number of steps used. Using 800 steps decreases the step size to h = 0.25. However, the increased step size doesn't alter any of the approximations. This shows RK4 is sufficiently accurate for steps sizes of 1 second.

**Parameter Study:**

The case study only focuses on a single landing solution for a single set of parameters. The goal of the following investigation is to identify other landing solutions by changing a single parameter, to understand the relationship between the parameters as they change, and to investigate the reliability of estimations made from a curve fit applied to the landing solution data. The parameter study requires running multiple trials to find several landing solutions. If a reliable curve fit is applied to the data, new landing solutions can be identified by simply substituting an independent variable into the equation for the curve. This process is much less time consuming than running multiple trials for a single landing solution.

For the following parameter study, the independent variable is the deceleration from thrust. The dependent variables are the landing time, and the starting velocity. The ignition altitude is held constant (r=1,781,870), and the planet's surface is still measured from the center of the moon (R=1,740,000).

The same technique used during the case study is used to find acceptable landing solutions. The tolerances for the landing solutions are set at r = ±5m for the altitude and v = ± 1m/s for the velocity. The tolerances are considered acceptable because of their negligible size compared to their initial values. All landing altitudes are $2.8 \times 10^{-4}$ % or less of the initial fixed altitude. All landing velocities are 0.25% or less of the trial's initial velocity. The data in Table#08 shows the 21 solutions collected using the RK4 integrator.

| **Deceleration from Thrust (T)** | **Landing Time (t)** | **Initial Velocity ($v_0$)** |
|---:|---:|---:|
| 3.5 | 210 | -400.8 |
| 3.55 | 208 | -406 |
| 3.6 | 205 | -411.12 |
| 3.65 | 202 | -416.18 |
| 3.7 | 200 | -421.18 |
| 3.75 | 198 | -426.1 |
| 3.8 | 195 | -431 |
| 3.85 | 193 | -435.8 |
| 3.9 | 191 | -440.6 |
| 3.95 | 189 | -445.3 |
| 4 | 187 | -450 |
| 4.05 | 185 | -454.6 |
| 4.1 | 183 | -459.2 |
| 4.15 | 181 | -463.72 |
| 4.2 | 180 | -468.26 |
| 4.25 | 178 | -472.66 |
| 4.3 | 176 | -477.1 |
| 4.35 | 175 | -481.44 |
| 4.4 | 173 | -485.8 |
| 4.45 | 172 | -490.1 |
| 4.5 | 170 | -494.31 |

Table#08: The table contains 21 landing solutions within the domain $3.5 \leq T \leq 4.5$. The deceleration from thrust, the landing time, and the initial velocity is show for each of the 21 soft touchdowns.

The data set includes the case study solution and 20 additional landing solutions. The domain for the data set is from 3.5 to 4.5 and points are spaced in intervals of 0.05. The domain and the intervals between points are very small. This should be considered when looking at the accuracy of curve fits.

With the exception of 2 landing solutions, landing velocity falls within the tolerance of $v=\pm 1$m/s. As the initial velocity of the lander increases, the lander's acceleration near the moon's surface changes too rapidly. At one time step, the lander's velocity is barely greater than 1m/s. Between the time steps, the lander changes direction, and at the following time step the velocity is greater than 1m/s. Both of the landing solutions were very close to being within the velocity tolerance. With more time and integration steps, this could be fixed.

The case study solution provides a good starting point for analyzing the relationship between each of the parameters. As the deceleration from thrust decreases, the landing times increase and the magnitude of the initial velocity decreases. The lander begins its retrorocket burn at the same altitude for each trial, but the deceleration from thrust decreases. The lander needs a smaller magnitude for initial velocity. Otherwise the lander will be traveling too fast and crash into the lunar surface. Decreasing the initial velocity means the lander is moving more slowly throughout its decent and therefore the landing time increases.

