Hybrid sparse elimination matrices and toric Sylvester forms.

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Abstract

The goal is the text is to show the construction of hybrid elimination matrices for sparse systems, following the development for multihomogeneous systems in [5]. This construction heavily depends on the computation of the local cohomology of the homogeneous coordinate ring of the underlying toric variety. We apply the results of [1], [4] and [17] for an explicit construction, and we discuss how to We understand these matrices in the toric geometry context and find a combinatorial interpretation of them. This results can be used for generalizing methods for solving overdetermined polynomial systems.

1 Introduction

Elimination theory can be described as the art of removing variables from a homogeneous polynomial system. This problem starts with the classical contributions of Cayley, Sylvester or Macaulay. The geometric interpretation of such procedure as a projection map can be found in the Main Theorem of Elimination [16, Chapter V]. As a consequence of this result, elimination theory becomes the art of finding matrices $\mathcal{M}$ such whose drop of the rank indicates the number of solutions of a given system.

However, once we consider sparse polynomial systems, things get a bit more complicated. The relation between the geometry of elimination and the algebraic meaning becomes unclear and one is forced to step into the world of geometric combinatorics [19] and toric geometry [12]. The motivation for the interplay between sparse elimination theory and toric geometry can be seen as follows.

Let M be a lattice of rank n and k a commutative ring and $M_{\mathbb{R}} = M \otimes \mathbb{R}$ the corresponding real vector space. Let $N = \mathrm{Hom}(M, \mathbb{Z})$ be its dual and $\mathbb{T}_N = N \otimes \mathbb{C}^{\times}$ the underlying torus. Let $F_0, \ldots, F_n$ be generic sparse polynomials with supports in some subsets $\mathcal{A}_i \subset M$ for $i = 0, \ldots, n$:

$$F_i^{\mathrm{Aff}} = \sum_{b \in \mathcal{A}_i} u_{i,b} x^b \in C^{\mathrm{Aff}} = k[u_{i,a}][x_1, \ldots, x_n] \quad a \in \mathcal{A}_i$$

and let $\Delta_i = \mathrm{conv}(\mathcal{A}_i) \subset M_{\mathbb{R}}$ be the Newton polytopes. Then, one can consider the toric variety X_Σ associated to the normal fan $\Sigma \subset N$ of the Minkowski sum $\Delta = \sum_{i=0}^{n} \Delta_i$. In particular, we write the Newton polytope in a facet presentation using the generators u_ρ of the rays (1-dimensional cones) $\rho \in \Sigma(1)$.

$$\Delta_i = \{m \in M_{\mathbb{R}} \quad \langle m, u_j \rangle \geq -a_{ij} \quad \rho_j \in \Sigma(1)\} \quad i = 0, \ldots, n$$

In order to take advantage of the graded structure given by this toric variety, we make use of the homogeneous coordinate ring, also known as the Cox ring [13], and consider the homogenization of the polynomials, by adding new variables $z_1, \dots, z_r$:

$$F_i = \sum_{m \in \mathcal{A}_i} u_{i,n} x^{\mathcal{F}m + a_i} \in C = k[u_{i,a}][x_1, \dots, x_n, z_1, \dots, z_r] \quad a \in \mathcal{A}_i$$

where $a_i \in \Sigma(1)$ is the vector of a_{ij} of the facet definition of the polytope and $\mathcal{F}$ is a matrix whose rows are the generators of the rays of Σ. Under such conditions, we say that $F_0 = \cdots = F_n = 0$ is a polynomial system in X_Σ and consider its defining ideal $I = \langle F_0, \dots, F_n \rangle$.

Definition 1.1. An elimination matrix $\mathcal{M}$ is a matrix with coefficients in $\mathbb{C}[u_{i,a} \quad a \in \mathcal{A}_i]$ such that for any specialization map $\rho : k[u_{i,a} \quad a \in \mathcal{A}_i] \to \mathbb{C}$ and a system:

$$f_i = \rho(F_i) \text{ for } i = 0, \dots, n$$

we have:

- The rank of $\rho(\mathcal{M})$ drops, if and only if, $f_0 = \cdots = f_n$ has a solution in X_Σ.
- If the number of solutions is finite in X_Σ and equals κ, then the corank of $\rho(\mathcal{M})$ is κ.

The first property corresponds to resultant theory and such matrices can be found in combinatorial ways using Canny-Emiris matrices [7], [8], [14]. The second property tells us a bit more of information and can be used for finding the cokernel of the matrix $\mathcal{M}$ and solving the system. The property is known to hold for matrices that correspond to strands of the Koszul complex in dense systems since [21], [20]. These matrices look like:

$$\mathbb{M}_\nu : \oplus S(-\alpha_i)_\nu \xrightarrow{(F_0, \dots, F_n)} S_\nu \quad \nu \in \mathrm{Cl}(X_\Sigma)$$

where $\alpha_0, \dots, \alpha_n$ are the classes corresponding to $F_0, \dots, F_n$ in $\mathrm{Cl}(X_\Sigma)$. These matrices are of Sylvester-type. Our main goal is to find a regions $\Theta^{\text{sylv}} \subset \mathrm{Cl}(X_\Sigma)$ such that for $\nu \in \Theta$, the matrix $\mathbb{M}_\nu$ is an elimination matrix. This region will be outside the support of (I^{sat}/I) where I^{sat} is the saturation of I with respect to the irrelevant ideal of the toric variety X_Σ. In particular, we will use the divisor $\delta = \sum_{i=0}^n \alpha_i - K_X$ for K_X being the canonical divisor of the toric variety, which is a vertex of the previous support.

