Correlations Redux

Ingram Olkin
Stanford University

Jeremy D. Finn
State University of New York at Buffalo

Correlational analysis is a cornerstone method of statistical analysis, yet most presentations of correlational techniques deal primarily with tests of significance. The focus of this article is obtaining explicit expressions for confidence intervals for functions of simple, partial, and multiple correlations. Not only do these permit tests of hypotheses about differences but they also allow a clear statement about the degree to which correlations differ. Several important differences of correlations for which tests and confidence intervals are not widely known are included among the procedures discussed. Among these is the comparison of 2 multiple correlations based on independent samples.

> One of the most meaningful phenomena we've discovered is that when you try to look closely at one thing, it is inextricably bound to many other things . . . .
>
> —John Muir

Although the correlation coefficient will shortly celebrate its centennial, questions about specific applications arise continually. In fact, the *Current Index of Statistics* (Institute for Mathematical Statistics, 1988, 1989, 1990, 1991, 1992) lists over 550 articles published in a 5-year span with *correlation* in the title. Most of these provide methods for testing hypotheses about correlations under various conditions. Test procedures can sometimes be transformed into procedures for obtaining confidence intervals, but this is not often done. The focus of this article is obtaining explicit expressions for confidence intervals, in addition to tests of significance, for functions of simple, partial, and multiple correlations. With these results, the investigator can address the oft-neglected question "How much of an effect is there?"—not just whether the effect is statistically significant.

All of the procedures address commonly asked questions about the effects of adding a variable to a predictor set, deleting a predictor variable, or the performance of different predictors or predictor sets in explaining criterion variation. Many basic regression textbooks construct F statistics from two multiple correlations or from residual mean squares to test the effect of adding or deleting predictor variables. In this article, the large-sample theory is used to derive confidence intervals on the improvement or decrement in predictability as well as tests of significance.

We emphasize the importance of a transition from testing hypotheses about correlations to a confidence interval approach. In some ways, regression analysis may provide a more insightful understanding of data. Thus, we do not advocate correlational analysis over regression analysis. Rather, we note that many researchers have a stronger intuitive sense about correlations than about regressions. For example, it is known that the Scholastic Achievement Test mathematics and verbal scores are nationally correlated at .66 and that the Wechsler Adult Intelligence Scale correlates about .80 to .85 with the Stanford-Binet Intelligence Scale and with the Raven's Progressive Matrices tests. Some researchers may prefer correlations because a single correlation provides a succinct summary of a complex analysis, for example, assessing the combined effects of several predictor variables or the impact of controlling for one or more covariates.

A variety of approaches have been taken to significance tests and confidence intervals for a single correlation coefficient. Most elementary textbooks include a test that the population product–moment correlation is zero. Under the null hypothesis the test statistic, $r(n-2)^{1/2}/(1-r^2)^{1/2}$, has a t distribution with $n-2$ degrees of freedom. A variety of alternatives are available to obtain a confidence interval. David (1938) has provided a graphical procedure for a confidence interval given r and n. This procedure is obtained from the noncentral distribution of r and is exact although subject to the inexactness of reading a graph.

Tests and confidence intervals can also be obtained using Fisher's z transformation, $z = \tfrac{1}{2}\log[(1+r)/(1-r)]$. Fisher's statistic is approximately normally distributed, even if the sam-

Ingram Olkin, Department of Statistics and School of Education, Stanford University; Jeremy D. Finn, School of Education, State University of New York at Buffalo.

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Correspondence concerning this article should be addressed to Ingram Olkin, Department of Statistics, Stanford University, Stanford, California 94305-4065. Electronic mail be sent via Internet to iolkin@playfair.stanford.edu.

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ple size is small or if the value of ρ is extreme, with mean $\zeta = \frac{1}{2}\log[(1+\rho)/(1-\rho)]$ and variance approximately $1/(n-3)$. Fisher's z can be used to obtain a confidence interval for ζ, which can then be transformed back to a confidence interval for ρ. This approach is recommended for tests and intervals on a single correlation coefficient but does not generalize readily to confidence intervals for differences of correlations.

When finding a confidence interval for a single correlation coefficient, there is the choice of using the sample correlation coefficient r or any one-to-one function $h(r)$ because an interval on $h(r)$ can be transformed back to an interval on r. However, if we wish to develop a confidence interval for the difference of correlations, the interval $h(\rho_1) - h(\rho_2)$ does not transform to an interval on $\rho_1 - \rho_2$. Consequently, the sample difference $r_1 - r_2$ must be used directly as the basis of an interval for $\rho_1 - \rho_2$. This problem does not occur in tests of hypotheses because the hypothesis H: $\rho_1 = \rho_2$ is equivalent to the hypothesis H: $h(\rho_1) = h(\rho_2)$.

The alternatives available for testing hypotheses about differences have bred an extensive literature on the topic. The review by Steiger (1980) contains a useful summary of strengths and weaknesses of several commonly used tests for differences including those based on Fisher's transformation and others based on the large-sample theory. Steiger noted that the latter depend on the assumption of asymptotic normality of correlations and require large n's, especially if ρ's are close to 1. At the same time, the review gives no attention to obtaining point or interval estimates of differences. Steiger and Browne (1984) presented a general framework for tests of differences of simple, partial, and multiple correlations based on asymptotics. Although the theory developed in their article could be used to construct confidence intervals (as we did), this step was omitted.

In an earlier article (Olkin & Finn, 1990), we presented two tests for the equality of correlations from a k-variate normal distribution. The first was for comparing the performance of $x_1, x_2, \ldots, x_k$ in predicting x_0; the hypothesis addressed was H: $\rho_{01} = \rho_{02} = \cdots = \rho_{0k}$. Confidence intervals for the difference between one predictor and another were not presented but could be easily derived from the results given.

The Olkin and Finn (1990) article also presented a test and confidence interval for comparing ρ_{12}, ρ_{34}, and ρ_{56} when these are based on a single sample from a multivariate normal population. The procedure is useful for repeated measures studies to determine if the correlation of two measures differs from one occasion to another or from one experimental condition to another. In the original article, it was used to see if the correlation of two physiological measures increased as the participants grew from childhood through adolescence to adulthood. The procedures in Olkin and Finn depend on the covariance of two sample correlations, which was obtained by Pearson and Filon (1898; see also Olkin, 1966), for example, the covariance of r_{01} with r_{02} (first procedure) and the covariance of r_{12} with r_{34} (second procedure).

In this article, tests and confidence intervals are discussed for five models:

Model A: Determining whether an additional variable provides an improvement in predicting the criterion: $\rho^2_{0(12)} - \rho^2_{01}$. This comparison shows whether an additional variable x_2 provides an improvement over x_1 alone in predicting x_0. The test for the equality of the squared multiple correlation $\rho^2_{0(12)}$ and the squared simple correlation ρ^2_{01} is equivalent to comparing a full model with a reduced model in regression analysis, commonly tested with an F statistic. However, the confidence interval has not generally been documented.

