University of BRISTOL

Machine Learning

Dirichlet Processes

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Introduction

Guest lectures

Dr. Tom Diethe, Amazon Research Cambridge

Lectures: 4 and 5th of December

Optimisation

- Classic optimisation

$$\hat{x} = \text{argmin}_x f(x)$$

- Much more common

$$\hat{x} = \text{argmin}_x \text{black-box}(x)$$

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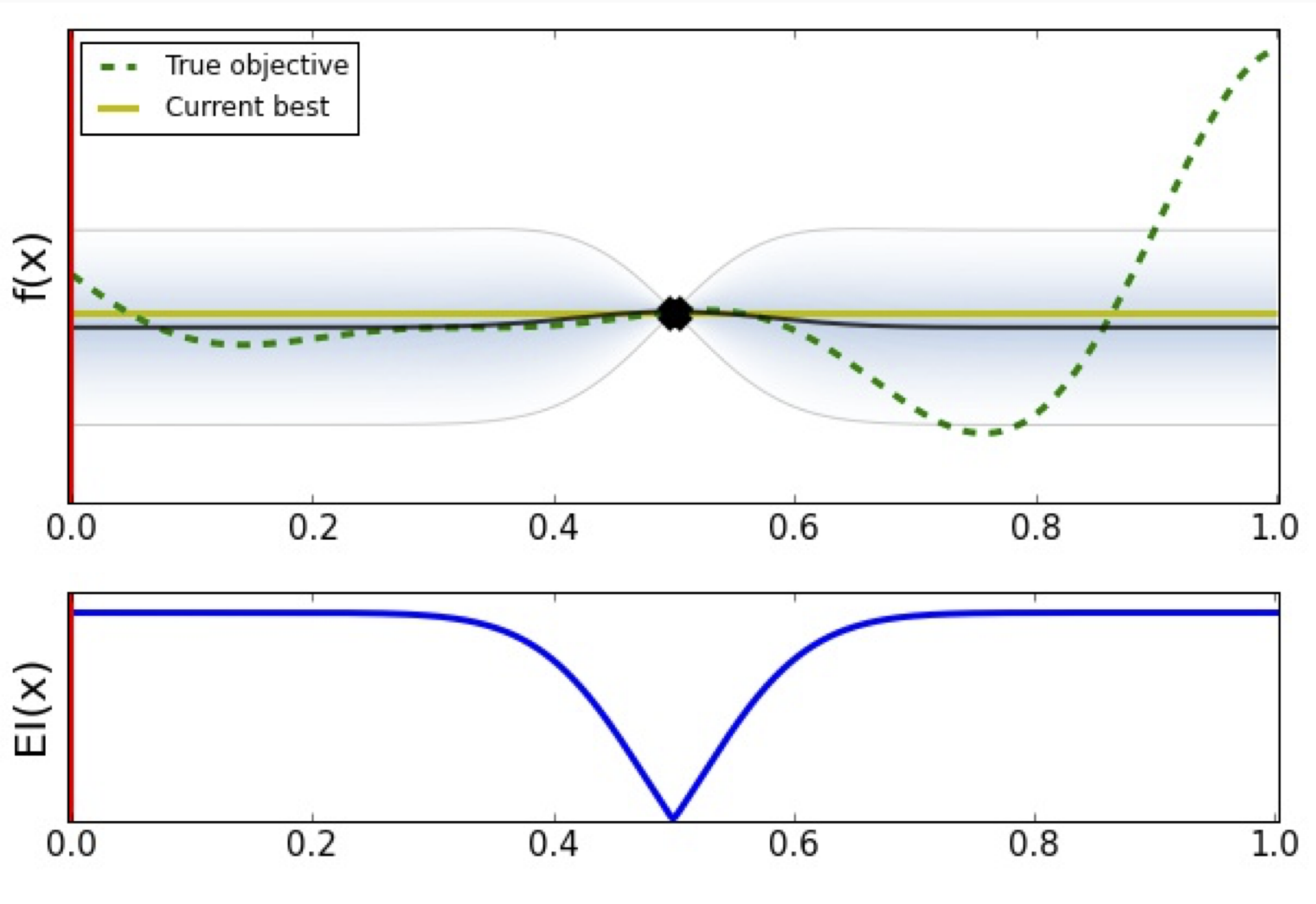
- the test is noisy
- *can we use machine learning to do this for us?*

Bayesian Optimisation [1]



[1] Slides courtesy of Javier Gonzales

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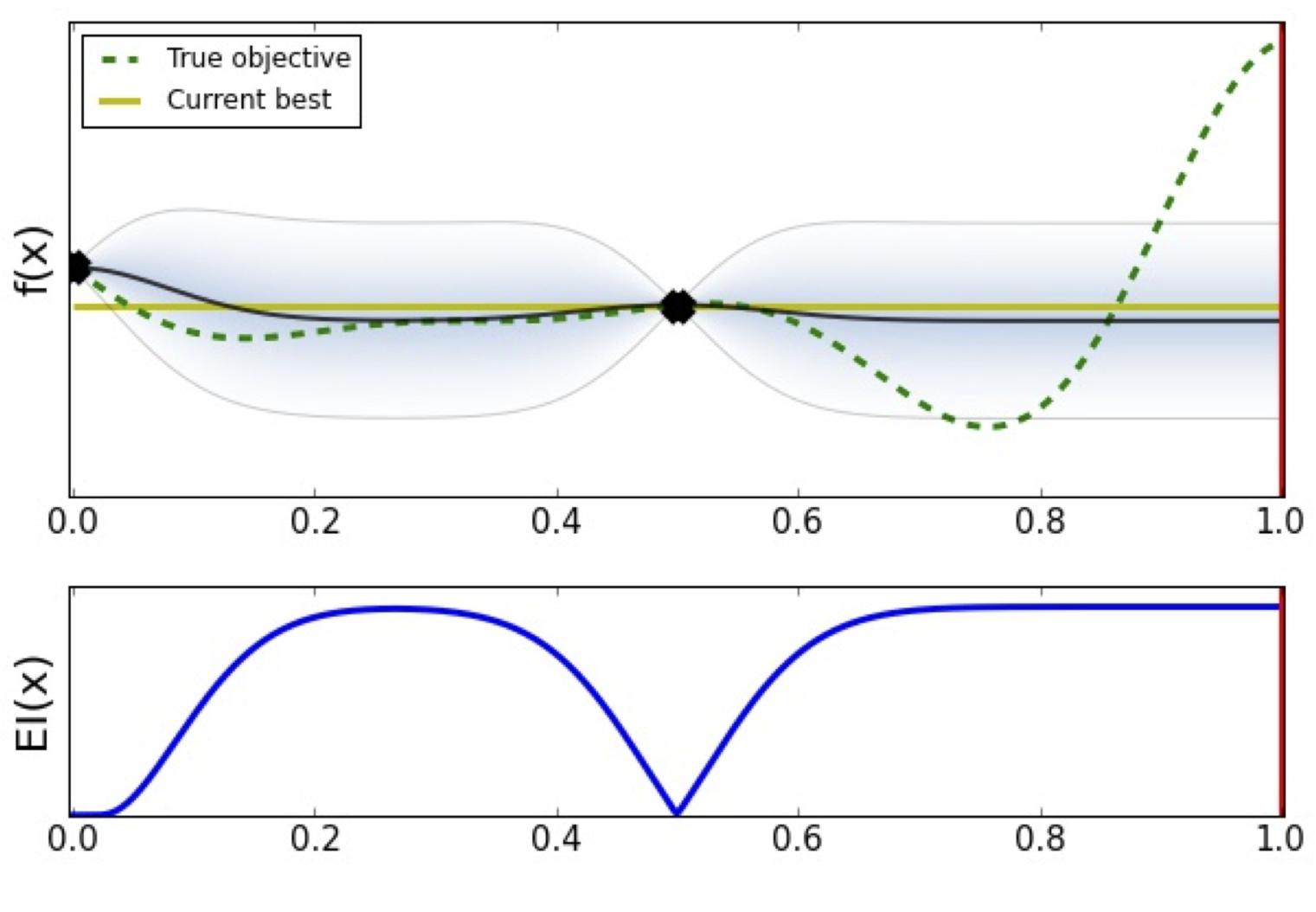

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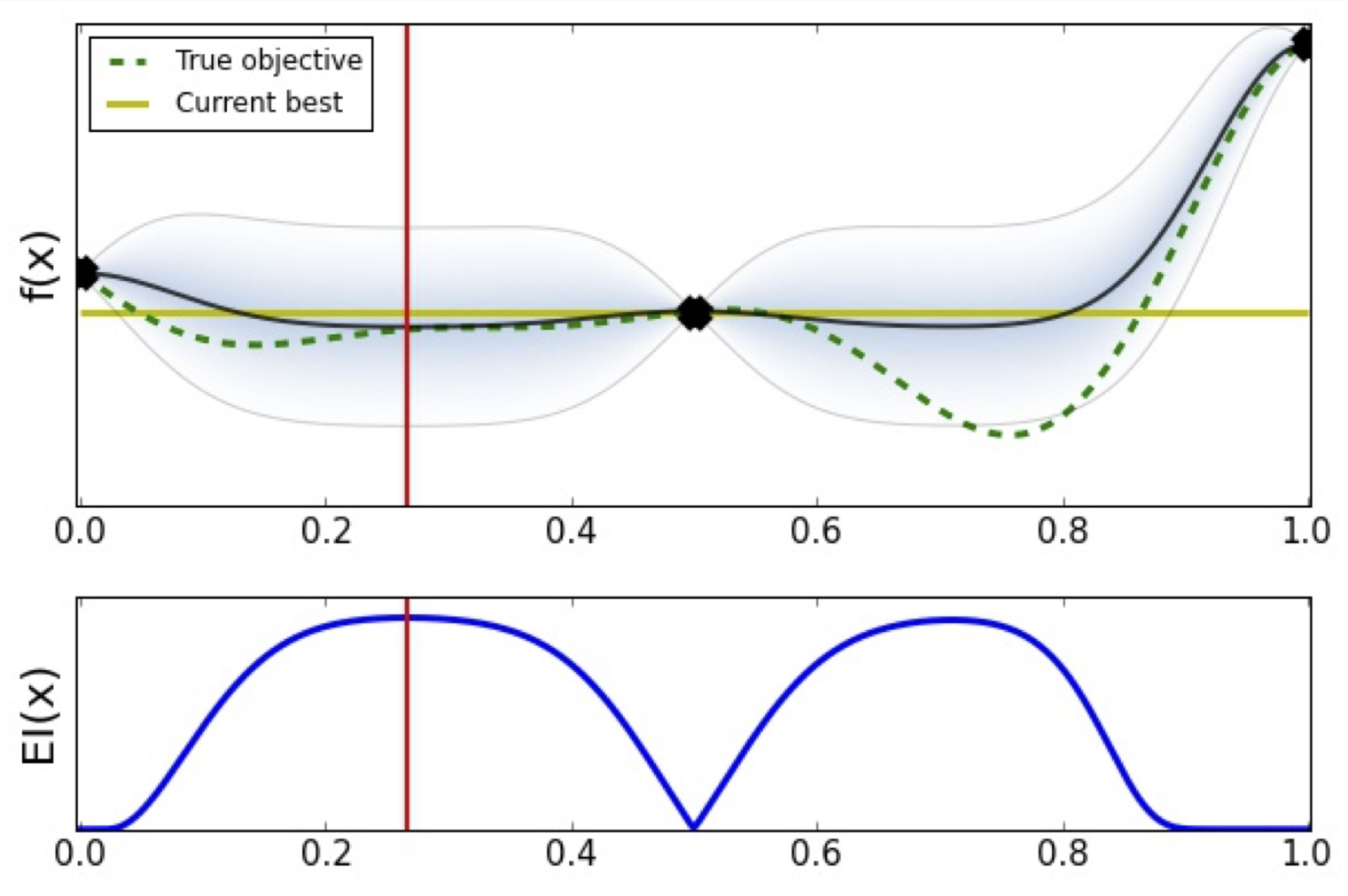

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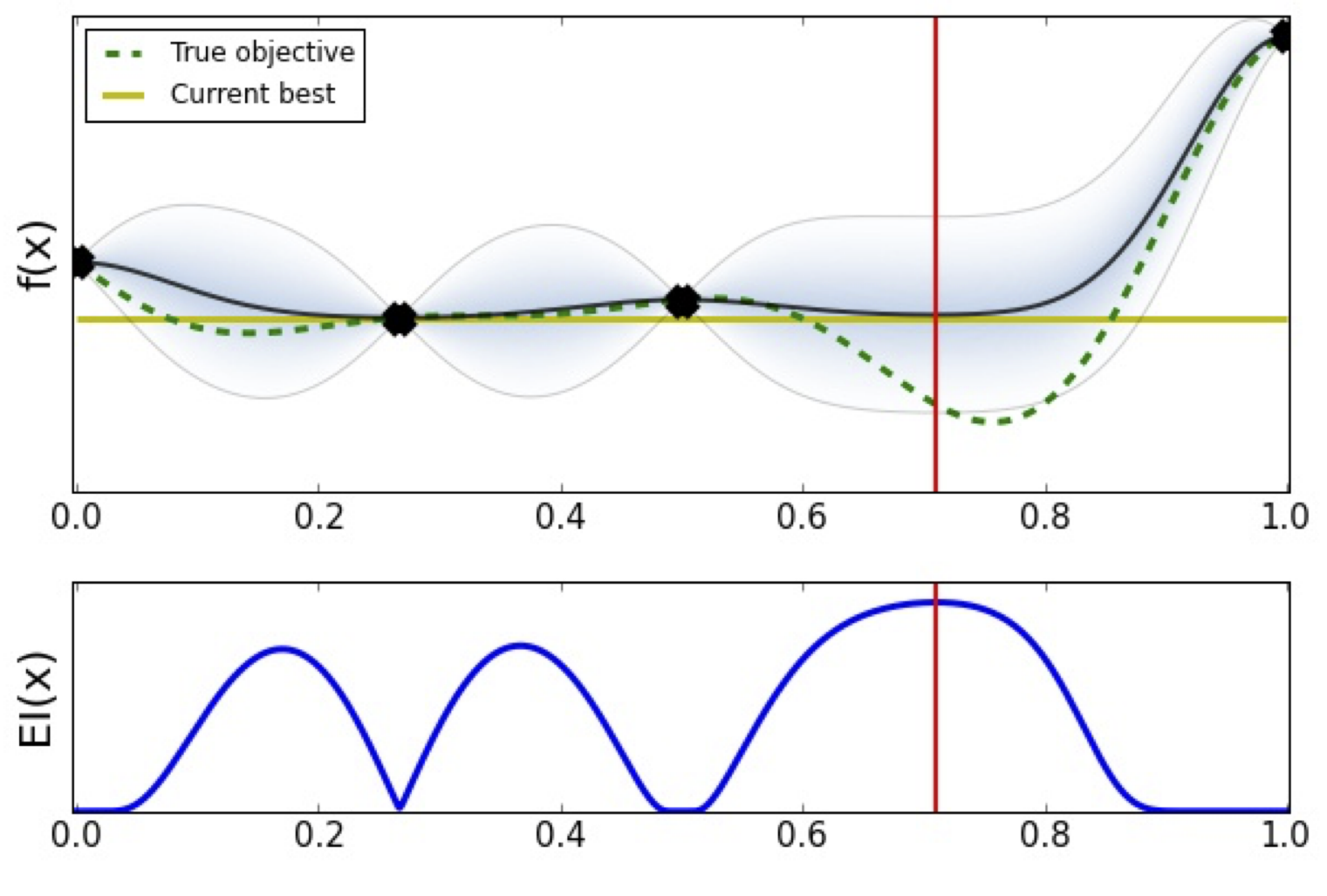

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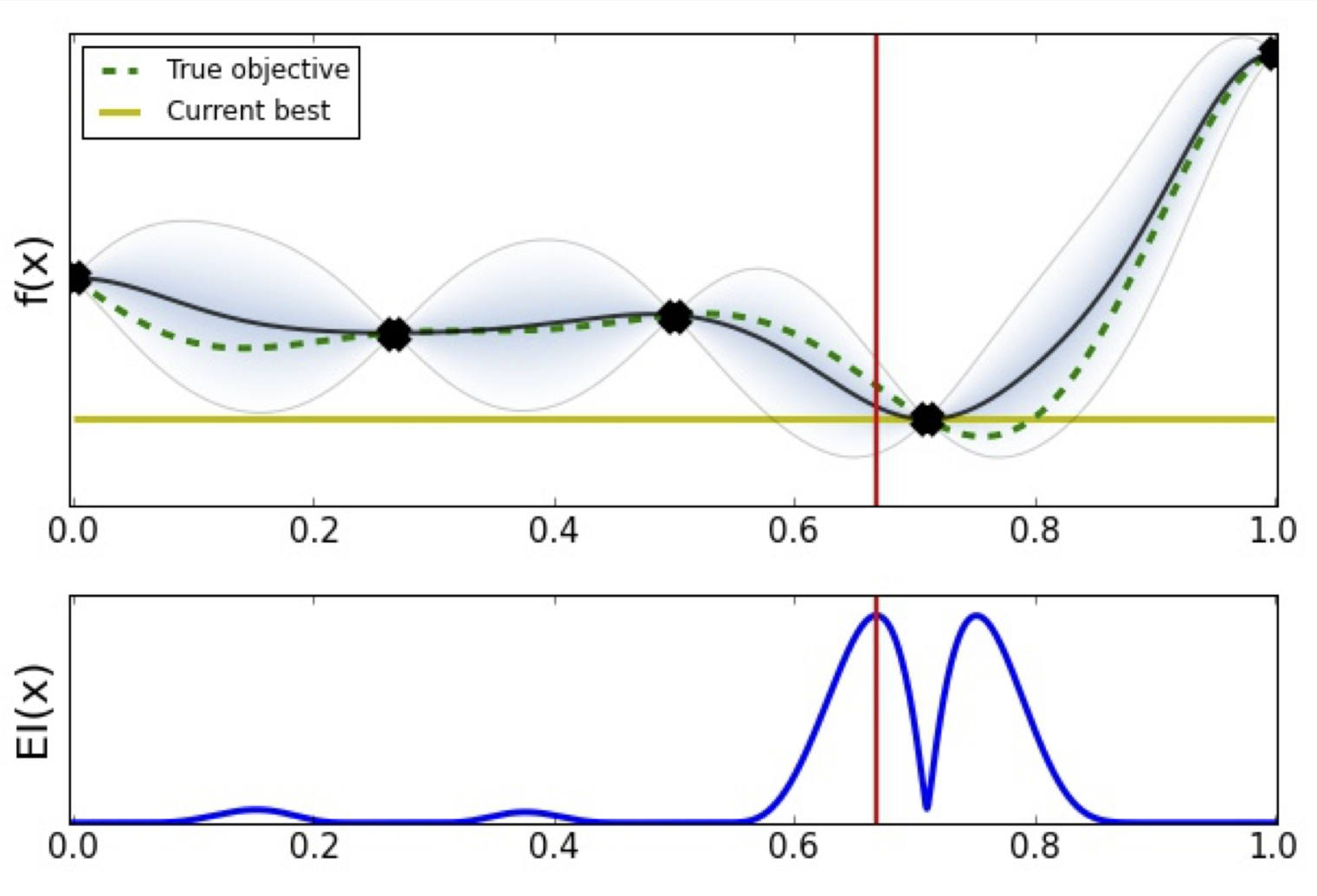

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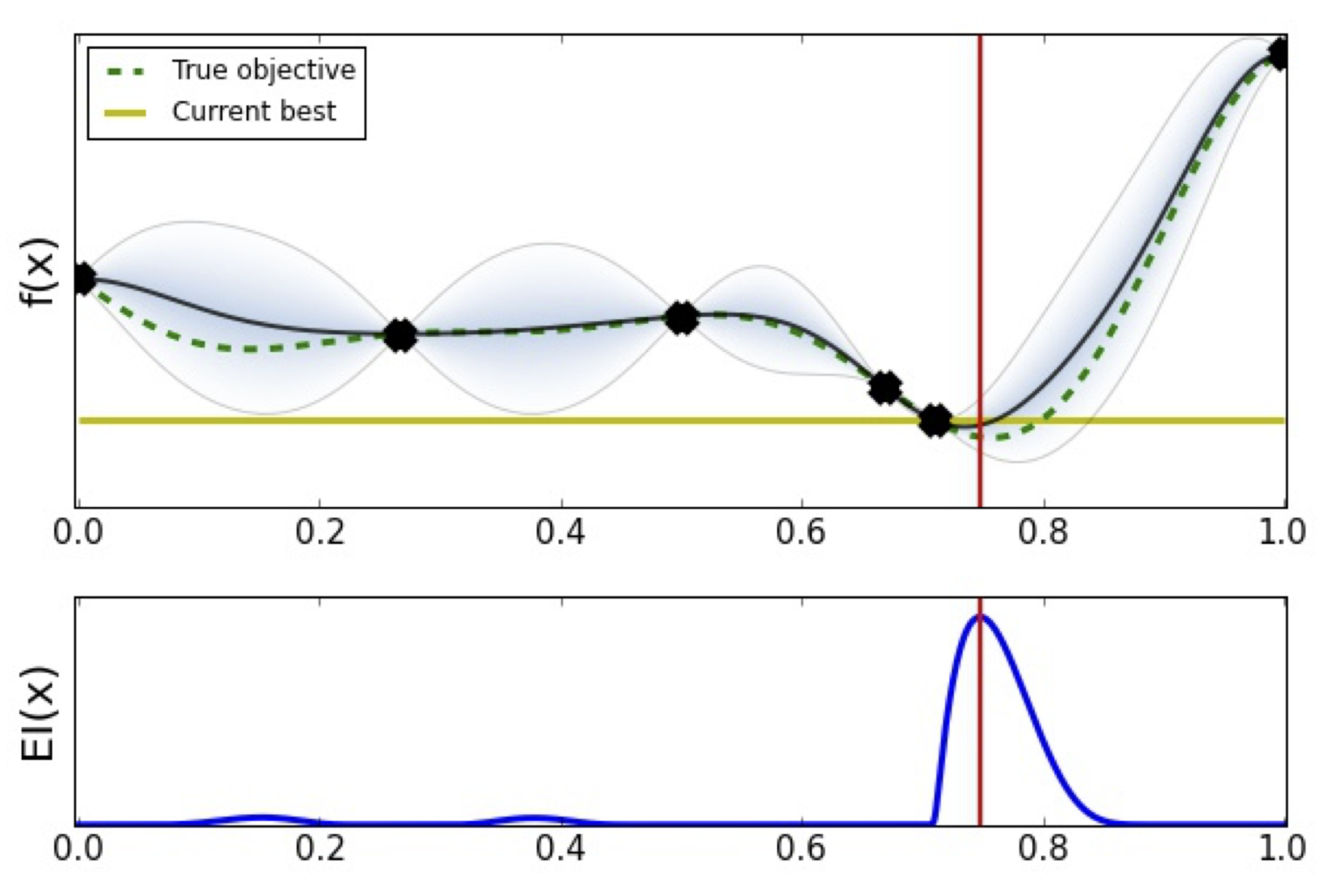