On the other hand, it is possible to build elimination matrices in some region $\Theta \subset \mathrm{Cl}(X_\Sigma)$ that may intersect with the support of (I^{sat}/I). In such region, we will be able to prove the duality property:

$$(I^{\text{sat}}/I)_\nu = \mathrm{Hom}_A((C/I)_{\delta-\nu}, A) = \mathrm{Hom}_A((C)_{\delta-\nu}, A)$$

which will allow us to find a basis of $(I^{\text{sat}}/I)_\nu$ by duality to a monomial basis of $(C)_{\delta-\nu}$. The construction of such basis generalizes the Jacobian construcion from [10] and [9] and depends on the fact that for each $\mu \in \mathbb{Z}^{\Sigma(1)}$ corresponding to a lattice point in the polytope associated to $C_{\delta-\nu}$, there is a decomposition of the polynomials as:

$$F_i = x^{\mu_1+1} F_{i1} + \dots x^{\mu_n+1} F_{in} + z_1^{\mu_{n+1}+1} \cdots z_{n+r}^{\mu_{n+r}+1} F_{i(n+1)}$$

which gives an element $\mathrm{sylv}_\mu = \det(F_{ij})$ which is well defined in $(I^{\text{sat}}/I)_\nu$. We will prove that:

$$x^\beta \, \mathrm{sylv}_\mu = \begin{cases} \mathrm{sylv}_0 & \mu = \beta \\ 0 & \text{otherwise} \end{cases}$$

and imply that $\{\mathrm{sylv}_\mu\}_{\mu\in C_\nu}$ form a basis of $(I^{\mathrm{sat}}/I)_\nu$. This will allow us to give other elimination matrices:

$$\mathbb{H}_\nu : \oplus S(-\alpha_i)_\nu \oplus (I^{\mathrm{sat}}/I)_{\delta-\nu} \xrightarrow{(F_0,\ldots,F_n,\mathrm{sylv}_\mu)} S_\nu \quad \nu\in\Theta$$

We will give a combinatorial interpretation of the Sylvester forms construction through the Batyrev-Borisov theorem [2] and see how this generalizes the lifting constructions of resultant theory [6], [15].

However, the explicit computation of Θ depends heavily on the computation of the local cohomology groups $H^i_{\mathfrak{b}}(S)$ of the Cox ring $S = k[x_1,\ldots,x_n,z_1,\ldots,z_r]$. For the case of multiprojective spaces, it is possible to use the Mayer-Vietoris sequence, in the fashion of [4]. For a general toric varieties, we can give some examples through comparing with sheaf and reduced cohomology and using the computations of [1] or [17].

2 Toric geometry in sparse elimination

The advantage of using toric geometry is that we can make good use of the combinatorial information encoded in the fan Σ.

Definition 2.1. The fan Σ is smooth, if for every cone $\sigma\in\Sigma$, its minimal generators are part of a basis of N.

Lemma 2.1. X_Σ is smooth, if and only if, Σ is smooth.

Proof. [11, Theorem 3.1.19] □

Firstly, it is possible to define a short exact sequence:

$$0\to M \xrightarrow{F} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\pi} \mathrm{Cl}(X_\Sigma)\to 0$$

where F is an $n\times|\Sigma(1)|$ matrix whose columns are the generators of the rays in $\Sigma(1)$ and π is chosen accordingly to be a cokernel matrix. The polytopes Δ_i can be seen as elements of $\mathbb{Z}^{\Sigma(1)}$ using the vectors (a_{ij}) for $\rho_j\in\Sigma(1)$. Two polytopes that map to the same class in $\mathrm{Cl}(X_\Sigma)$ are translations of each other.

Remark 2.1. For sake of simplicity, we will always choose π to have an identity block:

$$\begin{pmatrix} M & \mathrm{Id}_n\end{pmatrix}.$$

The second advantage is the possibility of using the homogeneous coordinate ring of X_Σ from [13], also known as the Cox ring. This is a ring defined as $S=k[x_\rho \quad \rho\in\Sigma(1)]$ and comes with the grading given by $\mathrm{Cl}(X_\Sigma)$ as:

$$S=\oplus_{\alpha\in\mathrm{Cl}(X_\Sigma)}S_\alpha \quad S_\alpha = H^0(X_\Sigma,\mathcal{O}_\Sigma(D))$$

In particular, we use the following notation for the variables of Cox ring. Choose $\sigma\in\Sigma$ to be a maximal smooth cone and consider $x_1,\ldots,x_n$ to be the variables associated to $\rho\subset\sigma(1)$ and $z_1,\ldots,z_r$ to be the rest of variables where r is the rank of $\mathrm{Cl}(X_\Sigma)$, which is free under the hypothesis that Σ is smooth.

The previous short exact sequence also allows us to homogenize the polynomials F_i in the Cox ring with respect to the Newton polytope Δ_i.

Finally, the fan also gives the information of the irrelevant ideal $\mathfrak{b}$, which is defined as:

$$\mathfrak{b} = \langle x^{\overline{\sigma}} \quad \sigma \in \Sigma(n) \rangle \quad x^{\overline{\sigma}} = \prod_{\rho \notin \sigma} x_\rho.$$

Using this ideal, we will be able to define $\mathfrak{b}$-torsion of a graded S-module B to be:

$$\Gamma_{\mathfrak{b}}(B) = \{a \in B \quad \mathfrak{b}^k a \quad k \in \mathbb{N}\}$$

and the local cohomology modules to be the derived functors of $B \to \Gamma_{\mathfrak{b}}(B)$. In particular, we are interested in $B = C/I$ for $I = \langle F_0, \dots, F_n \rangle$ which satisfies:

$$H^0_{\mathfrak{b}}(B) = I^{\text{sat}}/I \quad I^{\text{sat}} = (I : \mathfrak{b}^\infty)$$

Moreover, the irrelevant ideal has a primary decomposition, which will be useful in terms of understanding the local cohomology.

Definition 2.2. A subset of the rays $P \subset \Sigma(1)$ is called a primitive collection if it satisfies:

- $P \not\subset \sigma$ for $\sigma \in \Sigma(n)$.
- If $P' \subset P$ there is $\sigma \in \Sigma(n)$ such that $P \subset \sigma$.

Lemma 2.2.

$$\mathfrak{b} = \cap_P \langle x_\rho \, \rho \in P \rangle$$

where P runs over all the primitive families of Σ.

3 Cech-Koszul spectral sequences

$F_0, \dots, F_n$ define a regular sequence in C_σ being the localization at $x^{\overline{\sigma}}$ of C.

