Analysis of Numerical Root-finding methods

Carl De Vries Calculus II – Honors - 2013

Introduction

Objectives

- Understand the theory behind four fundamental methods:
  - Bisection method
  - False Position method
  - Newton-Raphson method
  - Secant method
- Address termination criteria.
- Validate our code.

Background

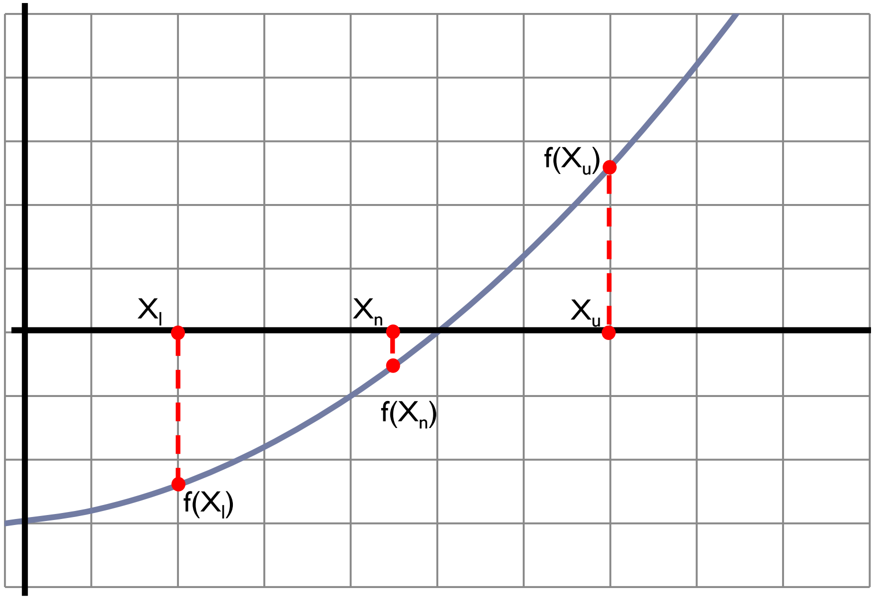
- Applications of numerical methods
  - Actuarial sciences
  - Plotting spacecraft trajectories
  - Calculating the values of financial instruments

Background

- Find the x values where f(x) = 0.
- Initial guesses and bounds are made by graphing the solution.
- Methods are divided into two families.
  - Bracketing methods
  - Open methods

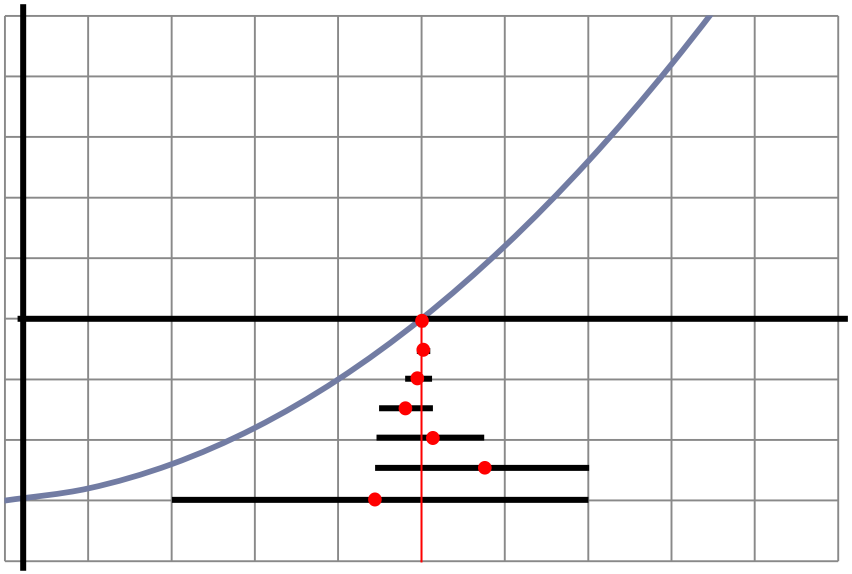
Bracketing methods

- Bracketing methods require both an upper bound and a lower bound.
- Decrease the width of the interval $[x_l, x_u]$ around the root.
- Find the interval containing the root after each iteration.
- Bracketing methods always converge.