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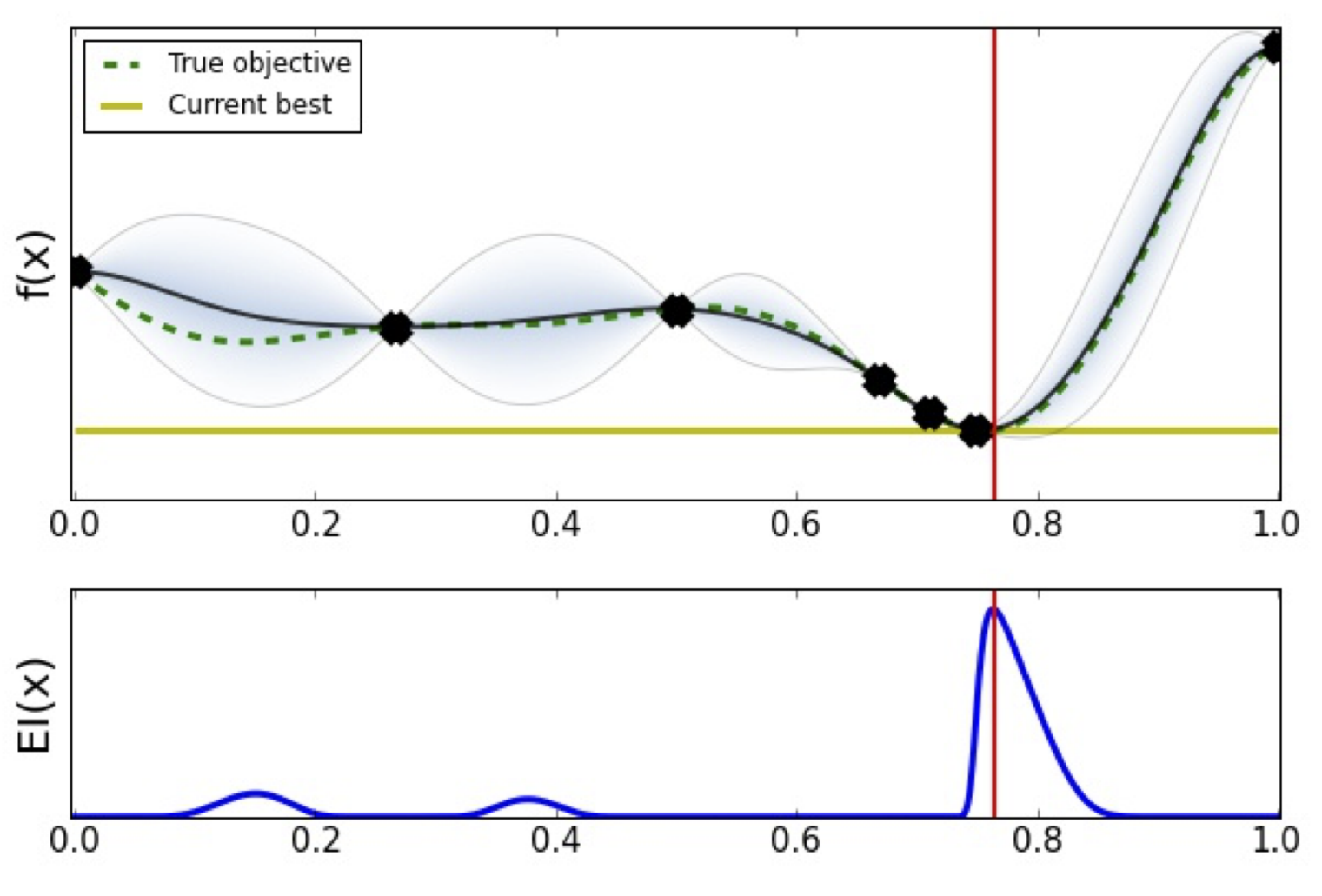

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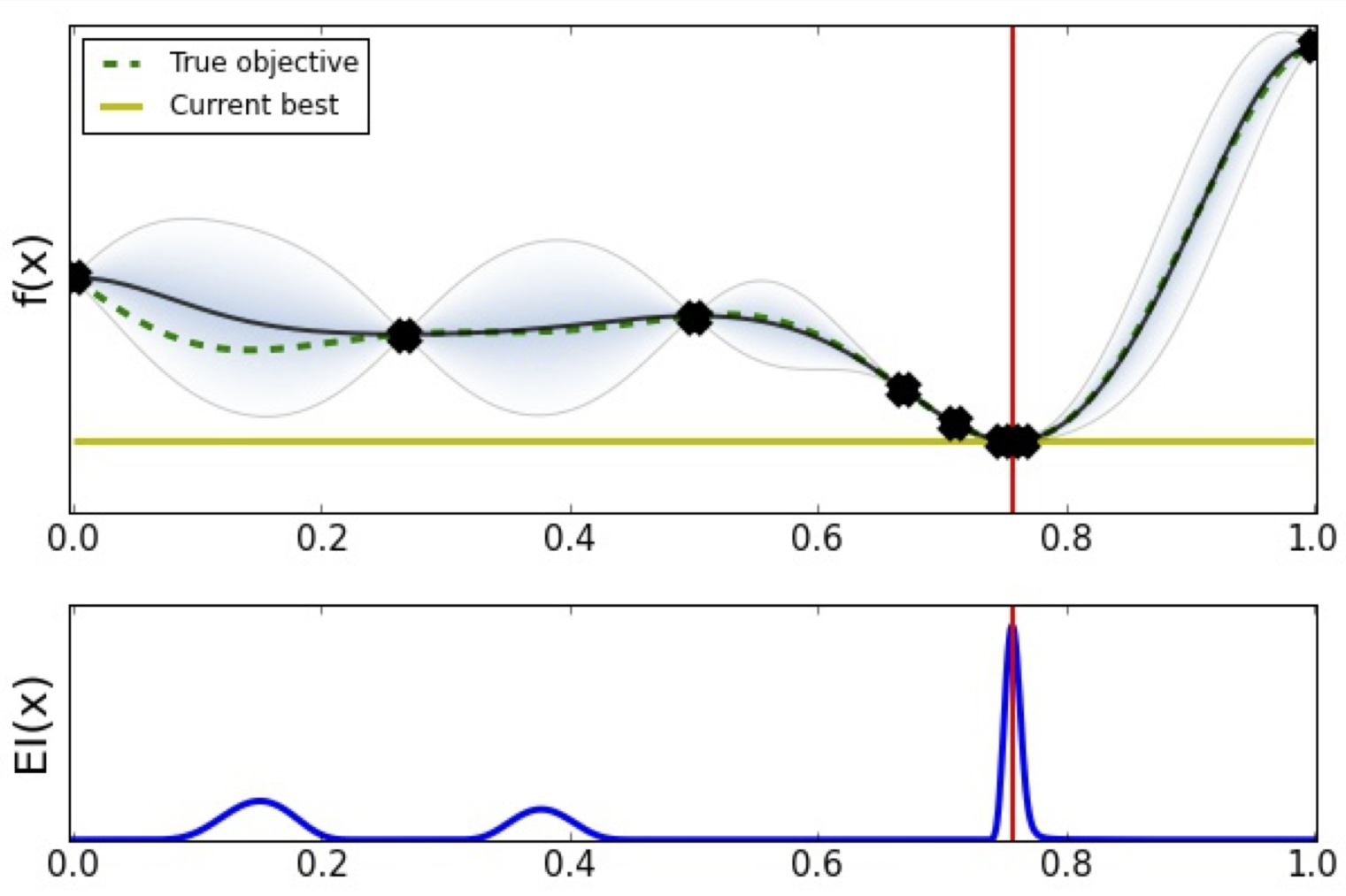

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  - we can get gradients of αx

Non-parametrics

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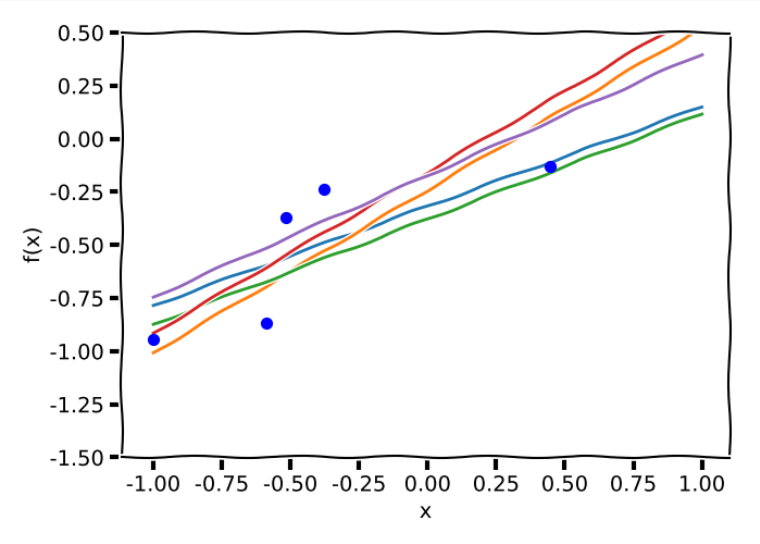
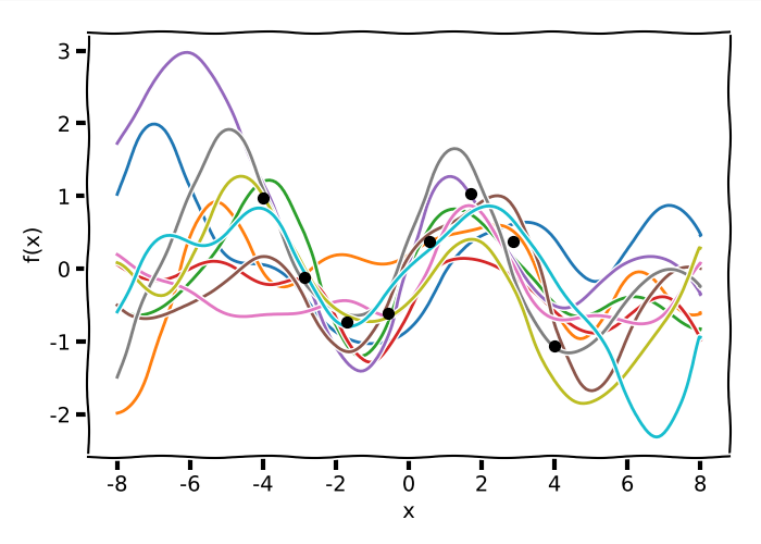




Parametric model

- Number of parameters fixed with respect to sample size
- fixed parameter space

Non-parametric model

- Number of parameters grows with sample size
- ∞ -dimensional parameter space

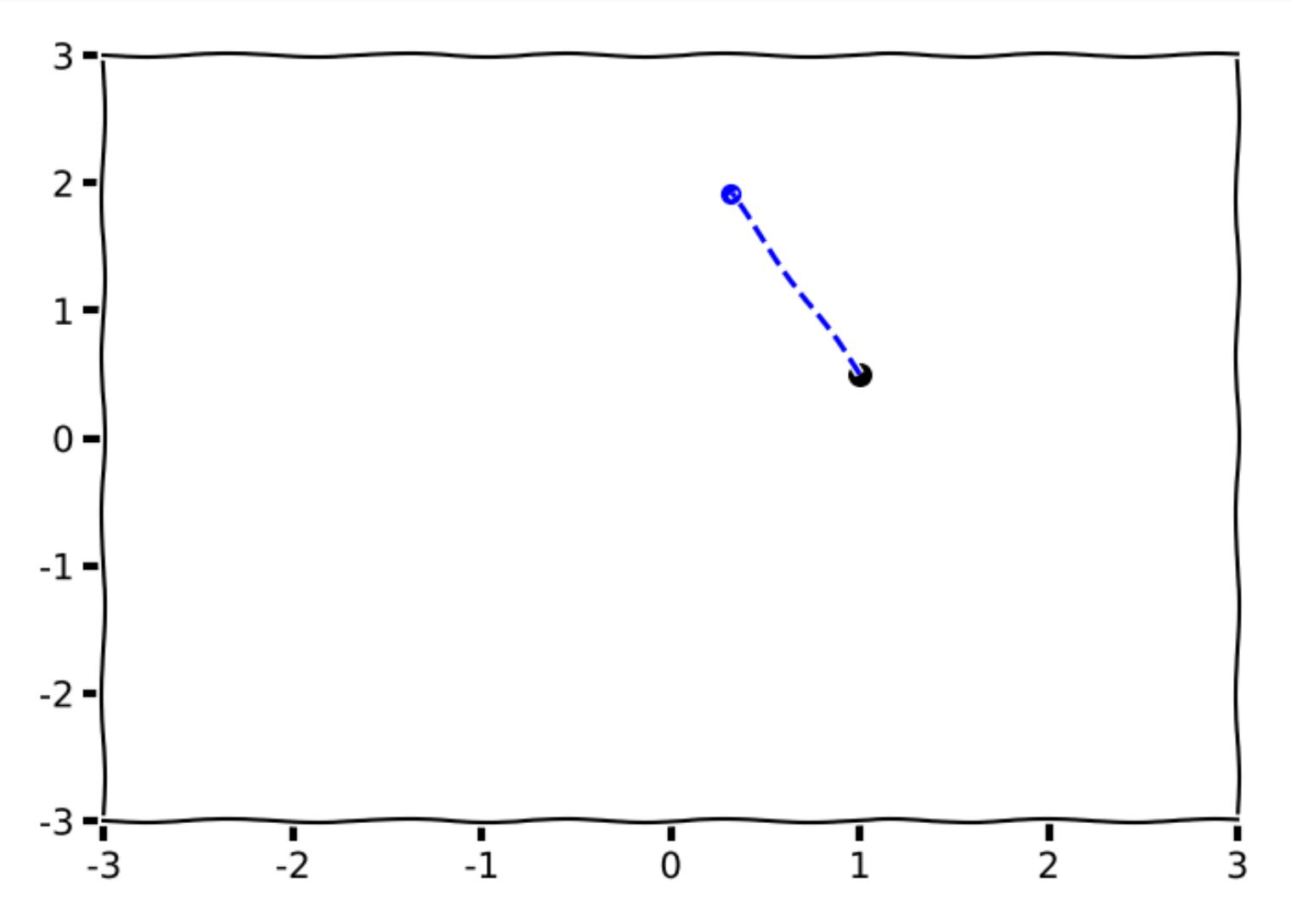
Nearest Neighbour

- Training data: $\{\mathbf{x}_i, y_i\}_{i=1}^{N}$
- Test data: $\{\mathbf{x}_i\}_{i=1}^{M}$
- Inference

$$\hat{i} = \text{argmin}_i D(\mathbf{x}_*, \mathbf{x}_i)$$

- Complexity grows with number of training data
- Does not generalise at all

Nearest Neighbour



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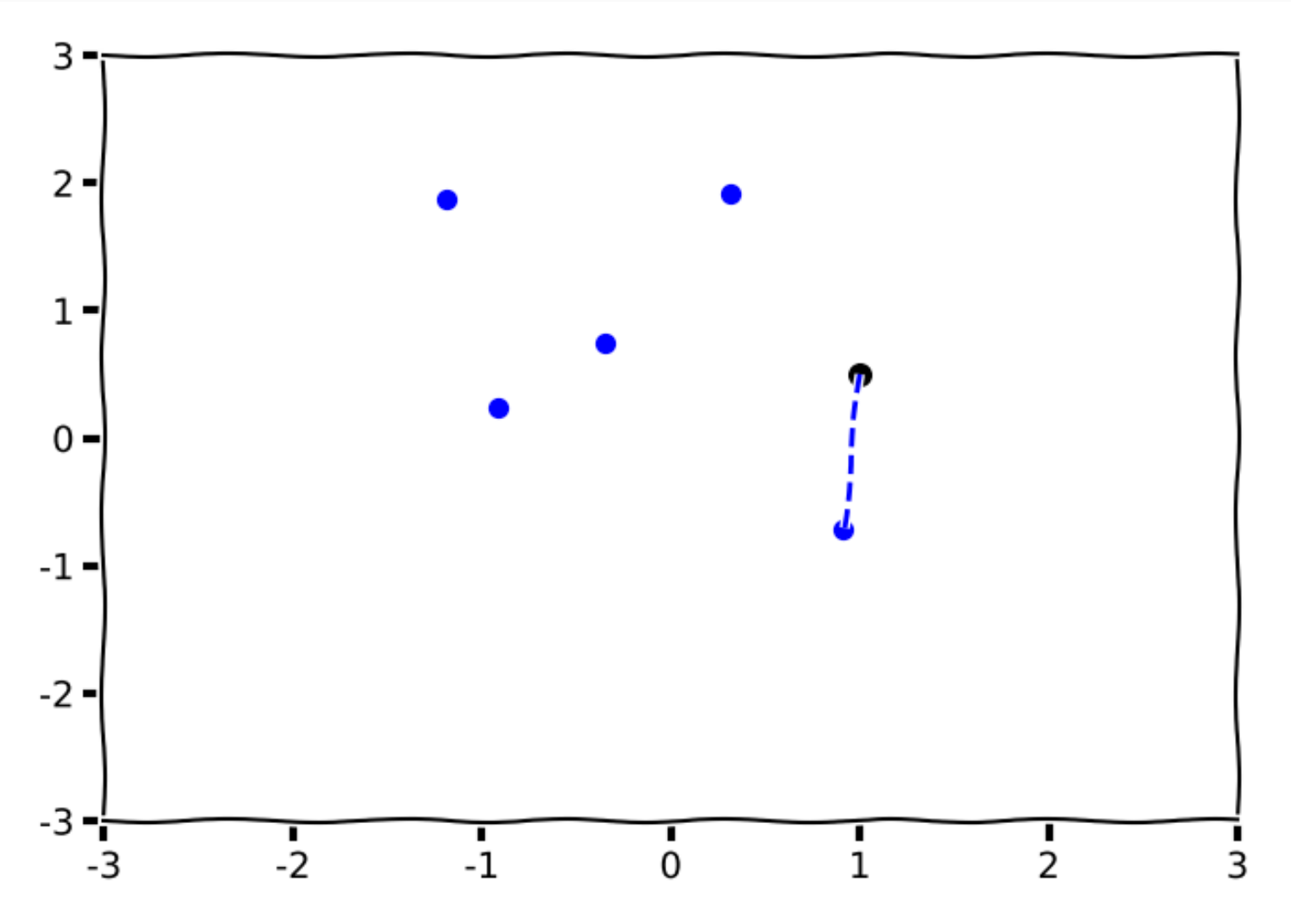

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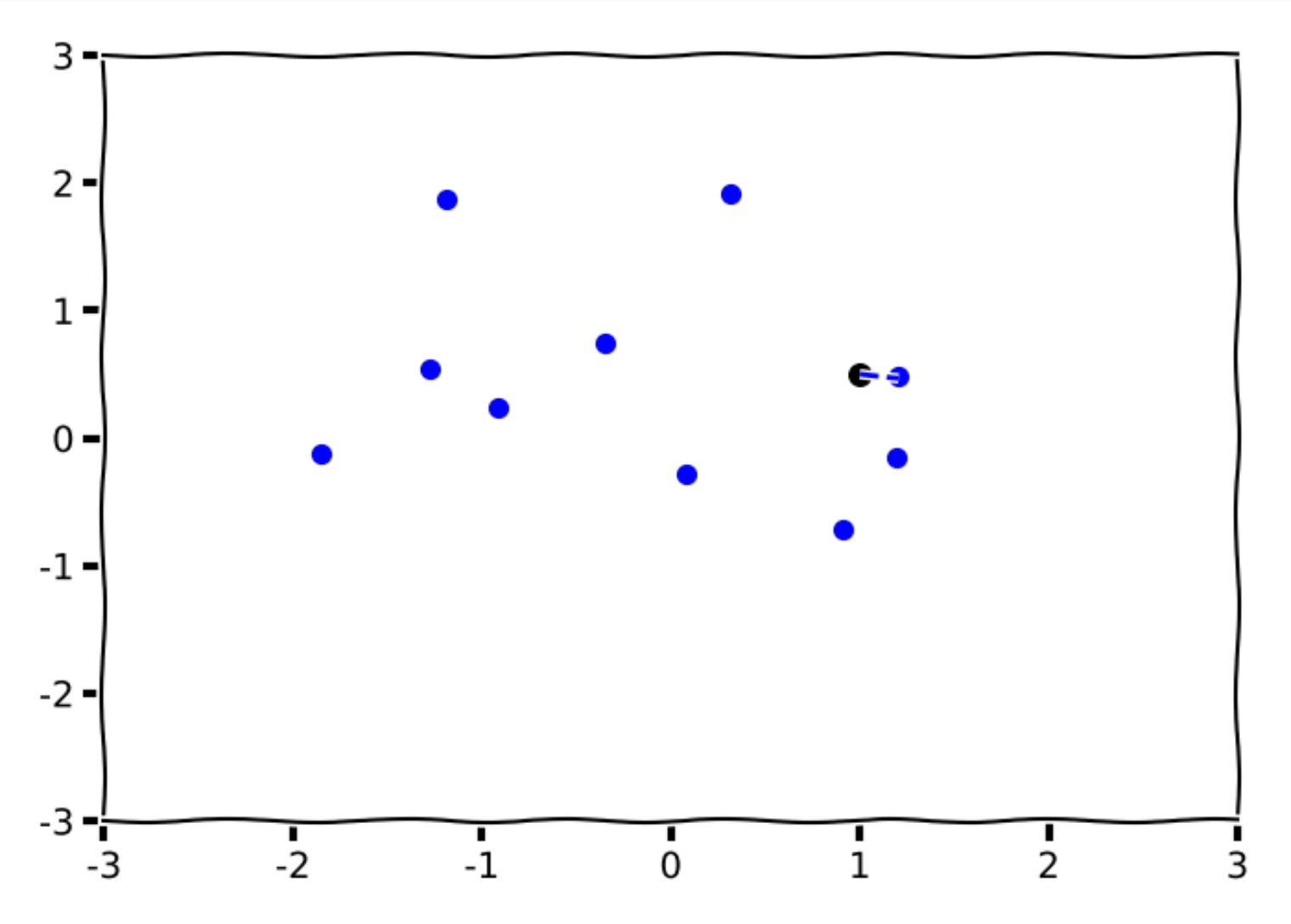