Model B: Deciding which of two variables adds more to the prediction of the criterion: $\rho^2_{0(12)} - \rho^2_{0(13)}$. This comparison shows whether the pair of predictors x_1, x_2 or the pair x_1, x_3 is more effective in predicting criterion x_0. It is a comparison of two squared multiple correlations of a single criterion with two different sets of predictor variables.

Model C: Determining the effect of a third variable on the association of two others: $\rho_{01} - \rho_{01\cdot 2}$. This comparison examines the effect of variable x_2 on the relationship of x_1 and x_0. It is a comparison of a simple correlation ρ_{01} with the partial correlation adjusted for the effects of x_2; that is, does x_2 explain why predictor x_1 is related to the criterion x_0?

Model D: Deciding which of two variables has a stronger effect on the association of two others: $\rho_{01\cdot 2} - \rho_{01\cdot 3}$. This comparison examines whether the effect of x_2 on the relationship of x_1 and x_0 is the same as the effect of x_3. It is a comparison of two partial correlations between x_0 and x_1—one adjusted for the effects of x_2 and the other adjusted for the effects of x_3.

Model E: Deciding if a given set of predictors performs equally well in two separate populations: $\rho^2_I - \rho^2_{II}$. This comparison shows whether a given set of predictors $(x_1, x_2, \ldots, x_k)$ performs equally well in two independent samples of data. For example, does a battery of preplacement tests predict job performance equally well for White and minority applicants? It is a comparison of two squared multiple correlations based on separate populations. Neither the test of significance nor the confidence interval is widely documented for this situation. All too often, conclusions are drawn about predictability in two populations without any formal statistical comparison of the multiple correlations.

The procedures presented in this article for Models A–D are shown for a total of three or four variables including the criterion. Each model can be extended to higher dimensions, although the computations are more extensive. Appendix A illustrates the generalization of Model B for additional variables.

The confidence intervals and tests obtained in this article are based on the large-sample theory. The results for Models A–D are based on the computation of covariances of unusual combinations of correlations and rely heavily on theoretical developments in Olkin and Siotani (1976) and Hedges and Olkin (1983). All of the procedures have the same general form, namely, if r_A and r_B are two sample correlations to be compared, and ρ_A and ρ_B are their corresponding population values, then the large-sample distributional form for the comparison is

$$[(r_A - r_B) - (\rho_A - \rho_B)] \sim N(0, \sigma^2_\infty), \tag{1}$$

where

$$\sigma^2_\infty = \mathrm{var}(r_A) + \mathrm{var}(r_B) - 2\mathrm{cov}(r_A, r_B)$$

is the asymptotic (large-sample) variance of the difference of the two correlations, which depends on the population correlations. It is generally the covariance term that is most cumbersome to determine. A $100(1-\alpha)\%$ confidence interval can then be constructed,

$$r_A - r_B \pm c\hat{\sigma}_\infty, \quad (2)$$

where c is the standard normal deviate $z_{\alpha/2}$ and $\hat{\sigma}_\infty$ is an estimate of σ_∞ in which sample values replace population values. The hypothesis H: $\rho_A - \rho_B = \delta$ is rejected at significance level α if δ is not contained in the confidence interval at a $100(1-\alpha)\%$ confidence level.

The variances and covariances of correlations used in Models A–D are given in Olkin and Siotani (1976) and in Hedges and Olkin (1983). The three results required in this article are (a) the variance of sample correlation r_{ij},

$$\operatorname{var}(r_{ij}) = (1 - \rho_{ij}^2)^2/n; \quad (3)$$

(b) the covariance of two correlations in which there is a common variable,

$$\operatorname{cov}(r_{ij}, r_{ik}) = [\tfrac{1}{2}(2\rho_{jk} - \rho_{ij}\rho_{ik})(1 - \rho_{ij}^2 - \rho_{ik}^2 - \rho_{jk}^2) + \rho_{jk}^3]/n; \quad (4)$$

and (c) the covariance of two sample correlations that do not involve any variables in common,

$$\operatorname{cov}(r_{ij}, r_{kl}) = [\tfrac{1}{2}\rho_{ij}\rho_{kl}(\rho_{ik}^2 + \rho_{il}^2 + \rho_{jk}^2 + \rho_{jl}^2) + \rho_{ik}\rho_{jl} + \rho_{il}\rho_{jk} - (\rho_{ij}\rho_{ik}\rho_{il} + \rho_{ji}\rho_{jk}\rho_{jl} + \rho_{ki}\rho_{kj}\rho_{kl} + \rho_{li}\rho_{lj}\rho_{lk})]/n. \quad (5)$$

Algebraically, Equation 5 reduces to Equation 4 if one variable is common to both sets, for example, if variable i and variable l are the same measure. Equation 4 reduces to Equation 3 if $j = k$. Large-sample estimates are obtained by replacing each population ρ in Equations 3, 4, and 5 with corresponding sample values. We evaluate these formulas explicitly for particular combinations of variables in the discussions that follow.

Equations 3–5 provide the essential ingredients for the procedures that follow. In addition, note that Models A–D involve differences of simple, partial, and multiple correlations, that is, composites of correlations. Thus, we use a general form for the large-sample variance of functions of a set of correlations. The general form (stated for three correlations only) is

$$\operatorname{var}_\infty f(r_{12}, r_{13}, r_{23}) = \mathbf{a}\Phi\mathbf{a}'.$$

Matrix Φ is the variance–covariance matrix of r_{12}, r_{13}, and r_{23}, with elements defined by Equations 3–5. Vector $\mathbf{a}$ contains a set of coefficients that differ depending on the particular function of correlations to be evaluated. We give the specific elements of $\mathbf{a}$ and Φ for each model as they are discussed.

Technically, if $f(r_{12}, r_{13}, r_{23})$ is a function of the three correlations, then vector $\mathbf{a}$ consists of the partial derivatives

$$\mathbf{a} = \left(\frac{\partial f}{\partial r_{12}}, \frac{\partial f}{\partial r_{13}}, \frac{\partial f}{\partial r_{23}}\right). \quad (6)$$

This result is obtained from the classic Taylor expansion, often called the delta method (e.g., see Cramér, 1946; Rao, 1973). The evaluation of Equation 6 is illustrated for Model A in Appendix B.

Data: A Study of Teenage Use of Abusable Substances

The data to illustrate these techniques were collected as part of a longitudinal study of the use of alcohol, cigarettes, and marijuana among urban school children (Iannotti & Bush, 1992). This study was begun with fourth- and fifth-grade data in 1988. For this review, we assume that the purpose of the investigation is to understand the antecedents of the use of abusable substances among eighth-grade African American students.

