UNIVERSITY OF CAMBRIDGE

Introduction to Inference

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Previous Session

- What does it mean to learn from data?

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- What does it mean to learn from data?
- What are the conclusions we can draw?

This Session

- Formalise ML

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- Probability Theory as one approach to encode beliefs

Statistical Learing Theory

Learning Theory

- $\mathcal{F}$ space of functions

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- $\mathcal{A}$ learning algorithm

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- $\mathcal{S} = \{(x_1, y_1), \ldots, (x_N, y_N)\}$

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Learning Theory

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- $\mathcal{S} = \{(x_1, y_1), \ldots, (x_N, y_N)\}$
- $\mathcal{S} \sim P(\mathcal{X} \times \mathcal{Y})$
- $\ell(\mathcal{A}_{\mathcal{F}}(\mathcal{S}), x, y)$ loss function

Statistical Learning

$$e(\mathcal{S}, \mathcal{A}, \mathcal{F}) = \mathbb{E}_{P(\{\mathcal{X},\mathcal{Y}\})}\left[\ell(\mathcal{A}_{\mathcal{F}}(\mathcal{S}), x, y)\right]$$

Statistical Learning

$$
\begin{aligned}
e(\mathcal{S}, \mathcal{A}, \mathcal{F}) &= \mathbb{E}_{P(\{\mathcal{X},\mathcal{Y}\})}\left[\ell(\mathcal{A}_{\mathcal{F}}(\mathcal{S}), x, y)\right] \\
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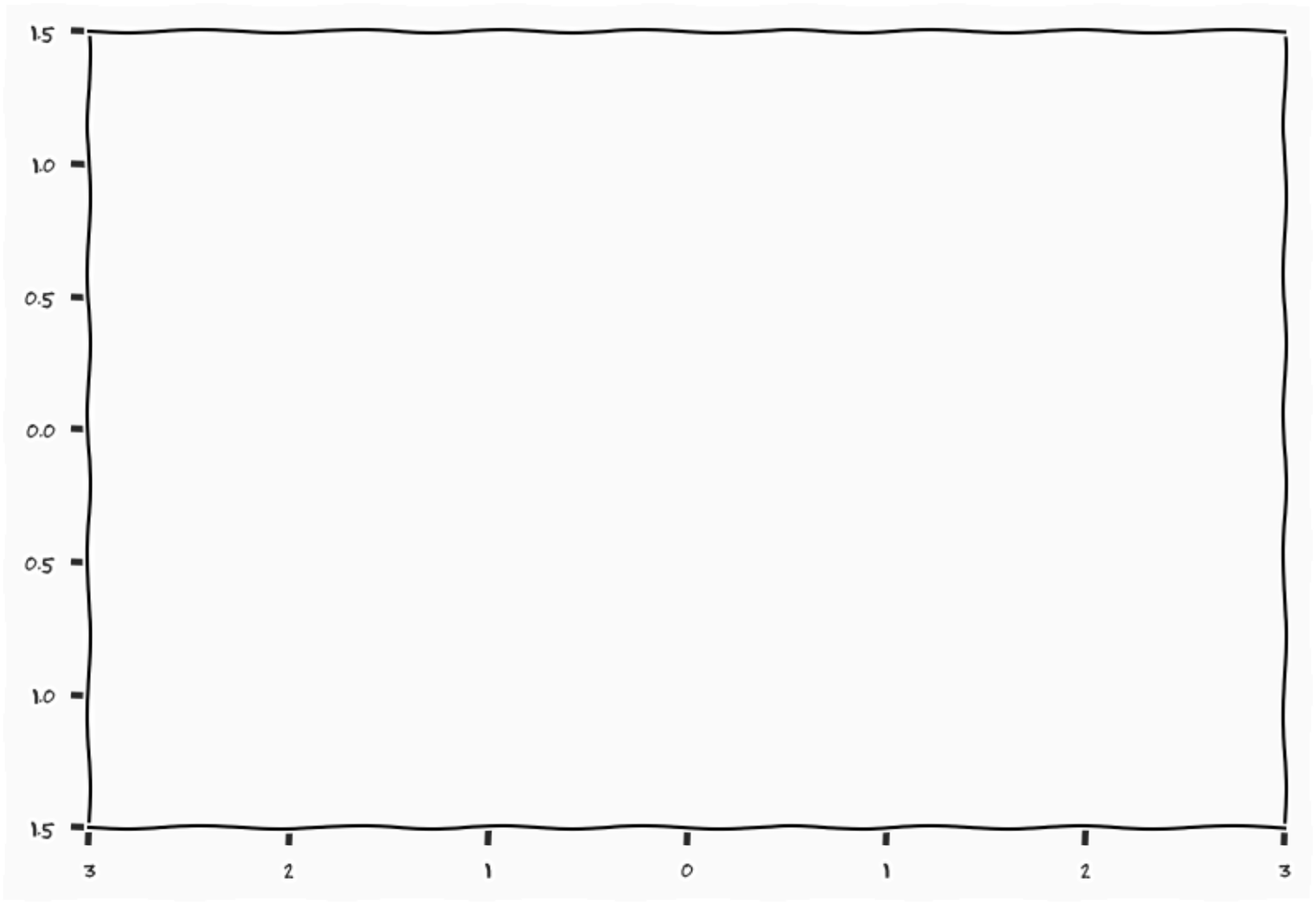
&= \int \ell(\mathcal{A}_{\mathcal{F}}(\mathcal{S}), x, y) p(x, y) \mathrm{d}x \mathrm{d}y \\
&\approx \frac{1}{M} \sum_{n=1}^{M} \ell(\mathcal{A}_{\mathcal{F}}(\mathcal{S}), x_n, y_n)
\end{aligned}
$$

No Free Lunch

We can come up with a combination of $\{\mathcal{S}, \mathcal{A}, \mathcal{F}\}$ that makes $e(\mathcal{S}, \mathcal{A}, \mathcal{F})$ take an arbitary value

Example



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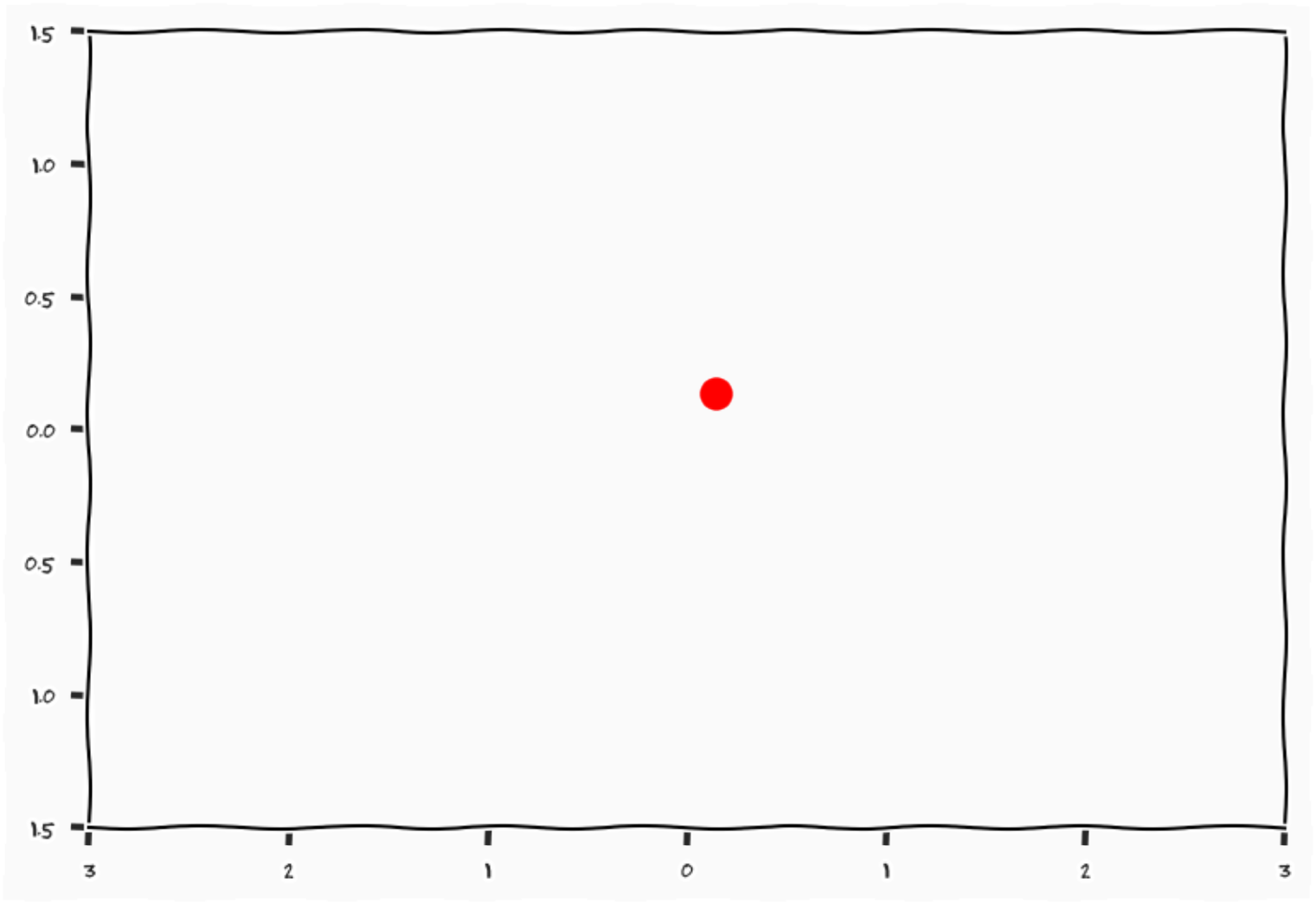

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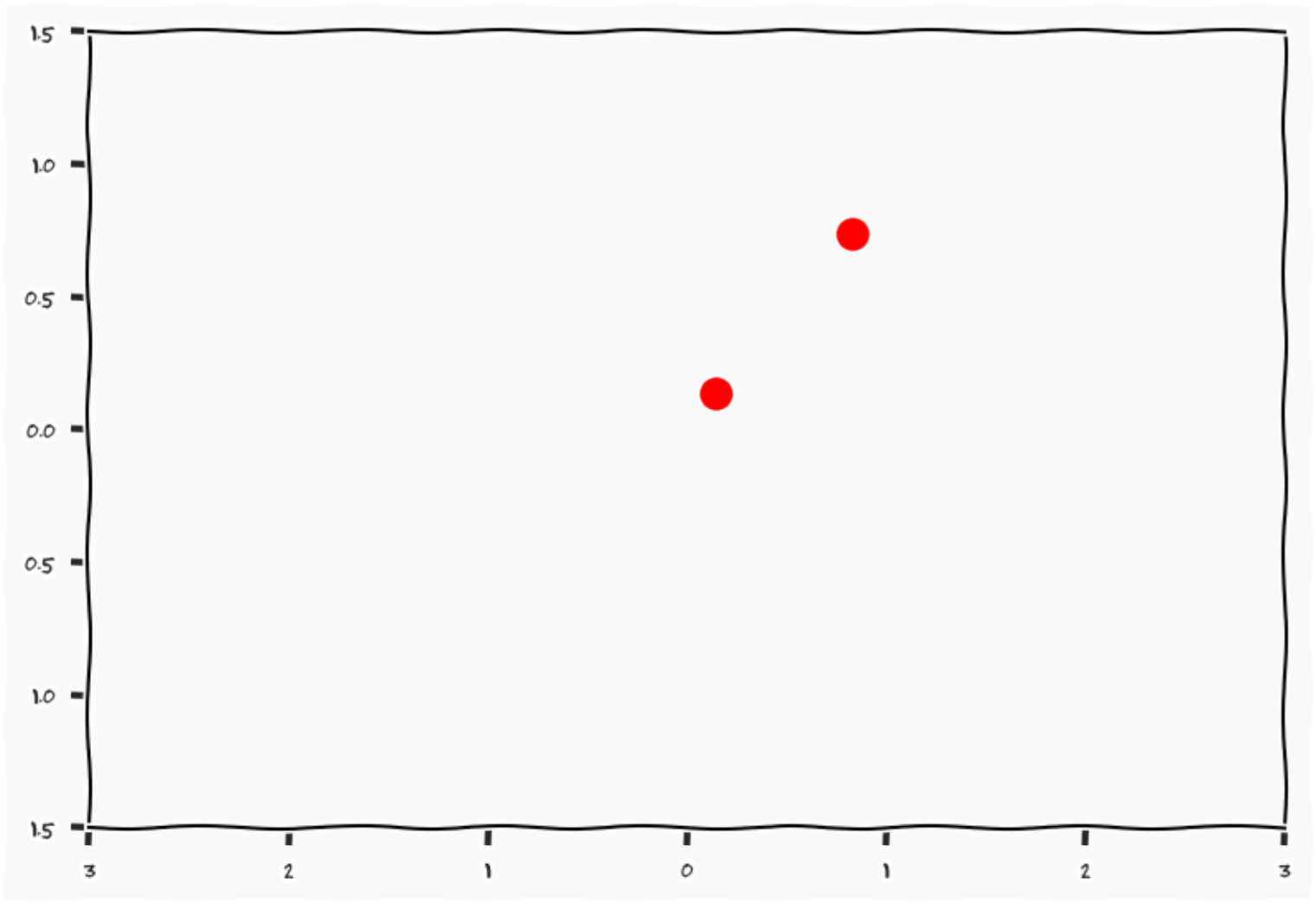

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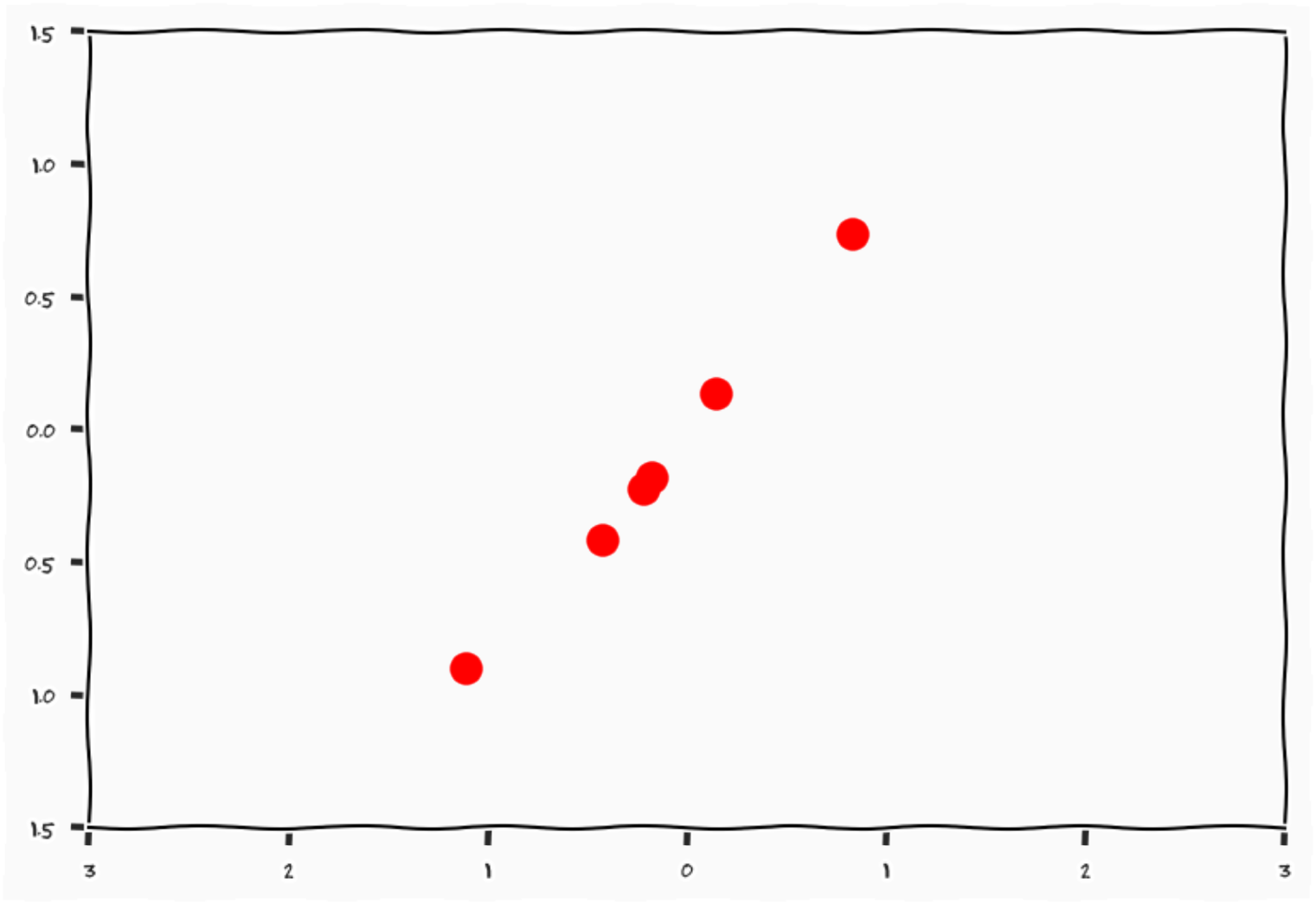

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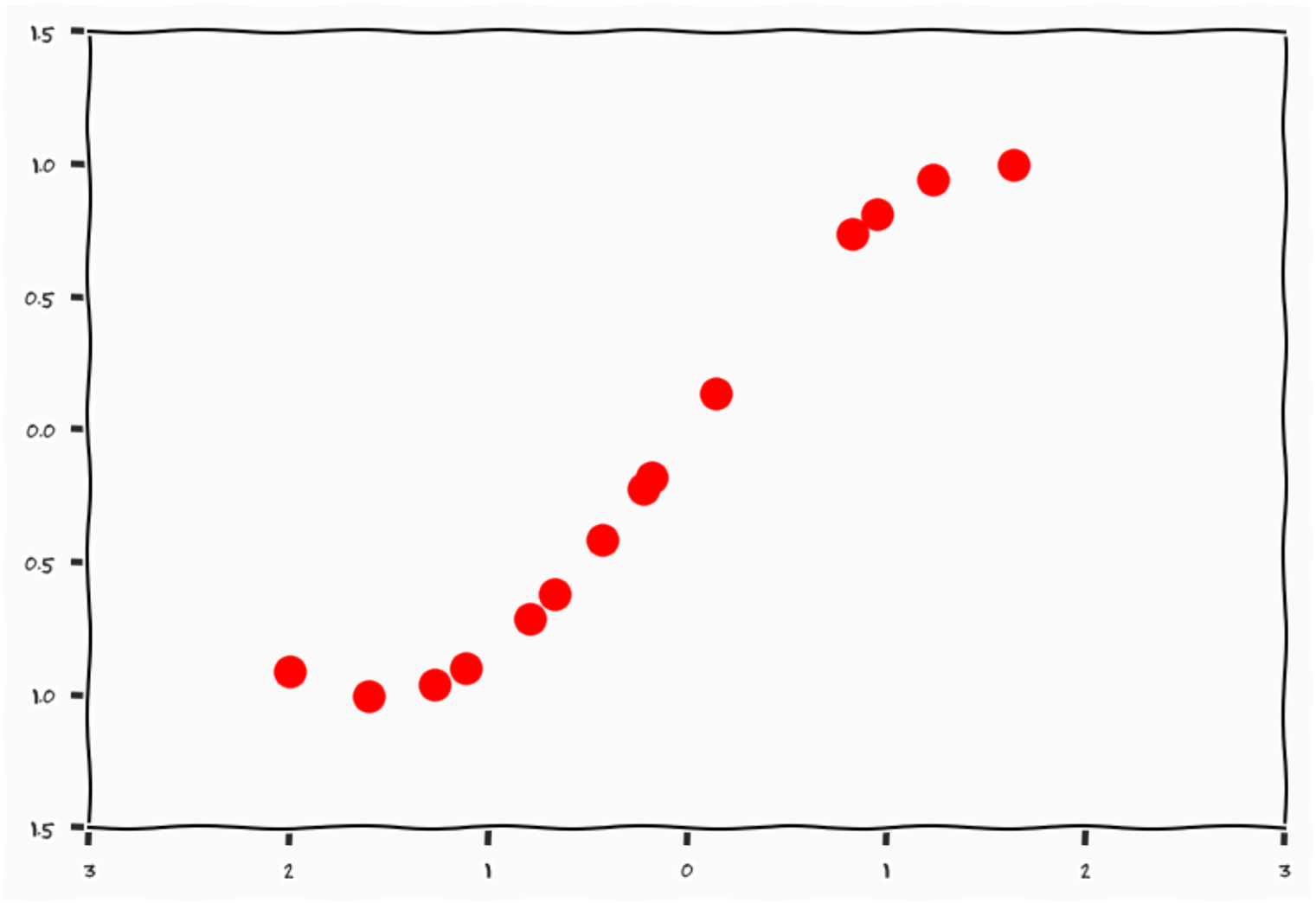

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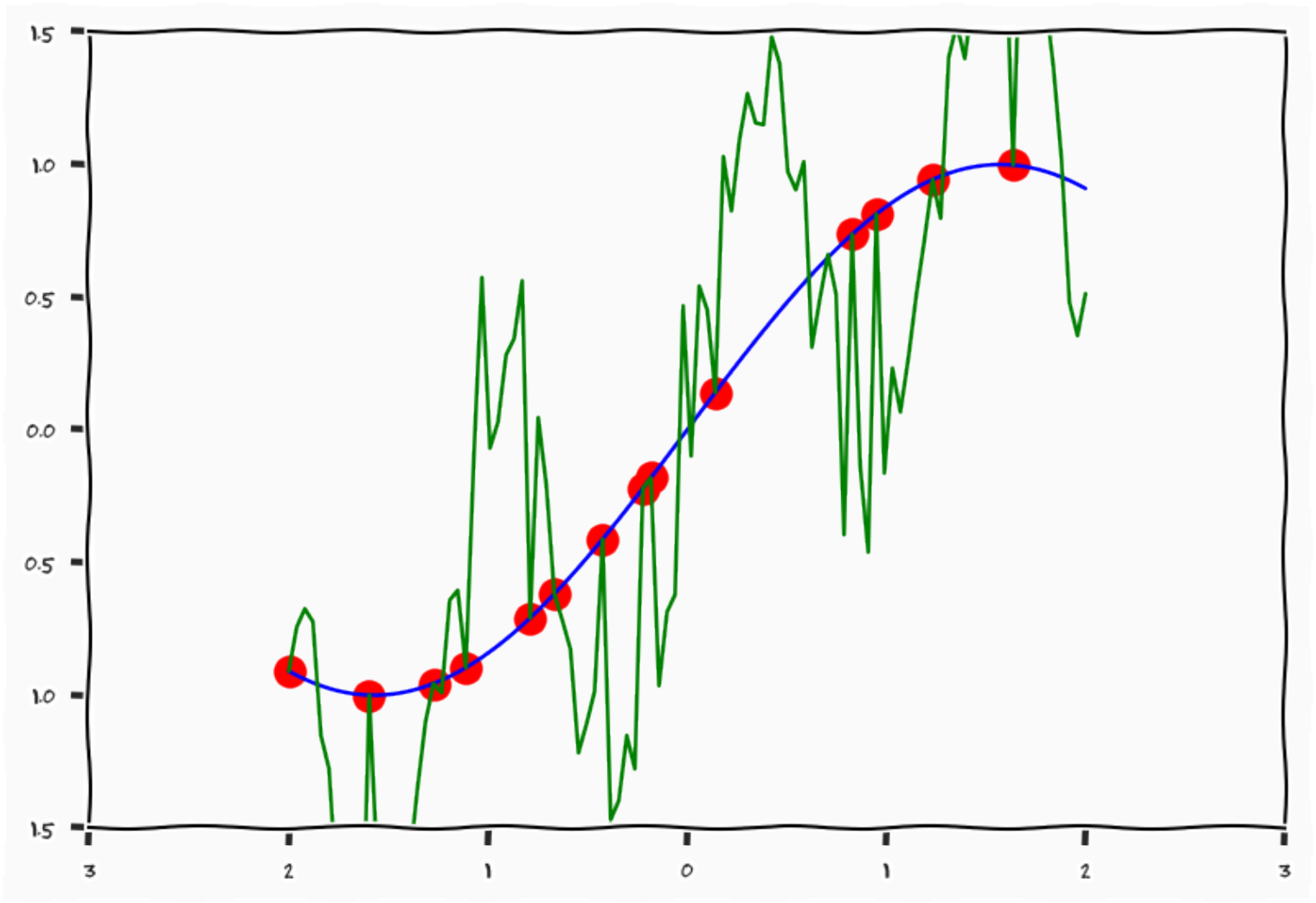
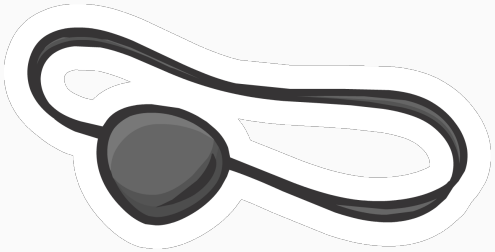