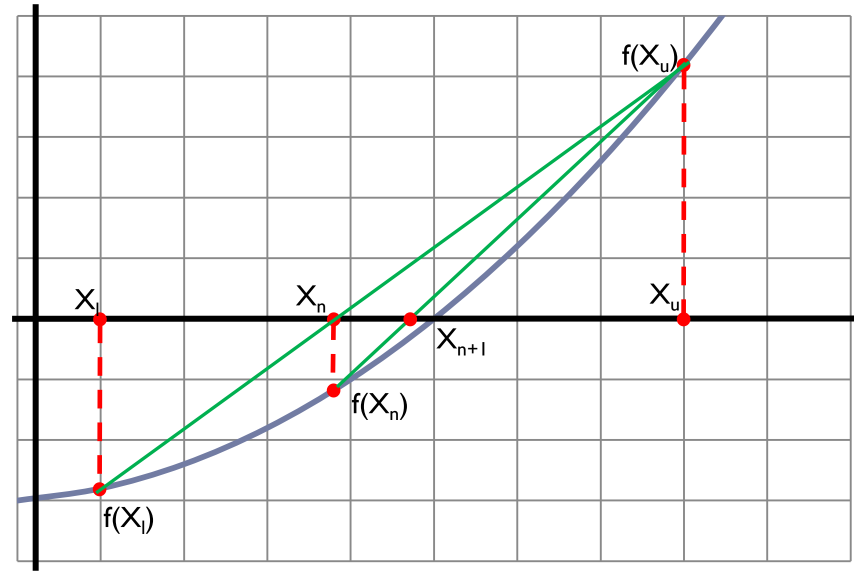
Bisection method

- Decrease the interval by half.
- The approximation has no direct relation to the function.
- Approximations may oscillate about or recede from the root.
- Approximation equation:
  - $x_n = \left(\frac{1}{2}\right)(x_l + x_u)$



False Position method

- A secant line can be visualized between the upper and lower bounds.
- Starting equation:
  - $\frac{f(x_l)-0}{x_l-x_n} = \frac{f(x_u)-0}{x_u-x_n}$
- Approximation equation:
  - $x_n = \frac{f(x_l)(x_u)-f(x_u)(x_l)}{f(x_l)-f(x_u)}$

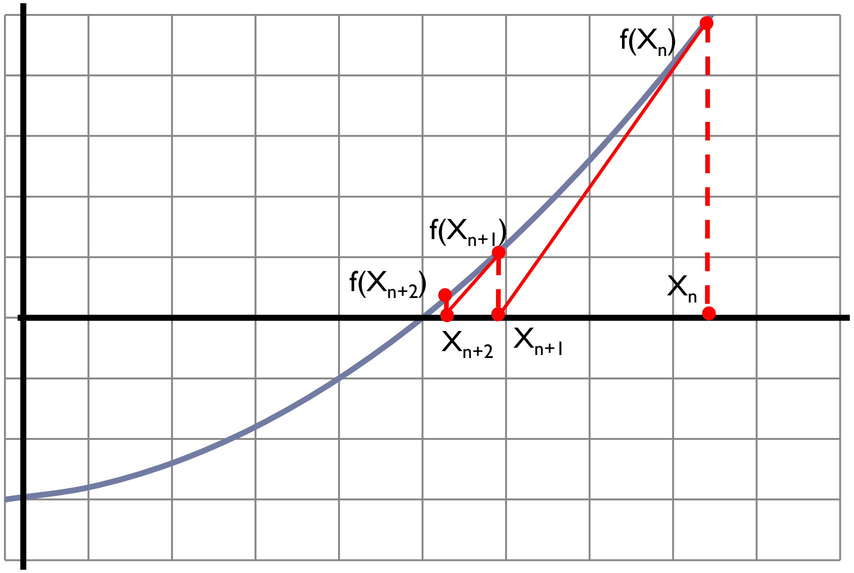


Open Methods

- Open methods may only require a single initial guess.
- The initial guesses aren’t required to surround the root.
- Open methods tend to converge faster than bracketing methods.
- The approximations may diverge from the root.

