Persuasion and Information Aggregation in Elections *

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Abstract

This paper studies a large majority election with voters who have heterogeneous, private preferences and exogenous private signals. We show that a Bayesian persuader can implement any state-contingent outcome in some equilibrium by providing additional information. In this setting, without the persuader's information, a version of the Condorcet Jury Theorem holds (Feddersen and Pesendorfer, 1997). Persuasion does not require detailed knowledge of the voters' private information and preferences: the same additional information is effective across environments. The results require almost no commitment power by the persuader. Finally, the persuasion mechanism is effective also in small committees with as few as 15 members.

In most elections, a voter's ranking of outcomes depends on her information. For example, a shareholder's view of a proposed merger depends on her belief regarding its profitability, and a legislator's support of proposed legislation depends on her belief regarding its effectiveness. An interested party that has private information may utilize this fact by strategically releasing information to affect voters'

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behavior. Examples of interested parties holding and strategically releasing relevant information for voters are numerous: in a shareholder vote, the management may strategically provide information regarding the merger through presentations and conversations; similarly, lobbyists provide selected information to legislators to influence their votes.

We are interested in the scope of such "persuasion" (Kamenica and Gentzkow, 2011) in elections. We study this question in the canonical voting setting by Feddersen and Pesendorfer (1997): There are two possible policies (outcomes)—A and B. Voters' preferences over policies are heterogeneous and depend on an unknown state, α or β, in a general way (some voters may prefer A in state α, some prefer A in state β, and some "partisans" may prefer one of the policies independently of the state). The preferences are drawn independently across voters and are each voters' private information. In addition, all voters privately receive information in the form of a noisy signal. The election determines the outcome by a simple majority rule.

In this setting, Feddersen and Pesendorfer (1997) have shown that within a broad class of "monotone" preferences and conditionally i.i.d. private signals, all equilibrium outcomes of large elections are equivalent to the outcome with a publicly known state ("information aggregation"). We restate their result as a benchmark in Theorem 1.

We ask the following question: can a manipulator ensure that a majority supports his favorite policy—potentially state-dependent—in a large election by providing *additional* information to the voters? Formally, the manipulator can choose and commit to any joint distribution over states and signal realizations that are then privately observed by the voters. In particular, the manipulator's additional signal is required to be independent of the voters' exogenous private signals and their individual preferences (it is an "independent expansion"). The previous result by Feddersen and Pesendorfer (1997) suggests a limited scope for persuasion because, if voters simply ignored the additional information, the outcome would be "as if" the state were known, and, hence, the information provided by the manipulator would be worthless.

Our main result (Theorem 4) shows that, perhaps surprisingly, within the same class of monotone preferences and for any state-contingent policy, there exists an independent expansion of the voters' exogenous i.i.d. signal and an equilibrium that ensures that the targeted policy is supported by a majority with probability close to one when the number of voters is large. Thus, just by providing additional information, a manipulator can implement, for example, a targeted policy that is, in every state, the opposite of the outcome with full information.

The additional information affects the voters' behavior in two ways: directly, by changing their beliefs about the state, and indirectly, by affecting their inference from being "pivotal" for the election outcome. While the direct effect is limited by the well-known "Bayesian-consistency" requirement of beliefs, the pivotal inference turns out to have no such constraint.

To explain the effectiveness of persuasion, we first consider the case in which all information of the voters comes from a manipulator ("monopolistic persuasion"). To invert the full information outcome, the manipulator can choose an information structure in which, roughly speaking, signals are of two possible qualities: *revealing* or *obfuscating*. When the signal is revealing, all voters observe the same signal, a in state α and b in state β. The signal is revealing with probability $1 - \varepsilon$. Thus, when $\varepsilon = 0$, the election leads to the full information outcome.

However, with probability ε, the signal is obfuscating. In this case, in both states, almost all voters receive an uninformative signal z while a few voters receive an "erroneous" signal, that is, they receive a in β and b in α. Hence, in this situation, a and b carry the opposite meaning from before.

What matters for the persuasion logic is that voters react to the closeness of the election. The closeness of the election tells voters something about the quality of the information of the others, and, in this way, also something about the quality of their own signal. In the equilibrium that we construct, a close election will imply that the signal of the others is of low quality (obfuscating), meaning that almost all received signal z, and, in this case, the meaning of an otherwise strong signal a in favor of α will be different and interpreted as being in favor of β, and vice versa for b.

A numerical example with 15 voters illustrates the persuasion logic. The construction uses the exact same fixed-point argument as the general analysis, showing that the same mechanism is already effective in small elections; see Section 4.3. Thus, even though we utilize large numbers in our formal statements, our results may also be relevant for committees with a small or intermediate number of members.

We argue the robustness of the persuasion logic by addressing common concerns regarding the sender's commitment power, equilibrium coordination of the receivers, and the dependence of the mechanism on details of the environment.

We show that the sender needs very little commitment power. To model partial commitment, we follow the existing literature (see e.g. Lipnowski, Ravid, and Shishkin (2019)): The sender is committed only with probability $1-\chi$, and, with probability χ, he is free to send any signals. We show that the sender can persuade a large electorate (Proposition 1) even for arbitrarily small $\chi > 0$.

The manipulated equilibrium has desirable properties that may facilitate the coordination on this equilibrium. First, the equilibrium is "attracting." In particular, its "basin of attraction" for the iterated best response dynamic is essentially the full set of strategy profiles: if we begin with almost any strategy profile and consider, first, the voters' best response to it and then the voters' best response to this best response, then the resulting strategy profile is arbitrarily close to the manipulated equilibrium when the number of voters is large (Proposition 3). Nevertheless, we show that, given the information structure, there is also one other equilibrium that yields the full-information outcome (Theorem 3). Second, the behavior in the manipulated equilibrium is based on a simple line of reasoning. In particular, voters will only need to interpret their own signal conditional on it being "obfuscating," and behave optimally given this interpretation (akin to so-called "sincere voting"). By contrast, any other equilibrium hinges on detailed calculations of pivotal likelihoods.[1]

We show that the same information structure can be used uniformly across many environments, that is, the signal does not need to be tailored to the details of the game (Wilson, 1987, "Wilson doctrine"): for any target policy, there exists a single information structure that implements the policy for any prior about the state and preference distribution of the voters that satisfy a weak condition; see Proposition 2.

In the second part of the paper, we consider the setting in which voters already have access to exogenous information of the form studied in Feddersen and Pesendorfer (1997). We show that, by adding information with the same signal structure as before to the exogenous information, the manipulator can still persuade the voters effectively to elect any state-contingent policy (Theorem 4). In particular, the additional signal structure also does not need to be finely tuned to the details of the environment and is effective independent of the voters' private information. Furthermore, it is sufficient if the sender has partial information about the state in the form of a private signal (Section 7.1).

In Section 6, we provide a stylized application to media markets. This serves two purposes: First, we show that the main results of the paper can also be

[1] In Section 4.6.4, we also briefly discuss a model with some behavioral types who do not condition on being pivotal, following Kawai and Watanabe (2013).

obtained in a setting with normally distributed voter information. Second, within the application, we can discuss concrete strategies of information manipulation.

In Section 8, we discuss the paper's contribution to the existing literature on information aggregation in elections and on voter persuasion, especially the work by Wang (2013), Alonso and Câmara (2016), Chan, Gupta, Li, and Wang (2019) and Bardhi and Guo (2018). This literature observed in particular that, with multiple receivers, the conditioning on being pivotal weakens the Bayesian consistency constraint. In contrast to this prior work, which assumes that the voters' preferences and information are commonly known, we allow for heterogeneous, privately known preferences and exogenous information. On the one hand, this allows capturing the canonical environment by Feddersen and Pesendorfer (1997) in which, otherwise, equilibrium implies the full-information outcome. On the other hand, the persuasion mechanism here is distinct from the persuasion logic when voters' preferences are commonly known and voters can be targeted individually, as illustrated in an example in Section 7.2. Moreover, we show that voter persuasion is robust in several dimensions (limited commitment, equilibrium coordination, and detail-freeness).

We note two broader implications of our analysis. First, it may be difficult for an outside observer to make a "robust" prediction. If an observer knows that voters have access to at least the information assumed in Feddersen and Pesendorfer (1997), but cannot exclude that voters have access to additional information of the type discussed here, then no outcome can be excluded as an equilibrium prediction. Second, if one interprets an information structure with a small ε as a small departure from common knowledge, our result adds another observation to the literature on the effects of strategic uncertainty (Weinstein and Yildiz, 2007).

The proof for the main result with a monopolistic sender, Theorem 2, is in the main body and the appendix. The proofs for the other results are sketched here; details are relegated to an online appendix.

1 Model

There are $2n+1$ voters (or citizens), two policies, A and B, and two states of the world, $\omega \in \{\alpha, \beta\}$. The prior probability of α is $\Pr(\alpha) \in (0,1)$.

Voters have heterogeneous preferences. A voter's preference is described by a type $t = (t_\alpha, t_\beta) \in [-1,1]^2$, with t_ω being the utility of A in ω. The utility of B is normalized to zero; so, t_ω is the difference of the utilities from A and B in ω.

The types are independently and identically distributed across voters according to a cumulative distribution function $G : [-1,1]^2 \to [0,1]$, with a strictly positive, continuous density g. The own type is the private information of the voter.

An *information structure* π is a finite set of signals S and a joint distribution of signal profiles and states that is independent of G. The conditional distribution is exchangeable with respect to the voters. In particular, there is a finite number of substates $\{\alpha_j\}_{j=1,\ldots,N_\alpha}$ and $\{\beta_j\}_{j=1,\ldots,N_\beta}$, such that the signals are independently and identically distributed conditional on the substates.[2] Abusing notation slightly, $\Pr(\omega_j|\omega)$ and $\Pr(s_i|\omega_j)$ denote the corresponding probabilities of the substates and the individual signal s_i, conditional on a substate. Thus, the probability of the signal profile $\mathbf{s} = (s_i)_{i=1,\ldots,2n+1} \in S^{2n+1}$ is

$$\Pr(\mathbf{s}|\omega) = \sum_j \Pr(\omega_j|\omega) \prod_{i=1,\ldots,2n+1} \Pr(s_i|\omega_j). \tag{1}$$

The observed signal is the private information of the voter.

We can show our main results already with a simple class of information structures with just two substates—$\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$—and three conditionally independent signals in each substate—$s \in \{a, b, z\}$; this information structure is illustrated in Figure 1.

The voting game is as follows: First, nature draws the state, the profile of preferences types $\mathbf{t}$, and the profile of signals $\mathbf{s}$ according to G and π. Second, after observing her type and signal, each voter simultaneously submits a vote for A or B. Finally, the submitted votes are counted and the majority outcome is selected. This defines a Bayesian game.

A strategy of a voter is a function $\sigma : S \times [-1,1]^2 \to [0,1]$, where $\sigma(s,t)$ is the probability that a voter of type t with signal s votes for A.

We consider only weakly undominated strategies. In particular, we require that

$$\begin{aligned} \sigma(s,t) &= 0 \quad \text{for all} \quad t = (t_\alpha, t_\beta) < (0,0), \\ \sigma(s,t) &= 1 \quad \text{for all} \quad t = (t_\alpha, t_\beta) > (0,0), \end{aligned} \tag{2}$$

where $t > (0,0)$ and $t < (0,0)$ are *partisans* who prefer A and B, respectively, independently of the state. Given our full support assumption on G, this rules

[2]The Hewitt-Savage-de Finetti theorem states that, for any exchangeable *infinite* sequence of random variables $(X_i)_{i=1}^{\infty}$ with values in some set X, there exists a random variable Y, such that the random variables X_i are independently and identically distributed conditional on Y.

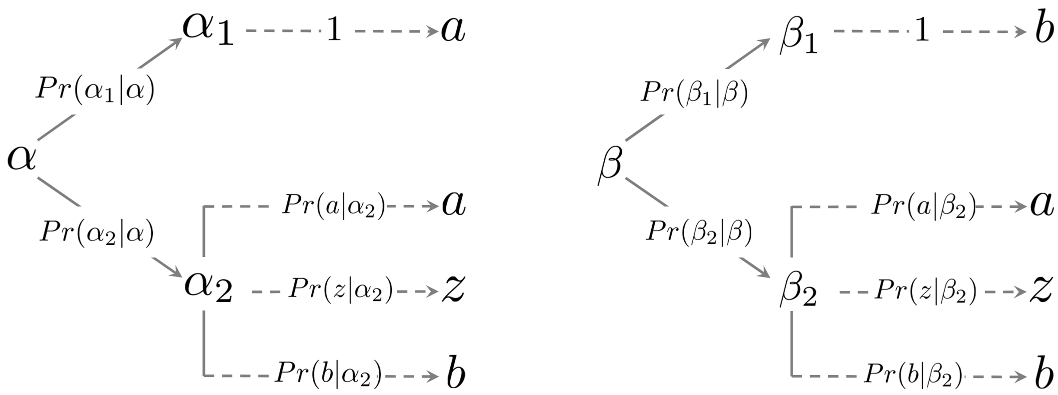

Figure 1: The main class of information structures considered in this paper. Each state ω has two substates $\{\omega_1, \omega_2\}$, occurring with conditional probabilities $\Pr(\omega_j|\omega)$. Conditional on the substate ω_j, the distribution of the signals $s_i \in \{a, z, b\}$ is independent and identical with the marginal probabilities denoted by $\Pr(s|\omega_j)$ (these marginals are degenerate in α_1 and β_1) .

out degenerate strategies for which either $\sigma(s,t) = 1$ for all (s,t) or $\sigma(s,t) = 0$ for all (s,t). Here, and in the following, we ignore zero measure sets when writing "*for all*".

From the viewpoint of a given voter and given any strategy σ' used by the other voters, the pivotal event piv is the event in which the realized types and signals of the other $2n$ voters are such that exactly n of them vote for A and n for B. In this event, if she votes A, the outcome is A; if she votes B, the outcome is B. In any other event, the outcome is independent of her vote. Thus, a strategy is optimal if and only if it is optimal conditional on the pivotal event.

Let $\Pr(\alpha|s, \text{piv}; \sigma')$ denote the posterior probability of α conditional on s and conditional on *being pivotal*, given the measure induced by the nondegenerate strategy σ'. The strategy σ is a best response to σ' if and only if

$$\Pr(\alpha|s, \text{piv}; \sigma') \cdot t_\alpha + (1 - \Pr(\alpha|s, \text{piv}; \sigma')) \cdot t_\beta > 0 \Rightarrow \sigma(s,t) = 1, \tag{3}$$

and

$$\Pr(\alpha|s, \text{piv}; \sigma') \cdot t_\alpha + (1 - \Pr(\alpha|s, \text{piv}; \sigma')) \cdot t_\beta < 0 \Rightarrow \sigma(s,t) = 0, \tag{4}$$

that is, a voter supports A if the expected value of A conditional on being pivotal is strictly positive, and a voter supports B otherwise. Note that indifference holds only for a set of types that has zero measure. For all other types, the best response is pure. It follows that there is no loss of generality to consider pure strategies with $\sigma(s,t) \in \{0,1\}$ for all (s,t).

Thus, a symmetric, undominated, and pure Bayes-Nash equilibrium of $\Gamma(\pi)$ is a strategy $\sigma : S \times [-1,1]^2 \to \{0,1\}$ that satisfies (2), (3), and (4), with $\sigma' = \sigma$. We refer to such a strategy simply as an *equilibrium.*

2 Preliminary Observations

2.1 Inference from the Pivotal Event

When making an inference from being pivotal, voters ask which state is more likely conditional on a tie, with exactly n voters supporting A and n supporting B. It is intuitive that a tie is evidence in favor of the substate in which the election is closer to being tied in expectation. Thus, conditional on being pivotal, a voter updates toward the substate in which the expected vote share is closer to $\frac{1}{2}$. We now verify this simple intuition and introduce some notation along the way.

For a strategy σ, the probability that a voter supports A in substate ω_j is

$$q(\omega_j;\sigma) = \sum_{s\in S} \Pr(s|\omega_j) \Pr_G(\{t : \sigma(s,t) = 1\}), \tag{5}$$

where $q(\omega_j;\sigma)$ is the *expected vote share* of A.

Given that the signals and the types of the voters are independent conditional on the substate, the probability of a tie in the vote count is

$$\Pr(\text{piv}|\omega_j;\sigma) = \binom{2n}{n} (q(\omega_j;\sigma))^n (1 - q(\omega_j;\sigma))^n. \tag{6}$$

For any two substates ω_j and $\hat{\omega}_l$, the likelihood ratio of being pivotal is

$$\frac{\Pr(\text{piv}|\omega_j;\sigma)}{\Pr(\text{piv}|\hat{\omega}_l;\sigma)} = \left(\frac{q(\omega_j;\sigma)(1-q(\omega_j;\sigma))}{q(\hat{\omega}_l;\sigma)(1-q(\hat{\omega}_l;\sigma))}\right)^n. \tag{7}$$

Using the conditional independence, the posterior likelihood ratio of any two substates conditional on a signal s and the event that the voter is pivotal is

$$\frac{\Pr(\omega_j|\text{piv},s;\sigma)}{\Pr(\hat{\omega}_l|\text{piv},s;\sigma)} = \frac{\Pr(\omega_j)}{\Pr(\hat{\omega}_l)} \frac{\Pr(s|\omega_j)}{\Pr(s|\hat{\omega}_l)} \frac{\Pr(\text{piv}|\omega_j;\sigma)}{\Pr(\text{piv}|\hat{\omega}_l;\sigma)}. \tag{8}$$

We record the intuitive fact that voters update toward the substate in which the vote share is closer to $1/2$, that is, the substate in which the election is closer to being tied in expectation.

Claim 1 *Take any two substates ω_j and $\hat{\omega}_l$, and any strategy σ for which* $\Pr(\text{piv}|\hat{\omega}_l;\sigma) \in (0,1)$*; if*

$$\left| q(\omega_j;\sigma) - \frac{1}{2} \right| < \left| q(\hat{\omega}_l;\sigma) - \frac{1}{2} \right|, \tag{9}$$

then

$$\frac{\Pr(\text{piv}|\omega_j;\sigma)}{\Pr(\text{piv}|\hat{\omega}_l;\sigma)} > 1. \tag{10}$$

Proof. The function $q(1-q)$ has an inverse u-shape on $[0,1]$ and is symmetric around its peak at $q = \frac{1}{2}$. So, $\left|q - \frac{1}{2}\right| < \left|q' - \frac{1}{2}\right|$ implies that $q(1-q) > q'(1-q')$. Thus, it follows from (7) that (9) implies (10). ■

2.2 Pivotal Voting

Given any strategy profile σ' used by the others, the vector of posteriors conditional on piv and s is denoted as

$$\boldsymbol{\rho}(\sigma') = (\Pr(\alpha|s,\text{piv};\sigma'))_{s\in S}. \tag{11}$$

This vector of posteriors is a sufficient statistic for the unique best response to σ' for all nonpartisan voter types; see (3) and (4).

Thus, given some arbitrary vector of beliefs $\mathbf{p} = (p_s)_{s\in S}$, let $\sigma^{\mathbf{p}}$ be the unique undominated strategy that is optimal if a voter with a signal s believes the probability of α to be p_s. That is, for all (s,t),

$$\sigma^{\mathbf{p}}(s,t) = 1 \Leftrightarrow p_s \cdot t_\alpha + (1-p_s)\cdot t_\beta > 0, \tag{12}$$

and (2) holds for the partisans.

The strategy σ is a best response to σ' if and only if $\sigma = \sigma^{\mathbf{p}}$ for $\mathbf{p} = \boldsymbol{\rho}(\sigma')$. Thus, σ^* is an equilibrium if and only if $\sigma^* = \sigma^{\boldsymbol{\rho}(\sigma^*)}$. Conversely, an equilibrium can be described by a vector of beliefs $\mathbf{p}^*$ that is a fixed point of $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$, that is

$$\mathbf{p}^* = \boldsymbol{\rho}(\sigma^{\mathbf{p}^*}); \tag{13}$$

meaning, the belief $\mathbf{p}^*$ corresponds to an equilibrium if, when voters behave optimally given $\mathbf{p}^*$ (i.e., vote according to $\sigma^{\mathbf{p}^*}$), the posterior conditional on being pivotal is again $\mathbf{p}^*$.

Equation (13) provides an equilibrium existence argument: the expression $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ defines a finite-dimensional mapping $[0,1]^{|S|} \to [0,1]^{|S|}$ from beliefs $\mathbf{p}$ into posterior beliefs $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$, and this mapping is continuous.[3] Thus, an application of Kakutani's theorem implies the existence of a fixed point $\mathbf{p}^*$ that solves (13).[4] The strategy $\sigma^{\mathbf{p}^*}$ is an equilibrium.[5]

The possibility of writing equilibria in terms of posteriors enables us to connect our model and results to the Bayesian persuasion literature.

2.3 Aggregate Preferences

A central object of the analysis is the *aggregate preference function*,

$$\Phi(p) := \Pr_G(\{t : p \cdot t_\alpha + (1-p) \cdot t_\beta > 0\}), \tag{14}$$

which maps a belief $p \in [0,1]$ to the probability that a random type t prefers A under p. The function Φ proves useful to express expected vote shares: if a strategy σ is optimal given beliefs $\mathbf{p}$—i.e., $\sigma = \sigma^{\mathbf{p}}$— then the expected vote share of outcome A in substate ω_j is

$$q(\omega_j; \sigma) = \sum_{s \in S} \Pr(s|\omega_j)\Phi(p_s). \tag{15}$$

Figure 2 illustrates Φ. Given p, the dashed (blue) line corresponds to the plane of indifferent types $t = (t_\alpha, t_\beta)$ with $p \cdot t_\alpha + (1-p) \cdot t_\beta = 0$. Voters having types to the north-east prefer A given p, and Φ is the measure of such types under G. The indifference plane has a slope $-\frac{p}{1-p}$, and a change in p corresponds to a rotation of it. Given that G has a continuous density, it follows that the function Φ is continuous in p. Given that G has a strictly positive density on $[-1,1]^2$, we also have that

$$0 < \Phi(p) < 1 \quad \text{for all} \quad p \in [0,1]. \tag{16}$$

As observed earlier, voters having types t in the north-east quadrant prefer A

[3]To see why $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is continuous in $\mathbf{p}$, first, note that (12) implies that $\Pr_G(\{t : \sigma^{\mathbf{p}}(s,t) = 1\})$ is continuous in $\mathbf{p}$ since G has a continuous density. Second, $q(\omega_j; \sigma^{\mathbf{p}})$ is continuous in $\Pr_G(\{t : \sigma^{\mathbf{p}}(s,t) = 1\})$, given (5). Third, $\rho(\sigma^{\mathbf{p}})$ is continuous in $q(\omega_j; \sigma^{\mathbf{p}})$, given (6) and (8).

[4]The ability to write an equilibrium as a finite-dimensional fixed point via (13) is a significant advantage. Similar reductions to finite dimensional equilibrium beliefs have been used in related voting settings previously (see Bhattacharya, 2013; Ahn and Oliveros, 2012).

[5]Note that, because of the partisans, $\sigma^{\mathbf{p}^*}$ is non-degenerate.

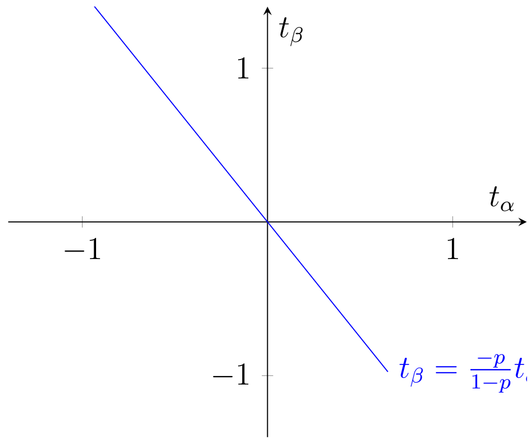

Figure 2: The plane of indifferent types is $t_\beta = \frac{-p}{1-p}t_\alpha$ for any given belief $p = \Pr(\alpha) \in (0, 1)$.

for all beliefs and voters having types t in the south-west quadrant always prefer B (*partisans*). Voters having types t in the south-east quadrant prefer A in state α and B in β (*aligned voters*), and voters having types t in the north-west quadrant prefer B in state α and A in β (*contrarian voters*).

We assume throughout the paper that the distribution of types is sufficiently rich so that there is a belief p for which a majority prefers A and a belief p' for which a majority prefers B,[6] that is,

$$\Phi\left(p'\right) < \frac{1}{2} < \Phi\left(p\right). \tag{17}$$

3 Large Elections: Basic Results

We consider a sequence of elections along which the electorate's size n grows. For each $2n+1$, we fix some strategy profile σ_n and calculate the probability that a policy $x \in \{A, B\}$ wins the support of the majority of the voters in state ω, denoted $\Pr\left(x|\omega;\sigma_n,n\right)$. We are interested in the limit of $\Pr\left(x|\omega;\sigma_n^*,n\right)$, as $n \to \infty$, for equilibrium sequences $(\sigma_n^*)_{n\in\mathbb{N}}$. We first state a central observation regarding the inference from being pivotal in large elections; we then show how this observation implies the "modern" Condorcet Jury Theorem (CJT), which we restate as a benchmark.

[6]Otherwise, the analysis is trivial. If, for all beliefs $p \in [0, 1]$, in expectation a majority prefers A, then, for any information structure, the vote share of A is larger than $\frac{1}{2}$, and A wins in every large election.

3.1 Inference in Large Elections

As a first step, we study the properties of the inference from being pivotal in a large election. We show that Claim 1 extends in an extreme form as the electorate grows large ($n \to \infty$): The event that the election is tied is infinitely more likely in the (sub-)state in which the election is closer to being tied in expectation. In fact, the likelihood ratio of the pivotal event diverges exponentially fast.

Because we want to allow the information structure to depend on n, we also include π_n in the argument. The set of substates remains fixed.

