Curves 2A.13

Geometric Tools for Computer Graphics

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Contents

1 Exercise

Consider the curve parameterized by the function $\gamma : [0, \pi] \to \mathbb{R}^2$ defined as follows:

$$\gamma(t) = (x(t), y(t)) = (5\cos(3t)\cos t, 5\cos(3t)\sin t)$$

1.1 Compute the distance to the origin from an arbitrary point of the curve. Which is the maximum value of these distances? Find the points of the curve achieving this maximum value.

The distance $d(t)$ of an arbitrary point is given by:

$$d(t) = \sqrt{x(t)^2 + y(t)^2}$$

e.g.

$$d(2) = \sqrt{x(2)^2 + y(2)^2} = \sqrt{\left(5\cos(2\times 3)\cos(2)\right)^2 + \left(5\cos(2\times 3)\sin(2)\right)^2} = 4.79$$

Given the parametrization of the curve, the maximum distance is calculated as:

$$d(t)^2 = x(t)^2 + y(t)^2$$

$$d(t)^2 = \left(5\cos(3t)\cos(t)\right)^2 + \left(5\cos(3t)\sin(t)\right)^2$$
$$d(t)^2 = 25\cos^2(3t)\cos^2(t) + 25\cos^2(3t)\sin^2(t)$$
$$d(t)^2 = 25\cos^2(3t)(\cos^2(t) + \sin^2(t))$$

With $\cos^2(t) + \sin^2(t) = 1$ and $25\cos^2(3t) = 1$

$$d(t)^2 = 25\cos^2(3t) = 25 \cdot 1 = 25$$

$$d(t) = \sqrt{25} = 5$$

The maximum points are achieved when $cos(3t) = 1$:

$$3t = 0 \Rightarrow t = 0,$$

$$3t = \pi \Rightarrow t = \pi/3,$$
$$3t = 2\pi \Rightarrow t = 2\pi/3,$$

1.2 Compute the tangent vector to the curve at an arbitrary point. Verify that the tangent vector at the point corresponding to t = 0 is linearly dependent from the tangent vector to the circle entered at the origin and of radius 5 at point (5, 0).

The tangent vector to the curve at an arbitrary point is calculated as the derivative $\gamma'(t)$.

$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 5\cos(3t)\cos t \\ 5\cos(3t)\sin t \end{pmatrix}$$

$$\gamma'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 5(-3\sin(3t)\cos(t) + \cos(3t)(-\sin(t))) \\ 5(-3\sin(3t)\sin(t) + \cos(3t)\cos(t)) \end{pmatrix}$$

Simplifying:

$$\gamma'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -15\sin(3t)\cos(t) - 5\cos(3t)\sin(t) \\ -15\sin(3t)\sin(t) + 5\cos(3t)\cos(t) \end{pmatrix}$$

For $t = 0$ in the curve we get:

$$\gamma'(0) = \begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} -15\sin(0)\cos(0) - 5\cos(0)\sin(0) \\ -15\sin(0)\sin(0) + 5\cos(0)\cos(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

For $t = 0$ in the tangent of a circle with $radius = 5$:

$$C(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} R\cos(t) \\ R\sin(t) \end{pmatrix}$$

$$C'(0) = \begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 5(-\sin(0)) \\ 5(\cos(0)) \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$\gamma'(0)$ and $C'(0)$ are linearly dependent since they are scalars of each other. More specifically, they are the same vector.

1.3 Use change of reference systems to give a parametrization of a copy of the given curve placed in a disc of radius 5 centered at point $(-1, 3)$, so that both curves are tangent at point $(2, -1)$.

$$s = \begin{cases} e_1 = (1, 0, 0) \\ e_2 = (0, 1, 0) \\ e_3 = (0, 0, 1) \end{cases}$$

$$p = (2, -1)$$
$$C_0 = (-1, 3)$$

To generate the new basis we follow this steps:

1. Get the vector that goes from the center of the circle to the curve point p.

$$C_v = p - C_0 = (2, -1) - (-1, 3) = (3, -4)$$

2. Generate a tangent vector using C_v and the perpendicular vector method (-b,a).

$$C_t = (4, 3)$$

3. We now create the new vectors e'_n using C_v and the tangent vector:

$$s' = \begin{cases} e'_1 = \frac{1}{5}(4, 3, 0) \\ e'_2 = \left(\frac{-3}{5}, \frac{4}{5}, 0\right) \\ e'_3 = (0, 0, 1) \end{cases}$$