As the deceleration from thrust increases, the landing times decrease, and the magnitude of the initial velocities increase. The increased deceleration from thrust causes the lander's velocity during the decent to change more rapidly, therefore the initial velocity must increase to compensate so the lander doesn't reverse direction before it reaches the lunar surface. The increase in initial velocity means the lander is traveling faster throughout its decent and thus, decreases the landing time.

Now that the relationship between parameters is defined, estimations can be checked by determining if they follow the expected relationship. To begin the estimation process, the starting velocity and landing time data is plotted against deceleration from thrust. Each dependent variable is plotted against deceleration from thrust separately. Figure#09 and Figure#10 are graphs of the data set with additional numerical points used for interpolation analysis. Figure #11 and Figure#10 show the same two plots again, but compare extrapolations along the best fit curve to numerical solutions. The equations for the best fit curves are applied using the Microsoft Excel spreadsheet application. The best fit curves applied are 2$^{nd}$ order polynomials (quadratic functions). The equations are as follows.

**Landing Time vs. Thrust best fit equation:**

$$t(T) = 13.32T^2 - 146.27T + 558.9$$

**Initial Velocity vs. Deceleration best fit equation:**

$$v(T) = 9.6619T^2 - 170.64T + 77.986$$

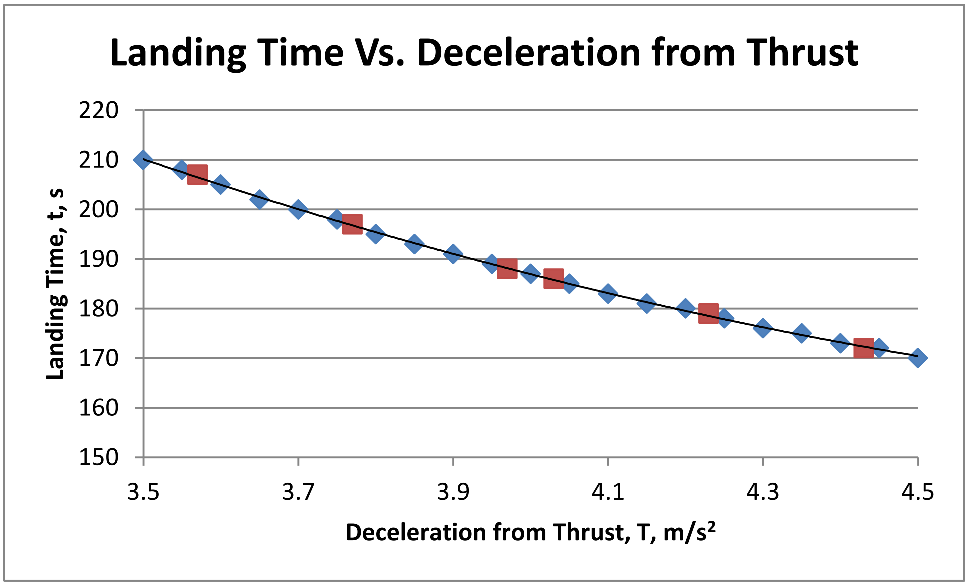
Microsoft Excel uses a value called the coefficient of determination (i.e. $R^2$) to show how closely the curve fit matches the data set it's applied to. A $R^2$ value near 1 (or -1) means the curve is a good fit to the data set. The $R^2$ value for Landing Time vs. Deceleration from Thrust is 0.9995 while the $R^2$ value for Initial Velocity vs. Deceleration from Thrust is 1.

The estimations for the parameter study fall into two categories: interpolations and extrapolations. Interpolations are estimations made within the domain of the data set while extrapolations are made outside the domain. These estimations are compared to numerical solutions from the RK4 integrator. The RK4 integrator provides the best approximations for the model. These approximations are considered the theoretical values when calculating the percent error for the interpolations and extrapolations.