Claim 2 *Consider any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$, any sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$, and any two substates ω_j and $\hat{\omega}_l$ for which $\Pr(\text{piv}|\hat{\omega}_l;\sigma,n,\pi_n) \in (0,1)$ for all n. If*

$$\lim_{n\to\infty}\left|q(\omega_j;\sigma_n,\pi_n)-\frac{1}{2}\right| < \lim_{n\to\infty}\left|q(\hat{\omega}_l;\sigma_n,\pi_n)-\frac{1}{2}\right|, \tag{18}$$

then, for any $d \geq 0$,

$$\lim_{n\to\infty}\frac{\Pr(\text{piv}|\omega_j;\sigma_n,\pi_n)}{\Pr(\text{piv}|\hat{\omega}_l;\sigma_n,\pi_n)}n^{-d}=\infty. \tag{19}$$

Proof. Let

$$k_n = \frac{q(\omega_j;\sigma_n,\pi_n)}{q(\hat{\omega}_j;\sigma_n,\pi_n)}\frac{(1-q(\omega_j;\sigma_n,\pi_n))}{(1-q(\hat{\omega}_j;\sigma_n,\pi_n))}.$$

From (7), the left-hand side of (19) is $\frac{(k_n)^n}{n^d}$. If (18) holds, then $\lim_{n\to\infty} k_n > 1$, because of the properties of $q(1-q)$ (inverse u-shaped around $1/2$). Therefore, $\lim_{n\to\infty}(k_n)^n = \infty$. Moreover, $(k_n)^n$ diverges exponentially fast and, hence, dominates the denominator n^d, which is polynomial. ■

3.2 Benchmark: Condorcet Jury Theorem

The model embeds a special case of the canonical voting game by Feddersen and Pesendorfer (1997) with a binary state. In the following, we restate their full-information equivalence result, assuming, at first, that signals are binary with $S = \{u, d\}$.

As in Feddersen and Pesendorfer (1997), we assume that the signals are independently and identically distributed across voters conditional on the state $\omega \in \{\alpha, \beta\}$.[7] This corresponds to the case of an information structure π^c with

[7] Feddersen and Pesendorfer (1997) assume the existence of subpopulations and allow the

a single substate in each state; in the following, we identify the substate with this state. The probabilities $\Pr(s|\omega;\pi^c)$ for $s \in \{u, d\}$ and $\omega \in \{\alpha, \beta\}$ satisfy

$$1 > \Pr(u|\alpha;\pi^c) > \Pr(u|\beta;\pi^c) > 0\,; \tag{20}$$

that is, signal u is indicative of α, and signal d is indicative of β. We further assume that

$$\Phi(p) \text{ is strictly increasing in } p. \tag{21}$$

We say that the aggregate preference function is *monotone*.[8] Monotonicity (21) and (17) together imply that $\Phi(0) < \frac{1}{2} < \Phi(1)$; thus, the *full information outcome* is A in α and B in β.

Theorem 1 *Feddersen and Pesendorfer (1997), Bhattacharya (2013).*
Suppose that Φ is strictly increasing. Then, for every sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} \Pr\left(A|\alpha;\sigma_n^*,\pi^c,n\right) = 1,$$
$$\lim_{n\to\infty} \Pr\left(B|\beta;\sigma_n^*,\pi^c,n\right) = 1.$$

The proof of Theorem 1 is standard. We state it in the Online Appendix for completeness and reference. The main observation is that the election must be equally close to being tied in both states,

$$\lim_{n\to\infty} q(\alpha;\sigma_n^*) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - q(\beta;\sigma_n^*). \tag{22}$$

This follows in three steps. First, voters with a signal u believe state α to be more likely than voters with a signal d do. Since the probability of signal u is higher in α, this, (15), and the monotonicity of Φ imply a larger vote share of A in α,

$$\forall n \in \mathbb{N} : q\left(\alpha;\sigma_n^*\right) > q\left(\beta;\sigma_n^*\right). \tag{23}$$

Second, in equilibrium, voters do not become certain of one of the states conditional on being tied. To see why, suppose that voters become certain the state is α. That is, $\Pr(\alpha|\text{piv};\sigma_n^*) \overset{n\to\infty}{\to} 1$. Then, in both states, the vote shares would be

signal distributions to vary across these; this is not critical. Moreover, they assume a continuum of states ω. Bhattacharya (2013) nests a binary-state version of their model. The binary state version here is a special case of the model in Bhattacharya (2013).

[8] Bhattacharya (2013) says the distribution of preferences satisfies "Strong Preference Monotonicity" if (21) holds. He shows that monotonicity is necessary for the Condorcet Jury Theorem. If monotonicity fails, there are parameters and equilibria that do not imply the full information outcome.

close to $\Phi(1)$ for n sufficiently large; thus, given (23), for all n sufficiently large,

$$\Phi(1) > q\left(\alpha;\sigma_n^*\right) > q\left(\beta;\sigma_n^*\right) > \frac{1}{2}. \tag{24}$$

Equation (24) means that the election is closer to being tied in β. In this case, Claim 1 implies that voters update toward β conditional on being pivotal—a contradiction to the voters becoming certain of state α.

Third, since voters must not become certain of the state conditional on being pivotal, it must be that the margins of victory are equal and (22) holds. Otherwise, Claim 2 would imply that voters become certain of the state in which the election is closer to being tied.

Finally, (22) and (23) imply $\lim_{n\to\infty} q(\alpha;\sigma_n^*) > \frac{1}{2} > \lim_{n\to\infty} q(\beta;\sigma_n^*)$; thus, in a large election, A wins in α and B wins in β, as claimed. The proof provides the detailed argument following this outline.

Theorem 1 holds more generally for *any* sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$ for which the signals are independent and identically distributed conditional on the state $\omega \in \{\alpha,\beta\}$ (i.e., there is a single substate) and for which signals do not become uninformative—that is,

$$\exists s \in S : \lim_{n\to\infty} \Pr(s|\pi_n) > 0 \text{ and } \lim_{n\to\infty} \frac{\Pr(s|\alpha;\pi_n)}{\Pr(s|\beta;\pi_n)} \neq 1. \tag{25}$$

Theorem 1' *Suppose Φ is strictly increasing. Then, for every sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$ with a single substate and satisfying (25) and for every sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}}$,*

$$\begin{aligned} \lim_{n\to\infty} \Pr\left(A|\alpha;\sigma_n^*,\pi_n,n\right) &= 1, \\ \lim_{n\to\infty} \Pr\left(B|\beta;\sigma_n^*,\pi_n,n\right) &= 1. \end{aligned}$$

4 Monopolistic Persuasion

We now consider the case of a sender who aims to affect the election outcome by providing information to voters, and voters have no other source of information on their own. Thus, the sender is the monopolist for information. This is the case studied in much of the literature on persuasion.

When the sender provides no information, the election outcome is trivially the outcome preferred by the majority at the prior, as determined by $\Phi\left(\Pr\left(\alpha\right)\right)$. The sender can also implement the full information outcome with public signals by revealing the state. What else can the sender implement?

For example, could the sender implement a constant policy that is the opposite of what the voters prefer at the prior? Or could the sender even implement the inverse of the full information outcome? Clearly, to implement these policies, the sender must provide some information to the voters. And, in fact, to implement the inverse of the full information outcome, the sender must provide sufficient information for the voters to be able to collectively distinguish the two states. On the other hand, the CJT suggests that providing information to voters may easily lead to the full information outcome, thereby suggesting that the possibility of persuasion is limited.

4.1 Result: Full Persuasion

Formally, we study what policies can be implemented in an equilibrium of a large election for some choice of π. This determines the set of feasible policies for a strategic sender.

The choice of the information structure π affects voters by affecting the posteriors $(\Pr(\alpha|s, \text{piv}; \sigma, \pi))_{s\in S}$. There are two effects of π. First, there is a *direct effect*; π pins down how voters learn from their signal. This effect is known from the work on persuasion. Second, there is an *indirect effect* of π because it affects the inference of the voters from being pivotal.

We show that there is no constraint on the set of feasible policies. For any state-dependent policy and for large n, there is an information structure π_n and an equilibrium σ_n for which the targeted policy wins with a probability close to one in the respective state.[9]

Theorem 2 *Take any Φ and any prior $\Pr(\alpha) \in (0,1)$: for every state-dependent policy $(x(\alpha), x(\beta)) \in \{A, B\}^2$, there exists a sequence of signal structures $(\pi_n)_{n\in\mathbb{N}}$ and equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}}$, such that*

$$\lim_{n\to\infty} \Pr(x(\alpha)|\alpha; \sigma_n^*, \pi_n, n) = 1,$$
$$\lim_{n\to\infty} \Pr(x(\beta)|\beta; \sigma_n^*, \pi_n, n) = 1.$$

In the following, we first provide a proof for a special case of the theorem in Section 4.2 , and we then illustrate it with a numerical example in Section 4.3. In

[9]The sender can also implement any stochastic policy by "mixing" over information structures in the appropriate manner.

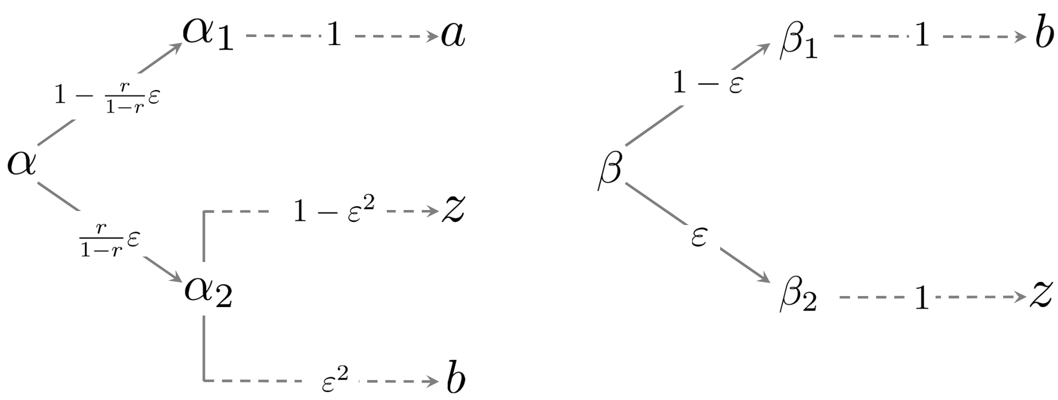

Figure 3: The information structure π_n^r with $\varepsilon = \frac{1}{n}$ and $r \in (0, 1)$.

Section 4.4, we discuss a general insight for persuasion in elections that underlies the result. Finally, we provide the proof for the general case in Section 4.5.

4.2 Proof: Constant Policy

This section proves Theorem 2 for the case in which Φ is monotonically increasing and the targeted policy is A in both states (i.e., Φ satisfies (21) and $(x(\alpha), x(\beta)) = (A, A)$). We further assume a uniform prior in order to simplify the algebra, setting $\Pr(\alpha) = \frac{1}{2}$.

4.2.1 The Information Structure

We specialize the general information structure introduced in the model section to the one defined in Figure 3. Setting $\varepsilon = \frac{1}{n}$, the information structure has a single free parameter, $r \in (0, 1)$, and we denote it by π_n^r.

As ε vanishes for large n, the signals are almost public in the following sense: conditional on observing any signal s, a voter believes that every other voter has received the same signal with a probability close (or equal) to one.

Furthermore, the signals a and b reveal the state (almost) perfectly. The signal z contains only limited information since $r \in (0, 1)$. When observing the signal z, a voter knows that the substate must be either α_2 or β_2. Moreover, given that a voter receives z with a probability close to one in either substate, we have (recall the uniform prior),

$$\lim_{n\to\infty} \Pr(\alpha|z;\pi_n^r) = \lim_{n\to\infty} \Pr(\alpha|\{\alpha_2,\beta_2\},\pi_n^r) = r. \tag{26}$$

4.2.2 Voter Inference

Clearly, for signal a,

$$\Pr(\alpha|a,\text{piv};\sigma_n,\pi_n^r) = 1. \tag{27}$$

Hence, in state α_1, when all voters receive a, the probability that a random citizen votes A is $\Phi(1) > \frac{1}{2}$. It follows from the weak law of large numbers that, in any equilibrium, A is elected with probability converging to 1 in state α_1.

In state β_1, all voters receive b. Conditional on the signal b alone, state β is more likely. The remaining part of this section shows that the indirect effect from the inference of being pivotal can dominate, such that there is an equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$ for which

$$\lim_{n\to\infty} \Pr(\alpha|b,\text{piv};\sigma_n^*,\pi_n^r) = 1. \tag{28}$$

The proof relies on two claims. First, consider the signal z and the inference about the relative likelihood of α_2 and β_2. We show that, for *any* strategy used by the other voters, the pivotal event contains no information regarding the relative probability of α_2 and β_2 as the electorate grows large.

Claim 3 *Given any $r \in (0,1)$ and any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$,*

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2;\sigma_n,\pi_n^r)}{\Pr(\text{piv}|\beta_2;\sigma_n,\pi_n^r)} = 1. \tag{29}$$

The proof is in the Appendix in Section A. The pivotal event contains no information since the distribution of signals is almost identical in the two substates α_2 and β_2 (and the distribution of preference types is identical by construction). Therefore, for any strategy σ, the distribution of votes must be almost identical in the two substates; in particular, the probability of a tie is also almost the same in the two substates.[10]

Claim 3 and (26) imply, in particular, that for any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$,

[10]The probability that *all* voters receive signal z in state α_2 is $(1-\frac{1}{n^2})^{2n}$ and $\lim_{n\to\infty}(1-\frac{1}{n^2})^{2n} = 1$, recalling that $\lim_{n\to\infty}(1-\frac{1}{n}\frac{1}{d})^{2n} = e^{-\frac{2}{d}}$. This observation is the critical step in the proof in the appendix.

$$\lim_{n\to\infty} \Pr(\alpha|z, \text{piv}; \sigma_n, \pi_n^r) = r. \tag{30}$$

Therefore, the sender can "steer" the behavior of voters with signal z by choosing r.

Next, we consider signal b and the voters' inference regarding the relative likelihood of α_2 and β_1. We show that, for this signal, the inference from the signal is dominated by the inference from being pivotal if the election is closer to being tied in state α_2 than in state β_1:

Claim 4 *Take any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} |q(\sigma_n; \alpha_2, \pi_n^r) - \frac{1}{2}| < \lim_{n\to\infty} |q(\sigma_n; \beta_1, \pi_n^r) - \frac{1}{2}|; \tag{31}$$

then,

$$\lim_{n\to\infty} \frac{\Pr(\alpha|b, \text{piv}; \sigma_n, \pi_n^r)}{\Pr(\beta|b, \text{piv}; \sigma_n, \pi_n^r)} = \infty. \tag{32}$$

Proof. The posterior likelihood ratio is

$$\begin{aligned}
\frac{\Pr(\alpha|b, \text{piv}; \sigma_n, \pi_n^r)}{\Pr(\beta|b, \text{piv}; \sigma_n, \pi_n^r)} &= \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\alpha_2|\alpha, \pi_n^r)}{\Pr(\beta_1|\beta, \pi_n^r)} \frac{\Pr(b|\alpha_2; \pi_n^r)}{\Pr(b|\beta_1; \pi_n^r)} \frac{\Pr(\text{piv}|\alpha_2; \sigma_n, \pi_n^r)}{\Pr(\text{piv}|\beta_1; \sigma_n, \pi_n^r)} \\
&= \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r\frac{1}{n}}{1-(1-r)\frac{1}{n}} \frac{\frac{1}{n^2}}{1} \frac{\Pr(\text{piv}|\alpha_2; \sigma_n, \pi_n^r)}{\Pr(\text{piv}|\beta_1; \sigma_n, \pi_n^r)} \\
&\approx \frac{\Pr(\text{piv}|\alpha_2; \sigma_n, \pi_n^r)}{\Pr(\text{piv}|\beta_1; \sigma_n, \pi_n^r)} n^{-3}.
\end{aligned} \tag{33}$$

For the approximation on the last line we used that the prior is uniform. Given (31), equation (32) follows from applying Claim 2 for $d = 3$. ■

Thus, for any sequence of strategies that satisfies (31), the critical posterior with signal b satisfies the desired property (28).

4.2.3 Fixed Point Argument

By the richness assumption on Φ (see (17)), there is some $\hat{r}$ such that $\Phi(\hat{r}) = \frac{1}{2}$. We will show that, for the information structure $\pi_n^{\hat{r}}$ and n large enough, there is an equilibrium in which A receives a strict majority of votes in both states in expectation.

The basic idea is this: The choice of $\hat{r}$ and (30) imply that the vote shares in states α_2 and β_2 are close to $\Phi(\hat{r}) = \frac{1}{2}$. Moreover, in equilibrium, it will be the case that A receives a strict majority of votes in state β_1. Hence, the election is closer to being tied in α_2 than in β_1. Therefore, by Claim 4, voters with signal b become convinced that the state is α; thus, the vote share of A in β_1 is close to $\Phi(1) > \frac{1}{2}$.

Recall that equilibrium is equivalently characterized by a vector of beliefs, $\mathbf{p}^* = (p^*_a, p^*_z, p^*_b)$, such that $\mathbf{p}^* = \boldsymbol{\rho}\left(\sigma^{\mathbf{p}^*}\right)$; see (13). Now, for any $\delta > 0$, let

$$\mathrm{B}_\delta = \left\{\mathbf{p} \in [0,1]^3 \,|\, |\mathbf{p}- (1, \hat{r}, 1)| \leq \delta\right\},$$

so that B_δ is the set of beliefs at most δ away from $(1, \hat{r}, 1)$. Take any $\mathbf{p} \in \mathrm{B}_\delta$ and the corresponding strategy $\sigma^{\mathbf{p}}$. Since $\Phi(1) > \frac{1}{2}$, this means that A receives a strict majority of votes in the states α_1 and β_1 for δ small enough. In the states α_2 and β_2, (almost) all voters observe signal z, so $q(\alpha_2; \sigma^{\mathbf{p}}, \pi^{\hat{r}}_n) \approx \Phi(\hat{r})$ and $q(\beta_2; \sigma^{\mathbf{p}}, \pi^{\hat{r}}_n) \approx \Phi(\hat{r})$. Since $\Phi(\hat{r}) = \frac{1}{2}$, the vote share for A is approximately $\frac{1}{2}$.

Now, we show that our two previous claims (Claim 3 and Claim 4) imply that—given $\sigma^{\mathbf{p}}$—the posterior conditional on being pivotal is again in B_δ, for any $\mathbf{p} \in \mathrm{B}_\delta$, any sufficiently small δ, and any sufficiently large n:

Claim 5 *For any δ sufficiently small, there exists $n(\delta)$ s.t., for all $n \geq n(\delta)$,*

$$\forall \mathbf{p} \in B_\delta : \boldsymbol{\rho}\left(\sigma^{\mathbf{p}}; \pi^{\hat{r}}_n, n\right) \in B_\delta. \tag{34}$$

Proof. Take any $\mathbf{p} \in \mathrm{B}_\delta$ and its corresponding behavior $\sigma^{\mathbf{p}}$. For the posterior following signal a it is immediate that, for all δ and n,

$$\boldsymbol{\rho}_a\left(\sigma^{\mathbf{p}}; \pi^{\hat{r}}_n, n\right) = 1; \tag{35}$$

see (27). Secondly,

$$\lim_{n\to\infty} \boldsymbol{\rho}_z\left(\sigma^{\mathbf{p}}; \pi^{\hat{r}}_n, n\right) = \hat{r}, \tag{36}$$

follows from Claim 3 for all δ; see (30).

Finally, for δ small enough and n large enough, the election is closer to being tied in α_2 than in β_1,

$$\forall \mathbf{p} \in \mathrm{B}_\delta\text{:}\;\; |q(\alpha_2; \sigma^{\mathbf{p}}, \pi^{\hat{r}}_n) - \frac{1}{2}| < |q(\beta_1; \sigma^{\mathbf{p}}, \pi^{\hat{r}}_n) - \frac{1}{2}|. \tag{37}$$

To see why, note that for n large enough, $q(\alpha_2; \sigma^{\mathbf{p}}, \pi^{\hat{r}}_n) \approx \Phi(p_z)$ and $q(\beta_1; \sigma^{\mathbf{p}}, \pi^{\hat{r}}_n) = \Phi(p_b)$ since almost all voters receive z in α_2 and all voters receive b in β_1. In addition, by the continuity of Φ, for δ small enough, we have that $\Phi(p_z) \approx \Phi(\hat{r})$ and $\Phi(p_b) \approx \Phi(1)$. Finally, (37) follows then from $\Phi(\hat{r}) = \frac{1}{2}$ and $\Phi(1) > \frac{1}{2}$.

Now, it follows from (37) and from Claim 4 that

$$\lim_{n\to\infty} \boldsymbol{\rho}_b\left(\sigma^{\mathbf{p}}; \pi_n^{\hat{r}}, n\right) = 1. \tag{38}$$

Thus, the claim follows from (35), (36), and (38). ■

Since $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is continuous in $\mathbf{p}$ by the arguments after (13), it follows from (34) and Kakutani's theorem that there exists a fixed point $\mathbf{p}_n^* \in B_\delta$ for all n large enough. By the arguments from the proof of Claim 5,

$$\lim_{n\to\infty} \mathbf{p}_n^* = (1, \hat{r}, 1), \tag{39}$$

see (35), (36), and (38). Finally, for the corresponding sequence of equilibrium strategies, $(\sigma^{\mathbf{p}_n^*})_{n\in\mathbb{N}}$, the policy A wins in both states; this follows from (39), which implies that voters with signals a and b are supporting A with a probability converging to $\Phi(1) > \frac{1}{2}$, and from the weak law of large numbers.

This completes the proof of the theorem for the special case in which Φ is monotone, the targeted policy is A in both states, and the prior is uniform. When the prior is not uniform, the only piece of the argument that needs to be adjusted is the choice of r. For a general prior $\Pr(\alpha) \neq \frac{1}{2}$, the value of r should be such that

$$\frac{\Pr(\alpha) r}{\Pr(\alpha) r + (1 - \Pr(\alpha))(1 - r)} = \hat{r}, \tag{40}$$

with $\Phi(\hat{r}) = \frac{1}{2}$.

4.3 Numerical Example with 15 voters

We provide an example and show that persuasion is effective when there are at least $2n + 1 = 15$ voters. For this example, suppose that G is such that $\Phi(p) = p$ for all $p \in [0, 1]$.[11] Further, we set $\Pr(\alpha) = \frac{1}{3}$. Now, consider the information structure $\tilde{\pi}_n^r$ with $r = \frac{2}{3}$ from Figure 4, which is as π_n^r from Figure 3, but with the signal b replaced by signal a.

[11]In Section C.1 of the Online Appendix, we provide an explicit example of a preference distribution G that induces $\Phi(p) = p$ for all p. Since, therefore, $\Pr(t : t_\alpha > 0, t_\beta < 0) = 1$, the example fails the assumption that G has a strictly positive density on $[-1, 1]^2$. This simplifies the presentation and one can find a nearby example with full support.

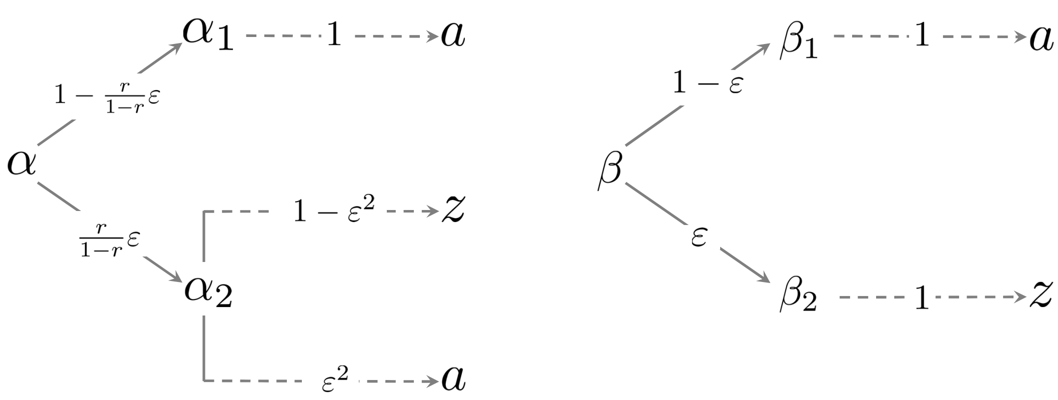

Figure 4: The information structure $\tilde{\pi}_n^r$ with $\varepsilon = \frac{1}{n}$ and $r \in (0,1)$. The parameter $r \in (0,1)$ controls the posterior after z.

In Section C.1 of the Online Appendix, we show that under these primitives, when there are at least 15 voters, there is an equilibrium σ_n^* for which A is elected with a probability larger than 99.9% in the states α_1 and β_1. Therefore, the overall probability of A being elected exceeds $0.999\Big[\Pr(\alpha)(1-\frac{r}{1-r}\frac{1}{n}) + \Pr(\beta)(1-\frac{1}{n})\Big]$, which is larger than 80% when there are at least $2n+1 = 15$ voters.

To show the result, recall that equilibrium is equivalently characterized by a vector of induced priors $\mathbf{p} = (p_a, p_z)$ satisfying (13). We show that under the specified primitives, when $n \geq 7$, the best response maps beliefs $\mathbf{p} = (p_a, p_z) \in [0,1]^2$ for which $p_a \geq 0.95$ and $p_z \in [0.32, 0.68]$ to beliefs satisfying the same inequalities. Then, an application of Kakutani's theorem yields an equilibrium belief $\mathbf{p}_n^* = (p_a^*, p_z^*)$ with $p_a^* \geq 0.95$ and $p_z^* \in [0.32, 0.68]$. The corresponding equilibrium $\sigma(\mathbf{p}_n^*)$ is such that voters with an a-signal vote A with a probability of at least 95%. Thus, we can utilize the exact same theoretical argument here for the example with small numbers as we do in our general analysis.

4.4 Bayesian Consistency Constraints in Elections

As noted, voters' behavior is determined by their critical belief, $\Pr(\alpha|s, \text{piv}; \sigma, \pi)$, implying a close connection to the standard information design and persuasion model. The signal structure π affects voters' beliefs in two ways – directly via the inference from s and indirectly via the inference from being pivotal. Bayesian consistency is understood to constrain a sender's ability to affect the signal inference by choice of π; however, the indirect effect is much less constrained.