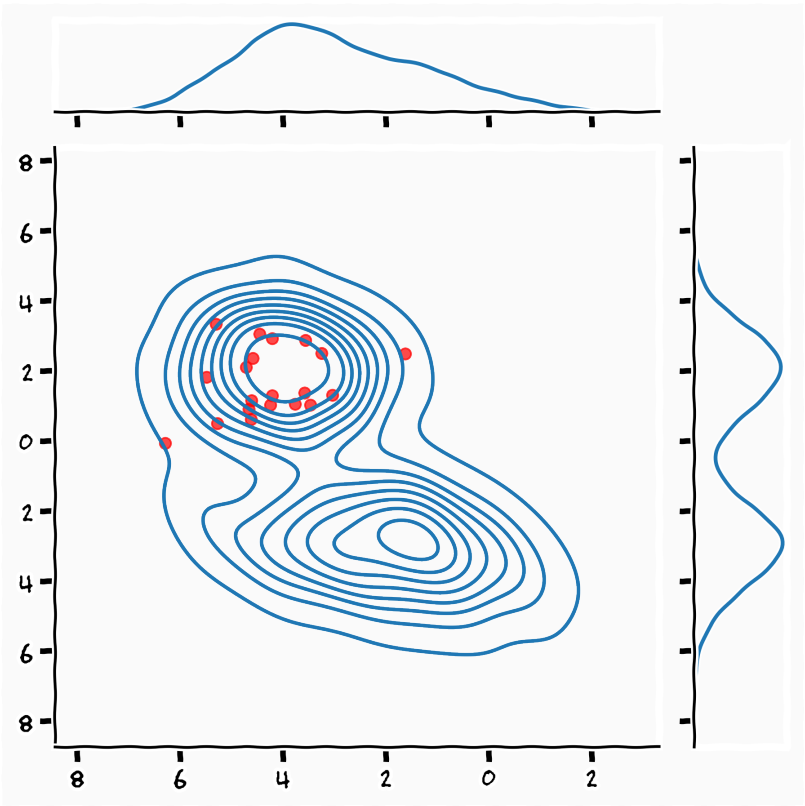
Assumptions: Algorithms



Statistical Learning

$$\mathcal{A}_{\mathcal{F}}(\mathcal{S})$$

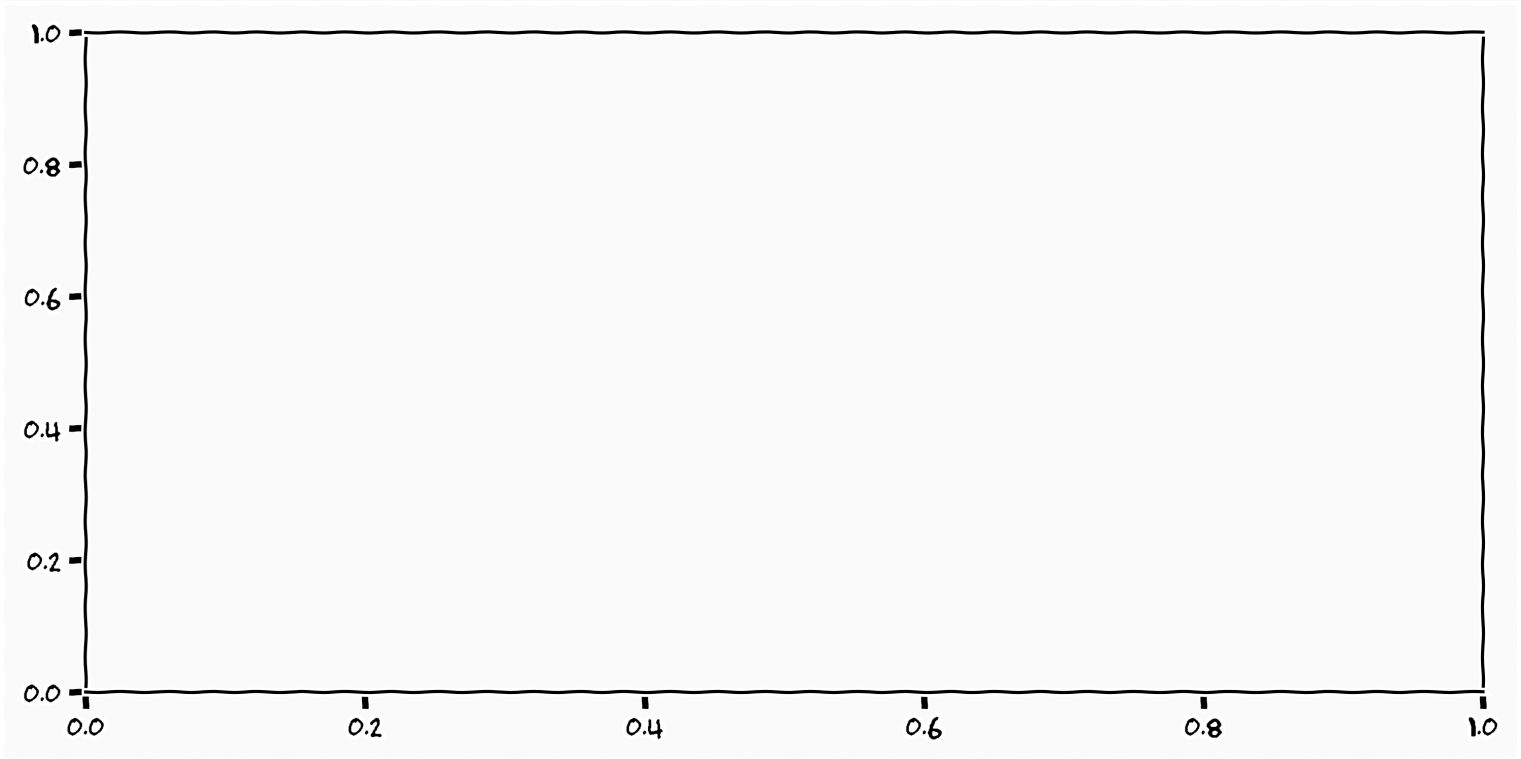
Assumptions: Biased Sample



Statistical Learning

$$\mathcal{A}_{\mathcal{F}}(\mathcal{S})$$

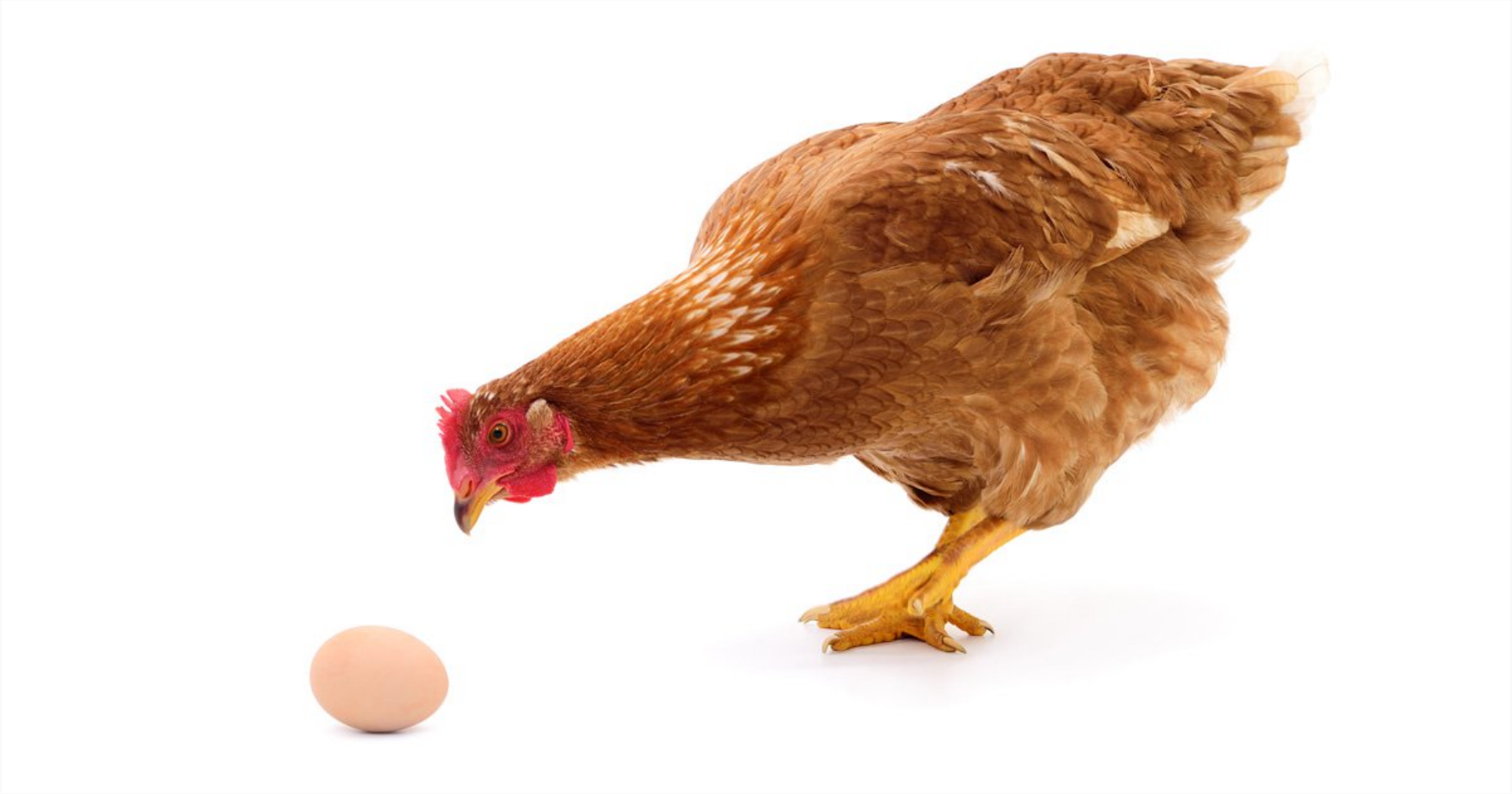
Assumptions: Hypothesis space



Statistical Learning

$$\mathcal{A}_{\mathcal{F}}(\mathcal{S})$$

Chicken and Egg



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    - The opposite is true
    - The best possible model has infinite support (nothing is excluded) but very focused mass
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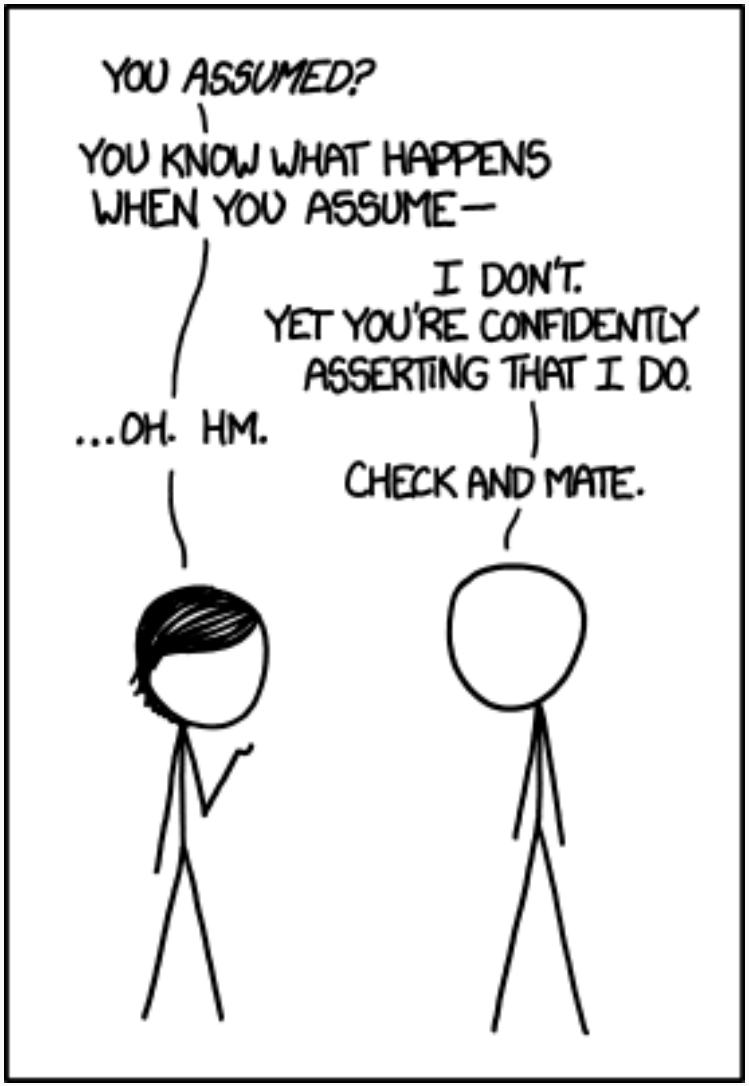
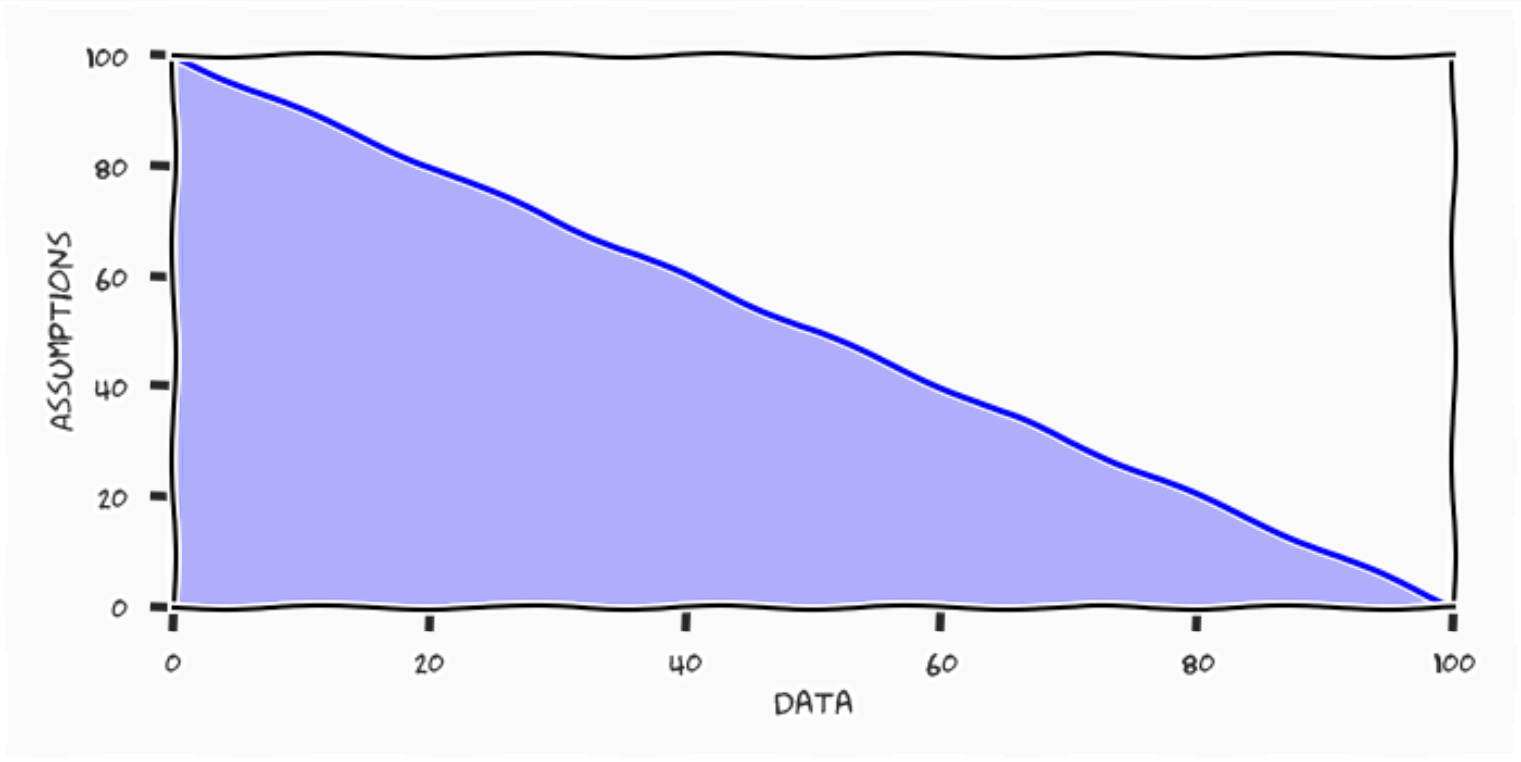
- Intepretations of observations are relative to our assumptions
- *Your solution can only ever be interpreted in the light of your assumptions*

Assumptions/Beliefs



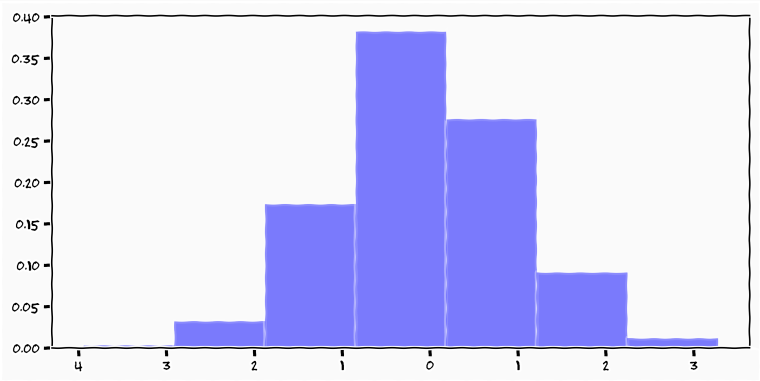
Data and Knowledge

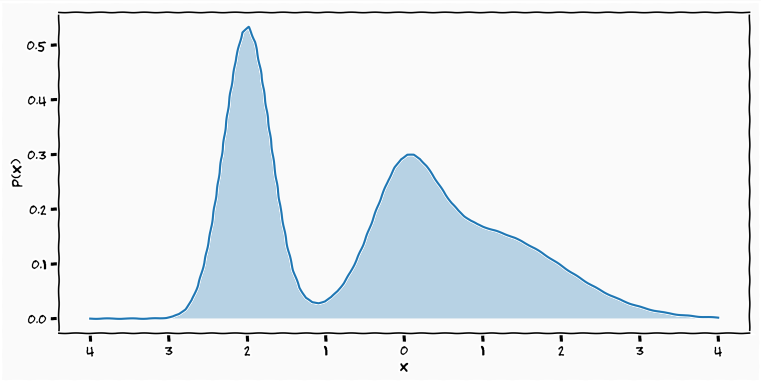


Statistical Modelling

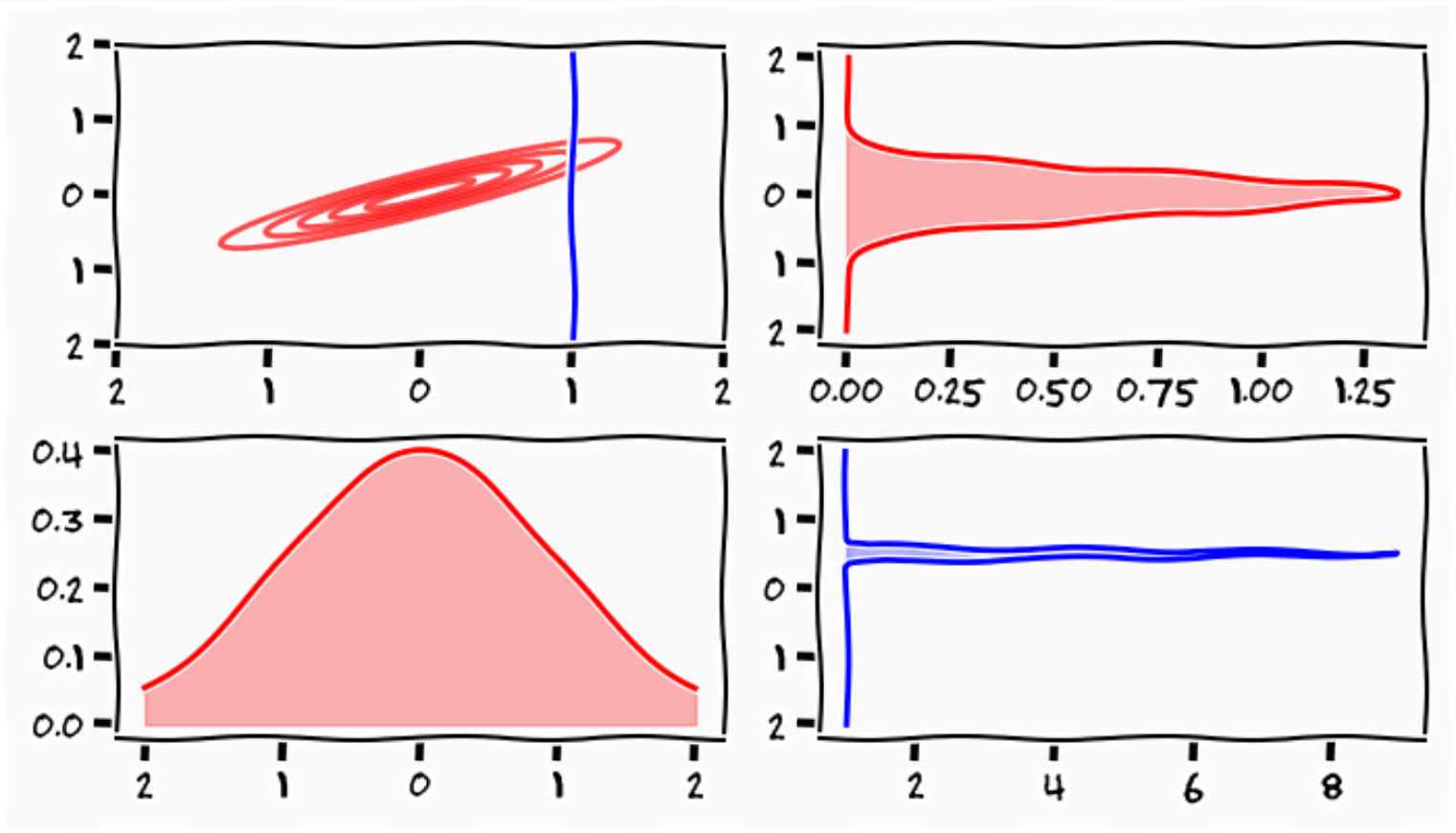
Encoding Beliefs

Probabilities





Basic Probabilities



Variables

Deterministic Variable

Code

```
int x = 3;
float y = 3.14;
```

Stochastic Variable

$$x \sim p(x)$$

$$y \sim \mathcal{N}(0, 1)$$

Rules of Probablity

Sum Rule

$$p(x) = \sum_{\forall y \in \mathcal{Y}} p(x, y)$$

Product Rule

$$p(x, y) = p(x \mid y)p(y)$$

Baye's "Rule"

$$p(x, y) = p(y|x)p(x)$$

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Baye's "Rule"

$$
\begin{aligned}
p(x, y) &= p(y|x)p(x) \\
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p(x|y)p(y) &= p(y|x)p(x) \\
p(x|y) &= \frac{p(y|x)p(x)}{p(y)}
\end{aligned}
$$

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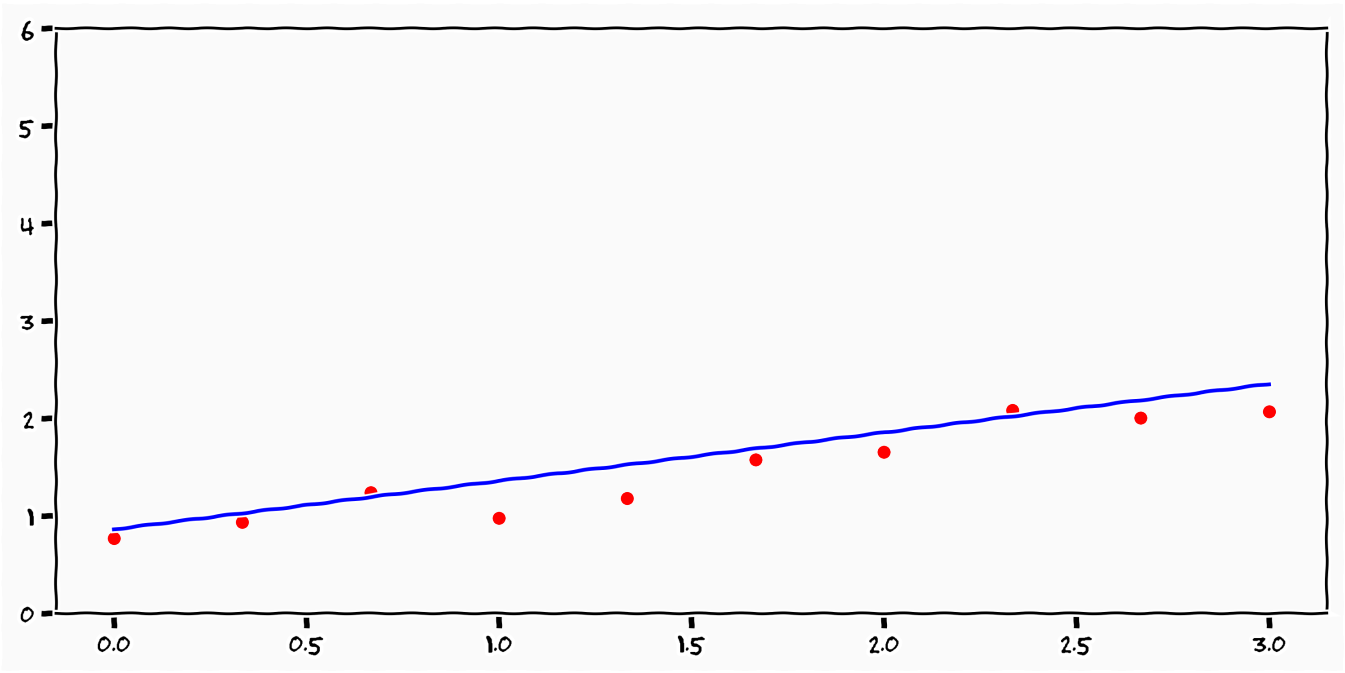
p(x|y)p(y) &= p(y|x)p(x) \\
p(x|y) &= \frac{p(y|x)p(x)}{p(y)} \\
&= \frac{p(y|x)p(x)}{\sum_x p(y|x)p(x)}
\end{aligned}
$$

Why get a tattoo?