Interpolations were the first technique used to check the reliability of the curve fit. The accuracy of the curve inside the data set was determined by interpolating 6 points. The points selected are symmetrical about the case study solution (T=4). Moving outward, the interpolations are spaced in intervals of 0.2. Table #09 and Figure #09 compare the landing time interpolations to the RK4 solutions.

| Deceleration from Thrust | Curve fit Landing  Time | Numerical Landing Time | % Error Time |
|---|---|---|---|
| 3.57 | 206.478168 | 207 | 0.25% |
| 3.77 | 196.777928 | 197 | 0.11% |
| 3.97 | 188.143288 | 188 | 0.08% |
| 4.03 | 185.760688 | 186 | 0.13% |
| 4.23 | 178.511328 | 179 | 0.27% |
| 4.43 | 172.327568 | 172 | 0.19% |

Table#09: The table shows the landing times interpolated from the best fit curve compared to the landing times generated using the RK4 numerical integrator. The percent errors for the interpolations are shown in column four.

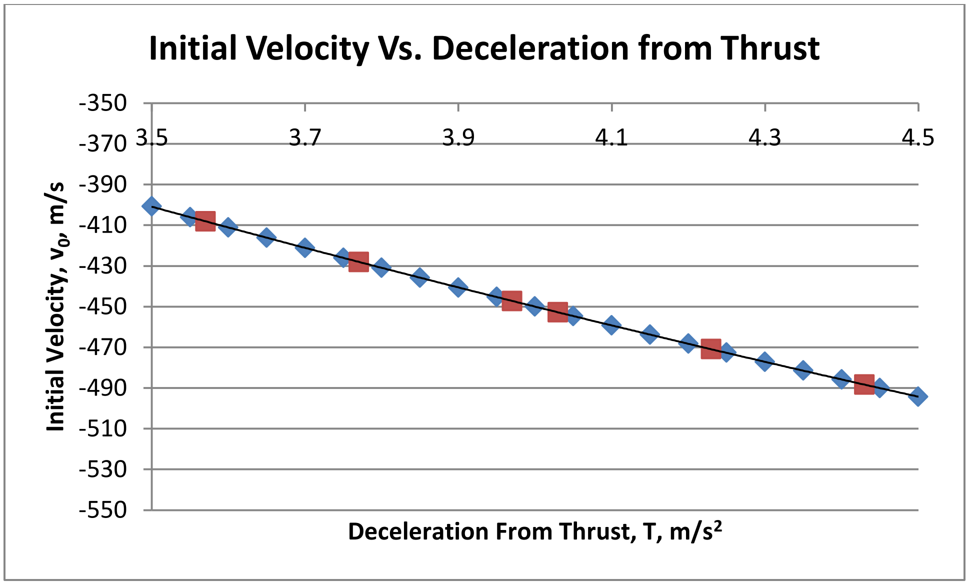



Figure#09: The graph compares the landing times from the original data set (diamonds and best fit curve) to the numerical landing time solutions from Table#09.

Table #10 and Figure #10 compare the landing time interpolations to the RK4 solutions.

| Deceleration from Thrust | Curve fit Initial Velocity | Numerical Initial Velocity | % Error Velocity |
|---|---|---|---|
| 3.57 | -408.06 | -408.05 | 0.00% |
| 3.77 | -428.00 | -428.08 | 0.02% |
| 3.97 | -447.17 | -447.22 | 0.01% |
| 4.03 | -452.78 | -452.76 | 0.00% |
| 4.23 | -470.94 | -470.89 | 0.01% |
| 4.43 | -488.34 | -488.35 | 0.00% |

Table#10: The table shows the initial velocities interpolated from the best fit curve compared to the initial velocities used to generate landing solutions using the RK4 numerical integrator. The percent errors for the interpolations are shown in column 4.