Bayesian consistency—or the law of iterated expectation—requires that

$$\Pr(\alpha) = \sum_{s \in S} \left[\Pr(\mathrm{s}, \mathrm{piv}) \Pr(\alpha | s, \mathrm{piv}) + \Pr(\mathrm{s}, \neg\mathrm{piv}) \Pr(\alpha | s, \neg\mathrm{piv}) \right], \tag{41}$$

where $\Pr(\alpha|s, \neg\mathrm{piv}; \sigma, \pi)$ is the posterior conditional on not being pivotal; we omitted (σ, π). With a single voter, $\Pr(\mathrm{piv}) = 1$, and so the expected critical belief is constrained to be the prior. However, with many voters, $\Pr(\mathrm{piv})$ becomes small, and, consequently, (41) imposes only a small constraint.

The effectiveness of "pivotal persuasion" has been observed before in a setting with known preferences and no private information by the voters; see our discussion of the related literature in Section 7.2; especially Chan, Gupta, Li, and Wang (2019) and Bardhi and Guo (2018).

Intuitively, what matters is that voters react to the closeness of the election. The closeness of the election tells voters something about the information of others, and, in this way, about the quality of the signal structure. The quality of the signal structure, in turn, affects the meaning of the own information.

In our construction, one may interpret the signal structure π^r as releasing either a high quality signal—in substates $\{\alpha_1, \beta_1\}$—or a low quality signal—in substates $\{\alpha_2, \beta_2\}$. The closeness of the election depends on the signal quality. In particular, when the quality of the signal structure is high, all voters observe the same revealing signal and the election is far from close. Conversely, when the election is close, this is because the quality of the signal is low. In this case, most voters learn that the signal quality is low but some may receive erroneous messages. In particular, when the election is close and the signal quality is low, the meaning of a b signal changes from being indicative of β to being an erroneous signal indicative of α.

The pivotal voting model considers the extreme case in which voters react perfectly to the closeness of the election; it illustrates the effectiveness of persuasion in this case. In Section 4.6.4, we discuss a model variant with some behavioral types who do not condition on being pivotal.

4.5 Sketch of the Proof: General Policy

Now, we allow for non-monotone Φ and show that the sender can implement any intended state-dependent policy, including the one that inverts the full-information outcome.

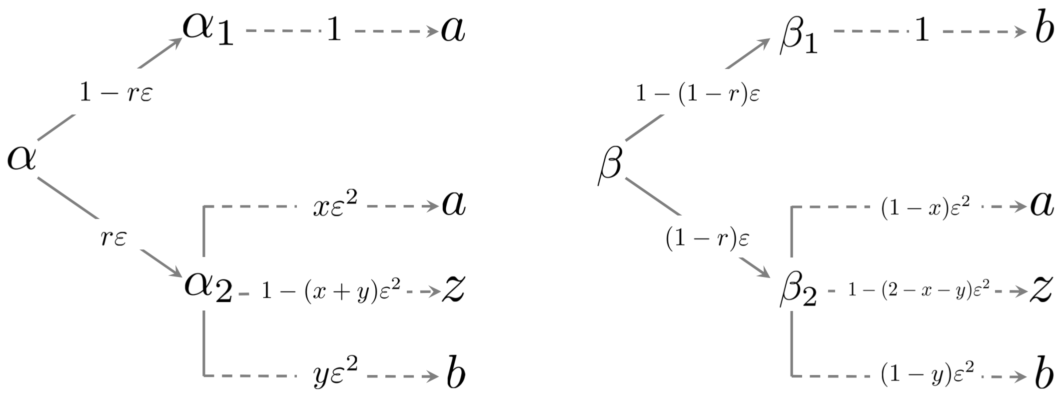

Figure 5: The information structure $\pi_n^{x,r,y}$ with $\varepsilon = \frac{1}{n}$ and $(x, r, y) \in [0, 1]^3$. The parameter r controls the posterior after z and the parameters x and y control the beliefs after a and b, respectively, conditional on being in substate α_2 or β_2.

For this, we consider the information structure depicted in Figure 5. The signals are (almost) public, similar to the information structure in the previous section from Figure 3. Moreover, as before, the signals a and b reveal the state (almost) perfectly. The signal z contains only limited information since $r \in (0, 1)$. When observing the signal z, a voter knows that the substate must be either α_2 or β_2, and her belief conditional on signal z is given by

$$\lim_{n\to\infty} \frac{\Pr(\alpha|z; \pi_n^{x,r,y})}{\Pr(\beta|z; \pi_n^{x,r,y})} = \lim_{n\to\infty} \frac{\Pr(\alpha|\{\alpha_2, \beta_2\}; \pi_n^{x,r,y})}{\Pr(\beta|\{\alpha_2, \beta_2\}; \pi_n^{x,r,y})} = \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r}. \tag{42}$$

We prove Theorem 2 by showing that by choosing the parameters $(x, r, y) \in [0, 1]^3$ appropriately, the sender can implement almost any belief μ_α in state α and any belief μ_β in state β as $n \to \infty$, in the sense that, with probability close to one, almost all voters will have such beliefs conditional on being pivotal.

Lemma 1 *Let $\hat{r}$ solve $\Phi(\hat{r}) = \frac{1}{2}$ and suppose $\hat{r} \notin \{0, 1\}$. Take any $(\mu_\alpha, \mu_\beta) \in [0, 1]^2$ with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\mu_\beta) \neq \frac{1}{2}$ and choose $(x, r, y) \in [0, 1]^3$ as the solutions* *to*[12]

$$\begin{aligned}
\frac{\hat{r}}{1-\hat{r}}\frac{x}{1-x} &= \frac{\mu_\alpha}{1-\mu_\alpha}, & (43)\\
\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r} &= \frac{\hat{r}}{1-\hat{r}}, & (44)\\
\frac{\hat{r}}{1-\hat{r}}\frac{y}{1-y} &= \frac{\mu_\beta}{1-\mu_\beta}. & (45)
\end{aligned}$$

Then, there exists a sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}} = (\pi_n^{x,r,y})_{n\in\mathbb{N}}$ such that

$$\begin{aligned}
\lim_{n\to\infty} \Pr\left(\alpha|\text{piv}, a; \sigma_n^*, \pi_n\right) &= \mu_\alpha, & (46)\\
\lim_{n\to\infty} \Pr\left(\alpha|\text{piv}, z; \sigma_n^*, \pi_n\right) &= \hat{r}, & (47)\\
\lim_{n\to\infty} \Pr\left(\alpha|\text{piv}, b; \sigma_n^*, \pi_n\right) &= \mu_\beta. & (48)
\end{aligned}$$

The lemma is proven in the Online Appendix in Section C.2, using ideas similar to those used earlier. First, as before, voters with signals z do not update conditional on being pivotal as $n \to \infty$ in any equilibrium, and r is then chosen such that, in substates α_2 and β_2, the vote share of A is close to $\frac{1}{2}$ in every equilibrium. Second, we show that there are equilibria in which voters with signals a and b behave according to the beliefs μ_α and μ_β. By the choice of the beliefs, with this behavior, there is either a strict majority for A or B in the substates α_1 and β_1; thus, the election is closer to being tied in α_2 and β_2 than in α_1 and β_1. Thus, conditional on being pivotal, voters with signals a and b believe that they are in substates α_2 and β_2, and, interpreting their signals conditional on these substates, their critical posteriors are as given in the lemma.

The lemma implies Theorem 2: the richness assumption (17) states that there is a belief p for which a majority prefers A in expectation and a belief p' for which a majority prefers B in expectation—that is, $\Phi(p) > \frac{1}{2} > \Phi(p')$. Thus, given belief p', it follows from the weak law of large numbers that B is elected with probability converging to one. Given belief p, it follows from the weak law of large numbers that A is elected with probability converging to one. Hence, the sender can implement any state-contingent policy $(x_\alpha, x_\beta) \in \{A, B\}^2$ by implementing belief p in any state ω for which $x_\omega = A$ and by implementing belief p' in any state for which $x_\omega = B$.

[12]For $\mu_\alpha = 1$, let $x = 1$, and for $\mu_\beta = 1$, let $y = 1$ such that the following equations hold in the extended reals, using the convention that $\frac{1}{0} = \infty$.

4.6 Robustness

In this section, we discuss the robustness of the persuasion result in Theorem 2. In particular, we ask: Can the sender be persuasive even if his commitment power is limited? Can he be persuasive if he does not know the exact details of the environment? How "stable" is the equilibrium? Are there other equilibria?

4.6.1 Persuasion with Partial Commitment

We relax the assumption that the sender can perfectly commit to an information structure. To model partial commitment, we follow Lipnowski, Ravid, and Shishkin (2019), Min (2017), and Fréchette, Lizzeri, and Perego (2019). The sender announces an information structure but is committed to the announced information structure only with probability $\chi \in (0,1)$; otherwise, he can freely release any signal profile from its support.

Formally, we assume that, given some targeted state-dependent policy $(x(\alpha), x(\beta)) \in \{A, B\}^2$, the sender's payoff is one if the targeted policy is implemented and zero otherwise. An information structure π with signal set S, a no-commitment strategy of the sender $\psi^* : \{\alpha, \beta\} \to \Delta(S^{2n+1})$, and a voter strategy σ^* form a χ-*equilibrium* (Lipnowski, Ravid, and Shishkin, 2019) if ψ^* is a best response by the sender given that the voters follow the strategy σ^* and σ^* is a voting equilibrium given that the sender commits to π with probability χ and otherwise sends signals according to ψ^*.

Perhaps somewhat surprisingly, it turns out that the sender needs almost no commitment power: He can persuade voters whenever n is large for *any* $\chi > 0$, no matter how small.

Proposition 1 *Suppose that the sender is committed with some probability* $\chi > 0$. *Then, for every preference distribution* Φ, *every prior* $\Pr(\alpha) \in (0,1)$, *and every state-dependent policy* $(x(\alpha), x(\beta)) \in \{A, B\}^2$, *there exists a sequence of* χ-*equilibria* $(\pi_n, \psi_n^*, \sigma_n^*)_{n\in\mathbb{N}}$, *such that*

$$\lim_{n\to\infty} \Pr(x(\alpha)\,|\alpha; \pi_n, \psi_n^*, \sigma_n^*, n) = 1,$$
$$\lim_{n\to\infty} \Pr(x(\beta)\,|\beta; \pi_n, \psi_n^*, \sigma_n^*, n) = 1.$$

The following discussion proves the proposition. We consider, first, the constant target policy A, that is, $x(\alpha) = x(\beta) = A$. As noted in Section 4.3, with full commitment, this policy can be implemented by the information structure $\tilde{\pi}_n^r$

from Figure 4, which sends signal a with a probability of one in substates α_1 and β_1. Given $\tilde{\pi}_n^{\hat{r}}$ with $\phi(\hat{r}) = \frac{1}{2}$, there are voting equilibria σ_n^* in which, following signal a, the vote share of A is strictly larger than $1/2$, whereas, after z, the vote share is equal to $1/2$. Now, take any $\chi \in (0,1)$. Given the voting behavior σ_n^*, the best-response ψ^* of the sender is to send signal a to all voters because signal a leads to a higher vote share for A than signal z. Now, it turns out that, for any $\chi > 0$ and n large enough, there is a signal structure π_n^χ such that π_n^χ, χ, and ψ^* jointly imply the exact same distribution over signals as the original information structure $\tilde{\pi}_n^r$.[13] Hence, the original voting behavior σ_n^* is a best response to π_n^χ and ψ^*. In other words, $(\pi_n^\chi, \psi^*, \sigma_n^*)$ form a χ-equilibrium that implements $x(\alpha) = x(\beta) = A$ as $n \to \infty$.

The construction shows that one can find such π_n^χ whenever $\chi > \max\left(\frac{1}{n}\frac{r}{1-r}, \frac{1}{n}\right)$. Thus, the required commitment power is vanishing at rate $1/n$. The key observation is that $\tilde{\pi}_n^r$ is already sending the sender's preferred signal a to all voters with probability close to 1 in both states.

Second, consider the targeted policy $(x(\alpha), x(\beta)) = (B, A)$ that inverts the full-information outcome. Let $\pi_n^{(x,r,y)}$ be the information structure from Figure 5. By Lemma 1 and the subsequent discussion, there are parameters (x, r, y) and equilibria σ_n^* that implement the targeted policy. The voting behavior is such that, after a, the vote share of A is strictly smaller than $1/2$, after z it is equal to $1/2$, and after b it is strictly larger than $1/2$. Given this voting behavior and the targeted policy, the sender's best response ψ^* is to send the signal a to all voters when the state is α and b to all voters when the state is β (recall that a majority of the voters are voting B with an a signal and A with a b signal). Finally, for any $\chi > 0$ and n large enough, one can construct a modified information structure π_n^χ in the same way as before such that π_n^χ, χ, and ψ^* jointly imply the same signal distribution as $\pi_n^{(x,r,y)}$; so, σ_n^* is a best response, proving the existence of a χ-equilibrium that implements $(x(\alpha), x(\beta)) = (B, A)$.

Numerical Example (continued). Recall the example with $\Phi(p) = p$ and

[13] The sender's information structure π_n^χ is constructed as follows: in α, he sends the signal a to all voters with probability $(1 - \frac{r(\alpha)}{1-r(\alpha)}\frac{1}{n})$, where $r(\alpha)$ solves $\chi(1 - \frac{r(\alpha)}{1-r(\alpha)}\frac{1}{n}) + (1-\chi) = 1 - \frac{\hat{r}}{1-\hat{r}}\frac{1}{n}$; in β, he sends the signal a to all voters with probability $(1 - \frac{r(\beta)}{1-r(\beta)}\frac{1}{n})$, where $r(\beta)$ solves $\chi(1 - \frac{r(\beta)}{1-r(\beta)}\frac{1}{n}) + (1-\chi) = 1 - \frac{1}{n}$. In α, otherwise, each voter receives a signal z randomly with probability $1 - \frac{1}{n^2}$, and a signal a with probability $\frac{1}{n^2}$; in β, otherwise, all voters receive the public signal z. This construction is feasible if $\chi > \max\left(\frac{1}{n}\frac{\hat{r}}{1-\hat{r}}, \frac{1}{n}\right)$, which ensures that $(1 - \frac{r(\alpha)}{1-r(\alpha)}\frac{1}{n})$ and $(1 - \frac{r(\beta)}{1-r(\beta)}\frac{1}{n})$ are in $(0,1)$.

$\Pr(\alpha) = \frac{1}{3}$. Given full commitment and the information structure $\tilde{\pi}_n^r$ with $r = \frac{2}{3}$, when there are at least $2n+1 = 15$ voters, there are equilibria $\sigma(\mathbf{p}_n^*)$ such that the constant policy A is elected with a probability larger than 80%. The construction of $\sigma(\mathbf{p}_n^*)$ shows that, following signal a, the vote share of A is strictly larger than 0.95, whereas, after z, the vote share is in $[0.3, 0.7]$. Therefore, the sender's best response to $\sigma(\mathbf{p}_n^*)$ is to send the public signal a, i.e. $\psi^*(\alpha) = \psi^*(\beta) = (a, \ldots, a)$. Again, for any $\chi > \frac{1}{n}\frac{r}{1-r} = \frac{3}{n}$, there is a signal structure π_n^χ such that π_n^χ, χ, and ψ^* jointly imply the exact same distribution over signals as the original information structure $\tilde{\pi}_n^r$; so $\sigma(\mathbf{p}_n^*)$ is a continuation equilibrium, proving the existence of a χ_n-equilibrium where A is elected with a probability larger than 80% when $2n+1 \geq 15$ and $\chi > \frac{3}{n}$.

4.6.2 Robustness: Detail-Freeness

In this section, we show that to persuade the voters, the signal structure does not need to be finely tuned to the details of the environment. Suppose that the prior and the preference distribution are such that

$$|\Phi(0) - \frac{1}{2}| > |\Phi(\Pr(\alpha)) - \frac{1}{2}|, \tag{49}$$

$$|\Phi(1) - \frac{1}{2}| > |\Phi(\Pr(\alpha)) - \frac{1}{2}|; \tag{50}$$

therefore, when the citizens vote optimally given their beliefs, the election is closer to being tied when they are uninformed and hold the prior belief relative to when they know the state.

Proposition 2 *Take $r = 1$ and $(x, y) \in \{0, 1\}^2$. For any prior and preference distribution satisfying (49) and (50), there is a sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given the sequence of signal structures $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, a; \sigma_n^*) = x, \tag{51}$$

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, z; \sigma_n^*) = \Pr(\alpha), \tag{52}$$

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, b; \sigma_n^*) = y. \tag{53}$$

The proposition implies that the sender can implement any policy using a single signal structure that works uniformly across the large set of priors and preference distributions satisfying (49) and (50). For example, the constant policy A is implemented by choosing $x = y = 1$, which leads to an equilibrium in which

A has a vote share $\Phi(1)$ as the election becomes large.

The proof is in the Online Appendix in Section C.3. The basic idea is that, given this signal, the vote shares are close to $\Phi(\Pr(\alpha))$ in states α_2 and β_2. Hence, by assumptions (49) and (50), if voters behave according to the posteriors x and y in states α_1 and β_1, the election is closer to being tied in α_2 and β_2 than in α_1 and β_1. Thus, just as before, conditional on being pivotal, voters with signals a and b believe that they are in states α_2 and β_2, and—interpreting their signals conditional on these substates—their critical posteriors are as given in the proposition.

A similar argument implies that the signal structure from Lemma 1 is also effective when the actual environment is slightly different: When the prior and Φ is slightly different from the one used to calculate (x, r, y), then there is still an equilibrium close-by with critical beliefs that are close to μ_α, $\hat{r}$, and μ_β, provided that vote shares at the critical beliefs imply that the election is still closer to being tied in states α_2 and β_2 than in states α_1 and β_1.

Random Signal Quality. Note that the signal from Proposition 2 matches the description in the introduction. In particular, we can swap the timing in the description of the signal. Rather than choosing the "quality" of the signal after the state of nature has realized, one can first choose randomly whether the signal is "revealing" or "obfuscating" and then, if it is revealing, send a signal corresponding to the realized state of nature to all voters (as in substates α_1 and β_1), and, if it is obfuscating, send the signals z or b in α and z or a in β (as in substates α_2 and β_2 when $x = 0$ and $y = 1$).

4.6.3 Robustness: Basin of Attraction

We show that, for a large set of initial strategies, an iterated best response leads quickly to the "manipulated equilibrium" of Theorem 2 described earlier.

Let (μ_α, μ_β) be any pair of beliefs with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\mu_\beta) \neq \frac{1}{2}$. By Lemma 1, there is a sequence of information structures $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$ and equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ that implements the pair of beliefs as $n \to \infty$, in the sense that, with probability close to 1, almost all voters will have such beliefs conditional on being pivotal. Hence, by choosing (μ_α, μ_β) appropriately, a sender can implement any desired policy. The next result shows that, for almost any strategy σ, the twice-iterated best response is arbitrarily close to σ_n^* when n is large, in the sense that the posteriors conditional on being tied are close to (μ_α, μ_β).

First, let us define the twice-iterated best response: Take any belief $\mathbf{p}$ and the strategy $\sigma^{\mathbf{p}}$ that is optimal given these beliefs. Then, $\sigma^{\boldsymbol{\rho}(\sigma^{\mathbf{p}})}$ is the best response to $\sigma^{\mathbf{p}}$ and is optimal given the beliefs

$$\boldsymbol{\rho}^1(\mathbf{p}) = \boldsymbol{\rho}(\sigma^{\mathbf{p}}), \tag{54}$$

where $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is the vector of the posteriors conditional on the pivotal event and the signals. In the same way, $\sigma^{\boldsymbol{\rho}(\sigma^{\boldsymbol{\rho}^1(\mathbf{p})})}$ is the best response to $\sigma^{\boldsymbol{\rho}^1(\mathbf{p})}$ (so it is the twice-iterated best response to $\sigma^{\mathbf{p}}$) and is optimal given the beliefs

$$\boldsymbol{\rho}^2(\mathbf{p}) = \boldsymbol{\rho}(\sigma^{\boldsymbol{\rho}^1(\mathbf{p})}). \tag{55}$$

Proposition 3 shows that for almost any $\mathbf{p}$, we have $|\boldsymbol{\rho}^2(\mathbf{p}) - (\mu_\alpha, \hat{r}, \mu_\beta)| < \epsilon$ when n is sufficiently large. This means that the twice-iterated best response is arbitrarily close to the manipulated equilibrium σ_n^* since the equilibrium is consistent with the belief $\boldsymbol{\rho}(\sigma_n^*) \approx (\mu_\alpha, \hat{r}, \mu_\beta)$; see (13).

Proposition 3 *Take any beliefs $(\mu_\alpha, \mu_\beta) \in [0,1]^2$ with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\mu_\beta) \neq \frac{1}{2}$ and the corresponding information structures $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$ from Lemma 1.*

For any $\delta > 0$, there is some $B \subset [0,1]^3$ with Lebesgue-measure of at least $1-\delta$ and some $\bar{n} \in \mathbb{N}$ such that, for all $n \geq \bar{n}$,

$$\forall \mathbf{p} \in B : |\boldsymbol{\rho}^2(\mathbf{p}) - (\mu_\alpha, \hat{r}, \mu_\beta)| < \delta. \tag{56}$$

The proof is in Section C.4 in the Online Appendix. The proof also implies that, for "almost any" strategy σ—even those that are not optimal given some belief $\mathbf{p}$—the twice-iterated best reply is arbitrarily close to the manipulated equilibrium σ_n^* when n is large, where the genericity requirement is with respect to the induced vote shares and given by condition (100), replacing $\sigma^{\mathbf{p}}$ by σ.

Simple Reasoning. Proposition 3 illustrates that a simple reasoning underlies the manipulated equilibrium σ_n^*. The result loosely relates to the concepts of level k-thinking and level-k-implementability (De Clippel, Saran, and Serrano, 2019). The proposition implies that, for almost any strategy (a "behavioral anchor"), the strategies that are consistent with level-2-thinking are close to the manipulated equilibrium. In this sense, any state-dependent target policy $(x(\alpha), x(\beta)) \in \{A, B\}^2$ is level-2-implementable.[14]

[14] De Clippel, Saran, and Serrano (2019) consider a different notion of level-2-implementability

4.6.4 Persuasion with Behavioral Types

The pivotal voting model considers the extreme case where voters react perfectly to the closeness of the election when interpreting their information and illustrates the effectiveness of persuasion in this case. The empirical literature has indeed provided evidence for strategic voting behavior.[15] However, while there is often a significant fraction of the voters that are shown to act strategically, others behave "sincerely".[16] We follow this literature—in particular the modeling in Kawai and Watanabe (2013)—and consider an alternative model in which citizens have not only a preference type but also a behavioral type. Each citizen is a "sincere voter" with a probability $\kappa \geq 0$, and, in that case, votes A only if $pt_\alpha + (1-p)t_\beta \geq 0$ for $p = \Pr(\alpha|s;\pi)$, where s is her private signal and π the information structure. Otherwise, with probability $1-\kappa$, a voter is a "pivotal voter" as in the analysis before.

For concreteness, we discuss the effect of sincere voters in the setup of the numerical example from Section 4.3 with $\Phi(p) = p$, the information structure $\tilde{\pi}_n^r$ from Figure 4, and with $\Pr(\alpha) = \frac{1}{3}$ and $r = \frac{2}{3}$. Here, signal a contains almost no information; therefore, the vote share of A among the sincere voters is roughly $\frac{1}{3}$. When n is large, this implies that the previous persuasion arguments continue to work when $\kappa\frac{1}{3} + (1-\kappa) > \frac{1}{2}$, which holds if $\kappa < \frac{3}{4}$—roughly in line with estimates from the empirical literature.[17]

The full analysis of a model with sincere voters may be worthwhile for future research, especially when considering the implementation of the inverse of the full information outcome, that is, $x(\alpha) = B$ and $x(\beta) = A$.

that demand that there is *some* behavioral anchor such that *any* profile of strategies that are level-1-consistent or level-2-consistent for this anchor implement a given social choice function. Here, almost any strategy can be such an anchor.

[15]See e.g. Guarnaschelli, McKelvey, and Palfrey (2000).

[16]See e.g. Kawai and Watanabe (2013), who provide estimates of strrategic voters ranging from 63.4% to 84.9%, or Esponda and Vespa (2014), who provide estimates between 20% and 50%. There are also other behavioral models of voting, such as ethical voting (Feddersen and Sandroni, 2006) or expressive voting that could be interesting as well.

[17]We revisit the example with sincere voters numerically in the Online Appendix in Section F. Given the parameters $\Phi(p) = p$ and $\Pr(\alpha) = \frac{1}{3}$, suppose that the fraction of sincere voters is $\kappa = 40\%$. Under these primitives, we show that when there are at least 170 voters, there is an equilibrium σ_n^* for which A is elected with a probability larger than 99.9% in the states α_1 and β_1.

4.6.5 Other Equilibria

Proposition 3 shows that the basin of attraction of the iterated best response of an arbitrarily small neighbourhood of the manipulated equilibria consists of almost all strategies when n is large enough. However, this still leaves open the possibility that there are other equilibria, such that if we begin exactly at such a strategy profile, the best response dynamic stays there. In the working paper version, (Heese and Lauermann, 2019, Theorem 4),[18] we show that this is indeed the case. There exists another equilibrium and that equilibrium is not "manipulated" but implements the full information outcome as $n \to \infty$. We restate the result here:

Theorem 3 *Let Φ be strictly increasing. For all information structures $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$ with $(x,r,y) \in (0,1)^3$, there exists an equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$ for which the full information outcome is elected as $n \to \infty$,*

$$\lim_{n\to\infty} \Pr(A|\alpha; \sigma_n^*, \pi_n, n) = 1,$$
$$\lim_{n\to\infty} \Pr(B|\beta; \sigma_n^*, \pi_n, n) = 1.$$

Intuition. Note that the signal π_n almost always sends an (almost) perfectly revealing signal when n is large. Hence, there is a sequence of strategies (e.g. given by sincere voting) for which the full-information outcome is elected as $n \to \infty$. The question then is whether such a sequence of strategies can be an equilibrium sequence. The theorem shows that, whenever Φ is monotone, the answer is yes. This is easy to see in the extreme case in which voters have a common type t, and, hence, have common interests. A result of McLennan (1998) shows that, with common interest, the utility maximizing symmetry strategy is a symmetric equilibrium. Hence, for this case, the existence of a sequence of strategies that yields the full-information outcome immediately implies the existence of an equilibrium sequence that yields it as well.