Cech-Koszul $\mathcal{C}^\bullet_{\mathfrak{b}}(K_\bullet(f, S))$.

$$\begin{array}{ccccc}
\mathcal{C}^0_{\mathfrak{b}}(K_{n+1}(f,S)) & \mathcal{C}^0_{\mathfrak{b}}(K_n(f,S)) & \mathcal{C}^0_{\mathfrak{b}}(K_{n-1}(f,S)) & \cdots & \mathcal{C}^0_{\mathfrak{b}}(K_0(f,S)) \\
\mathcal{C}^1_{\mathfrak{b}}(K_{n+1}(f,S)) & \mathcal{C}^1_{\mathfrak{b}}(K_n(f,S)) & \mathcal{C}^1_{\mathfrak{b}}(K_{n-1}(f,S)) & \cdots & \mathcal{C}^1_{\mathfrak{b}}(K_0(f,S)) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathcal{C}^n_{\mathfrak{b}}(K_{n+1}(f,S)) & \mathcal{C}^n_{\mathfrak{b}}(K_n(f,S)) & \mathcal{C}^n_{\mathfrak{b}}(K_{n-1}(f,S)) & \cdots & \mathcal{C}^n_{\mathfrak{b}}(K_0(f,S)) \\
\mathcal{C}^{n+1}_{\mathfrak{b}}(K_{n+1}(f,S)) & \mathcal{C}^{n+1}_{\mathfrak{b}}(K_n(f,S)) & \mathcal{C}^{n+1}_{\mathfrak{b}}(K_{n-1}(f,S)) & \cdots & \mathcal{C}^{n+1}_{\mathfrak{b}}(K_0(f,S))
\end{array}$$

Start horizontally, the second page is:

$$\begin{array}{ccccc} H^0_{\mathfrak{b}}(H_{n+1}(f,S)) & H^0_{\mathfrak{b}}(K_n(f,S)) & H^0_{\mathfrak{b}}(K_{n-1}(f,S)) & \cdots & I^{\mathrm{sat}}/I \\ 0 & 0 & 0 & \cdots & H^1_{\mathfrak{b}}(K_0(f,S)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & H^n_{\mathfrak{b}}(K_0(f,S)) \\ 0 & 0 & 0 & \cdots & H^{n+1}_{\mathfrak{b}}(K_0(f,S)) \end{array}$$

Lemma 3.1. Let C_σ be the ring C localized at the variables x

$$\begin{array}{ccccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H^n_{\mathfrak{b}}(S(-\sum(\alpha_i,\beta_i))) & H^n_{\mathfrak{b}}(\oplus_{j,k}S(-(\alpha_j+\alpha_k,\beta_j+\beta_k))) & \cdots & \cdots & H^n_{\mathfrak{b}}(S/I) \\ H^{n+1}_{\mathfrak{b}}(S(-\sum(\alpha_i,\beta_i))) & H^{n+1}_{\mathfrak{b}}(\oplus_{j,k}S(-(\alpha_j+\alpha_k,\beta_j+\beta_k))) & \cdots & \cdots & H^{n+1}_{\mathfrak{b}}(S/I) \end{array}$$

Theorem 3.1. (Grothendieck-Serre) X_Σ smooth in $\overline{C} = \rho(C)$ and M a $\overline{C}$-module.

$$\mathrm{HF}(M,\nu) = \mathrm{HP}(M,\nu) + \sum_{i=0}^{d}(-1)^i \dim_k H^i_{\mathfrak{b}}(M)_\nu$$

4 Sylvester forms and duality

Following [9], we are able to write the polynomials F_i for $i = 0,1,2$ as:

$$F_i = x_1 F_{i,x_1} + x_2 F_{i,x_2} + z_1 z_2 F_{i,z}$$

This decomposition is only possible if $F_i \in \mathfrak{b}$. Consider the determinant:

$$\mathrm{sylv}_0 = \det\begin{pmatrix} F_{0,x_1} & F_{0,x_2} & F_{0,z} \\ F_{1,x_1} & F_{1,x_2} & F_{1,z} \\ F_{2,x_1} & F_{2,x_2} & F_{2,z} \end{pmatrix}$$

which we call a Sylvester form. It has the following properties.

Proposition 4.1. sylv_0 does not depend in the decomposition when seen in $(I^{\mathrm{sat}}/I)_\delta$ and defines a nonzero element.

Remark 4.1. This non-zero element defines a toric residue in the sense of [9].

Theorem 4.1. (Batyrev-Borisov) Let $D = \sum_\rho a_\rho D_\rho$ and Δ be the associated polytope which has dimension n in the complete toric variety X_Σ. If D is nef then:

- $H^p(X_\Sigma, \mathcal{O}(-D)) = 0$ if $p \neq n$.
- $H^n(X, \mathcal{O}(-D)) = \oplus_{m \in \mathrm{Relint}(\Delta) \cap M} \mathbb{C}\chi^{-m}$

Proof. [11, Theorem 9.2.7]. □

By looking at the diagram for the local cohomology of $\mathcal{H}_r$ and comparing it to Batyrev-Borisov result, one can see that the smallest 2-dimensional polytope Δ corresponding to a divisor D in the nef cone such that $H^2(X, \mathcal{O}(-D)) \neq 0$ is the anti-canonical divisor $-K_X$. This characterizes the anti-canonical divisor as corresponding to the smallest maximal dimensional polytope having a lattice point in the relative interior.

Denote by Δ_Σ the polytope associated to the anticanonical divisor.

Lemma 4.1. Δ_Σ has a single interior lattice point, which in homogeneous coordinates corresponds to $x_0 \cdots x_n z_1 \cdots z_r$, if and only if, the anticanonical divisor is Cartier and ample.

Proof. Consequence of [11, Theorem 8.3.4]. □

These type of toric varieties are called Gorenstein Fano and our Hirzebruch surfaces are an example.

As a direct consequence of the comparison between the two Cech-Koszul sequences, we found that for some of the $\mu \in \mathrm{Pic}(X_\Sigma)$:

$$(I^{\mathrm{sat}}/I)_\mu \cong \mathrm{Hom}((C)_{\delta-\mu}, A)$$

In the case of $\mu = \delta$, this corresponds to the proved case and both are dimension 1 vector spaces as in [9]. For the cases where $\mu \neq \delta$, consider, the basis of $C_{\delta-\mu}$ given by monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n} z_1^{\alpha_{n+1}} \cdots z_1^{\alpha_{n+r}}$. Consider the projection $\pi : \mathbb{Z}^{n+r} \to \mathbb{Z}^r$, and as all the previous monomials correspond to the same graded piece C_μ, we have $\overline{\alpha} = \pi(\alpha) \in \mathrm{Pic}(X_\sigma)$ and we impose:

$$0 \geq \overline{\alpha}_r < \min a_{i,j}$$

Under such hypothesis, for each $\alpha \in \mathbb{Z}^{\Sigma(1)}$ we are able to write a decomposition:

$$F_i = x_1^{\alpha_1+1} F_{i,1} + x_2^{\alpha_2+1} F_{i,2} + \cdots + z_1^{\alpha_{n+1}+1} \cdots z_r^{\alpha_{n+r}+1} F_{i,n+1}$$

Definition 4.1. The Sylvester form sylv_α is defined as the polynomial:

$$\mathrm{sylv}_\alpha = \det(F_{i,1})$$

and has degree $\delta - \overline{\alpha}$.

