Problem Set 1: Introduction and Practice on Proofs
CS254, Fall 2019, Anna Rafferty

Solution: Please do not post these solutions or share them with anyone outside this class. Thanks!

You are welcome to talk to others about the problems on this problem set, but you should **read and attempt each problem prior do discussing it with others.** The write up must be your own work. That means that **you should write your answers by yourself, without looking at notes from sessions with collaborators, and you must be able to explain any answer you write down.**

Due date: All problems are due at **9PM on Wednesday, 18 September 2019**. Look at the homework handout on Moodle for how to turn them in. **The files must be named as described on the homework handout - note that the only problems you'll need to submit on Courses are numbered 04, 05, and 06.** Remember to also fill out the collaboration form on Moodle; this form must be turned in for each problem set in order to get credit for your work.

1. Read the syllabus and other documents handed out in class on Monday.

2. Complete the background survey on Moodle.

3. Introduce yourself on Piazza - look for the "Introductions" thread.

4. One of my goals for this course is that every student has the opportunity to learn and to make mistakes, since making mistakes is one of the most effective ways to learn. In class, we often brainstorm possible ways of constructing a machine or developing a proof, and that works most effectively when students share lots of ideas - including those they aren't yet quite sure if they'll work. I encourage you to volunteer answers in small groups or to the full class even if that's not something you typically feel comfortable doing; that discomfort is often the first step to gaining deeper understanding of the content. One of the barriers to this type of engagement can be feeling like you're "not a math person" or are "just not good at proofs." While this class isn't a math class, it has lots of abstract ideas and many of the techniques we use are mathematical. Thus, I want to address these barriers up front: there's no evidence that there are "math people" and "non math people." In fact, lots of neuroscience research demonstrates that actually, anyone can learn math - working hard to develop effective problem solving strategies and believing that you can learn has a much bigger impact than genetics. Watch this video about the connections between mistakes, beliefs, and learning - Jo Boaler TED Talk excerpt - and then read these two webpages describing two types of mindsets and how to move towards a growth mindset - Growth Mindset and Changing your mindset. For your response to this question, write a letter to a future student where you talk about a struggle you've had in learning and how you overcame this struggle. How did it make you feel when you struggled, and what did overcoming this struggle teach you? What advice would give to this future student when they face their own struggles? You can bring in some of the mindset ideas in your advice if you would like. Your letter doesn't need to be more than a paragraph or two - the goal isn't to force you to write a lot, but to give you an opportunity to reflect on your past learning struggles and how you've worked through them in order to give you ideas for how to work through struggles you might face in this class.

5. Let Σ be an alphabet, and define a string in Σ^* as being either (i) ϵ or (ii) aw where $a \in \Sigma$ and $w \in \Sigma^*$. For a string $x \in \Sigma^*$, x^R denotes the *reverse* of the string x. Formally we define the reverse of a string x as follows:

   - If $x = \epsilon$, then $x^R = x$.
   - If $x = ax'$ for $a \in \Sigma$, $x' \in \Sigma^*$, then $x^R = (x')^R a$.

   Use structural induction to prove that for all strings $x \in \Sigma^*$, $(x^R)^R = x$. In your proof, you should use only the definition of a string in Σ^* and the definition of reverse. Don't skip steps in your proof - show me all the details. Make sure your proof is readable - it should clearly state what quantity you're performing induction on, what type of induction you're using, what part is the base case and what part is the inductive case, and within the inductive case, what the inductive hypothesis is. If you're using the definition of reverse, make clear what string is being reversed. If you want a review of structural induction, check out the optional reading about this topic on Moodle.

   Solution:

   As we discussed in class, one route for this one is to use the property shown in the handout on moodle: For all strings x, y, $(xy)^R = y^R x^R$. After more consideration, I think this is the simplest way to prove this, as noted on

Piazza; my apologies for initially leading you astray, and thanks for some good discussions in office hours. So, here's what that route looks like:

To prove that for all strings $x \in \Sigma^*$, $(x^R)^R = x$, we'll perform structural induction on strings x.

Base case $x = \epsilon$: If $x = \epsilon$, then:

$$\begin{aligned} (x^R)^R &= (\epsilon^R)^R && \text{definition of } x \\ &= (\epsilon)^R && \text{definition of reverse of } \epsilon \\ &= \epsilon && \text{definition of reverse of } \epsilon \\ &= x && \text{definition of } x \end{aligned}$$

Inductive case: We'll show that our inductive hypothesis $(w^R)^R = w$ implies that $(x^R)^R$ for $x = aw$ for $a \in \Sigma$, $w \in \Sigma^*$.