5 Persuasion of Privately Informed Voters

Recall the binary information structure from the Condorcet Jury Theorem, defined by the signal probabilities $\Pr(s|\omega)_{\omega\in\{\alpha,\beta\}}$ for $s \in \{u, d\}$ such that (20) holds. We will think of this as exogenous private information that is held by the voters and denote this information structure by π^c. We say that an information structure π

[18] The working paper is publicly available here https://ideas.repec.org/p/bon/boncrc/crctr224_2019_128.html.

with signal set S is an *independent expansion* of π^c if it is the product of π^c and some additional signal structure π^p that is exchangeable, as before.[19]

We think of the expansion as resulting from additional information π^p that is provided by a sender to voters who also receive private signals from π^c. By considering only independent expansions, we do not allow the sender's signal to condition directly on the realization of π^c. As before, we also do not allow the sender to elicit the voters' private information (the preference type and the signal). We assume that the preferences of the voters are such that the aggregate preference function Φ is strictly increasing so that the CJT holds (Theorem 1) and, without an additional signal, the unique equilibrium outcome is the full information outcome as the electorate grows large.

What outcomes can the sender implement when the voters have exogenous signals? How should the sender communicate with the voters? Clearly, to implement any policy other than the full information outcome, the sender has to communicate with the voters in some way. Consider a sender who communicates with *public signals* $s_2 \in S_2$, meaning, that the signals are commonly received by all the voters.[20] When the voters receive a public signal s_2, this shifts the common belief from the prior $\Pr(\alpha)$ to $\Pr(\alpha|s_2)$. Since the CJT holds for any common prior, in the subgame following any public signal, the full information outcome is elected with probability converging to one, as $n \to \infty$.[21] So, to implement any outcome other than the full information outcome, the sender has to communicate privately with the voters.

5.1 Result: Full Persuasion

The following theorem shows that there exists an independent expansion of the private information of the voters that allows implementing any state-dependent policy—even the policy that inverts the full-information outcome.

[19] More formally, π is an independent expansion if there exists an information structure π^p with signal set S_2 and substates $\{\alpha_1, \dots, \alpha_{N_\alpha}\}$ and $\{\beta_1, \dots, \beta_{N_\beta}\}$ such that $S = \{u, d\} \times S_2$ and

$$\Pr(\mathbf{s}|\omega_j; \pi) = \Pr(\mathbf{s}_1|\omega; \pi^c)\Pr(\mathbf{s}_2|\omega_j; \pi^p) \tag{57}$$

for all $\omega_j \in \{\alpha_1, \dots, \alpha_{N_\alpha}\} \cup \{\beta_1, \dots, \beta_{N_\beta}\}$ and all $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2) \in (\{u, d\} \times S_2)^{2n+1}$.

[20] Alonso and Câmara (2016) have studied persuasion with public signals when voters do not have exogenous private signals.

[21] To be precise, the CJT only applies to any non-degenerate prior $\Pr(\alpha) \in (0, 1)$. However, if the sender reveals the state publicly, such that $\Pr(\alpha|s) \in \{0, 1\}$, trivially, the full-information outcome is elected as $n \to \infty$.

Theorem 4 *Take any exogenous private signals π^c of the voters satisfying (20) and any strictly increasing Φ. For every state-dependent policy $(x(\alpha), x(\beta)) \in \{A, B\}^2$, there exists a sequence of independent expansions $(\pi_n)_{n\in\mathbb{N}}$ of π^c and equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given $(\pi_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \Pr(x(\alpha)|\alpha; \sigma_n^*, \pi_n, n) = 1,$$
$$\lim_{n\to\infty} \Pr(x(\beta)|\beta; \sigma_n^*, \pi_n, n) = 1.$$

The next two sections provide an extensive sketch of the arguments establishing the theorem. In particular, the original signals from the previous section are sufficient. That is, π_n^r, as in Figure 3, can be chosen as an additional signal to implement equilibria in which A wins in both states, and $\pi_n^{x,r,y}$ from Figure 5, with $x = 0$ and $y = 1$, can be chosen to implement a policy that inverts the full-information outcome. Thus, the sender does not need to know whether agents have private information, or how much private information they have. The same signal structure works uniformly across environments.

5.2 Sketch of the Proof: Constant Policy

We show that the same signal structure π_n^r from Figure 3 leads to an equilibrium in which A wins in both states—even when voters have private signals.

The critical observation in the proof is that the vote shares in α_2 and β_2 are uniquely determined across all equilibria and parameters by an equal-margin-of-victory condition.

Claim 6 *Let Φ be strictly increasing. Suppose that the additional information is given by π_n^r, as in Figure 3. Then, there is some M with*

$$0 < M < \Phi(1) - \frac{1}{2} \tag{58}$$

such that, for every $r \in (0,1)$ and every equilibrium sequence (σ_n^*) given π_n^r,

$$\lim_{n\to\infty} q(\sigma_n^*; \alpha_2, \pi_n^r) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - q(\sigma_n^*; \beta_2, \pi_n^r) = M. \tag{59}$$

For the proof, see Section D.2 in the Online Appendix. The idea is the following: Given π_n^r, in substates α_2 and β_2, every voter receives the additional signal z with probability converging to one. Voters who received z know that either α_2 or β_2 holds and that almost all other voters received a signal z as well. Hence, from their perspective, it is close to common knowledge that the game is close to a game with a binary state and binary signals π^c, as in the original setting of the CJT. Recall that the proof of the CJT showed that the election must be equally close to being tied in expectation; see (22). The same arguments implies (59) here.

Now, one can show that there is a sequence of equilibria in which the vote share of A in state β_1 approaches its maximum, $\Phi(1)$, and thus

$$\lim_{n\to\infty} q(\sigma_n^*;\beta_1,\pi_n)-\frac{1}{2}=\Phi(1)-\frac{1}{2}. \tag{60}$$

Comparing (59) and (60), in this equilibrium sequence, the election is closer to being tied in α_2 than in β_1. Hence, it follows from Claim 2 that

$$\lim_{n\to\infty}\frac{\Pr\left(\text{piv}|\beta_1;\sigma_n^*\right)}{\Pr\left(\text{piv}|\alpha_2;\sigma_n^*\right)}=0. \tag{61}$$

Moreover, it also follows from Claim 2 that the inference from the pivotal event dominates the direct inference from the signal.[22] So, a voter with additional signal $s_2=b$ becomes convinced that the state is α_2 for either realization of the private signal $s_1\in\{u,d\}$,

$$\lim_{n\to\infty}\Pr\left(\alpha|\text{piv},s_1,s_2=b;\sigma_n^*\right)=1. \tag{62}$$

Since all voters observe the additional signal $s_2=b$ in state β_1, it follows that the vote share converges to $\Phi(1)$, as claimed in (60). Finally, it is clear that such an equilibrium sequence leads to outcome A in both states with probability converging to one.

Note that the basic idea here is similar to the one in Section 4.2 without the private signal. Here, Claim 6 pins down behavior in states α_2 and β_2, analogously to the implication of the previous Claim 3. Then, there is an equilibrium in which A receives a strict majority in β_1. In both settings, the equilibrium is supported by the fact that the election is closer to being tied in α_2 than in β_1, so that, conditional on being pivotal, voters with signal b become convinced that the state is α_2.

5.3 Sketch of the Proof: General Policy

The signal $\pi_n^{x,r,y}$ from Figure 5 can again be used to implement any intended policy by the appropriate choice of $(x,r,y)\in[0,1]^3$. The proof of this general

[22] See the proof of the analogous Claim 4.

result utilizes a lemma analogous to the previous Lemma 1, stated as Lemma 3 in the Online Appendix.

In particular, as before, the policy that inverts the full-information outcome can be implemented by choosing the additional signal with $x = 0$, $y = 1$, and any *arbitrary* $r \in (0, 1)$: for any such choice, we show that there is a sequence of equilibria (σ_n^*) in which the posterior probabilities conditional on being pivotal and the additional signals a and b are close to zero and one, respectively. Moreover, since the private signals are boundedly informative, it follows that, for $s_1 \in \{a, b\}$,

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s_1, s_2 = a; \sigma_n^*) = 0, \tag{63}$$

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s_1, s_2 = b; \sigma_n^*) = 1. \tag{64}$$

Thus, since all voters observe signals a and b in the substates α_1 and β_1, respectively, the equilibrium vote shares converge to $\Phi(0) < 1/2$ and $\Phi(1) > 1/2$, with the inequalities from Φ satisfying (17). Therefore, the weak law of large numbers implies that B wins in state α_1 and A wins in state β_1, thereby establishing the existence of an equilibrium that inverts the full-information outcome.

5.4 Robustness of Theorem 4

Partial Commitment. When the sender is only partially committed and free to send any signal with probability χ, then full persuasion is still possible for arbitrarily small $\chi > 0$, even if voters have exogenous information. The argument is the same as in Section 4.6.1, namely, the implementing signal structure is already sending the sender's preferred signal with probability close to 1.

Detail Freeness. Can the sender persuade the voters even when he does not know the exact details of the environment? We argue that Proposition 2 from the monopolistic sender setting holds in an even more general form when the voters hold exogenous private signals: here, to be able to persuade the voters, it is sufficient that the sender knows that Φ satisfies the monotonicity condition (21) and the richness assumption (17).

Specifically, the sender can release information to the voters such that his target policy is implemented uniformly, for any prior $\Pr(\alpha) \in (0, 1)$, any exogenous information π^c of the voters satisfying (20), and any aggregate preference function Φ satisfying (17) and (21). This is possible simply by choosing the parameters of the general signal $\pi_n^{x,r,y}$ with x and y in $\{0, 1\}$ and any arbitrary $r \in (0, 1)$. Any such information structure implements a targeted policy uniformly as outlined before.

In a sense, the conditions for uniform implementability are weaker here than in Proposition 2 in which we also required a condition on the prior. This is perhaps surprising if one thinks of the voters' exogenous information as a constraint on the sender. The reason it holds is that, with exogenous private information, the relevant "induced prior" after signal z, i.e., $\Pr(\alpha|\text{piv}, z)$, adjusts endogenously to ensure the equal-margin condition.

Belief Implementation. As in the monopolistic sender scenario, we provide a result more general than Theorem 4: In the spirit of the literature on Bayesian persuasion, we show that, for large electorates, there is a set of "implementable" posterior belief distributions, including arbitrarily extreme beliefs. This result is stated in Lemma 3 in the Online Appendix, and corresponds to Lemma 1 for the monopolistic sender. However, when voters have exogenous information, not all beliefs are implementable; this is consequential when the sender is only partially informed, as discussed in Section 7.1.

Basin of Attraction. The results from Section 4.6.3 regarding the basin of attraction of the manipulated equilibria for the case of a monopolistic sender do not extend when voters have exogenous private information.[23]

Other Equilibria. We conjecture that there always also exists a sequence of equilibria yielding the full-information outcome, as in the case of the monopolistic sender (see Theorem 3). However, so far, we have not been able to prove this result for the case with private signals.

6 Media Markets: An Application

We provide a stylized application to media markets. This serves two purposes: first, we show that the main results of the paper can also be obtained in a setting with normally distributed voter information. Second, within the application, we can discuss concrete strategies of information manipulation.

[23]Instead, one can show the following: Let the sender release the information $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$ to the voters as in Lemma 3. When the electorate is large enough, for almost any initial strategy, under the iterated best response, the voter behavior after signal z jumps back and forth indefinitely from voting approximately according to $\sigma^{\mathbf{p}}$, with $\mathbf{p} = \Pr(\alpha|s)_{s\in\{a,z,b\}}$, to voting approximately as if one of the states is known to be the true state. We omit the proof.

6.1 Benchmark: Exogenous Media

A media firm sends a message $m = \theta$ in α and $m = -\theta$ in β for some $\theta > 0$. Voters perceive the message with noise. That is, each voter receives a private signal $s = m + \epsilon$, where ϵ is drawn independently from a standard normal distribution; see Figure 6.[24] Each individual voter thus only holds partial information about the state. Since the private signals s satisfy the monotone likelihood ratio property, a higher signal leads to a higher belief in the likelihood of α.

For concreteness, suppose that voters have a common preference type: All voters prefer A in α and B in β. They receive a payoff of $t_\alpha = 1$ if A is elected in α and a payoff of $t_\beta = -1$ if A is elected in β; payoffs are normalized to zero if B is elected. Hence, voters prefer the policy that matches the state they believe to be more likely.

When there are $2n + 1$ voters, equilibrium can be described by a cutoff signal s_n^* that makes the voters indifferent: all voters with a signal $s > s_n^*$ assign a higher probability to α than β and vote A and voters with a signal $s < s_n^*$ assign a higher probability to β than α and vote B. One can show that $s_n^* \to 0$ as $n \to \infty$, for all priors. Figure 6 illustrates the limit equilibrium. For the cutoff zero, a majority of citizens votes A in α (shaded area) and B in β. For general preference distribu-

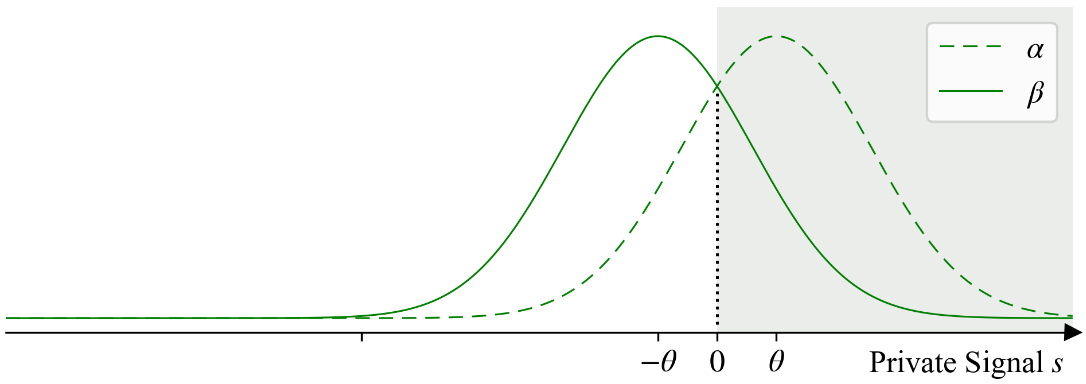

Figure 6: The density of the voters' private signals in states α and β. The shaded are shows the share of citizens voting A when n is large.

tions, in this benchmark, all equilibrium sequences aggregate information perfectly when preferences are monotone (that is, Φ is strictly increasing). This follows from arguments similar to the ones used for Theorem 1; see also Bhattacharya (2013).

[24]Gaussian information is a common way to model that voters have noisy perceptions of political statements or states; see, e.g., the literature on electoral competition (Matějka and Tabellini, 2016) or on media markets (Galperti and Trevino, 2018; Chen and Suen, 2019).

6.2 Media Slant

We show that a monopolistic media firm can persuade voters to elect any constant target policy by sometimes shifting the message towards one extreme ("slant").

With probability $1-\chi$, the media firm sends the same message as before, $m = \theta$ in α and $m = -\theta$ in β. With probability $\chi > 0$, the media firm sends a shifted message, $m = \theta - d$ in α and $m = -\theta - d$ in β, for some $d \neq 0$. One may think of the shift as resulting from journalists being incentivised by interested parties. Figure 7 illustrates the signal distribution when the shift is relatively large, $d > 2\theta$, so that $\theta - d < -\theta$. It shows the signal distribution in each state, when the shifted message is sent (substates α_2 and β_2) and when the normal message is sent (substates α_1 and β_1).

Again, for concreteness, suppose that voters have a common preference type $t = (t_\alpha, t_\beta) = (1, -1)$; so, voters prefer the policy that matches the state.

Take the target policy A and consider a slant $d > 2\theta$ as in Figure 7. One can show that for large n, there is an equilibrium in which voters use a cutoff strategy. Denoting the signal cutoff by s_n^*: all voters with a signal $s > s_n^*$ vote A and voters with a signal $s < s_n^*$ vote B. Further, one can show that $s_n^* \to -d$ as $n \to \infty$. The shaded area in Figure 7 illustrates the voting behavior in the limit. Critically, the median signal is to the right of $-d$ in all substates except β_2. Thus, a majority votes A in all these substates.

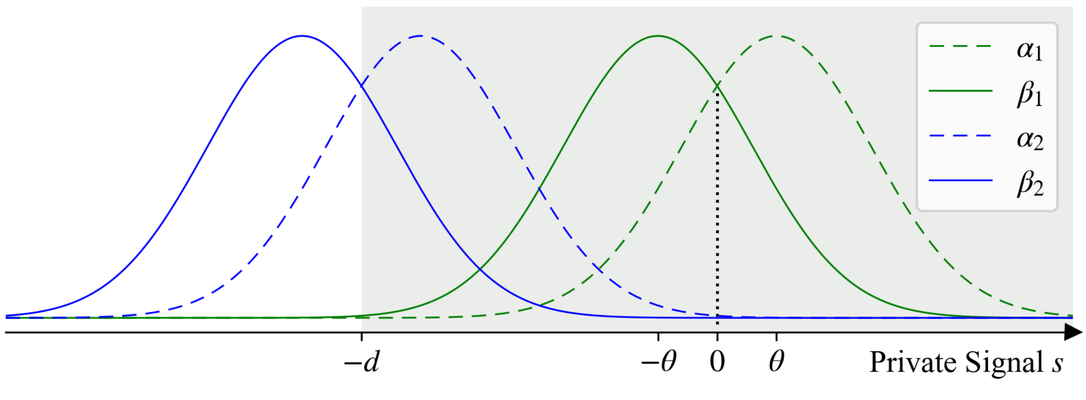

Figure 7: The density of the voters' private signals in the four substates when there is no slant (α_1 and β_1) and when there is a slant $-d$ (α_2,β_2).

The logic of the equilibrium is similar to the analysis in Section 5.2: For the cutoff $-d$, the election is closer to being tied in α_2 and β_2 than in α_1 and β_1. Hence, one can show that, conditional on being pivotal, voters believe that the state is in $\{\alpha_2, \beta_2\}$. Further, the margins of victory in α_2 and β_2 are asymptotically the same, that is, (59) holds.

In Appendix E, we consider the situation when voters have monotone preferences, that is, Φ is strictly increasing. Specifically, in Theorem 5, we show that for every target policy $x \in \{A, B\}$, d can be chosen such that there is an equilibrium sequence for which x is elected with a probability of at least $1 - \chi$ as $n \to \infty$.

6.3 Multiple Media Firms

We show that one media firm sending shifted messages can affect equilibrium outcomes even when voters receive additional information from an exogenous media firm. This means that manipulation of just a subset of the media landscape, for example, by incentivising some journalists to tilt their stories, may have an effect on elections.

Suppose that there is an exogenous "honest" media firm that sends a message as in the benchmark, $m_1 = \theta$ in α and $m_1 = -\theta$ in β. A second media firm sends in both states the same message, independently, with probability $1 - \chi$. However, it sends a shifted message with probability $\chi > 0$, that is, $m_2 = \theta - d$ in α and $m_2 = -\theta - d$ in β for some $d \neq 0$. Each voter receives two private signals, $s_1 = m_1 + \epsilon_1$ and $s_2 = m_2 + \epsilon_2$, where ϵ_1 and ϵ_2 are drawn independently from a standard normal distribution.

We show that in this setting, when preferences are monotone, for any target policy $x \in \{A, B\}$, the second media firm can chose d so that there is an equilibrium sequence where x is elected with probability larger $1-\chi$ as $n \to \infty$ (Theorem 6 in Appendix E).

7 Remarks and Extensions

Here, we collect some extensions and remarks for the setup with privately informed voters.

7.1 Partially Informed Sender

In the working paper version, Heese and Lauermann (2019), we consider a sender who does not know the state $\omega \in \{\alpha, \beta\}$.[25] Instead, the sender receives a private signal m. Conditional on the private signal m, the sender can release signals to the voters that are coarsenings of m.

[25] Available at https://ideas.repec.org/p/bon/boncrc/crctr224_2019_128.html.

Suppose that the sender's signal is binary, $m \in \{\ell, h\}$. Then, we show the following: If the sender is the monopolistic information provider (voters receive no private information), then the sender can implement any policy as a function of the own signal, i.e., for any $(x(\ell), x(h)) \in \{A, B\}^2$, the sender can ensure that a majority votes for $x(\ell)$ given the information released to voters after the own signal ℓ and for $x(h)$ after the own signal h. This is, in fact, implied by the analysis of the current paper. To see this, note that the sender's own signal m simply assumes the role of the state of nature ω in the current setting, and we can "integrate out" the state to rewrite the voters' preferences in terms of $\{\ell, h\}$.

However, when the voters have private information as well, the analysis is more subtle. Suppose voters observe an exogenous private signal π^c as in the CJT setting and the sender can release additional information in the form of a coarsening of the own noisy signal. For this case we show that, whenever the sender's own information is sufficiently precise relative to π^c, then again the sender can implement any policy as a function of the own signal, $(x(\ell), x(h)) \in \{A, B\}^2$; see Heese and Lauermann (2019, Theorem 7). For example, if the voters' signals $\{u, d\}$ are symmetric across states, then it is sufficient that the sender's own information is at least as informative as the joint signal of two voters (in the Blackwell sense).[26]

7.2 Known Preferences: Targeted Persuasion

When the types of the voters are known to a potential sender, voters can be "targeted" with recommendations; formally, a revelation principle applies saying that any equilibrium is equivalent to a recommendation policy that will be followed by the voters. Below, we show that when the preference types are known, there is a simple way in which the sender can persuade the voters to elect a constant policy via private recommendations.[27] We also show that, with known preferences, the possibility of persuasion is unaffected by the presence of a private signal of the voters.

Suppose that the voters' preference types $t^i = (t^i_\alpha, t^i_\beta)$ are commonly known for any $i \in \{1, \ldots, 2n+1\}$. The voters receive exogenous private signals as in the setting of the CJT (Section 3.2). The following result extends when these

[26] This is shown in Heese and Lauermann (2019, Remark 2). The key step in the proof is the observation that, when the sender's signal is sufficiently precise, then the sender can induce beliefs that are "implementable" in the sense of Lemma 3 from the current Online Appendix.

[27] This has been observed by Chan, Gupta, Li, and Wang (2019) and in Bardhi and Guo (2018) in similar settings. Therefore, the main parts of these papers consider settings with voting costs ("expressive voting") and unanimity, respectively.

exogenous signals are uninformative. Suppose that the voters $1, \ldots, m$ prefer A in α and B in β—that is $t^i_\alpha > 0$ and $t^i_\beta < 0$—and without loss let $m > n$. The remaining voters $m+1, \ldots, 2n+1$ prefer B in α and A in β, that is $t^i_\alpha < 0$ and $t^i_\beta > 0$.[28]

The following recommendation policy implements the outcome A with probability of at least $1-\epsilon$ in an equilibrium, for arbitrarily small $\epsilon > 0$: in both states, with probability $1-\epsilon$, all voters receive the recommendation "vote A" (signal a). In state α, with the remaining probability ϵ, a random subset of size $n+1$ of the voters $1, \ldots, m$ receives the recommendation "vote A" and the remaining n voters receive the recommendation "vote B" (signal b). In state β, with the remaining probability $\epsilon > 0$, a random subset of size $n+1$ of the voters $1, \ldots, m$ receives b and the remaining n voters receive a.

Voting A after an a-signal and B after a b-signal constitutes an equilibrium: Given this strategy, denoted by σ, voters $i \in \{1, \ldots, m\}$ with an a-signal are only pivotal in α, and voters $i \in \{1, \ldots, m\}$ with a b-signal are only pivotal in β—that is $\Pr(\alpha|\text{piv}, a, i \leq m; \sigma) = 1$ and $\Pr(\alpha|\text{piv}, b, i \leq m; \sigma) = 0$. Hence, voting A after a and B after b is a strict best response for any voter $i \in \{1, \ldots, m\}$. Voters $i \in \{m+1, \ldots, 2n+1\}$ are never pivotal if the other voters follow the recommendations. Hence, following the recommendation is a also best response for them, and, therefore, σ is an equilibrium. Since with probability $1-\epsilon$ all citizens vote A, given σ, the recommendation policy implements the outcome A with a probability of at least $1-\epsilon$.

Note how the signal structure above is finely tuned to the details of the setting. By way of contrast, we show that persuasion is effective even if information cannot be tailored to a specific preferences profile. In fact, we show that information does not even need to be tailored to the distribution of preferences. The mechanism driving persuasion is fundamentally different from the one described here. This difference may be most salient with exogenous private information where the equilibrium behavior of the voters adjusts endogenously to maintain the critical "equal-margin condition" across environments.

7.3 Bayes Correlated Equilibria

The Bayes correlated equilibria given some exogenous information structure π^c are the Bayes-Nash equilibria that arise from expansions π of π^c (see Bergemann and Morris (2016) for the definition of an expansion and the characterization of

[28]The example can be extended to include "partisans".

Bayes correlated equilibria). In terms of Bayes correlated equilibria, Theorem 4 means that for any state-dependent outcome function $(x(\alpha), x(\beta)) \in \{A, B\}^2$, there exists a sequence of Bayes correlated equilibria given π^c that leads to this outcome as $n \to \infty$.

8 Related Literature

Voter Persuasion Literature. The paper is related to work on information design in general (see Bergemann and Morris (2019) for a survey), especially with multiple receivers (e.g., Mathevet, Perego, and Taneva (2020)).

Previous work on persuasion in an election context has studied persuasion in settings in which the preferences of the voters are commonly known and voters have no access to exogenous private signals. The previous work has considered public signals by the sender (Alonso and Câmara, 2016), persuasion with conditionally independent private signals by the sender (Wang, 2013), and targeted persuasion with private signals by the sender (Bardhi and Guo, 2018; Chan, Gupta, Li, and Wang, 2019). We discussed persuasion when the preferences of the voters are known in Section 7.2, and we showed how the persuasion mechanism and its logic are quite different.