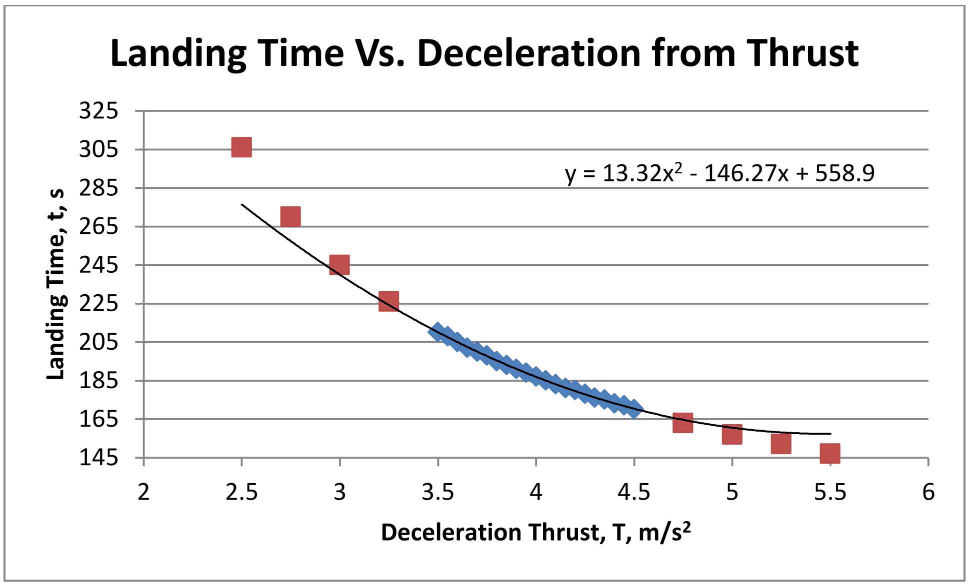

Figure#10: The graph compares the initial velocities from the original numerical data set (diamonds and best fit curve) to the numerical landing time solutions from Table#10.

Figure#09 and Figure#10 show the interpolations for both landing times and initial velocities are very accurate. The percent errors for all landing time interpolations are 0.27% or less (Table#09) while the initial velocity percent error is always 0.02% or less (Table#10). The graphs also show the interpolations (squares points) closely follow the data set and best fit curve. The landing time interpolations have a slightly higher percent error. The $R^2$ value for landing time is 0.005 less than the value for initial velocity which may explain the slightly higher percent error. There is no trend showing growth for interpolations as the points move away from T=4. The percent difference for both landing time and initial velocity appear to increase between the inner and middle interpolations. Then for the outer interpolations, the percent difference decreases. Some landing solutions used to create the curve fit had landing altitudes and landing velocities near 1,740,000m and 0m/s, while others barely remained within the tolerances of ±5m and ±1m/s. Because some points used for the curve fit are more accurate, the interpolations made near those points may be more accurate as well.

The next test involved extrapolating points along the best-fit curve. The accuracy of the curve outside of the data set was determined at 8 points. The domain for the extrapolation data is between $2.5 \leq t < 3.5$ to the left of the data set and $4.5 < t \leq 5.5$ to the right. The coefficient of determination has less meaning outside the data set because there are no data points. For the same reason, it's typically more difficult to obtain accurate extrapolations. Table #11 and Figure #11compare the landing time extrapolations to the RK4 solutions.

| Deceleration from Thrust | Curve fit Landing Time | Numerical Landing Time | % Error Time |
|---|---|---|---|
| 5.5 | 157.35 | 147 | 7.04% |
| 5.25 | 158.12 | 152 | 4.03% |
| 5 | 160.55 | 157 | 2.26% |
| 4.75 | 164.65 | 163 | 1.01% |
| 3.25 | 224.22 | 226 | 0.79% |
| 3 | 239.97 | 245 | 2.05% |
| 2.75 | 257.39 | 270 | 4.67% |
| 2.5 | 276.48 | 306 | 9.65% |

Table#11: The table contains landing times extrapolated from the best fit curve compared to the numerical landing time solutions generated from the RK4 integrator. The percent errors between the extrapolation and numerical solutions are shown in column 4.