Theorem 4.2.
$$x^\beta \operatorname{sylv}_\alpha = \begin{cases} \operatorname{sylv}_0 & \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

Proof. From the decomposition, we have that:

$$x_j^{\alpha_j+1} \operatorname{sylv}_\alpha = \det \begin{pmatrix} \cdots & x_j^{\alpha_j+1} F_{0,j} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & x_j^{\alpha_j+1} F_{0,j} & \cdots \end{pmatrix} = \det \begin{pmatrix} \cdots & F_0 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & F_n & \cdots \end{pmatrix} \in I$$

Therefore, if $\beta \neq \alpha$ there is $j = 1, \ldots, n+r$ such that $\beta_j > \alpha_j$ and therefore $x_j^{\alpha_j+1}$ divides x^β:

$$x^\beta \operatorname{sylv}_\alpha = \frac{x^\beta}{x_j^{\alpha_j+1}} x_j^{\alpha_j+1} \operatorname{sylv}_\alpha \in I$$

On the other hand, we have:

$$x^\beta \operatorname{sylv}_\beta = x_1^{\beta_j} \cdots x_n^{\beta_n} z_1^{\beta_{n+1}} z_r^{\beta_{n+r}} \det(F_{ij}) = \det(x_j^{\beta_j} F_{ij})$$

but at the same time, the decomposition:

$$F_i = x_1 x_1^{\alpha_1} F_{i,1} + x_2 x_2^{\alpha_2} F_{i,2} + \cdots + z_1 \cdots z_r z_1^{\alpha_{n+1}} \cdots z_r^{\alpha_{n+r}} F_{i,n+1}$$

gives the Sylvester form sylv_0, implying the equality □

Remark 4.2. The duality also follows from the Global transformation law of [9, Theorem 0.2]. If we consider the toric residue as:

$$\operatorname{Res}_F(H) = \operatorname{Res}(\frac{H\Omega}{F_0 \cdots F_n})$$

where Ω is the Euler form. Then, the global transformation law implies that for a decomposition $G_j = \sum_{i=0} A_{ij} F_j$, we have:

$$\operatorname{Res}_G(H) = \operatorname{Res}_F(H \det(A_{ij}))$$

$$x_1 \implies F_i = x_1^2 F_{i,1} + x_2 F_{i,2} + z_1 z_2 F_{i,3}$$

$$z_1 \implies F_i = x_1 F_{i,1} + x_2 F_{i,2} + z_1^2 z_2 F_{i,3}$$

4.1 The hypotheses of the Global transformation law

Let Σ be simplicial and choose a maximal cone $\sigma \in \Sigma(n)$. Let $\alpha \in \operatorname{Pic}(X_\Sigma)$ and pick Δ_α be a representative polytope such that:

$$a_j = 0 \quad \rho_j \in \sigma$$

Lemma 4.2. Let $\alpha \in \operatorname{Pic}(X_\Sigma)$ such that $0 \geq \alpha_j < \min_i \alpha_{ij}$. The following are equivalent:

- $C_\alpha = (C/I)_\alpha$.

- If F has degree α for every $\mu \in \mathbb{Z}^{\Sigma(1)}$ such that $\pi(\mu) = \alpha$, there is a decomposition:

$$F = x_1^{\mu_1+1}\overline{F}_1 + \cdots + x_n^{\mu_n+1}\overline{F}_n + z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1}\overline{F}_{n+1}$$

In order to prove this, which will be the hypotheses that we can put on

In fact, as proved in []

A Sylvester decomposition for the pair $(F, \overline{\mu})$ is a vector $(F_1, \ldots, F_{n+1})$ such that for every element $\mu \in \mathbb{Z}^{\Sigma(1)}$ in the class of $\overline{\mu}$ we have:

$$F = x_1^{\mu_1+1}F_1 + \ldots x_n^{\mu_n+1}F_n + z_1^{\mu_{n+1}+1} \cdots z_r^{\mu_{n+r}+1}F_{n+1}$$

Lemma 4.3. For $\mu \in \mathrm{Pic}(X_\Sigma)$, a Sylvester decomposition if and only if $0 \leq \mu_{n+j} < a_j$ for $j = 1, \ldots, r$.

Now that we know the duality and the conditions needed for it, we can interpret Sylvester forms as the lattice points in the interior of a lattice, using Batyrev-Borisov theorem. In particular, for $\alpha \in \Theta$, we have:

$$(I^{\mathrm{sat}}/I)_{\delta-\alpha} = \mathrm{Hom}((C/I)_\alpha, A) = H_{\mathfrak{b}}^{n+1}(S(-\sum \alpha_i))_{\delta-\alpha} = H^n(X_\Sigma, \mathcal{O}_\Sigma(\delta - \alpha - \sum_i \alpha_i)) =$$

$$H^n(X_\Sigma, \mathcal{O}_\Sigma(-K_X - \alpha)) = \oplus_{m \in \mathrm{Relint}(P)}\mathbb{C}\chi^{-m}$$

where P is the polytope associated to the divisor $K_X + \alpha$.

5 Hybrid elimination matrices

6 Examples: Hirzebruch surfaces

6.1 The local cohomology of the Hirzebruch surfaces

We follow [1] for the description of the local cohomology of S using tempting subsets, but other references could give us a similar results by using Mayer-Vietoris (as done in [4] for multihomogeneous polynomials) or the supports of local cohomology in [17]. This description is based in finding tempting subsets $\mathcal{R} \subset \Sigma(1)$ which are subsets of the rays that satisfy that their geometric realization is not acyclic [1, Definition 5.1]. Without entering in much details, each of these subsets define regions of $\mathrm{Cl}(X_\Sigma)$ as:

$$\mathcal{M}_{\mathbb{Z}}^{\mathcal{R}} = \pi(\mathbb{Z}_{\geq 0}^{\Sigma(1)-\mathcal{R}} \times \mathbb{Z}_{\leq -1}^{\mathcal{R}}) \quad \pi : \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X_\Sigma)$$

Then, they prove [1, Proposition 5.5] that a class $[D] \in \mathrm{Cl}(X_\Sigma)$ is in $\mathcal{M}_{\mathbb{Z}}^{\mathcal{R}}$, if and only if, it lies in the same class as some divisor $\sum_\rho \lambda_\rho D_\rho$ with $\mathcal{R} = \{\rho \in \Sigma(1) \quad \lambda_\rho < 1\}$, if and only if the sheaf cohomology cohomology $H^i(X_\Sigma, D) \neq 0$ for some i (calculating it through the reduced cohomology of its geometric realization).