"Model"

$$y = f(x)$$

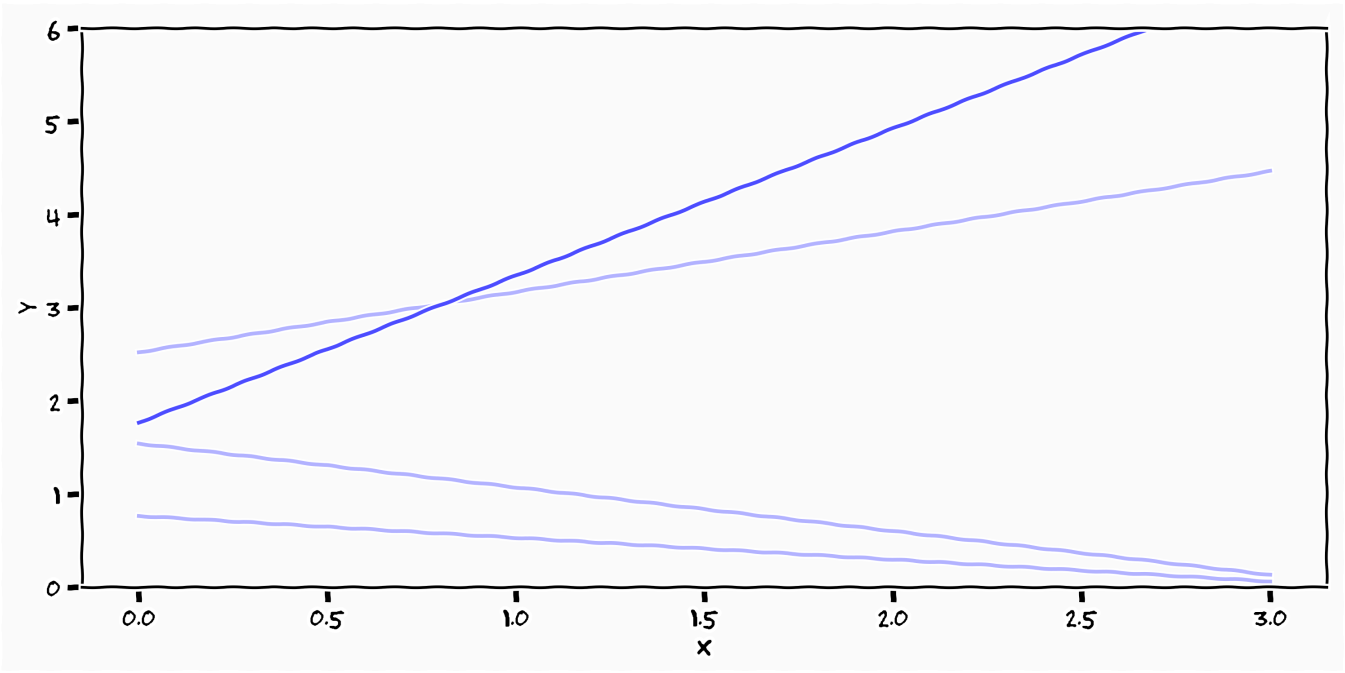
Likelihood



$$p(y \mid f, x)$$

How strongly do I believe that the data y that I see comes from the function f at input x

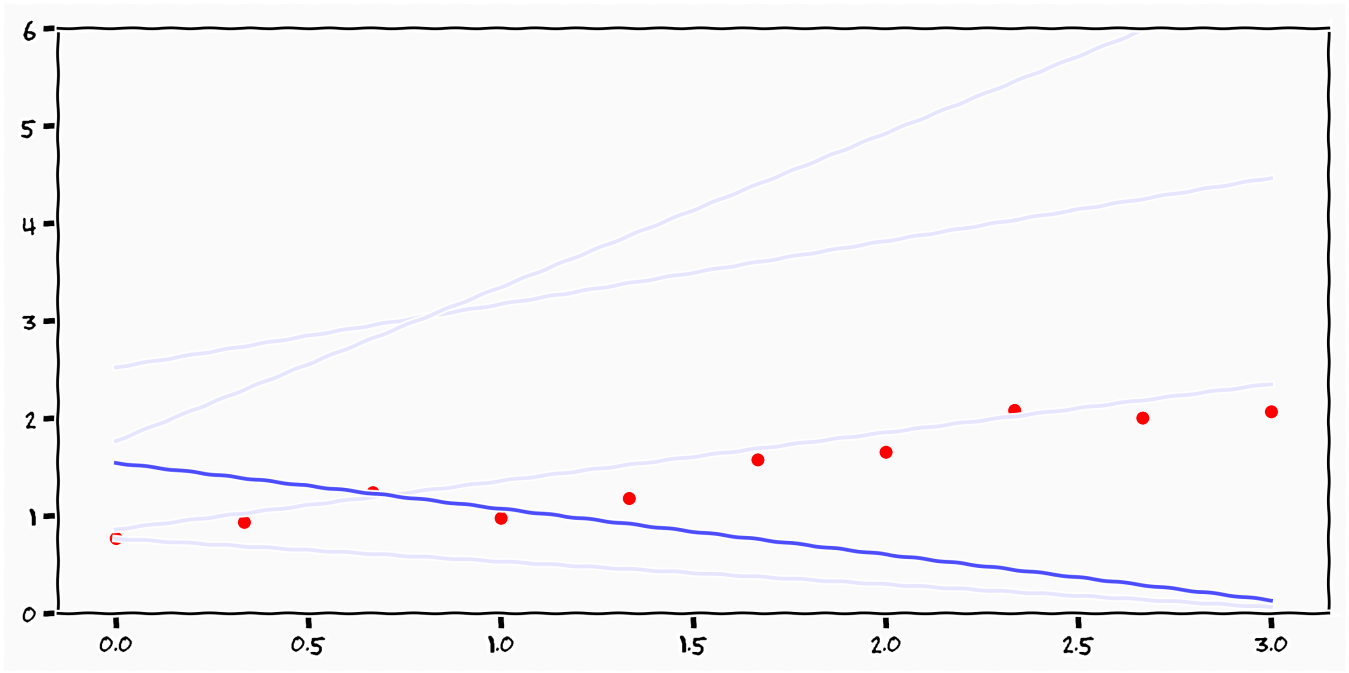
Prior



$$p(f)$$

Before I have seen the data how much do I believe in each possible function?

Posterior



$$p(f \mid y)$$

Given that I have seen observations y what is my updated belief about the function f

Bayes Rule

$$\underbrace{p(f \mid y)}_{\text{Posterior}} = \frac{\overbrace{p(y \mid f, x)}^{\text{Likelihood}} \overbrace{p(f)}^{\text{Prior}}}{\underbrace{p(y)}_{\text{Marginal Likelihood}}}$$

Model we define `Likelihood` and `Prior`

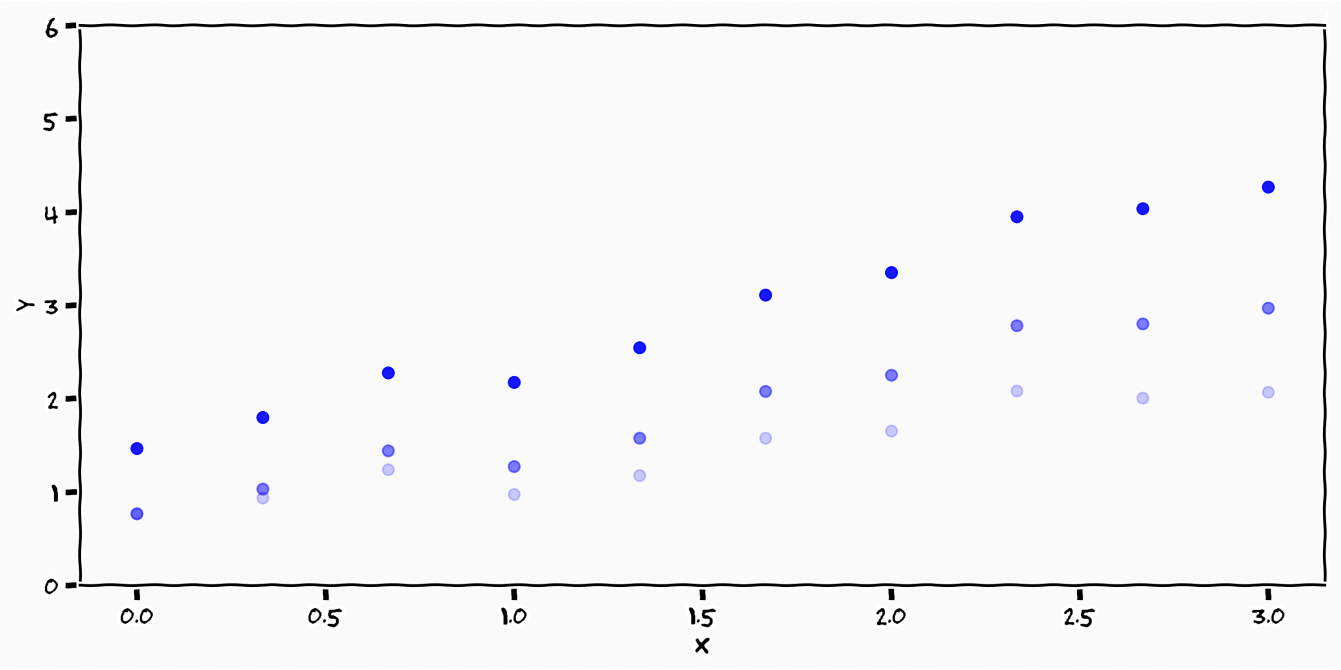
Inference we compute `Posterior` through *marginalisation* of our beliefs

Marginalisation

$$p(y) = \int p(f, y) \mathrm{d}f = \int p(y|f)p(f) \mathrm{d}f$$

- Marginalisation accounts for all your belief in a variable
- Importantly it does not "remove" the effect of the variable
- Marginalisation is an expectation over a conditional distribution

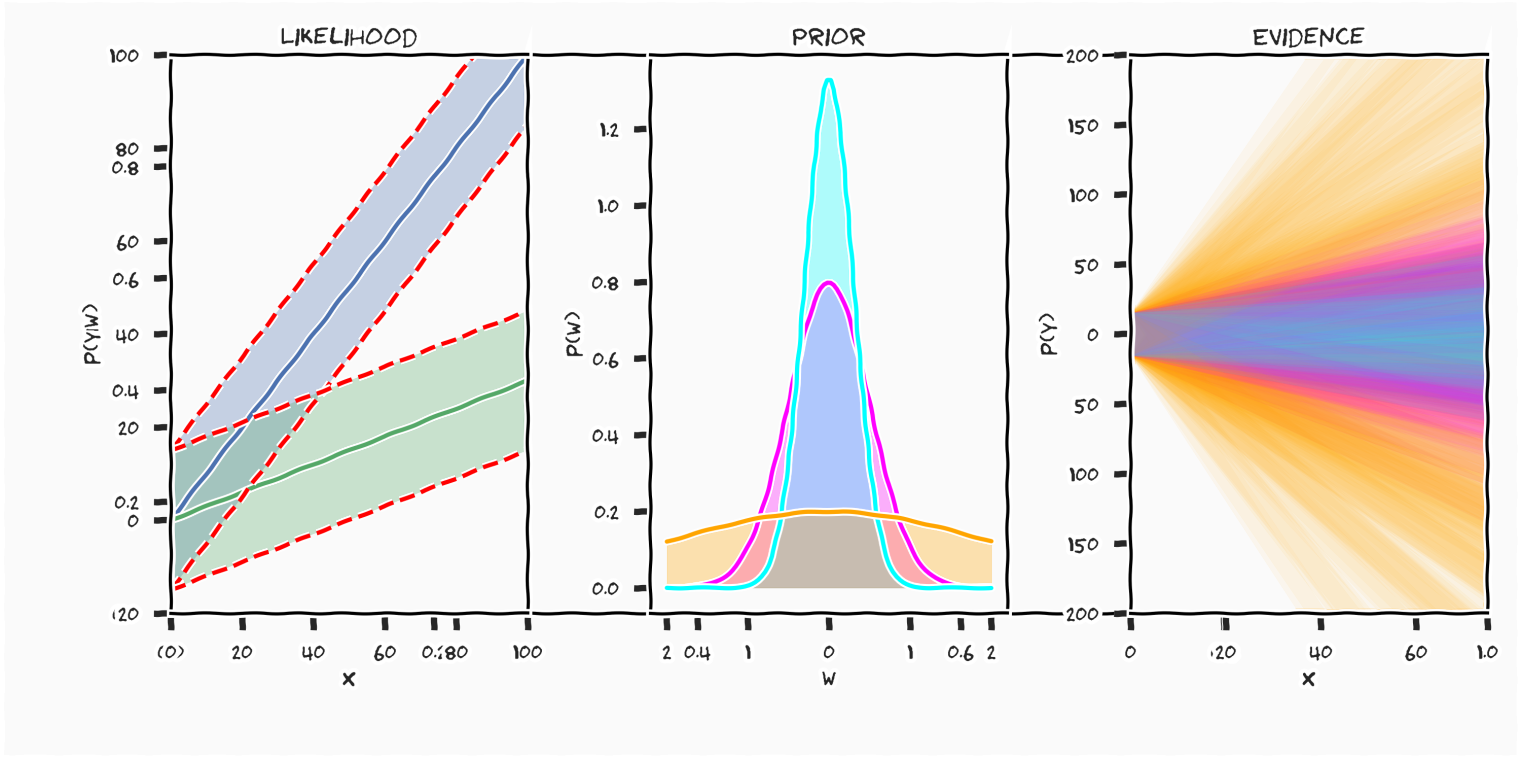
Marginal Likelihood/Evidence



$$p(y) = \int p(y \mid f) p(f) \mathrm{d}f$$

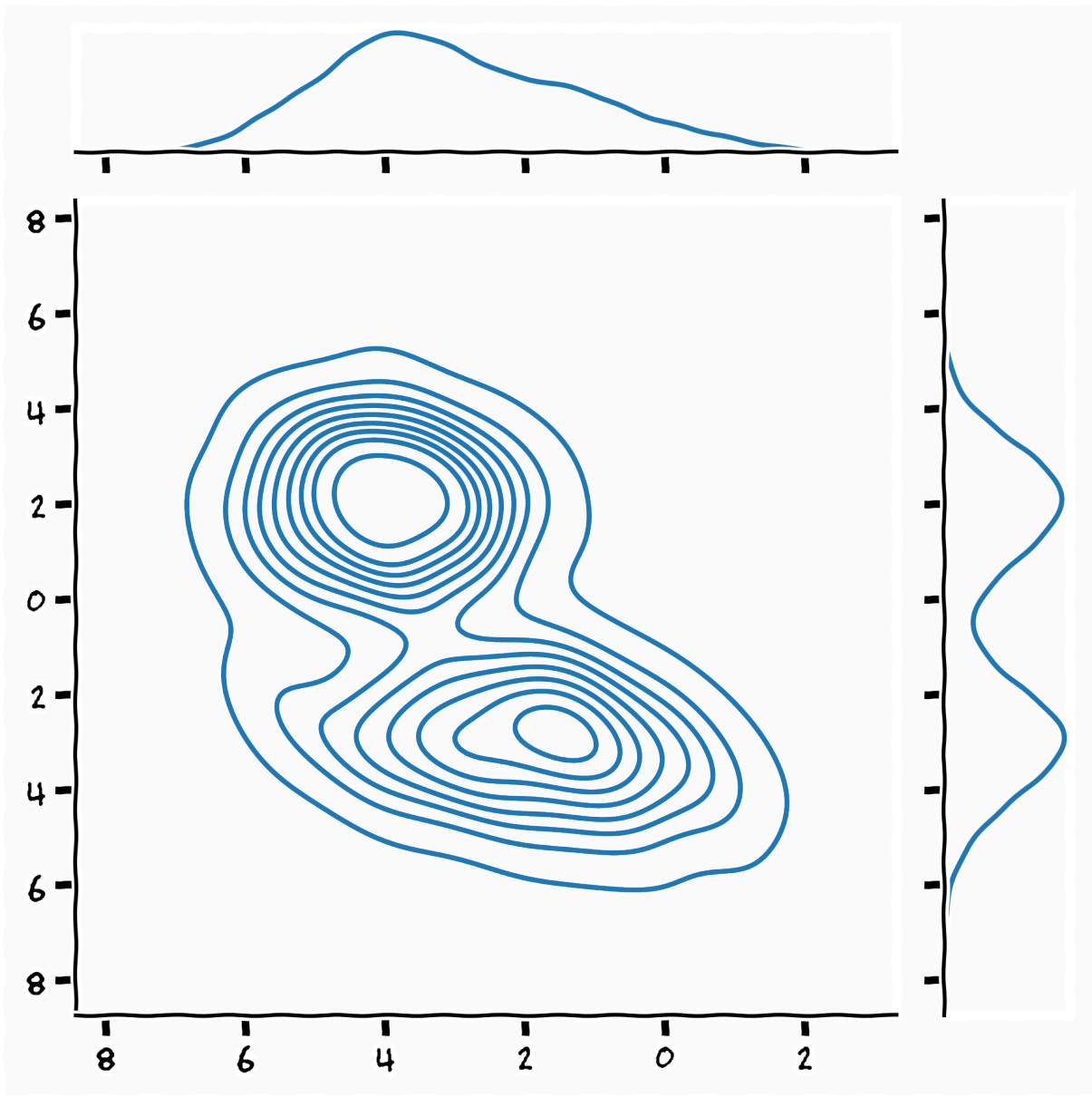
Given different data sets y_i how much evidence do they provide of the model?

Regression Model



$$y = x \cdot w \pm 15$$

Marginal Distribution



Marginalisation

Next time you want to give your friends a compliment, tell them that you have completely marginalised them from your life

Bernoulli Trial

Coin Toss[1]

- We want to figure out if a coin is biased based on data
- We will toss the coin N times

$$\mathcal{D} = \{x_1, x_2, \ldots, x_N\}$$

- "Biasedness" of coin μ

[1] how much do you need to bend a coin to make it biased

Inference

$$p(\mu \mid \mathcal{D}) = \frac{p(\mathcal{D} \mid \mu)p(\mu)}{p(\mathcal{D})}$$

Beliefs/Assumptions

$$p(\mathcal{D} \mid \mu) = \prod_{n=1}^{N} p(x_n \mid \mu)$$

- We completely trust our capability to see the side the coin landed on

Beliefs/Assumptions

$$p(\mathcal{D} \mid \mu) = \prod_{n=1}^{N} p(x_n \mid \mu)$$

- We completely trust our capability to see the side the coin landed on
- Tossing the coin does not significantly change the coin

Bernoulli Distribution

- Distribution over binary random variable $x \in \{0, 1\}$

$$p(x = 1|\mu) = \mu$$

- Due to binary outcome

$$p(x = 0|\mu) = 1 - \mu$$

- Distribution

$$\text{Bern}(x|\mu) = \mu^x (1 - \mu)^{1-x}$$

Conjugate Prior

Posterior

I want a prior distribution, such that when interacting with the likelihood to reach the posterior, the posterior takes the same functional form as the prior"

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Posterior

I only want the data to be able to change the degree of my belief not its fundamental form

Conjugate Priors

$$\text{Beta}(\mu|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1},$$

- The `Beta` distribution is the conjugate prior of the `Bernoulli` likelihood parameter μ

Posterior

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&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{\sum_i x_i+a-1}(1-\mu)^{\sum_i (1-x_i)+b-1}
\end{aligned}
$$

Posterior II

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})} = \frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{\sum_i x_i + a - 1}(1-\mu)^{\sum_i (1-x_i)+b-1}}{p(\mathcal{D})}$$

$$p(\mathcal{D}) = \int p(\mathcal{D} \mid \mu)p(\mu)\mathrm{d}\mu$$

- we still do not know the normaliser/evidence?
- conjugacy means that we know its form

Posterior III

$$p(\mu|\mathcal{D}) \propto \mu^{\underbrace{\sum_i x_i + a - 1}_{a_n - 1}} (1-\mu)^{\underbrace{\sum_i (1 - x_i) + b - 1}_{b_n - 1}}$$

- we know the normaliser to a `Beta` distribution
- now we can avoid computing the integral

Posterior III

$$p(\mu|\mathcal{D}) = \frac{\Gamma(a_n + b_n)}{\Gamma(a_n)\Gamma(b_n)}\mu^{a_n}(1-\mu)^{b_n-1}$$

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Posterior III

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p(\mu|\mathcal{D}) &= \frac{\Gamma(a_n + b_n)}{\Gamma(a_n)\Gamma(b_n)}\mu^{a_n}(1-\mu)^{b_n-1} \\
&= \mathsf{Beta}(\mu|a_n, b_n)
\end{aligned}
$$

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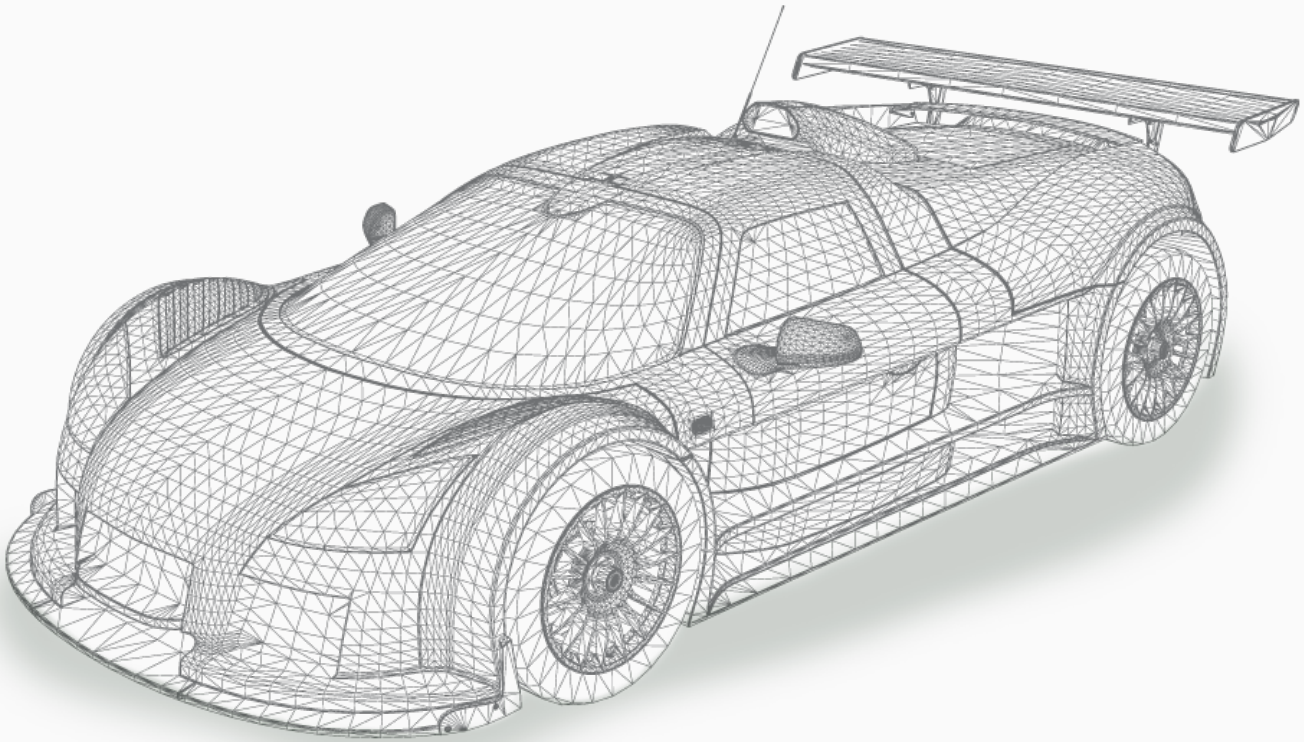
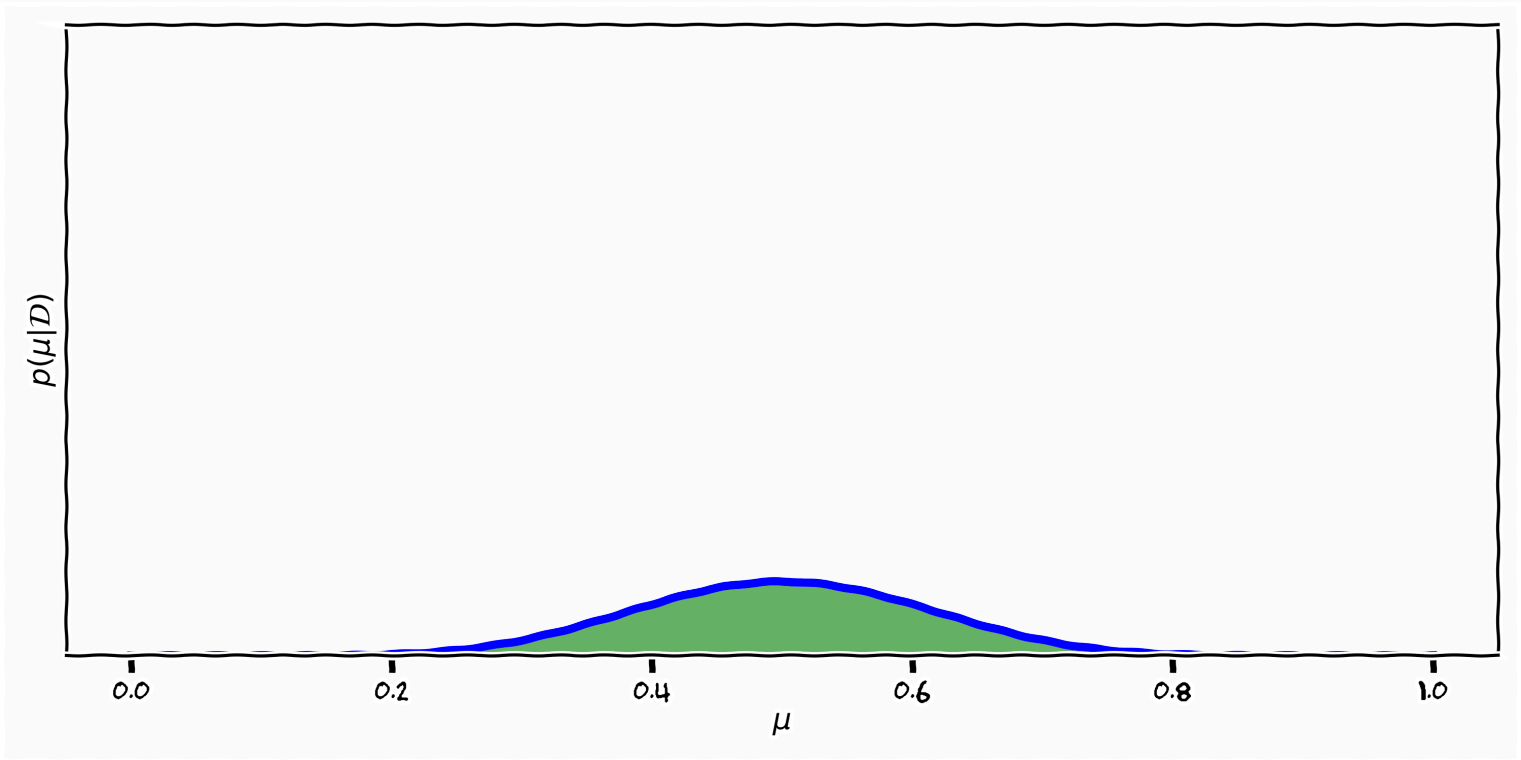
Why its awesome to work in Computer Science!



