

Dealing with Separation in Logistic Regression Models

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paper, data, and code at
crain.co/research

Dealing with Separation in Logistic Regression Models

The prior matters a lot,
so choose a good one.

The prior matters a lot,
so choose a good one.

43 times larger

The prior matters a lot,
so choose a good one.

43^{million} times larger

The prior matters a lot,

so choose a good one.

The prior matters a lot,

1. in practice

so choose a good one.

The prior matters a lot,

1. in practice
2. in theory

so choose a good one.

The prior matters a lot,

1. in practice

2. in theory

so choose a good one.

3. concepts

The prior matters a lot,

1. in practice

2. in theory

so choose a good one.

3. concepts

4. software

The Prior Matters
in Practice





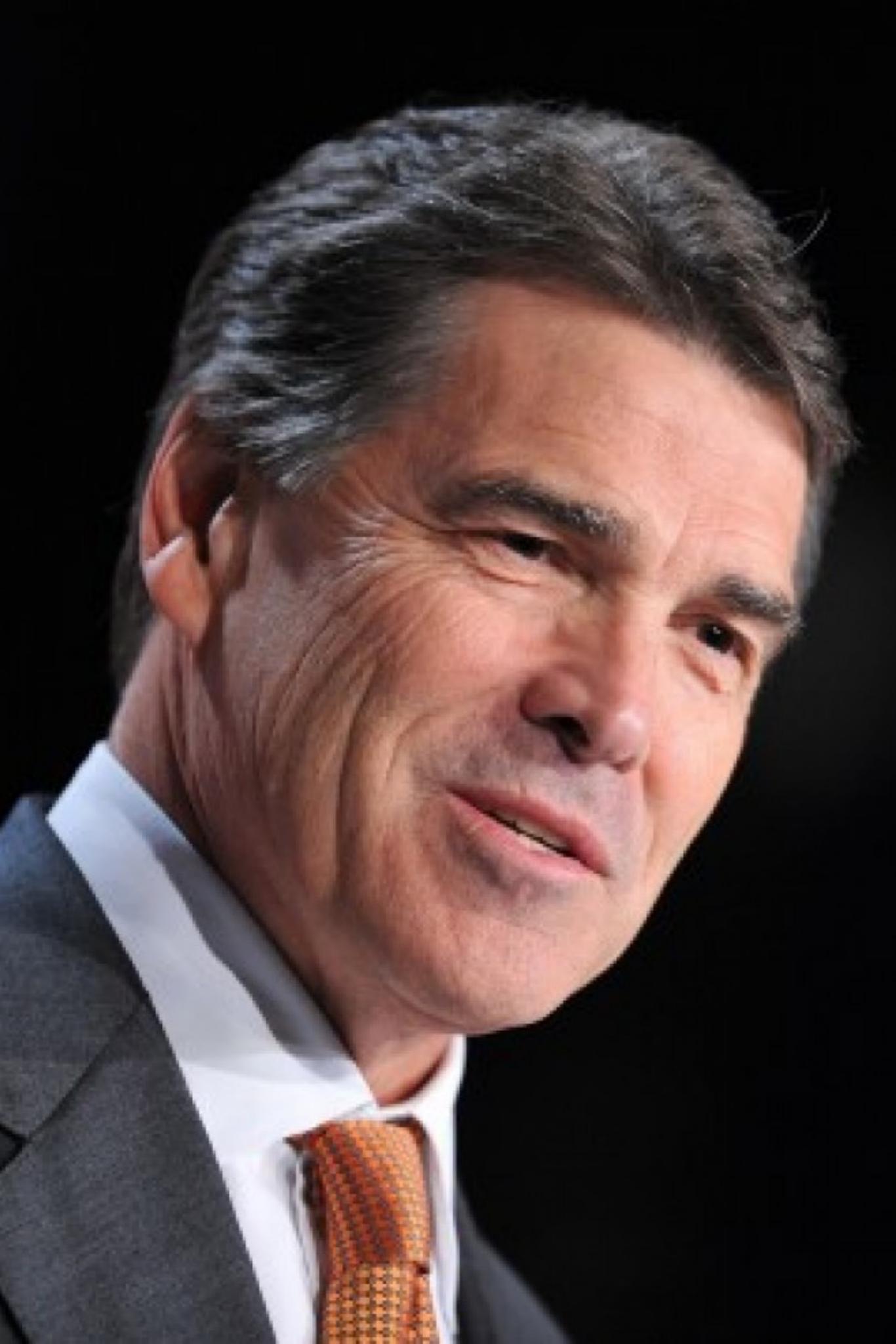


2 million

3,000

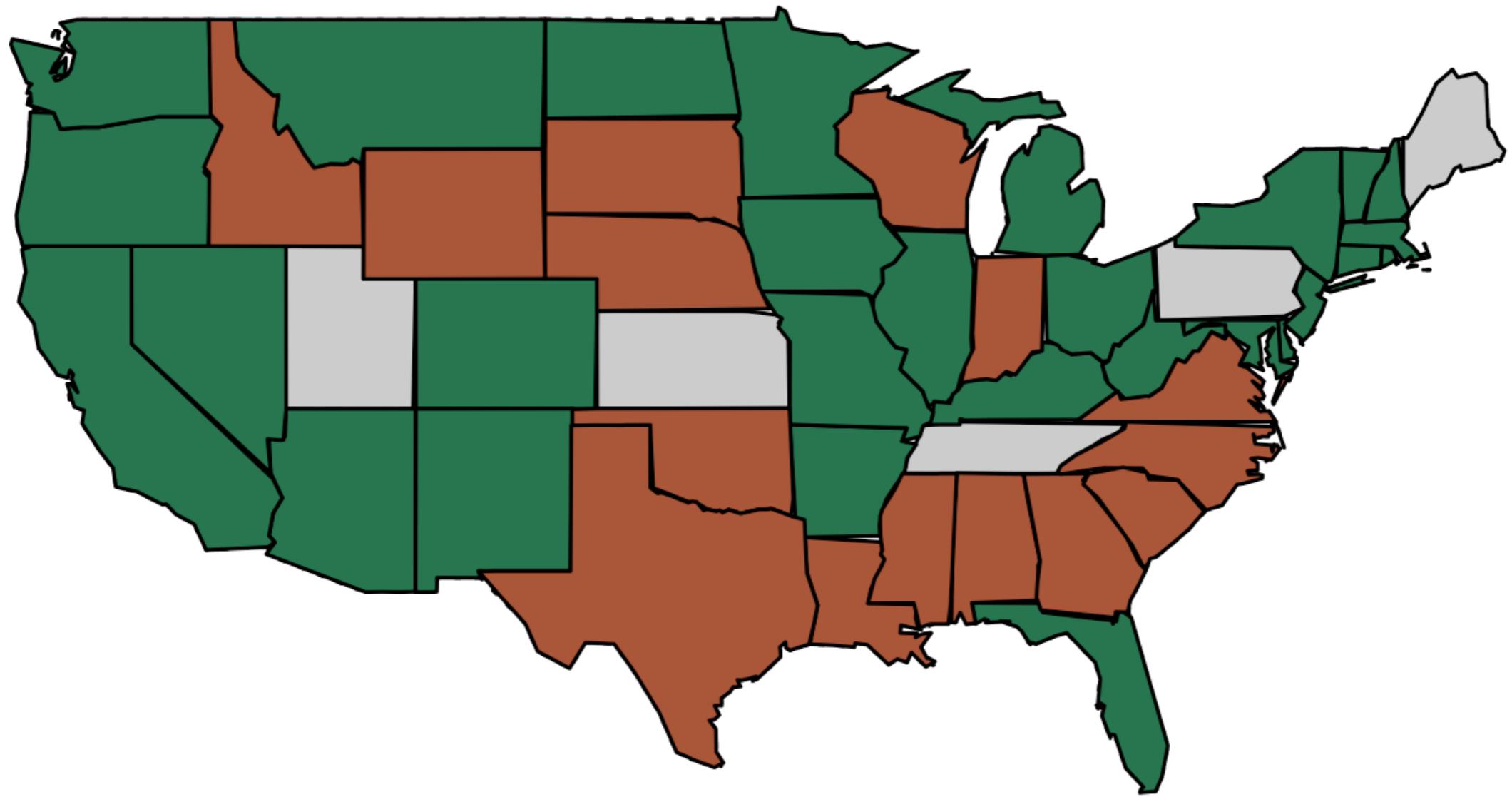
100%

90%



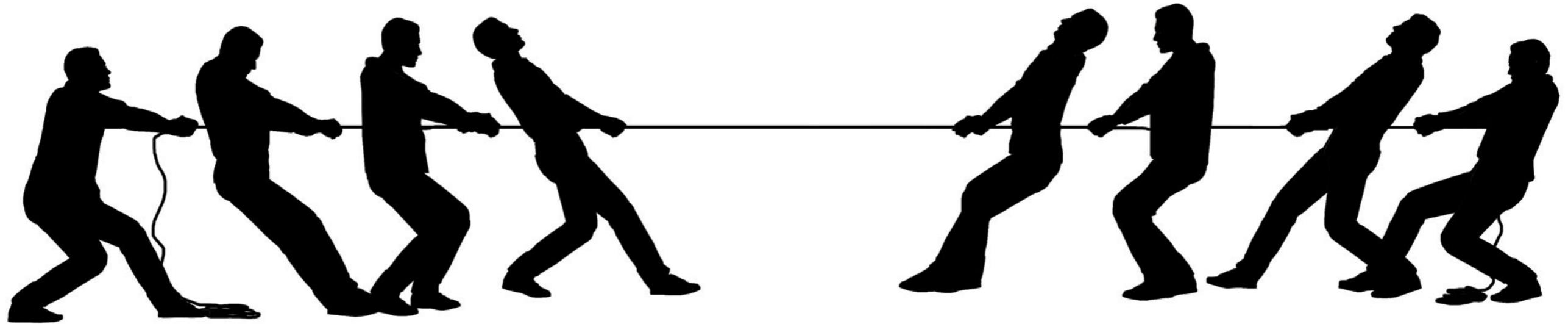
“To expand this program is not
unlike adding a thousand
people to the Titanic.”