Remember the definition of primitive collections.

Definition 6.1. A set $\mathcal{P} \subset \Sigma(1)$ is called a primitive collection if $\mathcal{P} \not\subset \sigma(1)$ for any $\sigma \in \Sigma(n)$ but for any subset $\mathcal{P}' \subset \mathcal{P}$ there is a cone $\sigma \in \Sigma(n)$ such that $\mathcal{P}' \subset \sigma(1)$.

These collections are related to the irrelevant ideal by the primary decomposition:

$$\mathfrak{b} = \cap_{\mathcal{P}} \langle x_\rho \quad \rho \in \mathcal{P} \rangle$$

When we find primitive collections that never intersect to each other, we say that the fan Σ splits. The local cohomology for such cases can be derived from [1], as well as when the Picard number is 2 or 3. For spliting fans, such as the fans corresponding to Hirzebruch surfaces, but also the multiprojective case, the description of the tempting subsets is as easy as finding the primitive subsets of the ideal $\mathfrak{b}$. If $\mathcal{P}_1, \dots, \mathcal{P}_s$ are the primitive subsets, then:

$$\mathcal{R} \subset \Sigma(1) \text{ is tempting} \iff \mathcal{R} = \cup_{j \in J} \mathcal{P}_j \quad J \subset \{1, \dots, s\}$$

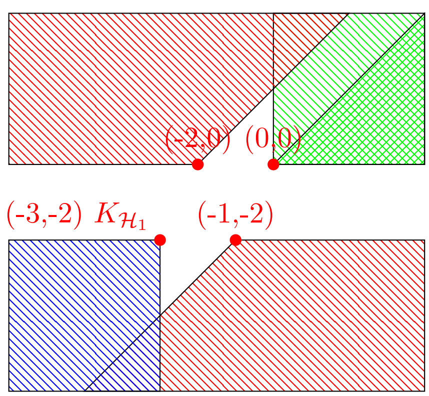
$$\mathcal{R} = \emptyset \quad (a,b) \in \mathcal{M}^{\mathcal{R}} \iff \exists (u,v) \quad (u, v, a - rv - u, b - v) \text{ all positive} \iff (a,b) \geq 0$$

$$\mathcal{R} = \{\rho_1, \rho_2, \rho_3, \rho_4\} \quad (a,b) \in \mathcal{M}^{\mathcal{R}} \iff \exists (u,v) \quad (u, v, a - rv - u, b - v) \text{ all negative} \iff (a,b) \leq (-2-r, -2)$$

$$\mathcal{R} = \{\rho_1, \rho_3\} \quad (a,b) \in \mathcal{M}^{\mathcal{R}} \iff \exists (u,v) \quad (u, v, a - rv - u, b - v) \text{ only } u, a - rv - u \text{ negative} \iff b \geq 0 \quad a - rb \leq -$$

$$\mathcal{R} = \{\rho_2, \rho_4\} \quad (a,b) \in \mathcal{M}^{\mathcal{R}} \iff \exists (u,v) \quad (u, v, a - rv - u, b - v) \text{ only } v, b - v \text{ negative} \iff \iff b \leq -2 \quad a - rb \geq$$

Using the reduced cohomology as well, the picture that they get for $\mathcal{H}_1$ is the following.



where the green region indicates the nonvanishing of $H^0(X_\Sigma, D)$, the red region indicates the support of $H^1(X_\Sigma, D)$ and the blue region indicates the nonvanishing of $H^2(X_\Sigma, D)$. The ray in the middle of the green region indicates the separation between the nef and the effective cone.

For sake of simplification, we will write these cones as:

Remark 6.1. Of course, this picture depends on the choice of a projection matrix π, but we already made this choice in the first section. It is important to remark that the vertex of the blue cone corresponds to the canonical divisor of $K_{\mathcal{H}_r}$. Therefore, the symmetry between the regions corresponds to toric Serre duality [11, Theorem 9.2.10].

In other words, the sheaf cohomology groups for $D = \sum a_\rho D_\rho$ with representative $(\alpha, \beta) \in \mathrm{Pic}(\mathcal{H}_1)$ can be described as:

$$H^0(\mathcal{H}_1, D) = 0 \iff \alpha < 0 \text{ or } \beta < 0$$

$$H^1(\mathcal{H}_1, D) = 0 \iff (\beta > -3 \text{ or } \alpha - \beta < 3) \text{ and } (\beta < 0 \text{ or } \alpha - \beta > -2)$$

$$H^2(\mathcal{H}_1, D) = 0 \iff \alpha > -2 \text{ or } \beta > -3$$

and using the relation between local and sheaf cohomology:

$$H^2_{\mathfrak{b}}(S) = (-2, 0) + \mathrm{Cone}((1,1), (-1,0)) \cup (-1,-2) + \mathrm{Cone}((1,0), (-1,-1))$$

$$H^3_{\mathfrak{b}}(S)_{(\alpha,\beta)} = (-2,-3) + \mathrm{Cone}((-1,0), (0,-1))$$

and $H^0_{\mathfrak{b}}(S)_{(\alpha,\beta)} = H^1_{\mathfrak{b}}(S)_{(\alpha,\beta)} = 0$ as a consequence of the identification $S_{(\alpha,\beta)} = H^0(\mathcal{H}_1, D)$.

6.2 The Cech-Koszul complex

Consider the Cech-Koszul complex for three polynomials in $\mathcal{H}_1$ and denote by $(\alpha_0, \beta_0), (\alpha_1, \beta_1), (\alpha_2, \beta_2)$ their degrees in $\mathrm{Pic}(\mathcal{H}_r)$. If we start taking homologies horizontally, the second page that we get is:

$$\begin{array}{ccccccc}
* & \longrightarrow & * & \longrightarrow & * & \longrightarrow & H^0_{\mathfrak{b}}(S/I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^1_{\mathfrak{b}}(S/I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^2_{\mathfrak{b}}(S/I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^3_{\mathfrak{b}}(S/I)
\end{array}$$

All the zeroes appear as the F_i form a regular sequence in the rings C_σ corresponding to the maximal cones of Σ. For the Hirzebruch case, the proof is exactly the same as in [5, Lemma 3.4].

