

Social Learning with State-Dependent Observations

Carl Heese *

November 26, 2019

In this note, I study a variant of the canonical binary-state binary-action social learning model (Bikhchandani et al. [1992]). An individual would like to choose an action only in the high state. She observes the choices of previous decision-maker when making her own decision. Importantly, the likelihood of observing the action of previous decision-maker depends on the state. I show that when observing the action is more likely in the low state, the individual faces an inference problem: does she observe many actions because the state is high and previous decision-makers had private information about this or because the state is low and previous actions are more visible. In this situation, learning is confounded (Smith and Sørensen [2000]).

Model

The state of the world is either high $s = H$ or low $s = L$. There is a common prior belief, w.l.o.g $\Pr(H) = \Pr(L) = \frac{1}{2}$. An infinite sequence of individuals $n = 1, 2, \dots, \infty$ arrives in an exogeneous order. Each individual n receives a private signal and computes his private belief $p_n \in (0, 1)$ using Bayes rule. Given the state $s \in \{H, L\}$, the private belief process $\langle p_n \rangle$ is i.i.d. with conditional c.d.f. F_s . We assume that F_s is differentiable for $s \in \{H, L\}$, and that the densities f_s satisfy the (strict) monotone likelihood ratio property, have full support on \mathbb{R} and that

$$\lim_{p \rightarrow 1} \frac{f_H(p)}{f_L(p)} = \infty, \quad \lim_{p \rightarrow 0} \frac{f_H(p)}{f_L(p)} = 0. \quad (1)$$

*Institute for Microeconomics, University of Bonn, Lennéstrasse 43, 53113 Bonn, Germany; heese@uni-bonn.de

We assume that

$$F_H(p) < F_L(p) \quad \text{for any } p \in (0, 1). \quad (2)$$

Every individual n makes a choice $a_n \in \{0, 1\}$. Each individual receives a payoff of 1 if his action matches the state, and otherwise a payoff of zero. Given that an individual n chooses $a_n = 1$, nature decides if the 1-action of n is publicly observable or not. With probability p_s all individuals $m > n$ observe that $a_n = 1$. No individual can observe n 's action if $a_n = 0$. Let $\rho_H < \rho_L$.¹ Let b_n be 1 if $a_n = 1$ and the 1-action of n is observable, and otherwise let b_n be zero.

Decision Problem of an Individual

Before acting, an individual n observes his private belief p_n and the history h of observable 1-actions. Let

$$q_n(h) = \frac{\Pr(h|H)}{\Pr(h|H) + \Pr(h|L)}.$$

Applying Bayes rule implies a posterior belief r_n of n in terms of p and $q(h)$ given by

$$\frac{r_n}{1 - r_n} = \frac{q_n}{1 - q_n} \frac{p_n}{1 - p_n}.$$

(W.l.o.g.) an individual n chooses $a_n = 1$ if $r_n \geq \frac{1}{2}$. Let $l_n = \frac{1 - q_n}{q_n}$.

Steady States of the Public Belief Process

Suppose that the state is H . The likelihood ratio process l_n is a martingale conditional on state H (see Doob [1953]). Also, $\langle l_n \rangle$ converges almost surely to a random variable $l_\infty = \lim_{n \rightarrow \infty} l_n$ with $\text{supp}(l_\infty) \subseteq [0, \infty)$. This follows from

¹Suppose that the decision of agents is if to order at a restaurant or not. Then, the assumption $\rho_H < \rho_L$ loosely captures the idea that the service at a low quality restaurant might be slower than that of a high quality restaurant. Thus, the visibility of previous customers (agents who chose 1) is higher to future agents. More generally, the assumption $\rho_H \neq \rho_L$ could represent any type of state-dependent visibility of previous actions.

the Martingale Convergence Theorem for nonnegative, perhaps unbounded random variables (see Breiman, Theorem 5.14). Note that $(\langle l_n, b_n \rangle)$ is a Markov process on $\mathbb{R} \times \{0, 1\}$ with transitions $l_{n+1} = \phi(b_n, l_n)$ given by

$$\begin{aligned} \phi(1, l_n) &= l_n \frac{\Pr(r_n \geq \frac{1}{2} | H, l_n) \rho_H}{\Pr(r_n \geq \frac{1}{2} | L, l_n) \rho_L} \quad \text{with probability} \quad \Pr(r_n \geq \frac{1}{2} | \omega, l_n) p_H, \\ \phi(0, l_n) &= l_n \frac{1 - (\Pr(r_n \geq \frac{1}{2} | H, l_n) \rho_H)}{1 - (\Pr(r_n \geq \frac{1}{2} | L, l_n) \rho_L)} \quad \text{with probability} \quad 1 - \Pr(r_n \geq \frac{1}{2} | \omega, l_n) p_H. \end{aligned} \quad (3)$$