— July 2012



politics

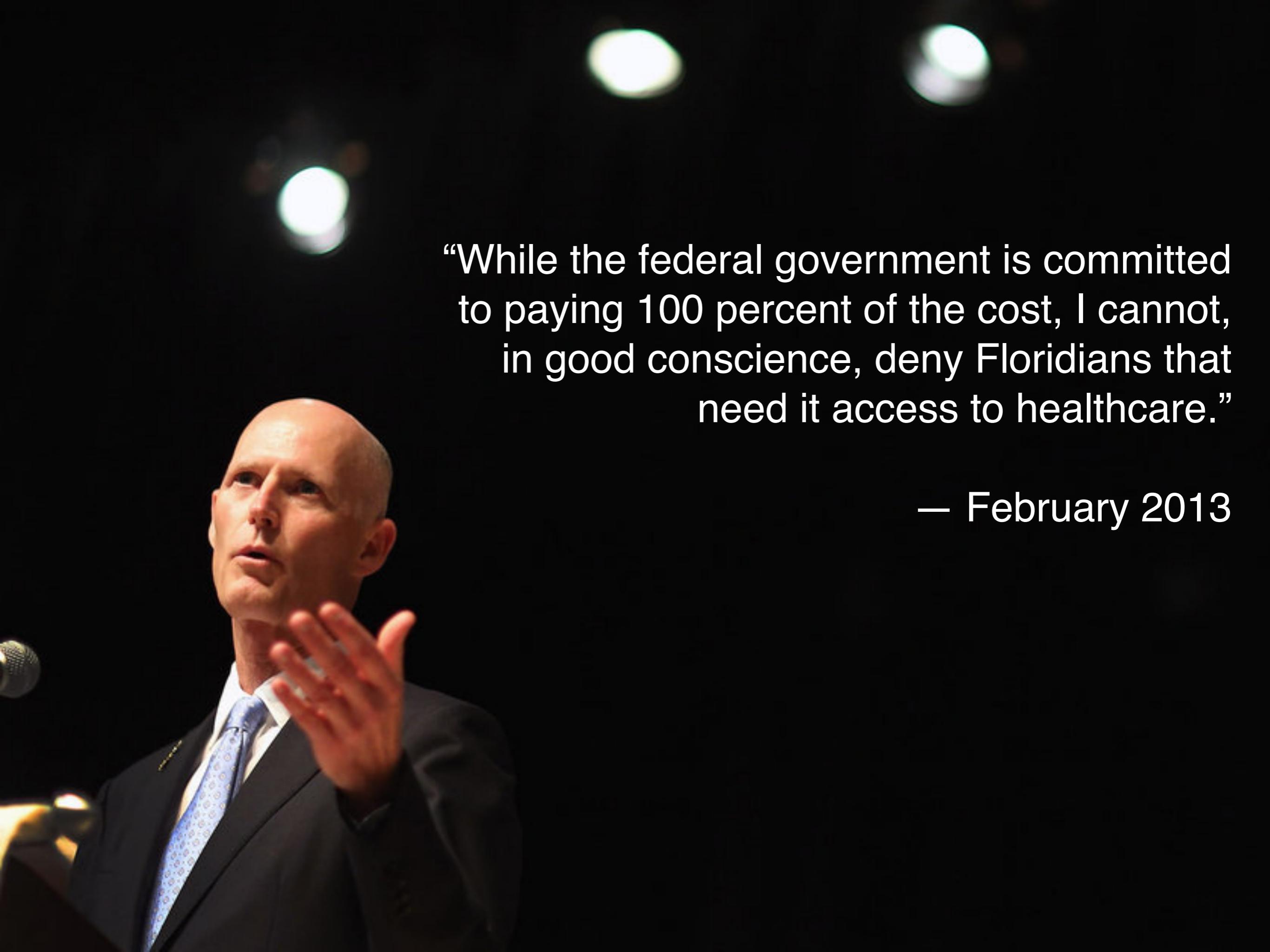
need



“Obamacare is going to be horrible for patients. It’s going to be horrible for taxpayers. It’s probably the biggest job killer ever.”

– October 2010



A photograph of a man with a shaved head, wearing a dark suit, white shirt, and patterned tie. He is gesturing with his right hand while speaking into a microphone. The background is dark with three bright spotlights visible.

