Formularium Digital Signal Processing

- **f(t) is periodiek met periode T**

Definitie

$$f(t) = \sum_{n=0}^{\infty} \{ a_n.\cos(n.\omega_0.t) + b_n.\sin(n.\omega_0.t)\}$$

Coëfficiënten

$$a_0 = \frac{1}{T}.\int_{-T/2}^{T/2} f(\tau)\, d\tau$$

$a_n = \frac{2}{T}.\int_{-T/2}^{T/2} f(\tau).\cos(n.\omega_0.\tau)\,.d\tau$ met n = 1,2,3…

$b_n = \frac{2}{T}.\int_{-T/2}^{T/2} f(\tau).\sin(n.\omega_0.\tau)\,.d\tau$ met n = 1,2,3…

- **Convolutie**

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k].h[k]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau).h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau).x(t-\tau)\, d\tau$$

- **Fourier reeksontwikkeling**

x(t) is periodiek met een pulsatie ω_0

$$X[n] = \frac{1}{T}.\int_{0}^{T} x(t).e^{-j.n.\omega_0.t}\,.dt$$

$$x(t) = \sum_{n=-\infty}^{\infty} X[n].e^{+j.n.\omega_0.t}$$

- **Fouriertransformatie**

$$X(\omega) = \int_{-\infty}^{\infty} x(t).\ e^{-j.\omega.t}.dt$$

$$x(t) = \frac{1}{2.\pi}.\int_{-\infty}^{\infty} X(\omega).e^{+j.\omega.t}.d\omega$$

- **DTFT**

$$X(e^{j.\omega}) = \sum_{n=-\infty}^{\infty} x[n].e^{-j.\omega.n}$$

- **IDTFT**

$$x[n] = \frac{1}{2.\pi}\int_{-\pi}^{+\pi} X(e^{j.\omega}).e^{+j.\omega.n}.d\omega$$

- **Fouriertabellen**

Table 3.1: Symmetry relations of the discrete-time Fourier transform of a complex sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{ca}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{ca}[n]$	$jX_{im}(e^{j\omega})$

Note: $X_{cs}(e^{j\omega})$ and $X_{ca}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{cs}[n]$ and $x_{ca}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

Table 3.2: Symmetry relations of the discrete-time Fourier transform of a real sequence.

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

Table 3.3: Commonly used discrete-time Fourier transform pairs.

Sequence	Discrete-Time Fourier Transform
$\delta[n]$	1
$1, \ (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$\mu[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\alpha^n \mu[n], \quad (\lvert\alpha\rvert < 1)$	$\frac{1}{1-\alpha e^{-j\omega}}$
$(n+1)\alpha^n \mu[n], \ (\lvert\alpha\rvert < 1)$	$\frac{1}{(1-\alpha e^{-j\omega})^2}$
$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \ (-\infty < n < \infty)$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$

Table 3.4: Discrete-time Fourier transform theorems.

Theorem	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-reversal	$g[-n]$	$G(e^{-j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation-in-frequency	$ng[n]$	$j\frac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\, d\theta$
Parseval's Relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\, d\omega$	

- **DFT tabellen**

Table 3.5: General properties of the DFT.

Type of Property	Length-N Sequence	N-point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o \rangle_N]$	$W_N^{kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o n} g[n]$	$G[\langle k - k_o \rangle_N]$
Duality	$G[n]$	$Ng[\langle -k \rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n - m \rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N} \sum_{m=0}^{N-1} G[m]H[\langle k - m \rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$	

Table 3.6: Symmetry properties of the DFT of a complex sequence.

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k \rangle_N]$
$x^*[\langle -n \rangle_N]$	$X^*[k]$
$\text{Re}\{x[n]\}$	$X_{\text{pcs}}[k] = \frac{1}{2}\{X[\langle k \rangle_N] + X^*[\langle -k \rangle_N]\}$
$j\,\text{Im}\{x[n]\}$	$X_{\text{pca}}[k] = \frac{1}{2}\{X[\langle k \rangle_N] - X^*[\langle -k \rangle_N]\}$
$x_{\text{pcs}}[n]$	$\text{Re}\{X[k]\}$
$x_{\text{pca}}[n]$	$j\,\text{Im}\{X[k]\}$

Note: $x_{\text{pcs}}[n]$ and $x_{\text{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\text{pcs}}[k]$ and $X_{\text{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

Table 3.7: Symmetry properties of the DFT of a real sequence.