If we start taking homologies vertically, we get a first page which is:

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2_{\mathfrak{b}}(S(-\sum(\alpha_i, \beta_i)) & \longrightarrow & H^2_{\mathfrak{b}}(\oplus_{j<k} S(-(\alpha_j + \alpha_k, \beta_j + \beta_k)) & \longrightarrow & H^2_{\mathfrak{b}}(\oplus_i S(-(\alpha_i, \beta_i)) & \longrightarrow & H^2_{\mathfrak{b}}(S) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^3_{\mathfrak{b}}(S(-\sum(\alpha_i, \beta_i)) & \longrightarrow & H^3_{\mathfrak{b}}(\oplus_{j<k} S(-(\alpha_j + \alpha_k, \beta_j + \beta_k)) & \longrightarrow & H^3_{\mathfrak{b}}(\oplus_i S(-(\alpha_i, \beta_i)) & \longrightarrow & H^2_{\mathfrak{b}}(S)
\end{array}$$

Proposition 6.1.

$$(H^2_{\mathfrak{b}}(\oplus_{j<k}S(-(\alpha_j+\alpha_k,\beta_j+\beta_k)) = \cup_{j<k}H^2_{\mathfrak{b}}(S(-(\alpha_j+\alpha_k,\beta_j+\beta_k)) =$$

$$\cup_{j<k}(-2+\alpha_j+\alpha_k,\beta_j+\beta_k)+\mathrm{Cone}((1,1),(-1,0))\cup(-1+\alpha_j+\alpha_k,-2+\beta_j+\beta_k)+\mathrm{Cone}((1,0),(-1,-1))$$

$$(H^2_{\mathfrak{b}}(\oplus_{j<k}S(-(\alpha_i,\beta_i)) = \cup_i H^2_{\mathfrak{b}}(S(-(\alpha_i,\beta_i)) =$$

$$\cup_i(-2+\alpha_i,\beta_i)+\mathrm{Cone}((1,1),(-1,0))\cup(-1+\alpha_i,-2+\beta_i)+\mathrm{Cone}((1,0),(-1,-1))$$

The comparison between the two spectral sequences gives that for ν not in any of these supports, we have:

$$(I^{\mathrm{sat}}/I)_\mu \cong \mathrm{Hom}((C/I)_{\delta-\mu}, A)$$

In this region, minus the region where $(I^{\mathrm{sat}}/I)_\mu = 0$, we have elimination matrices of the tpye $\mathbb{M}_\nu$. We now want to understnad a basis of $(I^{\mathrm{sat}}/I)_{\delta-\mu}$ generated by Sylvester forms which is dual to the basis that we have in C_μ. We start at $\mu = 0$.

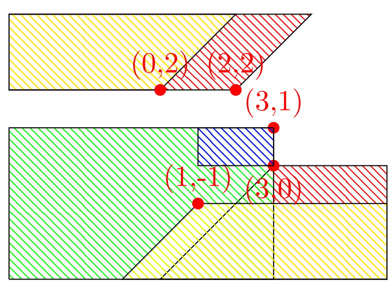
Example 6.1. Suppose that $(\alpha_i,\beta_i) = (2,1)$ which correspond to the homogenized polynomials:

$$F_0 = a_0z_1^2z_2 + a_1x_1z_1z_2 + a_2x_1^2z_2 + a_3x_2z_1 + a_4x_1x_2$$

$$F_1 = b_0z_1^2z_2 + b_1x_1z_1z_2 + b_2x_1^2z_2 + b_3x_2z_1 + b_4x_1x_2$$

$$F_2 = c_0z_1^2z_2 + b_1x_1z_1z_2 + b_2x_1^2z_2 + b_3x_2z_1 + c_4x_1x_2$$

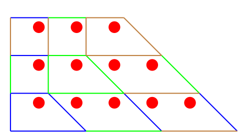
Then the diagram of possible ν looks like:



6.3 Examples of hybrid elimination matrices in $\mathbf{H}_1$

In the Example 0.3, we have all the regions of vanishing of local cohomology that give us elimination matrices, in particular the vectors $(4,2), (3,2), (3,1), (2,1)$ give us such matrices. Let's see how each of this matrices looks like:

- $\nu = (4,2)$. This corresponds to the formula of Canny-Emiris, for a mixed subdivision of the polytopes that may look like:



  and gives the graded map $\oplus_{i=0,1,2}C_{(2,1)} \to C_{(4,2)}$ and the matrix:

$$\begin{pmatrix}
a_0 & a_1 & a_2 & 0 & 0 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0\\
0 & a_0 & a_1 & a_2 & 0 & 0 & a_3 & a_4 & 0 & 0 & 0 & 0\\
0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & a_3 & a_4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & 0 & a_3 & a_4 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & 0 & a_3 & a_4\\
b_0 & b_1 & b_2 & 0 & 0 & b_3 & b_4 & 0 & 0 & 0 & 0 & 0\\
0 & b_0 & b_1 & b_2 & 0 & 0 & b_3 & b_4 & 0 & 0 & 0 & 0\\
0 & 0 & b_0 & b_1 & b_2 & 0 & 0 & b_3 & b_4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 & 0 & b_3 & b_4 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 & 0 & b_3 & b_4\\
c_0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & 0 & 0 & 0 & 0 & 0\\
0 & c_0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & 0 & 0 & 0 & 0\\
0 & 0 & c_0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & c_0 & c_1 & c_2 & 0 & c_3 & c_4 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & c_0 & c_1 & c_2 & 0 & c_3 & c_4
\end{pmatrix}$$

- $\nu = (3,2)$. This is still a Sylvester-type formula as $\mathbb{M}_\nu$ but it does not appear in a full applocation of Canny-Emiris (similarly to exact multihomogeneous formulas as in [18], [3] or the incremental approach in [7]). The map is $\oplus_{i=0,1,2} C_{(1,1)} \to C_{(3,2)}$ and the matrix:

$$\begin{pmatrix}
a_0 & a_1 & a_2 & 0 & 0 & a_3 & a_4 & 0 & 0\\
0 & a_0 & a_1 & a_2 & 0 & 0 & a_3 & a_4 & 0\\
0 & 0 & a_0 & a_1 & a_2 & 0 & 0 & a_3 & a_4\\
b_0 & b_1 & b_2 & 0 & 0 & b_3 & b_4 & 0 & 0\\
0 & b_0 & b_1 & b_2 & 0 & 0 & b_3 & b_4 & 0\\
0 & 0 & b_0 & b_1 & b_2 & 0 & 0 & b_3 & b_4\\
c_0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & 0 & 0\\
0 & c_0 & c_1 & c_2 & 0 & 0 & c_3 & c_4 & 0\\
0 & 0 & c_0 & c_1 & c_2 & 0 & 0 & c_3 & c_4
\end{pmatrix}$$

  which turns out to be an exact Sylvester-type formula.