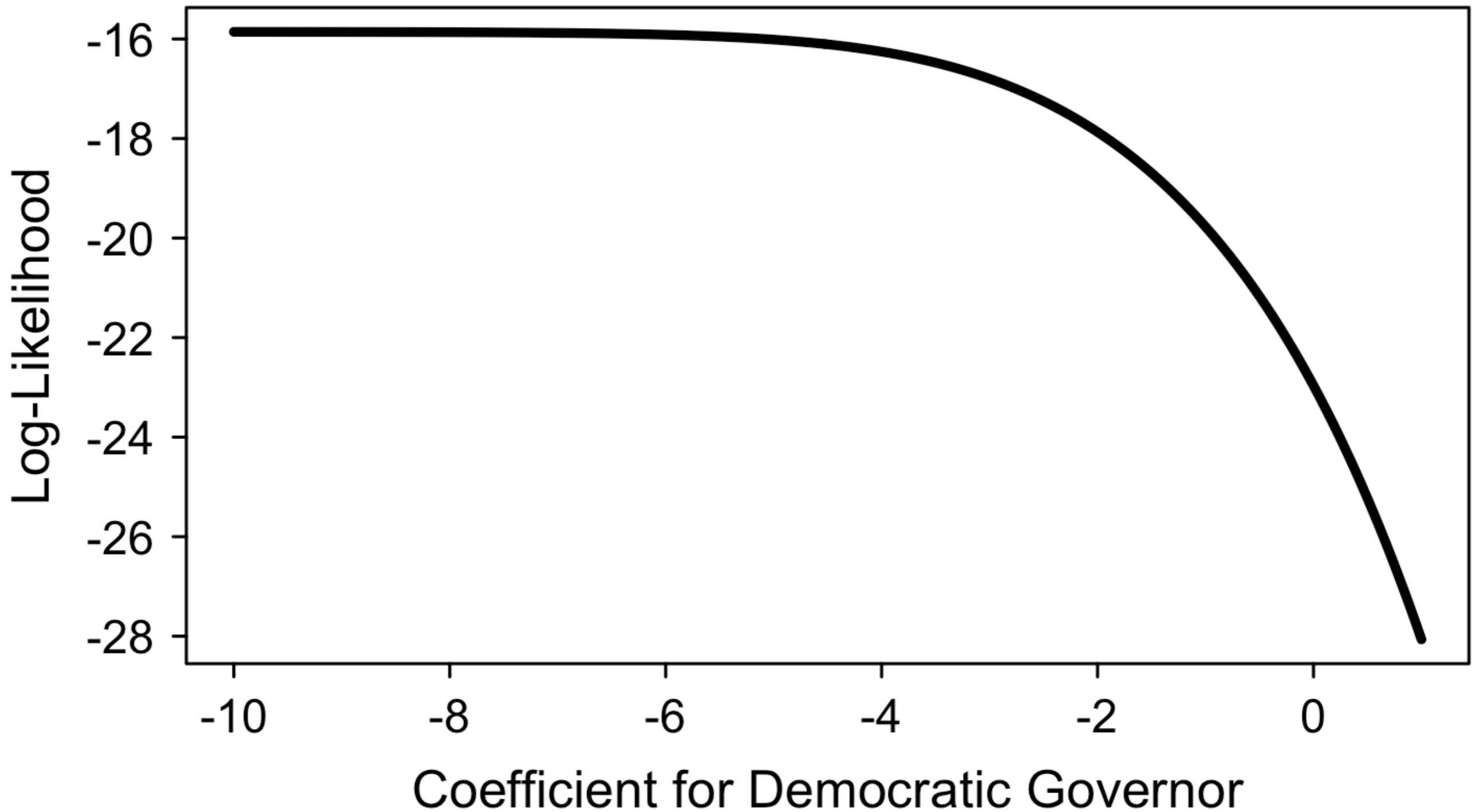
“While the federal government is committed to paying 100 percent of the cost, I cannot, in good conscience, deny Floridians that need it access to healthcare.”

— February 2013

In the tug-of-war between politics and need,
which one wins?

<i>Variable</i>	<i>Coefficient</i>	<i>Confidence Interval</i>
Democratic Governor	-20.35	[-6,340.06; 6,299.36]
% Uninsured (Std.)	0.92	[-3.46; 5.30]
% Favorable to ACA	0.01	[-0.17; 0.18]
GOP Legislature	2.43	[-0.47; 5.33]
Fiscal Health	0.00	[-0.02; 0.02]
Medicaid Multiplier	-0.32	[-2.45; 1.80]
% Non-white	0.05	[-0.12; 0.21]
% Metropolitan	-0.08	[-0.17; 0.02]
Constant	2.58	[-7.02; 12.18]

	Doesn't Oppose	Opposes
Republican	14	16
Democrat	20	0



<i>Variable</i>	<i>Coefficient</i>	<i>Confidence Interval</i>
Democratic Governor	-26.35	[-126,979.03; 126,926.33]
% Uninsured (Std.)	0.92	[-3.46; 5.30]
% Favorable to ACA	0.01	[-0.17; 0.18]
GOP Legislature	2.43	[-0.47; 5.33]
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↑
unreasonable

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↑ ↑

unreasonable *useless*

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unreasonable useless

This is a failure of maximum likelihood.