During the 1990–1991 school year, questionnaires and sociometric measures were administered to about 7,700 sixth- and seventh-grade students in the Washington, DC, area. The same questionnaires were administered again in 1991–1992 to about 6,600 students in seventh and eighth grades. Each year, over 90% of the sample were African American students. The data for the present study consist of measures on 1,415 African American students who were included in the seventh-grade sample in 1990–1991 and who were also in the eighth grade sample in 1991–1992.

Students were assured of confidentiality at the outset and did not identify themselves on the answer sheets. (The refusal rate was less than one half of 1%.) The primary measures of the study were based on responses to three questions: "How old were you when you had your *first sip* of beer, wine, wine cooler, or other alcoholic drink?," "How old were you when you tried your *first puff* of a cigarette?," and "How old were you when you tried your *first puff* of marijuana?" In each case, one response option is "never have." An abusable substance score was created by summing the number of substances (cigarettes, alcohol, or marijuana) that the individual had tried. The score, ranging from 0 to 3, is called USE-7 for students in seventh grade and USE-8 for students in eighth grade. USE-8 is the primary criterion variable in this analysis.

Perceived friends' use (FRIENDS-8) was assessed by questions that asked students to indicate the number of friends, up to a maximum of four, who were using alcohol, cigarettes, or marijuana. The resulting scale ranged from 0 to 12. Perceived family use (FAMILY-7) is the number of abusable substances, out of three, that were used by any member of the student's family. Classroom use (CLASS-8) is the average substance use reported by all members of the individual's class, excluding the student. Finally, a rough index of socioeconomic status (SES) was obtained by dividing the number of persons living in the student's home by the total number of bedrooms.

The variance–covariance matrix and correlation matrix among all six measures are given in Table 1. The bottom section of the table gives the estimate of the variance of each correlation by substituting sample values in Equation 3. In the following illustrations, calculations were performed from correlations carried to six decimals; computations based on the correlations in Table 1 may differ slightly from those presented here.

Distribution Theory and Applications

Model A: Determining Whether an Additional Variable Provides an Improvement in Predicting the Criterion

Because of the well-established importance of peers in influencing the behavior of adolescents, we wish to determine whether the family's use of abusable substances makes any additional contribution to predicting substance use in eighth grade, above and beyond peers' use. For this illustration, the variables are x_0 = USE-8, x_1 = FRIENDS-8, and x_2 = FAMILY-7.

Table 1
Dispersions Among Abusable Substance Measures

Dispersion matrices	Variable 1	2	3	4	5	6
Variance–covariance matrix						
1. USE-8	0.6261					
2. USE-7	0.3752	0.6272				
3. FAMILY-7	0.1381	0.1656	0.7722			
4. FRIENDS-8	1.0897	0.7775	0.4971	0.1038		
5. CLASS-8	0.0387	0.0197	0.0003	0.1214	0.0616	
6. SES	0.0647	0.0449	0.0903	0.2606	0.0122	1.3422
Correlation matrix[a]						
1. USE-8	—					
2. USE-7	0.599	—				
3. FAMILY-7	0.199	0.238	—			
4. FRIENDS-8	0.433	0.309	0.178	—		
5. CLASS-8	0.197	0.100	0.001	0.154	—	
6. SES	0.071	0.049	0.089	0.071	0.043	—
Sample variances of correlations[b]						
1. USE-8	—					
2. USE-7	0.4114	—				
3. FAMILY-7	0.9226	0.8900	—			
4. FRIENDS-8	0.6598	0.8183	0.9377	—		
5. CLASS-8	0.9240	0.9801	0.9999	0.9532	—	
6. SES	0.9901	0.9952	0.9843	0.9900	0.9964	—

Note. USE-8 = use in eighth grade; USE-7 = use in seventh grade; FAMILY-7 = perceived family use in seventh grade; FRIENDS-8 = perceived friends' use in eighth grade; CLASS-8 = classroom use in eighth grade; SES = socioeconomic status.

[a] Two-tailed significance points as follows: for $\alpha = .10$, $r = .0437$; for $\alpha = .05$, $r = .0521$; for $\alpha = .02$, $r = .0618$; for $\alpha = .01$, $r = .0685$. Values obtained by substituting significance points of t_{n-2} in $t = r[(n-2)/(1-r^2)]^{1/2}$ and solving for r.

[b] $\widehat{\mathrm{VAR}}(r_{ij}) = (1 - r_{ij}^2)^2/n$. The values in the table do not include the multiplier $1/n$ to preserve significant digits. Each entry is multiplied by 1/1,415.

This procedure compares the squared multiple correlations $\rho^2_{0(12)}$ with the squared simple correlation ρ^2_{01}, using estimates $r^2_{0(12)}$, r^2_{01}, and $\hat{\sigma}^2_\infty = \widehat{\mathrm{var}}(r^2_{0(12)} - r^2_{01})$. The variance of the difference is

$$\mathrm{var}(r^2_{0(12)} - r^2_{01}) = \mathbf{a}\Phi\mathbf{a}'.$$

Both the vector $\mathbf{a}$ and the covariance matrix Φ are functions of the correlations among x_0, x_1, and x_2. The symmetric matrix of population correlations is

$$P = \begin{pmatrix} 1 & \rho_{01} & \rho_{02} \\ & 1 & \rho_{12} \\ & & 1 \end{pmatrix}.$$

From these correlations, the vector $\mathbf{a} = (a_1, a_2, a_3)$ is

$$a_1 = \frac{2\rho_{12}}{1-\rho_{12}^2}(\rho_{01}\rho_{12} - \rho_{02}), \quad a_2 = \frac{2}{1-\rho_{12}^2}(\rho_{02} - \rho_{01}\rho_{12}),$$

$$a_3 = \frac{2}{(1-\rho_{12}^2)^2}(\rho_{12}\rho_{01}^2 + \rho_{12}\rho_{02}^2 - \rho_{01}\rho_{02} - \rho_{01}\rho_{02}\rho_{12}^2).$$

The derivation of these elements, applying Equation 6, is given in Appendix B.

The matrix of variances and covariances among the sample correlations is

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ & \phi_{22} & \phi_{23} \\ & & \phi_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{var}(r_{01}) & \mathrm{cov}(r_{01}, r_{02}) & \mathrm{cov}(r_{01}, r_{12}) \\ & \mathrm{var}(r_{02}) & \mathrm{cov}(r_{02}, r_{12}) \\ & & \mathrm{var}(r_{12}) \end{pmatrix}.$$

The sample correlation matrix for USE-8, FRIENDS-8, and FAMILY-7 obtained from Table 1 is

$$R = \hat{P} = \begin{pmatrix} 1.000 & 0.433 & 0.199 \\ & 1.000 & 0.178 \\ & & 1.000 \end{pmatrix}.$$

The estimate of ρ^2_{01} is $r^2_{01} = .188$. The estimate of $\rho^2_{0(12)}$ is

$$\hat{\rho}^2_{0(12)} = \frac{r^2_{01} + r^2_{02} - 2r_{01}r_{02}r_{12}}{1 - r^2_{12}} = .203,$$

and the difference is $r^2_{0(12)} - r^2_{01} = .015$.