Experiments



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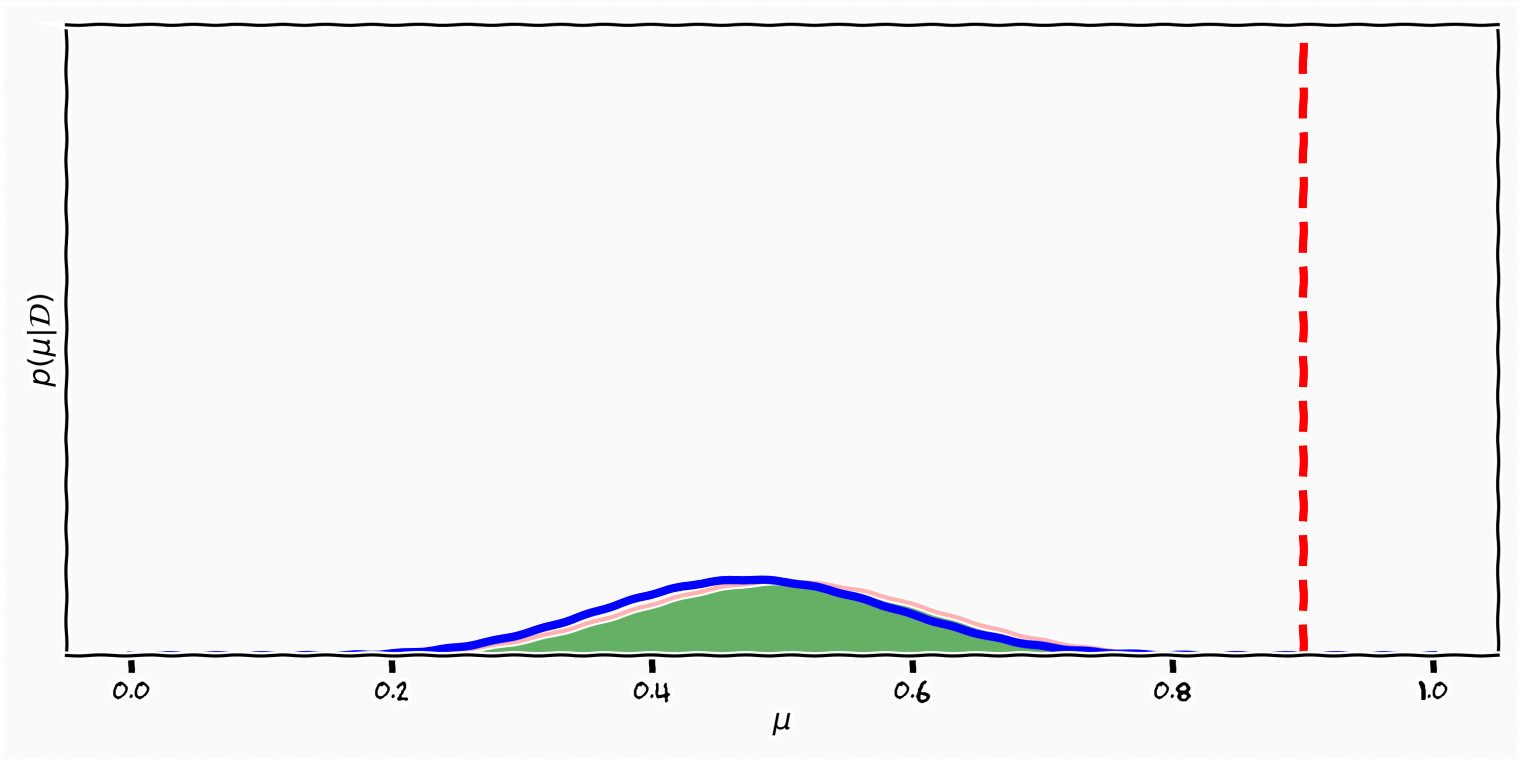
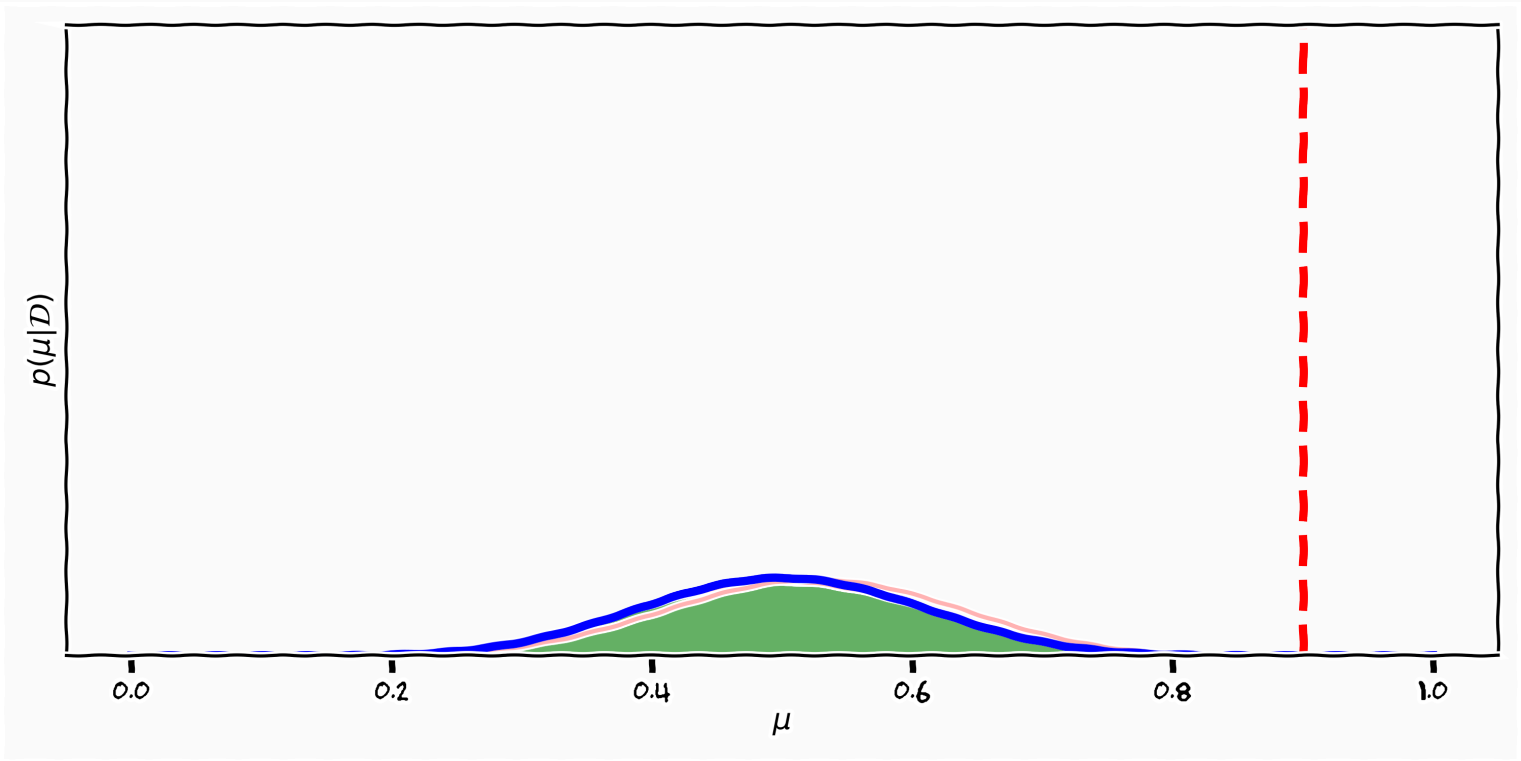


Experiments



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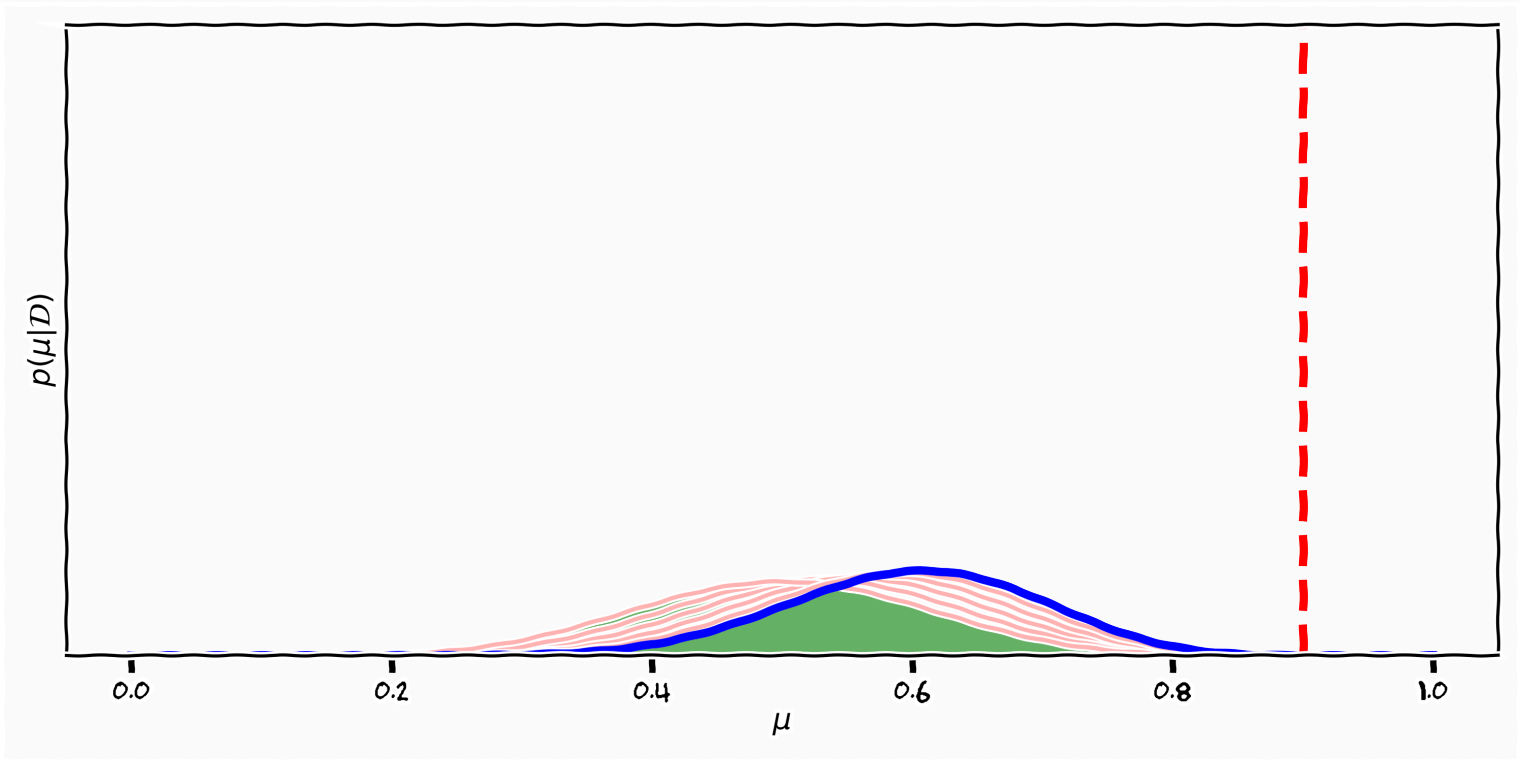

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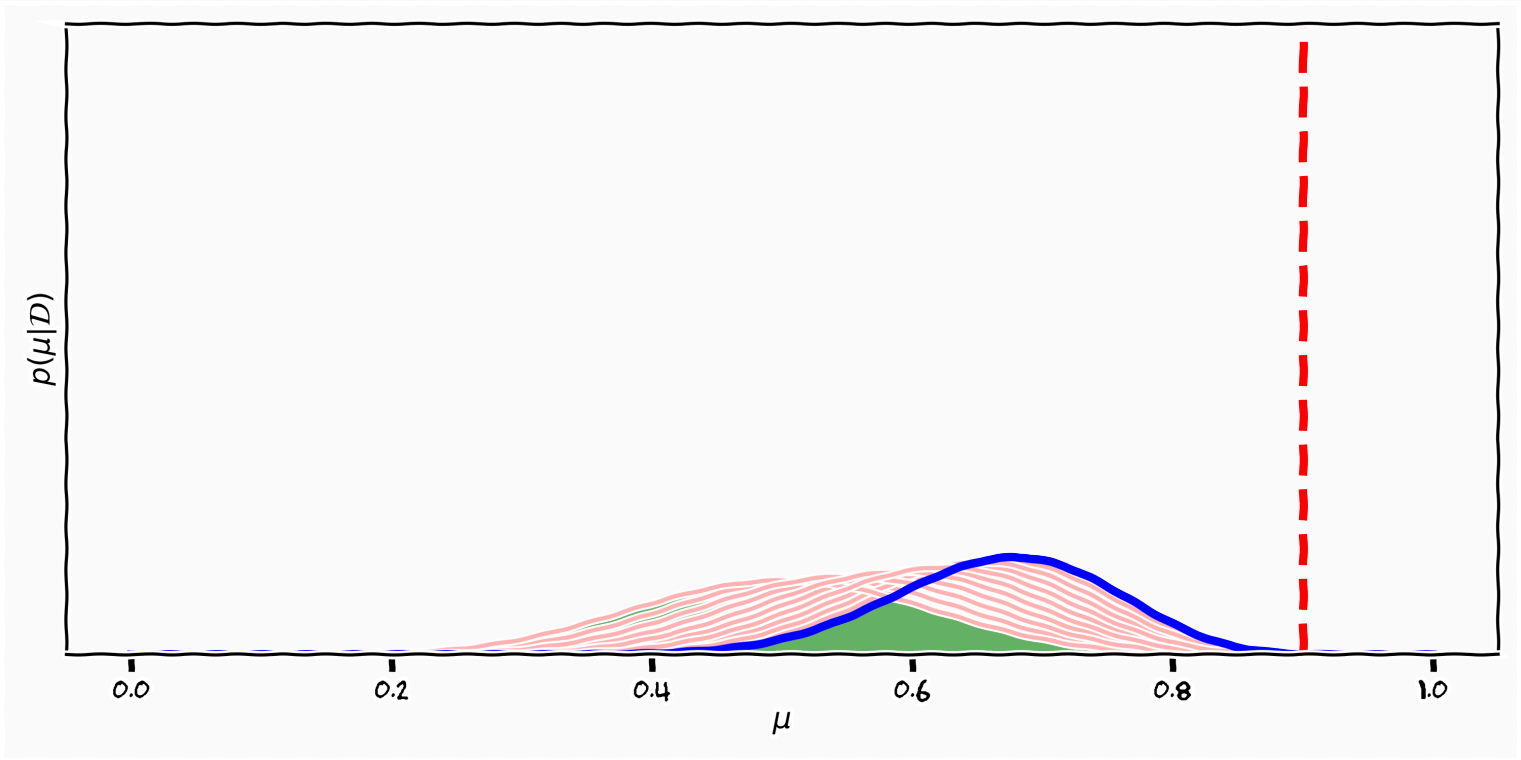

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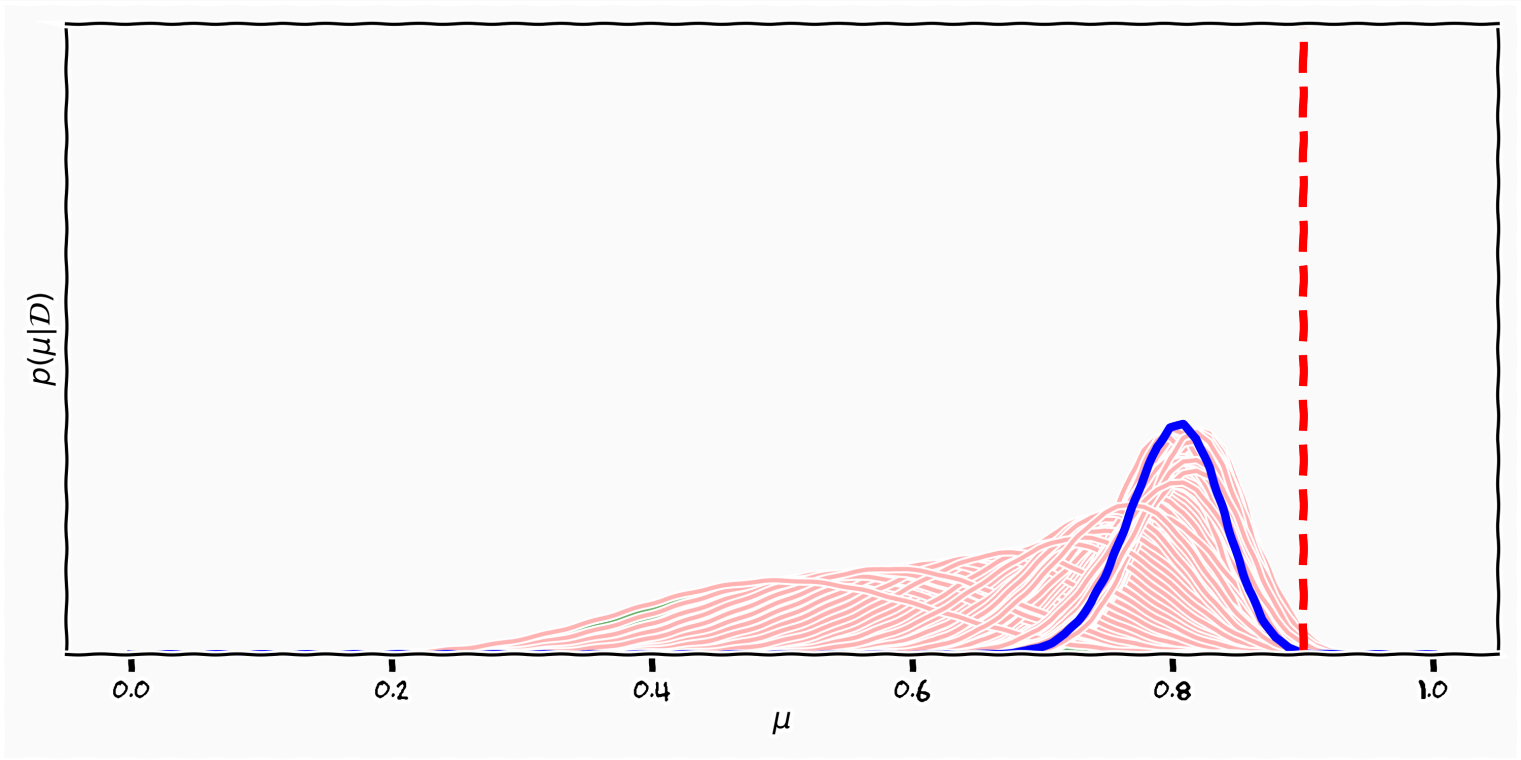

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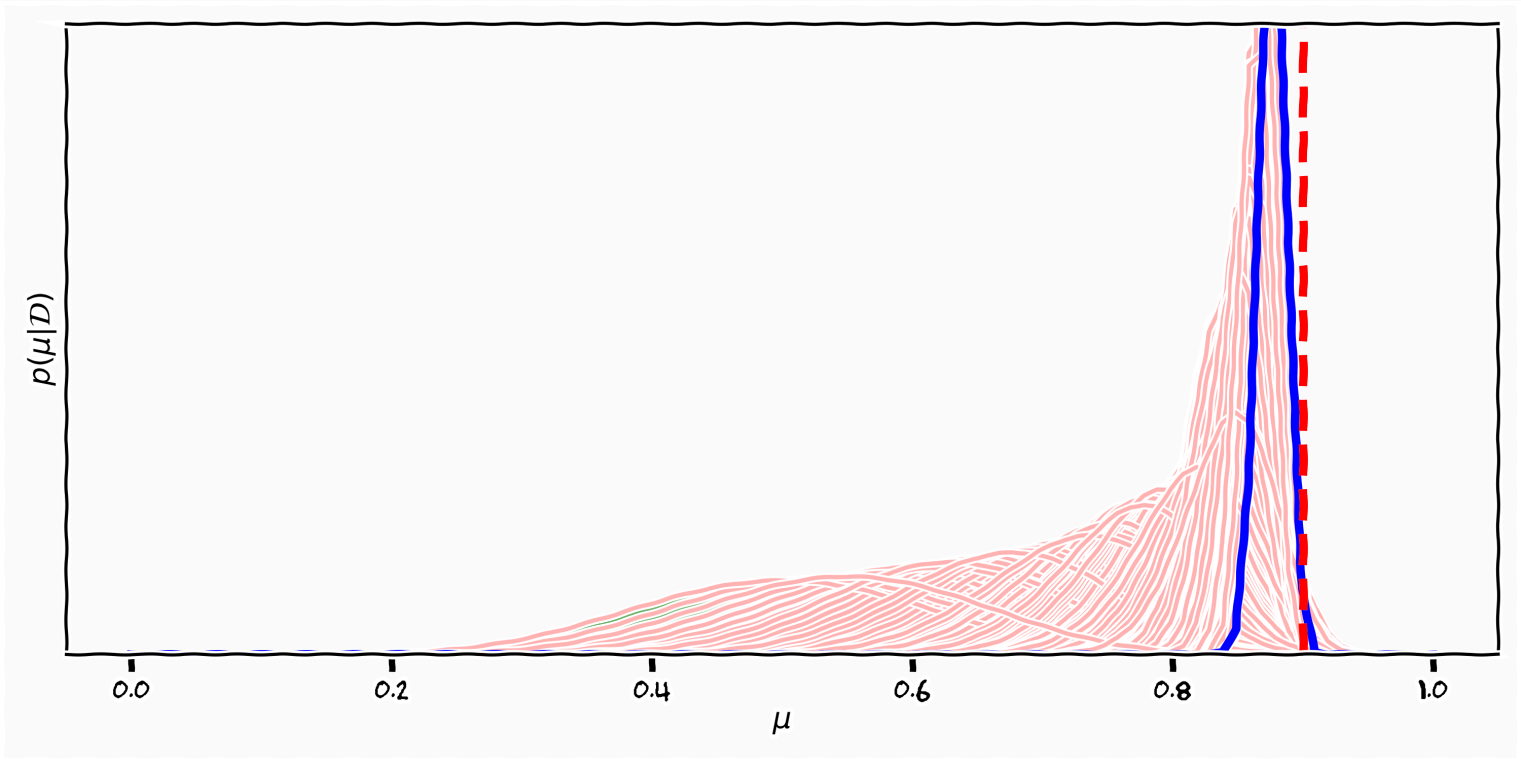

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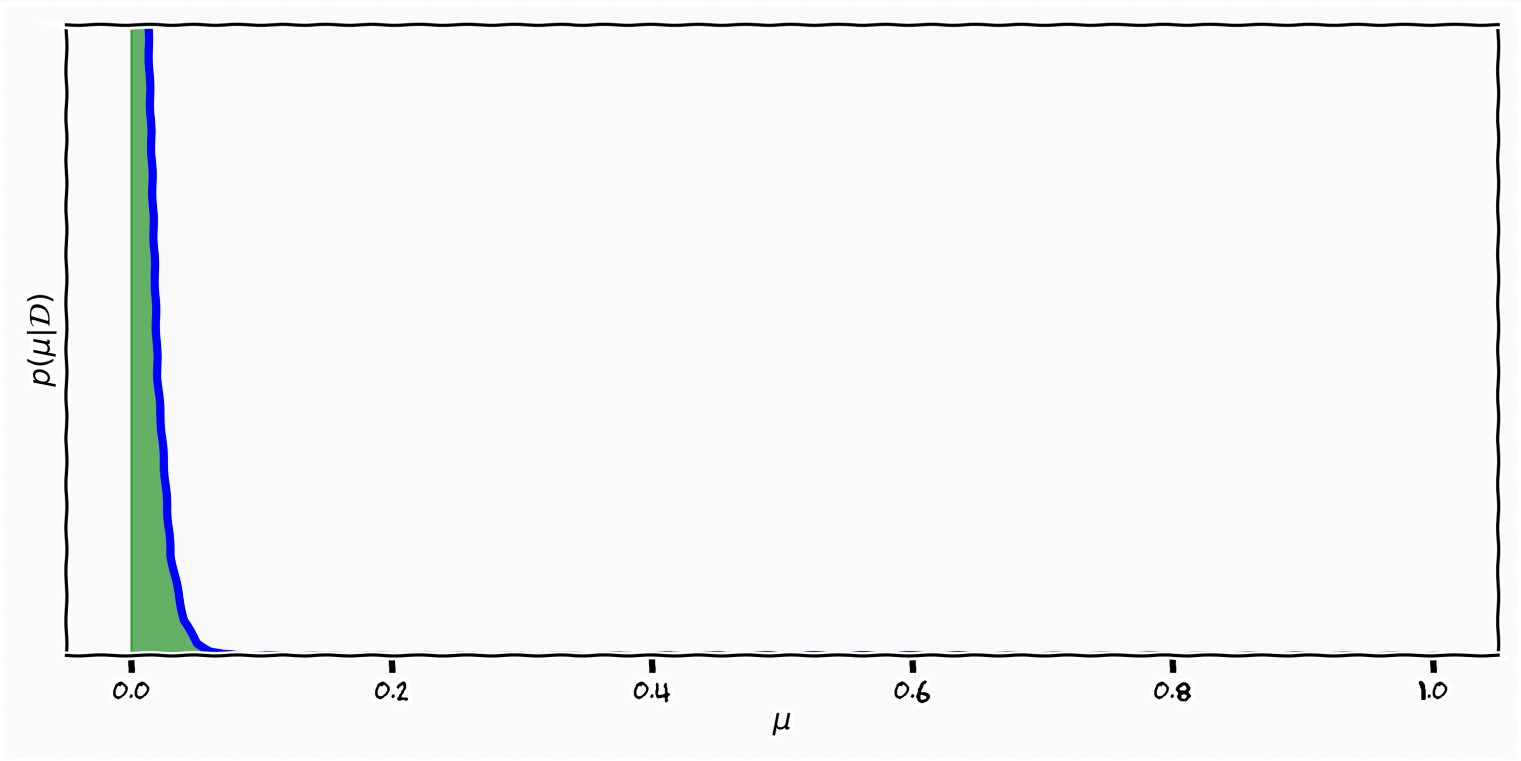

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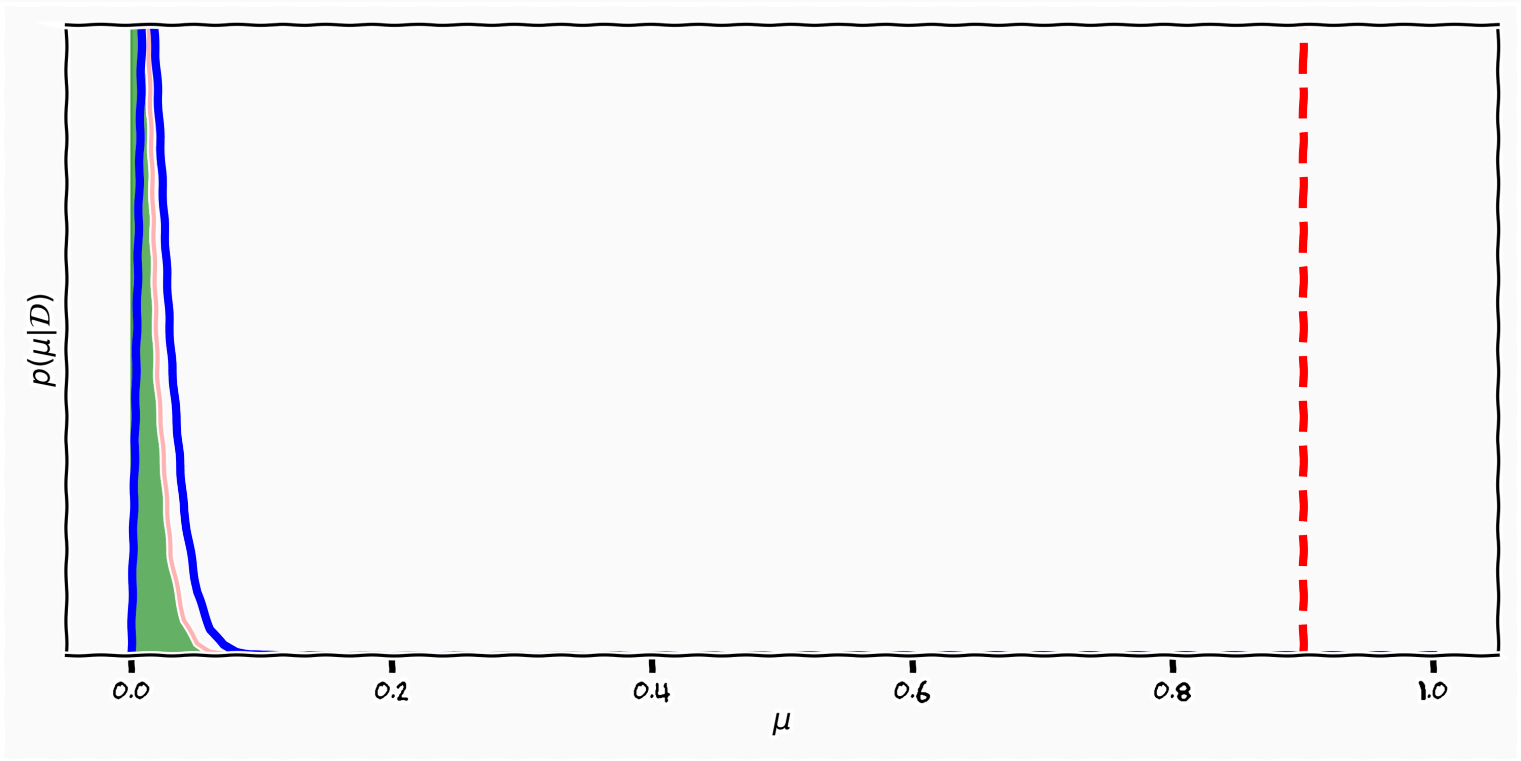