Jeffreys' Prior

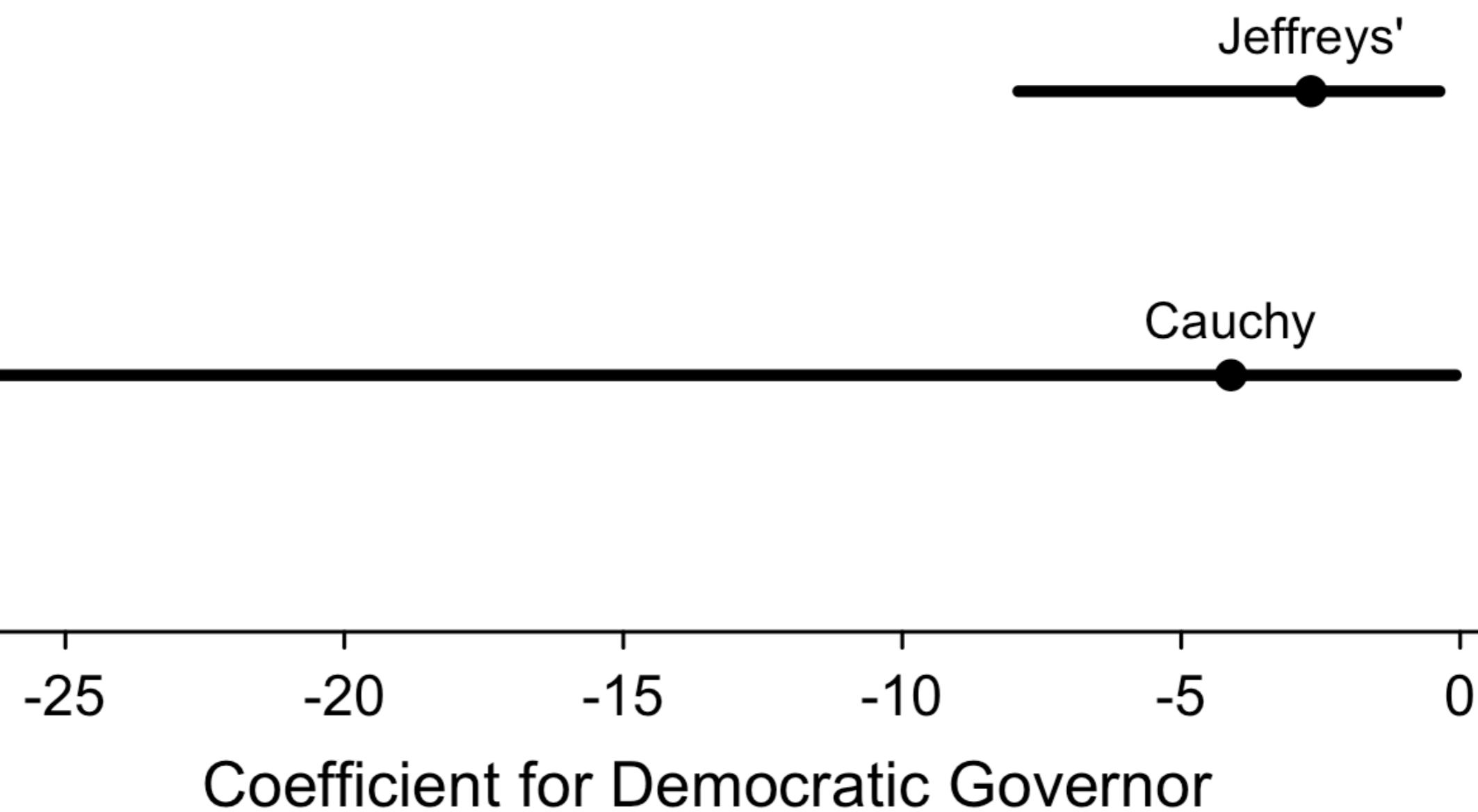
Zorn (2005)