- $\nu = (3,1)$. This corresponds to $\nu = \delta$ and in this case, we will be introducing a Sylvester form. This form is sylv_0 and can be computed as before, by a determinant that we write as:

$$\mathrm{sylv}_0 = \det\begin{pmatrix}
a_1 z_1 z_2 + a_2 x_1 z_2 + a_4 x_2 & a_3 z_1 & a_0 z_1\\
b_1 z_1 z_2 + b_2 x_1 z_2 + b_4 x_2 & b_3 z_1 & b_0 z_1\\
c_1 z_1 z_2 + c_2 x_1 z_2 + c_4 x_2 & c_3 z_1 & c_0 z_1
\end{pmatrix} = [130] z_1^3 z_2 + [230] x_1 z_1^2 z_2 + [430] x_2 z_1^2$$

  where

$$[ijk] = \begin{pmatrix}
a_i & a_j & a_k\\
b_i & b_j & b_k\\
c_i & c_j & c_k
\end{pmatrix}$$

and a matrix that would look like:

$$\begin{pmatrix} a_0 & a_1 & a_2 & 0 & a_3 & a_4 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & a_3 & a_4 \\ b_0 & b_1 & b_2 & 0 & b_3 & b_4 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & b_3 & b_4 \\ c_0 & c_1 & c_2 & 0 & c_3 & c_4 & 0 \\ 0 & c_0 & c_1 & c_2 & 0 & c_3 & c_4 \\ [130] & [230] & 0 & 0 & [430] & 0 & 0 \end{pmatrix}$$

- Finally, for $\nu = (2,1)$ we introduce two different Sylvester forms and we find a small matrix of the form:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ [013] & [023]+[014] & [024] & 0 & 0 \\ [023] & [024]+[123] & [124] & 0 & 0 \end{pmatrix}$$

6.4 Another example

Let's consider now the following system. $\mathcal{A}_0 = \{(0,0),(1,0),(0,1),(1,1)\}$ and $\mathcal{A}_1 = \mathcal{A}_2 = \{(0,0),(1,0),(0,1)\}$. The underlying toric variety is the blowup $\mathrm{Bl}_{[1:0]\times[1:0]}(\mathbb{P}^1 \times \mathbb{P})$. The Newton polytopes define nef divisors in this variety. The short exact sequence is:

$$0 \to \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^5 \xrightarrow{B} \mathbb{Z}^3 \to 0$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The rays of the fan are $\{\rho_1,\rho_2,\rho_3,\rho_4,\rho_5\}$ The tempting subsets are also described in [1, Lemma 8.3] and they are:

$$\mathcal{R} \in \Big\{\emptyset, \{\rho_1,\rho_3\}, \{\rho_3,\rho_4\}, \{\rho_2,\rho_4\}, \{\rho_2,\rho_5\}, \{\rho_1,\rho_4\}, \{\rho_1,\rho_2,\rho_5\}\Big\}$$

$$\Big\{\{\rho_1,\rho_3,\rho_4\}, \{\rho_1,\rho_3,\rho_5\}, \{\rho_2,\rho_3,\rho_5\}, \{\rho_2,\rho_4,\rho_5\}, \{\rho_1,\rho_2,\rho_3,\rho_4,\rho_5\}\Big\}$$

We take representatives $(0,0,a,b,c) \in \mathbb{Z}^5$ and:

$$(a,b,c) \in \mathcal{M}_\emptyset \iff (a,b,c) \geq (0,0,0)$$

$$(a,b,c) \in \mathcal{M}_{1,2,3,4,5} \iff (a,b,c) \leq (-2,-2,-3)$$

$$(a,b,c) \in \mathcal{M}_{1,3} \iff (u,v,a-u,b-v,c-u-v) \iff a \leq -2, b \geq 0$$

7 Examples: Picard rank 3

Example 7.1. Let $M = \mathbb{Z}^2$ and let $\mathcal{A}_0 = \mathcal{A}_1 = \{(0,0),(1,0),(0,1),(1,1)\}$ and $\mathcal{A}_2 = \{(0,0),(1,0),(0,1)\}$ be the supports of three polynomials F_0, F_1, F_2. These polynomials represent the global sections of three basepoint free, nef, Cartier divisors D_0, D_1, D_2 in the smooth toric variety $X_\Delta = \mathrm{Bl}_{(0)}(\mathbb{P}^1 \times \mathbb{P}^1)$. Their classes in the Picard group are $[D_0] = [D_1] = (1,1,2)$ and $[D_2] = (1,1,1)$. The canonical divisor is given by the class of (2,2,3). When we homogenize in the toric sense (in the Cox ring $C = \mathbb{C}[z, x_0, x_1, y_0, y_1]$), we get that our polynomials (with generic coefficients) are:

$$\overline{F}_0 = a_0 x_0 y_0 z^2 + a_1 x_1 y_0 z + a_2 x_0 y_1 z + a_3 x_1 y_1$$

$$\overline{F}_1 = b_0 x_0 y_0 z^2 + b_1 x_1 y_0 z + b_2 x_0 y_1 z + b_3 x_1 y_1$$

$$\overline{F}_2 = c_0 x_0 y_0 z + c_1 x_1 y_0 + c_2 x_0 y_1$$

Let $I \subset \mathbb{C}[z, x_0, x_1, y_0, y_1][a, b, c]$ be generated by F_0, F_1, F_2. The regularity, in the sense of all the classes $\alpha = [D] \in \mathrm{Pic}(X_\Delta)$ such that:

$$I_\alpha = (I : B^{\mathrm{inf}})_\alpha$$

is given by:

$$\delta + \mathbb{N}^\rho \quad \rho = |\,\mathrm{Pic}(X_\Delta)|$$

$\delta = \sum_{i=0}^{2} [D_i] - [\omega]$ where the canonical class is the degree of $z x_0 x_1 y_0 y_1$ which is $(2,2,3)$. Therefore $\delta = (1,1,1)$.

For $\nu = (2,2,3)$ in the regularity, we have Sylvester-type elimination matrices:

$$C(1,1,1) \oplus C(1,1,1) \oplus C(1,1,2) \to C(2,2,3) \quad \begin{pmatrix} a_0 & a_1 & 0 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & 0 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & 0 & a_2 & a_3 \\ b_0 & b_1 & 0 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & 0 & b_2 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & 0 & b_2 & b_3 \\ c_0 & c_1 & 0 & c_2 & 0 & 0 & 0 & 0 \\ 0 & c_0 & c_1 & 0 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_0 & c_1 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 0 & c_0 & c_1 & 0 & c_2 \end{pmatrix}$$

A minor of this matrix, chosen intelligently (row content), corresponds to the Canny-Emiris formula.