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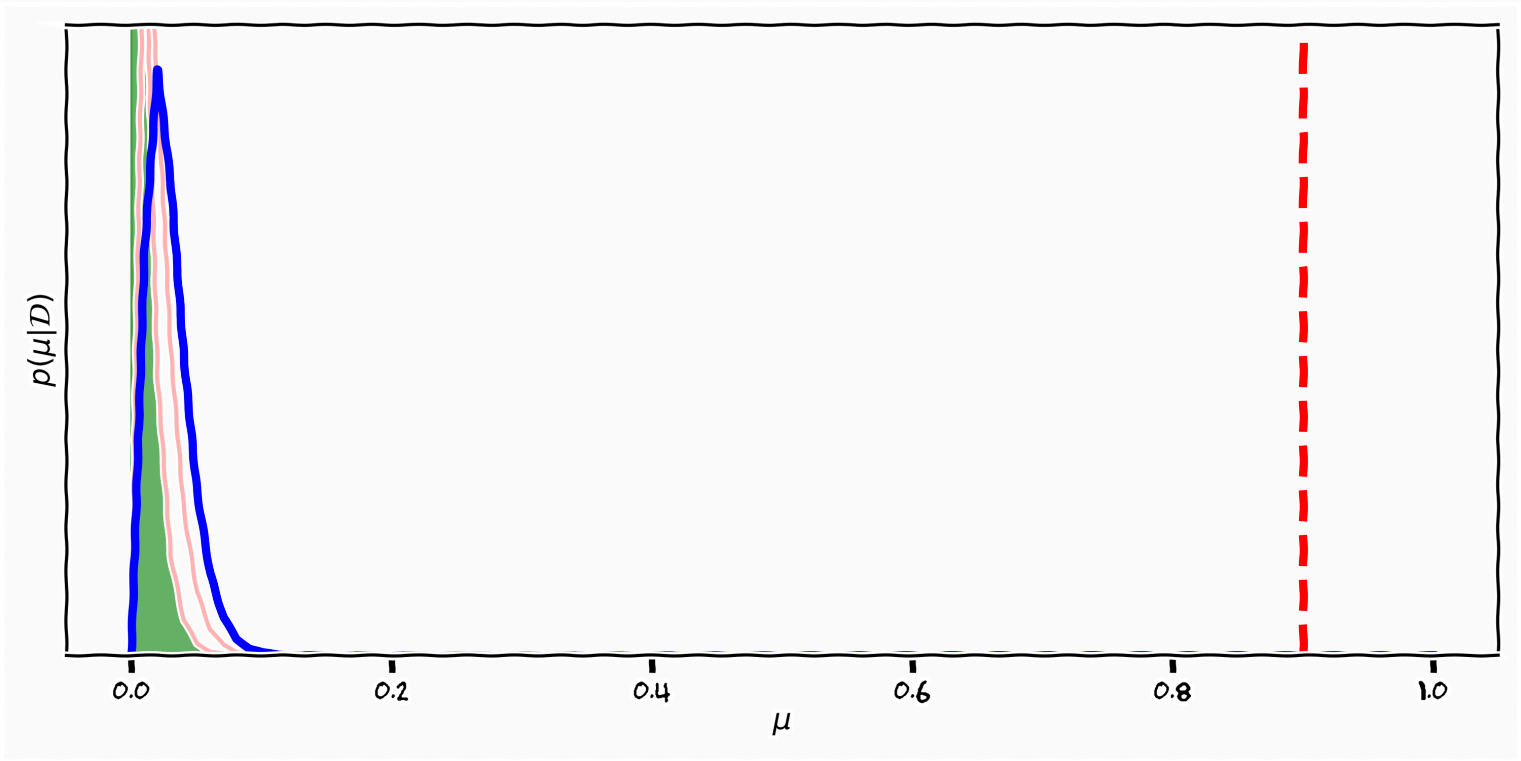

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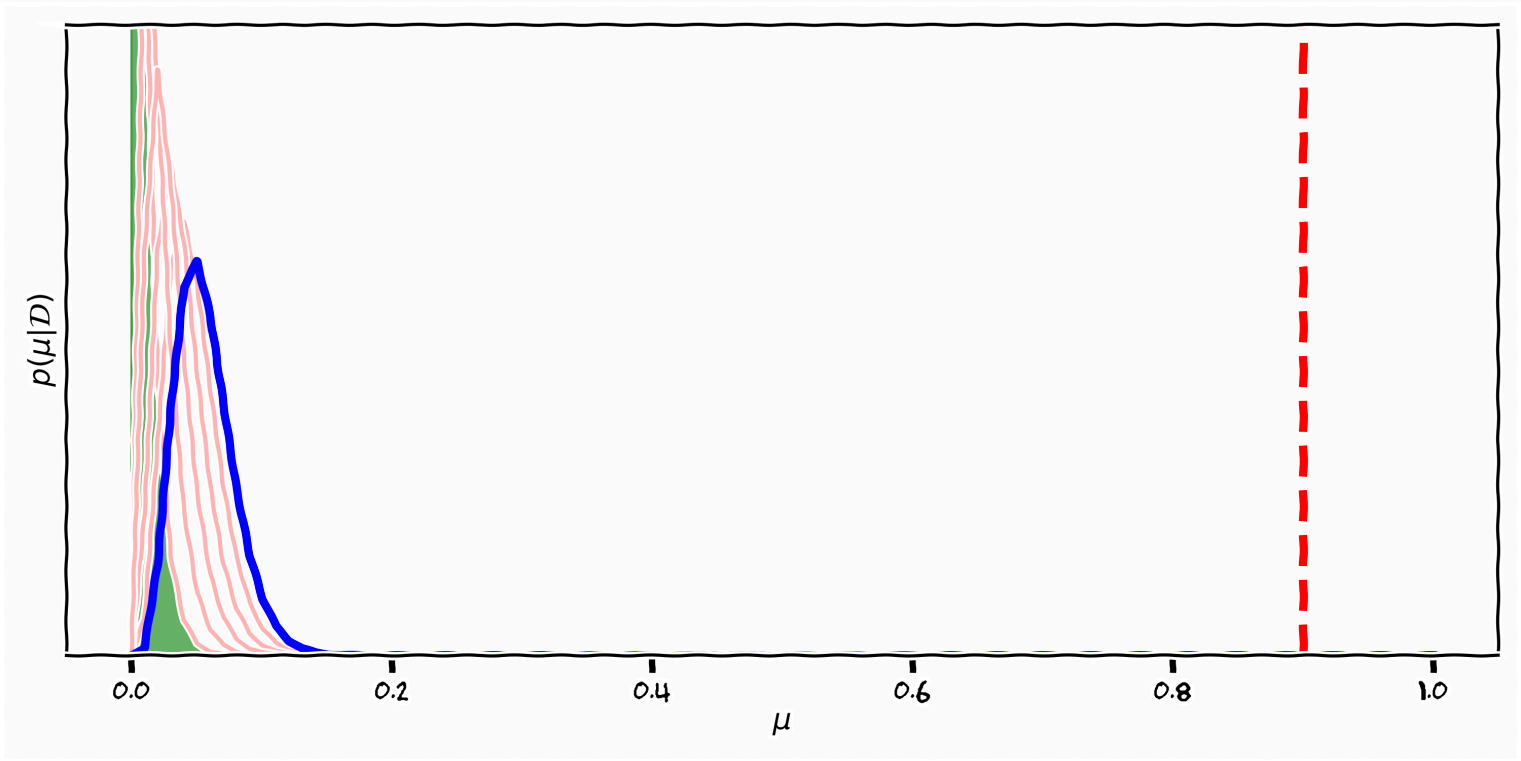

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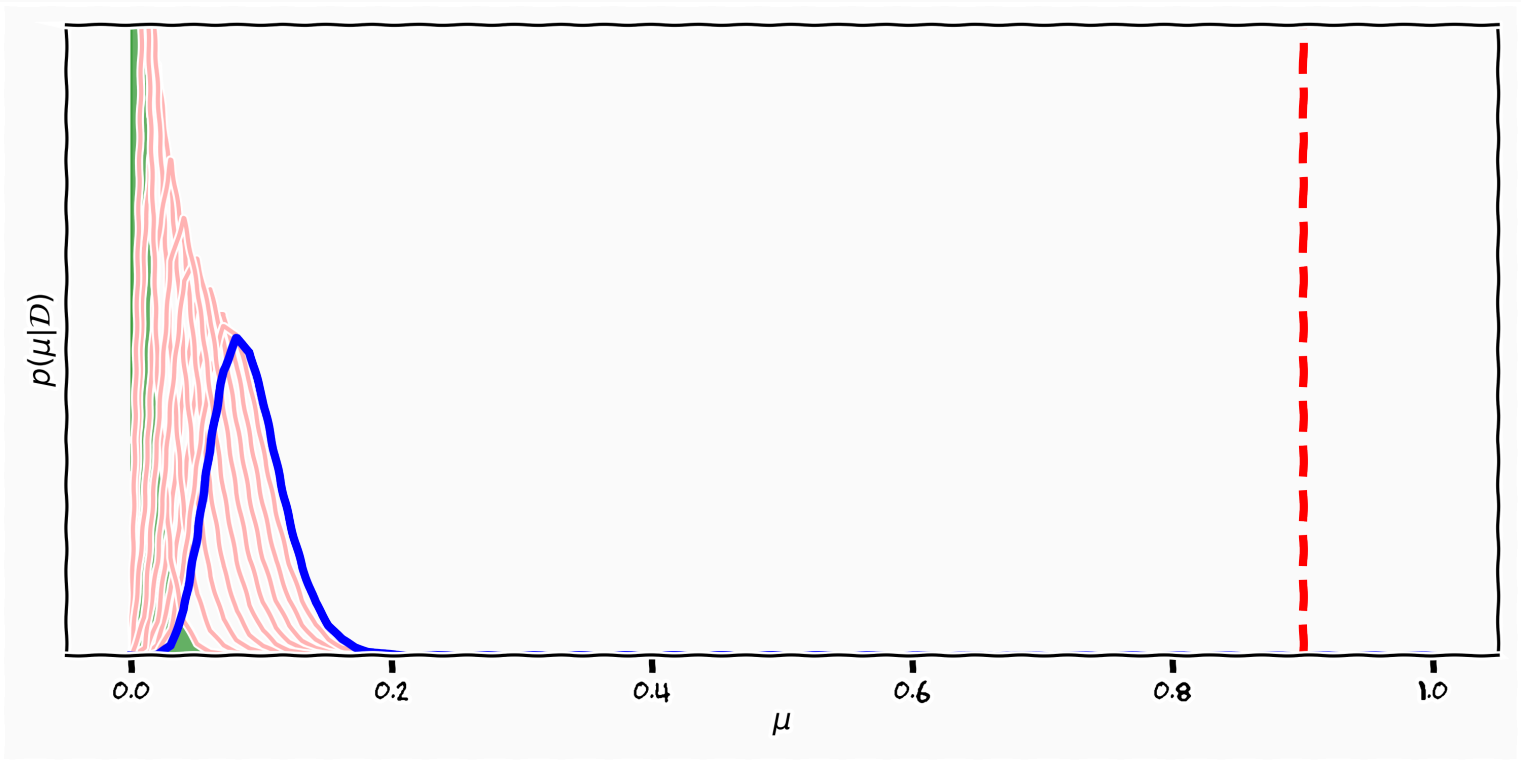

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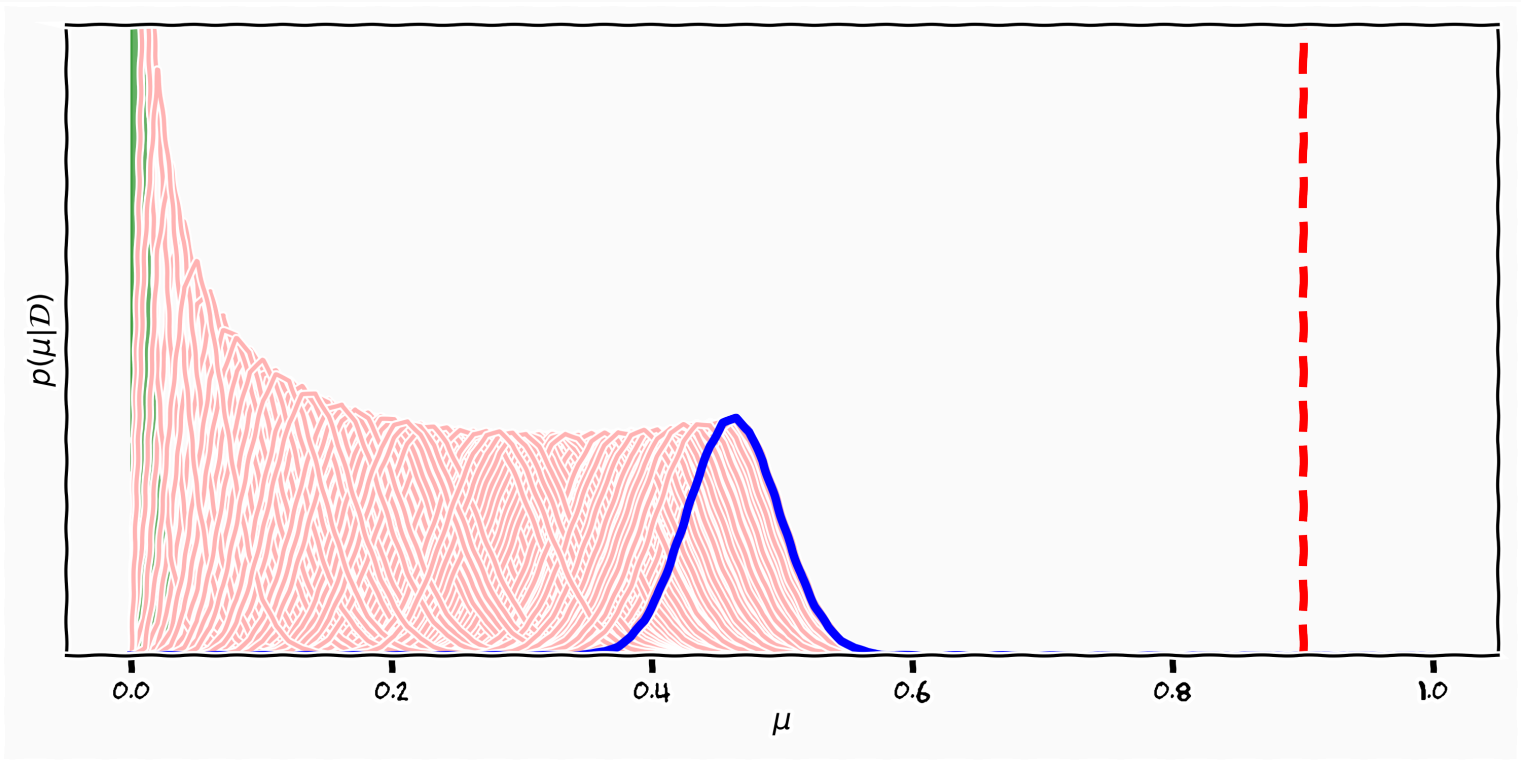

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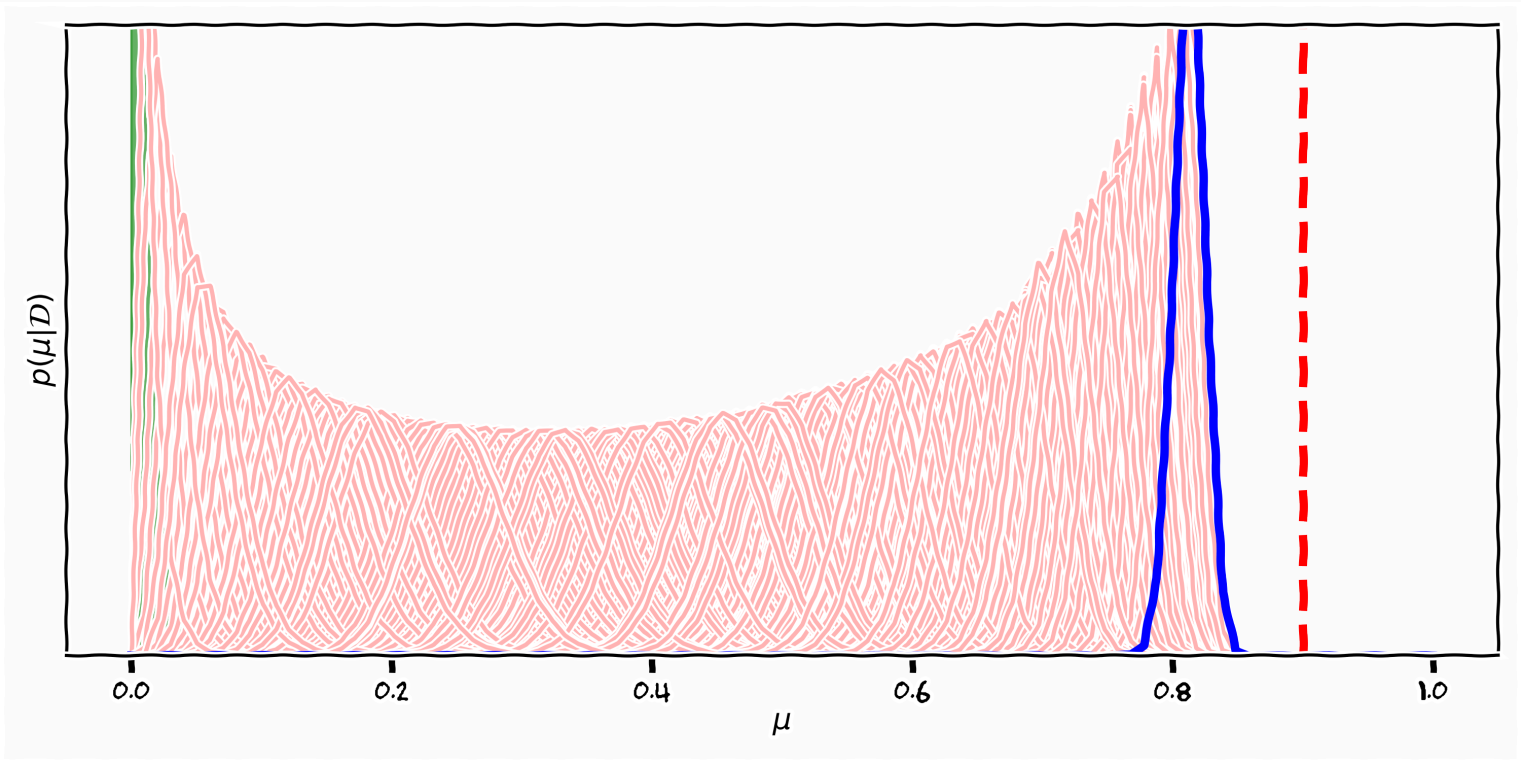

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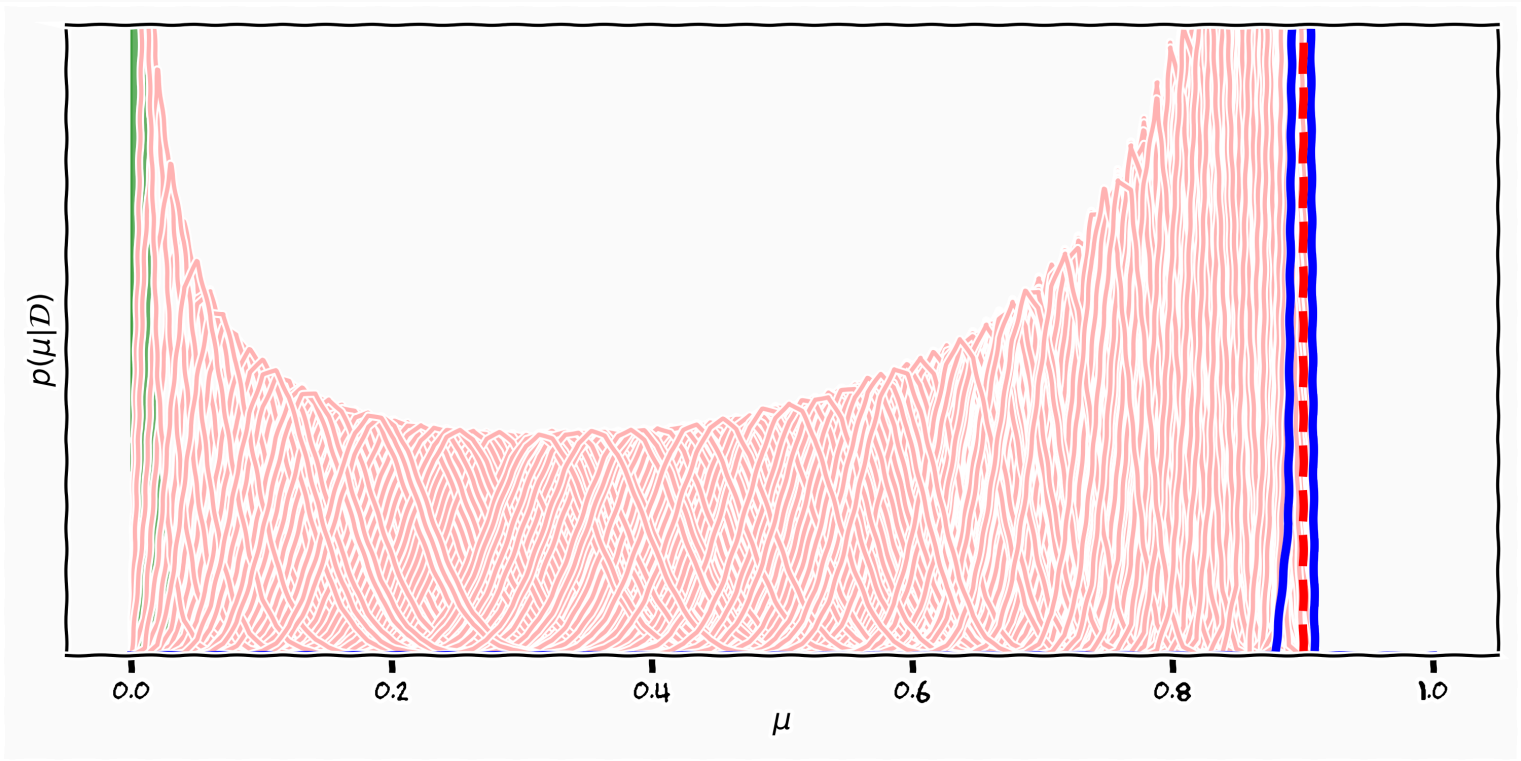

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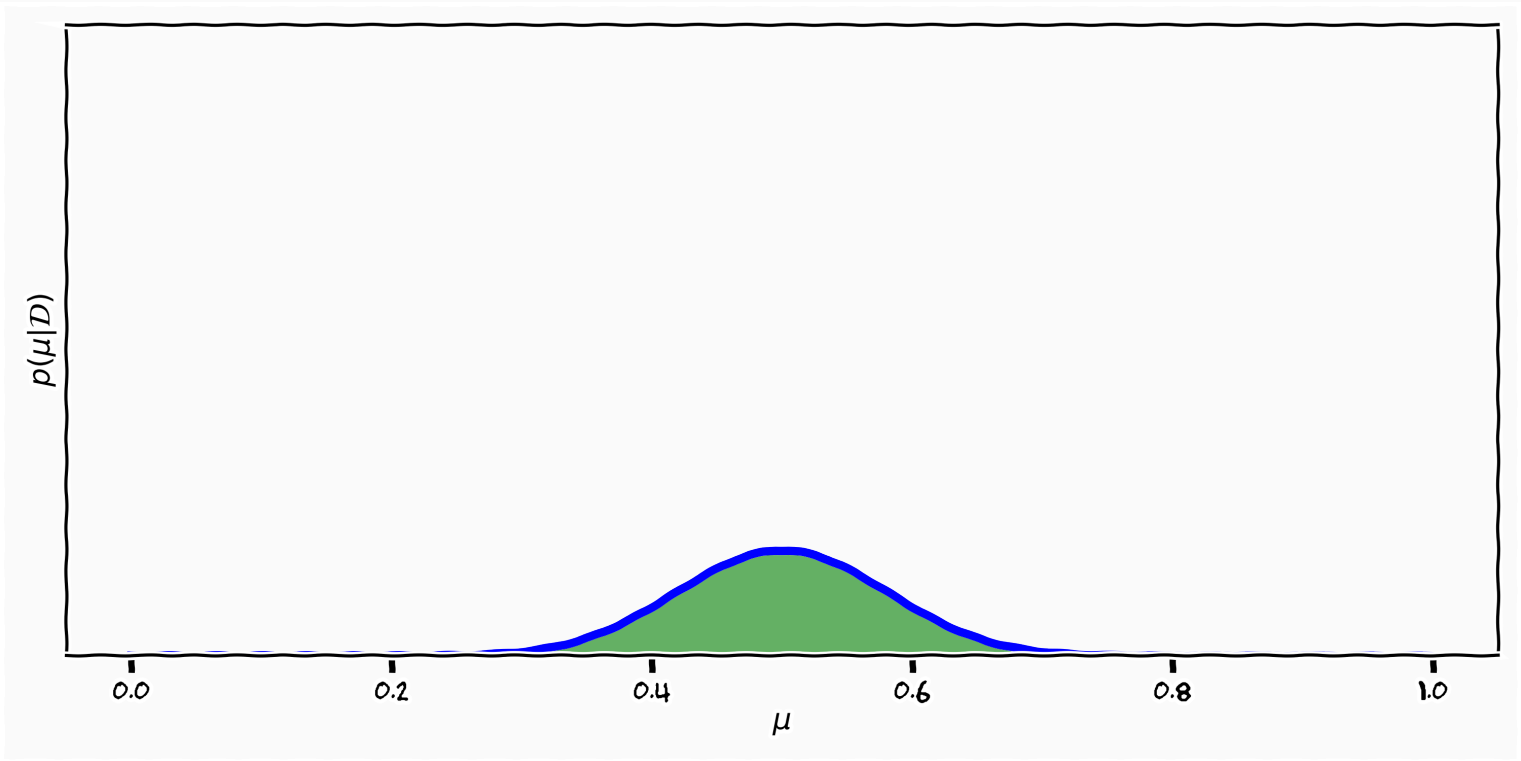

