Matmek mandatory assignment

1.2.3

Want to show that

$$u(t,x,y) = \exp\big(\iota\big(k_x x + k_y y - \omega t\big)\big)$$

is a solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

We compute and find that for the above expression of u we have

$$\frac{\partial^2 u}{\partial t^2} = (-\iota\omega)^2 u = -\omega^2 u,$$

$$\frac{\partial^2 u}{\partial x^2} = (\iota k_x)^2 u = -k_x^2 u,$$

and

$$\frac{\partial^2 u}{\partial y^2} = \left(\iota k_y\right)^2 u = -k_x^2 u.$$

So overall we have

$$c^2 \nabla^2 u = c^2\left(-k_x^2 - k_y^2\right)u = -c^2 \|k\|^2 u = -\omega^2 u = \frac{\partial^2 u}{\partial t^2},$$

as we wanted to show (ie. u given by the above expression is a solution to the wave equation).

1.2.4

We have the following discretization

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2 \left[\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}\right]$$

Rearranging, we find:

$$\begin{aligned}\left(u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}\right) &= C^2\left[\left(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n\right) + \left(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n\right)\right] \\ &= C^2\left[u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n - 4u_{i,j}^n\right]\end{aligned}$$

We insert for

$$u_{i,j}^n = \exp(\iota(kh(i+j) - \tilde{\omega} n \Delta t))$$

and divide by $u_{i,j}^n$ and get

$$\exp(-\iota\tilde{\omega}\Delta t) - 2 + \exp(\iota\tilde{\omega}\Delta t) = C^2[2\exp(\iota kh) + 2\exp(-\iota kh) - 4].$$

This further rewrites to:

$$2\cos(\tilde{\omega}\Delta t) - 2 = C^2[4\cos(kh) - 4].$$

We now assume $C = \frac{1}{\sqrt{2}}$. dividing both sides by 2, we get:

$$\cos(\tilde{\omega}\Delta t) - 1 = \cos(kh) - 1,$$

so overall we have the condition

$$\cos(\tilde{\omega}\Delta t) = \cos(kh).$$

we have a solution for

$$\tilde{\omega}\Delta t = kh,$$

ie.

$$\tilde{\omega} = \frac{kh}{\Delta t} = k\frac{c}{C}.$$

Furthermore, we have

$$\left\|\vec{k}\right\| = \sqrt{k_x^2 + k_y^2} = \sqrt{2}k.$$

Using this and again inserting for $C = \frac{1}{\sqrt{2}}$, we find that

$$\tilde{\omega} = \left\|\vec{k}\right\|c = \omega.$$

1.2.6

Check the readme in the top folder!