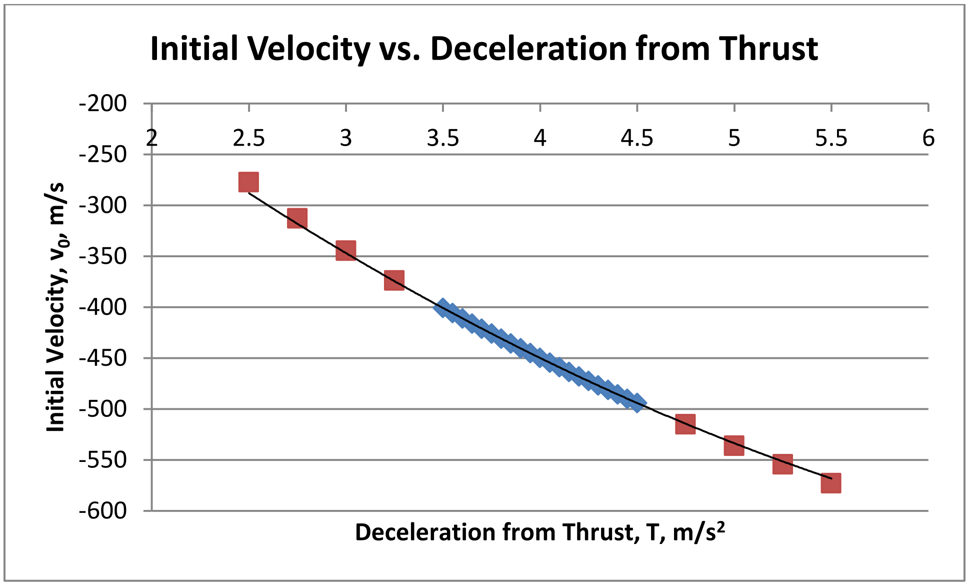



Figure#11: The graph shows the landing times from the original data set (diamonds and best fit curve) plotted alongside the numerical extrapolations from Table#11 (square points).

Table #12 and Figure #12compare the landing time extrapolations to the RK4 solutions.

| Deceleration from Thrust | Curve fit Initial Velocity | Numerical Initial Velocity | % Error Velocity |
|---|---|---|---|
| 5.5 | -568.26 | -572.78 | 0.79% |
| 5.25 | -551.57 | -554.2 | 0.47% |
| 5 | -533.67 | -535.98 | 0.43% |
| 4.75 | -514.56 | -515.05 | 0.10% |
| 3.25 | -374.54 | -373.78 | 0.20% |
| 3 | -346.98 | -344.63 | 0.68% |
| 2.75 | -318.21 | -312.78 | 1.74% |
| 2.5 | -288.23 | -277.3 | 3.94% |

Table#12: The table contains the initial velocities extrapolated from the best fit curve compared to numerical solutions obtained from the RK4 integrator. Column 4 shows the percent errors between the extrapolated values and numerical solutions.




Figure#12: The graph shows the initial velocities from the data set (diamonds and best fit curve) compared to the numerical extrapolations in Table#12 (square points).

The magnitude of the slope appears to be decreasing for the numerical solutions. Sometimes there were a small range of initial velocities which resulted in acceptable landing solutions for a given thrust value. When gathering data, the first solution which met the tolerance requirement was used. The points were approximated by moving outward from the value T = 4. This means for ranges of acceptable solutions, the solution closer to the initial velocity and landing time for T = 4 was used. The differences in these values were small, but may cause the curve to appear slightly more flat than if other solutions in the range were used.