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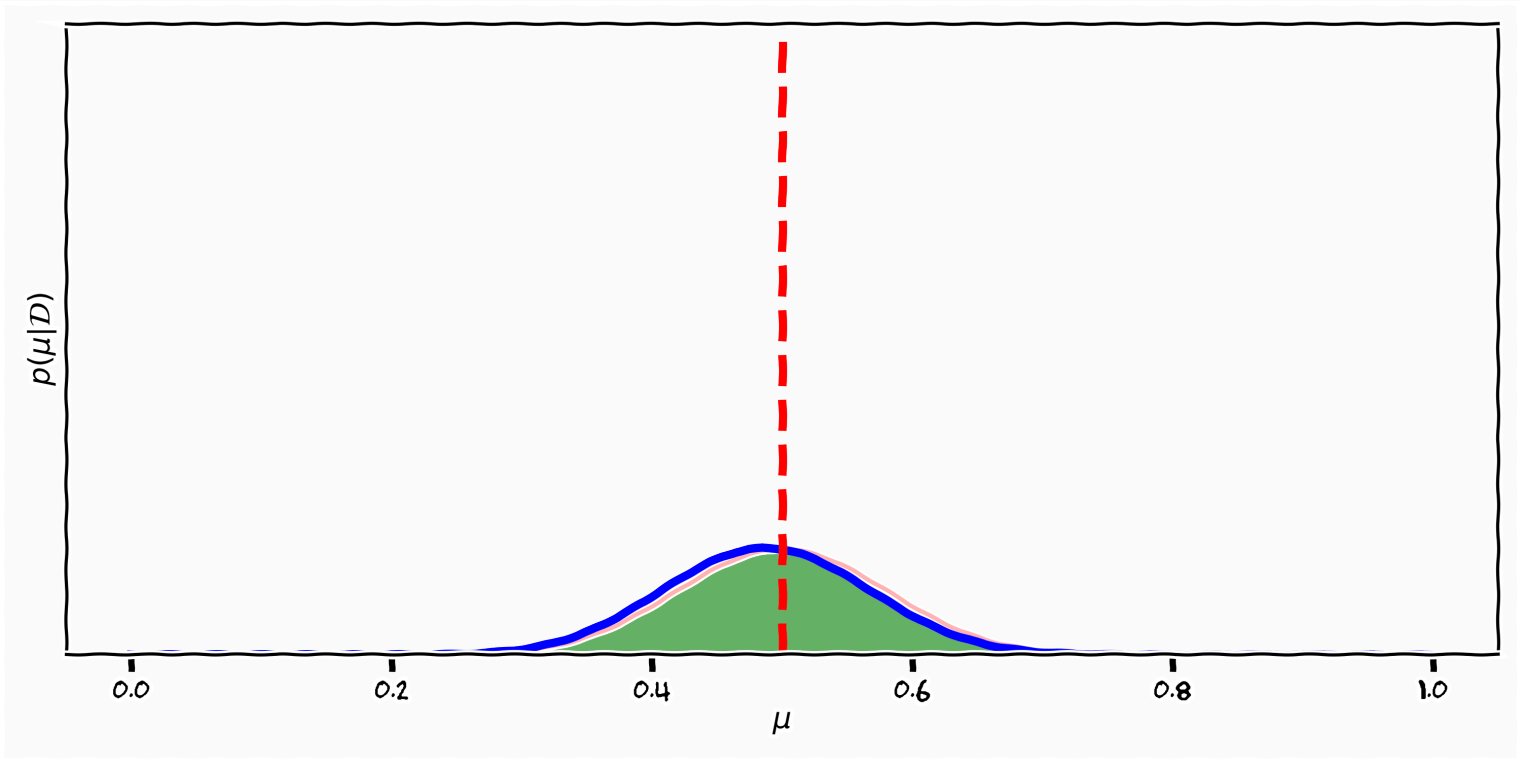

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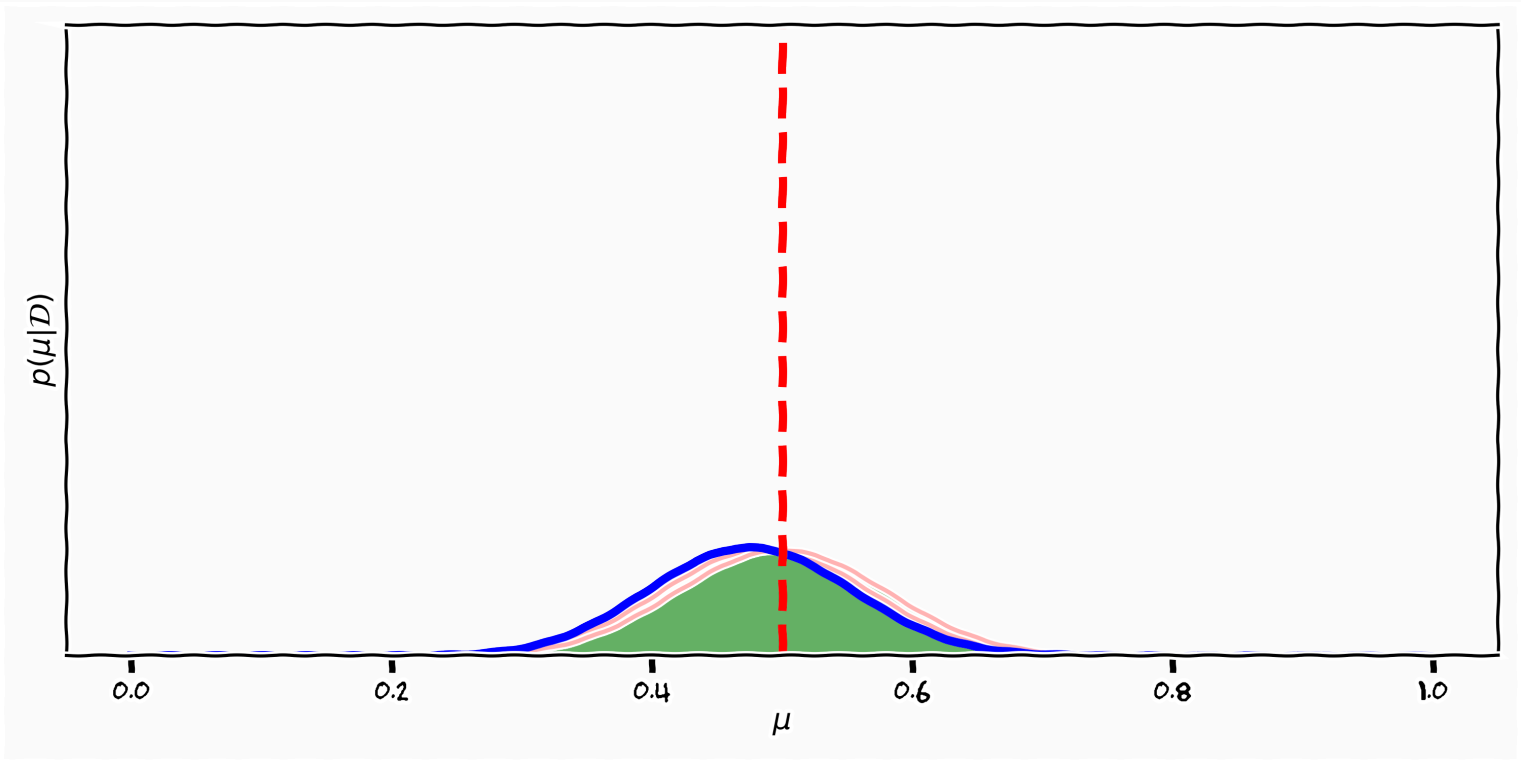

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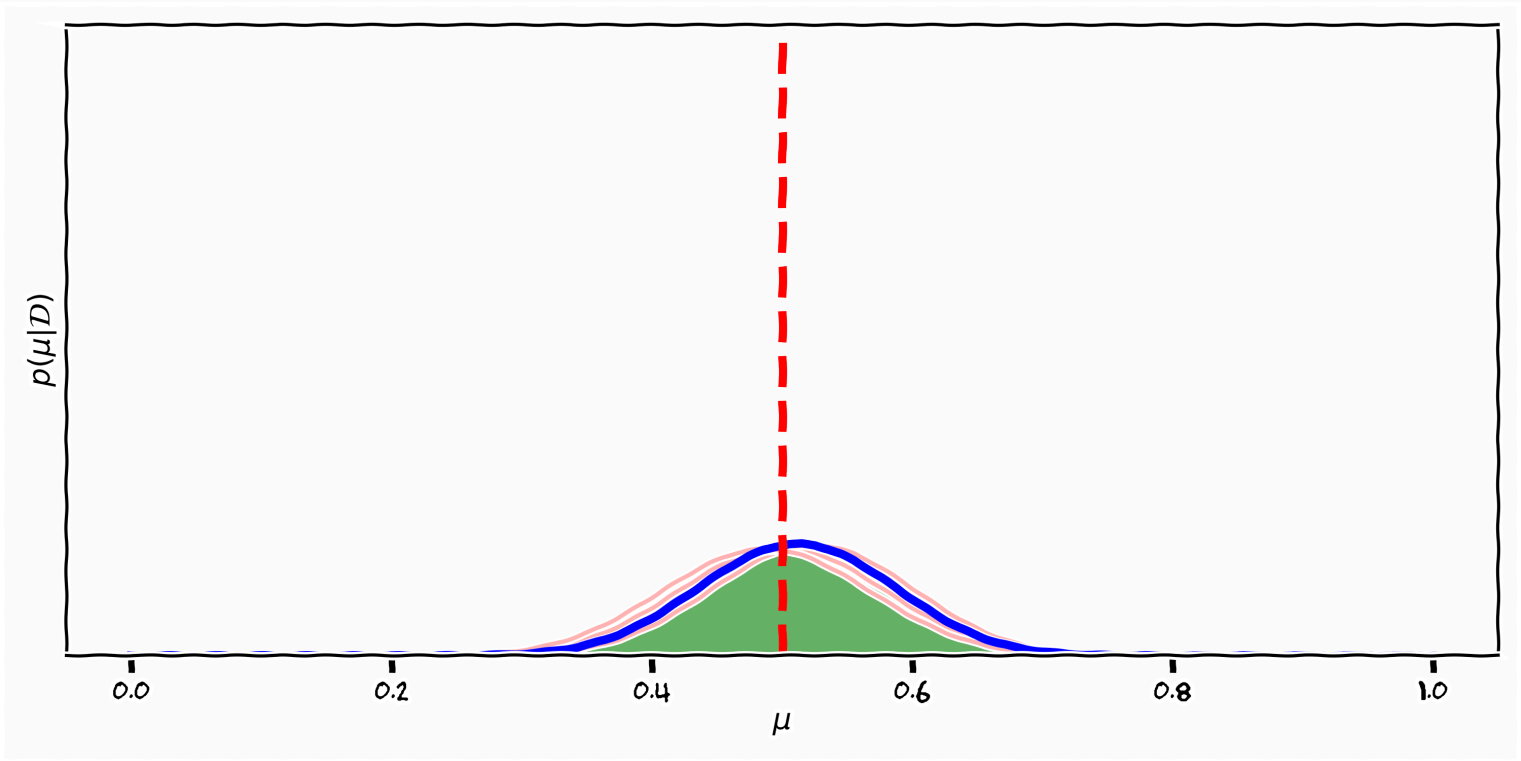

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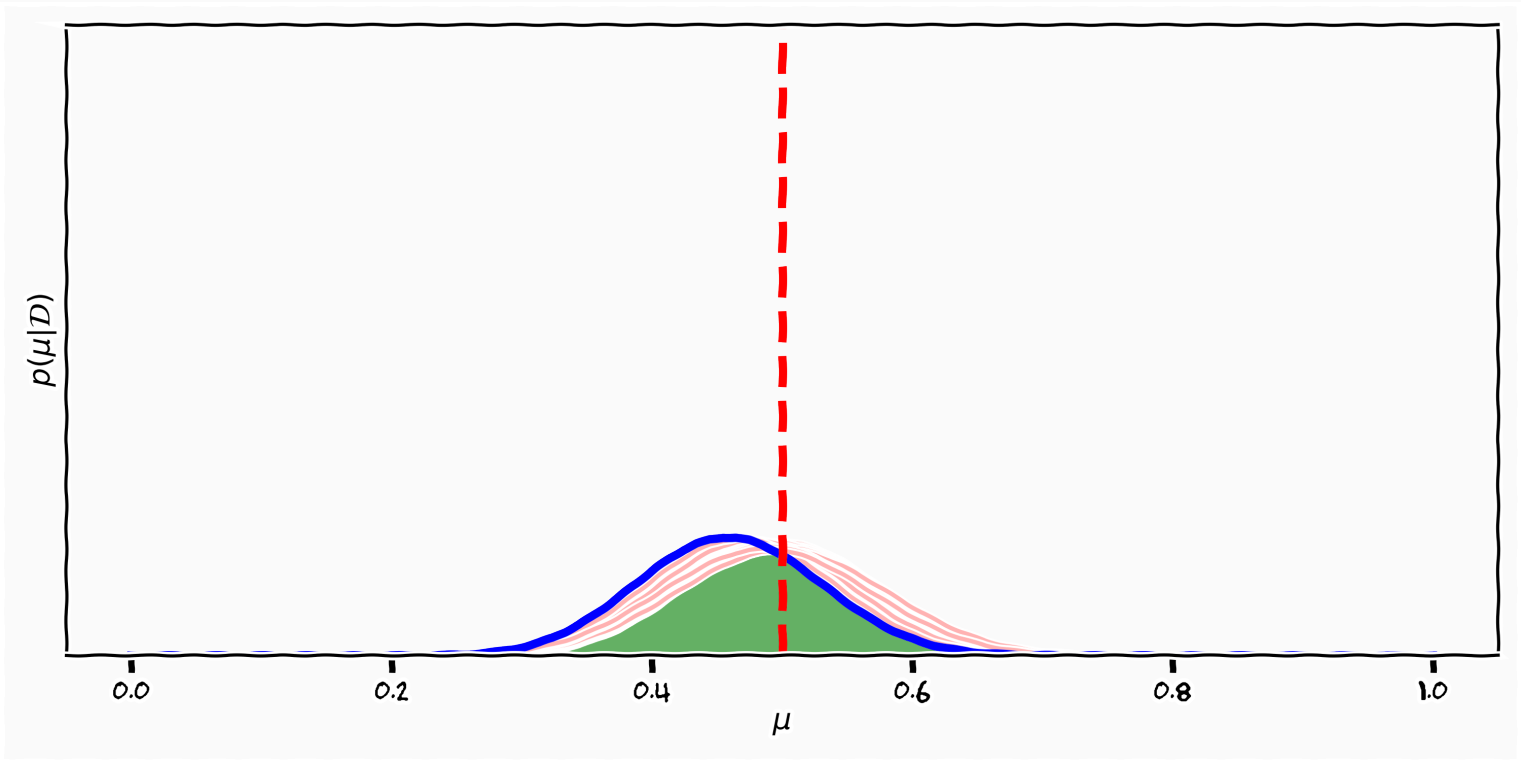

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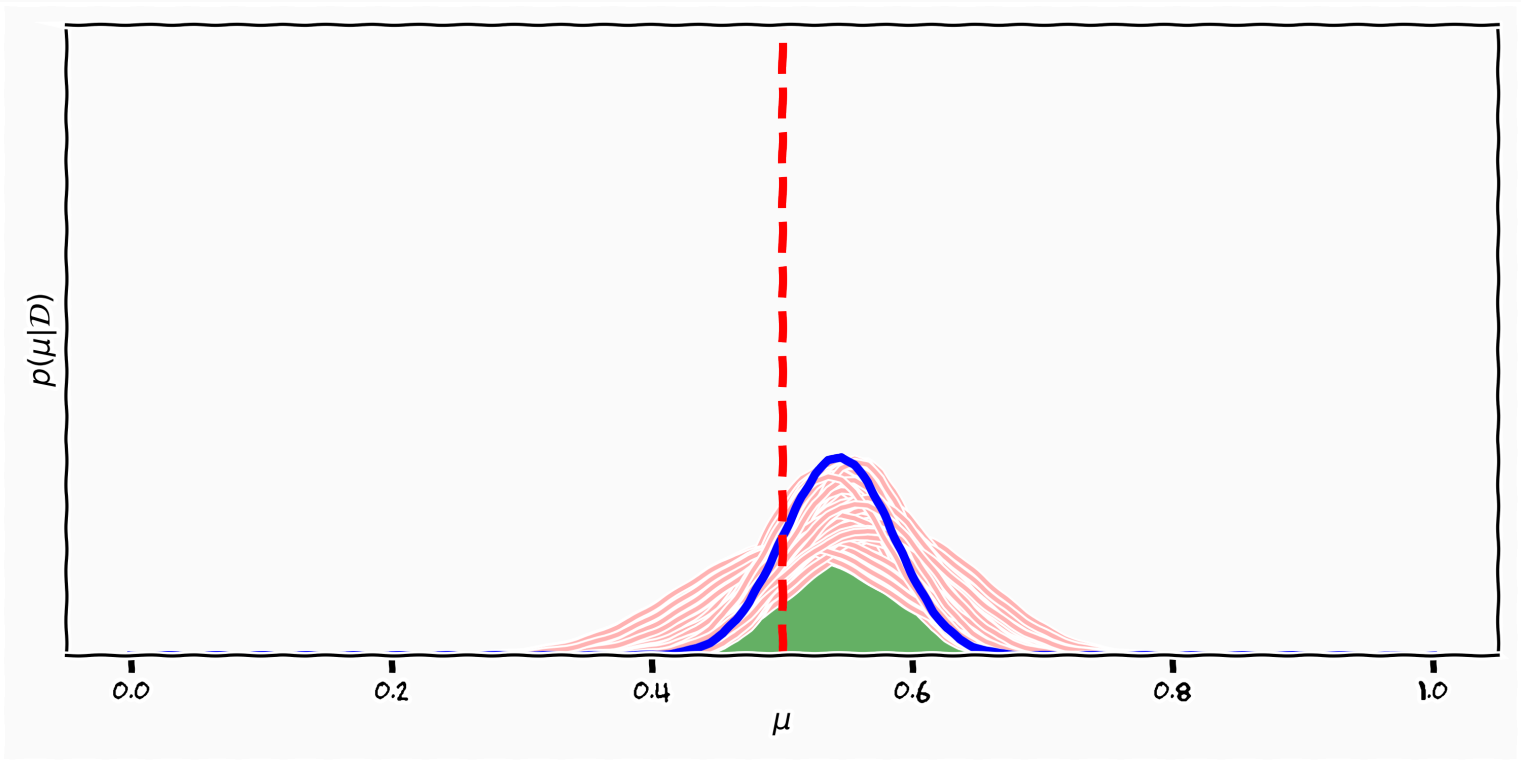

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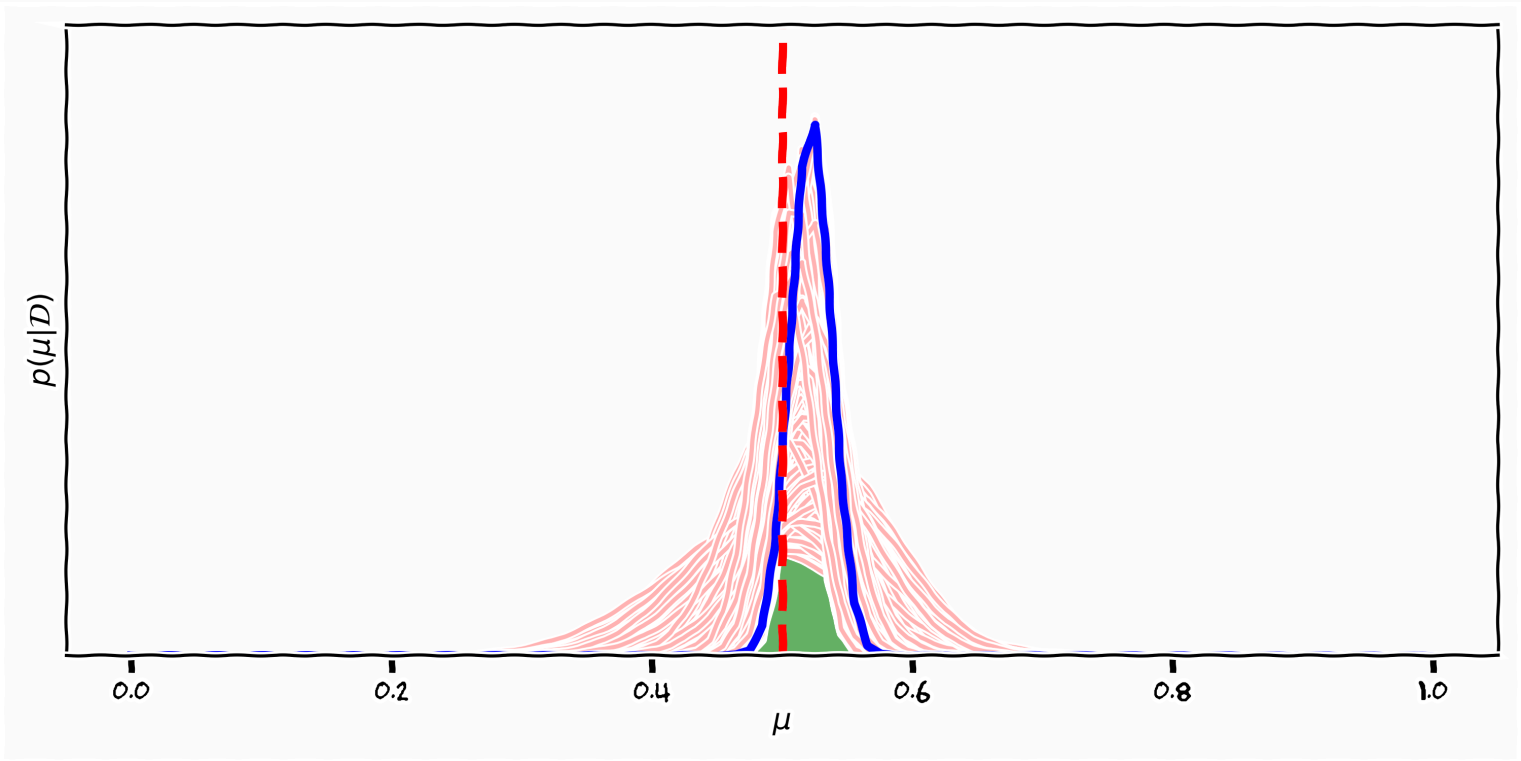
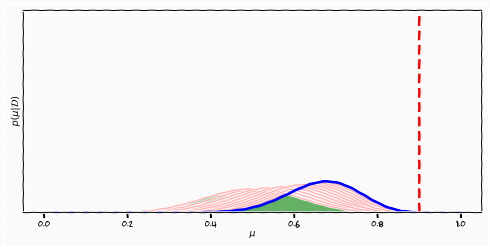
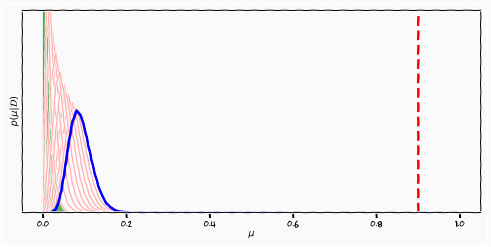
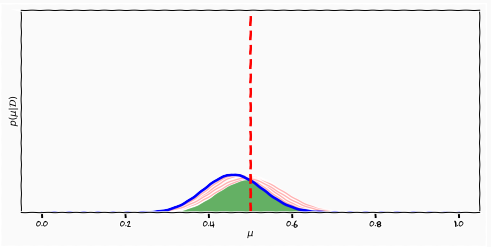