If we choose $\nu = \delta = (1,1,2)$, it is possible to construct an hybrid elimination matrix, through the introduciton of Sylvester forms. In this case, the sylvester form $\mathrm{sylv}_{[(0,0),(0,0),(0,0,0)]}$ corresponds to the toric residue in the sense of CCD. It comes from the decomposition.

$$\overline{F}_0 = x_0 y_0 z (a_0 z) + x_1 (a_1 y_0 z + a_3 y_1) + y_1 (a_2 x_0 z)$$

$$\overline{F}_1 = x_0 y_0 z (b_0 z) + x_1 (b_1 y_0 z + b_3 y_1) + y_1 (b_2 x_0 z)$$

$$\overline{F}_2 = x_0 y_0 z (c_0) + x_1 (c_1 y_0) + y_1 (c_2 x_0)$$

and the determinant:

$$\begin{vmatrix} a_0z & a_1y_0z + a_3y_1 & a_2x_0z \\ b_0z & b_1y_0z + b_3y_1 & b_2x_0z \\ c_0 & c_1y_0 & c_2x_0 \end{vmatrix} = [012]x_0y_0z^2 + [032]x_0y_1z$$

where:

$$[ijk] = \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix} \text{ with } i,j,k = 0,1,2,3 \quad c_3 = 0$$

In such context, there is an elimination matrix:

$$C(0,0,0)^2 \oplus C(0,0,1) \oplus A \to C(1,1,2) \qquad \begin{pmatrix} a_0 & a_1 & a_1 & a_2 \\ b_0 & b_1 & b_1 & b_2 \\ c_0 & c_1 & c_1 & 0 \\ [012] & 0 & [032] & 0 \end{pmatrix}$$

Let N be a lattice and let Σ be a complete fan. We will denote by $\sigma \in \Sigma(n)$ the maximal cones of the fan and by $\rho \in \Sigma(1)$ the rays of the fan.

Definition 7.1. Let

For every ray $\rho \in \Sigma(1)$, there is a torus-invariant divisor D_ρ corresponding to an orbit There is a toric variety X_Σ thhat can be constructed from the information on the fan. The homogeneous coordinate ring of the toric variety, most commonly known as the Cox ring, corresponds to the ring $\mathbb{C}[x_\rho\, \rho \in \Sigma(1)]$ with a grading in $A_{n-1}(X_\Sigma)$ given by the short exact sequence:

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X_\Sigma)$$

Moreover, there is an ideal $B(\Sigma)$ generated by the monomials $\prod_{\rho \notin \sigma} x_\rho$ for $\sigma \in \Sigma(n)$. This ideal is called the irrelevant ideal.

References

[1] Klaus Altmann et al. *Immaculate line bundles on toric varieties*. 2018. arXiv: `1808.09312 [math.AG]`.

[2] Victor V. Batyrev and Lev A. Borisov. *On Calabi-Yau Complete Intersections in Toric Varieties*. 1994. arXiv: `alg-geom/9412017 [alg-geom]`.

[3] M. R. Bender et al. "Koszul-type determinantal formulas for families of mixed multilinear systems". In: (2021). arXiv: `2105.13188 [math.AC]`.

[4] Nicolás Botbol. *The implicit equation of a multigraded hypersurface*. 2011. arXiv: `1007.3437 [math.AC]`.

[5] Laurent Busé, Marc Chardin, and Navid Nemati. *Multigraded Sylvester forms, Duality and Elimination Matrices*. 2021. arXiv: `2104.08941 [math.AC]`.

[6] J. F. Canny and I. Z. Emiris. *An efficient algorithm for the sparse mixed resultant*. 1993.

[7] J. F. Canny and I. Z. Emiris. *Efficient Inceremtal Algorithms for the Sparse Resultant and the Mixed Volume*. Aug. 1995. DOI: `10.1006/jsco.1995.1041`.

[8] J. F. Canny and P. Pedersen. *An Algorithm for the Newton Resultant*. USA, 1993.

[9] Eduardo Cattani, David Cox, and Alicia Dickenstein. *Residues in Toric Varieties*. 1995. arXiv: `alg-geom/9506024 [alg-geom]`.

[10] Eduardo Cattani, Alicia Dickenstein, and Bernd Sturmfels. *Residues and Resultants*. 1997. arXiv: `alg-geom/9702001 [alg-geom]`.

[11] D. Cox, J. B. Little, and H. Schenck. *Toric Varieties*. 2012. DOI: `https://doi.org/10.1365/s13291-012-0048-9`.

[12] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Soc., 2011. ISBN: 9780821884263. URL: `https://books.google.fr/books?id=eXLGwYD4pmAC`.

[13] David Cox. "The homogeneous coordinate ring of a toric variety". In: (Nov. 1992).

[14] C. D'Andrea, G. Jeronimo, and M. Sombra. *The Canny-Emiris conjecture for the sparse resultant*. 2020. arXiv: `2004.14622 [math.AC]`.

[15] Carlos D'Andrea and Ioannis Z. Emiris. *Hybrid Sparse Resultant Matrices for Bivariate Systems*. London, Ontario, Canada, 2001. URL: `https://doi.org/10.1145/384101.384106`.

[16] D. Eisenbud et al. *The Geometry of Schemes*. Graduate Texts in Mathematics. Springer, 2000. ISBN: 9780387986388. URL: `https://books.google.fr/books?id=BpphspzsasEC`.

[17] David Eisenbud, Mircea Mustata, and Mike Stillman. *Cohomology on Toric Varieties and Local Cohomology with Monomial Supports*. 2000. arXiv: `math/0001159 [math.AG]`.

[18] I. Z. Emiris and A. Mantzaflaris. *Multihomogeneous resultant formulae for systems with scaled support*. New York, NY, USA, 2009. DOI: `10.1145/1576702.1576724`. URL: `https://doi.org/10.1145/1576702.1576724`.

[19] M. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. 1994.

[20] Jean-Pierre Jouanolou. "Formes d'inertie et résultant: Un formulaire". In: *Advances in Mathematics* 126 (Mar. 1997), pp. 119–250. DOI: `10.1006/aima.1996.1609`.

[21] Daniel Lazard. "Resolution des Systemes d'Equations Algebriques". In: *Theor. Comput. Sci.* 15 (1981), pp. 77–110.