4. To get the translation for the new base, we transform the position $\gamma(0) = (5, 0)$ to the new rotation and translate it to p:

$$R = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$s'_o = p - \left(R \times \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \right) = (-2, -4)$$

5. Finally, we create the homogeneous matrix:

$$T = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} & 0 & -2 \\ \frac{3}{5} & \frac{4}{5} & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The $\gamma(t)$ curve that fulfills the requirements will be:

$$\gamma(t) = \begin{pmatrix} 5\cos(3t)\cos(t) \\ 5\cos(3t)\sin(t) \\ 0 \\ 1 \end{pmatrix}$$

$$\gamma_{transformed}(t) = T \cdot \gamma(t)$$

Where $\gamma'_{\text{transformed}}(0) = (-3, 4)$ is tangent to $C_t = (4, 3)$

2 Images

To improve readability, I decided to place explanatory images after the operations to help on the visualization on some parts of the problem. The object we are working with looks like this. The parametrization presents a faster frequency on one component (cosine) than the other (sine) which makes the function to generate this smaller loops. The radius in front of the cosine will represent the amplitude of the final curve.

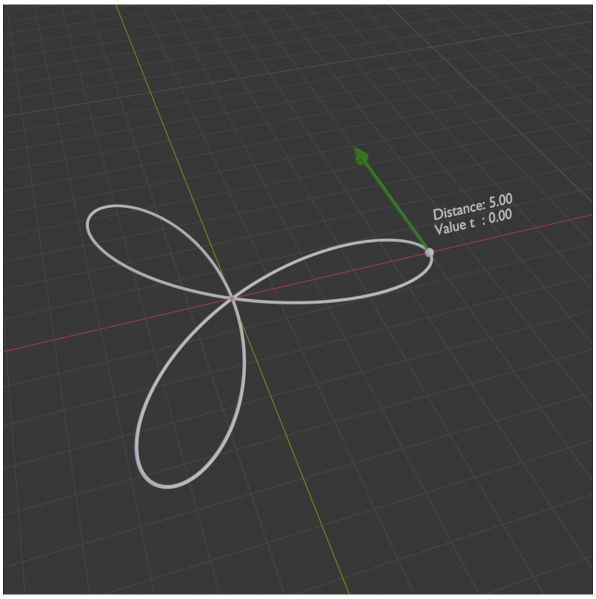

Figure 1: Base object

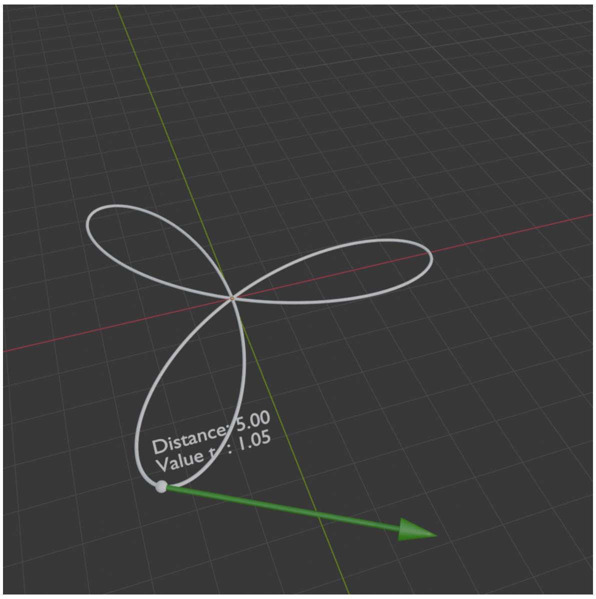

Figure 2: Tangent pi/3

Changing the frequency will, of course increment or decrees the number of loops.