Summary



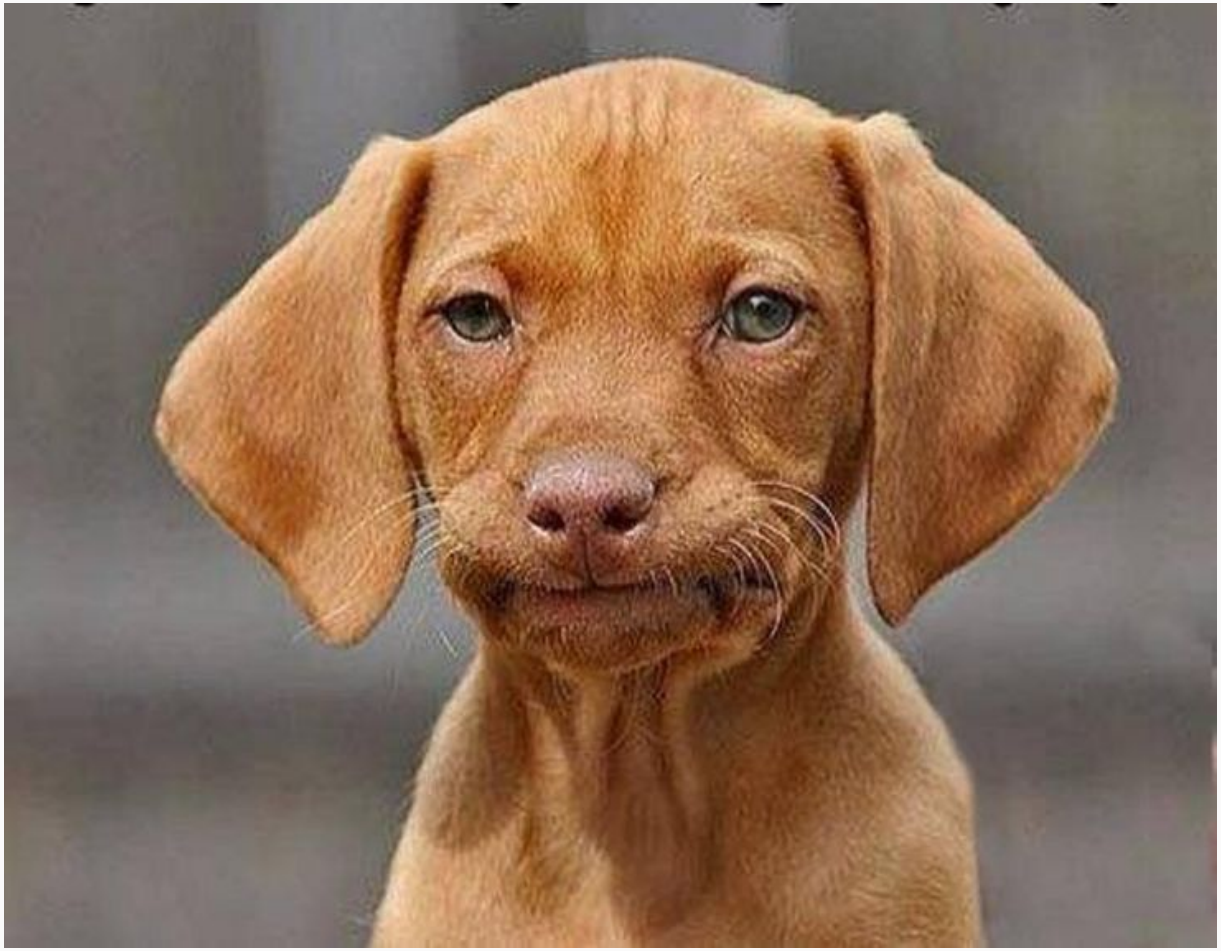




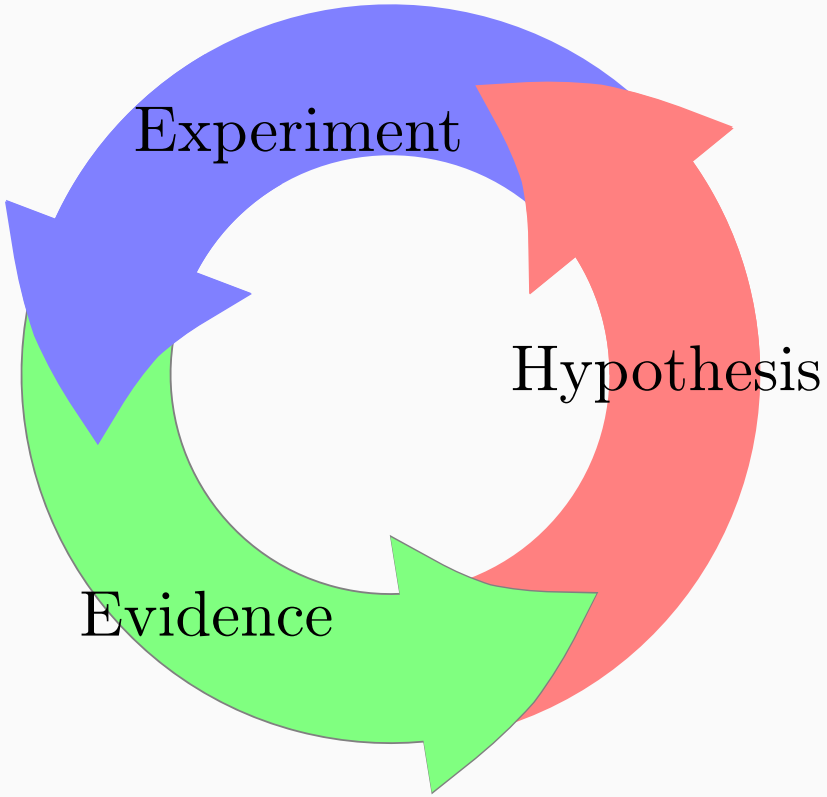
Summary

- It is impossible to differentiate solutions without assumptions
- Our "solution" can only be interpreted in the light of our assumptions
- Posterior inference "balances" the information in the data with the assumptions

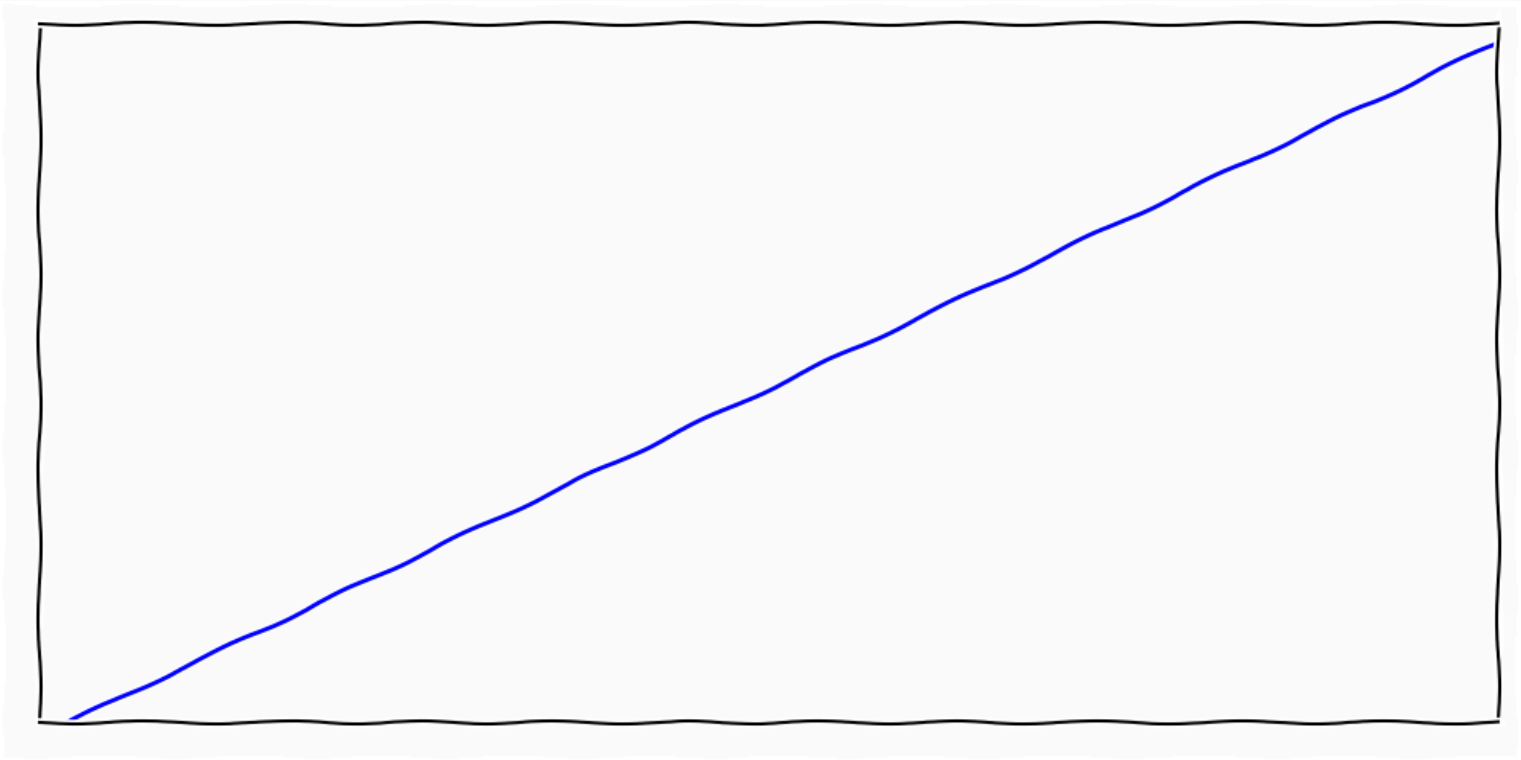
Reflections



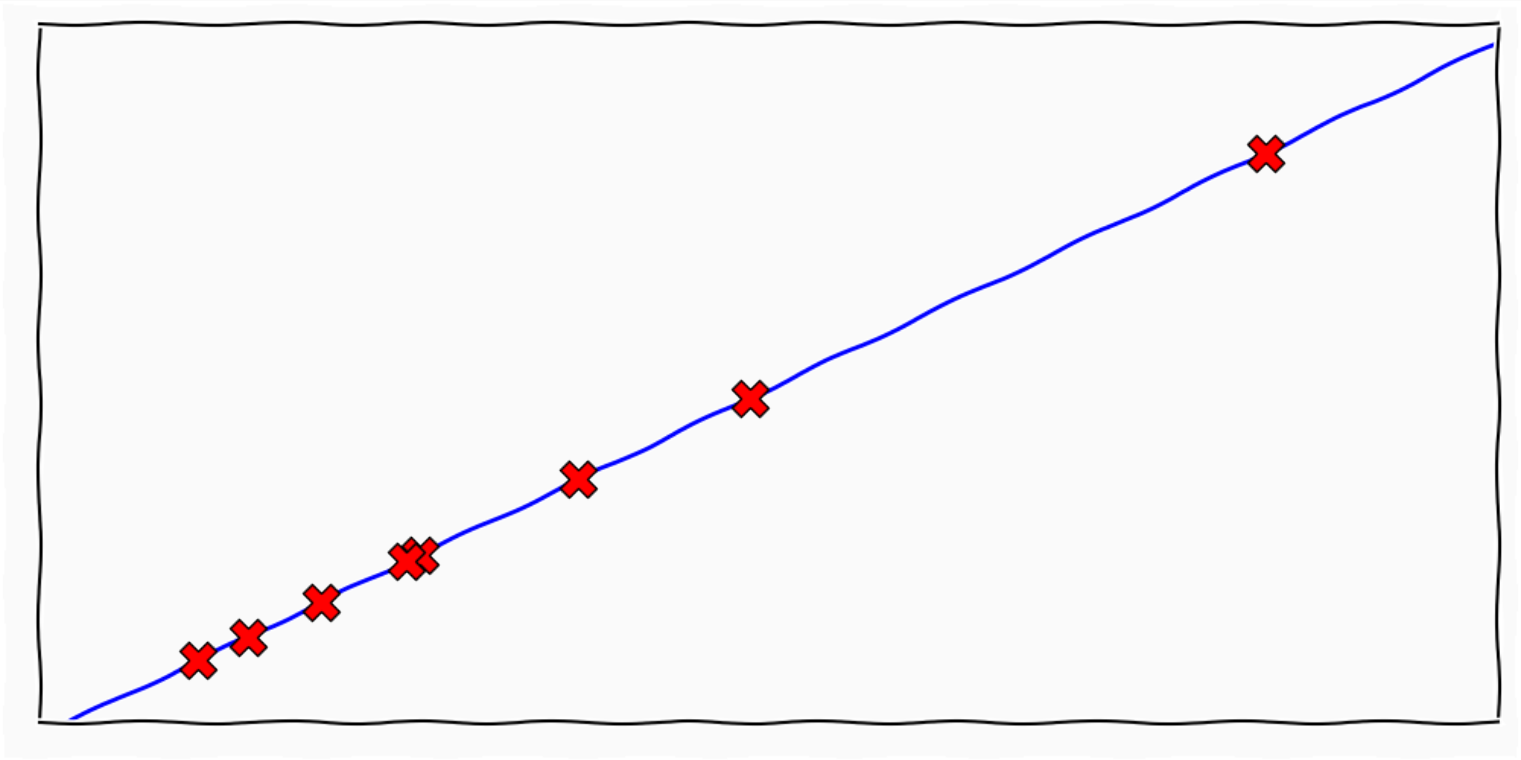
Science?



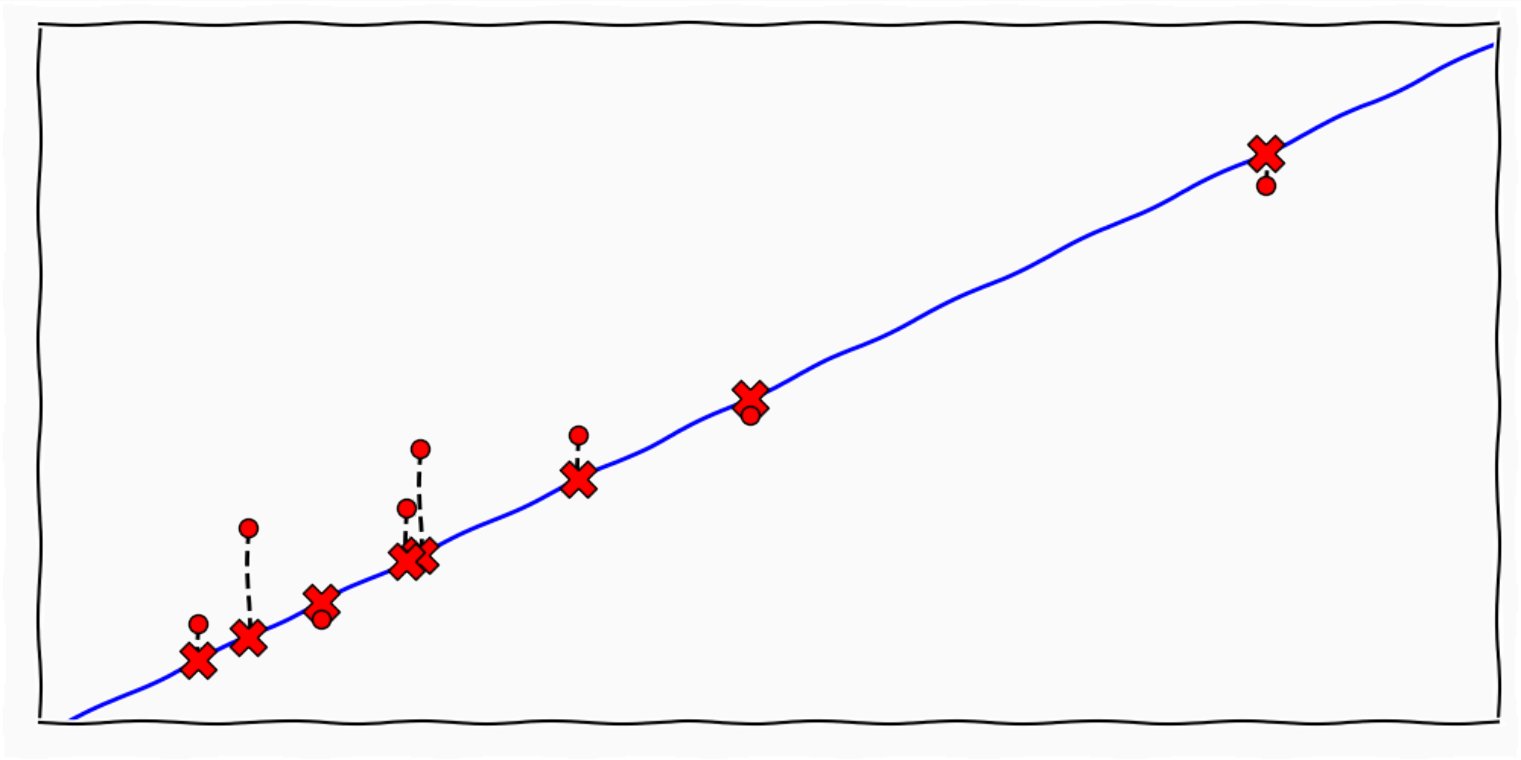
Linear Regression: Modelling



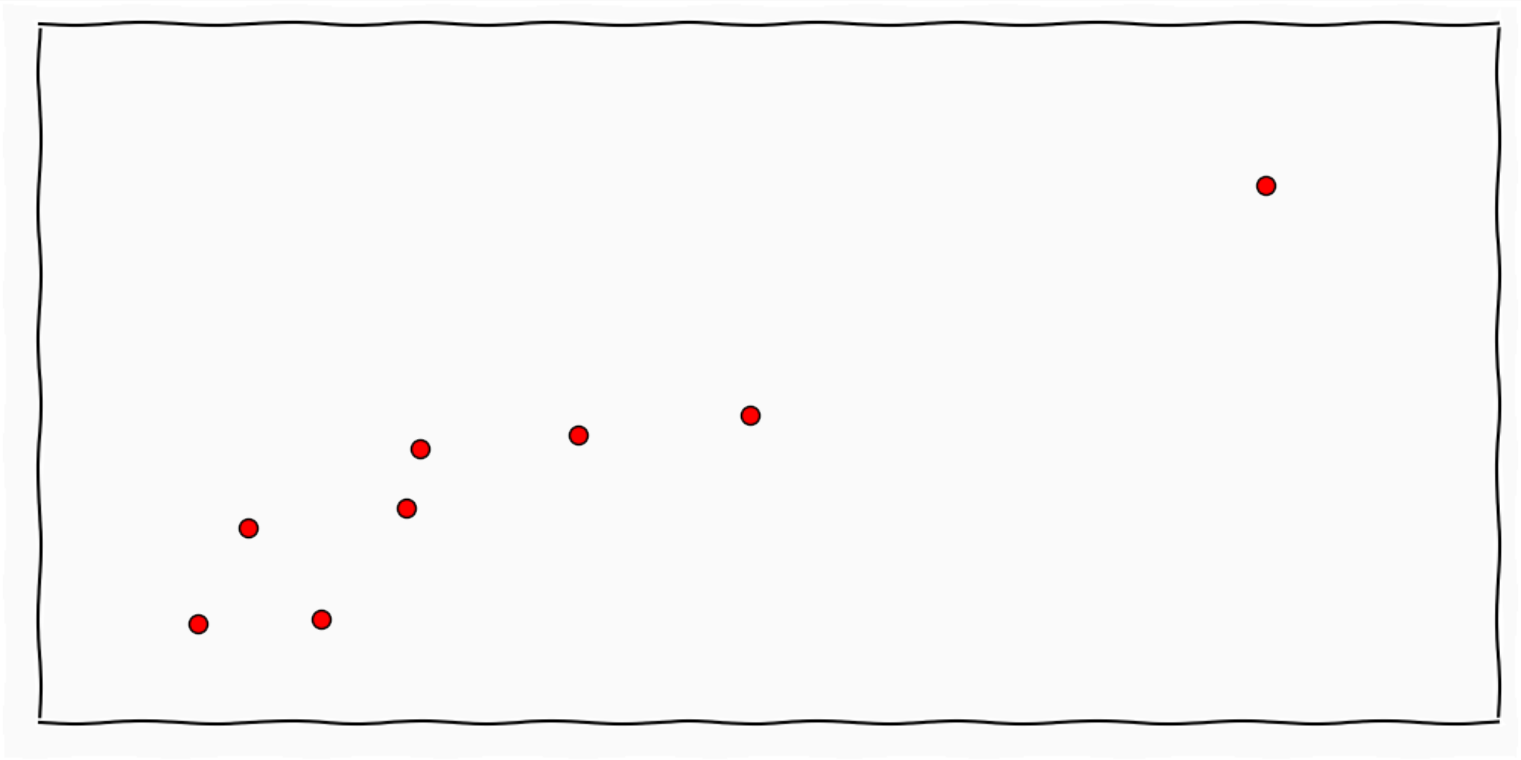
Linear Regression: Modelling



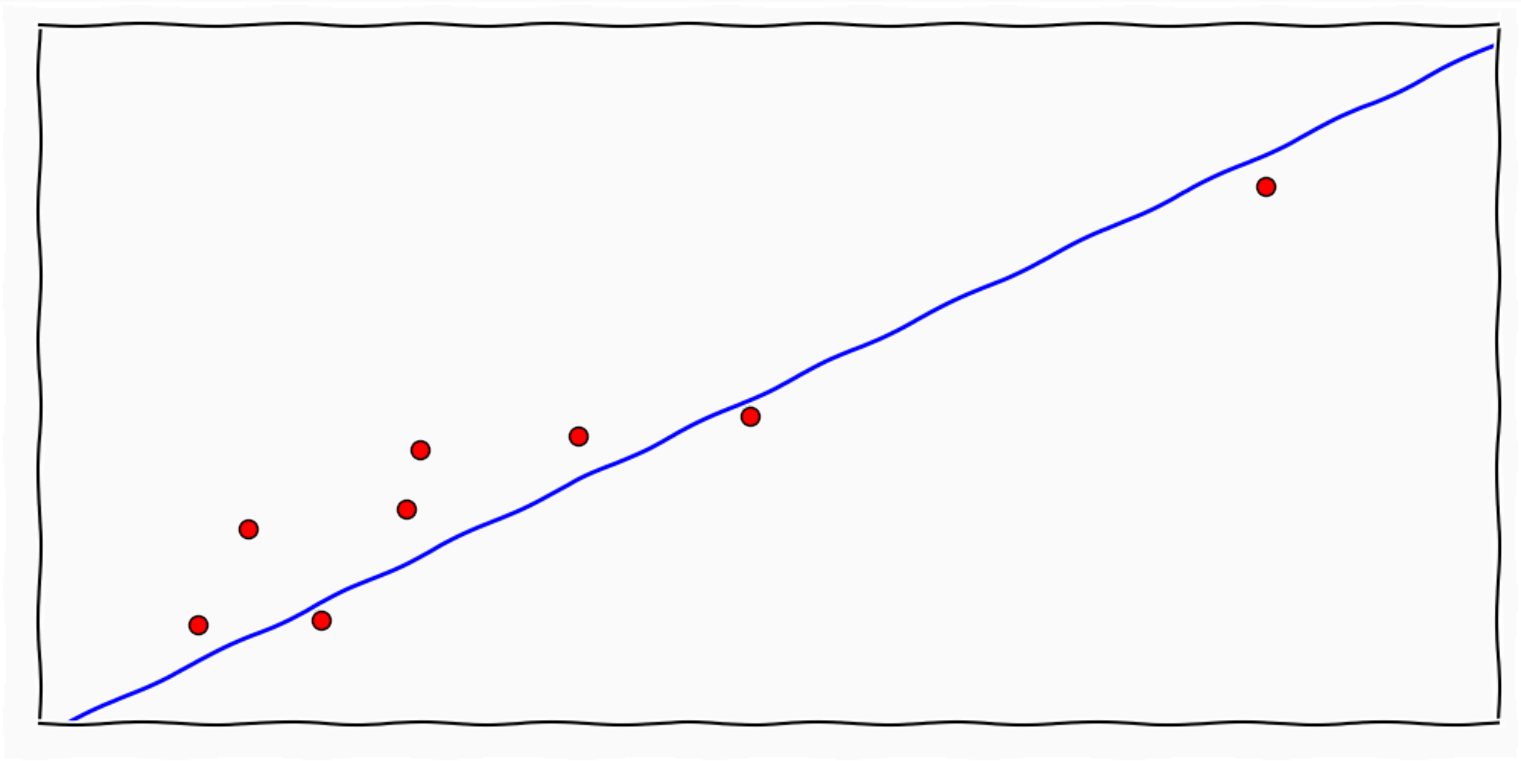
Linear Regression: Modelling



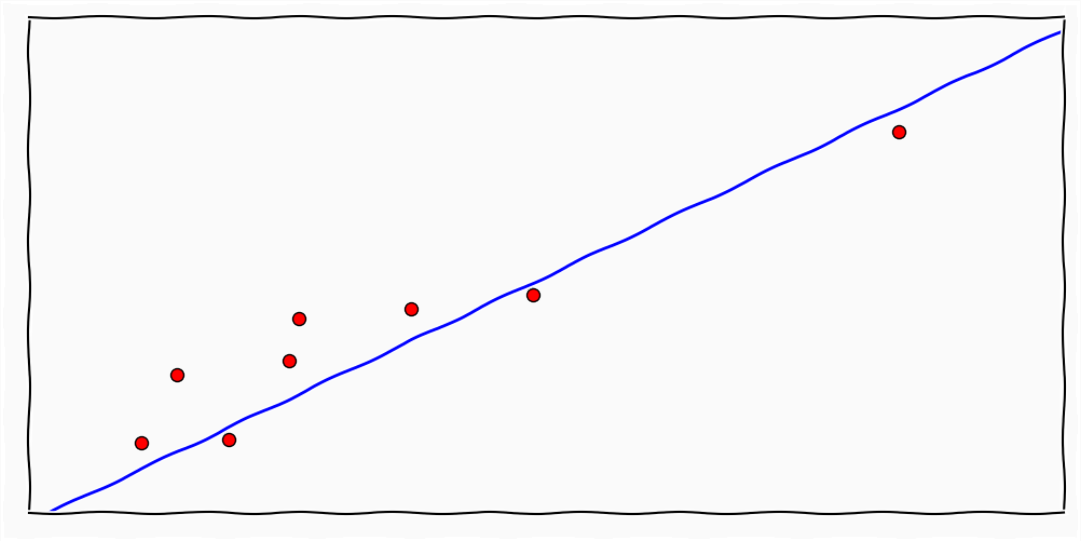
Linear Regression: Modelling



Linear Regression: Modelling

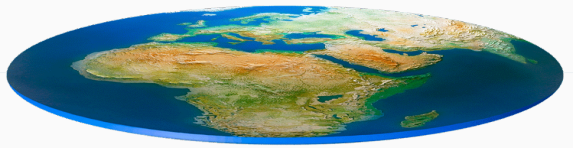


Linear Regression: Modelling



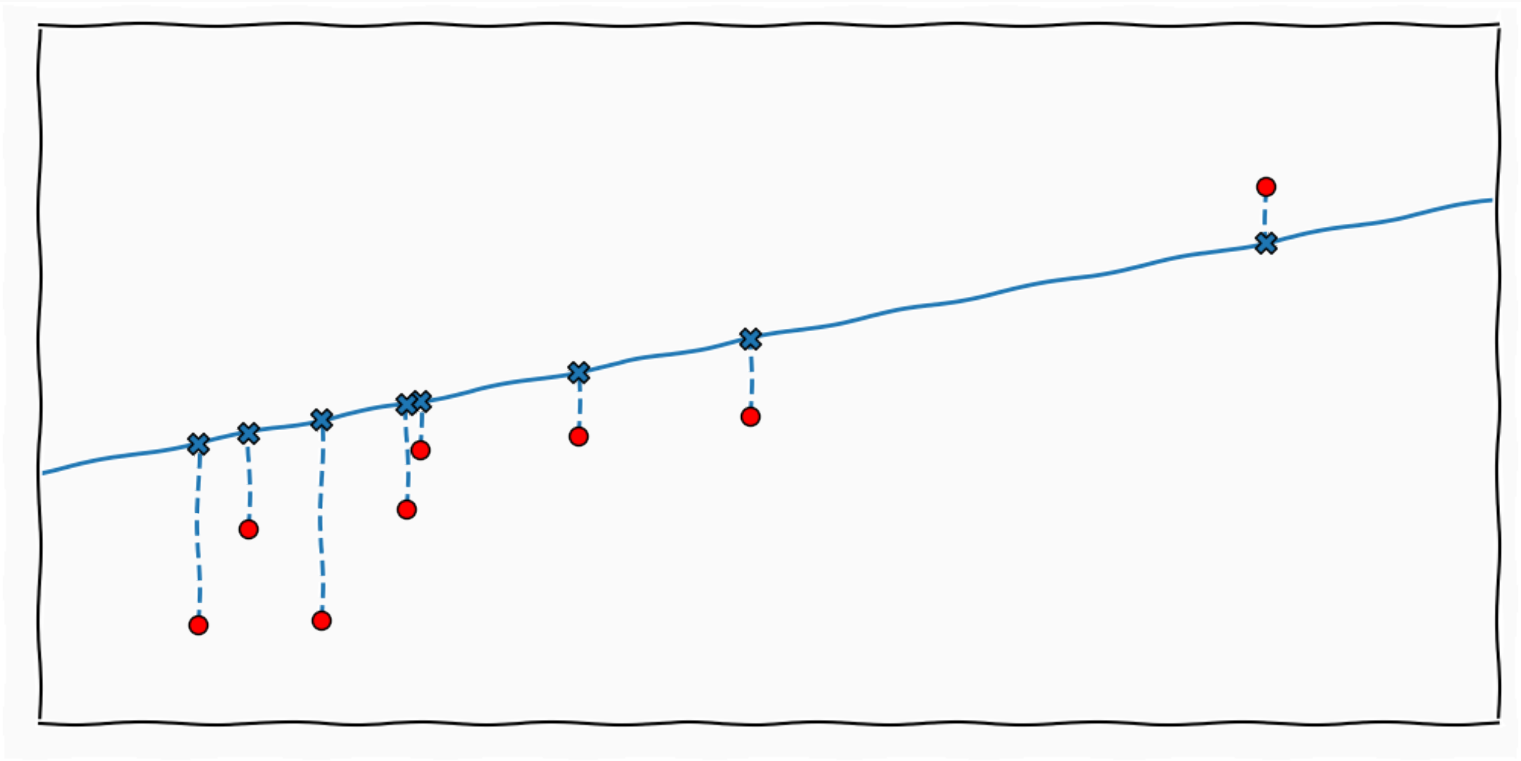
$$y_i = \mathbf{w}^{\mathrm{T}}\mathbf{x}_i + \epsilon_i$$
$$\epsilon_i \sim \mathcal{N}(0, \beta^{-1})$$