We'll first show a lemma: For a single character $a \in \Sigma$, $a = a^R$:

$$\begin{aligned} a^R &= (a\epsilon)^R && \text{string identity} \\ &= \epsilon^R a && \text{definition of reverse for a string} a\epsilon \\ &= \epsilon a && \text{definition of reverse of} \epsilon \\ &= a && \text{string identity} \end{aligned}$$

Our inductive hypothesis is that $(w^R)^R = w$:

$$\begin{aligned} (x^R)^R &= ((aw)^R)^R && \text{definition of } x \\ &= (w^R a)^R && \text{definition of } (aw)^R \\ &= a^R (w^R)^R && \text{property shown in handout for strings } w^R \text{ and } a^R \\ &= a (w^R)^R && \text{lemma above} \\ &= aw && \text{inductive hypothesis} \end{aligned}$$

6. Let the alphabet $\Sigma = \{0, 1\}$ and consider strings consisting only of characters in Σ. Define the function $\#1(w)$ as a function that computes the number of 1s in the string w. For example, if $w = 10001$, $\#1(w) = 2$. We can define this function recursively as follows:

   For a string w:

   - If $w = \epsilon$, then $\#1(w) = 0$.
   - If $w \neq \epsilon$, then $w = 1s$ or $w = 0s$ for some string s. If $w = 1s$, then $\#1(w) = 1 + \#1(s)$. If $w = 0s$, then $\#1(w) = \#1(s)$.

   Prove that given strings v and w, $\#1(vw) = \#1(v) + \#1(w)$. As above, show all of your work, do not skip steps, and make sure your proof is clear and understandable to the reader. You may use without proof some or all of the following: the associative property of string concatenation, associativity of addition, the additive property of addition ($a + 0 = 0 + a = a$ for numeric a), and the identity property of string concatenation ($\epsilon t = t\epsilon = t$ for any string t). Hint: Use structural induction.

Solution: We will perform structural induction on v, and we'll assume w is any string. We have three cases: either $v = \epsilon$, $v = 1s$ for a string s, or $v = 0s$ for a string s. We handle the first case in the base case, and the other two in the inductive case.

Base case $v = \epsilon$. Then $\#1(vw) = \#1(\epsilon w) = \#1(w)$ by the properties of string concatenation ($\epsilon s = s$ for any string s). Continuing from $\#1(w)$, $\#1(w) = 0 + \#1(w)$ by additive identity, and $0 + \#1(w) = \#1(\epsilon) + \#1(w)$ according to the given definition of $\#1$. Thus, $\#1(vw) = \#1(\epsilon) + \#1(w) = \#1(v) + \#1(w)$ when $v = \epsilon$.

Inductive case 1 in which $v = 0s$: Assume the inductive hypothesis holds for s: $\#1(sw) = \#1(s) + \#1(w)$. We seek to show that $\#1(vw) = \#1((0s)w) = \#1(0s) + \#1(w)$. We proceed as follows:

$$
\begin{aligned}
\#1(vw) &= \#1((0s)w)) && \text{definition of } v \\
&= \#1(0(sw)) && \text{associativity} \\
&= 0 + \#1(sw) && \text{definition of } \#1 \\
&= 0 + \#1(s) + \#1(w) && \text{inductive hypothesis} \\
&= \#1(0s) + \#1(w) && \text{definition of } \#1 \\
&= \#1(v) + \#1(w) && \text{definition of } v
\end{aligned}
$$

Thus, if $v = 0s$, $\#1(vw) = \#1(v) + \#1(w)$.

Inductive case 2 in which $v = 1s$ proceeds similarly: Let w be some string, and assume the inductive hypothesis holds for s: $\#1(sw) = \#1(s) + \#1(w)$. We seek to show that $\#1(vw) = \#1((1s)w) = \#1(1s) + \#1(w)$:

$$
\begin{aligned}
\#1(vw) &= \#1((1s)w)) && \text{definition of } v \\
&= \#1(1(sw)) && \text{associativity} \\
&= 1 + \#1(sw) && \text{definition of } \#1 \\
&= 1 + \#1(s) + \#1(w) && \text{inductive hypothesis} \\
&= \#1(1s) + \#1(w) && \text{definition of } \#1 \\
&= \#1(v) + \#1(w) && \text{definition of } v
\end{aligned}
$$

Thus, we have shown that for both possible ways of forming v, $\#1(vw) = \#1(v) + \#1(w)$. We have proven the inductive case, and so we have that for any v and w, $\#1(vw) = \#1(v) + \#1(w)$.