Coefficient for Democratic Governor

Cauchy Prior

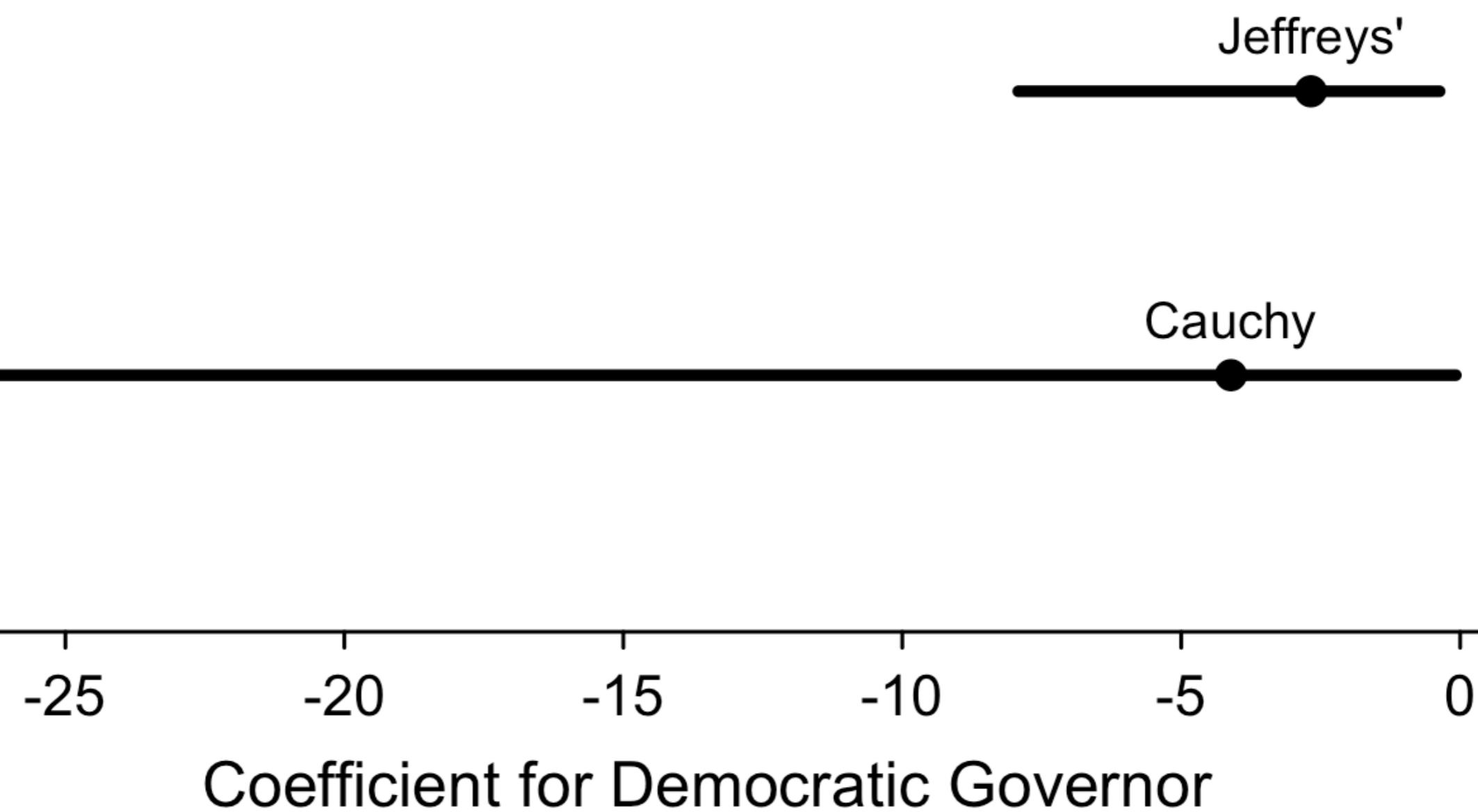
Gelman et al. (2008)