In contrast to the existing literature, we revisit the general voting setting of Feddersen and Pesendorfer (1997) with private preferences: In this setup, as a consequence of the Condorcet Jury Theorem, there is no scope for persuasion with public signals and also no scope for persuasion with conditionally independent private signals; see Theorem 1.

More generally, most of the Bayesian persuasion literature assumes that the sender has extensive knowledge of the environment; in particular, perfect knowledge about the state and receiver types is typically assumed.[29] In this paper, the informational requirements for persuasion are significantly weaker. We allow for private preferences and exogenous private signals of the receivers; we also consider the case in which the sender has incomplete information regarding the prior probabilities of the state, the distribution of the private preference types of the voters, or the distribution of the private signals of the voters (see Section 4.6.2 and Section 5.4). In the working paper version, Heese and Lauermann (2019), we consider the case in which the sender's information regarding the state is incomplete (see

[29]Exceptions are Guo and Shmaya (2019) and Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017), who study persuasion of a single, privately informed receiver.

Section 7.1). Further, we show that persuasion requires almost no commitment power.

Several other papers study how groups can be influenced through strategic information transmission, but are less closely related: For example, Kerman, Herings, and Karos (2020) study targeted persuasion via private signals when the sender is restricted to use signals that induce the voters to behave sincerely; compare to the discussion of targeted persuasion in Section 7.2. Levy, Moreno de Barreda, and Razin (2018) study persuasion of voters with correlation neglect. Schipper and Woo (2019) study the persuasion of voters who are unaware of certain features. Schnakenberg (2015) studies a cheap talk setting in which an expert tries to manipulate a voting body. Salcedo (2019) studies persuasion of subgroups of receivers via private messages in a setting in which each receiver's payoff depends only on his own action and the state.

More distantly related is work on the design of an elicitation mechanism to obtain information from multiple experts for an adversary to use (Gerardi, McLean, and Postlewaite, 2009; Feng and Wu, 2019).

Information Aggregation Literature. Voting theory has identified several circumstances in which information may fail to aggregate. We discuss the studies that are most closely related: Feddersen and Pesendorfer (1997) (Section 6) show that an invertibility problem causes a failure when there is aggregate uncertainty with respect to the preference distribution conditional on the state. We have already mentioned that Bhattacharya (2013) shows that information may fail to aggregate when preference monotonicity is violated.

In a pure common-values setting, Mandler (2012) shows that a failure can occur when there is aggregate signal uncertainty conditional on the state. There is a sense in which such aggregate uncertainty is necessary for a failure of information aggregation, in the sense that if there is a single substate, the CJT applies (Theorem 1 and the subsequent discussion). Here, as in his model, the voters' updating about the signal distribution of others conditional on a close election is important.[30] Note that the pure common-values assumption implies that this setting is a special case of a setting in which the individual voters' preference type is known (discussed in Section 7.2). In contrast to Mandler (2012), we consider a setting in which voters do not have common values; rather than perturbing that

[30] Uncertainty regarding the signal distribution and updating about it is also central in Acharya and Meirowitz (2017), in which aggregate uncertainty supports sincere voting. Other recent contributions on the conditions for information aggregation are Kosterina (2019) and Barelli, Bhattacharya, and Siga (2019).

original signal, we study the effect of an additional signal in the canonical setting by Feddersen and Pesendorfer (1997).

Additional related models, which show that elections perform poorly in aggregating information are include Razin (2003), Acharya (2016), Ekmekci and Lauermann (2019), Ali, Mihm, and Siga (2018) and Bhattacharya (2018).

9 Conclusion

In the canonical voting setting by Feddersen and Pesendorfer (1997), information aggregation may be upset by an interested sender who provides additional information to the voters. We have shown how an interested sender can exploit strategic voters by manipulating their inference from the election being close. In equilibrium, the closeness of the election tells voters something about the quality of the information of the other voters, and, because signals are correlated, also about the quality of their own signal. This way the sender can "steer" the meaning of signals and make voters elect even the inverse of the full-information outcome.

What is particularly striking about the result is its robustness; almost no commitment power of the sender is required. The resulting equilibrium is simple and selected by an iterated best response dynamic. The sender does not need precise knowledge of the environment ("detail-freeness"). In fact, the same information structure that implements a given policy in the monopolistic sender setting also implements the policy when voters have private information. Even a manipulator with very limited knowledge about the state itself can persuade a large electorate.

We discuss concrete information strategies within an application to media markets. For example, we show that a media firm can persuade voters by broadcasting news that is biased towards one extreme relative to the exogenous media.

Conceptually, our results also mean that equilibrium outcomes in the setting by Feddersen and Pesendorfer (1997) can be hard to predict for an outside observer without precise knowledge of the voters' information. The outside observer must be able to exclude the possibility that voters have access to additional information of the form discussed here.

Information aggregation has also been studied in (double-) auctions, a setting that shares some features with elections. An interesting question may be whether, in auctions, information aggregation is an "informationally robust" prediction or whether bidders having additional information can also upset it. Information design in auction settings has been studied by Bergemann, Brooks, and Morris

(2016), Du (2018), and Yamashita et al. (2016), among others, but mostly with a focus on revenue and efficiency.

The pivotal voting model considers the extreme case in which voters react perfectly to the closeness of the election when interpreting their information and illustrates the effectiveness of persuasion in this case. One may conjecture that, in a setting in which voters react less sensitively, persuasion is still effective but, presumably, less so. We provide some initial observations on this conjecture here.

Appendix

A Monopolistic Persuasion: Proof of Claim 3

Without loss of generality, suppose σ_n is such that $q(\alpha_2\,;\sigma_n)(1-q(\alpha_2\,;\sigma_n)) < q(\beta_2;\sigma_n)(1-q(\beta_2;\sigma_n))$ for all n. It follows directly from (7) that

$$\lim_{n\to\infty}\frac{\Pr(\text{piv}|\alpha_2;\sigma_n,\pi_n)}{\Pr(\text{piv}|\beta_2;\sigma_n,\pi_n)} \leq 1. \tag{65}$$

We now show that the reverse inequality also holds and thereby finish the proof of the lemma. For this, we show the following. There exists some $L>0$ and $M>0$ such that, for all n and all σ_n satisfying the ordering above,

$$\frac{\Pr(\text{piv}|\alpha_2;\sigma_n,\pi_n)}{\Pr(\text{piv}|\beta_2;\sigma_n,\pi_n)} \geq \left(1-\frac{L}{Mn^2}\right)^n. \tag{66}$$

First, it follows from (15) that the expected vote share for A in α_2 differs from the expected vote share for A in β_2 maximally by the probability that b is observed in α_2, that is, by $\varepsilon^2=\frac{1}{n^2}$; so,

$$|q(\alpha_2\,;\sigma_n)-q(\beta_2\,;\sigma_n)| \leq \epsilon^2, \tag{67}$$

for all n. Second, recall that $\Phi(0) < q(\omega_j;\sigma) < \Phi(1)$ for any strategy and any substate ω_j, and note that the derivative of $h(q)=q(1-q)$ is bounded by some $L>0$ on the compact interval $[\Phi(0),\Phi(1)]$. These observations taken together imply that

$$h(q(\beta_2\,;\sigma_n))\left|\frac{h(q(\alpha_2\,;\sigma_n))}{h(q(\beta_2\,;\sigma_n)}-1\right| = |h(q(\alpha_2\,;\sigma_n))-h(q(\beta_2\,;\sigma_n))| \leq L\epsilon^2. \tag{68}$$

for all n. Since $0 < \Phi(0) < q(\alpha_2\,;\sigma_n) < \Phi(1)$ and h is inverse U-shaped with maximum at $\frac{1}{2}$, this bound implies

$$\frac{h(q(\alpha_2\,;\sigma_n))}{h(q(\beta_2\,;\sigma_n)} \geq 1 - \frac{L}{h(q(\beta_2\,;\sigma_n))n^2} \geq 1 - \frac{L}{Mn^2} \tag{69}$$

for $M = \min\left(h(\Phi(0)), h(\Phi(1))\right)$ and all n. Now, (66) follows from (7).

Finally, since $\lim_{n\to\infty}(1 - \frac{L}{Mn^2})^n = 1$, (66) implies that

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2;\sigma_n,\pi_n)}{\Pr(\text{piv}|\beta_2;\sigma_n,\pi_n)} \geq 1. \tag{70}$$

To see why $\lim_{n\to\infty}(1-\frac{L}{Mn^2})^n = 1$, note that $\lim_{n\to\infty}(1-\frac{L}{Mn^2})^{2n} = \lim_{n\to\infty}(1-\frac{\sqrt{L}}{\sqrt{M}n})^{2n}(1+\frac{\sqrt{L}}{\sqrt{M}n})^{2n} = e^{2\sqrt{\frac{L}{M}}}e^{-2\sqrt{\frac{L}{M}}} = e^0 = 1$ where we used $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$. This finishes the proof of Claim 3.

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Online Appendix

B Proof of the Condorcet Jury Theorem

Step 1 *For all n and every equilibrium σ_n^*, the vote share of A is larger in α than in β,*

$$0 < q(\beta; \sigma_n^*, n) < q(\alpha; \sigma_n^*, n) < 1. \tag{71}$$

This ordering of the vote shares follows from the likelihood ratio ordering of the signals. In particular, recall the expression (8) for the posterior likelihood ratio of two states conditional on a given voter's signal s and the event that the voter is pivotal,

$$\frac{\Pr(\alpha|s, \text{piv}; \sigma_n^*, n)}{1 - \Pr(\alpha|s, \text{piv}; \sigma_n^*, n)} = \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\text{piv}|\alpha; \sigma_n^*, n)}{\Pr(\text{piv}|\beta; \sigma_n^*, n)} \frac{\Pr(s|\alpha; \pi^c)}{\Pr(s|\beta; \pi^c)}, \tag{72}$$

where $\Pr(\text{piv}|\beta; \sigma_n^*, n) > 0$ because σ_n^* is nondegenerate by (2). Therefore, $\frac{\Pr(u|\alpha;\pi^c)}{\Pr(u|\beta;\pi^c)} > \frac{\Pr(d|\alpha;\pi^c)}{\Pr(d|\beta;\pi^c)}$ implies that $\Pr(\alpha|u, \text{piv}; \sigma_n^*, n) > \Pr(\alpha|d, \text{piv}; \sigma_n^*, n)$. Now, (71) follows from (15) and the monotonicity of Φ. Intuitively, the expected posterior in state α is higher and this translates into a larger set of types preferring A given the monotonicity of Φ.

Step 2 *Voters cannot become certain of the state conditional on being pivotal, that is, the inference from the pivotal event must remain bounded,*

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha; \sigma_n^*, n)}{\Pr(\text{piv}|\beta; \sigma_n^*, n)} \in (0, \infty), \tag{73}$$

for every convergent subsequence in the extended reals.

Suppose not and suppose instead, for example, that conditional on being pivotal, voters become convinced that the state is β, i.e., $\eta = \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha;\sigma_n^*,n)}{\Pr(\text{piv}|\beta;\sigma_n^*,n)} = 0$. This would imply $\lim_{n\to\infty} \Pr(\alpha|s, \text{piv}; \sigma_n^*, n) = 0$ for $s \in \{u, d\}$. Then, given $\Phi(0) < \frac{1}{2}$, a strict majority would support B in both states. However, the election is then closer to being tied in state α and voters would update toward state α conditional on being pivotal, in contradiction to $\eta = 0$.

Formally, if $\eta = 0$ for some converging subsequence, then $\lim_{n\to\infty} q(\omega; \sigma_n^*) = \Phi(0) < \frac{1}{2}$ for $\omega \in \{\alpha, \beta\}$. Therefore, for large enough n, (71) implies that $q(\beta; \sigma_n^*) < q(\alpha; \sigma_n^*) < 1/2$. Now, Claim 1 implies that voters update toward state α, that is, $\frac{\Pr(\text{piv}|\alpha;\sigma_n^*,n)}{\Pr(\text{piv}|\beta;\sigma_n^*,n)} \geq 1$, in contradiction to $\eta = 0$.

Step 3 *In every equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$, the limit of the vote share of A is larger in α than in β,*

$$\lim_{n\to\infty} q(\alpha;\sigma_n^*) > \lim_{n\to\infty} q(\beta;\sigma_n^*). \tag{74}$$

From (73) and (72), we have that the limits of the posteriors conditional on being pivotal and $s \in \{u, d\}$ are interior and hence ordered,

$$0 < \lim_{n\to\infty} \Pr(\alpha|d, \text{piv}; \sigma_n^*, n) < \lim_{n\to\infty} \Pr(\alpha|u, \text{piv}; \sigma_n^*, n) < 1.$$

Now, (74) follows from (15) since Φ is strictly increasing.

Step 4 *The election is equally close to being tied in expectation, that is, (22) holds:*

$$\lim_{n\to\infty} q(\alpha;\sigma_n^*) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - q(\beta;\sigma_n^*).$$

Since voters must not become certain conditional on being pivotal by (73), Claim 2 requires that

$$\lim_{n\to\infty} \left| q(\alpha;\sigma_n^*) - \frac{1}{2} \right| = \lim_{n\to\infty} \left| q(\beta;\sigma_n^*) - \frac{1}{2} \right|. \tag{75}$$

Given the ordering of the limits of the vote shares from (74), the equation (75) implies (22).

It follows from Step 4 and (74) that

$$\lim_{n\to\infty} q(\alpha;\sigma_n^*) > \frac{1}{2} > \lim_{n\to\infty} q(\beta;\sigma_n^*).$$

Therefore, by the weak law of large numbers, A wins in state α with probability converging to 1 as $n \to \infty$ and B wins in state β with probability converging to 1 as $n \to \infty$. This proves Theorem 1.

Sketch of the proof of Theorem 1'. To see why the theorem is true, note that, given the binary state, the signals can be taken to be ordered by the monotone likelihood ratio, without loss of generality. For any fixed information structure π and any equilibrium σ_n^*, it then follows from (72) that the distribution of posteriors $\Pr(\alpha|\text{piv}, s; \sigma_n^*, \pi, n)$ in the state α (as implied by the distribution over s) first order stochastically dominates the distribution of posteriors $\Pr(\alpha|\text{piv}, s; \sigma_n^*, \pi, n)$ in the state β. Then, given that Φ is monotone, it follows from (15) that the vote shares satisfy the ordering (71). From (71) onward none of the arguments use that the signals are binary.

By the same line of argument, Theorem 1 holds even when we allow the information structure π_n with a single substate to vary with n (keeping the signal set S fixed), as long as the limit information structure is not completely uninformative.

C Monopolistic Persuasion

C.1 Numerical Example

Note that one example of a distribution G on $[0,1]\times[-1,0]$ that induces a uniform distribution of 'thresholds of doubt', i.e. Φ with $\Phi(p)=p$ for all $p\in[0,1]$ is given by the density[31]

$$g(t_\alpha,t_\beta)=\begin{cases}\sqrt{1+(\frac{t_\beta}{t_\alpha})^2}^{-1}\cdot(2\cdot\int_{|t_\alpha|>|t_\beta|}\sqrt{1+(\frac{t_\beta}{t_\alpha})^2}^{-1}dt)^{-1} & \text{if} \quad \frac{-t_\beta}{t_\alpha-t_\beta}\leq\frac{1}{2},\\ \sqrt{1+(\frac{t_\alpha}{t_\beta})^2}^{-1}\cdot(2\cdot\int_{|t_\alpha|>|t_\beta|}\sqrt{1+(\frac{t_\beta}{t_\alpha})^2}^{-1}dt)^{-1} & \text{if} \quad \frac{-t_\beta}{t_\alpha-t_\beta}\geq\frac{1}{2}.\end{cases}$$

We utilize the following auxiliary result.

Lemma 2 *Consider any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ and any sequence of information structures $(\pi_n)_{n\in\mathbb{N}}$ with a common set of substates across n. Then, for any substates $\omega_i,\omega'_j\in\{\alpha_1,\ldots,\alpha_{N_\alpha}\}\cup\{\beta_1,\ldots,\beta_{N_\beta}\}$ and any $n\in\mathbb{N}$,*

$$\frac{\Pr(\text{piv}|\omega_i;\sigma_n,\pi_n)}{\Pr(\text{piv}|\omega'_j;\sigma_n,\pi_n)}=\Big[1+\frac{(q(\omega'_j;\sigma^{\mathbf{P}})-\frac{1}{2})^2-(q(\omega_i;\sigma^{\mathbf{P}})-\frac{1}{2})^2}{\frac{1}{4}-(q(\omega'_j;\sigma^{\mathbf{P}})-\frac{1}{2})^2}\Big]^n \tag{76}$$

Proof. Let $x_n=q(\omega_i;\sigma_n)-\frac{1}{2}$ and $y_n=q(\omega'_j;\sigma_n)-\frac{1}{2}$. Then,

$$\begin{aligned}\frac{q(\omega_i;\sigma_n)(1-q(\omega_i;\sigma_n))}{q(\omega'_j;\sigma_n)(1-q(\omega'_j;\sigma_n))} &= \frac{(\frac{1}{2}+x_n)(\frac{1}{2}-x_n)}{(\frac{1}{2}+y_n)(\frac{1}{2}-y_n)}\\ &= \frac{\frac{1}{4}-y_n^2+y_n^2-x_n^2}{\frac{1}{4}-y_n^2}\\ &= 1+\frac{y_n^2-x_n^2}{\frac{1}{4}-y_n^2}\end{aligned}$$

The claim follows from (8). ■

Fixed Point Argument.

[31] To see why, note that for each $t>>0$, $d(t)=\Big[\sqrt{1+(\frac{t_\alpha}{t_\beta})^2}\Big]$ is the length of the indifference plane of t. By setting the density of types proportional to $\frac{1}{d(t)}$, integrating over each indifference plane gives the same number such that types are uniformly distributed across indifference planes.

Consider a belief $\mathbf{p} = (p_a, p_z)$ with

$$p_a \geq 0.95, \tag{77}$$
$$p_z \in [0.32, 0.68]. \tag{78}$$

Consider the information structure $\tilde{\pi}_n^r$ from Figure 4. Then, we have the following bounds for $n \geq 7$:

$$\begin{aligned} q(\omega_1; \sigma^{\mathbf{P}}, n) &\geq 0.95 \quad \text{for} \quad \omega_1 \in \{\alpha_1, \beta_1\}, & (79)\\ q(\alpha_2; \sigma^{\mathbf{P}}, n) &> 0.3 & (80)\\ q(\beta_2; \sigma^{\mathbf{P}}, n) &\leq 0.7. & (81) \end{aligned}$$

In the following, we omit the dependence on $\sigma^{\mathbf{P}}$ and on π_n most of the time.

Step 1 *For any $n \in \mathbb{N}$ and any $\omega_1 \in \{\alpha_1, \beta_1\}, \omega_2' \in \{\alpha_2, \beta_2\}$,*

$$\frac{\Pr(\text{piv}|\omega_2')}{\Pr(\text{piv}|\omega_1)} \geq (4.4)^n \tag{82}$$

Indeed,

$$\begin{aligned} & \frac{\Pr(\text{piv}|\omega_2')}{\Pr(\text{piv}|\omega_1)} \\ \geq\ & [1 + \min_{\omega_1, \omega_2'} \frac{(q(\omega_1; \sigma^{\mathbf{P}}) - \frac{1}{2})^2 - (q(\omega_2'; \sigma^{\mathbf{P}}) - \frac{1}{2})^2}{\frac{1}{4} - (q(\omega_1; \sigma^{\mathbf{P}}) - \frac{1}{2})^2}]^n \\ \geq\ & (1 + (\frac{(\frac{9}{20})^2 - (\frac{4}{20})^2}{\frac{1}{4} - (\frac{9}{20})^2})) \\ \geq\ & (1 + \frac{65}{19})^n \\ \geq\ & (4.42)^n. \end{aligned} \tag{83}$$

where we used Lemma 2 for the inequality on the second line.

Step 2 *For $n \geq 7$: $\rho_a(\sigma^{\mathbf{P}}) \geq 0.95$, and $\rho_z(\sigma^{\mathbf{P}}) \in [0.32, 0.68]$.*

First,

$$\begin{aligned} \frac{\rho_a(\sigma^{\mathbf{P}})}{1 - \rho_a(\sigma^{\mathbf{P}})} &\geq \frac{p_0}{1 - p_0} \frac{\Pr(\alpha_2|\alpha) \Pr(b|\alpha_2) \Pr(\text{piv}|\alpha_2)}{\Pr(\beta_1|\beta) \Pr(b|\beta_1) \Pr(\text{piv}|\beta_1)} \\ &\geq \frac{1}{2} \frac{\frac{2}{n} \frac{1}{n^2}}{(1 - \frac{1}{n})} (4.42)^n \\ &\geq 82 \quad \text{for} \quad n \geq 7. \end{aligned}$$

where we used (83) for the inequality on the second line. Hence, for $n \geq 7$,

$$\rho(\sigma^{\mathbf{P}})_a \geq \frac{82}{83} > 0.95. \tag{84}$$

Second,

$$\begin{aligned}
\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} &\leq [1 + \frac{|(q(\beta_2; \sigma^{\mathbf{P}}) - \frac{1}{2})^2 - (q(\alpha_2; \sigma^{\mathbf{P}}) - \frac{1}{2})^2|}{\frac{1}{4} - (q(\beta_2; \sigma^{\mathbf{P}}) - \frac{1}{2})^2}]^n \\
&\leq (1 + \frac{\frac{1}{n^4} + \frac{1}{n^2}}{\frac{1}{4} - \frac{16}{400}})^n \\
&\leq 2. \quad \text{for} \quad n \geq 7.
\end{aligned}$$

where we used Lemma 2 for the inequality on the first line. For the inequality on the second line, we used that z is sent with probability $1 - \frac{1}{n^2}$ in both α_2 and β_2 such that the difference in the squared margins of victory cannot exceed $(x + \frac{1}{n^2})^2 - x^2 \leq \frac{2x}{n^2} + \frac{1}{n^4}$ where x is the minimum margin of victory in the states α_2, β_2. Finally, the inequality follows since the margin of victory in both α_2 and β_2 is bounded by 0.2. So,

$$\begin{aligned}
\frac{\rho_z(\sigma^{\mathbf{P}})_z}{1 - \rho_z(\sigma^{\mathbf{P}})} &= \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_2|\beta)} \frac{\Pr(z|\alpha_2)}{\Pr(z|\beta_2)} \frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \\
&= (1 - \frac{1}{n^2}) \frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \\
&\leq 2 \quad \text{for} \quad n \geq 7.
\end{aligned}$$

Consequently, for all $n \geq 7$,

$$\rho(\sigma^{\mathbf{P}})_z \quad \leq \quad \frac{2}{3}. \tag{85}$$

Third,

$$\begin{aligned}
\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} &\geq (1 - \frac{|(q(\beta_2; \sigma^{\mathbf{P}}) - \frac{1}{2})^2 - (q(\alpha_2; \sigma^{\mathbf{P}}) - \frac{1}{2})^2|}{\frac{1}{4} - (q(\beta_2; \sigma^{\mathbf{P}}) - \frac{1}{2})^2} \\
&\geq (1 - \frac{\frac{1}{n^4} + \frac{1}{n^2}}{\frac{1}{4} - \frac{16}{400}})^n \\
&\geq 0.48 \quad \text{for} \quad n \geq 7.
\end{aligned} \tag{86}$$

So, for all $n \geq 7$,

$$\begin{aligned} \frac{\rho(\sigma^{\mathbf{p}})_z}{1-\rho(\sigma^{\mathbf{p}})_z} &= (1-\frac{1}{n^2})\frac{\Pr(\text{piv}|\alpha_2;\sigma^{\mathbf{p}})}{\Pr(\text{piv}|\beta_2;\sigma^{\mathbf{p}})} \\ &\geq 0.471. \end{aligned}$$

This gives for all $n \geq 7$,

$$\rho(\sigma^{\mathbf{p}})_z \geq \frac{0.471}{1+0.471} \geq 0.32. \tag{87}$$

The claim follows from (84), (85)), and (87).

Step 3 *For $n \geq 7$, there is an equilibrium σ_n^* which satisfies (79) - (81).*

It follows from Step 2 that, for any $n \geq 7$, the continuous map that sends $\mathbf{p}$ to $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is a self-map on the set of beliefs that satisfy (77) - (78). It follows from the Kakutani fixed point theorem that there exists fixed points $\mathbf{p}_n^*$ that satisfy (77) - (78). The corresponding strategies $\sigma^{\mathbf{p}_n^*}$ are equilibria (compare to (13)) and they satisfy (79) - (81).

Step 4 *Given the equilibrium σ_n^* for $n \geq 7$, the probability that A is elected is larger than* 80%.

Evaluation of the binomial distribution shows that $\Pr(\mathcal{B}(2n+1,x)) > n) \geq 0.999999$ if $n \geq 7$ and $x \geq 0.95$. Hence, given σ_n^*, A is elected with probability larger than 99.9% in the states α_1 and β_1. Finally, the claim follows since the probability of the substate being α_1 or β_1 is $\Pr(\alpha)(1-\frac{r}{1-r}\frac{1}{n})+\Pr(\beta)(1-\frac{1}{n})$, which is larger than 0.8 when $n \geq 7$, given that $\Pr(\alpha) = \frac{1}{3}$ and $r = \frac{2}{3}$. The fourth step finishes the calculations for the example.

C.2 Proof of Lemma 1

C.2.1 Preliminaries: Voter Inference

The basic arguments of the previous discussion of the voters' inference from Section 4.2.2 extend to the general case.

Consider the signal z and the inference about the relative likelihood of α_2 and β_2. As in Claim 3, for *any* strategy used by the other voters, the pivotal event contains no information about the relative probability of α_2 and β_2 as the electorate grows large.

Claim 7 *Given any parameters $(x, r, y) \in [0, 1]^3$ and any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$,*

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2; \sigma_n, \pi_n^{x,r,y})}{\Pr(\text{piv}|\beta_2; \sigma_n, \pi_n^{x,r,y})} = 1. \tag{88}$$

The arguments from the proof of the analogous Claim 3 hold verbatim with the required changes in notation; therefore, the proof is omitted. Claim 7 and (42) imply, in particular, that

$$\lim_{n\to\infty} \frac{\Pr(\alpha|z, \text{piv}; \sigma_n, \pi_n^{x,r,y})}{\Pr(\beta|z, \text{piv}; \sigma_n, \pi_n^{x,r,y})} = \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r}. \tag{89}$$

Next, we consider a signal $s \in \{a, b\}$ and the voters' inference about the relative likelihood of α and β. We show that, analogous to Claim 4, for this signal, the inference from the signal is dominated by the inference from being pivotal if the election is closer to being tied in states α_2 and β_2 than in the states α_1 and β_1.