A fixed point l of (3) satisfies for all $m \in \{0, 1\}$: either $\phi(m, l) = l$ or $\Pr(m | l) = 0$. Clearly, $l = 0$ is a fixed point of (3). Any interior fixed point $l^* > 0$ must satisfy

$$\begin{aligned} \Pr(r_n \geq \frac{1}{2} | H, l^*) \rho_H &= \Pr(r_n \geq \frac{1}{2} | L, l^*) \rho_L \\ \Leftrightarrow \frac{\rho_H}{\rho_L} &= \frac{\Pr(r_n \geq \frac{1}{2} | L, l^*)}{\Pr(r_n \geq \frac{1}{2} | H, l^*)}. \end{aligned} \quad (4)$$

Thus, at an interior fixed point, the inference from the private information of the previous decision-makers offsets exactly the inference from the state-dependence of observations, see (4).

Confounded Learning

Theorem 1 *When it is more likely to observe the action of a previous decision-maker in state L , i.e. $\rho_H < \rho_L$, then the public belief process $\langle q_n \rangle$ has a unique interior steady state $q^* \in (0, 1)$ in state H .*

Proof. Note that $r_n \geq \frac{1}{2} \Leftrightarrow \frac{1}{l_n} \frac{p_n}{1-p_n} \geq 1 \Leftrightarrow p_n \geq \frac{l_n}{1+l_n}$. Hence $\Pr(r_n \geq \frac{1}{2} | s, l) = 1 - F_s(\frac{l_n}{1+l_n})$. So, the function $\frac{\Pr(r_n \geq \frac{1}{2} | H, l)}{\Pr(r_n \geq \frac{1}{2} | L, l)} = \frac{1 - F_L(\frac{l_n}{1+l_n})}{1 - F_H(\frac{l_n}{1+l_n})}$ is continuous and it follows from the monotone likelihood ratio property that the function is strictly decreasing in l . So, any interior fixed point of (3) is unique. It follows from (1) and an application of l' Hospital's rule that

$$\lim_{l_n \rightarrow \infty} \frac{1 - F_L(\frac{l_n}{1+l_n})}{1 - F_H(\frac{l_n}{1+l_n})} = \lim_{p \rightarrow 1} \frac{f_L(p)}{f_H(p)} = 0. \quad (5)$$

Clearly,

$$\frac{1 - F_L(0)}{1 - F_H(0)} = 1. \quad (6)$$

It follows from the intermediate value theorem that the function $\frac{\Pr(r_n \geq \frac{1}{2} | H, l)}{\Pr(r_n \geq \frac{1}{2} | L, l)}$ is surjective on $(0, 1]$. Hence, it follows from the assumption that $\rho_H < \rho_L$ that there exists an interior fixed point l^* of (3). This finishes the proof of the theorem since $q^* = \frac{1}{1+l^*}$ is a steady state of the public belief process in state H . ■

Confounded Learning. Note that at the limit outcome l^* , agents cannot learn anything from the observations. In this sense learning is “confounded”. Confounded learning can also arise when preferences of agents are heterogeneous (see Smith and Sørensen [2000]).

Remark 1 *Theorem 4 in Smith and Sørensen [2000] shows that the fixed point l^* is locally stable if the continuation functions $\phi(b, l_n)$ are strictly increasing in l_n in a neighbourhood of l^* for $b \in \{0, 1\}$ and $\phi_l(b, l^*) \neq 1$ for some b . Locally stable means that there exists an open neighbourhood of l^* such that the process converges to l^* with positive probability once it enters this open neighbourhood.*

Benchmark: No State-Dependence of Observations

When $\rho_H = \rho_L = 1$, Smith and Sørensen [2000] show that asymptotic learning is complete. This can be easily seen from the fixed point equation (4). Note that (4) does not have a solution if $\rho_H = \rho_L$ since it follows from (2) that $\frac{\Pr(r_n \geq \frac{1}{2} | L, l)}{\Pr(r_n \geq \frac{1}{2} | H, l)} = \frac{1 - F_L(\frac{l_n}{1+l})}{1 - F_H(\frac{l}{1+l})} < 1$ for any $l \in (0, \infty)$. Hence, the set of fixed points of (3) is just the singleton $\{0\}$, and necessarily $\text{supp}(l_\infty) = \{0\}$.

References

- Sushil Bikhchandani, David Hirshleifer, and Ivo Welch. A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of political Economy*, 100(5):992–1026, 1992.
- Leo Breiman. Probability addison-wesley, 1968. *Google Scholar*.

Joseph L Doob. *Stochastic processes*, volume 7. Wiley New York, 1953.

Lones Smith and Peter Sørensen. Pathological outcomes of observational learning.
Econometrica, 68(2):371–398, 2000.