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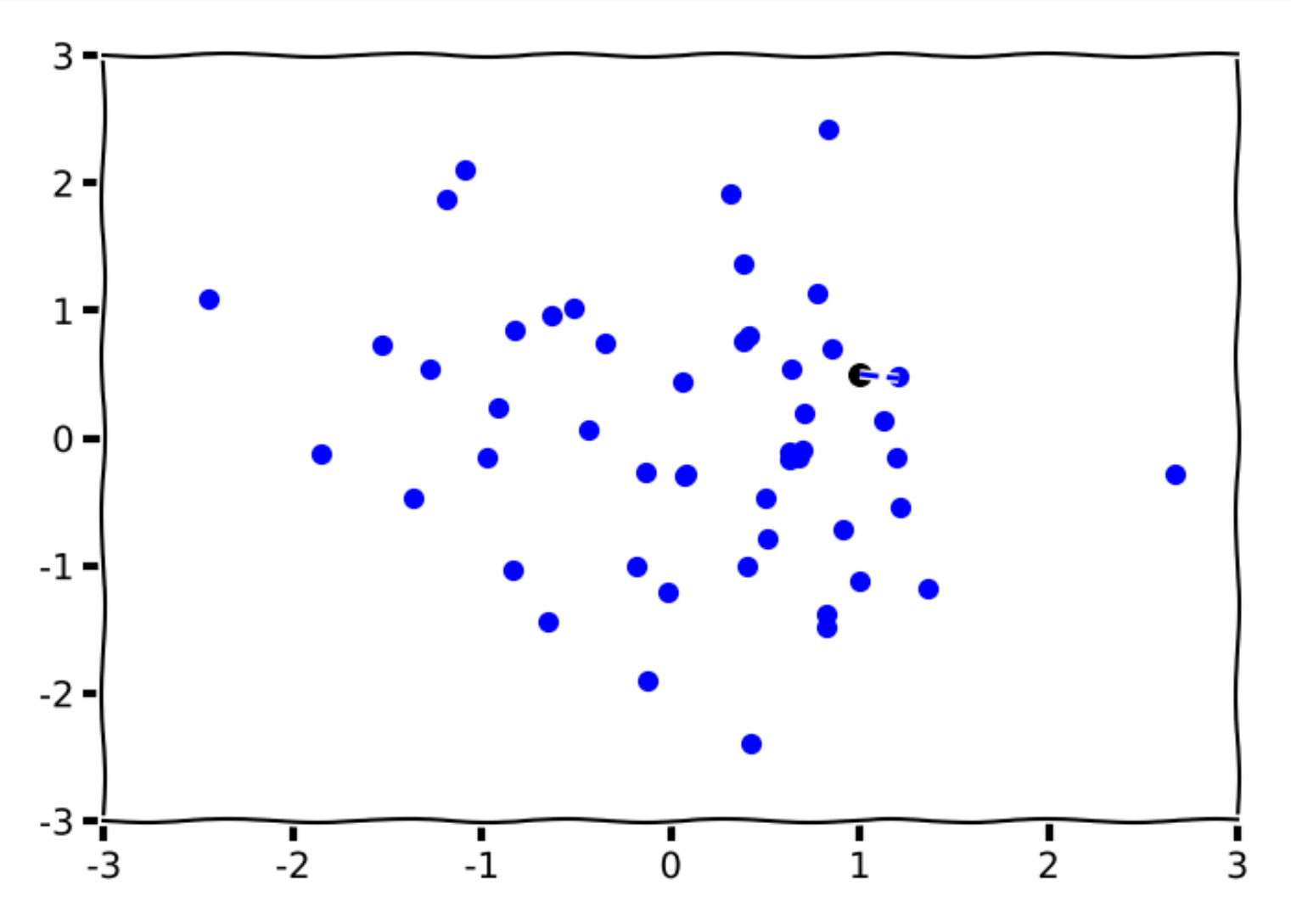
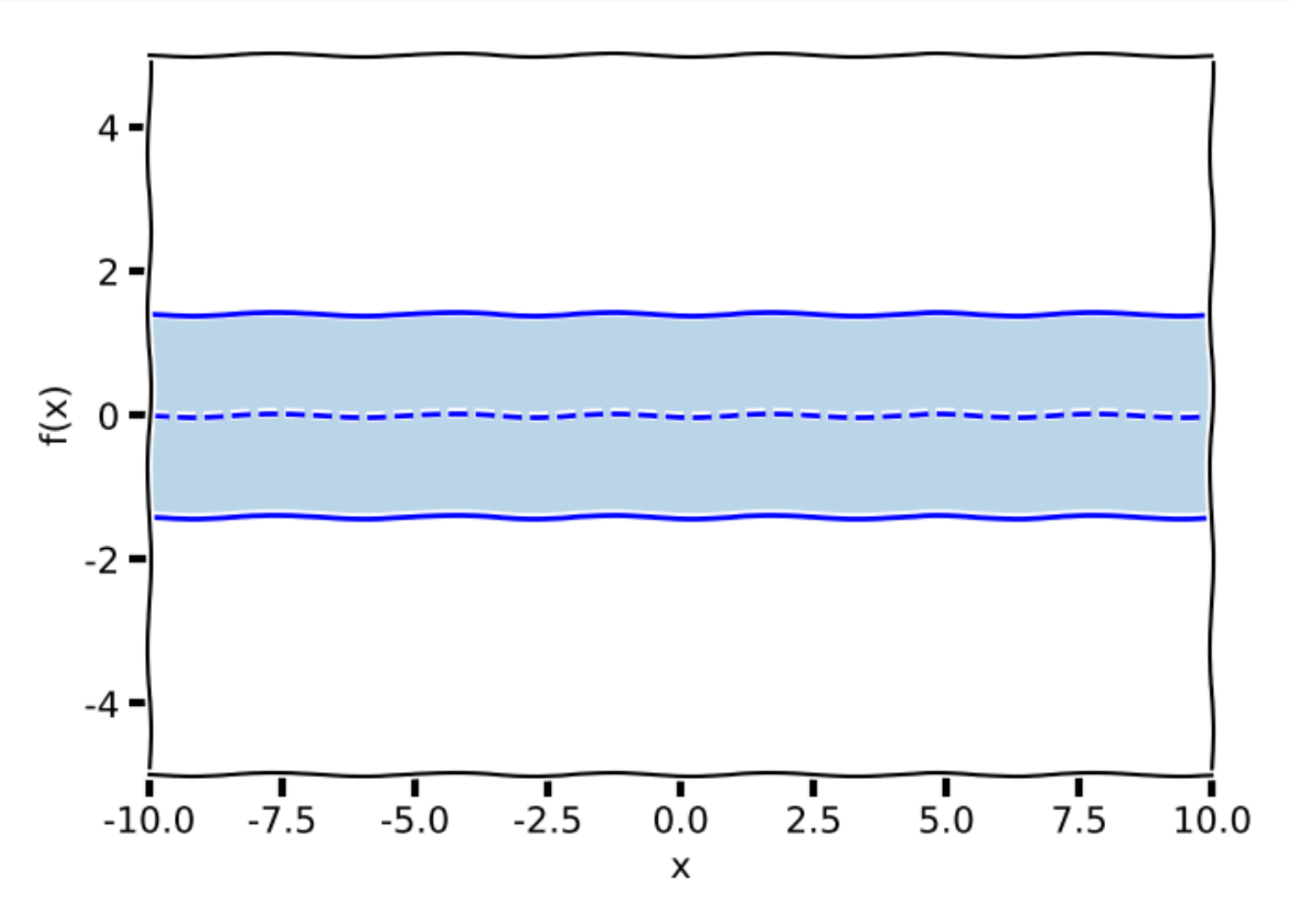


Gaussian Processes



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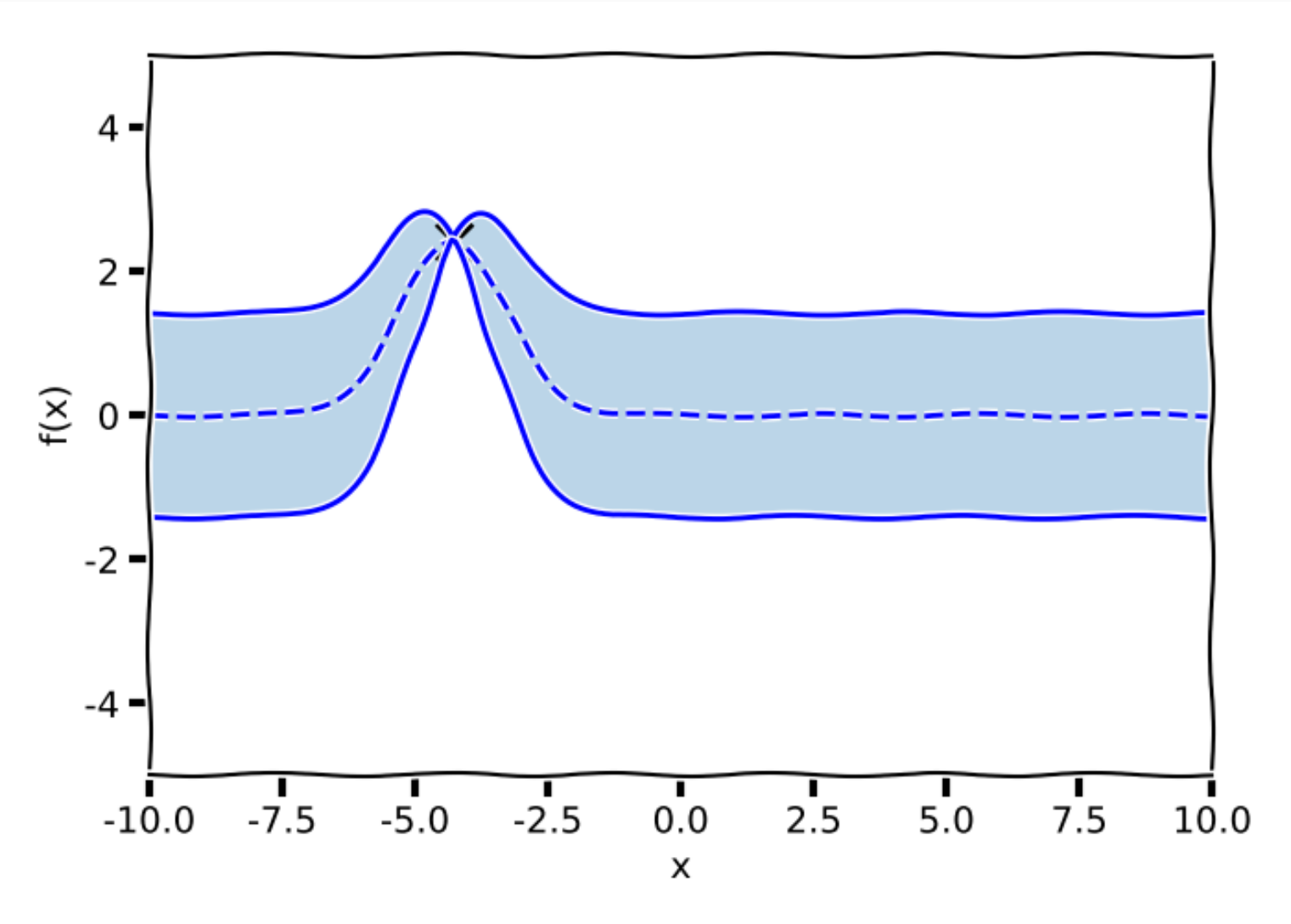

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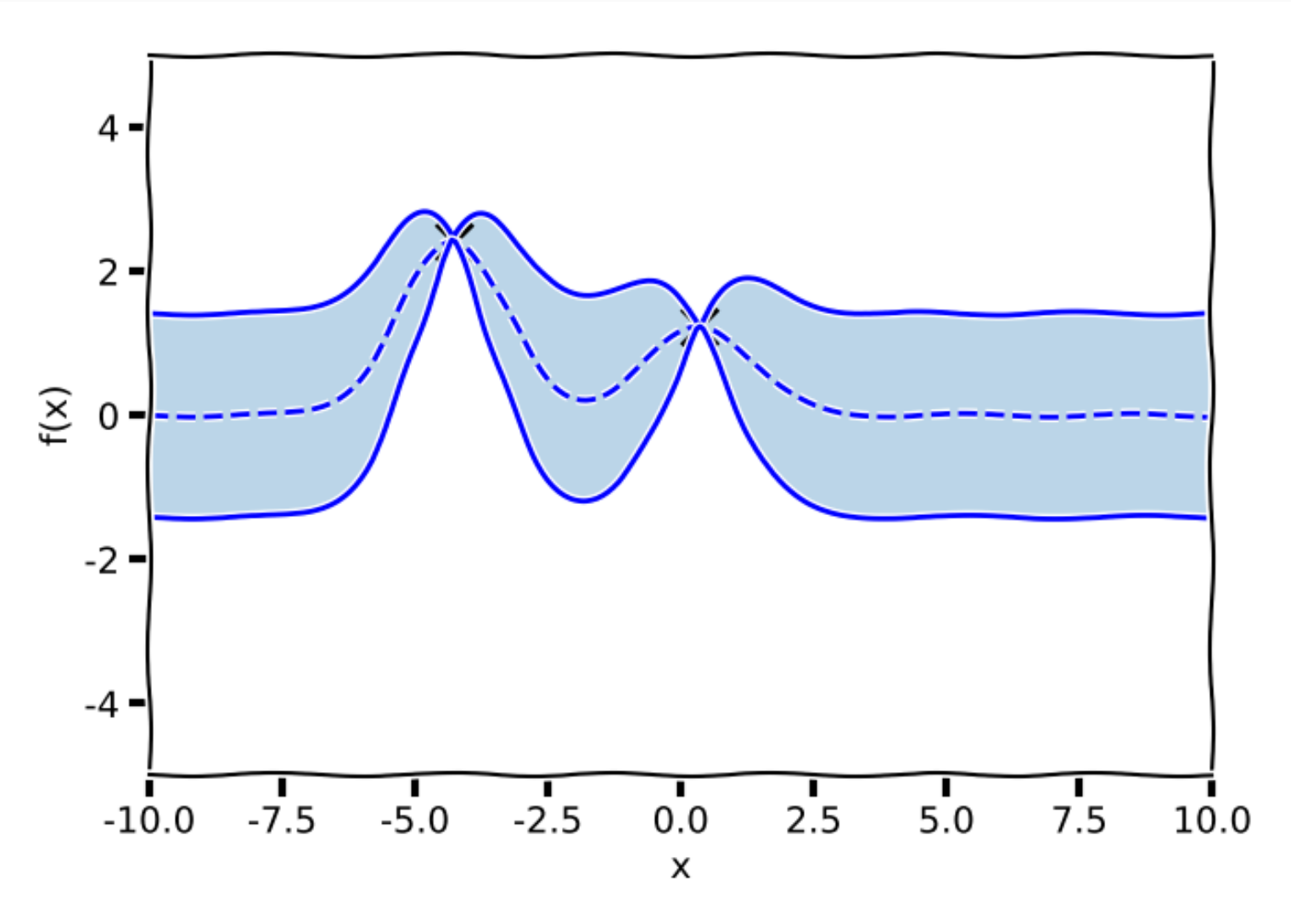

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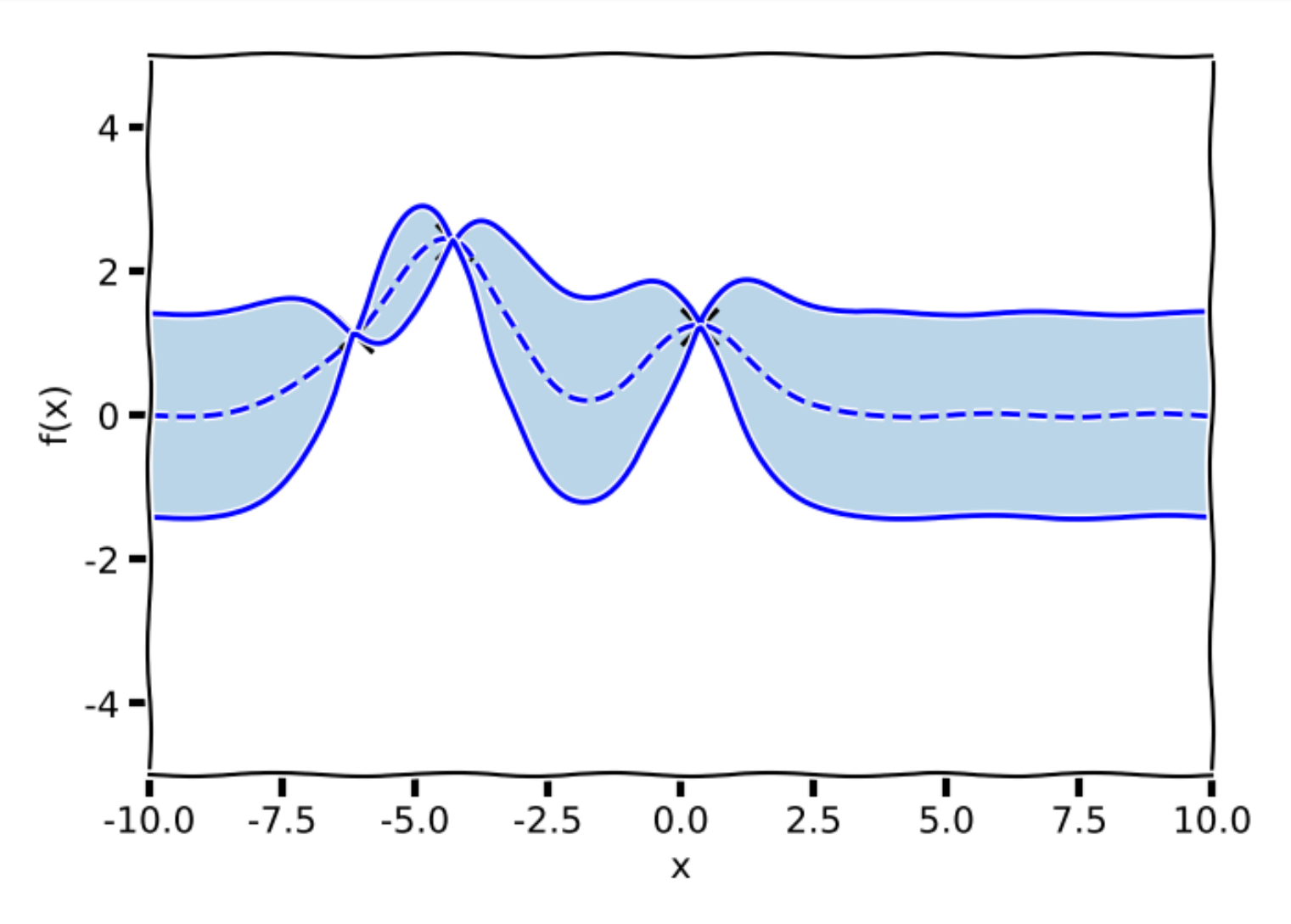

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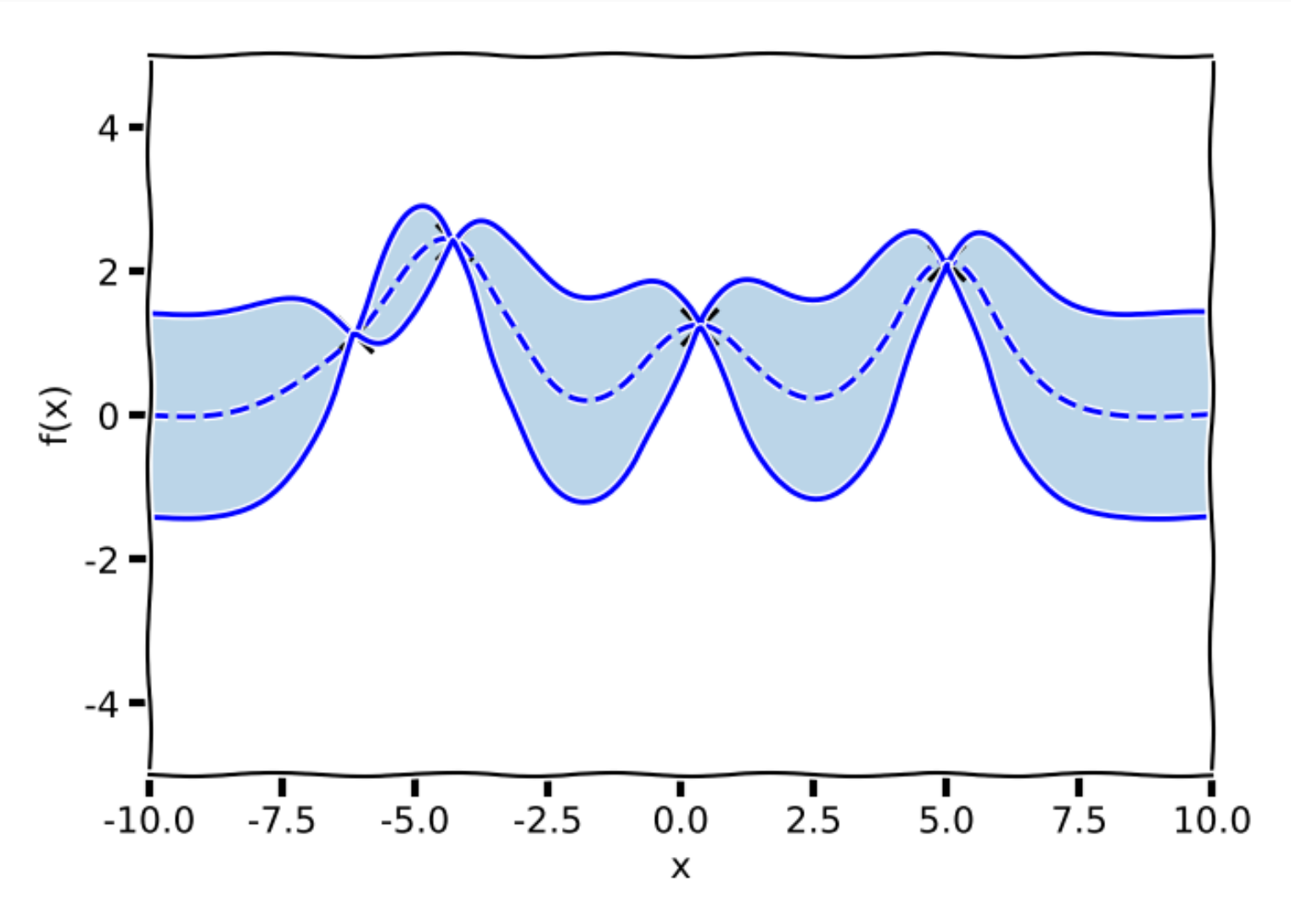