Claim 8 *Take any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ such that*

$$\begin{aligned} & \lim_{n\to\infty} \max_{\omega_2\in\{\alpha_2,\beta_2\}} |q(\sigma_n; \omega_2, \pi_n^{x,r,y}) - \frac{1}{2}| \\ < & \lim_{n\to\infty} \min_{\omega_1\in\{\alpha_1,\beta_1\}} |q(\sigma_n; \omega_1, \pi_n^{x,r,y}) - \frac{1}{2}|; \end{aligned} \tag{90}$$

then, for $s \in \{a, b\}$,

$$\lim_{n\to\infty} \frac{\Pr(\{\alpha_2, \beta_2\}|s, \text{piv}; \sigma_n, \pi_n^{x,r,y})}{\Pr(\{\alpha_1, \beta_1\}|s, \text{piv}; \sigma_n, \pi_n^{x,r,y})} = \infty. \tag{91}$$

The claim follows from the same arguments as Claim 4, and we omit this proof as well.

For any sequence of strategies that satisfies (90), Claims 7 and 8 imply that, for signal a,[32]

$$\begin{aligned} \lim_{n\to\infty} \frac{\Pr(\alpha|a, \text{piv}; \sigma_n, \pi_n^{x,r,y})}{\Pr(\beta|a, \text{piv}; \sigma_n, \pi_n^{x,r,y})} &= \frac{\Pr(\alpha_2|\{\alpha_2, \beta_2\}, a; \sigma_n, \pi_n^{x,r,y})}{\Pr(\beta_2|\{\alpha_2, \beta_2\}, a; \sigma_n, \pi_n^{x,r,y})} \\ &= \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{r}{1-r} \frac{x}{1-x} \end{aligned} \tag{92}$$

[32]Recall the convention $\frac{1}{0} = \infty$, such that, for $x = 1$, the following equalities hold in the extended reals.

and that for signal b,

$$\lim_{n\to\infty}\frac{\Pr(\alpha|b,\text{piv};\sigma_n,\pi_n^{x,r,y})}{\Pr(\beta|b,\text{piv};\sigma_n,\pi_n^{x,r,y})} = \frac{\Pr(\alpha_2|\{\alpha_2,\beta_2\},b;\sigma_n,\pi_n^{x,r,y})}{\Pr(\beta_2|\{\alpha_2,\beta_2\},b;\sigma_n,\pi_n^{x,r,y})} = \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{y}{1-y}. \tag{93}$$

C.2.2 Implementable Beliefs

We use that an equilibrium is equivalently characterized by a vector of beliefs, $\mathbf{p}^* = (p_a^*, p_z^*, p_b^*)$ such that $\mathbf{p}^* = \boldsymbol{\rho}\left(\sigma^{\mathbf{p}^*}\right)$; see (13). Take any $\delta > 0$ and let

$$\mathrm{B}_\delta = \left\{\mathbf{p} \in [0,1]^3 \,|\, |\mathbf{p}- (\mu_\alpha, r', \mu_\beta)| \le \delta\right\}, \tag{94}$$

so that B_δ is the set of beliefs at most δ away from $(\mu_\alpha, r', \mu_\beta)$.

We show that Claim 7 and 8 imply that there is a large set of belief triples $(\mu_\alpha, r', \mu_\beta)$ such that, given $\sigma^{\mathbf{p}}$, the posterior conditional on being pivotal is again in B_δ, for any $\mathbf{p} \in \mathrm{B}_\delta$, any sufficiently small δ and any sufficiently large n.[33]

Claim 9 *Let $(\mu_\alpha, \mu_\beta) \in [0,1]^2$ and $r' \in (0,1)$ with*

$$|\Phi(\mu_\alpha) - \frac{1}{2}| > |\Phi(r') - \frac{1}{2}| \;\; \text{and} \;\; |\Phi(\mu_\beta) - \frac{1}{2}| > |\Phi(r') - \frac{1}{2}|. \tag{95}$$

For any $\delta > 0$ small enough, there exists $n(\delta)$ such that for all $n \ge n(\delta)$,

$$\forall \mathbf{p} \in B_\delta : \boldsymbol{\rho}\left(\sigma^{\mathbf{p}}; \pi_n^{x,r,y}, n\right) \in B_\delta \tag{96}$$

for (x,r,y) being the solutions to $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{x}{1-x} = \frac{\mu_\alpha}{1-\mu_\alpha}$, $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{y}{1-y} = \frac{\mu_\beta}{\mu_\beta}$, and $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r} = \frac{r'}{1-r'}$.

Proof. Let $\pi_n = \pi_n^{x,r,y}$. Take any $\mathbf{p} \in B_\delta$ and consider the corresponding strategy $\sigma^{\mathbf{p}}$. The condition (95) implies that for δ small enough, the election is closer to being tied in the states α_2 and β_2 than in the states α_1 and β_1 in expectation as $n \to \infty$:

$$\forall \mathbf{p} \in \mathrm{B}_\delta: \quad \lim_{n\to\infty}\max_{\omega_2\in\{\alpha_2,\beta_2\}}|q(\omega_2;\sigma^{\mathbf{p}},\pi_n) - \frac{1}{2}| < \lim_{n\to\infty}\min_{\omega_1\in\{\alpha_1,\beta_1\}}|q(\omega_1;\sigma^{\mathbf{p}},\pi_n) - \frac{1}{2}|. \tag{97}$$

[33] In the following, we use the convention that dividing by zero yields a result of infinity such that formulas like $\frac{\Pr(\alpha)}{\Pr(\beta)}\frac{r}{1-r}\frac{x}{1-x} = \frac{\mu_\alpha}{1-\mu_\alpha}$ make sense for $\mu_\alpha \in \{0,1\}$.

To see why, note that for n large enough, $q(\alpha_2; \sigma^{\mathbf{p}}, \pi_n) \approx \Phi(p_z)$ and $q(\beta_2; \sigma^{\mathbf{p}}, \pi_n) \approx \Phi(p_z)$ since almost all voters receive z in α_2 and β_2. Also, $q(\alpha_1; \sigma^{\mathbf{p}}, \pi_n) = \Phi(p_a)$ since all voters receive a in α_1 and $q(\beta_1; \sigma^{\mathbf{p}}, \pi_n) = \Phi(p_b)$ since all voters receive b in β_1. In addition, by the continuity of Φ, for δ small enough, we have that $\Phi(p_z) \approx \Phi(r')$, $\Phi(p_a) \approx \Phi(\mu_\alpha)$ and $\Phi(p_b) \approx \Phi(\mu_\beta)$. Finally, (97) follows then from $\Phi(\hat{r}) = \frac{1}{2}$ and $\Phi(\mu_\omega) \neq \frac{1}{2}$ for $\omega \in \{\alpha, \beta\}$. Now, it follows from (97), Claim 8, and its implications (92) and (93) that

$$\lim_{n\to\infty} \boldsymbol{\rho}_a(\sigma^{\mathbf{p}}; \pi_n, n) = \mu_\alpha, \tag{98}$$
$$\lim_{n\to\infty} \boldsymbol{\rho}_b(\sigma^{\mathbf{p}}; \pi_n, n) = \mu_\beta. \tag{99}$$

for any $\delta > 0$ small enough. Thus, the claim follows from (89), (98) and (99). ∎

We finish the proof of Lemma 1. Let $r = \frac{\Pr(\alpha)\hat{r}}{\Pr(\alpha)\hat{r}+(1-\Pr(\alpha))(1-\hat{r})}$ with $\Phi(\hat{r}) = \frac{1}{2}$; see (40). Take any (μ_α, μ_β) with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\mu_\beta) \neq \frac{1}{2}$. Then, given Claim 9, $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is a self-map on B_δ for δ small enough and $n \geq n(\delta)$. Since $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is continuous in $\mathbf{p}$, it follows from Kakutani's theorem that there exists a fixed point $\mathbf{p}_n^* \in B_\delta$ for all n large enough, i.e., $\mathbf{p}_n^* = \boldsymbol{\rho}(\sigma^{\mathbf{p}_n^*})$ and the corresponding behavior $\sigma^{\mathbf{p}_n^*}$ forms a sequence of equilibria. Lemma 1 follows from (98) and (99).

C.3 Proof of Proposition 2

We provide the proof for the constant target policy A in both states, i.e., $(x(\alpha), x(\beta)) = (A, A)$. Let the sender use the information structures $\pi_n = \pi_n^{x,r,y}$ with $x = y = 1$ and $r = \frac{1}{2}$. It follows from Claim 9 that, for any Φ for which (49) and (50) hold, there is a δ small enough such that $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is a self-map on $B_\delta = \{\mathbf{p} \in [0,1]^3 : |\mathbf{p} - (1, \Pr(\alpha), 1)| \leq \delta\}$ for all n large enough.
Since $\boldsymbol{\rho}(\sigma^{\mathbf{p}})$ is continuous in $\mathbf{p}$, it follows from Kakutani's theorem that there exists a fixed point $\mathbf{p}_n^* \in B_\delta$ for all n large enough, i.e., $\mathbf{p}_n^* = \boldsymbol{\rho}(\sigma^{\mathbf{p}_n^*})$ and the corresponding behavior $\sigma^{\mathbf{p}_n^*}$ forms a sequence of equilibria that implements the beliefs $(\mu_\alpha, \mu_\beta) = (1, 1)$. Given $(\sigma^{\mathbf{p}_n^*})_{n\in\mathbb{N}}$, the policy A wins in both states; this follows since voters with an a and b-signal are supporting A with a probability converging to $\Phi(1) > \frac{1}{2}$ and from the weak law of large numbers. The other cases are analogous. This finishes the proof of the lemma.

C.4 Proof of Proposition 3 (Basin of Attraction)

Recall that for any strategy σ, the distance between the margin of victory in α_2 and β_2 is smaller than $\frac{2}{n^2}$ in expectation since the probability that a voter receives the signal z is at least $1-\frac{2}{n^2}$ in both the substates. Now, consider any belief $\mathbf{p} \in [0,1]^3$ such that under the corresponding strategy $\sigma^{\mathbf{p}}$ the margins of victory differ by at least $\delta > 0$ for any other pair of substates. The theorem follows from the following claim: we show that for any such belief $\mathbf{p}$, the twice-iterated response is δ-close to the manipulated equilibrium when n is large enough.

Claim 10 *Take any beliefs $(\mu_\alpha, \mu_\beta) \in [0,1]^2$ with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\mu_\beta) \neq \frac{1}{2}$ and the corresponding information structures $(\pi_n^{x,r,y})$ from Lemma 1.*

For any $\delta > 0$, there exists $\bar{n} \in \mathbb{N}$ s.t., for any $\mathbf{p} \in [0,1]^3$ for which

$$\left| |q(\omega_i, \sigma^{\boldsymbol{p}}, \pi_n) - \frac{1}{2}| - |q(\omega_j', \sigma^{\mathbf{p}}, \pi_n) - \frac{1}{2}| \right| > \delta, \tag{100}$$

for all $\omega_i \in \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ and $\omega_j' \in \{\alpha_1, \beta_1\}$ with $\omega_i \neq \omega_j'$, it holds that, for $n \geq \bar{n}$,

$$|\boldsymbol{\rho}^2(\mathbf{p}) - (\mu_\alpha, \hat{r}, \mu_\beta)| < \delta. \tag{101}$$

The claim implies Proposition 3 because δ can be chosen arbitrarily small.

Proof. Take any $\mathbf{p} \in [0,1]^3$ such that (100) holds and consider the corresponding behavior $\sigma^{\mathbf{p}}$. Denote the best response to $\sigma^{\mathbf{p}}$ by $\tilde{\sigma} = \sigma^{\boldsymbol{\rho}(\sigma^{\mathbf{p}};\pi_n,n)}$ and let $\pi_n = \pi_n^{x,\hat{r},y}$ with $x = \mu_\alpha$ and $y = \mu_\beta$. The critical step is to show that $\tilde{\sigma}$ satisfies (90), i.e., the expected margin of victory in the states α_1 and β_1 is larger than in the states α_2 and β_2. We show one part of (90), namely,

$$\lim_{n\to\infty} \max_{\omega_2 \in \{\alpha_2, \beta_2\}} |q(\tilde{\sigma}; \omega_2, \pi_n) - \frac{1}{2}| < \lim_{n\to\infty} |q(\tilde{\sigma}; \alpha_1, \pi_n) - \frac{1}{2}|. \tag{102}$$

The proof for the second part, the analogous statement where we replace α_1 by β_1, is verbatim with the required changes in notation. To prove (102), we distinguish two cases.

Case 1 $\lim_{n\to\infty} |q(\sigma^{\mathbf{p}}; \omega_2, \pi_n) - \frac{1}{2}| < \lim_{n\to\infty} |q(\sigma^{\mathbf{p}}; \alpha_1, \pi_n) - \frac{1}{2}|$.

Given (100), the difference is at least δ. Since almost all voters receive signal z in α_2 and β_2, the expected vote shares in α_2 and β_2 differ by much less than $\frac{\delta}{2}$ for n large enough. So, the expected margin of victory in α_1 is larger than the expected margin of victory in both α_2 and β_2 for n large enough. It follows from Claim 2 that for any $\omega_2 \in \{\alpha_2, \beta_2\}$ for which $\Pr(a|\omega_2; \pi_n, n) > 0$,

$$\lim_{n\to\infty} \frac{\Pr\left(\omega_2|\text{piv}, a; \sigma^{\mathbf{P}}, \pi_n\right)}{\Pr\left(\alpha_1|\text{piv}, a; \sigma^{\mathbf{P}}, \pi_n\right)} = \infty. \tag{103}$$

Since all voters receive a in α_1, it holds $q(\alpha_1; \tilde{\sigma}, \pi_n) = \Phi(\boldsymbol{\rho}_a(\sigma^{\mathbf{P}}))$. Since almost all voters receive z in α_2 and β_2 (see Figure 5), it holds $q(\alpha_2; \tilde{\sigma}, \pi_n) \approx \Phi(\boldsymbol{\rho}_z(\sigma^{\mathbf{P}}))$ and $q(\beta_2; \tilde{\sigma}, \pi_n) \approx \Phi(\boldsymbol{\rho}_z(\sigma^{\mathbf{P}}))$. It follows from (103) and Claim 7, which says that conditional on α_2 and β_2, there is nothing to be learned from the pivotal event, that, when a voter observes signal a, the inference from the signal probabilities in the states α_2 and β_2 pins down the limits of the beliefs conditional on being pivotal,

$$\begin{aligned}\lim_{n\to\infty} \Pr(\alpha|a, \text{piv}; \sigma^{\mathbf{P}}, \pi_n) &= \lim_{n\to\infty} \Pr(\alpha|a, \{\alpha_2, \beta_2\}; \sigma^{\mathbf{P}}, \pi_n) \\ &= \mu_\alpha; \end{aligned} \tag{104}$$

compare to (92). Finally, (102) follows from (104) and (89) together with $\Phi(\mu_\alpha) \neq \frac{1}{2}$ and $\Phi(\hat{r}) = \frac{1}{2}$. This finishes the first case.

Case 2 $\lim_{n\to\infty} |q(\sigma^{\mathbf{P}}; \omega_2, \pi_n) - \frac{1}{2}| > \lim_{n\to\infty} |q(\sigma^{\mathbf{P}}; \alpha_1, \pi_n) - \frac{1}{2}|$

Given (100), the difference is at least δ. Since almost all voters receive signal z in α_2 and β_2 (see Figure 5), the expected vote shares in α_2 and β_2 differ by much less than $\frac{\delta}{2}$ for n large enough. So, the expected margin of victory in α_1 is smaller than the expected margin of victory in both α_2 and β_2 for n large enough. It follows from Claim 2 that for $\omega_2 \in \{\alpha_2, \beta_2\}$,

$$\lim_{n\to\infty} \frac{\Pr\left(\text{piv}|\alpha_1; \sigma^{\mathbf{P}}, \pi_n\right)}{\Pr\left(\text{piv}|\omega_2; \sigma^{\mathbf{P}}, \pi_n\right)} = \infty. \tag{105}$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty} \frac{\rho_a(\sigma^{\mathbf{P}}; \pi_n, n)}{1 - \rho_a(\sigma^{\mathbf{P}}; \pi_n, n)} \\ \geq\ &\lim_{n\to\infty} \frac{\Pr(\alpha)\Pr(\alpha_1|\alpha)\Pr(a|\alpha_1)\Pr\left(\text{piv}|\alpha_1; \sigma^{\mathbf{P}}, \pi_n\right)}{\sum_{j=1,2}\Pr(\beta)\Pr(\beta_j|\beta)\Pr(a|\beta_j)\Pr(\text{piv}|\beta_j, a; \sigma^{\mathbf{P}}, \pi_n)}, \\ =\ &\frac{\Pr(\alpha)}{\Pr(\beta)} \frac{(1-\frac{r}{n^2})}{(1-r)\frac{1}{n}} \frac{1}{(1-x)\frac{1}{n^2}} \frac{\Pr\left(\text{piv}|\alpha_1; \sigma^{\mathbf{P}}, \pi_n\right)}{\Pr\left(\text{piv}|\beta_2; \sigma^{\mathbf{P}}, \pi_n\right)} \\ =\ &\infty, \end{aligned} \tag{106}$$

where the equality on the third line follows since the probability of signal a is zero in β_1 and where we used (105) for the equality on the last line.

We will show now that (106) implies (102): to see why, recall that for n large enough, $q(\alpha_2; \tilde{\sigma}, \pi_n) \approx \Phi(\boldsymbol{\rho}_z(\sigma^{\mathbf{P}}; \pi_n, n))$ and $q(\beta_2; \tilde{\sigma}, \pi_n) \approx \Phi(\boldsymbol{\rho}_z(\sigma^{\mathbf{P}}; \pi_n, n))$ since almost all voters receive z in α_2 and β_2. Also, $q(\alpha_1; \tilde{\sigma}, \pi_n) = \Phi(\boldsymbol{\rho}_a(\sigma^{\mathbf{P}}; \pi_n, n))$ since all voters receive a in α_1. In addition, we have that $\rho_z(\sigma^{\mathbf{P}}; \pi_n, n) \approx \hat{r}$ by (89) and $\rho_a(\sigma^{\mathbf{P}}; \pi_n, n) \approx 1$ by (106). Finally, (102) follows since $\Phi(\hat{r}) = \frac{1}{2}$ and since $\Phi(1) \neq \frac{1}{2}$. This finishes the second case.

Now, we finish the proof of Claim 10. Since we just showed that, given $\tilde{\sigma} = \sigma^{\boldsymbol{\rho}(\sigma^{\mathbf{P}}; \pi_n, n)}$, the expected margin of victory in α_1 and β_1 is larger than in α_2 and β_2, it follows from Claim 8 that

$$\lim_{n \to \infty} \frac{\Pr\left(\{\alpha_2, \beta_2\} | \text{piv}, s; \tilde{\sigma}, \pi_n, n\right)}{\Pr\left(\{\alpha_1, \beta_1\} | \text{piv}, s; \tilde{\sigma}, \pi_n, n\right)} = \infty \tag{107}$$

for any $s \in \{a, b\}$. It follows from (107) and Claim 7, which says that conditional on α_2 and β_2, there is nothing to be learned from the pivotal event, that, given $\tilde{\sigma}$; when a voter observes signal a, the inference from the signal probabilities in the states α_2 and β_2 pins down the limits of the beliefs conditional on being pivotal, such that (92) and (93) hold for $\sigma_n = \tilde{\sigma}$. This, together with (89) yields Claim 10.
■

D Persuasion of Privately Informed Voters

This section proves the following lemma that shows the "implementability" of a large set of beliefs by an appropriate choice of $(x, r, y) \in (0, 1)^3$.

Lemma 3 *Take any exogenous private signals π^c of the voters satisfying (20) and any strictly increasing Φ. There exist $0 < \lambda_\alpha < \lambda < \lambda_\beta < 1$ such that, for any $(\mu_\alpha, \mu_\beta) \in [0, 1]^2$ satisfying $\mu_\alpha \notin [\lambda_\alpha, \lambda]$ and $\mu_\beta \notin [\lambda, \lambda_\beta]$, when $(x, y) \in [0, 1]^2$ are given by*

$$\frac{x\lambda}{x\lambda + (1-x)(1-\lambda)} = \mu_\alpha, \tag{108}$$

$$\frac{y\lambda}{y\lambda + (1-y)(1-\lambda)} = \mu_\beta, \tag{109}$$

and $r \in (0,1)$, there exists a sequence of equilibria (σ_n^*) given $\pi_n^{x,r,y}$ such that

$$\lim_{n\to\infty} \Pr\left(\alpha|\text{piv}, s_2 = a; \sigma_n^*\right) = \mu_\alpha, \tag{110}$$
$$\lim_{n\to\infty} \Pr\left(\alpha|\text{piv}, s_2 = z; \sigma_n^*\right) = \lambda, \tag{111}$$
$$\lim_{n\to\infty} \Pr\left(\alpha|\text{piv}, s_2 = b; \sigma_n^*\right) = \mu_\beta. \tag{112}$$

In particular, $\mu_\alpha \in \{0,1\}$ and $\mu_\beta \in \{0,1\}$ satisfy the conditions of the lemma. This implies Theorem 4.

D.1 Preliminaries

We provide a compact representation of equilibrium as a belief vector, similar to before in (13). Given any strategy σ' used by the others, the vector of posteriors conditional on piv and the additional signal $s_2 \in S_2$ is denoted as

$$\hat{\boldsymbol{\rho}}(\sigma';\pi,n) = (\Pr(\alpha|s_2,\text{piv};\sigma',\pi))_{s_2\in S_2}, \tag{113}$$

and called the vector of *induced priors.*[34] It follows from the independence of the additional information and the exogenous information π^c that the vector of induced priors pins down the full vector of the critical beliefs: for any $s_2 \in S_2$ and any $s_1 \in \{u,d\}$,

$$\Pr(\alpha|s_1,s_2,\text{piv};\sigma',\pi) = \frac{\hat{\rho}_{s_2}(\sigma';\pi,n)\Pr(s_1|\alpha)}{\hat{\rho}_{s_2}(\sigma';\pi,n)\Pr(s_1|\alpha) + (1-\hat{\rho}_{s_2}(\sigma';\pi,n))\Pr(s_1|\beta)}. \tag{114}$$

Recall that the vector of beliefs $(\Pr(\alpha|s_1,s_2,\text{piv};\sigma',\pi))_{(s_1,s_2)\in\{u,d\}\times S_2}$ is a sufficient statistic for the unique best response to σ' for all types; see (11). Hence, the vector of induced priors pins down the best response for all types. Slightly abusing notation, for any $\mathbf{p} = (p_a,p_z,p_b) \in [0,1]^3$, we let $\sigma^{\mathbf{p}}$ be the unique strategy that is optimal given the induced prior $\mathbf{p}$, i.e., when a voter with signal (s_1,s_2) believes the probability of α is

$$\frac{p_{s_2}\Pr(s_1|\alpha)}{p_{s_2}\Pr(s_1|\alpha) + (1-p_{s_2})\Pr(s_1|\beta)}. \tag{115}$$

[34] We adopt the terminology from Bhattacharya (2013).

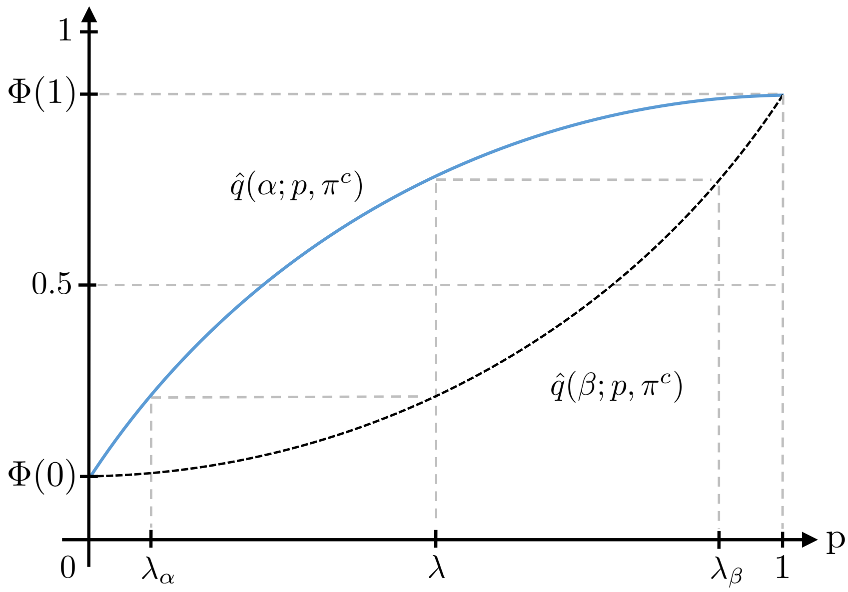

Figure 8: The function $\hat{q}(\alpha; p, \pi^c)$ of the implied vote share in state α and the function $\hat{q}(\beta; p, \pi^c)$ of the implied vote share in state β given an induced prior $p \in (0, 1)$.

Equilibrium can be equivalently characterized by a vector of induced priors $\mathbf{p}^* = (p_a^*, p_z^*, p_b^*)$ such that

$$\mathbf{p}^* = \hat{\boldsymbol{\rho}}(\sigma^{\mathbf{p}^*}; \pi, n); \tag{116}$$

as before; see (13).

For any induced prior $p \in (0, 1)$,

$$\hat{q}(\omega; p, \pi^c) = \sum_{s_1 \in \{u,d\}} \Pr(s_1|\omega; \pi^c)\Phi(\frac{p \Pr(s_1|\alpha)}{p \Pr(s_1|\alpha) + (1-p) \Pr(s_1|\beta)}), \tag{117}$$

is the probability that a voter with induced prior p draws a type t and a signal $s_1 \in S_1$ for which she votes for the outcome A in state ω. Figure 8 illustrates the functions $\hat{q}(\omega; p, \pi^c)$.