Likelihood

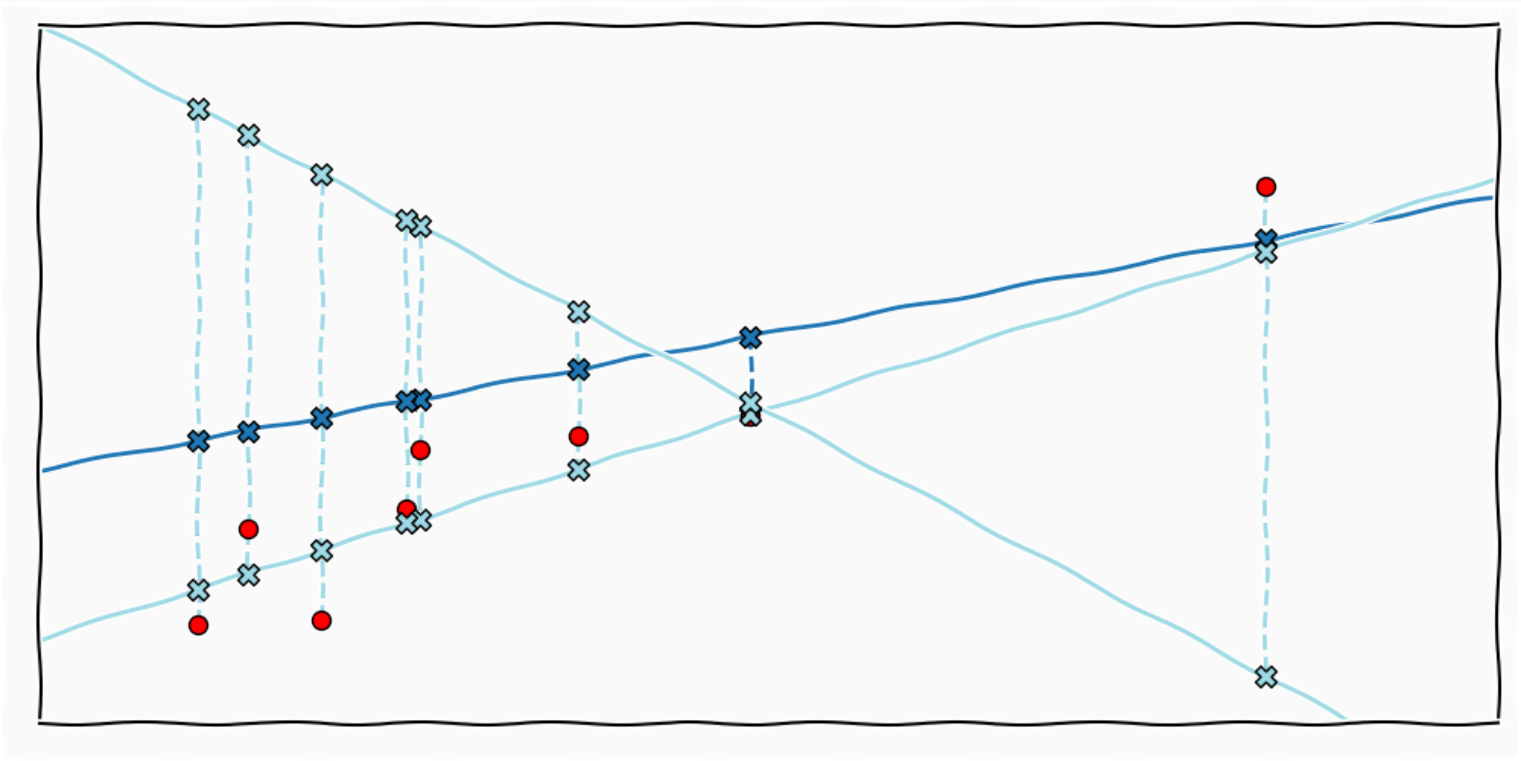


"Given the premise that the earth is flat, how supportive do I believe observations y are of this?"

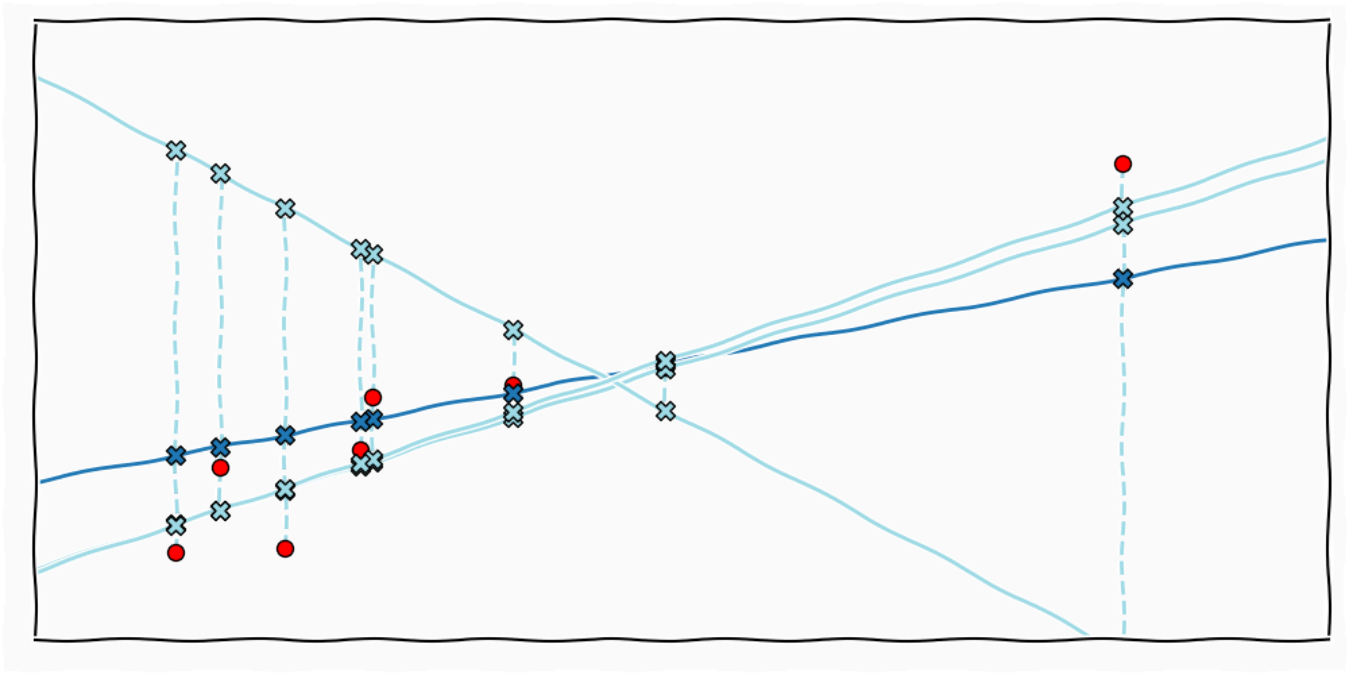
Linear Regression: Modelling



Linear Regression: Modelling

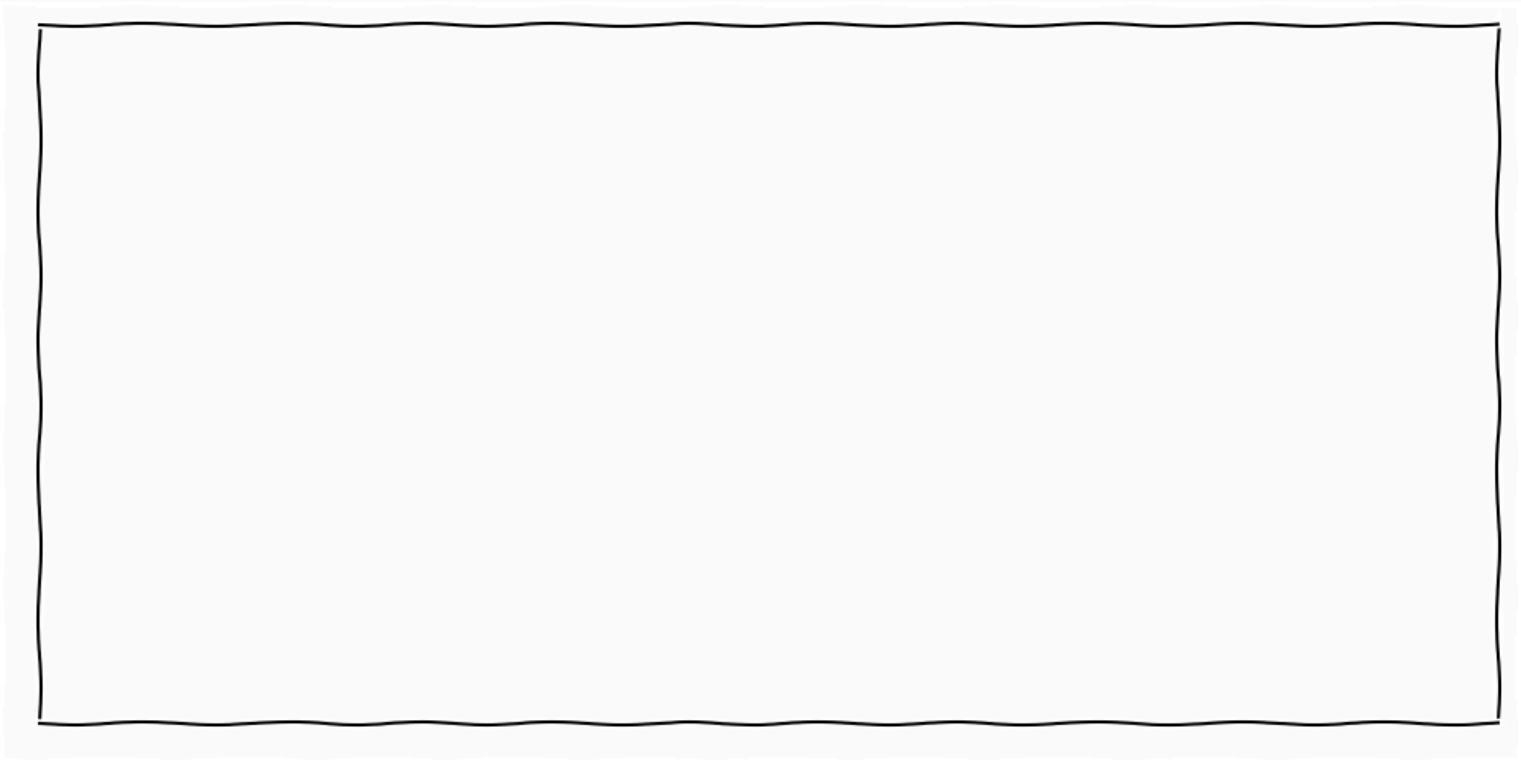


Linear Regression: Likelihood

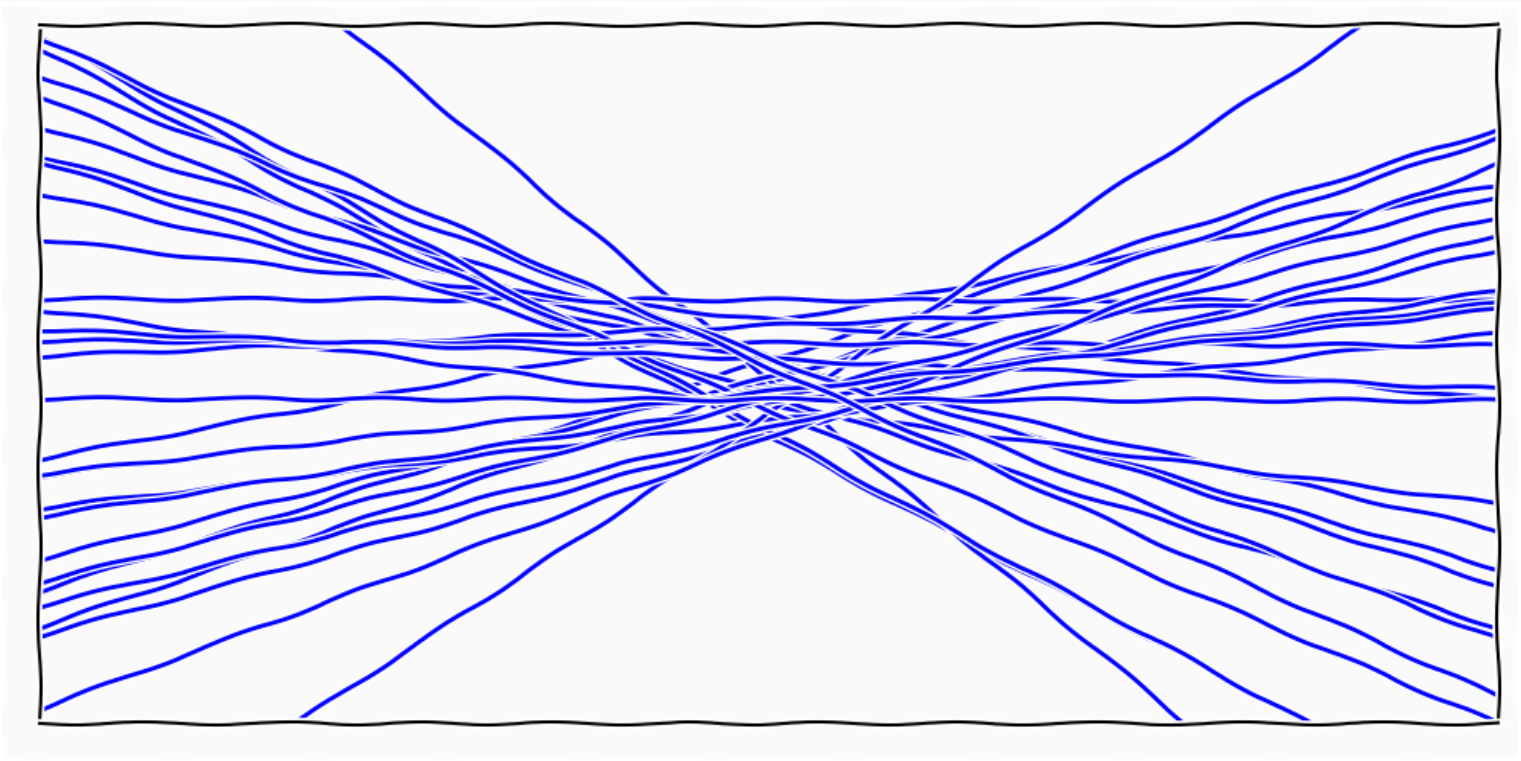


$$p(y|\mathbf{w}, \mathbf{x}) = \mathcal{N}(y|\mathbf{w}^{\mathrm{T}}\mathbf{x}, \beta^{-1}I)$$

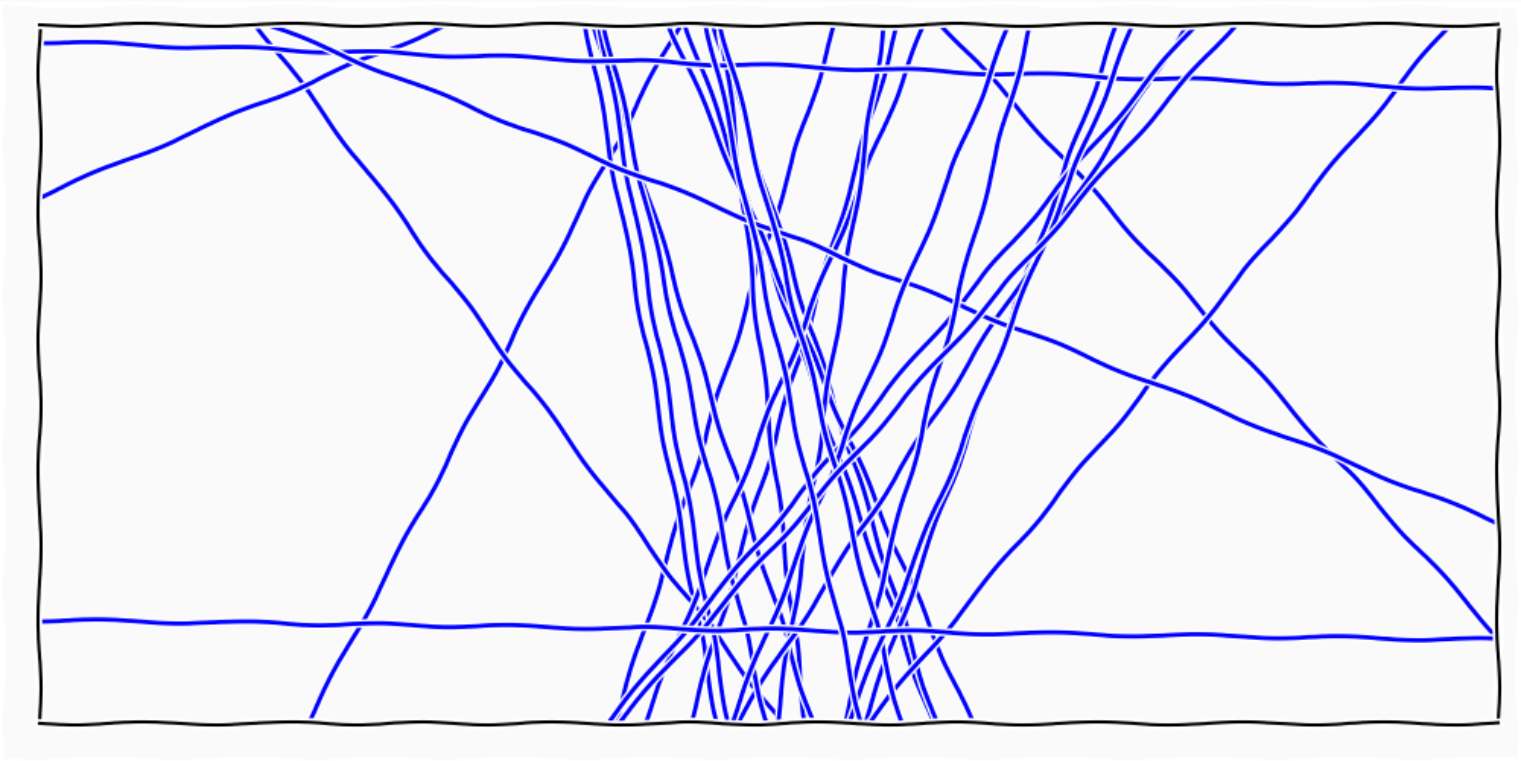
But I don't believe in a flat earth



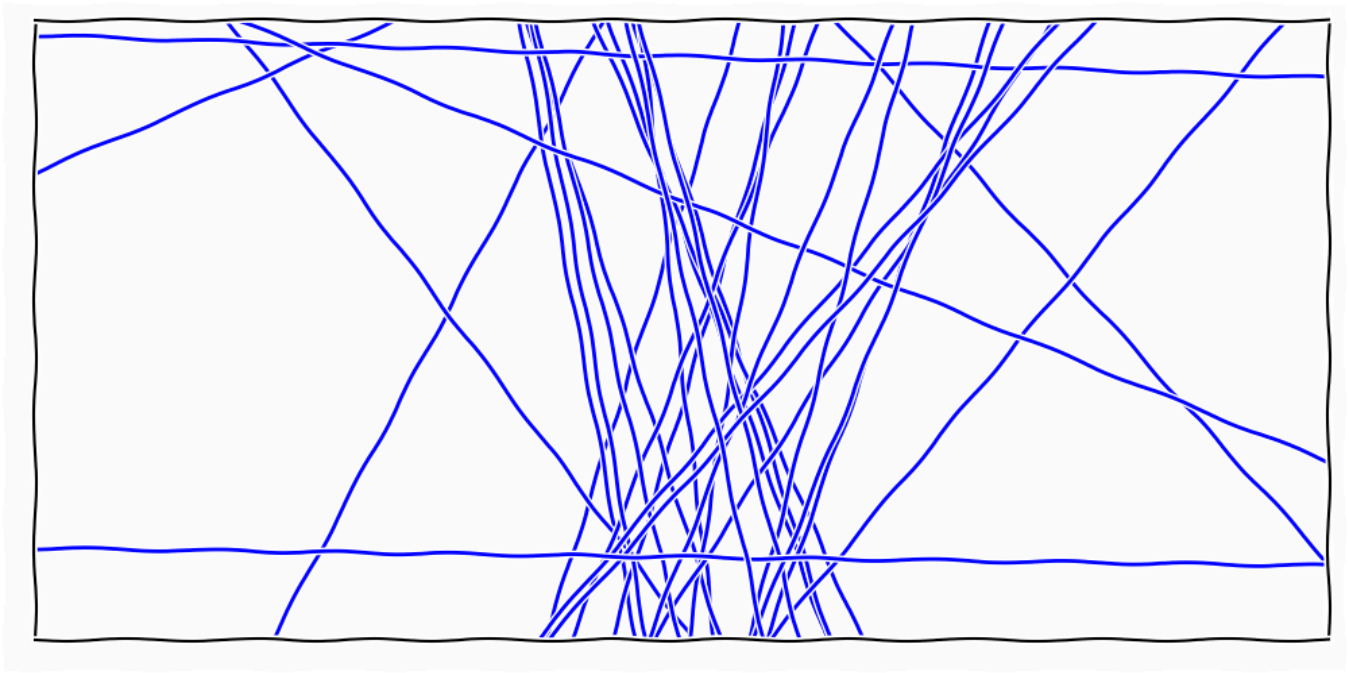
Prior



Prior

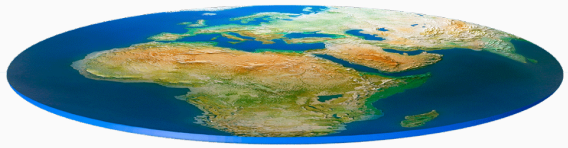


Prior



$$w \sim \mathcal{N}(0, 2)$$

Prior



"Well this is how much credibility belief I give to the hypothesis that the earth is flat"

Two opposing theories

That French Dude Again

It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge.
– Laplace Laplace, 1814

Semantics

$$p(w \mid y) = \frac{p(y \mid w)p(w)}{\int p(y \mid w)p(w)\mathrm{d}w}$$

Likelihood How much evidence is there in the data for a specific hypothesis

Prior What are my beliefs about different hypothesis

Posterior What is my updated belief after having seen data

Evidence What is my belief about any observations

Challenges

- Acquire knowledge
- Mathematically formulate knowledge
- Acquire data
- Inference
- Interpret Results

Conjugacy

Model and Inference

$$p(\mathcal{D}, \theta) = p(\mathcal{Y}, \mathcal{X}, \theta) = p(\mathcal{Y} \mid \mathcal{X}) p(\mathcal{X} \mid \theta) p(\theta)$$

$$p(\theta \mid \mathcal{D})$$

Inference

Churn the handle

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})} = \frac{p(\mathcal{D}|\mu)p(\mu)}{\underbrace{\int p(\mathcal{D}|\mu)p(\mu)\mathrm{d}\mu}_{\text{This is hard}}}$$

Inference

Churn the handle

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})} = \frac{p(\mathcal{D}|\mu)p(\mu)}{\underbrace{\int p(\mathcal{D}|\mu)p(\mu)\mathrm{d}\mu}_{\text{This is hard}}}$$

Conjugacy

- We know the functional form of the posterior

Inference

Churn the handle

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})} = \frac{p(\mathcal{D}|\mu)p(\mu)}{\underbrace{\int p(\mathcal{D}|\mu)p(\mu)\mathrm{d}\mu}_{\text{This is hard}}}$$

Conjugacy

- We know the functional form of the posterior
- We know that the posterior is proportional to the likelihood times the prior

Inference

Churn the handle

$$p(\mu|\mathcal{D}) = \frac{p(\mathcal{D}|\mu)p(\mu)}{p(\mathcal{D})} = \frac{p(\mathcal{D}|\mu)p(\mu)}{\underbrace{\int p(\mathcal{D}|\mu)p(\mu)\mathrm{d}\mu}_{\text{This is hard}}}$$

Conjugacy

- We know the functional form of the posterior
- We know that the posterior is proportional to the likelihood times the prior
- *Use these facts to avoid the integral*

Conjugate Priors

- Most distributions are parametetrised using exponentials
- Exponential family natural parametrisation

$$p(\mathbf{x}|\boldsymbol{\eta}) = h(\mathbf{x})g(\boldsymbol{\eta})e^{\boldsymbol{\eta}^{\mathrm{T}}\mathbf{u}(\mathbf{x})}$$

- Conjugate prior

$$p(\boldsymbol{\eta}|\boldsymbol{\chi}, \nu) = f(\boldsymbol{\chi}, \nu)g(\boldsymbol{\chi})^{\nu}e^{\nu\boldsymbol{\eta}^{\mathrm{T}}\boldsymbol{\chi}}$$

Conjugate Models

Conjugate Priors

Beyond Conjugacy

$$p(y) = \int p(y \mid x) p(x) \mathrm{d}x$$

- Analytically intractable
- Computationally intractable

Summary

Summary

Beliefs/Assumptions think of probabilities as quantifications of beliefs

Probability theory operationalise learning

Inference provides a semantic

Tomorrow

Part I A Story about three donkeys

Part II Machine Learning in a practical setting

Part III Practical session

References

References

 Laplace, Pierre Simon (1814). *Théorie analytique des probabilités*. Mme. Ve. Sourcier.