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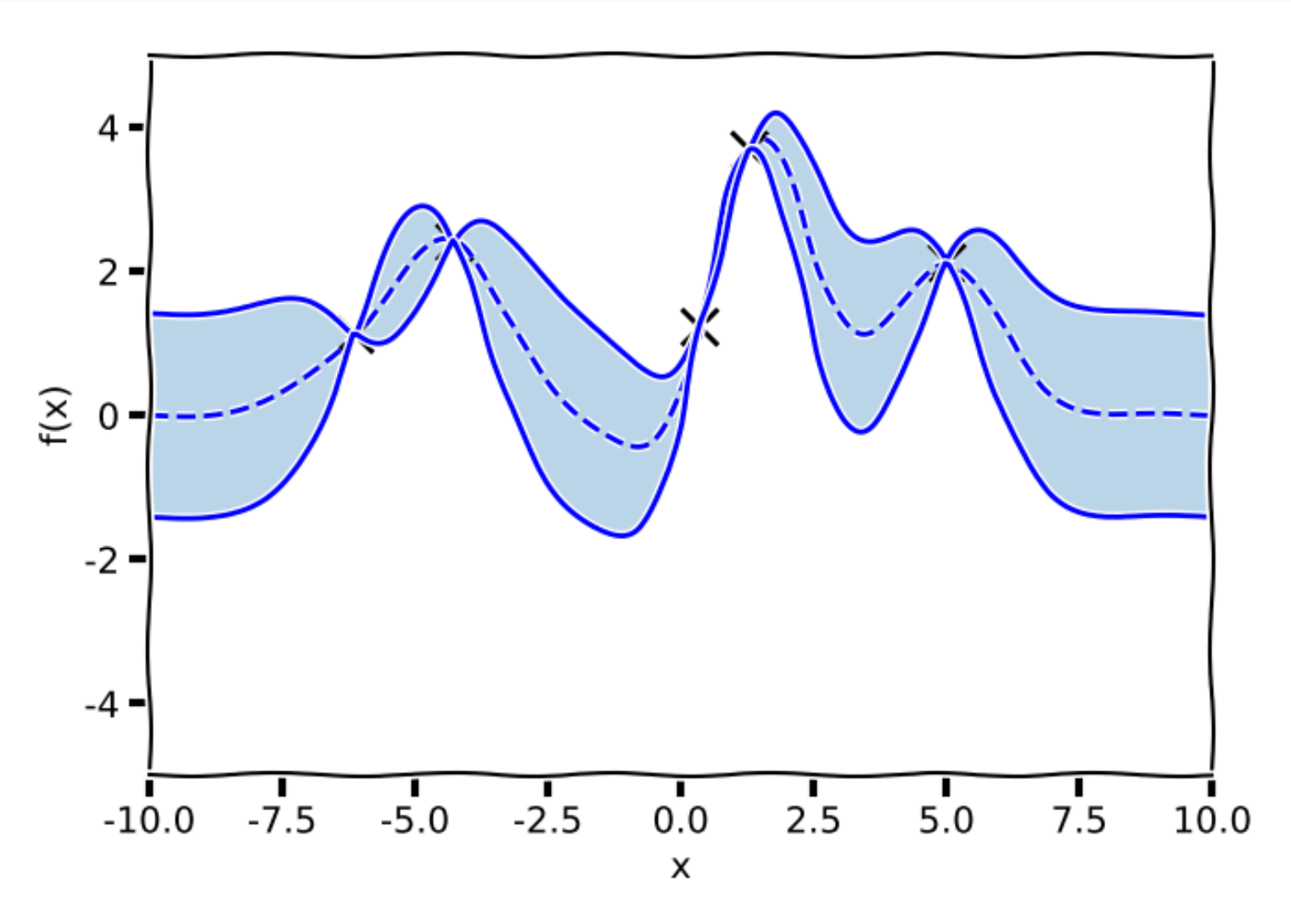

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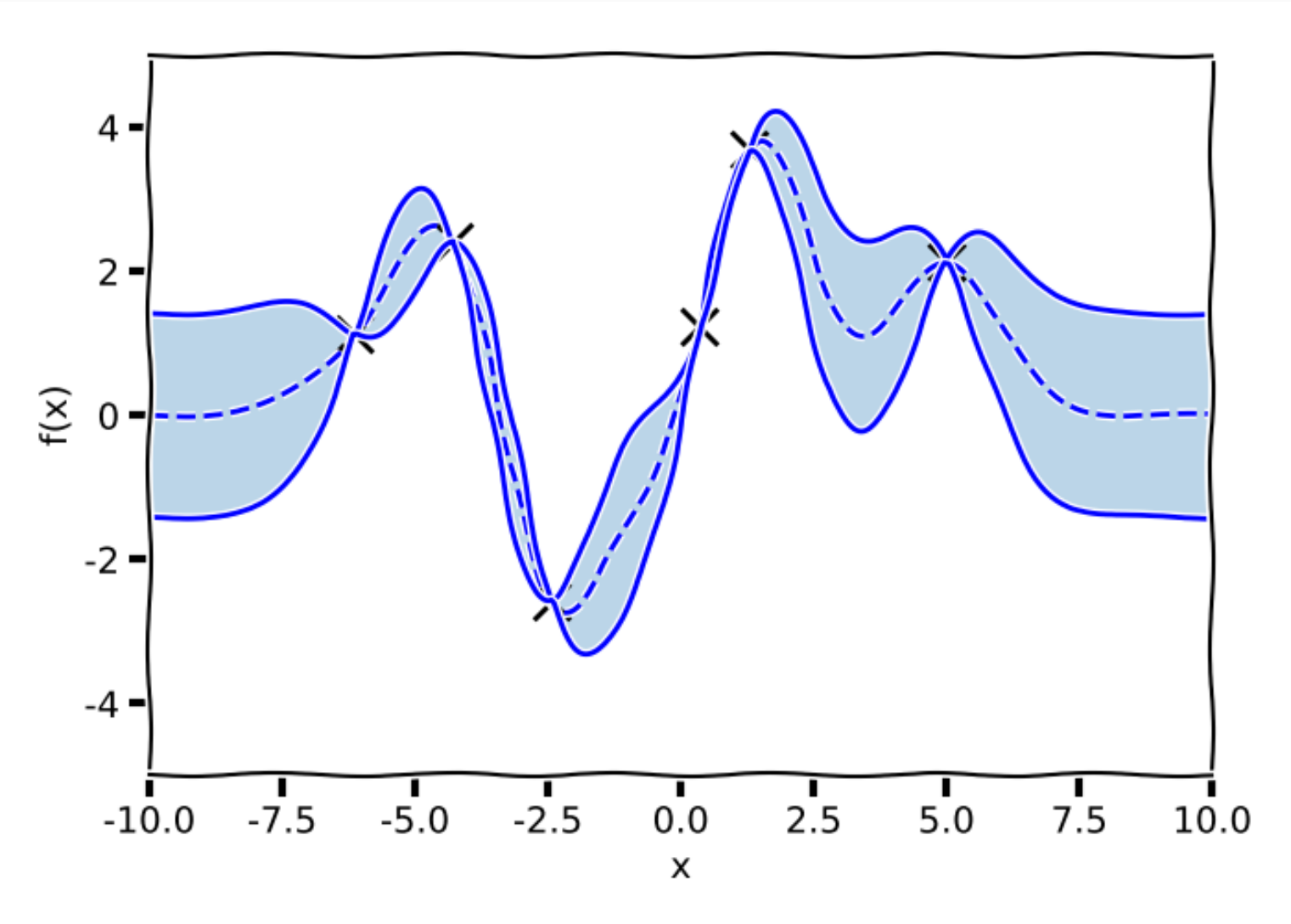

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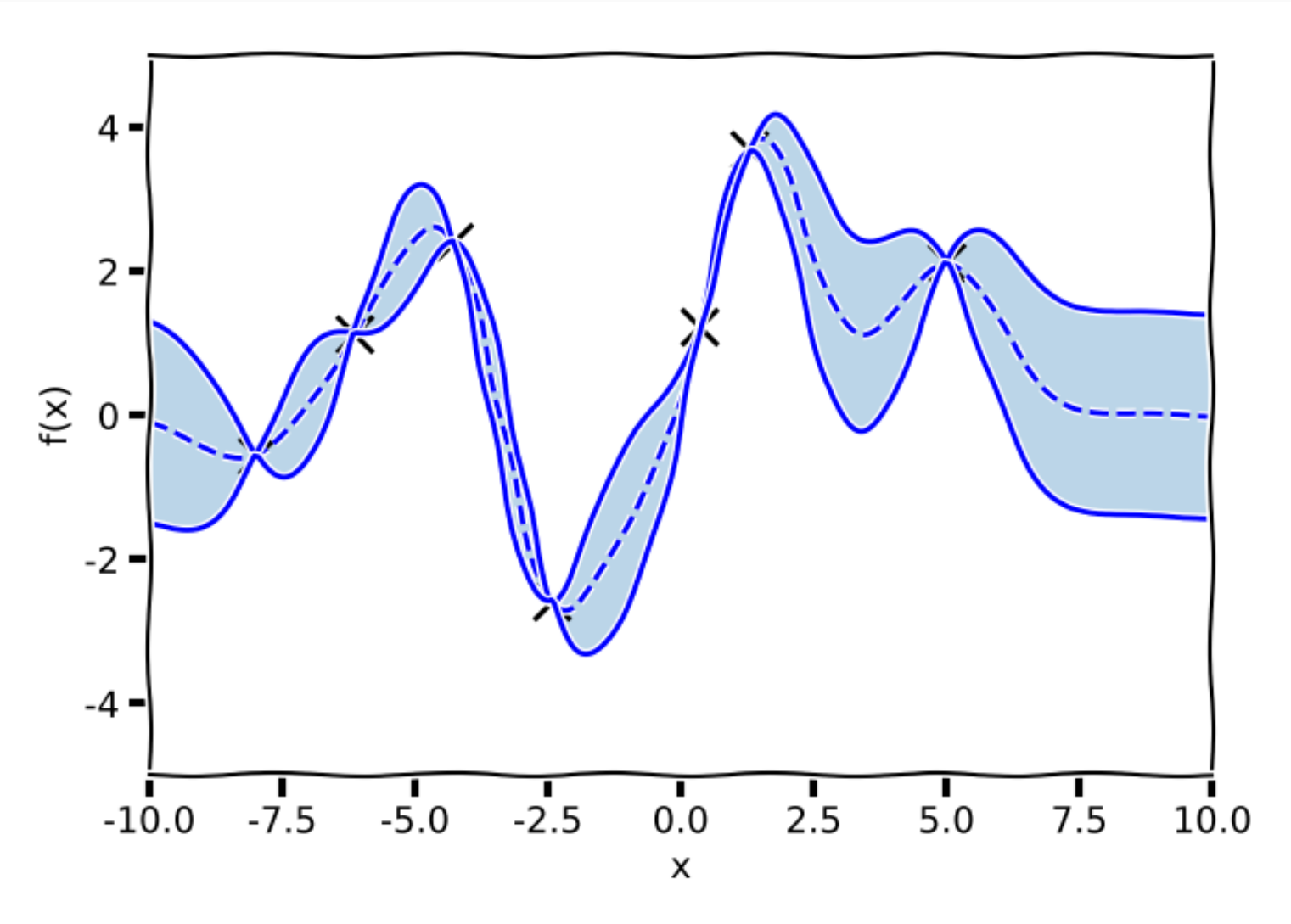

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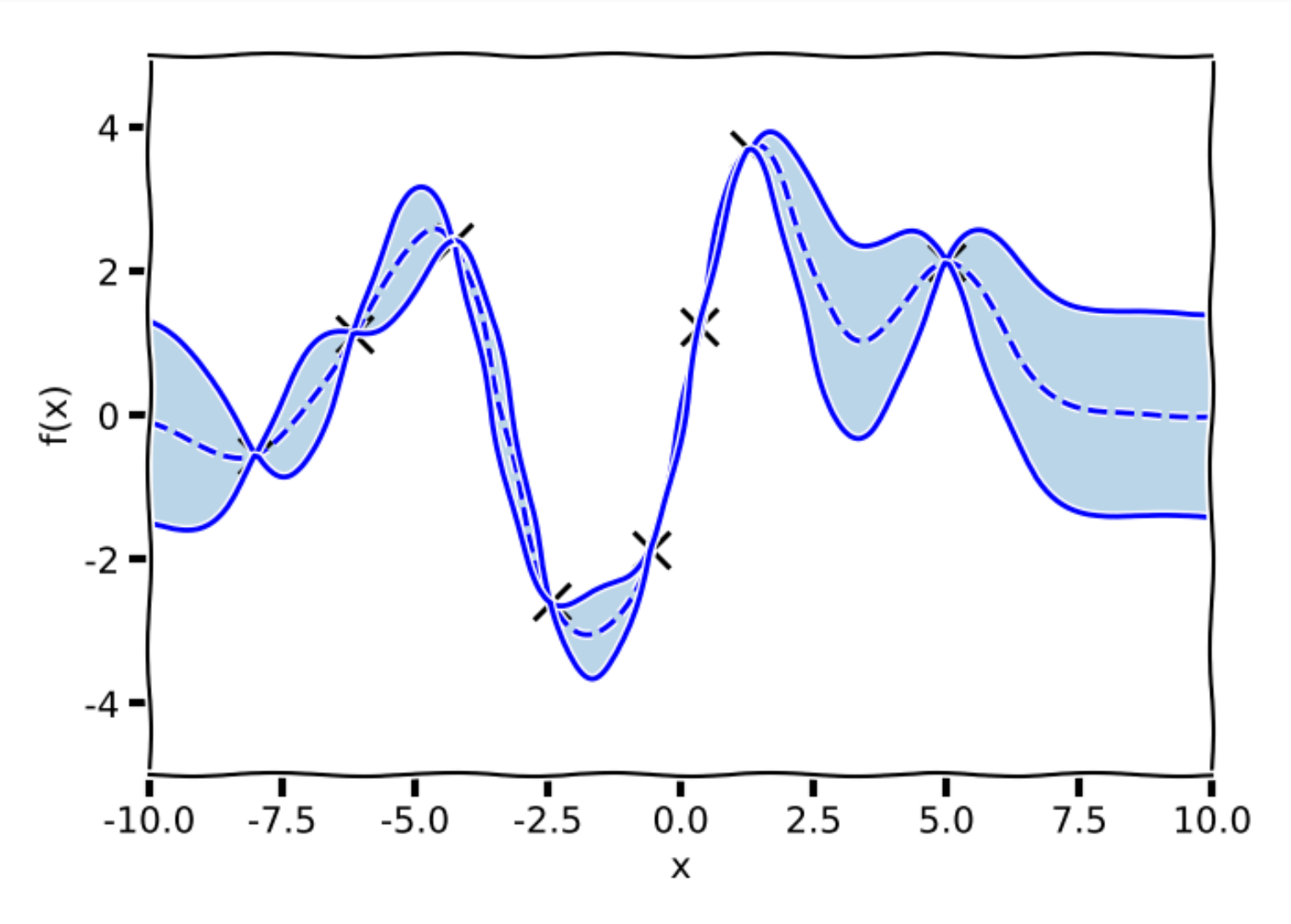

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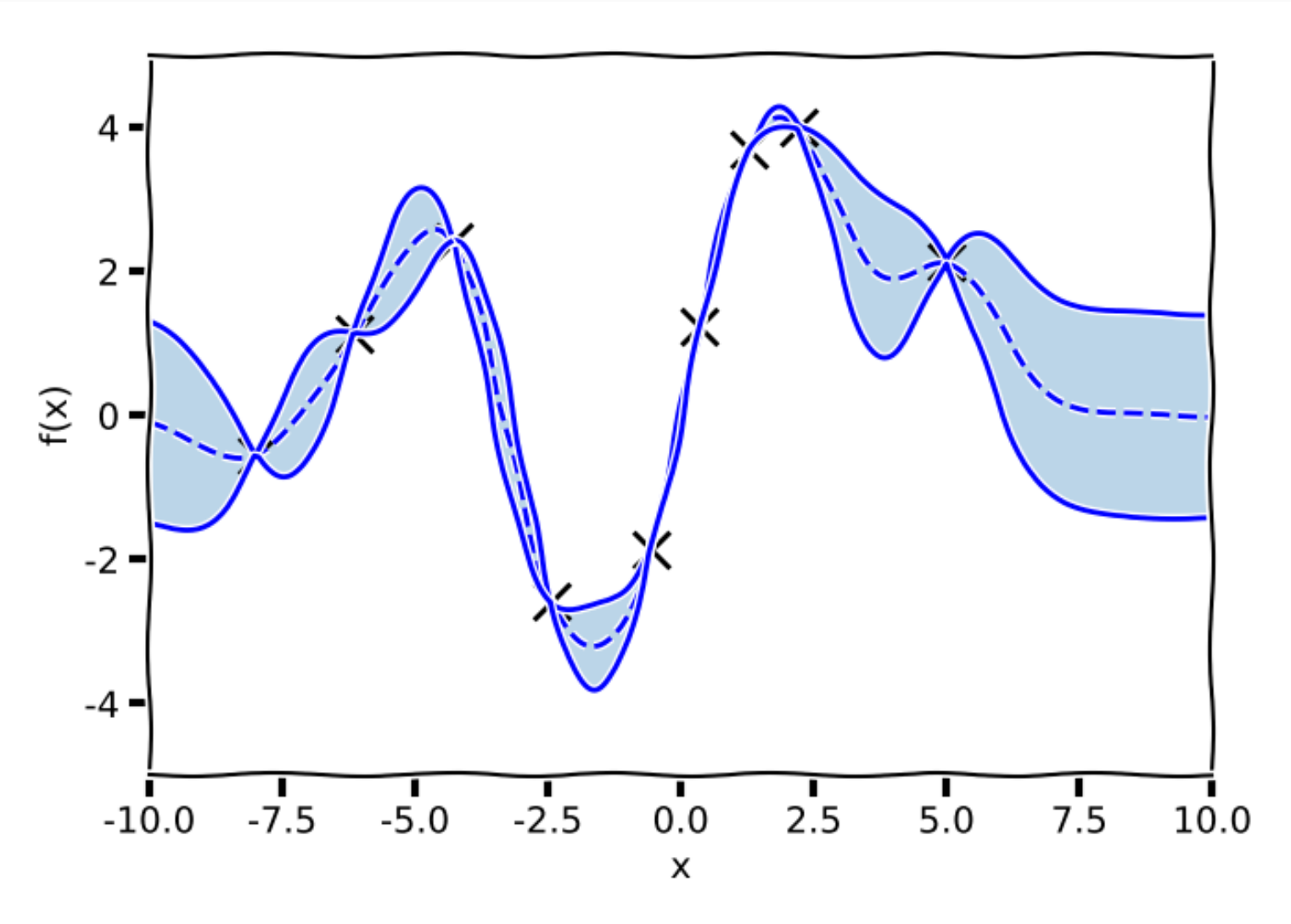

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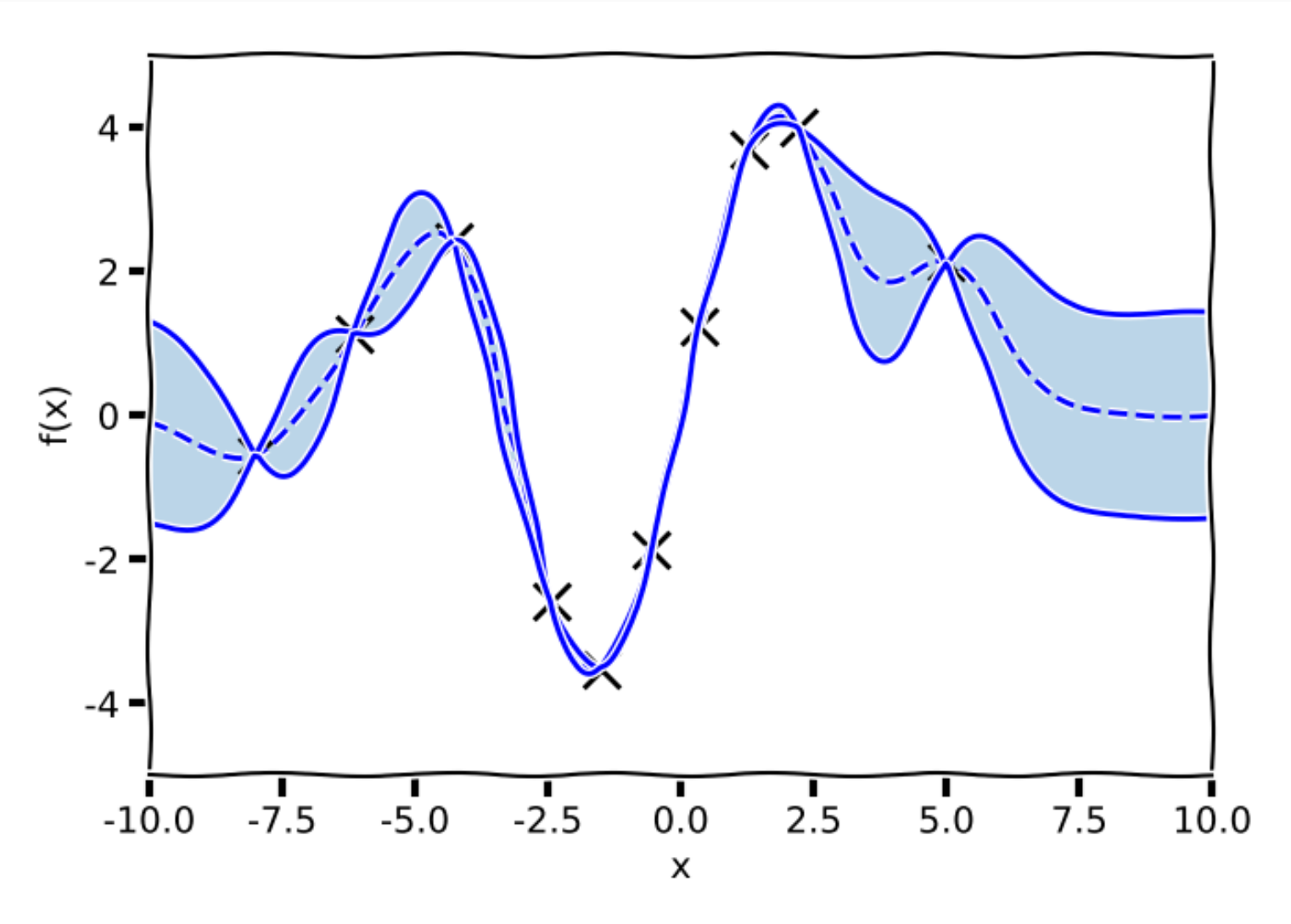