To obtain the variance estimate, the sample values in R are substituted in the expressions for a_1, a_2, and a_3. The resulting vector is $\hat{\mathbf{a}} = (-.0447, .2511, -.1032)$.

The variance–covariance matrix is obtained by evaluating Equations 3 and 4 for the correlations in P, that is, ρ_{01}, ρ_{02}, and ρ_{12}. The variances are

$$\phi_{11} = (1-\rho_{01}^2)^2/n, \quad \phi_{22} = (1-\rho_{02}^2)^2/n,$$
$$\phi_{33} = (1-\rho_{12}^2)^2/n. \qquad (7)$$

The covariances are

$$\phi_{12} = [\tfrac{1}{2}(2\rho_{12} - \rho_{01}\rho_{02})(1-\rho_{12}^2-\rho_{01}^2-\rho_{02}^2)+\rho_{12}^3]/n,$$
$$\phi_{13} = [\tfrac{1}{2}(2\rho_{02} - \rho_{01}\rho_{12})(1-\rho_{12}^2-\rho_{01}^2-\rho_{02}^2)+\rho_{02}^3]/n,$$
$$\phi_{23} = [\tfrac{1}{2}(2\rho_{01} - \rho_{02}\rho_{12})(1-\rho_{12}^2-\rho_{01}^2-\rho_{02}^2)+\rho_{01}^3]/n. \qquad (8)$$

Substituting sample values in these expressions, the estimated variance–covariance matrix is obtained from Table 1:

$$\hat{\Phi} = \frac{1}{1{,}415}\begin{pmatrix} .6598 & .1056 & .1265 \\ & .9226 & .3893 \\ & & .9377 \end{pmatrix}.$$

Consequently, $\hat{\mathbf{a}}\hat{\Phi}\hat{\mathbf{a}}' = .0481/1{,}415$ and $\hat{\sigma}_\infty = .0058$. The difference of the squared correlations is .015, and a 95% confidence interval for the difference $\rho_{0(12)}^2 - \rho_{01}^2$ is $r_{0(12)}^2 - r_{01}^2 \pm 1.96\hat{\sigma}_\infty = .015 \pm (1.96)(.0058)$, which yields the interval [.004, .027]. The family's use of abusable substances contributes to explaining use in eighth grade, above and beyond the effects of peers. The additional predictive power is small, however, amounting to no more than 2.7% of the criterion variation.

Model B: Deciding Which of Two Variables Adds More to the Prediction of the Criterion

Because friends' use of abusable substances is known to be an important antecedent of the individual's own use, we ask whether the additional effect of the family is greater or less than the additional effect of other classmates. For this illustration, the variables are x_0 = USE-8, x_1 = FRIENDS-8, x_2 = FAMILY-7, and x_3 = CLASS-8.

This procedure compares the squared multiple correlation $\rho_{0(12)}^2$ with $\rho_{0(13)}^2$ using estimates $r_{0(12)}^2$, $r_{0(13)}^2$, and $\hat{\sigma}_\infty^2 = \widehat{\text{var}}(r_{0(12)}^2 - r_{0(13)}^2)$. For this model, the variance of the difference is

$$\text{var}(r_{0(12)}^2 - r_{0(13)}^2) = \mathbf{a}\Phi\mathbf{a}'.$$

Both the vector $\mathbf{a}$ and the covariance matrix Φ are functions of the correlations among the variables x_0, x_1, x_2, and x_3. The symmetric matrix of correlations is

$$P = \begin{pmatrix} 1 & \rho_{01} & \rho_{02} & \rho_{03} \\ & 1 & \rho_{12} & \rho_{13} \\ & & 1 & \rho_{23} \\ & & & 1 \end{pmatrix}.$$

The vector $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$ has five components:

$$a_1 = \frac{\rho_{01}(\rho_{12}^2-\rho_{13}^2)+(\rho_{03}\rho_{13}-\rho_{02}\rho_{12})+\rho_{12}\rho_{13}(\rho_{02}\rho_{13}-\rho_{03}\rho_{12})}{(1-\rho_{12}^2)(1-\rho_{13}^2)},$$

$$a_2 = 2\left(\frac{\rho_{02}-\rho_{01}\rho_{12}}{1-\rho_{12}^2}\right), \quad a_3 = 2\left(\frac{\rho_{03}-\rho_{01}\rho_{13}}{1-\rho_{13}^2}\right),$$

$$a_4 = \frac{2}{(1-\rho_{12}^2)^2}(\rho_{12}\rho_{01}^2+\rho_{12}\rho_{02}^2-\rho_{12}^2\rho_{01}\rho_{02}-\rho_{01}\rho_{02}),$$

$$a_5 = \frac{-2}{(1-\rho_{13}^2)^2}(\rho_{13}\rho_{01}^2+\rho_{13}\rho_{03}^2-\rho_{13}^2\rho_{01}\rho_{03}-\rho_{01}\rho_{03}).$$

The variance–covariance matrix Φ of r_{01}, r_{02}, r_{03}, r_{12}, and r_{13}, in that order, is given in Table 2. The variance of r_{23} and the covariances of other correlations with r_{23} are not required for this model.

The sample correlation matrix for USE-8, FRIENDS-8, FAMILY-7, and CLASS-8 is

$$R = \hat{P} = \begin{pmatrix} 1.000 & 0.433 & 0.199 & 0.197 \\ & 1.000 & 0.178 & 0.154 \\ & & 1.000 & 0.001 \\ & & & 1.000 \end{pmatrix}.$$

The estimates of $\rho_{0(12)}^2$ and $\rho_{0(13)}^2$ are obtained from these values:

$$\hat{\rho}_{0(12)}^2 = \frac{r_{01}^2 + r_{02}^2 - 2r_{01}r_{02}r_{12}}{1-r_{12}^2} = .203,$$

$$\hat{\rho}_{0(13)}^2 = \frac{r_{01}^2 + r_{03}^2 - 2r_{01}r_{03}r_{13}}{1-r_{13}^2} = .205.$$

The difference is $r_{0(12)}^2 - r_{0(13)}^2 = .203 - .205 = -.002$.