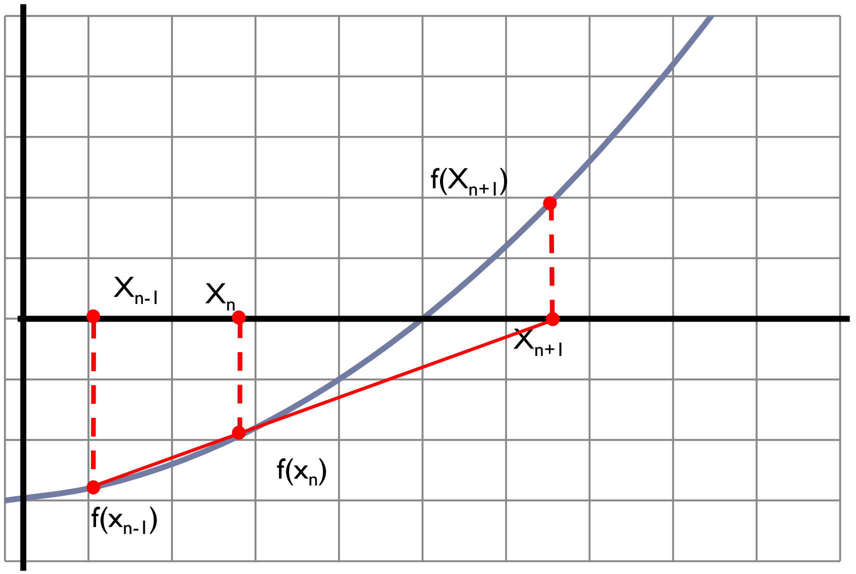
Newton-Raphson method

- Newton-Raphson requires an analytical derivative.
- The derivative cannot evaluate to zero.
- The method may give an answer when $|f(x_n)| < \varepsilon$, but doesn't actually cross the root.
- Starting equation:
  - $f'(x_n) = \frac{f(x_n)-0}{x_n - x_{n+1}}$
- Approximation equation:
  - $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$



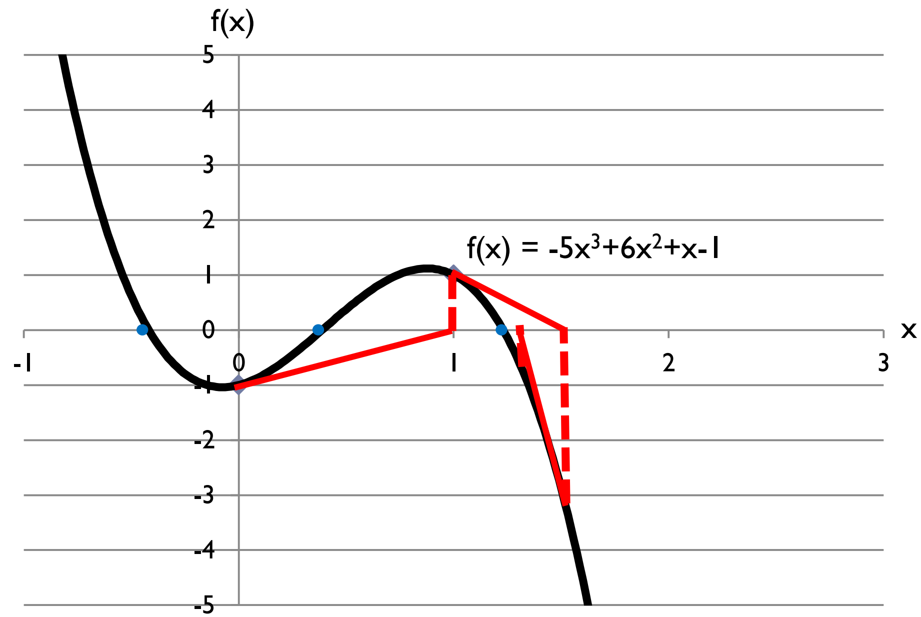
Secant method

- Useful when an solving for the analytical derivative is difficult.
- $f'(x_n)$ is approximately equal to a slope approximation of two near points.
- Staring Equation:
  - $x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_{n-1})-f(x_n)}{x_{n-1}-x_n}}$
- Approximation Equation:
  - $x_{n+1} = x_n - \frac{f(x_n)(x_{n-1}-x_n)}{f(x_{n-1})-f(x_n)}$