Since Φ is continuous and strictly increasing, it follows from (17) and the intermediate value theorem that there exists a unique belief λ such that the implied vote shares satisfy

$$\hat{q}(\alpha;\lambda,\pi^c) - \frac{1}{2} = \frac{1}{2} - \hat{q}(\beta;\lambda,\pi^c); \tag{118}$$

see Figure 8. Let $M = \hat{q}(\alpha;\lambda,\pi^c) - \frac{1}{2}$.

The boundaries λ_α and λ_β are such that all beliefs outside the intermediate intervals $[\lambda_\alpha, \lambda]$ and $[\lambda, \lambda_\beta]$ imply margins of victory that are larger than the ones implied by λ in *any* state $\omega \in \{\alpha, \beta\}$, i.e., larger than M. Formally, λ_α and λ_β are given by

$$q(\alpha;\lambda_\alpha,\pi^c) = q(\beta;\lambda,\pi^c), \tag{119}$$

$$q(\beta;\lambda_\beta,\pi^c) = q(\alpha;\lambda,\pi^c). \tag{120}$$

Figure 8 illustrates the boundaries λ_α and λ_β. For a belief $p > \lambda_\beta$,

$$\hat{q}\left(\beta;p,\pi_1\right) - \frac{1}{2} > M \tag{121}$$

Similarly, for $p > \lambda$,

$$\hat{q}\left(\alpha;p,\pi_1\right) - \frac{1}{2} > M \tag{122}$$

Note that when the exogenous information π^c of the voters becomes revealing (the signal likelihood ratios of d and u go to 0 and ∞, respectively), then

$$\lambda_\alpha \to 0, \text{ and } \lambda_\beta \to 1. \tag{123}$$

D.2 Proof of Claim 6

The Claim 6 in the main text is stated for the information structure π^r. Claim 11 below shows the analogous statement for the information structure $\pi^{x,r,y}$, noting (125). The same arguments imply Claim 6, and we will therefore omit its proof.

D.3 Voter Inference

We show that, when the sender provides additional information $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$, the induced prior after z—and thereby the margin of victory in the states α_2 and β_2—is the same across *all* equilibrium sequences and determined uniquely by the exogenous information π^c of the voters.

Claim 11 *Suppose the additional information is given by* $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$ *for some* $(x,y) \in [0,1]^2$ *and* $r \in (0,1)$*, and consider the induced sequence* $(\pi_n)_{n\in\mathbb{N}}$ *of independent expansions of* π^c*. For any equilibrium sequence* (σ_n^*) *given* (π_n)*,*

$$\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*, \pi_n, n) = \lambda. \tag{124}$$

Proof. The key idea is that, for any equilibrium sequence $(\sigma_n^*)_{n\in\mathbb{N}}$, the election is equally close to being tied in expectation in α_2 and β_2 as $n \to \infty$.

$$\lim_{n\to\infty} q(\sigma_n^*; \alpha_2, \pi_n) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - q(\sigma_n^*; \beta_2, \pi_n), \tag{125}$$

by arguments similar to those from the proof of the CJT; see (22).

Since almost all voters receive z in α_2 and β_2, the expected vote share in these states converges to the vote share implied by the induced prior after z; for $\omega_2 \in \{\alpha_2, \beta_2\}$,

$$\lim_{n\to\infty} q(\sigma_n^*; \omega_2, \pi_n) = \lim_{n\to\infty} \hat{q}(\omega; \hat{\rho}_z(\sigma_n^*; \pi_n, n), \pi^c). \tag{126}$$

Recall that λ is the unique induced prior such that the margins of victory are equal given the implied vote shares; see (118). So, (125) and (126) imply the claim, (124). It remains to show (125).

Step 1 *For all* n *and every equilibrium* σ_n^**, voters with a* (z,u)*-signal are more likely to vote* A *than voters with a* (z,d)*-signal when* n *is large enough, i.e.*

$$\Phi(\rho_{z,u}(\sigma_n^*)) > \Phi(\rho_{z,d}(\sigma_n^*)). \tag{127}$$

This ordering follows from the likelihood ratio ordering of the signals u and d, i.e., $\frac{\Pr(u|\alpha;\pi^c)}{\Pr(u|\beta;\pi^c)} > \frac{\Pr(d|\alpha;\pi^c)}{\Pr(d|\beta;\pi^c)}$, and the independence of $\pi_n^{x,r,y}$ and π^c. Using (115), we have $\Pr(\alpha|z,u,\text{piv};\sigma_n^*,\pi_n,n) > \Pr(\alpha|z,d,\text{piv};\sigma_n^*,\pi_n,n)$. Now, (127) follows from the monotonicity of Φ.

Step 2 *For all* n *and every equilibrium* σ_n^**, the vote share of* A *is at most* $\frac{1}{n^2}$ *smaller in* α_2 *than in* β_2*,*

$$q(\alpha_2; \sigma_n^*) - q(\beta_2; \sigma_n^*) \geq -\frac{1}{n^2} \tag{128}$$

For signals (a,b), the ordering may be the reverse of (127). However, in α_2 and β_2, the likelihood that a voter does not receive signal z is smaller than $\frac{1}{n^2}$. So, this follows from (15), given (20) and (127).

Step 3 *For every equilibrium sequence* (σ_n^*),

$$\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*;\pi_n,n) \notin \{0,1\}. \tag{129}$$

We have

$$\frac{\hat{\rho}_z(\sigma_n^*;\pi_n,n)}{1-\hat{\rho}_z(\sigma_n^*;\pi_n,n)} = \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(\alpha_2|\alpha;\pi_n)}{\Pr(\beta_2|\beta;\pi_n)}\frac{\Pr(\text{piv}|\alpha_2;\sigma_n^*,\pi_n,n)}{\Pr(\text{piv}|\beta_2;\sigma_n^*,\pi_n,n)}. \tag{130}$$

Suppose that $\lim_{n\to\infty}\hat{\rho}_z(\sigma_n^*;\pi_n,n)=0$. We show that this implies

$$\lim_{n\to\infty}\frac{\Pr(\text{piv}|\alpha_2;\sigma_n^*,\pi_n,n)}{\Pr(\text{piv}|\beta_2;\sigma_n^*,\pi_n,n)} \geq 1; \tag{131}$$

a contradiction. Since almost all voters receive z in α_2 and β_2 and since $\Phi(0)<\frac{1}{2}$, the hypothesis $\lim_{n\to\infty}\hat{\rho}_z(\sigma_n^*;\pi_n,n)=0$ implies that

$$\lim_{n\to\infty} q(\alpha_2,\sigma_n^*) = \lim_{n\to\infty} q(\beta_2,\sigma_n^*) < \frac{1}{2}. \tag{132}$$

Recall that $\Phi(0)<q(\omega_j;\sigma)<\Phi(1)$ for any strategy and any substate ω_j and note that the derivative of $h(q)=q(1-q)$ is bounded below by some Lipschitz constant $L>0$ on the compact interval $[\Phi(0),\Phi(1)]$. Hence, (128) implies

$$h(q(\beta_2,\sigma_n^*))(\frac{h(q(\alpha_2,\sigma_n^*))}{h(q(\beta_2,\sigma_n^*))}-1) = h(q(\alpha_2,\sigma_n^*))-h(q(\beta_2,\sigma_n^*)) \geq -\frac{L}{n^2}. \tag{133}$$

Recall that the function $h(q)=q(1-q)$ is inverse U-shaped with a peak at $q=\frac{1}{2}$ and note that it follows from (17) and Φ being strictly increasing that $0<\Phi(0)<\frac{1}{2}$ and $\Phi(1)>\frac{1}{2}$. Since $\Phi(0)<q(\beta_2;\sigma_n^*)<\Phi(1)$,

$$\frac{h(q(\alpha_2,\sigma_n^*))}{h(q(\beta_2,\sigma_n^*))} \geq 1-\frac{L}{h(q(\beta_2\,;\sigma_n))n^2} \geq 1-\frac{L}{Mn^2} \tag{134}$$

for $M=\min\left(h(\Phi(0)),h(\Phi(1))\right)$ and all n. It follows from (7) that $\frac{\Pr(\text{piv}|\alpha_2;\sigma_n^*,\pi_n,n)}{\Pr(\text{piv}|\beta_2;\sigma_n^*,\pi_n,n)} \geq (1-\frac{L}{Mn^2})^n$. Now, (131) follows since $\lim_{n\to\infty}(1-\frac{L}{Mn^2})^n=1$; see the analogous argument at the end of the proof of Claim 3. A similar argument excludes $\lim_{n\to\infty}\hat{\rho}_z(\sigma_n^*;\pi_n,n)=1$ (using the analogous bound to (128)). This finishes the proof of the step.

Step 4 *In every equilibrium sequence* $(\sigma_n^*)_{n\in\mathbb{N}}$*, the limit of the vote share of* A *is* *larger in α_2 than in β_2,*

$$\lim_{n\to\infty} q(\alpha_2; \sigma_n^*) > \lim_{n\to\infty} q(\beta_2; \sigma_n^*). \tag{135}$$

Since almost all voters receive z in α_2 and β_2, we have

$$\lim_{n\to\infty} q(\alpha_2; \sigma_n^*) = \lim_{n\to\infty} \hat{q}(\alpha; \hat{\rho}_z(\sigma_n^*, \pi_n, n)), \tag{136}$$

$$\lim_{n\to\infty} q(\beta_2; \sigma_n^*) = \lim_{n\to\infty} \hat{q}(\beta; \hat{\rho}_z(\sigma_n^*, \pi_n, n)). \tag{137}$$

From (129), the limits of the posteriors conditional being pivotal, the signal z and the signals $s \in \{u, d\}$ are interior, and hence, strictly ordered,

$$0 < \lim_{n\to\infty} \Pr(\alpha | z, d, \text{piv}; \sigma_n^*, \pi_n, n) < \lim_{n\to\infty} \Pr(\alpha | z, u, \text{piv}; \sigma_n^*, \pi_n, n) < 1. \tag{138}$$

Now, (135) follows from (136), (137), and (117), given (20), (138), and since Φ is strictly increasing.

We now finish the proof of Claim 11. It follows from (129) that voters must not become certain conditional on being pivotal and the substate being α_2 or β_2, i.e., $\lim_{n\to\infty} \Pr(\alpha | \{\alpha_2, \beta_2\}, \text{piv}; \sigma_n^*, \pi_n) \notin \{0, 1\}$. Hence, Claim 2 requires that

$$\lim_{n\to\infty} \left| q(\alpha_2; \sigma_n^*) - \frac{1}{2} \right| = \lim_{n\to\infty} \left| q(\beta_2; \sigma_n^*) - \frac{1}{2} \right|. \tag{139}$$

Given the ordering of the limits of the vote shares from (135), the equation (139) implies (125). As noted, this completes the proof of Claim 11. ∎

Consider a voter who received an additional signal $s_2 \in \{a, b\}$. The following result shows that the inference from the signals is dominated by the inference from the pivotal event if the election is closer to being tied in states α_2 and β_2 than in the states α_1 and β_1. The arguments are analogous to the ones from the proof of Claims 4 and 8; we therefore omit the proof.

Claim 12 *Suppose that the additional information is given by $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$ for some $(x, y) \in [0, 1]^2$ and $r \in (0, 1)$, and consider the corresponding sequence $(\pi_n)_{n\in\mathbb{N}}$ of independent expansions of π^c. Take any sequence of strategies $(\sigma_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \min_{\omega_1 \in \{\alpha_1, \beta_1\}} |q(\sigma_n; \omega_1, \pi_n) - \frac{1}{2}| > \lim_{n\to\infty} \max_{\omega_2 \in \{\alpha_2, \beta_2\}} |q(\sigma_n; \omega_2, \pi_n) - \frac{1}{2}|; \tag{140}$$

then, for any $s \in \{u,d\} \times \{a,b\}$,

$$\lim_{n\to\infty} \frac{\Pr(\{\alpha_2,\beta_2\}|s,\text{piv};\sigma_n,\pi_n)}{\Pr(\{\alpha_1,\beta_1\}|s,\text{piv};\sigma_n,\pi_n)} = \infty. \tag{141}$$

Now, take any sequence of *equilibria* $(\sigma_n^*)_{n\in\mathbb{N}}$ that satisfies (140). Claim 12 implies that

$$\lim_{n\to\infty} \frac{\Pr(\alpha|a,\text{piv};\sigma_n^*,\pi_n,n)}{\Pr(\beta|a,\text{piv};\sigma_n^*,\pi_n,n)} = \lim_{n\to\infty} \frac{\Pr(\alpha_2|a,\text{piv};\sigma_n^*,\pi_n,n)}{\Pr(\beta_2|a,\text{piv};\sigma_n^*,\pi_n,n)} \tag{142}$$

In the following formula, we omit the dependence on σ_n^* and π_n. Using Bayes' rule,[35]

$$\begin{aligned}\lim_{n\to\infty} \frac{\Pr(\alpha_2|a,\text{piv})}{\Pr(\beta_2|a,\text{piv})} &= \lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_2|\beta)}\frac{\Pr(a|\alpha_2)}{\Pr(a|\beta_2)}\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)}\\ &= \lim_{n\to\infty} \frac{\Pr(\alpha|\{\alpha_2,\beta_2\},\text{piv})}{\Pr(\beta|\{\alpha_2,\beta_2\},\text{piv})}\frac{\Pr(a|\alpha_2)}{\Pr(a|\beta_2)}.\end{aligned} \tag{143}$$

Note that $\lim_{n\to\infty} \hat{\rho}_z(\sigma_n^*;\pi_n,n) = \lim_{n\to\infty} \Pr(\alpha|\{\alpha_2,\beta_2\},\text{piv};\sigma_n^*,\pi_n,n)$ such that Claim 11 implies

$$\lim_{n\to\infty} \Pr(\alpha|\{\alpha_2,\beta_2\},\text{piv};\sigma_n^*,\pi_n,n) = \lambda. \tag{144}$$

Using (142), (143), (144), and the definition of the information structure $\pi_n^{x,r,y}$, we conclude

$$\lim_{n\to\infty} \frac{\Pr(\alpha|a,\text{piv};\sigma_n^*,\pi_n)}{\Pr(\beta|a,\text{piv};\sigma_n^*,\pi_n)} = \frac{x}{1-x}\frac{\lambda}{1-\lambda}. \tag{145}$$

Similarly, for the additional signal b,

$$\lim_{n\to\infty} \frac{\Pr(\alpha|b,\text{piv};\sigma_n^*,\pi_n,n)}{\Pr(\beta|b,\text{piv};\sigma_n^*,\pi_n,n)} = \frac{y}{1-y}\frac{\lambda}{1-\lambda}. \tag{146}$$

D.4 Fixed Point Argument

In this section, we prove Lemma 3, using the observations from the preceding section. Let us consider some belief $\mu_\alpha \notin [\lambda_\alpha,\lambda]$ and some belief $\mu_\beta \notin [\lambda,\lambda_\beta]$ with λ, λ_α, and λ_β given by (118), (119) and (120).

[35] As before, if $x = 1$, that is if $\Pr(a|\beta_2) = 0$, using the convention $\frac{1}{0} = \infty$, the following equality holds in the extended reals.

Recall from Section D.1 that equilibrium can be equivalently characterized by a vector of beliefs $\mathbf{p}^* = (p_a^*, p_z^*, p_b^*)$ such that $\mathbf{p}^* = \hat{\boldsymbol{\rho}}(\sigma^{\mathbf{p}^*}; \pi, n)$; see (116). Now, take any $\delta > 0$ and let

$$\mathrm{B}_\delta = \left\{ \mathbf{p} \in [0,1]^3 \mid |\mathbf{p} - (\mu_\alpha, \lambda, \mu_\beta)| \leq \delta \right\}.$$

Take any $\mathbf{p} \in \mathrm{B}_\delta$ and the corresponding strategy $\sigma^{\mathbf{p}}$. We define a constrained best-response function as its "truncation" to B_δ:

$$\hat{\rho}_a^{tr}(\sigma^{\mathbf{p}}) = \begin{cases} \mu_\alpha - \delta & \text{if} \quad \hat{\rho}_a(\sigma^{\mathbf{p}}) < \mu_\alpha - \delta, \\ \mu_\alpha + \delta & \text{if} \quad \hat{\rho}_a(\sigma^{\mathbf{p}}) > \mu_\alpha + \delta, \\ \hat{\rho}_a(\sigma^{\mathbf{p}}) & \text{else.} \end{cases} \tag{147}$$

The components $\hat{\rho}_z^{tr}$ and $\hat{\rho}_b^{tr}$ are defined in the analogous way. The function $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}})$ is continuous in $\mathbf{p}$ such that Kakutani's theorem implies that $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}})$ has a fixed point $\mathbf{p}^* \in \mathrm{B}_\delta$.

Any fixed point $\mathbf{p}^*$ of $\hat{\boldsymbol{\rho}}^{tr}$ is shown to be in the interior of B_δ when n is large enough and δ is small enough, i.e., $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}^*}) = \hat{\boldsymbol{\rho}}(\sigma^{\mathbf{p}^*})$:

Claim 13 *Consider any $\mu_\alpha \notin [\lambda_\alpha, \lambda]$ and any $\mu_\beta \notin [\lambda, \lambda_\beta]$. Consider the sequence of independent expansions $(\pi_n)_{n\in\mathbb{N}}$ of π^c with additional information $(\pi_n^{x,r,y})_{n\in\mathbb{N}}$ where $\mu_\alpha = \frac{x\lambda}{x\lambda+(1-x)(1-\lambda)}$ and $\mu_\beta = \frac{y\lambda}{y\lambda+(1-y)(1-\lambda)}$ and $r \in (0,1)$.*

For any $\delta > 0$ small enough, there exists $n(\delta) \in \mathbb{N}$ such that for all $n \geq n(\delta)$, any fixed point of $\hat{\boldsymbol{\rho}}^{tr}$ is in the interior of B_δ.

Proof. Pick some $\mathbf{p}$ for which p_z is on the boundary. We show $\mathbf{p}$ cannot be a fixed point for n large enough and δ small enough. First, suppose $p_z = \lambda - \delta$. Then, given σ and as $n \to \infty$, the margin of victory in α_2 is strictly smaller than the margin of victory in β_2, given the definition of λ; see (118). Hence, Claim 2 implies that $\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2;\sigma^{\mathbf{p}},\pi_n^{x,r,y},n)}{\Pr(\text{piv}|\beta_2,\sigma^{\mathbf{p}},\pi_n^{x,r,y},n)} = \infty$. This implies, $\lim_{n\to\infty} \hat{\rho}_z(\sigma^{\mathbf{p}}; \pi_n^{x,r,y}, n) = 1$. For any n large enough this contradicts $p_z = \lambda - \delta$ and so $\mathbf{p}$ is not a fixed point of $\hat{\boldsymbol{\rho}}^{tr}(\sigma^{\mathbf{p}})$. In the same way we can exclude that $p_z = \lambda + \delta$ for any n large enough. In general, the same argument implies that, for n large enough, for any fixed point $\mathbf{p}^*$,

$$\hat{\rho}_z(\sigma^{\mathbf{p}^*}) \approx \lambda. \tag{148}$$

Given the assumptions on μ_α, μ_β, we can choose $\delta > 0$ small enough such that, for any $\mathbf{p} \in \mathrm{B}_\delta$ and the corresponding behavior $\sigma^{\mathbf{p}}$, the expected margins of victory in the states α_2 and β_2 are strictly smaller than the expected margins of victory in the states α_1 and β_1, i.e., $\sigma^{\mathbf{p}}$ satisfies (140). Therefore, it follows from Claim 12 and (148) that (145) and (146) hold; hence, given the definition of $\hat{\boldsymbol{\rho}}$

$$\hat{\rho}_a(\sigma^{\mathbf{p}^*}) \approx \mu_\alpha, \tag{149}$$

$$\hat{\rho}_b(\sigma^{\mathbf{p}^*}) \approx \mu_\beta. \tag{150}$$

We conclude that any fixed point $\mathbf{p}^*$ of $\hat{\boldsymbol{\rho}}^{tr}$ is interior when δ is small enough and n is large enough. ■

Now, we finish the proof of Lemma 3. Note that the strategy $\sigma^{\mathbf{p}^*}$ corresponding to any interior fixed point $\mathbf{p}^*$ of $\hat{\boldsymbol{\rho}}^{tr}$ is an equilibrium. Therefore, Claim 13 implies the existence of a sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ for which (148), (149), and (150) hold. This finishes the proof of Lemma 3.

E Media Markets: An Application

Recall the setting from Section 6.2. With probability $1-\chi$, the media firm sends the message $m=\theta$ in α and $m=-\theta$ in β. With probability $\chi>0$, the media firm sends a shifted message, $m=\theta-d$ in α and $m=-\theta-d$ in β for some $d\neq 0$. Voters perceive the message m with noise, that is, each voter receives a private signal $s=m+\epsilon$ where ϵ is drawn independently from a standard normal distribution. The information structure of the voters thus has four substates $\omega_i\in\{\alpha_1,\beta_1,\alpha_2,\beta_2\}$, one for each of the four messages of the firm. Denote by $m(\omega_i)$ the message in substate ω_j; we label the substates so that

$$m(\omega_i)=\begin{cases}\theta & \text{if} \quad \omega_i=\alpha_1,\\ -\theta & \text{if} \quad \omega_i=\beta_1\\ \theta-d & \text{if} \quad \omega_i=\alpha_2,\\ -\theta-d & \text{if} \quad \omega_i=\beta_2.\end{cases} \tag{151}$$

Theorem 5 *Consider the setting from Section 6.2. Take any strictly increasing Φ satisfying (17). Let $d>2\theta$. For every constant policy $x\in\{A,B\}$, there is a sequence of equilibria $(\sigma_n^*)_{n\in\mathbb{N}}$ given d such that*

$$\lim_{n\to\infty}\Pr\left(x|\sigma_n^*,d,n\right)\geq 1-\chi. \tag{152}$$

We provide the proof for $x = A$. The case $x = B$ is analogous. Recall that the collection of critical beliefs $(\Pr(\alpha|s, \text{piv}; \sigma))_{s\in\mathbb{R}}$ is a sufficient statistic for the best response; see (3) and (4). For the purpose of this proof, it will turn out more convenient to conduct the analysis in the space of vectors of expected vote shares $(q(\omega_j; \sigma))_{\omega_j\in\{\alpha_1,\alpha_2,\beta_1,\beta_2\}}$ rather than the space of critical beliefs.[36] For any strategy σ, the vector $(q(\omega_j; \sigma))_{\omega_j\in\{\alpha_1,\alpha_2,\beta_1,\beta_2\}}$ is a sufficient statistic for the pivotal likelihood in each state, and therefore for $\Pr(\alpha|s, \text{piv}; \sigma)$, for any $s \in \mathbb{R}$. We conclude that the vote share vector is also a sufficient statistic for the best response. Given some arbitrary vector of vote shares $\mathbf{q}$, let $\mathbf{p}(\mathbf{q}) = (\Pr(\alpha|s, \text{piv}; \mathbf{q})_{s\in\mathbb{R}}$ be the collection of posteriors induced by $\mathbf{q}$ and $\sigma^{\mathbf{q}} = \sigma^{\mathbf{p}(\mathbf{q})}$ the unique best response in undominated strategies given $\mathbf{p}(\mathbf{q})$; compare to (12). A strategy σ^* is an equilibrium if and only if $\sigma^* = \sigma^{\mathbf{q}}$ for $\mathbf{q} = (q(\omega_j; \sigma^*))_{\omega_j\in\{\alpha_1,\alpha_2,\beta_1,\beta_2\}}$. Conversely, an equilibrium can be described by a vector of vote shares $\mathbf{q}^*$ that is a fixed point of

$$\mathbf{q} \to q(\omega_j; \sigma^{\mathbf{q}})_{\omega_j\in\{\alpha_1,\alpha_2,\beta_1,\beta_2\}}. \tag{153}$$

Take $\delta > 0$. For any $\mathbf{q}$, we consider a constrained variant of the map (153), denoted

$$\psi(\mathbf{q}) \to \hat{q}(\omega_j; \sigma^{\mathbf{q}})_{\omega_j\in\{\alpha_1,\alpha_2,\beta_1,\beta_2\}} \tag{154}$$

and defined sequentially across substates:

$$\begin{aligned}
\hat{q}(\beta_2; \sigma^{\mathbf{q}}) &= \min\left(\frac{1}{2}, q(\beta_2; \sigma^{\mathbf{q}})\right), \\
\hat{q}(\alpha_2; \sigma^{\mathbf{q}}) &= \max\left(\frac{1}{2}, q(\alpha_2; \sigma^{\mathbf{q}})\right),
\end{aligned}$$

and

$$\begin{aligned}
\hat{q}(\beta_1; \sigma^{\mathbf{q}}) &= \max\left(q(\beta_1; \sigma^{\mathbf{q}}), \hat{q}(\alpha_2; \sigma^{\mathbf{q}}) + \delta\right), \\
\hat{q}(\alpha_1; \sigma^{\mathbf{q}}) &= \max\left(q(\alpha_1; \sigma^{\mathbf{q}}), \hat{q}(\beta_1; \sigma^{\mathbf{q}}) + \delta\right).
\end{aligned}$$

[36] This is because there are infinitely many critical priors, given that the distribution of the signal s is continuous. In contrast, the vote shares vector $q(\omega_j; \sigma)$ is finite-dimensional. Further, the best response is not necessarily cutoff strategies (in s, for fixed t) and working with vote shares helps because it "integrates out".

By definition, ψ maps to the set of vectors $\mathbf{q} = (q(\omega_j))_{\omega\in\{\alpha,\beta\},j\in\{1,2\}}$ satisfying

$$q(\beta_2) \leq \frac{1}{2} \leq q(\alpha_2), \tag{155}$$
$$q(\alpha_2) + \delta \leq q(\beta_1), \tag{156}$$
$$q(\beta_1) + \delta \leq q(\alpha_1) \tag{157}$$

and we denote ψ_n^{res} its restriction to this domain, adding the subscript n for the dependence on the electorate size. Note that ψ_n^{res} is a continuous self-map, such that Kakutani's fixed point theorem implies that it has a fixed point.