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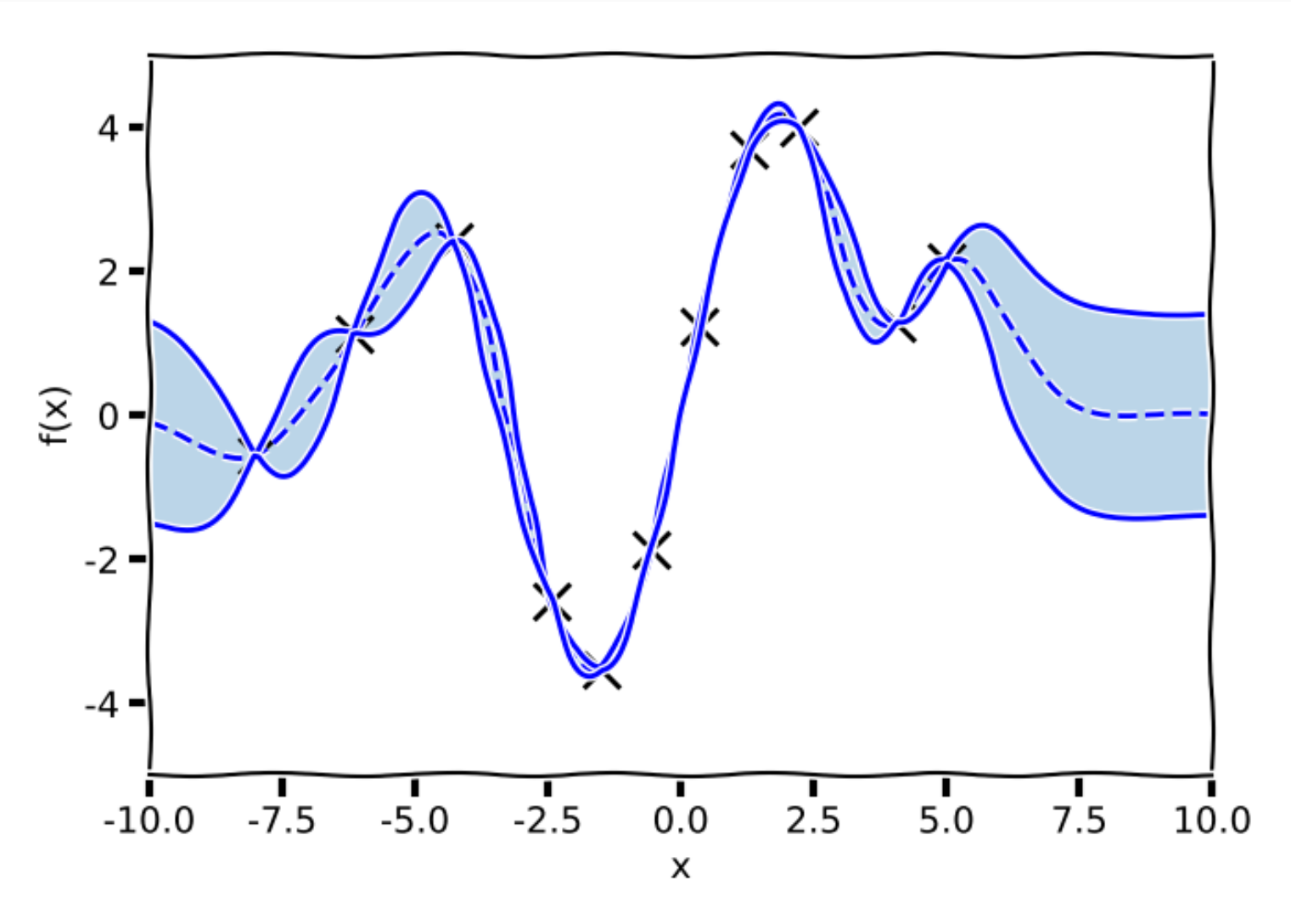

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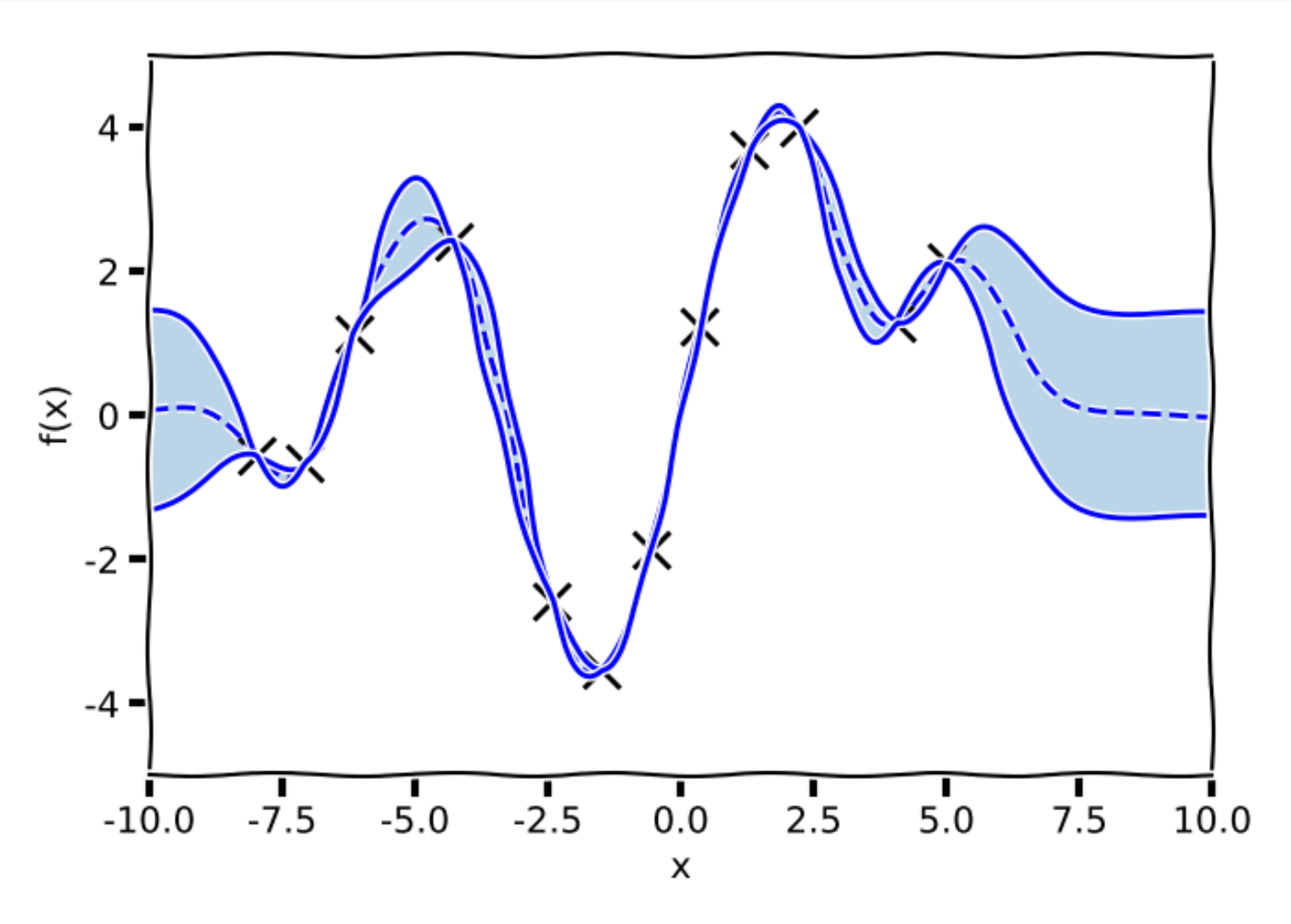

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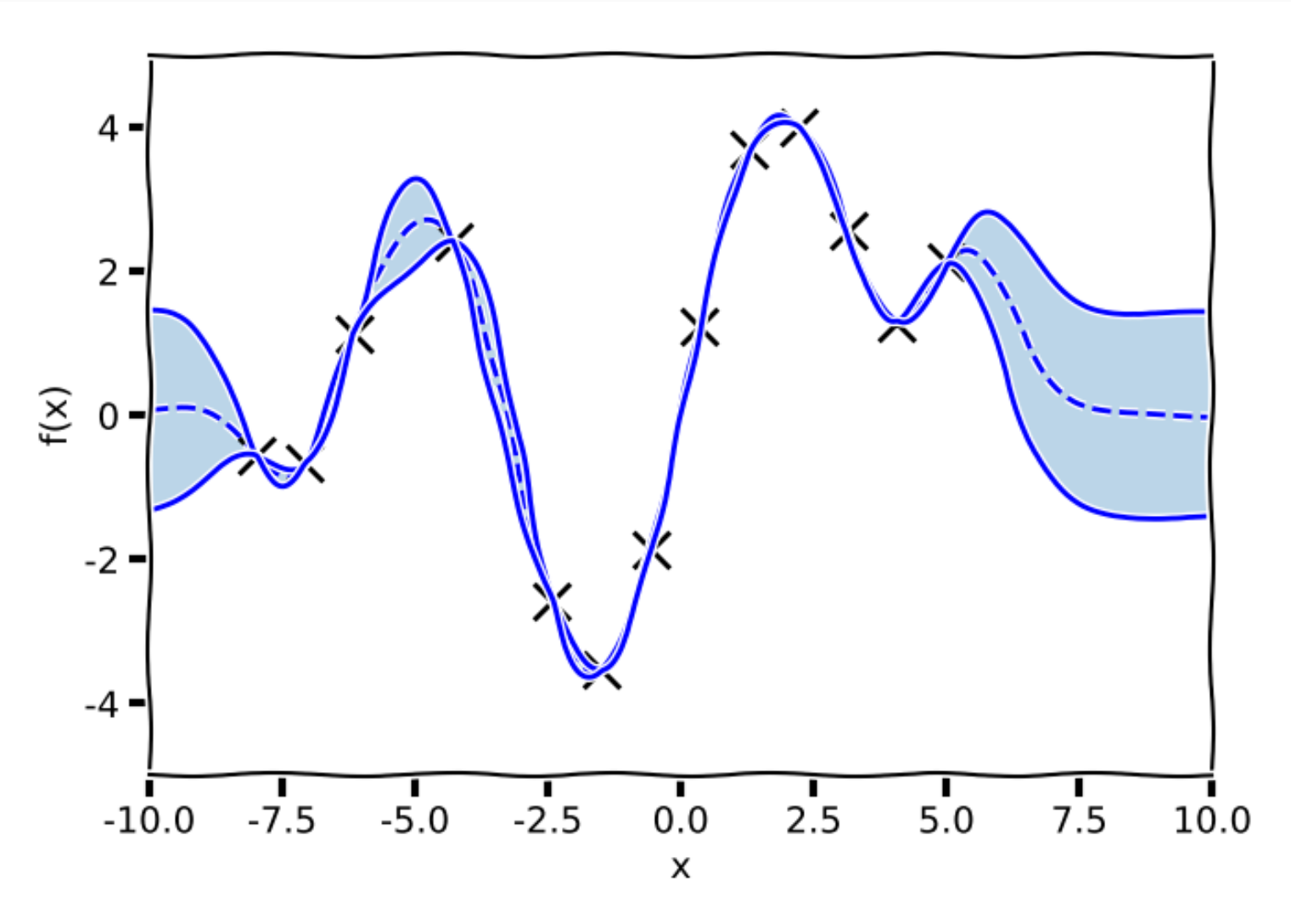

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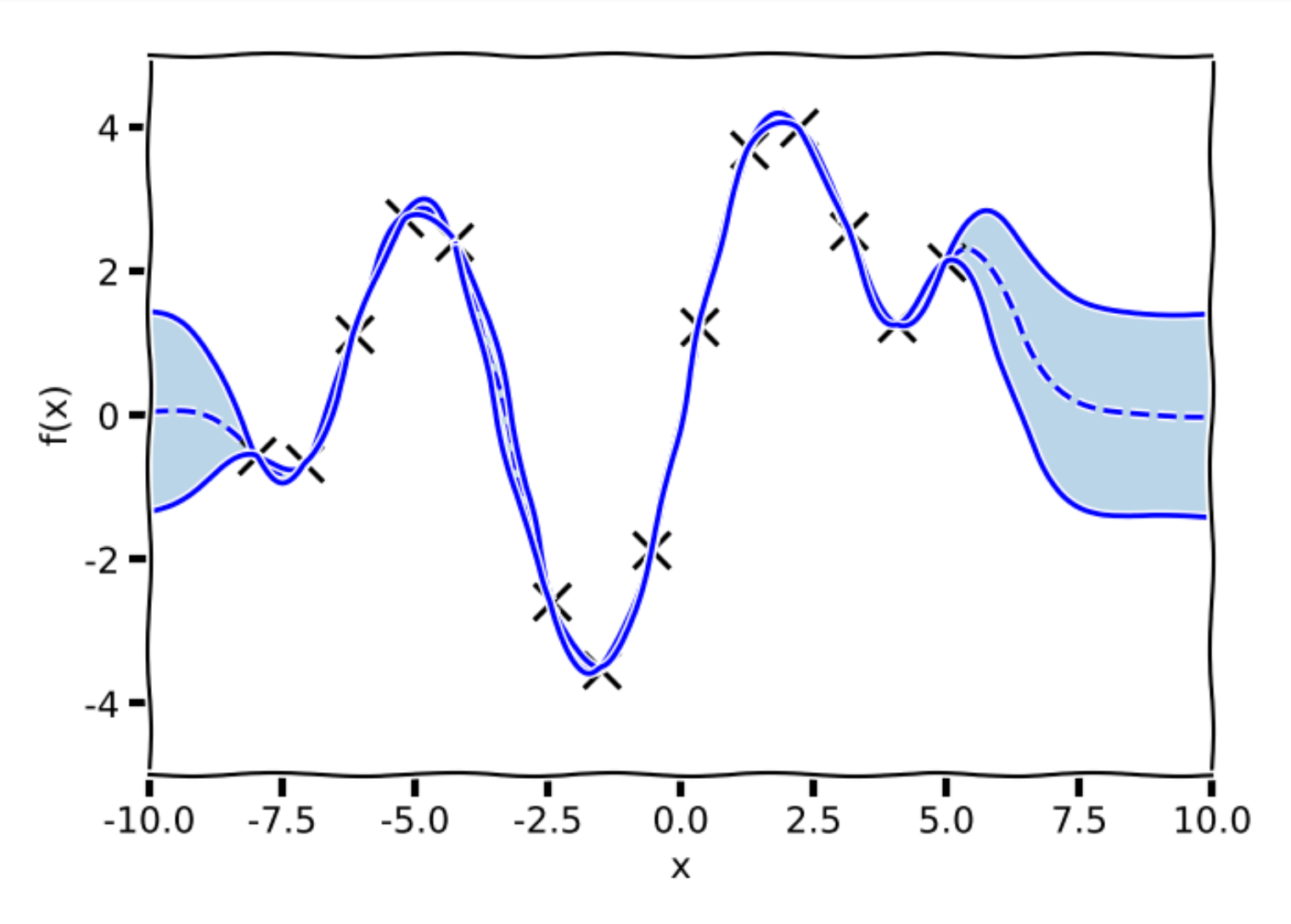

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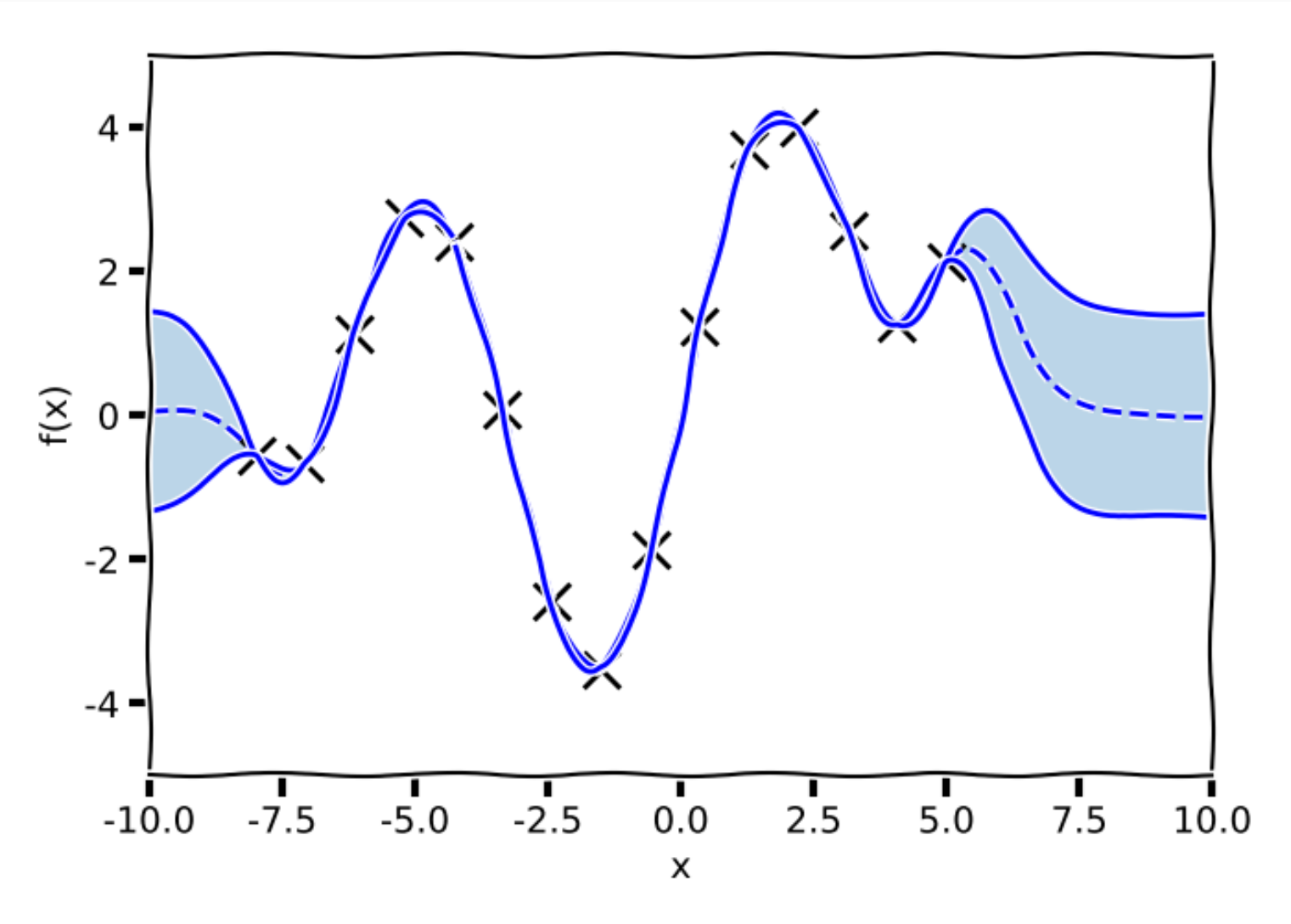

Bayesian non-parametrics

Definition (Bayesian Non-parametrics)

Is a Bayesian model on a ∞ -dimensional parameter space

Process $\rightarrow$ Distribution $\rightarrow$ value

- Each evaluation of a process is a distribution

$$\mathcal{N}(0, \Sigma) \sim \mathcal{N}(0, k(\mathbf{X}, \mathbf{X}))$$

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- Parametric models are definedy by distributions and non-parametric by processes

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- Formulate process
- Evaluate process at specific location $x \rightarrow$ distribution
- Evaluate distribution at any location y
- GP is defined over uncountable infinite space