The type of curve fit effects the estimation accuracy. This is especially true during extrapolation. As the distance from the end of the data set increases, accuracy decreases because the best fit curve and the true curve change at different rates. The best fit curves shown are $2^{nd}$ order polynomials. This means both graphs will have a critical point at some location along the curve. In this instance, both data sets have minimums. The $1^{st}$ derivative test reveals the minimum for the Landing Time vs. Deceleration curve is located on the graph at T = 5.49. The final extrapolation is at T = 5.5. This point is separated from the other extrapolations by the critical point. It's also impossible to see the trend of the data set without evaluating another point. Extrapolating a landing time at the point T = 6 generates the following data.

| Thrust | Estimated Landing Time | Numerical Landing Time | % Error Time |
|---|---|---|---|
| 6 | 160.8 | 138 | 16.52% |

Table#13: The table shows the landing time extrapolated from the best fit curve compared to the numerical landing time. Column 4 shows the percent error for the extrapolated value compared to the numerical solution.

**Conclusion:**

The data shows the best fit estimations will continue to diverge from the numerical solution at an increased rate due to the change in concavity. Therefore, it's assumed that any point beyond the critical point is not a reliable estimation. The same trend is seen for initial velocity when extrapolating outside the critical point at T=6.

Increasing the order of the best fit polynomial decreases the accuracy of extrapolations even short intervals from the data set. This trend is expected. As the number of changes in concavity grows, the curve will become more unpredictable; however, there is a possibility of finding a more accurate best fit.

Extrapolations stand to benefit the most from changing the best fit curve. As the extrapolations move away from the data set, all the best-fit curves, especially $3^{rd}$ through $6^{th}$ order polynomials, diverge from the true data set by some amount.

Full data sets are not complete, but preliminary trials show applying a power best-fit curve to the Landing Time vs. Deceleration from Thrust increases the accuracy of extrapolations. For Initial Velocity vs. Deceleration from Thrust, a logarithmic curve is providing better estimations.

After examining the data, the most notable results are the percent error for extrapolations and interpolations. These were expected. The percent error associated with the interpolations is very small. Estimations within the data set are extremely accurate and may even be considered accurate enough to serve as a numerical solution with no verification using the RK4 integrator. The extrapolations however, are not nearly as accurate. Extrapolations can be made only within small intervals outside the data set when a quadratic fit is applied.

Other curve fits besides quadratic can be applied to the data set. The data shows polynomials are not reliable for this application due to changes in concavity. As mentioned previously, preliminary trials show power curves may fit the Landing Time vs. Rocket Deceleration data the best. Whereas, a logarithmic curve fit appears to be more accurate for approximating solutions from the Initial Velocity vs. Rocket Deceleration data set. Power and logarithmic graphs don't have changes in concavity, but more research needs to be collected on these curve fits to validate these results.

Even though some curve fits increase the accuracy for extrapolations, the estimated values still don't fall within a reasonable tolerance when compared to the numerical solution. It's better to consider these points as approximate values to begin the numerical integration process. The results will then verify whether the curve fit estimation is a good approximation.

This means that interpolations should be made whenever possible. To a certain degree, and for certain data sets, this reduces the need to find the best curve fit. If the curve is smooth like it is for the lander problem most curves will fit the data set quite well. Setting up an equation so that all estimations are interpolations can be done by anticipating all the values that will be estimated and stretching the domain of the numerical data set to include those values. To keep the percent error down, additional landing solutions may need to be added to the data set that is curve fit.

When anticipating the values that will be estimated, identify the primary constraint for the problem. If landing solutions are needed for a set of rockets with various accelerations from thrust, use acceleration from thrust as the independent variable and ensure that the domain includes the acceleration that each rocket produces. It's possible for the primary constraint to be another parameter. For example, if the lander must reach the surface after or before a certain time, landing time should be used as the independent variable. The domain of the data set should range from the minimum time to the maximum time required.

Now that the conditions and best practices for obtaining accurate estimations with this model have been determined, this technique can be tested on other models. The model doesn't necessarily need to simulate the decent of a lander. The acceleration equation from the case is applicable to take-off and flight for aerospace vehicles as well. More advanced models may contain additional parameters affecting the vehicle. Further parameter studies could also be conducted on the new model to identify the relationship between the new set of parameters. When a model of sufficient accuracy is created, the numerical integration and curve fitting technique investigated in this paper can be used to make the process of finding landing or other information easier.

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