Divergence

- Open method approximations may diverge from the root completely.
- If multiple roots exist, the approximation may converge on a different root.
- Example using Newton-Raphson
  - Roots at
    - x = -0.419
    - x = 0.388
    - x = 1.230



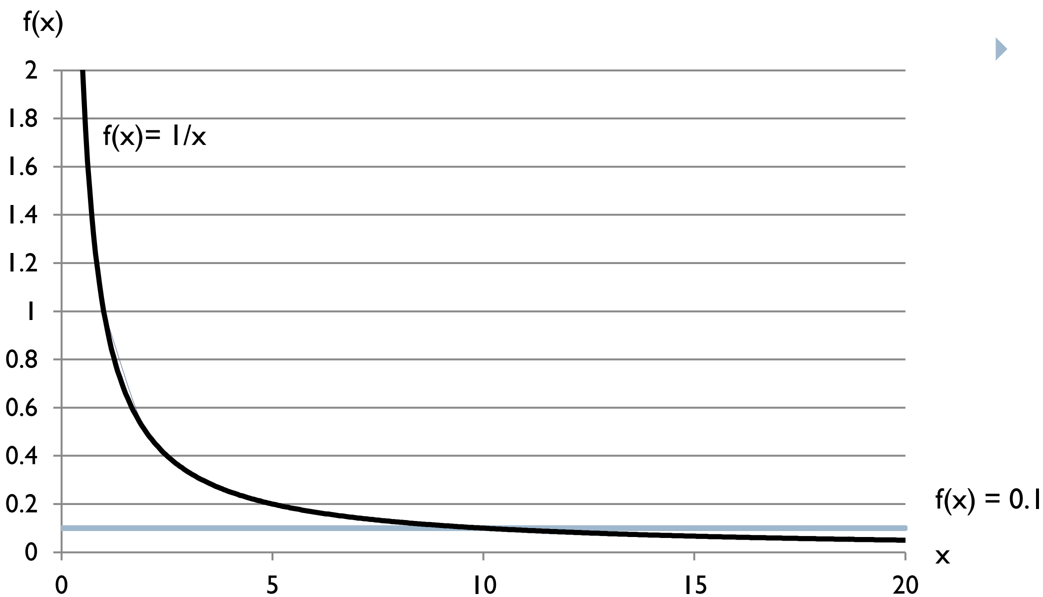
Stopping Criteria

- Determine when the approximation is accurate enough.
- Truncation error
- Round-off error
- Maximum Iterations
  - Epsilon is too small
  - The approximation diverges
  - Poor initial guess

Stopping Criteria

Option one

- $|f(x_n)| < \varepsilon$

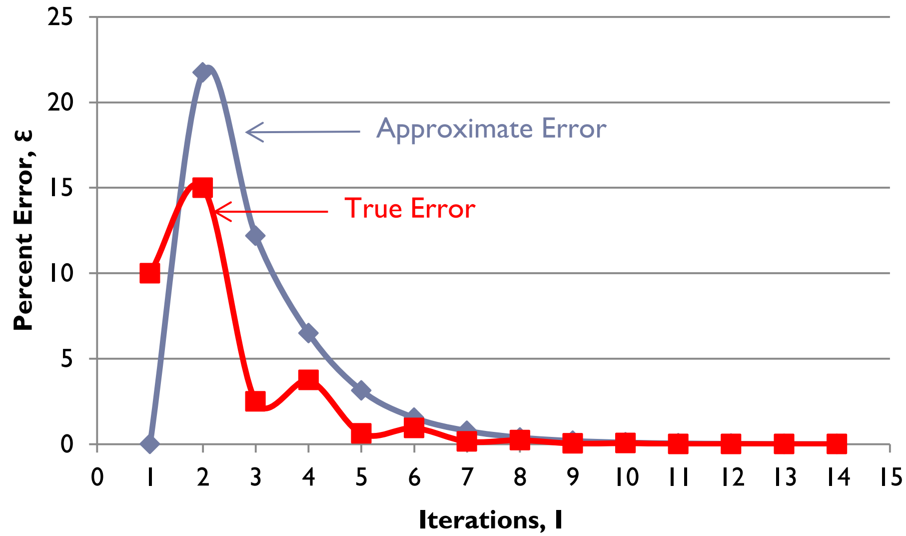


Option two

- $|f(x_n)| < \varepsilon_v$ and $x_u - x_l < \varepsilon_h$
- ε_v is a vertical measure
- ε_h is a horizontal measure
- The $\Delta x < \varepsilon$ condition is useful for functions which approach the x-axis asymptotically.