To obtain the variance estimate, the values in R are substituted in the expressions for $a_1, a_2, \ldots, a_5$, resulting in the vector $\hat{\mathbf{a}} = (-.0018, .2511, .2667, -.1032, .1101)$. The variance–covariance matrix is obtained by evaluating Equations 3, 4, and 5 for r_{01}, r_{02}, r_{03}, r_{12}, and r_{13}. The variances for r_{01}, r_{02}, and r_{12} are given in Equation 7; these are ϕ_{11}, ϕ_{22}, and ϕ_{44} in the present model. The remaining variances have the same form. The covariances of r_{01} with r_{02}, r_{01} with r_{12}, and r_{02} with r_{12} are given

Table 2
Covariance Matrix of r_{01}, r_{02}, r_{03}, r_{04}, *and* r_{05}

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} \\ & \phi_{22} & \phi_{23} & \phi_{24} & \phi_{25} \\ & & \phi_{33} & \phi_{34} & \phi_{35} \\ & & & \phi_{44} & \phi_{45} \\ & & & & \phi_{55} \end{pmatrix} = \begin{pmatrix} \text{var}(r_{01}) & \text{cov}(r_{01}, r_{02}) & \text{cov}(r_{01}, r_{03}) & \text{cov}(r_{01}, r_{12}) & \text{cov}(r_{01}, r_{13}) \\ & \text{var}(r_{02}) & \text{cov}(r_{02}, r_{03}) & \text{cov}(r_{02}, r_{12}) & \text{cov}(r_{02}, r_{13}) \\ & & \text{var}(r_{03}) & \text{cov}(r_{03}, r_{12}) & \text{cov}(r_{03}, r_{13}) \\ & & & \text{var}(r_{12}) & \text{cov}(r_{12}, r_{13}) \\ & & & & \text{var}(r_{13}) \end{pmatrix}.$$

in Equation 8; these are ϕ_{12}, ϕ_{14}, and ϕ_{24} in the present model. The remaining covariances, except for $\text{cov}(r_{02}, r_{13})$ and $\text{cov}(r_{03}, r_{12})$, have the same form. The two exceptions require Equation 5. Specifically,

$$\begin{aligned}\phi_{25} = \text{cov}(r_{02}, r_{13}) = [(\tfrac{1}{2})\rho_{02}\rho_{13}(\rho_{01}^2 + \rho_{03}^2 + \rho_{12}^2 + \rho_{23}^2)\\ + \rho_{01}\rho_{23} + \rho_{03}\rho_{12} - (\rho_{02}\rho_{01}\rho_{03} + \rho_{02}\rho_{12}\rho_{13}\\ + \rho_{01}\rho_{12}\rho_{13} + \rho_{03}\rho_{23}\rho_{13})]/n,\\ \phi_{34} = \text{cov}(r_{03}, r_{12}) = [(\tfrac{1}{2})\rho_{03}\rho_{12}(\rho_{01}^2 + \rho_{02}^2 + \rho_{13}^2 + \rho_{23}^2)\\ + \rho_{01}\rho_{23} + \rho_{02}\rho_{13} - (\rho_{03}\rho_{01}\rho_{02} + \rho_{03}\rho_{13}\rho_{23}\\ + \rho_{01}\rho_{13}\rho_{12} + \rho_{02}\rho_{23}\rho_{12})]/n. \quad (9)\end{aligned}$$

Substituting sample values in these expressions, the estimated variance–covariance matrix is

$$\hat{\Phi} = \frac{1}{1{,}415}\begin{pmatrix} .6598 & .1056 & .0870 & .1265 & .1302 \\ & .9226 & -.0169 & .3893 & .0052 \\ & & .9240 & .0066 & .3949 \\ & & & .9377 & -.0118 \\ & & & & .9532 \end{pmatrix}.$$

Consequently, $\hat{\mathbf{a}}\hat{\Phi}\hat{\mathbf{a}}' = .1461/1{,}415$ and $\hat{\sigma}_\infty = .0102$. The difference of the two squared multiple correlations is $-.002$, and a 95% confidence interval for $\rho_{0(12)}^2 - \rho_{0(13)}^2$ is $r_{0(12)}^2 - r_{0(13)}^2 \pm 1.96\hat{\sigma}_\infty = -.002 \pm (1.96)(.0102)$, which yields the interval $[-.022, .018]$. Because zero is contained in the interval, the hypothesis H: $\rho_{0(12)}^2 = \rho_{0(13)}^2$ would not be rejected by these data. Both friends' and parents' substance use, and friends' and classmates' substance use, predict eighth-grade substance use equally well. Both pairs of predictors account for approximately 20% of the variation in eighth-grade usage.

Model C: Determining the Effect of a Third Variable on the Association of Two Others

This model (and Model D) analyzes the extent to which the correlation of two variables x_0 and x_1 can be attributed to a third variable x_2. It is possible that an explanation for the correlation of x_0 with x_1 is provided by common features or behaviors reflected in x_2. If so, the partial correlation $\rho_{01\cdot2}$ will be smaller than the zero-order correlation ρ_{01}. Model C can also be used to examine whether x_2 acts as a suppressor variable with respect to x_0 and x_1. To the extent that this is the case, the correlation after adjustment for x_2 (i.e., $\rho_{01\cdot2}$) may be larger than the simple correlation ρ_{01}.

In this illustration, we explore the relationship of students' use of abusable substances in seventh grade with their families' use. Although the correlation is not particularly high ($r = .238$), we examine the extent to which it is attributable to SES common to parents and their children. The variables are x_0 = USE-7, x_1 = FAMILY-7, and x_2 = SES.

This procedure compares the simple correlation ρ_{01} with the partial correlation $\rho_{01\cdot2}$, using estimates r_{01}, $r_{01\cdot2}$, and $\hat{\sigma}_\infty^2 = \widehat{\text{var}}(r_{01} - r_{01\cdot2})$. The variance is

$$\text{var}(r_{01} - r_{01\cdot2}) = \mathbf{a}\Phi\mathbf{a}'/[(1 - \rho_{02}^2)(1 - \rho_{12}^2)],$$

where the vector

$$\mathbf{a} = \left(1 - \sqrt{(1 - \rho_{02}^2)(1 - \rho_{12}^2)}, \frac{\rho_{01}\rho_{02} - \rho_{12}}{1 - \rho_{02}^2}, \frac{\rho_{01}\rho_{02} - \rho_{02}}{1 - \rho_{02}^2}\right).$$

P is the symmetric matrix of correlations among x_0, x_1, and x_2, and Φ is the matrix of variances and covariances among the sample correlations. Both of these have the same form as in Model A, and the elements of Φ are given by Equations 7 and 8.

The sample correlation matrix for USE-7, FAMILY-7, and SES is

$$R = \hat{P} = \begin{pmatrix} 1.000 & 0.238 & 0.044 \\ & 1.000 & 0.089 \\ & & 1.000 \end{pmatrix}.$$

The estimate of ρ_{01} is $r_{01} = .238$. The estimate of $\rho_{01\cdot2}$ is

$$\hat{\rho}_{01\cdot2} = \frac{r_{01} - r_{02}r_{12}}{\sqrt{(1 - r_{02}^2)(1 - r_{12}^2)}} = .235.$$

Although the difference .003 is very small, we proceed with the analysis to illustrate the method.