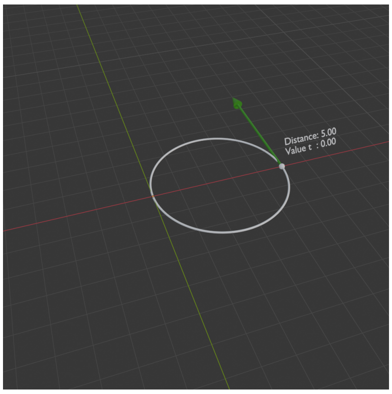

Figure 3: 1 loop

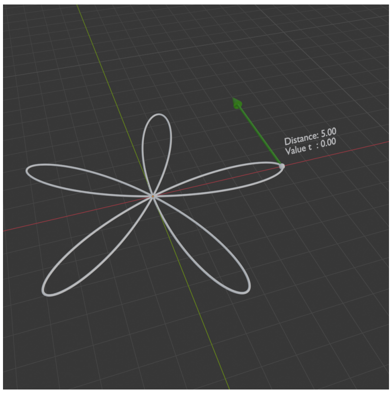

Figure 4: 5 loops

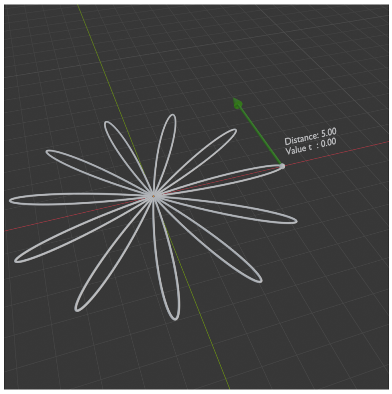

Figure 5: 11 loops

The following image 6 shows visually how the vector from the circle and the curve are the same:

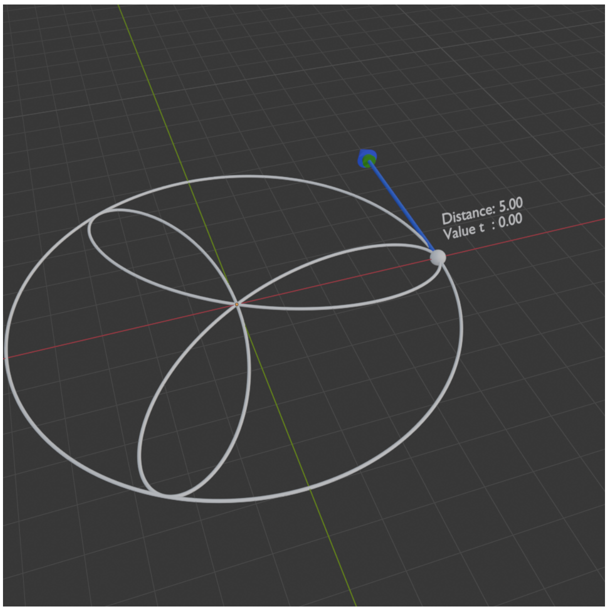

Figure 6: Tangent vectors are equal.

Visualizing the tangent vectors 7 after the transformation allows us to observe how both tangents are 90º.

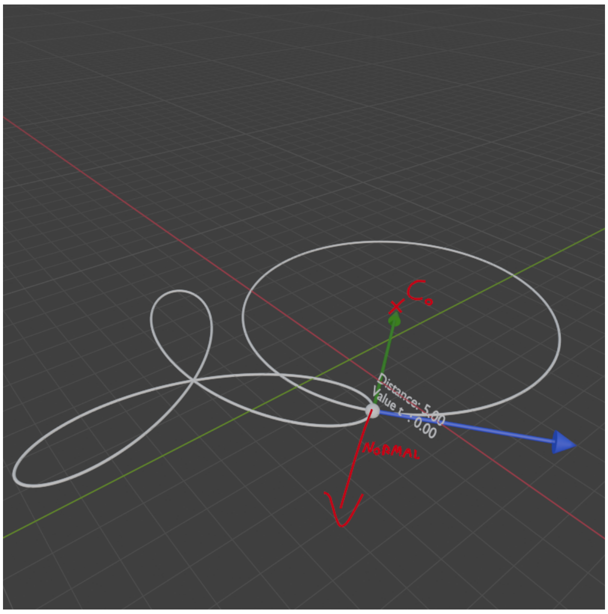

Figure 7: Tangent vectors