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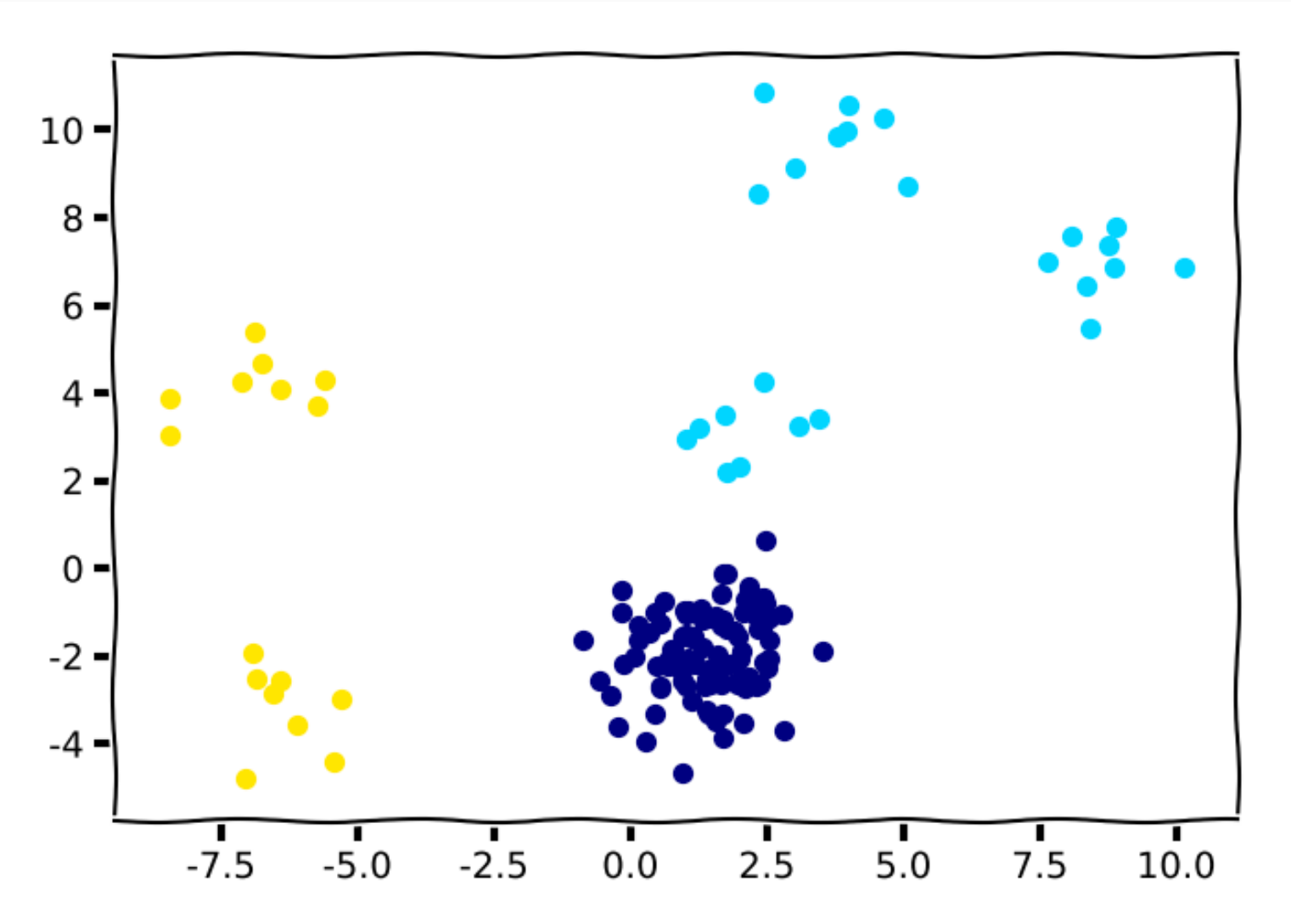
- GP is defined over uncountable infinite space
- *What about countable objects?*

Gaussian Mixture Model



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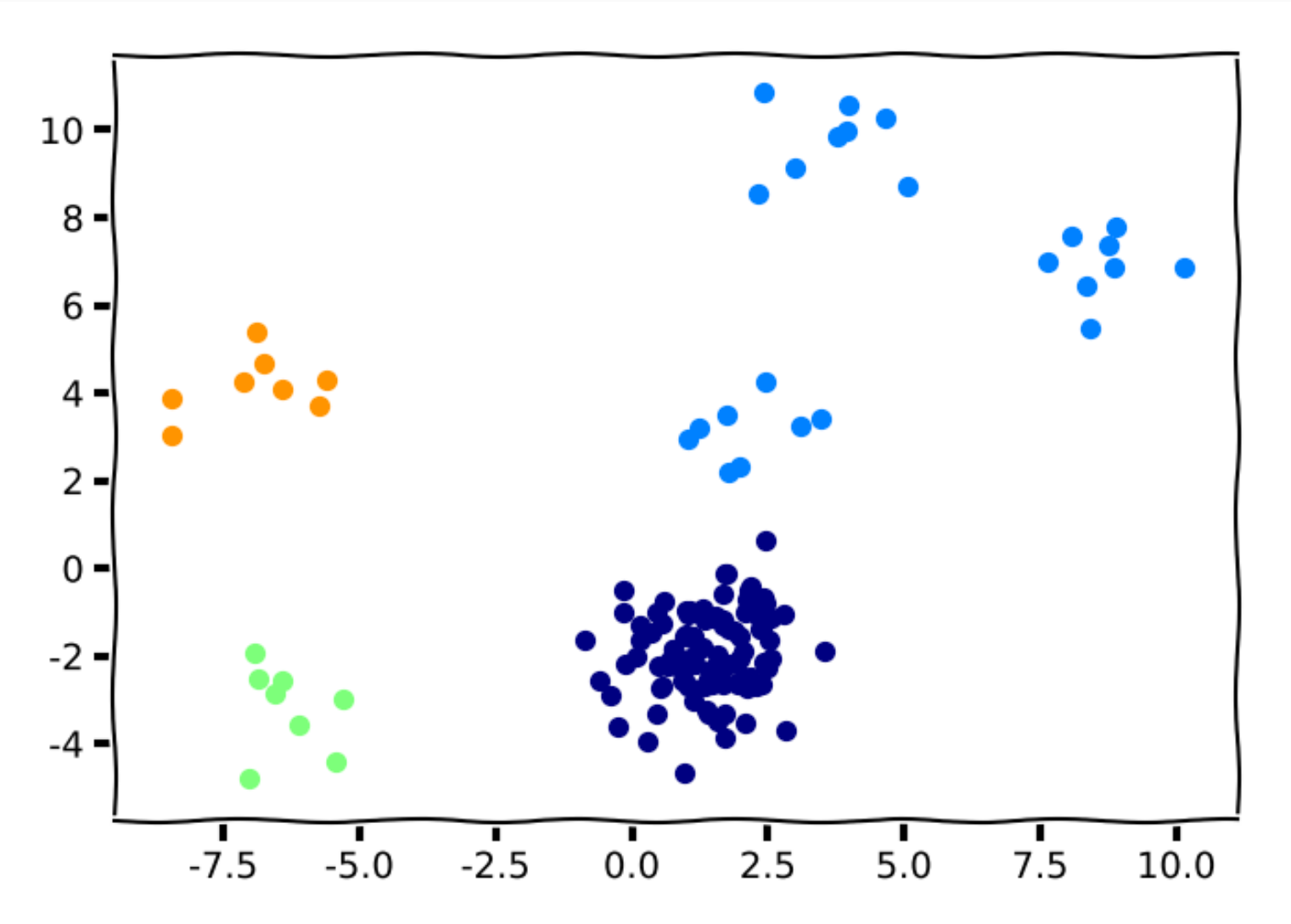

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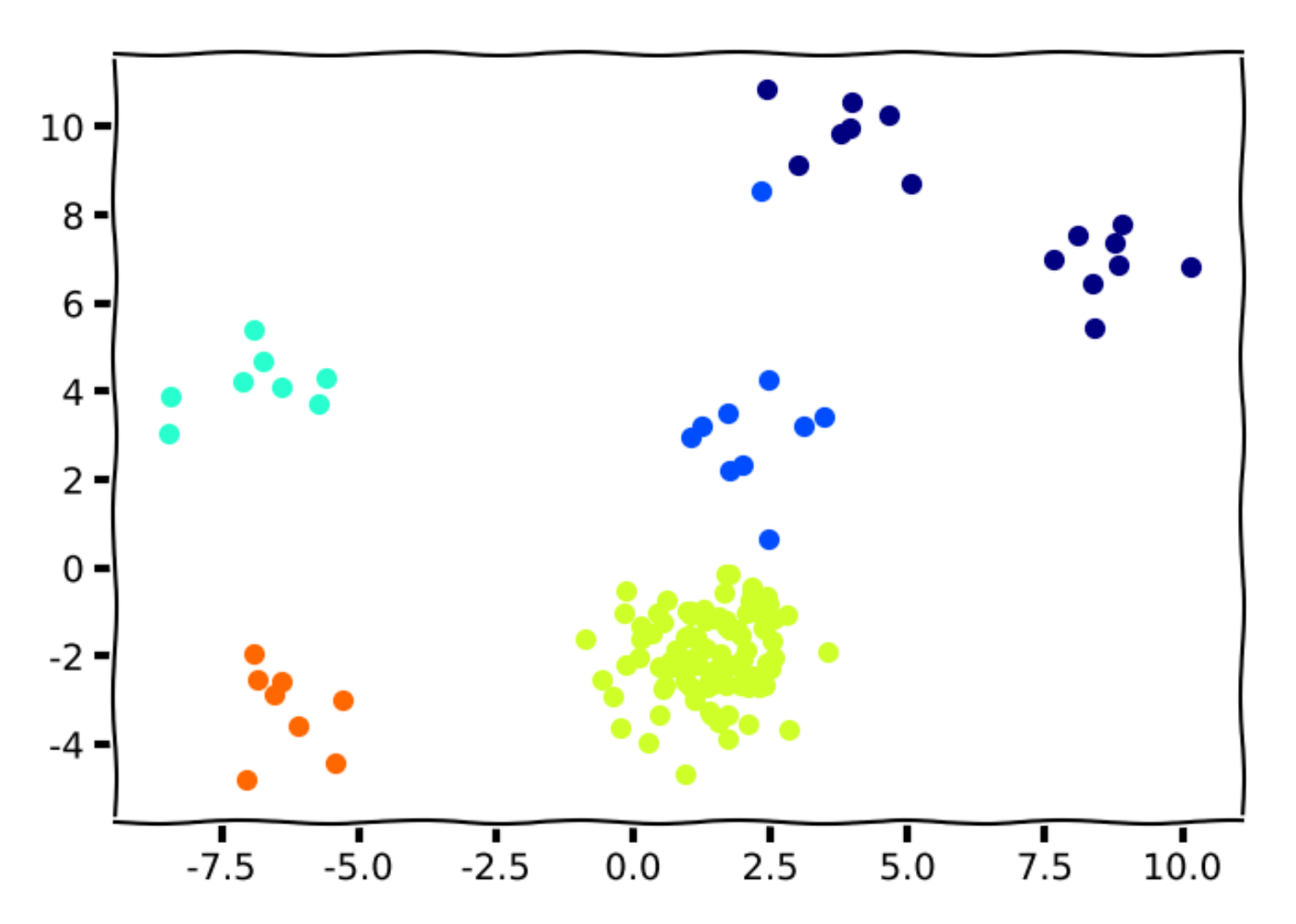

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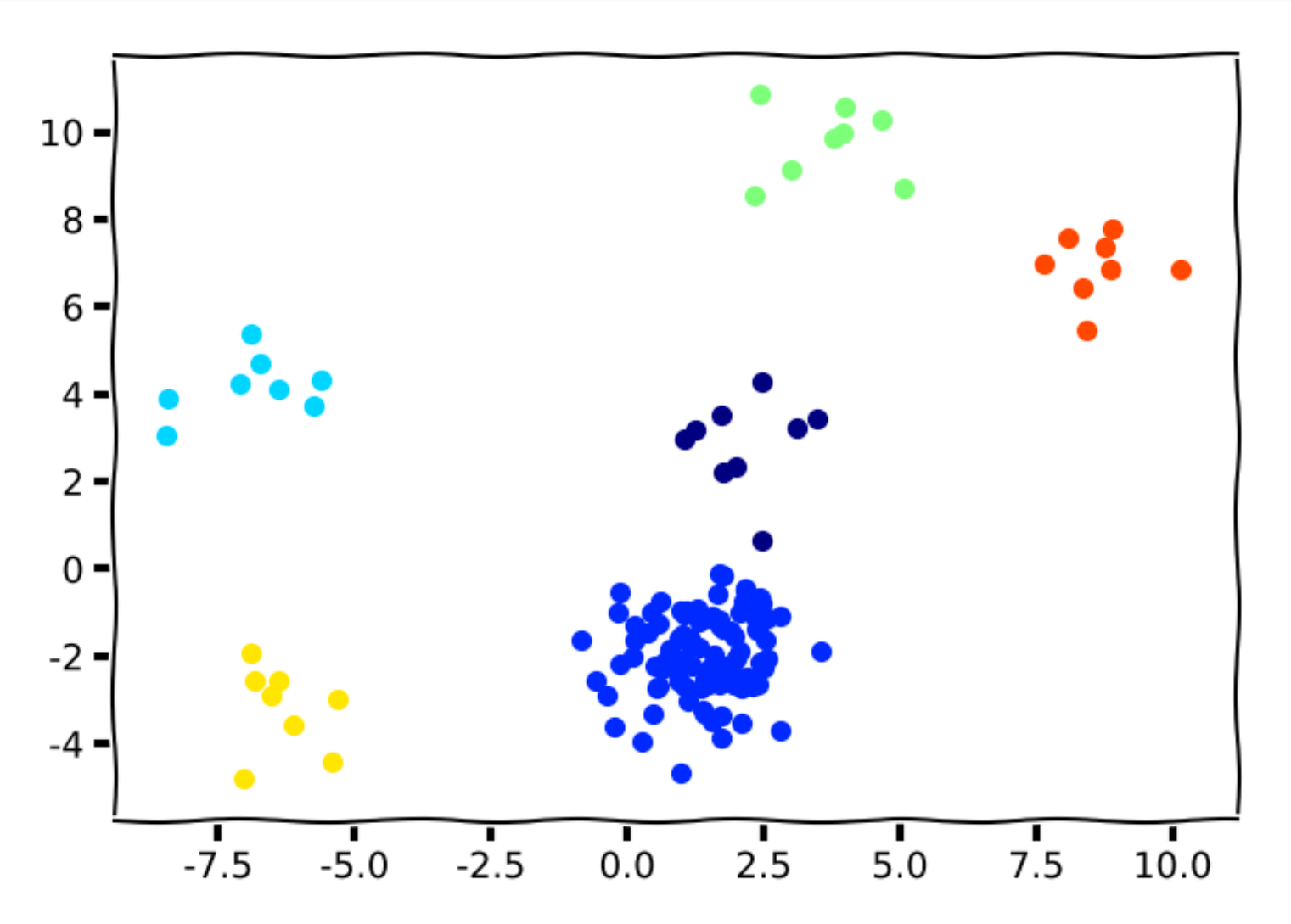

Gaussian Mixture Models

$$p(\mathbf{X}) = \sum_{k=1}^{K} p(\mathbf{X}|k)p(k) = \sum_{k=1}^{K} \mathcal{N}(\mathbf{X}|\boldsymbol{\mu}_k, \Sigma_k)p(k)$$

- Represent the probability of $\mathbf{X}$ as a combination or *mixture* of distributions
- What should K be?
- Can we make K infinite?

Gaussian Process

$$p(\mathbf{y}|\mathbf{X},\theta) = \int p(\mathbf{y}|f)p(f|\mathbf{X},\theta)\mathrm{d}f$$

Infinite Mixture Model

$$p(\mathbf{X}) = \sum_{k=1}^{\infty} p(\mathbf{X}|k)p(k) = \sum_{k=1}^{\infty} \mathcal{N}(\mathbf{X}|\boldsymbol{\mu}_k, \Sigma_k)p(k)$$

Generative Models

- When we build models we describe how the data has been generated

Unsupervised Linear Regression

1. Sample (pick) a weight

Clustering

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1. Sample cluster identity

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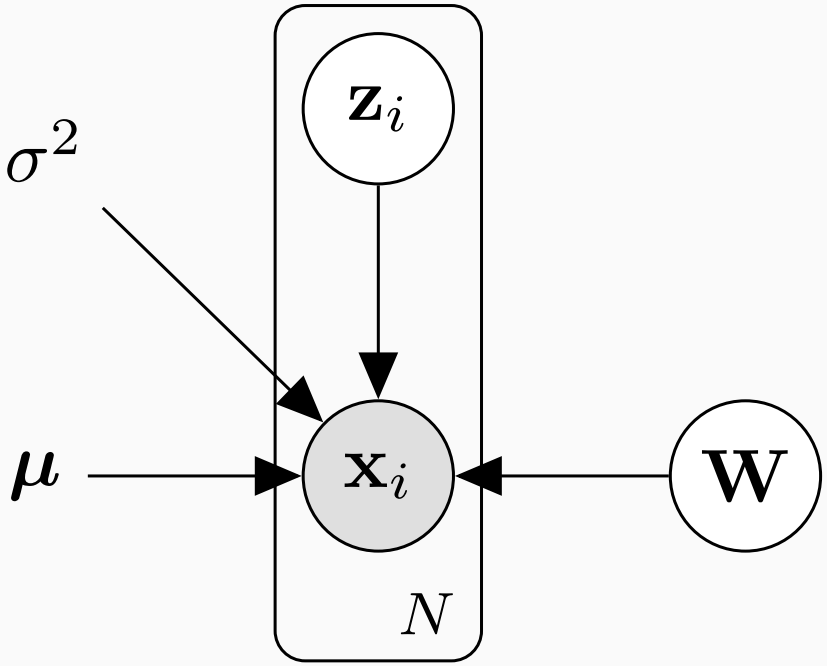
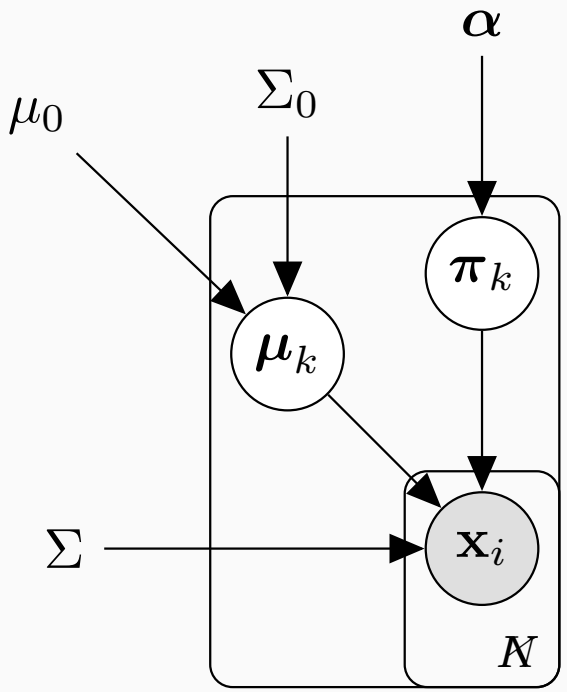
Clustering

1. Sample cluster identity
2. Sample point from cluster

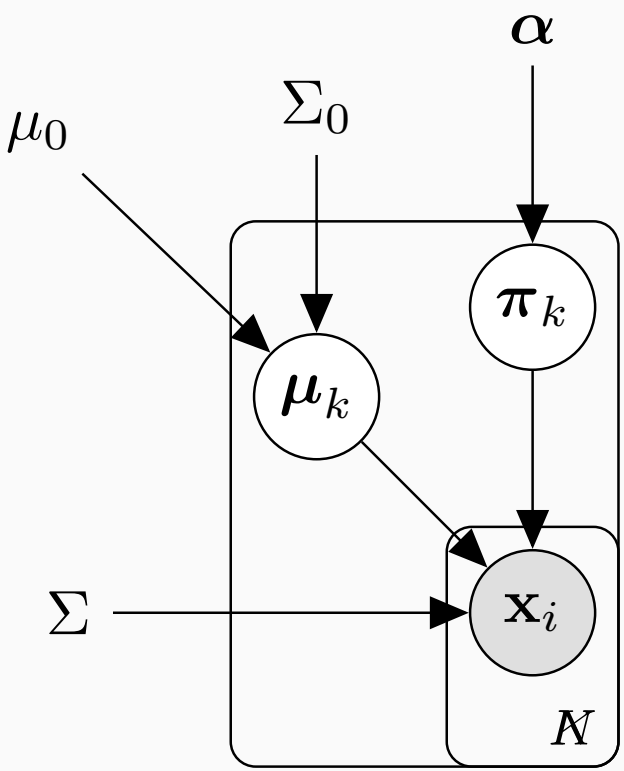
Generative Models





- Graphical model clearly shows generative proceedure

Gaussian Mixture Model



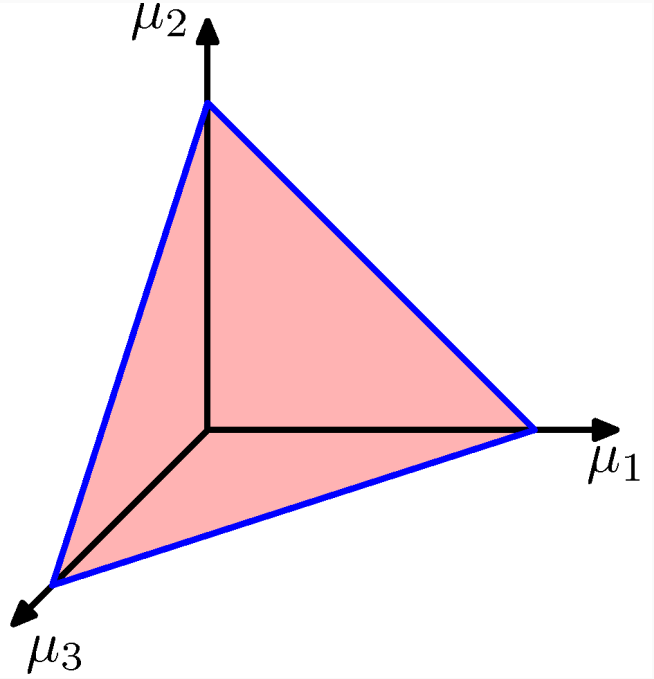
1. Sample proportions
2. Sample cluster id given proportions
3. Sample cluster mean
4. Sample data

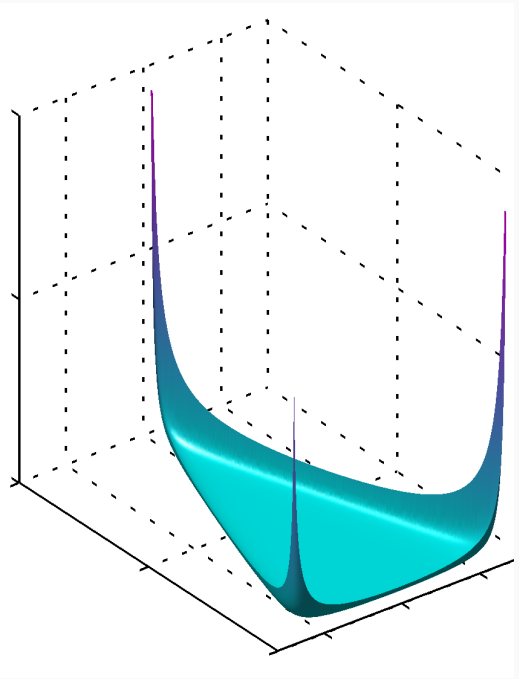
Distributions over partitionings [1] Ch 2.2.0

$$\text{Mult}(m_1, m_2, \ldots, m_k | \boldsymbol{\mu}, N) = \binom{N}{m_1, m_2, \ldots, m_K} \prod_{k=1}^{K} \mu_k^{m_k}$$