Substituting sample values in the expressions for $\mathbf{a}$ and Φ,

$$\hat{\mathbf{a}} = (.00514, -.07725, -.03742)$$

and

$$\hat{\Phi} = \frac{1}{1{,}415}\begin{pmatrix} .8900 & .0780 & .0361 \\ & .9952 & .2334 \\ & & .9843 \end{pmatrix}.$$

Consequently, $\hat{\mathbf{a}}\hat{\Phi}\hat{\mathbf{a}} = .00861/1{,}415$ and $\hat{\sigma}_\infty = \sqrt{(.00861/1{,}415)/.9898} = .0025$.

The sample difference is $.238 - .235 = .003$, and a 95% confidence interval for the difference $\rho_{01} - \rho_{01\cdot2}$ is $r_{01} - r_{01\cdot2} \pm 1.96\hat{\sigma}_\infty = .003 \pm (1.96)(.0025)$, which yields the interval $[-.0017, .0080]$. We conclude that SES does not explain why a seventh grader's use of abusable substances is related to reported use by his or her family. The hypothesis H: $\rho_{01} = \rho_{01\cdot2}$ is supported.

Model D: Deciding Which of Two Variables Has a Stronger Effect on the Association of Two Others

This model asks whether the correlation of two variables, x_0 and x_1, is explained better by the third variable x_2 or x_3. It involves a comparison of the partial correlation of x_0 and x_1 adjusted for the effects of x_2 with the partial correlation of x_0 and x_1 adjusted for x_3. In our illustration, we examine the relative stability of the use of abusable substances from seventh grade to eighth grade ($r = .599$). We ask whether this stability is explained better by the effect of family members as models during this period of time or by the possible influence of friends' usage. The variables are x_0 = USE-8, x_1 = USE-7, x_2 = FAMILY-7, and x_3 = FRIENDS-8.

This procedure compares the partial correlation $\rho_{01\cdot2}$ with $\rho_{01\cdot3}$, using estimates $r_{01\cdot2}$, $r_{01\cdot3}$, and $\hat{\sigma}_\infty^2 = \widehat{\text{var}}(r_{01\cdot2} - r_{01\cdot3})$. For this model, the variance is $\text{var}(r_{01\cdot2} - r_{01\cdot3}) = \mathbf{a}\Phi\mathbf{a}'$, where the vector $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$ has elements

$$a_1 = \frac{\rho_{01} - \rho_{02}\rho_{12}}{\sqrt{(1-\rho_{02}^2)(1-\rho_{12}^2)}} - \frac{\rho_{01} - \rho_{03}\rho_{13}}{\sqrt{(1-\rho_{03}^2)(1-\rho_{13}^2)}},$$

$$a_2 = \frac{\rho_{01}\rho_{02} - \rho_{12}}{(1-\rho_{02}^2)^{3/2}(1-\rho_{12}^2)^{1/2}},$$

$$a_3 = \frac{\rho_{01}\rho_{03} - \rho_{13}}{(1-\rho_{03}^2)^{3/2}(1-\rho_{13}^2)^{1/2}},$$

$$a_4 = \frac{\rho_{01}\rho_{12} - \rho_{02}}{(1-\rho_{02}^2)^{1/2}(1-\rho_{12}^2)^{3/2}},$$

$$a_5 = \frac{\rho_{01}\rho_{13} - \rho_{03}}{(1-\rho_{03}^2)^{1/2}(1-\rho_{13}^2)^{3/2}}.$$

P is the symmetric matrix of correlations among x_0, x_1, x_2, and x_3, and Φ is the variance–covariance matrix of $r_{01}, r_{02}, r_{03}, r_{12}$, and r_{13}, in that order (see Model B).

The sample correlation matrix for USE-8, USE-7, FAMILY-7, and FRIENDS-8 is

$$R = \hat{P} = \begin{pmatrix} 1.000 & 0.599 & 0.199 & 0.433 \\ & 1.000 & 0.238 & 0.309 \\ & & 1.000 & 0.178 \\ & & & 1.000 \end{pmatrix}.$$

The estimates of $\rho_{01\cdot 2}$ and $\rho_{01\cdot 3}$ can be obtained from these values:

$$\hat{\rho}_{01\cdot 2} = \frac{r_{01} - r_{02}r_{12}}{\sqrt{(1-r_{02}^2)(1-r_{12}^2)}} = .579,$$

$$\hat{\rho}_{01\cdot 3} = \frac{r_{01} - r_{03}r_{13}}{\sqrt{(1-r_{03}^2)(1-r_{13}^2)}} = .542.$$

The difference is $r_{01\cdot 2} - r_{01\cdot 3} = .579 - .542 = .037$.

To obtain the variance estimate, the sample values in R are substituted in the expression for a. The resulting vector is $\hat{\mathbf{a}} =$ (.0370, −.1301, −.0710, −.0626, −.3202). The variance–covariance matrix is obtained by evaluating Equations 3, 4, and 5 for $r_{01}, r_{02}, r_{03}, r_{12}$, and r_{13} as for Model B. The variances of r_{01}, r_{02}, and r_{12} are given by Equation 7; the remaining variances have the same form. The covariances of r_{01} with r_{02}, r_{01} with r_{12}, and r_{02} with r_{12} are given by Equation 8. The remaining covariances have the same form except for cov(r_{02}, r_{13}) and cov(r_{03},r_{12}) that are given by Equation 9.

Substituting sample values in these expressions, the estimated variance–covariance matrix is

$$\hat{\Phi} = \frac{1}{1{,}415}\begin{pmatrix} .4114 & .1108 & .0937 & .0773 & .2034 \\ & .9226 & .1056 & .5284 & .9515 \\ & & .6598 & .0672 & .4053 \\ & & & .8900 & .1209 \\ & & & & .8183 \end{pmatrix}.$$

Consequently, $\hat{\mathbf{a}}\hat{\Phi}\hat{\mathbf{a}}' = .1431/1{,}415$ and $\hat{\sigma}_\infty = .0101$. The difference of the two partial correlations is .037, and a 95% confidence interval for $\rho_{01\cdot 2} - \rho_{01\cdot 3}$ is $r_{01\cdot 2} - r_{01\cdot 3} \pm 1.96\hat{\sigma}_\infty = .037 \pm (1.96)(.0101)$, which yields the interval [.017, .057]. Because zero is not contained in this interval, the hypothesis H: $\rho_{01\cdot 2} = \rho_{01\cdot 3}$ would be rejected. The effect of friends' substance use on the stability of use of a student between seventh grade and eighth grade is greater than the effect of family use on year-to-year stability. The difference is small but statistically reliable.

Model E: Deciding If a Given Set of Predictors Performs Equally Well in Two Separate Populations

This model examines the extent to which the multiple correlation of x_0 with $x_1, x_2, \ldots, x_k$ differs between one sample of observations and a second independent sample. It is useful for determining, for example, the extent to which college placement examinations predict grade averages equally well for White and minority students or whether employment examinations predict job performance equally well for any two groups (e.g., males and females). Indeed, in the prediction of college success, there is a history of research comparing regression equations that draw conclusions about different multiple correlations but without the benefit of formal tests or interval estimates (e.g., Stanley & Porter, 1967; Temp, 1971). The question of differential predictability can be addressed through regression models that include interactions with a group. However, if the interaction is significant, then separate regressions should be estimated, and point and interval estimates of the difference of the squared multiple correlations will be informative.

