Study Problems for the Midterm, Math 104 A [1]

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1. Suppose you have a quadrature $Q_h[f]$ to approximate the definite integral

$$I[f] = \int_a^b f(x)dx \tag{1}$$

and you know that for sufficiently small h the error $E_h[f] = I[f] - Q_h[f]$ satisfies

$$E_h[f] = c_4 h^4 + R(h), \tag{2}$$

where c_4 is a constant and $R(h)/h^4 \to 0$ as $h \to 0$.

   (a) What is the rate of convergence of Q_h and what does it mean for the error (if h is halved what happens to the corresponding error)?

   (b) Use (2) to find a computable estimate of the error, $\tilde{E}[f]$.

   (c) Give a way to check that if h is sufficiently small for that estimate of the error to be accurate.

   (d) Use $\tilde{E}[f]$ (or equivalently Richardson's extrapolation) to produce a more accurate quadrature from Q_h.

2. Suppose that we would like to approximate $\int_0^1 f(x)dx$ by

$$Q[f] = \int_0^1 P_2(x)dx, \tag{3}$$

where $P_2(x)$ is the polynomial of degree at most two which interpolates f at 0, 1/2, and 1.

   (a) Write $P_2(x)$ in Lagrange form and prove that

$$Q[f] = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]. \tag{4}$$

   (b) Consider now a general interval $[a, b]$ and the integral $\int_a^b f(x)dx$. Do the change of variables $x = a + (b-a)t$ to transform the integral to one in $[0, 1]$ and use (4) to obtain the more general quadrature (Simpson's Rule):

$$Q^S[f] = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{5}$$

(c) Prove that the quadrature is exact, $Q^S[f] = I[f]$, when f is a polynomial of degree 3 or less.

(d) If we use an interpolating polynomial of much higher degree than 2 and equidistributed nodes, does the accuracy of the quadrature improve for a general continuous integrand? Explain.

3. Let V be a normed linear space and W a subspace of V. Let $f \in V$. Prove that the set of best approximations to f by elements in W is a convex set.

4. Let f and g in $C[0,1]$, α and β constants, and denote by $B_n f$ the Bernstein polynomial of f of degree n. Prove that

   (a) $B_n(\alpha f + \beta g) = \alpha B_n f + \beta B_n g$, i.e. B_n is a linear operator in $C[0,1]$.

   (b) If $f(x) \geq g(x)$ for all $x \in [0,1]$ then $B_n f(x) \geq B_n g(x)$ for all $x \in [0,1]$, i.e. B_n is a monotone operator.

5. Write down the simple Bézier cubic curve $B_3(t)$, $t \in [0,1]$ for the four control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ and prove that

   (a) $B_3(0) = \mathbf{P}_0$ and $B_3(1) = \mathbf{P}_3$.

   (b) The tangent line at $\mathbf{P}_0$ is the line connecting $\mathbf{P}_0$ to $\mathbf{P}_1$ and the tangent line at $\mathbf{P}_3$ is the line connecting $\mathbf{P}_2$ to $\mathbf{P}_3$.

6. (a) Compute an approximate value for $\ln 1.2$ given

$$\begin{aligned} \ln 1.0 &= 0, \\ \ln 1.1 &= 0.09531, \\ \ln 1.3 &= 0.26236. \end{aligned}$$

   (b) Find a bound for the error.

7. Consider the data $(0,1)$, $(1,3)$, $(2,7)$. Let $P_2(x)$ be the polynomial of degree at most two which interpolates these points.
   (a) Find the Lagrange form of $P_2(x)$.
   (b) Find the Newton's form of $P_2(x)$ (build the divided difference table).

8. Let $f(x) = 5^x$, then $f(0) = 1$ and $f(0.5) = \sqrt{5}$.
   (a) Find an approximation to $f(0.3)$ using linear interpolation.
   (b) Find a bound of the error.

9. Prove that if P is a polynomial of degree at most n and $x_0, \ldots, x_n$ are distinct nodes

$$\sum_{j=0}^{n} l_j(x) P(x_j) = P(x), \tag{6}$$

   where the $l_j(x)$ are the elementary Lagrange polynomials associated with the nodes $x_0, \ldots, x_n$.

10. Verify that for any three distinct points x_0, x_1, x_2

$$f[x_0, x_1, x_2] = f[x_2, x_0, x_1] = f[x_1, x_2, x_0]. \tag{7}$$

11. Let P_n be the interpolation polynomial of f at the nodes $x_0, \ldots, x_n$. Suppose we know $P_n(x) = 3x^n + Q(x)$, where Q is a polynomial of degree at most $n-1$. Find $f[x_0, \ldots, x_n]$.

12. Find the Hermite polynomial that interpolates the values $f(0) = 0$, $f'(0) = 0$, $f(1) = 1$, $f'(1) = 3$ and verify your answer.

13. (a) Find the interpolation polynomial $P_2(x)$ of $f(x) = x - 9^{-x}$ at $x_0 = 0$, $x_1 = 1/2$ and $x = 1$
(b) The equation $x - 9^{-x} = 0$ has a solution in $[0, 1]$. Find an approximation of the solution by solving the quadratic equation $P_2(x) = 0$.

14. Find a piecewise linear function that interpolates the points $(0, 1)$, $(1, 1)$, $(2, 0)$, $(3, 10)$.

15. True (T) or False (F).

   (a) (   ) $p(x) = 3x^2 - 2x - 5$ interpolates $(0, -5)$, $(1, -4)$, $(2, 0)$.

   (b) (   ) Let $f \in C^\infty[a, b]$. The higher the interpolation polynomial of f, the better its approximation to f.

   (c) (   ) The barycentric weights for the nodes $x_j = \cos(\pi j/n)$, $j = 0, 1, \ldots n$ are $\lambda_j^{(n)} = (-1)^j \binom{n}{j}$.

   (d) (   ) If $f \in C^\infty[-1, 1]$ and P_n is the interpolation polynomial of $f(x)$ at $x_j = -1 + j(2/n)$, $j = 0, \ldots, n$, then $\|f - P_n\|_\infty \to 0$ as $n \to \infty$.

   (e) (   ) If $f \in C^1[-1, 1]$ and P_n is the interpolation polynomial of f at $x_j = \cos(j\frac{\pi}{n})$, $j = 0, \ldots, n$, then $\|f - P_n\|_\infty \to 0$ as $n \to \infty$.