- Joint distribution over $m_1, m_2, \ldots, m_K$
- The parameter $\boldsymbol{\mu}$ says how likely each component is

Dirichlet Distribution

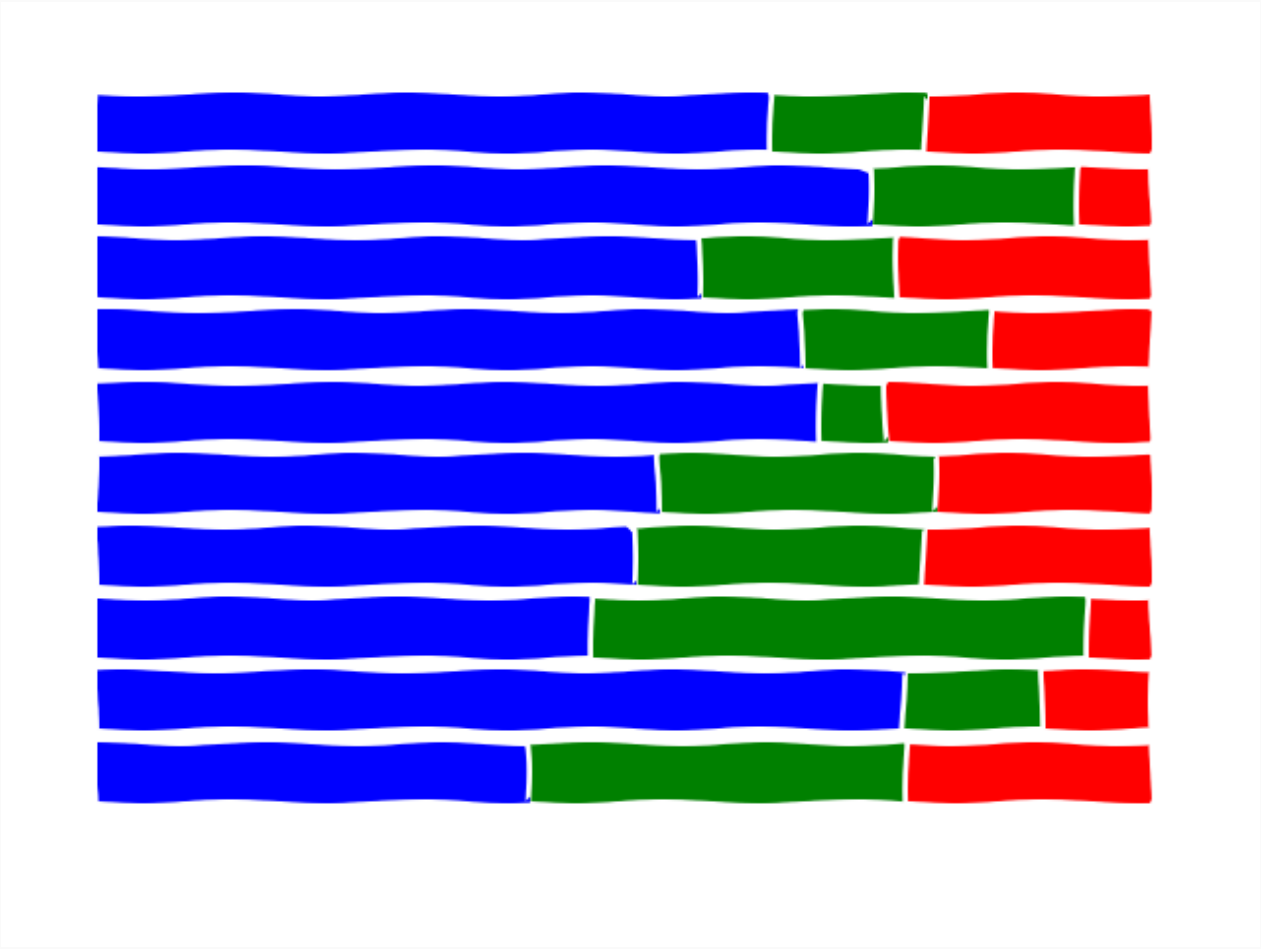




- Conjugate prior to multinomial

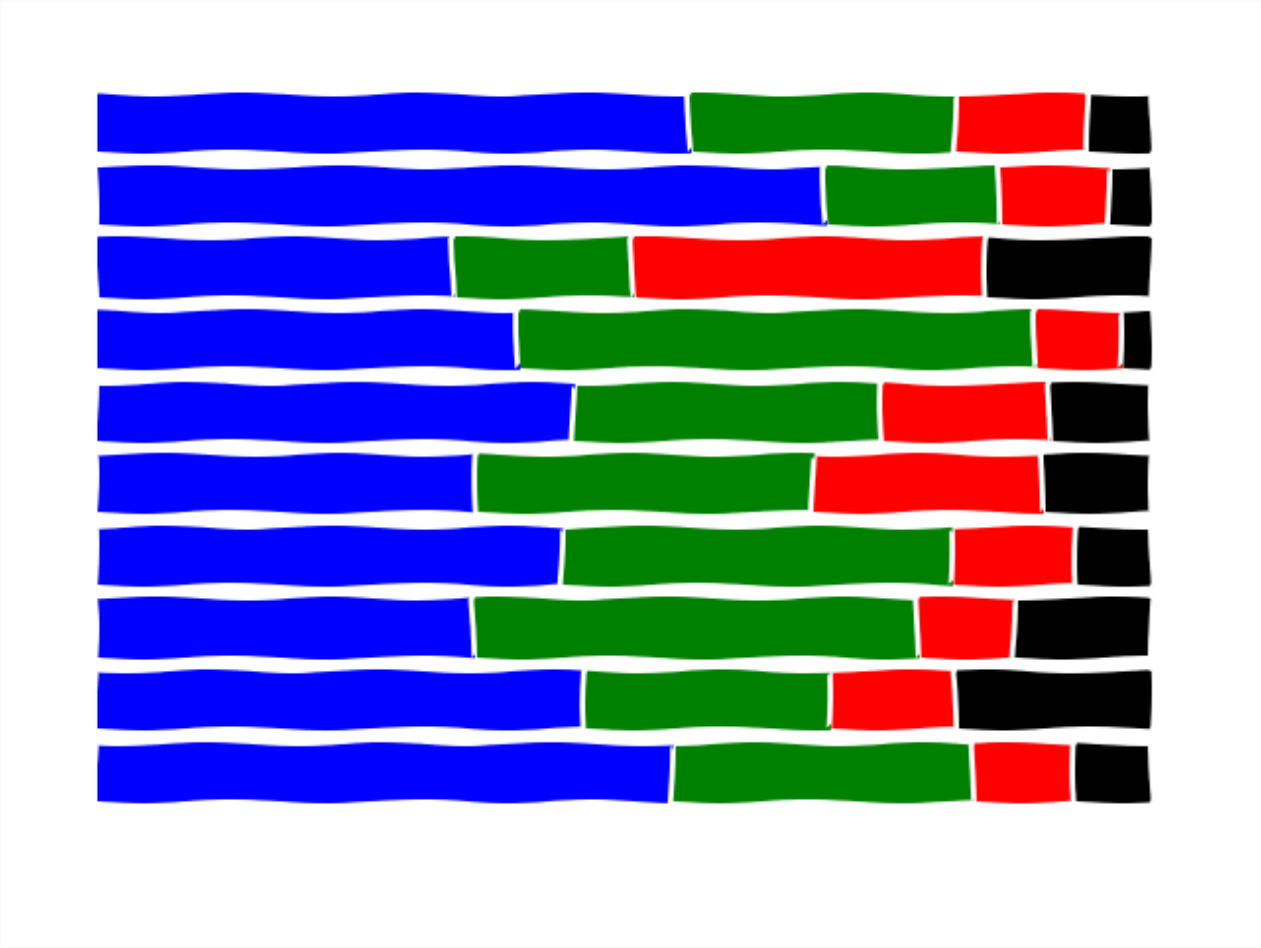
$$\mathrm{Dir}(\boldsymbol{\mu}|\boldsymbol{\alpha}) = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdot\ldots\cdot\Gamma(\alpha_K)}\prod_{k=1}^{K}\mu_k^{\alpha_k-1}$$

Dirichlet Distribution



$\mathrm{Dir}(10, 5, 3)$

Dirichlet Distribution



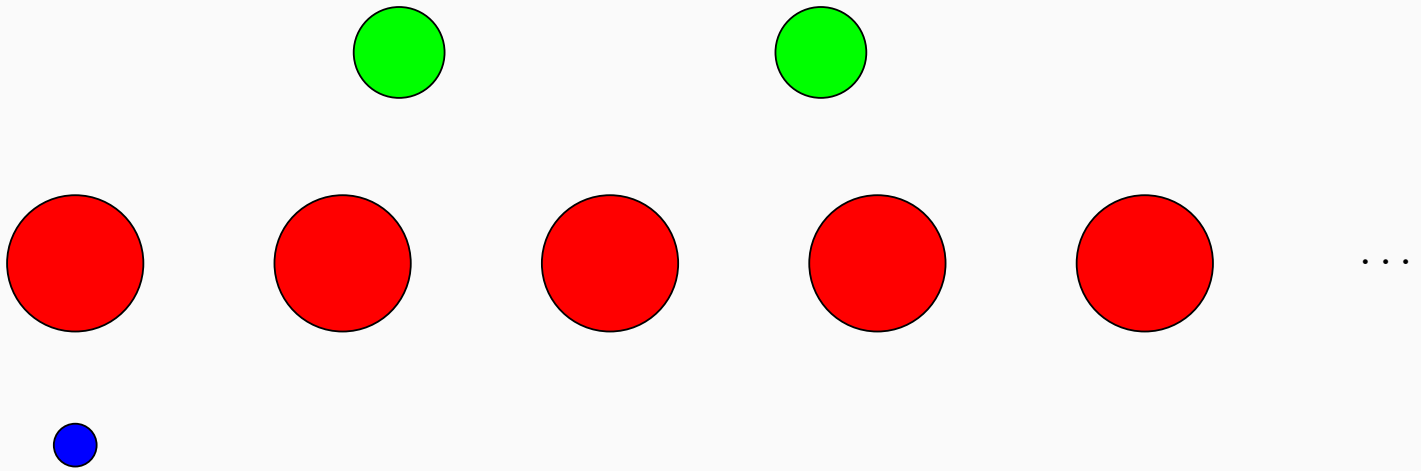
$\mathrm{Dir}(7, 5, 3, 2)$

Dirichlet Processes

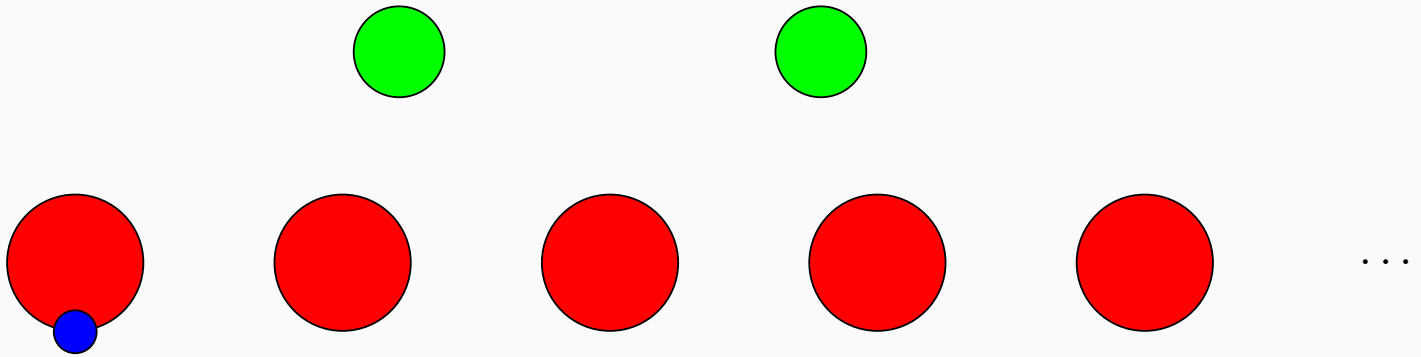
Dirichlet Process

- Is the infinite dimensional generalisation of a Dirichlet distribution
  - just as Gaussian process is of Gaussian distribution
- Generates a partitioning of (possibly) infinite number of elements
- Not as intuitive to write down
- Best written constructively