For this illustration, we ask whether eighth-grade male and female students are influenced equally by their families, friends, and classmates. The variables are x_0 = USE-8, x_1 = FAMILY-7, x_2 = FRIENDS-8, and x_3 = CLASS-8, but the data set is divided into two parts for male students and for female students, respectively. The predictive power of the three antecedents is compared for the two groups.

Two approaches can be taken to compare squared multiple correlations, ρ_I^2 and ρ_{II}^2, from separate populations, one using the asymptotic variance of sample values r_I^2 and r_{II}^2 and another using a variance-stabilizing transformation that parallels Fisher's z transformation for ordinary correlation coefficients. Both approaches can be taken using the correlations ρ_I and ρ_{II} or squared correlations ρ_I^2 and ρ_{II}^2. Because we focus on a confidence interval for the difference, we present results for the more commonly used proportion of explained variance index.

The variance of r^2 was originally found by Wishart (1931; see also Stuart & Ord, 1991, p. 103). If p is the total number of variates (i.e., predictors + 1),

$$\mathrm{var}(r^2) = [4\rho^2(1-\rho^2)^2(n-p)^2]/[(n^2-1)(n+3)],$$

within a margin of error on the order of $1/n^2$. For large samples, the variance is very close to

$$\mathrm{var}(r^2) \approx \frac{4}{n}\rho^2(1-\rho^2)^2\left[1 - \frac{(2p+3)}{n}\right].$$

When $2p + 3$ is small relative to n, as is the case in our example,

$$\mathrm{var}(r^2) \approx \frac{4}{n}\rho^2(1-\rho^2)^2. \tag{10}$$

If r_I^2 and r_{II}^2 are obtained from large independent samples of n_1 and n_2 observations, respectively, then the difference $r_I^2 - r_{II}^2$ is approximately normal with the variance

$$\operatorname{var}(r_I^2 - r_{II}^2) = \frac{4}{n_1}\rho_I^2(1-\rho_I^2)^2 + \frac{4}{n_2}\rho_{II}^2(1-\rho_{II}^2)^2. \quad (11)$$

The sample for the study of abusable substance use was composed of $n_1 = 654$ male students and $n_2 = 761$ female students. The squared multiple correlations for predicting USE-8 from FAMILY-7, FRIENDS-8, and CLASS-8 for male students and female students are $r_I^2 = .254$ and $r_{II}^2 = .193$, respectively. These were obtained by computer because scalar formulas for three-predictor multiple correlations are intractable. The difference in predictability is $.254 - .193 = .061$. To estimate the variance, sample values are substituted for ρ_I^2 and ρ_{II}^2 in Equation 11:

$$\widehat{\operatorname{var}}(r_I^2 - r_{II}^2) = \frac{4}{654}(.254)(1-.254)^2 + \frac{4}{761}(.193)(1-.193)^2 = .00153.$$

A 95% confidence interval for the difference $\rho_I^2 - \rho_{II}^2$ is $r_I^2 - r_{II}^2 \pm (1.96)\sqrt{.00153} = .061 \pm (1.96)(.039)$, which yields the interval $[-.015, .138]$. In general, boys' substance use in eighth grade is no more or no less related to usage by their family, friends, and classmates than girls' use is.

Note that a second approach to comparing ρ_I^2 with ρ_{II}^2 uses a variance-stabilizing transformation much like Fisher's z but applicable to r^2 rather than to r. The statistic

$$z^* = \log\left[\frac{1+\sqrt{r^2}}{1-\sqrt{r^2}}\right] = \log\left[\frac{1+r}{1-r}\right]$$

is asymptotically normal with expected value $E(z^*) = \zeta^* = \log[(1+\rho)/(1-\rho)]$ and $\operatorname{var}(z^*) = 16/n$. Whereas this approach leads to a test of $H: \rho_I^2 = \rho_{II}^2$, however, it does not readily result in a confidence interval for the difference of ρ_I^2 and ρ_{II}^2.

Conclusions

The correlation coefficient remains a mainstay of statistical analysis. Yet the use of correlations has been constrained because methods for testing a number of important hypotheses have not been widely accessible, and because techniques for obtaining point and interval estimates of differences between correlations have been all but nonexistent in applied fields. This article provides a remedy by presenting procedures for confidence intervals for five functions of simple, multiple, and partial correlations. We focus on estimation rather than tests of significance because far too much attention has been given to the issue of statistical significance, and far too little to the question, "How big is the effect?"

In comparison to tests of differences using Fisher's z transformation, the approach we presented yields point and interval estimates of differences as well as tests of significance. The sufficient statistics for our procedures are large-sample estimates of the variances and covariances of the zero-order correlations among the measures in the variable set. As noted by Steiger (1980), the question of a minimum requisite sample size, which also depends on the values of the ρ_{ij}, is complex. However, we recommend that the procedures be applied judiciously when sample sizes are moderate (e.g., $60 < n < 200$) and readily with larger samples.

The procedures described in this article can be expanded to more general forms. For example, Model B compares the additional predictive power of one variable (x_2) to that of another (x_3). With somewhat more algebra, the researcher can compare the additional predictive power of an entire set of number of predictor variables to that of a different set of predictors. Models that examine the effect of controlling for a particular measure (i.e., with a partial correlation) can be expanded to control for an entire set of measures. The theoretical basis for these general forms is given in Olkin and Siotani (1976) and in Hedges and Olkin (1983). However, their complexity can be daunting without computer software.

Though somewhat burdensome, the computations needed for the procedures given here are presented in their simplest forms to make them generally accessible and understood. A computer program to obtain the estimated variance–covariance matrix of a set of correlations could readily be programmed. In the meantime, we should not let our all-too-common reliance on ready-made computer packages thwart the use of statistical procedures most appropriate to a particular research study.

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Appendix A

Higher Dimensional Extension of Model B

Extensions to higher dimensions require considerably more detail and mathematical complexity. As an illustration, we provide a confidence interval for the difference between two squared multiple correlation coefficients, $\rho^2_{0(\alpha,\beta)} - \rho^2_{0(\alpha,\gamma)}$, in a single population; α, β, γ denote subsets of coefficients. This comparison shows the effect of predicting one variable using a common set of variables (denoted by α) together with different additional variables (denoted by β and γ, respectively).