Stopping Criteria

- Option three (approximate error)
  - $\left|\frac{x_n - x_{n-1}}{x_n}\right| * 100 = \epsilon_a$
  - $\epsilon_a < \epsilon$
- True error is always less than approximate error.
- If the approximate error dips below tolerance the true error is also within the tolerance.



Code Validation

Known solution

Equation: $x^2+x-30 = 0$

$(x+6)(x-5) = 0$

Roots at x = -6 and **x = 5**

Variables

- Stopping criteria three
  - $\epsilon = 0.01\%$
- Max Iterations = 20

Method	x_l	X_u	i	x_n
Bisection	2	7	14	5.000183105
False Position	2	7	6	4.999945306
	x_n	x_{n+1}		
Newton-Raphson	2	N/A	5	5
Secant	6	7	4	5.000000161

Code Validation

Case Study

- When a circuit changes state, the energy storage in capacitors and inductors oscillates.
- Find the resistance required to dissipate the fluctuations in charge.
- Use Bisection method
- Determined initial bounds by graphing the equation

Equation

Charge in the circuit as a function of time:

$$q(t) = q_0 e^{-Rt/2L} \cos(\sqrt{\frac{1}{LC} - (\frac{R}{2L})^2} * t)$$

q/q_0 = 0.01%

t = 0.05 seconds

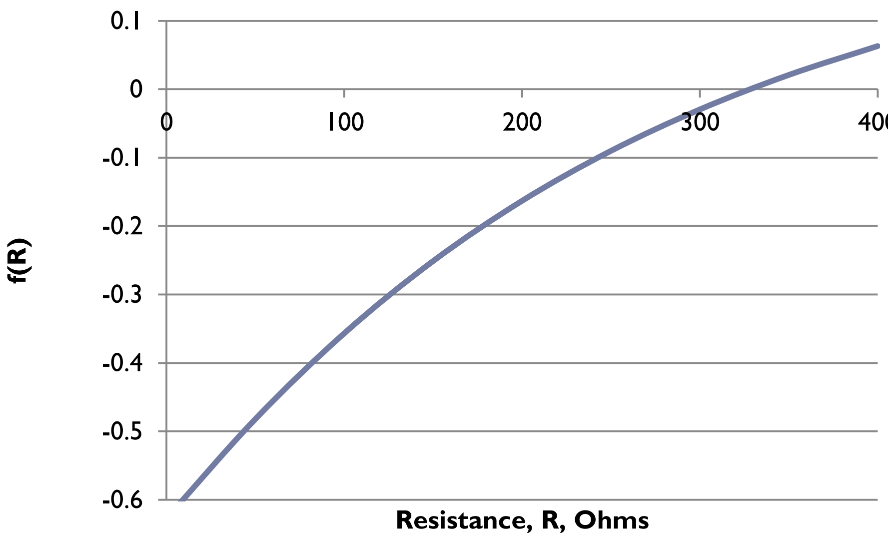
L = 5 Henries

C =10^{-4} Farads

Code Validation

The function rearranged in terms of resistance:

$$f(R) = e^{-0.005R}\cos\left(\sqrt{2000-0.01R^2}(0.05)\right) - 0.01$$



x_l	x_u	ϵ	Given x_n	Given Iterations	Calculated x_n	Calculated Iterations
0	400	0.0001%	328.1515	21	328.1515	21

Summary

- These methods are fundamental numerical methods.
- More advanced methods exist, but have their own pros and cons.
- The bracketing methods will always converge.
- Open methods may converge faster, but may also fail in several ways if:
  - $f'(x_n) = 0$
  - The approximations diverge or converge on a different root.
- Approximate error tends to be the best stopping criteria.