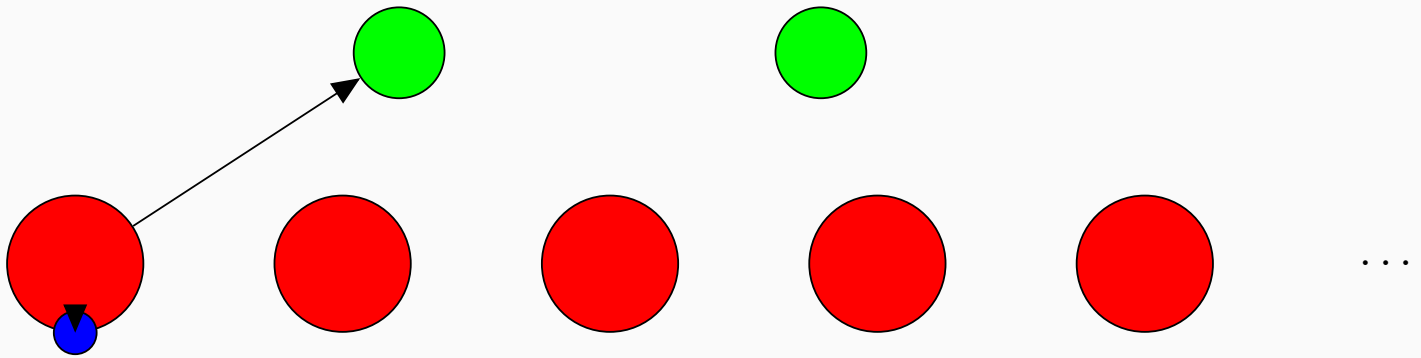
Chinese Resturant Process



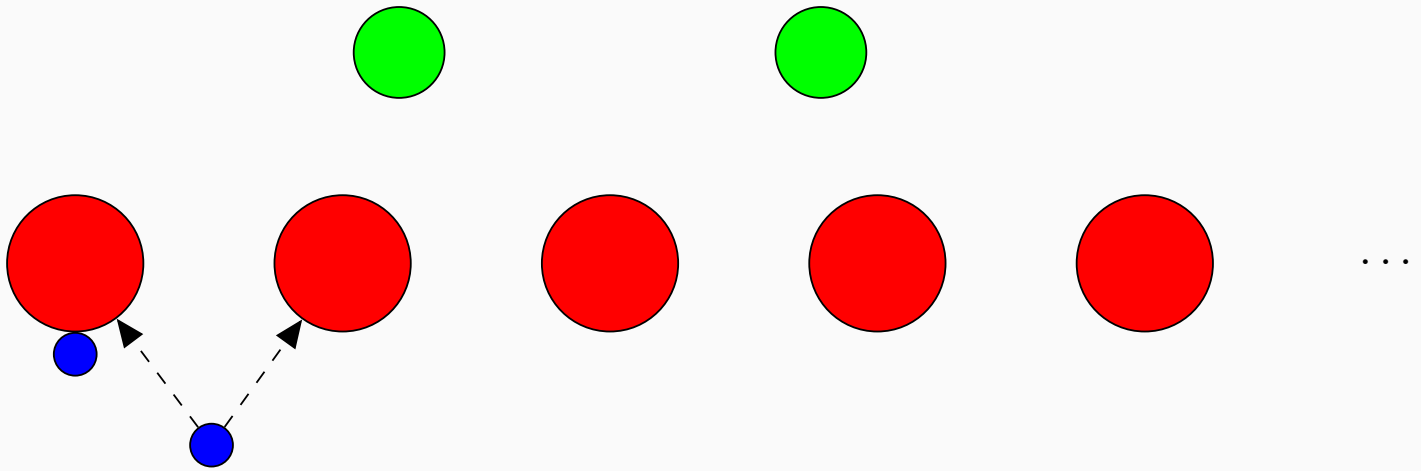
Chinese Resturant Process



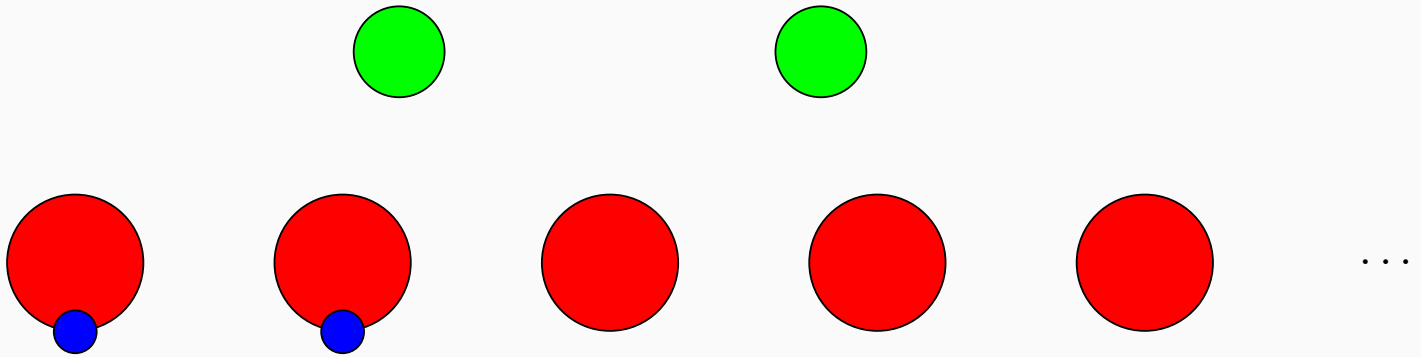
Chinese Resturant Process



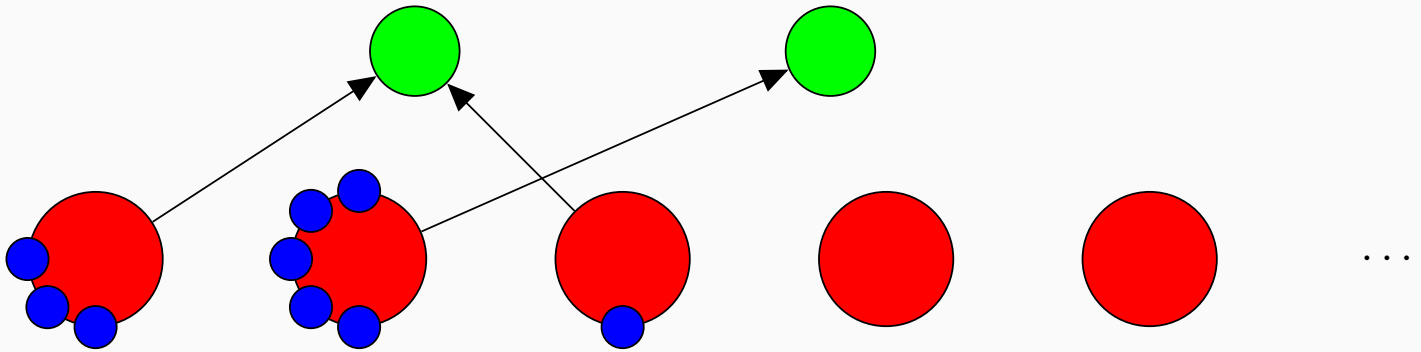
Chinese Resturant Process



Chinese Resturant Process

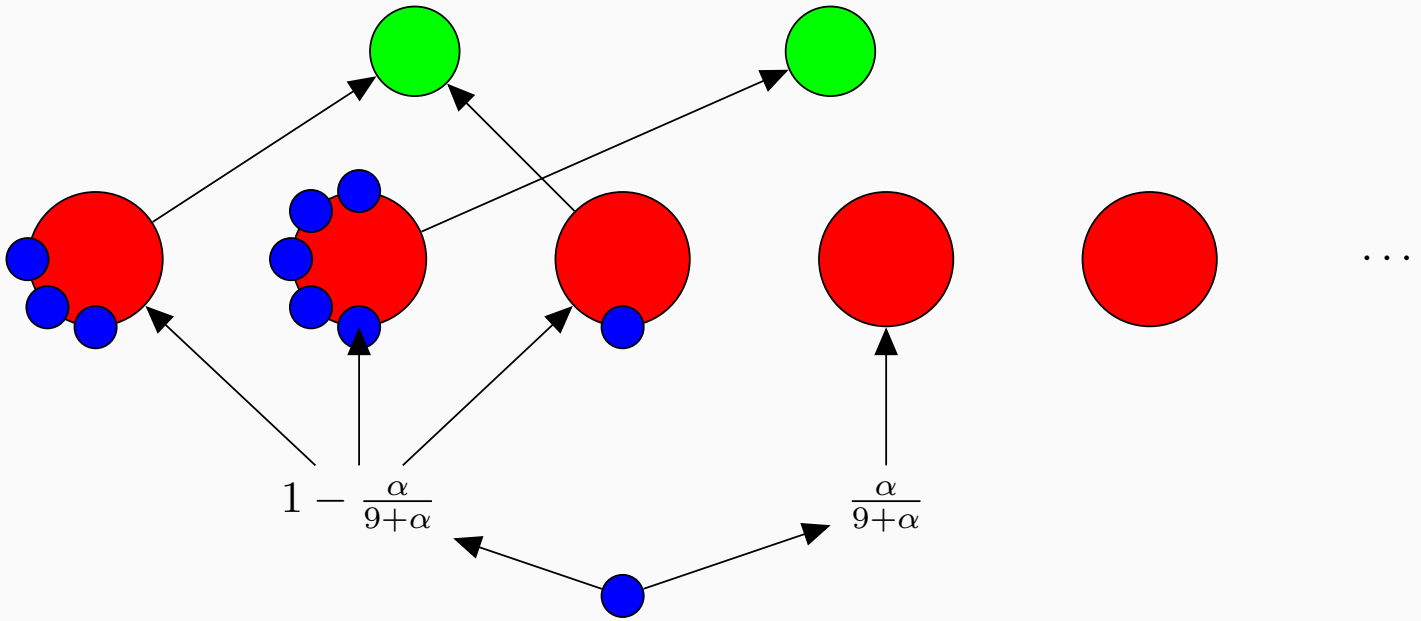


Chinese Resturant Process

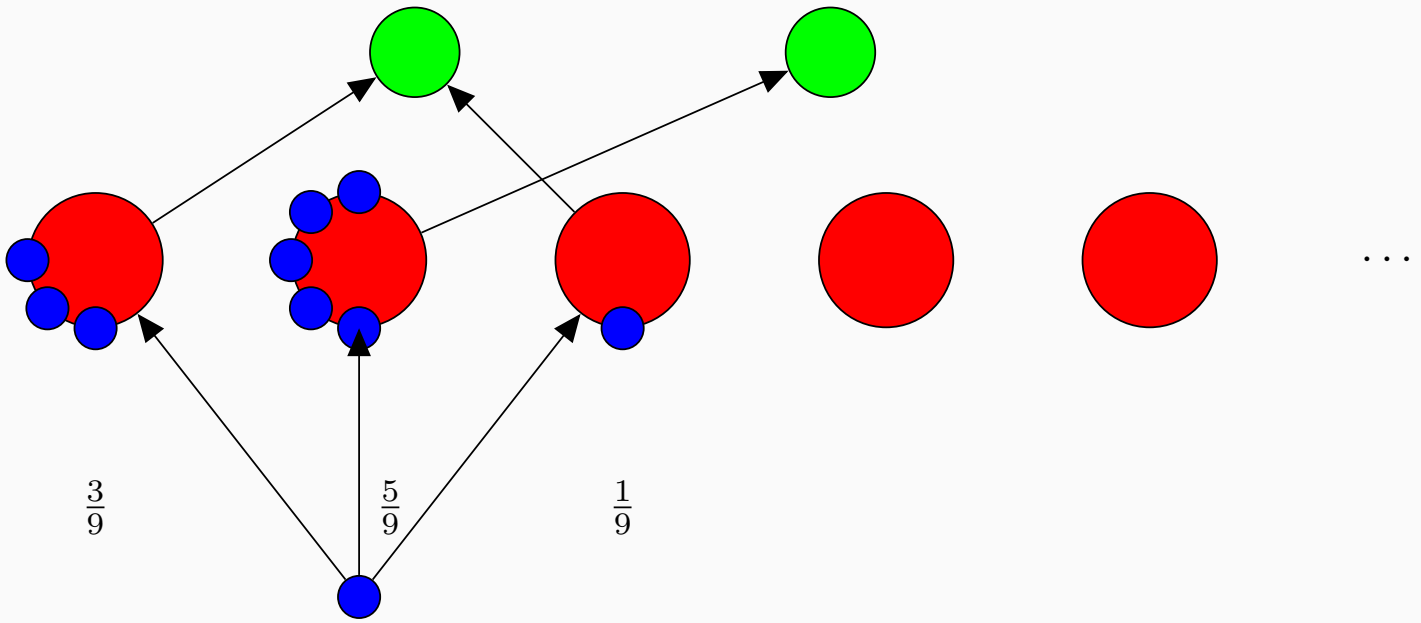


- Go to new table $\frac{\alpha}{N-1+\alpha}$
- If not choose table as $\frac{n_i}{N}$ where n_i number of diners at table *in*

Chinese Resturant Process



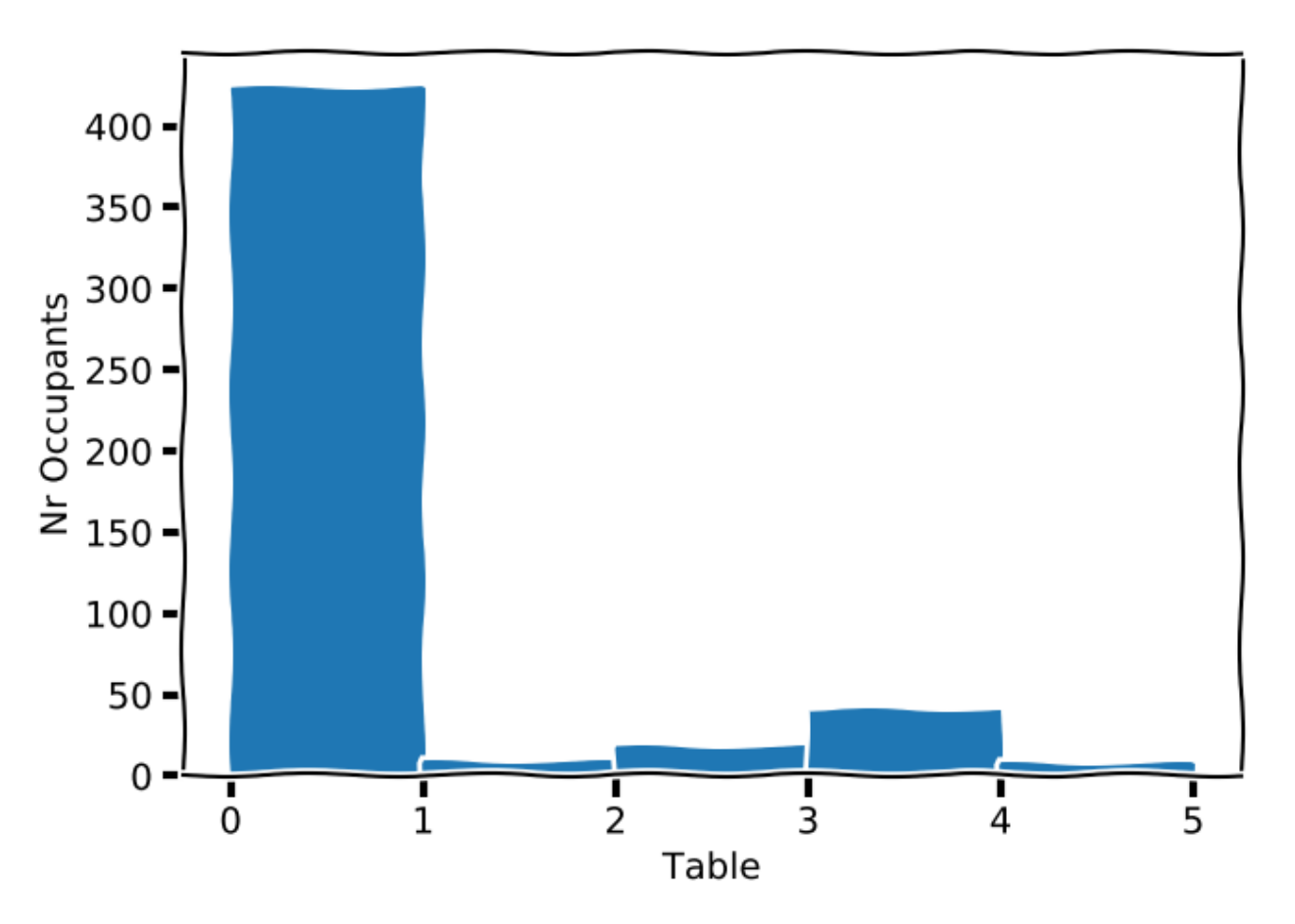
Chinese Resturant Process



Chinese Resturant Process Code

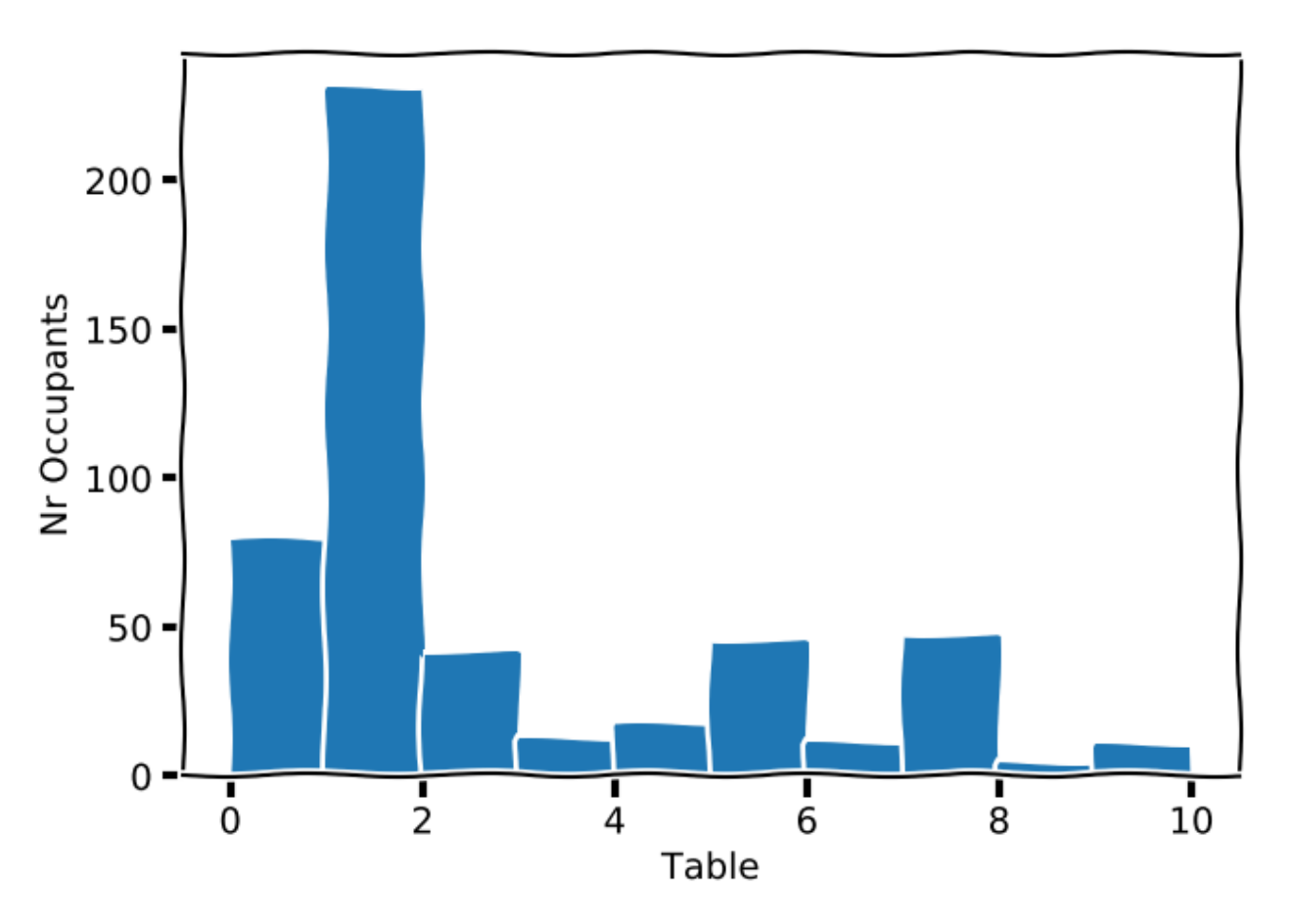
```
def chrp(N, alpha):
    table_assignments = np.zeros(N)
    next_open_table = 0
    for i in range(0,N):
        r = np.random.random()
        if r < (alpha/(i+alpha)):
            table_assignments[i] = next_open_table
            next_open_table += 1
        else:
            index = int(np.round((i-1)*np.random.random()))
            table_assignments[i] = table_assignments[index]

    return table_assignments
```

Chinese Resturant Process



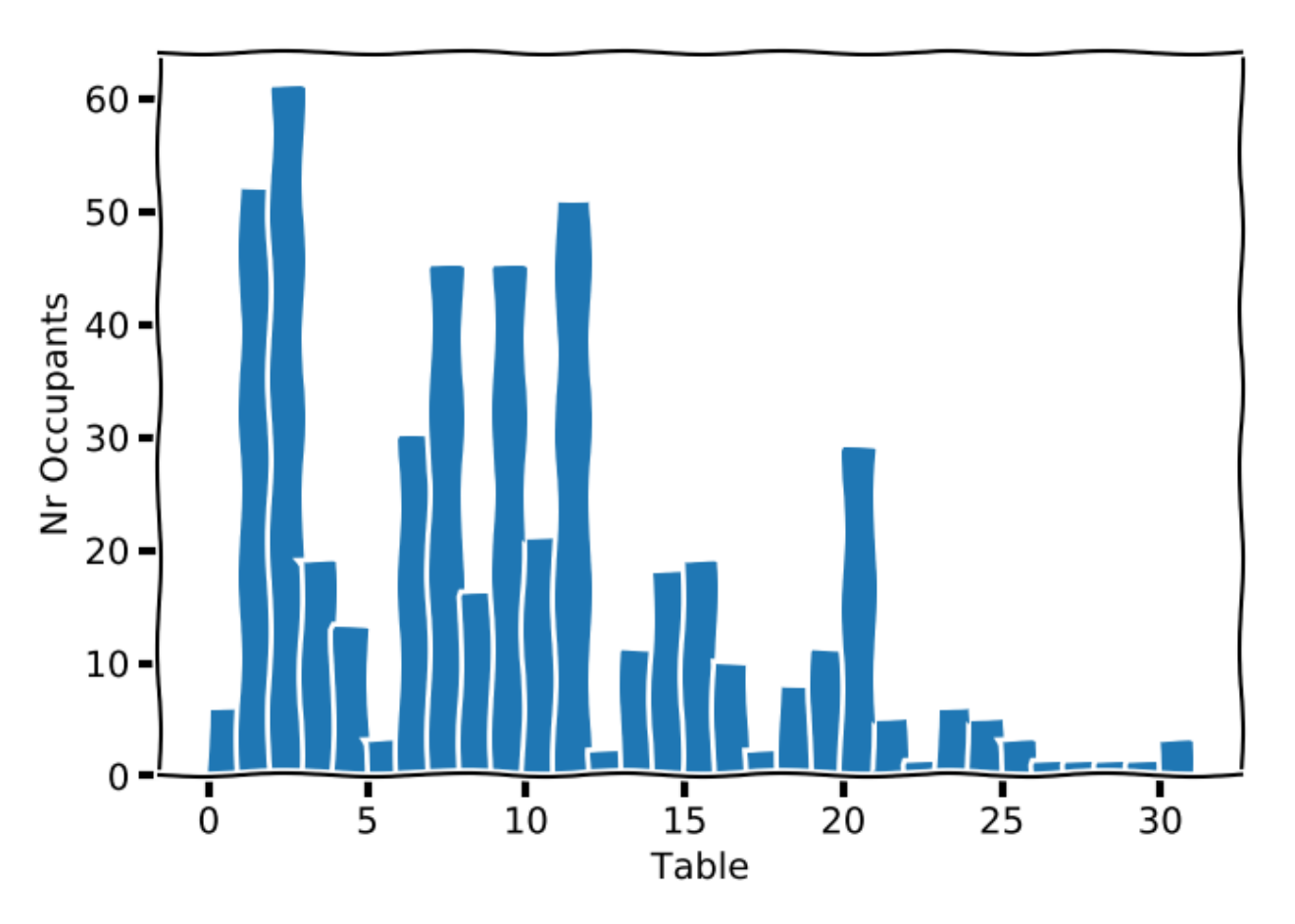
$N = 500 \quad \alpha = 1.0$

Chinese Resturant Process



$N = 500 \quad \alpha = 2.0$

Chinese Resturant Process



$N = 500 \quad \alpha = 10.0$

Chinese Resturant Process

N	α	μ_K	σ_K
500	3	14.3	8.81
500	5	21.6	15.64
500	10	39.1	27.29
500	20	64.3	63.8
500	100	180	69.69

Infinite Mixture Model

1. Pick first data-point

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2. Pick the first mixture

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4. Next data-point

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5. Associate with a new mixture or create new

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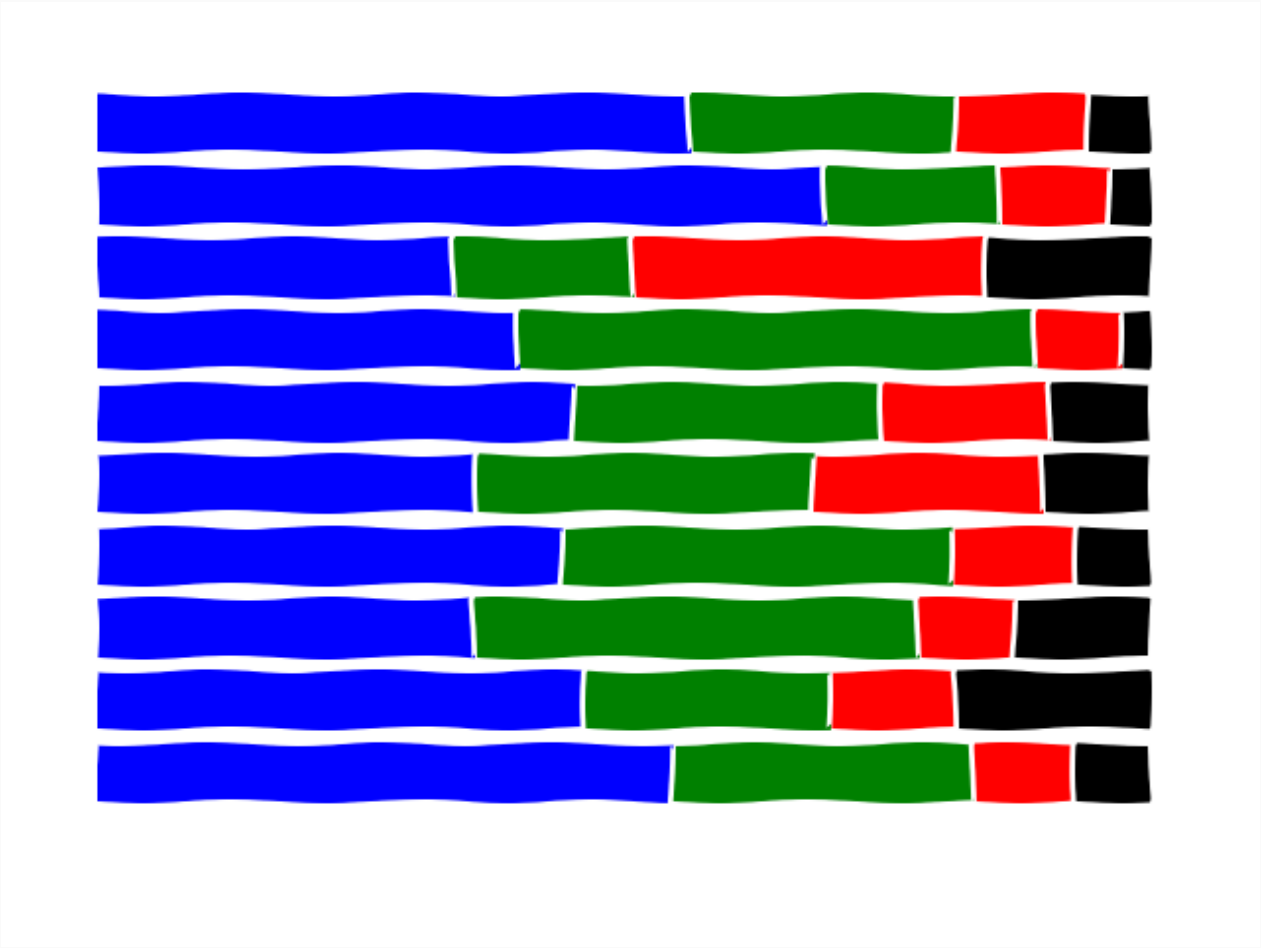
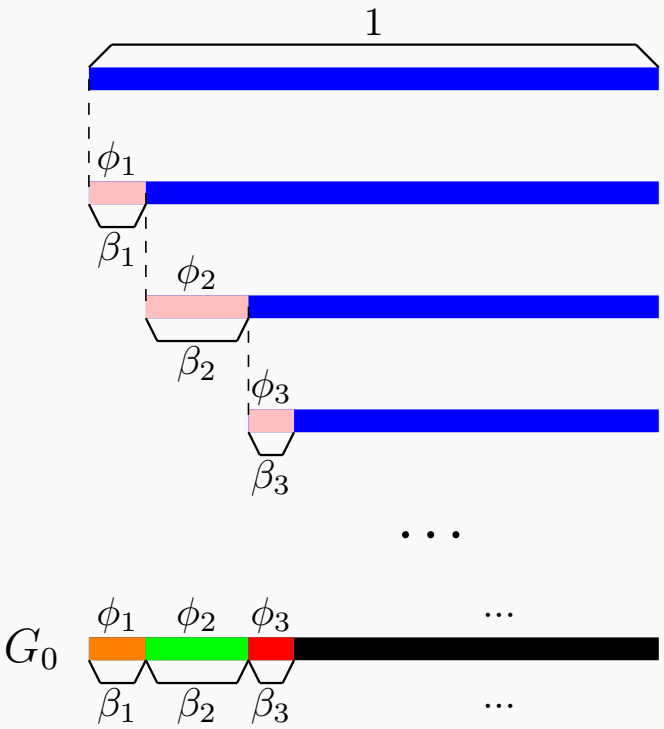
4. Next data-point
5. Associate with a new mixture or create new
6. If new, pick new parameters
7. Repeat

Dirichlet Distribution



$\mathrm{Dir}(7, 5, 3, 2)$

Stick-Breaking



$$G \sim \mathrm{DP}(\mathcal{H}, \alpha)$$

$$\hat{\beta}_k \sim \mathrm{Beta}(1, \alpha)$$

$$\beta_k = \hat{\beta}_k \prod_{l=1}^{k-1} (1 - \hat{\beta}_l)$$

$$\Phi \sim \mathcal{H}$$

$$G_0 = \sum_{k=1}^{\infty} \beta_k \Phi_k$$

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- We have derived a formulation that provides measure on any partition
  - We can now search for the configuration that is most likely: ML or MAP etc.
  - We can try to derive the posterior over the partition
- *We are so far only looking at how to formulate models*

Summary

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- Allows for models dealing with infinite partitions
    - Infinite Gaussian Mixture Models
- Take home message: generative models
- Tomorrow: Latent Dirichlet Allocation

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References


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