For example, if $\alpha = \{1, 2\}$, $\beta = \{3\}$, $\gamma = \{4\}$, then we would obtain a confidence interval for $\rho^2_{0(1,2,3)} - \rho^2_{0(1,2,4)}$. In general, a confidence interval is given by $r^2_{0(\alpha,\beta)} - r^2_{0(\alpha,\gamma)} \pm z_{\alpha/2}\hat{\sigma}_\infty/\sqrt{n}$, where

$$\sigma^2_\infty = \mathrm{var}\,(r^2_{0(\alpha,\beta)}) + \mathrm{var}\,(r^2_{0(\alpha,\gamma)}) - 2\,\mathrm{cov}\,(r^2_{0(\alpha,\beta)}, r^2_{0(\alpha,\gamma)}),$$

and its estimate $\hat{\sigma}^2_\infty$ is obtained by replacing population correlations by their corresponding sample correlations.

The variance term is

$$\mathrm{var}\,(r^2_{0(\alpha,\beta)}) = \rho^2_{0(\alpha,\beta)}(1 - \rho^2_{0(\alpha,\beta)})^2/n.$$

To obtain the covariance term we require the following definition. If ξ denotes a set of subscripts $\xi_1, \ldots, \xi_m$, we write $R(\xi)$ to be the determinant of the sample correlation matrix corresponding to these subscripts and $P(\xi)$ to be its population counterpart. For example, if $\xi = \{1, 5, 6\}$, then

$$R(\xi) = \begin{vmatrix} 1 & r_{15} & r_{16} \\ r_{15} & 1 & r_{56} \\ r_{16} & r_{56} & 1 \end{vmatrix}.$$

With this notation

$$\mathrm{cov}\,(r^2_{0(\alpha,\beta)}, r^2_{0(\alpha,\gamma)}) = \frac{\mathrm{cov}\,(R(\alpha,\beta), R(\alpha,\gamma))}{P(\alpha,\beta)P(\alpha,\gamma)}$$

$$+ \frac{P(0, \alpha, \beta)P(0, \alpha, \gamma)}{P^2(\alpha, \beta)P^2(\alpha, \gamma)}\,\mathrm{cov}\,(R(0, \alpha, \beta), R(0, \alpha, \gamma))$$

$$- \frac{P(0, \alpha, \beta)}{P^2(\alpha, \beta)P(\alpha, \gamma)}\,\mathrm{cov}\,(R(0, \alpha, \beta), R(\alpha, \gamma))$$

$$- \frac{P(0, \alpha, \gamma)}{P(\alpha, \beta)P^2(\alpha, \gamma)}\,\mathrm{cov}\,(R(\alpha, \beta), R(0, \alpha, \gamma)).$$

The covariance of two determinants $R(\xi)$ and $R(\eta)$, where ξ and η are subsets of subscripts, is

$$\mathrm{cov}\,(R(\xi), R(\eta)) = \frac{1}{n} \sum_{i,j \in \xi} \sum_{l,m \in \eta} \rho^{ij}\rho^{lm}\{\tfrac{1}{2}\rho_{ij}\rho_{lm}(\rho^2_{il} + \rho^2_{im} + \rho^2_{jl} + \rho^2_{jm})$$

$$+ \rho_{il}\rho_{jm} + \rho_{im}\rho_{jl} - \rho_{ij}\rho_{il}\rho_{im} - \rho_{ji}\rho_{jl}\rho_{jm} - \rho_{li}\rho_{lj}\rho_{lm} - \rho_{mi}\rho_{mj}\rho_{ml}\}, \quad \text{(B1)}$$

and ρ^{ij} are the elements of the inverse p^{-1} of the full correlation matrix P including all the subscripts (for details, see Hedges & Olkin, 1983).

For example, in the case $\alpha = \{1, 2\}$, $\beta = \{3\}$, $\gamma = \{4\}$,

$$\mathrm{var}\,(r^2_{0(1,2,3)}) = \rho^2_{0(1,2,3)}(1 - \rho^2_{0(1,2,3)})^2/n,$$

$$\mathrm{var}\,(r^2_{0(1,2,4)}) = \rho^2_{0(1,2,4)}(1 - \rho^2_{0(1,2,4)})^2/n,$$

$$\mathrm{cov}\,(r^2_{0(1,2,3)}, r^2_{0(1,2,4)}) = \frac{\mathrm{cov}\,(R(1, 2, 3), R(1, 2, 4))}{P(1, 2, 3)P(1, 2, 4)}$$

$$+ \frac{P(0, 1, 2, 3)P(0, 1, 2, 4)}{P^2(1, 2, 3)P^2(1, 2, 4)}\,\mathrm{cov}\,(R(0, 1, 2, 3), R(0, 1, 2, 4))$$

$$- \frac{P(0, 1, 2, 3)}{P^2(1, 2, 3)P(1, 2, 4)}\,\mathrm{cov}\,(R(0, 1, 2, 3), R(1, 2, 4)$$

$$- \frac{P(0, 1, 2, 4)}{P(1, 2, 3)P^2(1, 2, 4)}\,\mathrm{cov}\,(R(1, 2, 3), R(0, 1, 2, 4)),$$

for which, for example,

$$P(0, 1, 2, 3) = \begin{vmatrix} 1 & \rho_{01} & \rho_{02} & \rho_{03} \\ & 1 & \rho_{12} & \rho_{13} \\ & & 1 & \rho_{23} \\ & & & 1 \end{vmatrix},$$

with similar definitions for other subscripts.

We illustrate the computations with $\mathrm{cov}(R(0, 1, 2, 3), R(1, 2, 4))$. Here $\xi = \{0, 1, 2, 3\}$ and $\eta = \{1, 2, 4\}$, so the sum in Equation B1 is over all (i, j) as distinct elements in ξ and (l, m) as distinct elements in η.

(Appendix B follows on next page)

Appendix B

Evaluation of Coefficients for Model A

The evaluation of the coefficients denoted by the vector **a** in each model requires obtaining the derivatives of the function of the correlations with respect to each element. We illustrate this computation for Model A.

$$f(r_{01}, r_{02}, r_{12}) = r^2_{0(12)} - r^2_{01}$$

$$= \frac{r^2_{01} + r^2_{02} - 2r_{01}r_{02}r_{12}}{1 - r^2_{12}} - r^2_{01} = \frac{(r_{02} - r_{01}r_{12})^2}{1 - r^2_{12}},$$

$$\frac{\partial f}{\partial r_{01}} = \frac{2(r_{02} - r_{01}r_{12})}{1 - r^2_{12}}(-r_{12}) = \frac{2r_{12}(r_{01}r_{12} - r_{02})}{1 - r^2_{12}},$$

$$\frac{\partial f}{\partial r_{02}} = \frac{2(r_{02} - r_{01}r_{12})}{1 - r^2_{12}},$$

$$\frac{\partial f}{\partial r_{12}} = \frac{(1 - r^2_{12})2(r_{02} - r_{01}r_{12})(-r_{01}) - (r_{02} - r_{01}r_{12})^2(-2r_{12})}{(1 - r^2_{12})^2}$$

$$= \frac{2}{(1 - r^2_{12})^2}(r_{12}r^2_{01} + r_{12}r^2_{02} - r_{01}r_{02} - r_{01}r_{02}r^2_{12}).$$

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