Proof of Theorem 5.

Step 1 *For any sequence $(\mathbf{q}_n)_{n\in\mathbb{N}}$ of fixed points of ψ_n^{res},*

$$\lim_{n\to\infty} |q_n(\alpha_2) - \frac{1}{2}| = \lim_{n\to\infty} |\frac{1}{2} - q_n(\beta_2)|, \tag{158}$$

and there is $\bar{\lambda} \in (0,1)$, so that for all $s \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, s; \mathbf{q}_n, d, n)}{\Pr(\beta|\text{piv}, s; \mathbf{q}_n, d, n)} = \frac{\bar{\lambda}}{1-\bar{\lambda}} \frac{f(s|\alpha_2; d)}{f(s|\beta_2; d)}. \tag{159}$$

Proof. It follows from the analogue of Claim 2 for the posteriors $\mathbf{p}(\mathbf{q}_n)$ and from (155) - (157) that $\lim_{n\to\infty} \frac{\Pr(\text{piv}|\{\alpha_2,\beta_2\};\mathbf{q}_n,d,n)}{\Pr(\text{piv}|\{\alpha_1,\beta_1\};\mathbf{q}_n,d,n)} = \infty$. Hence,

$$\lim_{n\to\infty} \Pr(\{\alpha_1, \beta_1\}|\text{piv}; \mathbf{q}_n, d, n) = 0. \tag{160}$$

Denote by $f(s|\omega_i; d)$ the density of the voter's private signal s conditional on the substate ω_i. Given (160),

$$\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, s; \mathbf{q}_n, d, n)}{\Pr(\beta|\text{piv}, s; \mathbf{q}_n, d, n)} = \lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_2|\beta)} \frac{\Pr(\text{piv}|\alpha_2; \mathbf{q}_n, d, n)}{\Pr(\text{piv}|\beta_2; \mathbf{q}_n, d, n)} \frac{f(s|\alpha_2; d)}{f(s|\beta_2; d)} \tag{161}$$

Recall that the substate ω_2 in which the firm uses the slant $-d$ has the same probability in each state $\omega \in \{\alpha, \beta\}$, that is, $\frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_2|\beta)} = 1$. Given (161),

$$\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, s; \mathbf{q}_n, d, n)}{\Pr(\beta|\text{piv}, s; \mathbf{q}_n, d, n)} = \frac{\lambda}{1-\lambda} \frac{f(s|\alpha_2; d)}{f(s|\beta_2; d)}, \tag{162}$$

for $\frac{\lambda}{1-\lambda} = \lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\text{piv}|\alpha_2;s;\mathbf{q}_n,d,n)}{\Pr(\text{piv}|\beta_2;s;\mathbf{q}_n,d,n)}$.[37]

[37] We use the convention $\frac{1}{0} = \infty$.

Now, we show (158). Suppose that

$$\lim_{n\to\infty} |q_n(\alpha_2) - \frac{1}{2}| > \lim_{n\to\infty} |\frac{1}{2} - q_n(\beta_2)|. \tag{163}$$

Claim 2 implies $\lambda = 0$. Together with (162), this implies

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s; \mathbf{q}_n, d, n) = 0 \tag{164}$$

for all $s \in \mathbb{R}$. Given (164), the vote shares of the best response satisfy $\lim_{n\to\infty} q(\beta_2; \sigma^{\mathbf{q}_n}) = \lim_{n\to\infty} q(\alpha_2; \sigma^{\mathbf{q}_n}) = \Phi(0)$. Since $\Phi(0) < 1/2$ by assumption and since the constrained best response ψ restricts the vote shares in α_2 to be weakly larger than $1/2$, $\lim_{n\to\infty} \hat{q}(\alpha_2; \sigma^{\mathbf{q}_n}) = \frac{1}{2}$. Since $\mathbf{q}_n$ is a fixed point, $q_n(\alpha_2) = \hat{q}(\alpha_2; \sigma^{\mathbf{q}_n})$, so that $\lim_{n\to\infty} q_n(\alpha_2) - \frac{1}{2} = 0$, which yields a contradiction to (163).

Suppose that

$$\lim_{n\to\infty} |q_n(\alpha_2) - \frac{1}{2}| < \lim_{n\to\infty} |\frac{1}{2} - q_n(\beta_2)|. \tag{165}$$

Claim 2 implies $\lambda = 1$. Together with (162), this implies

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s; \mathbf{q}_n, d, n) = 1 \tag{166}$$

for all $s \in \mathbb{R}$. Given (166), the vote shares of the best response satisfy $\lim_{n\to\infty} q(\beta_2; \sigma^{\mathbf{q}_n}) = \lim_{n\to\infty} q(\alpha_2; \sigma^{\mathbf{q}_n}) = \Phi(1)$. Since $\Phi(1) > 1/2$ by assumption and since the constrained best response ψ restricts the vote shares in β_2 to be weakly smaller than $1/2$, $\lim_{n\to\infty} \hat{q}(\beta_2; \sigma^{\mathbf{q}_n}) = \frac{1}{2}$. Since $\mathbf{q}_n$ is a fixed point, $q_n(\beta_2) = \hat{q}(\beta_2; \sigma^{\mathbf{q}_n})$, so that $\lim_{n\to\infty} q_n(\beta_2) - \frac{1}{2} = 0$, which yields a contradiction to (165).

Now, we prove (159): Note that (159) follows from (162) if we establish that $\lambda \notin \{0, 1\}$. Suppose that $\lambda = 0$. We just gave an argument showing that this implies $\lim_{n\to\infty} q_n(\alpha_2) = \frac{1}{2}$ and $\lim_{n\to\infty} q(\beta_2; \sigma^{\mathbf{q}_n}) = \Phi(0)$. Since $\Phi(0) < 1/2$ and since $\mathbf{q}_n$ is a fixed point, $q_n(\beta_2) = q(\beta_2; \sigma^{\mathbf{q}_n})$ when n is large. Altogether, this implies $\lim_{n\to\infty} |q_n(\beta_2) - \frac{1}{2}| > \lim_{n\to\infty} |q_n(\alpha_2) - \frac{1}{2}|$, a contradiction to (158). Hence, $\lambda \neq 0$. Suppose that $\lambda = 1$. We just gave an argument showing that this implies $\lim_{n\to\infty} q_n(\beta_2) = \frac{1}{2}$ and $\lim_{n\to\infty} q(\alpha_2; \sigma^{\mathbf{q}_n}) = \Phi(1)$. Since $\Phi(1) > 1/2$ and since $\mathbf{q}_n$ is a fixed point, $q_n(\alpha_2) = q(\alpha_2; \sigma^{\mathbf{q}_n})$ when n is large. Altogether, this implies $\lim_{n\to\infty} |q_n(\beta_2) - \frac{1}{2}| < \lim_{n\to\infty} |q_n(\alpha_2) - \frac{1}{2}|$, a contradiction to (158). Hence $\lambda \neq 1$. ■

Step 2 *If $m(\omega_j) > m(\omega_i)$: For any sequence $(\mathbf{q}_n)_{n\in\mathbb{N}}$ of fixed points of ψ_n^{res}, $\lim_{n\to\infty} q(\omega_j;\sigma^{\mathbf{q}_n}) > \lim_{n\to\infty} q(\omega_i;\sigma^{\mathbf{q}_n})$.*

Proof. Take a sequence of fixed points $\mathbf{q}_n$. In the following, sometimes we drop the dependence on $\mathbf{q}_n$, d and n from the notation. Consider the limit vote shares of the best response,

$$\lim_{n\to\infty} q(\omega_i;\sigma^{\mathbf{q}_n}) = \lim_{n\to\infty} \int_{s\in\mathbb{R}} \Phi(\Pr(\alpha|\text{piv}, s; \mathbf{q}_n, d, n)) f(s|\omega_i; d) ds,$$

for $\omega_i \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$. We rewrite

$$\begin{aligned}\lim_{n\to\infty} q(\omega_i;\sigma^{\mathbf{q}_n}) &= \int_{s\in\mathbb{R}} \lim_{n\to\infty} \Phi(\Pr(\alpha|\text{piv}, s; \mathbf{q}_n, d, n)) f(s|\omega_i; d) ds,\\ &= \int_{s\in\mathbb{R}} \Phi(\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s; \mathbf{q}_n, d, n)) f(s|\omega_i; d) ds, \qquad (167)\end{aligned}$$

where we apply the dominated convergence theorem for the first equality and for the second equality we use that Φ is continuous.Recall that the signal distributions in the substates are normal distributions with the same variance and shifted mean. In particular, $f(s|\omega_i; d) = f(s + m(\omega_j) - m(\omega_i)|\omega_j; d)$ for substates $\omega_i \neq \omega_j$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} q(\omega_i) &= \int_{s\in\mathbb{R}} \Phi(\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s)) f(s + m(\omega_j) - m(\omega_i)|\omega_j; d) ds\\ &= \int_{s'\in\mathbb{R}} \Phi(\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s' - m(\omega_j) + m(\omega_i))) f(s'|\omega_j; d) ds' (168)\end{aligned}$$

substituting $s' = s + m(\omega_j) - m(\omega_i)$. Now,

$$\begin{aligned}\lim_{n\to\infty} q(\omega_j) &= \int_{s'\in\mathbb{R}} \Phi(\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s')) f(s|\omega_j) ds'\\ &> \int_{s'\in\mathbb{R}} \Phi(\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s' - m(\omega_j) + m(\omega_i))) f(s'|\omega_j) ds' \ (169)\end{aligned}$$

where the equality simply restates (167) and we claim that the inequality (169) holds if $m(\omega_j) > m(\omega_i)$. To see why (169) holds, recall (159), which says that there is $\bar{\lambda} \in (0,1)$ so that $\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv},s')}{\Pr(\beta|\text{piv},s')} = \frac{\bar{\lambda}}{1-\bar{\lambda}} \frac{f(s|\alpha_2;d)}{f(s|\beta_2;d)}$ for all $s' \in \mathbb{R}$. The likelihood ratio $\frac{f(s|\alpha_2;d)}{f(s|\beta_2;d)}$ is strictly increasing in s since the mean of the normal in α_2 is higher than the mean of the normal in β_2. Together, we obtain $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s') > \lim_{n\to\infty} \Pr(\alpha|\text{piv}, s' - m(\omega_j) + m(\omega_i))$ for all $s' \in \mathbb{R}$ if $m(\omega_j) > m(\omega_i)$. This implies the inequality (169) since we assumed that Φ is strictly increasing. Finally, (168) and (169) together show $\lim_{n\to\infty} q(\omega_j; \sigma^{\mathbf{q}_n}) > \lim_{n\to\infty} q(\omega_i; \sigma^{\mathbf{q}_n})$. ∎

Step 3 *For any sequence $(\mathbf{q}_n)_{n\in\mathbb{N}}$ of fixed points of ψ_n^{res},*

$$\lim_{n\to\infty} q(\alpha_2; \sigma^{\mathbf{q}_n}) > \frac{1}{2} > \lim_{n\to\infty} q(\beta_2; \sigma^{\mathbf{q}_n}). \tag{170}$$

Proof. Suppose that $\lim_{n\to\infty} q(\alpha_2; \sigma^{\mathbf{q}_n}) \leq 1/2$. Then, $\lim_{n\to\infty} \hat{q}(\alpha_2; \sigma^{\mathbf{q}_n}) = 1/2$ by the definition of $\hat{q}$. Given (151), $m(\alpha_2) > m(\beta_2)$, so that Step 2 implies $\lim_{n\to\infty} q(\beta_2; \sigma^{\mathbf{q}_n}) < 1/2$. But this means that (158) does not hold, , which cannot be. Hence, $\lim_{n\to\infty} q(\alpha_2; \sigma^{\mathbf{q}_n}) > 1/2$. Similarly, suppose that $\lim_{n\to\infty} q(\beta_2; \sigma^{\mathbf{q}_n}) \geq 1/2$. Then, $\lim_{n\to\infty} \hat{q}(\beta_2; \sigma^{\mathbf{q}_n}) = 1/2$. Since $m(\alpha_2) > m(\beta_2)$, Step 2 implies $\lim_{n\to\infty} q(\alpha_2; \sigma^{\mathbf{q}_n}) > 1/2$. But this means that (158) does not hold, which cannot be. Hence, $\lim_{n\to\infty} q(\beta_2; \sigma^{\mathbf{q}_n}) < 1/2$. ∎

Step 4 *For any sequence $(\mathbf{q}_n)_{n\in\mathbb{N}}$ of fixed points of ψ_n^{res}, there is $\bar{n} \in \mathbb{N}$ so that for $n \geq \bar{n}$, the fixed point $\mathbf{q}_n$ is interior.*

Proof. The assumption $d > 2\theta$ implies $\theta - d < -\theta$ so that the messages in the substates are ordered as

$$m(\beta_2) < m(\alpha_2) < m(\beta_1) < m(\alpha_1), \tag{171}$$

given (151). Applying Step 2,

$$\lim_{n\to\infty} q(\beta_2; \sigma^{\mathbf{q}_n}) < \lim_{n\to\infty} q(\alpha_2; \sigma^{\mathbf{q}_n}) < q(\beta_1; \sigma^{\mathbf{q}_n}) < q(\alpha_1; \sigma^{\mathbf{q}_n}). \tag{172}$$

So, the limit vote shares differ pair-wise by at least some $\delta' > 0$. We claim that $\delta' > 0$ does not depend on the bound $\delta > 0$ in the definition of ψ_n^{res}, so that we can choose $\delta < \delta'/2$. To see why, note that the limit vote share in ω_i only depends on $\bar{\lambda}$, on the message $m(\omega_i)$, and on the preference distribution Φ, given (162) and (167). Further, $\bar{\lambda}$ is fixed by the equal-margin condition (158). We conclude that differences in the limit vote shares do not depend on δ.

The inequalities (172) and (170) together with the lower bound $\delta' > 0$ and $\delta < \frac{\delta'}{2}$ imply that the vector of vote shares $q(\omega_i; \sigma^{\mathbf{q}_n})$ satisfies (155) - (157) when n is large enough. Since $\mathbf{q}_n$ is a fixed point of ψ_n^{res}, this implies that $q(\omega_i; \sigma^{\mathbf{q}_n}) = q_n(\omega_i)$ for all $\omega_i \in \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$, meaning that the fixed point of ψ_n^{res} is interior. ∎

We finish the proof of Theorem 5. Take any sequence of fixed points $(\mathbf{q}_n)_{n\in\mathbb{N}}$ of ψ_n^{res}. Step 4 implies that the fixed point $\mathbf{q}_n$ corresponds to an equilibrium $\sigma^{\mathbf{q}_n}$ when

n is sufficiently large. This shows the existence of an equilibrium sequence, and this sequence satisfies (170) and (172). It follows from the law of large numbers that policy A is elected with probability converging to 1 as $n \to \infty$ in each of the sub-states α_1, α_2, and β_1, which occur with a joint probability larger than $1 - \psi$. This proves the claim (173) of Theorem 5.■

E.1 Multiple Media Firms

Theorem 6 *Consider the setting from Section 6.3. Take any strictly increasing Φ satisfying (17). Let $d > 2\theta$. For every constant policy $x \in \{A, B\}$, there is a sequence of equilibria $(\sigma_n^*)_{n \in \mathbb{N}}$ given d such that*

$$\lim_{n \to \infty} \Pr(x | \sigma_n^*, d, n) \geq 1 - \chi. \tag{173}$$

The proof of Theorem 6 is almost verbatim to the proof of Theorem 5, and therefore omitted.

The key observation is that in the setting with multiple media firms, there are still the four relevant substates from the setting with a monopolistic media firm and we can work with the vector of vote shares in the same way as in the proof of Theorem 5 for the monopolistic case. Additionally, we note that for each substate ω_i and pair $s = (s_1, s_2)$,

$$f(s|\omega_i; d) \quad = \quad f(s_1|\omega_i) f(s_2|\omega_i; d)$$

since the signals s_1 and s_2 are independent conditional on the state; further, the relevant substitution in the analogue of (168) is $s_2' = s_2 + m_2(\omega_j) - m_2(\omega_i)$. That is, in the analogous step, we leverage that $f(s_2|\omega_i) = f(s_2 + m_2(\omega_j) - m_2(\omega_i)|\omega_j)$, which holds by the definition of the distribution of the private signal s_2 in the substates.

F Persuasion of Behavioral Types: Numerical Example

Consider a belief $\mathbf{p} = (p_a, p_z)$ and the corresponding strategy $\sigma^{\mathbf{p}}$. Consider the information structure $\pi_n = \tilde{\pi}_n^r$ from Figure 4. In the following, we omit the dependence on $\sigma^{\mathbf{p}}$ and on π_n most of the time. We have

$$\frac{\Pr(\alpha|z;\tilde{\pi}_n^r)}{\Pr(\beta|z;\tilde{\pi}_n^r)} = \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_2|\beta)}\frac{\Pr(z|\alpha_2)}{\Pr(z|\beta_2)} \tag{174}$$

$$= \frac{1}{2}2(1-\frac{1}{n^2}), \tag{175}$$

and

$$\frac{\Pr(\alpha|a;\tilde{\pi}_n^r)}{\Pr(\beta|a;\tilde{\pi}_n^r)} = \frac{\Pr(\alpha)}{\Pr(\beta)}\frac{\sum_{i=1,2}\Pr(\alpha_i|\alpha)\Pr(a|\alpha_i)}{\sum_{i=1,2}\Pr(\beta_i|\alpha)\Pr(a|\beta_i)} \tag{176}$$

$$= \frac{1}{2}\frac{(1-\frac{3}{n})+\frac{3}{n^3})}{1-\frac{1}{n}}. \tag{177}$$

Let $q(\omega_i;\sigma^{\mathbf{p}},\kappa,n)$ the expected vote share in a substate ω_i when there is a fraction $\kappa = 0.4$ of sincere voters, who vote according to $\sigma^{\mathbf{p}'}$ for $\mathbf{p}' = (\Pr(\alpha|a),\Pr(\alpha|z))$, and a fraction $1-\kappa = 0.6$ of pivotal voters who vote according to $\sigma^{\mathbf{p}}$. Similarly, $\rho(\sigma^{\mathbf{p}}) = (\Pr(\alpha|a,\text{piv};\sigma^{\mathbf{p}},\kappa),\Pr(\alpha|z,\text{piv};\sigma^{\mathbf{p}},\kappa))$ is the vector of induced priors given by the vote shares $q(\omega_i;\sigma^{\mathbf{p}},\kappa,n)$ through (6) and (7).

Fixed Point Argument. Take any $\mathbf{p} = (p_a,p_z)$ with

$$p_a \geq 0.99, \tag{178}$$

$$p_z \in [0.475, 0.525]. \tag{179}$$

For $n \geq 84$ and $\kappa = 0.4$, we have the following bounds for the expected vote share for policy A:

$$q(\omega_1;\sigma^{\mathbf{p}},\kappa,n) \geq \kappa 0.328 + (1-\kappa)0.99 \tag{180}$$

$$\geq 0.725 \quad \text{for} \quad \omega_1 \in \{\alpha_1,\beta_1\}, \tag{181}$$

$$q(\alpha_2;\sigma^{\mathbf{p}},\kappa,n) > 0.475 \tag{182}$$

$$q(\beta_2;\sigma^{\mathbf{p}},\kappa,n) \leq 0.525. \tag{183}$$

Step 1 *For any $n \geq 84$ and any $\omega_1 \in \{\alpha_1,\beta_1\}, \omega_2' \in \{\alpha_2,\beta_2\}$,*

$$\frac{\Pr(\text{piv}|\omega_2')}{\Pr(\text{piv}|\omega_1)} \geq (1.25)^n \tag{184}$$

The step follows from,

$$
\begin{aligned}
& \frac{\Pr(\text{piv}|\omega_2')}{\Pr(\text{piv}|\omega_1)} \\
\geq\ & [1+\min_{\omega_1,\omega_2'} \frac{(q(\omega_1;\sigma^{\mathbf{P}})-\frac{1}{2})^2-(q(\omega_2';\sigma^{\mathbf{P}})-\frac{1}{2})^2}{\frac{1}{4}-(q(\omega_1;\sigma^{\mathbf{P}})-\frac{1}{2})^2}]^n \\
\geq\ & (1+(\frac{(\frac{9}{40})^2-(\frac{1}{40})^2}{\frac{1}{4}-(\frac{9}{40})^2})) \\
\geq\ & (1+\frac{80}{319})^n \\
\geq\ & (1.25)^n.
\end{aligned} \tag{185}
$$

where we used Lemma 2 and that $0.725=\frac{29}{40}$, $0.475=\frac{19}{40}$, and $0.525=\frac{21}{40}$ for the inequality on the second line, and dropped κ and n from the notation for the vote shares.

Step 2 *For $n \geq 84$: $\rho_a(\sigma^{\mathbf{P}}) \geq 0.99$, and $\rho_z(\sigma^{\mathbf{P}}) \in [0.475, 0.525]$.*

First,

$$
\begin{aligned}
\frac{\rho_a(\sigma^{\mathbf{P}})}{1-\rho_a(\sigma^{\mathbf{P}})} &\geq \frac{p_0}{1-p_0}\frac{\Pr(\alpha_2|\alpha)\Pr(b|\alpha_2)\Pr(\text{piv}|\alpha_2)}{\Pr(\beta_1|\beta)\Pr(b|\beta_1)\Pr(\text{piv}|\beta_1)} \\
&\geq \frac{1}{3}\frac{\frac{3}{n}\frac{1}{n^2}}{(1-\frac{1}{n})}(1.25)^n \\
&\geq 100 \quad \text{for} \quad n \geq 84,
\end{aligned}
$$

where we used (83) for the inequality on the second line. Hence, for $n \geq 84$,

$$
\rho(\sigma^{\mathbf{P}})_a \geq \frac{100}{101} > 0.99. \tag{186}
$$

Second,

$$
\begin{aligned}
\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} &\leq [1+\frac{|(q(\beta_2;\sigma^{\mathbf{P}})-\frac{1}{2})^2-(q(\alpha_2;\sigma^{\mathbf{P}})-\frac{1}{2})^2|}{\frac{1}{4}-(q(\beta_2;\sigma^{\mathbf{P}})-\frac{1}{2})^2}]^n \\
&\leq (1+\frac{\frac{1}{n^4}+\frac{1}{n^2}}{\frac{1}{4}-\frac{81}{40^2}})^n \\
&\leq 1.08. \quad \text{for} \quad n \geq 84.
\end{aligned}
$$

where we used Lemma 2 for the inequality on the first line. For the inequality on the second line, we used that z is sent with probability $1-\frac{1}{n^2}$ in both α_2

and β_2 such that the difference in the squared margins of victory cannot exceed $(x+\frac{1}{n^2})^2 - x^2 \leq \frac{2x}{n^2} + \frac{1}{n^4}$ where x is the minimum margin of victory in the states α_2, β_2. Finally, the inequality follows since the margin of victory in both α_2 and β_2 is bounded by 0.2. So,

$$\begin{aligned}
\frac{\rho_z(\sigma^{\mathbf{P}})_z}{1-\rho_z(\sigma^{\mathbf{P}})} &= \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\alpha_2|\alpha)}{\Pr(\beta_2|\beta)} \frac{\Pr(z|\alpha_2)}{\Pr(z|\beta_2)} \frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \\
&= (1-\frac{1}{n^2})\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \\
&\leq 1.08 \quad \text{for} \quad n \geq 84.
\end{aligned}$$

Consequently, for all $n \geq 84$,

$$\rho(\sigma^{\mathbf{P}})_z \leq \frac{1.08}{2.08} < 0.52 \tag{187}$$

Third,

$$\begin{aligned}
\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} &\geq (1-\frac{|(q(\beta_2;\sigma^{\mathbf{P}})-\frac{1}{2})^2-(q(\alpha_2;\sigma^{\mathbf{P}})-\frac{1}{2})^2|}{\frac{1}{4}-(q(\beta_2;\sigma^{\mathbf{P}})-\frac{1}{2})^2} \\
&\geq (1-\frac{\frac{1}{n^4}+\frac{1}{n^2}}{\frac{1}{4}-\frac{81}{40^2}})^n \\
&\geq 0.934 \quad \text{for} \quad n \geq 84.
\end{aligned} \tag{188}$$

So, for all $n \geq 84$,

$$\begin{aligned}
\frac{\rho(\sigma^{\mathbf{P}})_z}{1-\rho(\sigma^{\mathbf{P}})_z} &= (1-\frac{1}{n^2})\frac{\Pr(\text{piv}|\alpha_2;\sigma^{\mathbf{P}})}{\Pr(\text{piv}|\beta_2;\sigma^{\mathbf{P}})} \\
&\geq 0.93.
\end{aligned}$$

This gives for all $n \geq 84$,

$$\rho(\sigma^{\mathbf{P}})_z \geq \frac{0.93}{1+0.93} \geq 0.48. \tag{189}$$

The claim follows from (186), (187)), and (189).

Step 3 *For $n \geq 84$, there is an equilibrium σ_n^* which satisfies (180) - (183).*

It follows from Step 2 that, for any $n \geq 84$, the continuous map that sends $\mathbf{p}$ to $\boldsymbol{\rho}(\sigma^{\mathbf{P}})$ is a self-map on the set of beliefs that satisfy (178) - (179). It follows from the Kakutani fixed point theorem that there exists fixed points $\mathbf{p}_n^*$ that satisfy

(178) - (179). The corresponding strategies $\sigma^{\mathbf{p}_n^*}$ are equilibria (compare to (13)) and they satisfy (180) - (183).

Step 4 *Given the equilibrium σ_n^* for $n \geq 84$, the probability that A is elected is larger than* 99.9%*.*

Evaluation of the binomial distribution shows that $\Pr(\mathcal{B}(2n+1,x)) > n) \geq 0.999999$ if $n \geq 84$ and $x \geq 0.725$. Hence, given σ_n^*, A is elected with probability larger than 99.9% in the states α_1 and β_1. Finally, the claim follows since these states occur with probability $(1 - \frac{r}{1-r}\frac{1}{n}) = (1 - \frac{2}{n})$ and $(1 - \frac{1}{n})$ respectively. The fourth step finishes the calculations for the example.