Length-N Sequence	**N-point DFT**
$x[n]$	$X[k] = \mathrm{Re}\{X[k]\} + j\,\mathrm{Im}\{X[k]\}$
$x_{pe}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{po}[n]$	$j\,\mathrm{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$ $\mathrm{Re}\,X[k] = \mathrm{Re}\,X[\langle -k\rangle_N]$ $\mathrm{Im}\,X[k] = -\,\mathrm{Im}\,X[\langle -k\rangle_N]$ $\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$ $\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x_{pe}[n]$ and $x_{po}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

Table 5.1: Symmetry properties of the DFT of a complex sequence.

Length-N Sequence	**N-point DFT**
$x[n] = x_{re}[n] + jx_{im}[n]$	$X[k] = X_{re}[k] + jX_{im}[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$x_{re}[n]$	$X_{cs}[k] = \frac{1}{2}\{X[k] + X^*[\langle -k\rangle_N]\}$
jx_{im}	$X_{ca}[k] = \frac{1}{2}\{X[k] - X^*[\langle -k\rangle_N]\}$
$x_{cs}[n]$	$X_{re}[k]$
$x_{ca}[n]$	$jX_{im}[k]$

Note: $x_{cs}[n]$ and $x_{ca}[n]$ are the circular conjugate-symmetric and circular conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{cs}[k]$ and $X_{ca}[k]$ are the circular conjugate-symmetric and circular conjugate-antisymmetric parts of $X[k]$, respectively.

Table 5.2: Symmetry properties of the DFT of a length-N real sequence.

Length-N Sequence	**N-point DFT**
$x[n] = x_{\mathrm{ev}}[n] + x_{\mathrm{od}}[n]$	$X[k] = X_{\mathrm{re}}[k] + jX_{\mathrm{im}}[k]$
$x_{\mathrm{ev}}[n]$	$X_{\mathrm{re}}[k]$
$x_{\mathrm{od}}[n]$	$jX_{\mathrm{im}}[k]$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$ $X_{\mathrm{re}}[k] = X_{\mathrm{re}}[\langle -k\rangle_N]$ $X_{\mathrm{im}}[k] = -X_{\mathrm{im}}[\langle -k\rangle_N]$ $\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$ $\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x_{\mathrm{ev}}[n]$ and $x_{\mathrm{od}}[n]$ are the circular even and circular odd parts of $x[n]$, respectively.

Table 5.3: DFT Theorems.

Theorem	**Length-N Sequence**	**N-point DFT**
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o\rangle_N]$	$W_N^{-kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o} g[n]$	$G[\langle k - k_o\rangle_N]$
Duality	$G[n]$	$Ng[\langle -k\rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n - m\rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k - m\rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert g[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert G[k]\rvert^2$	

- $\sum_{n=0}^{N} a^n = \frac{1- a^{N+1}}{1-a}$

- **DFT: x[n] is periodiek met periode N**

$$X[k] = \sum_{n=0}^{N-1} x[n].e^{-\frac{j.2.\pi.k.n}{N}}$$

- **IDFT**

$$x[n] = \frac{1}{N}.\sum_{k=0}^{N-1} X[k].e^{+\frac{j.2.\pi.k.n}{N}}$$

- **Circulaire convolutie**

$y_c[n] = g[n] Ⓝ h[n]$

$= \sum_{k=0}^{N-1} g[k].h[< n - k >_N]$

Als: $0 \leq n - k$:
$h[< n - k >_N] = h[n - k]$

Als: $n - k < 0$:
$h[< n - k >_N] = h[N + n - k]$

- **Z-getransformeerde**

$$X(\rho, e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n].\rho^{-n}.e^{-j.\omega.n}$$

Stel $z = \rho.e^{+j.\omega}$

$X(z) = \sum_{n=-\infty}^{\infty} x[n].z^{-n}$

Table 3.8: Some commonly used z-transform pairs.

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n\mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n\cos\omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$(r^n\sin\omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Table 6.1: Some commonly used z-transform pairs.

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n\mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\,\alpha^n\mu[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(n+1)\,\alpha^n\mu[n]$	$\dfrac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n\cos\omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$(r^n\sin\omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$