The Cauchy prior produces...

a confidence interval that is

250% wider



The Cauchy prior produces...

a coefficient estimate that is

50% larger

The Cauchy prior produces...

a risk-ratio estimate that is

43 million times larger

Different *default* priors
produce different results.

The Prior Matters
in Theory

For

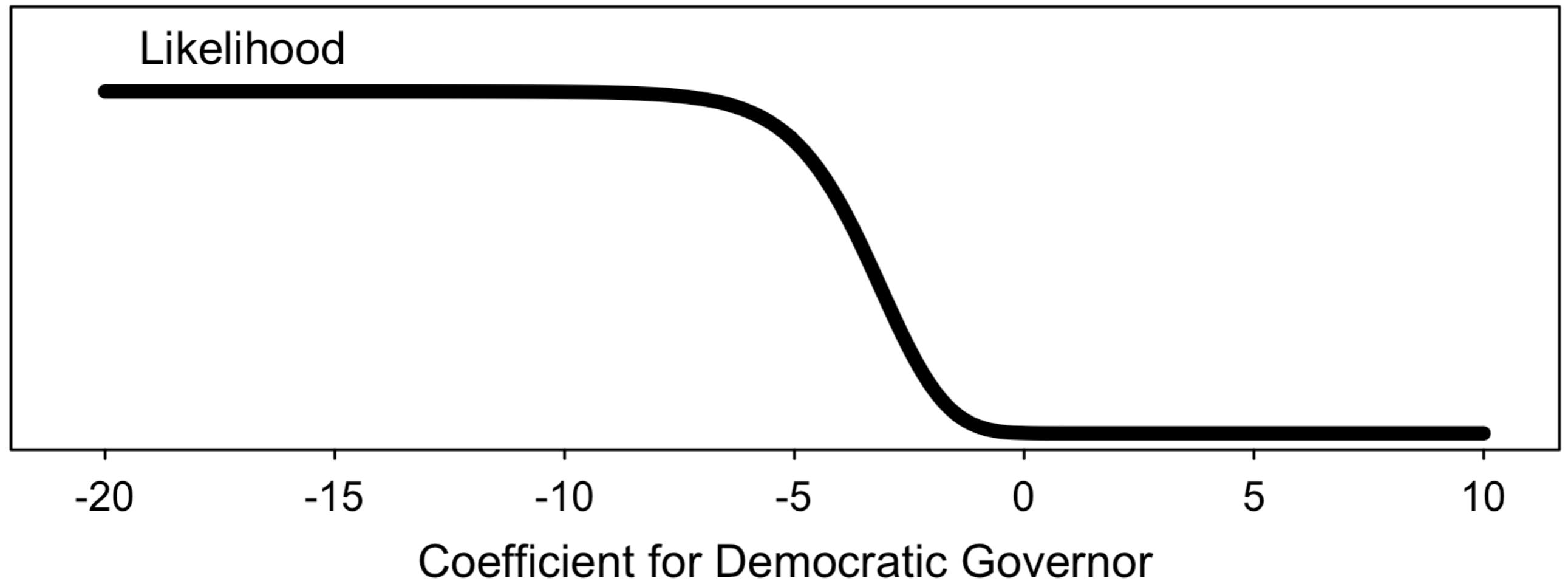
1. a monotonic likelihood $p(y|\beta)$ decreasing in β_s ,
2. a proper prior distribution $p(\beta|\sigma)$, and
3. a large, negative β_s ,

the posterior distribution of β_s is proportional to the prior distribution for β_s , so that $p(\beta_s|y) \propto p(\beta_s|\sigma)$.

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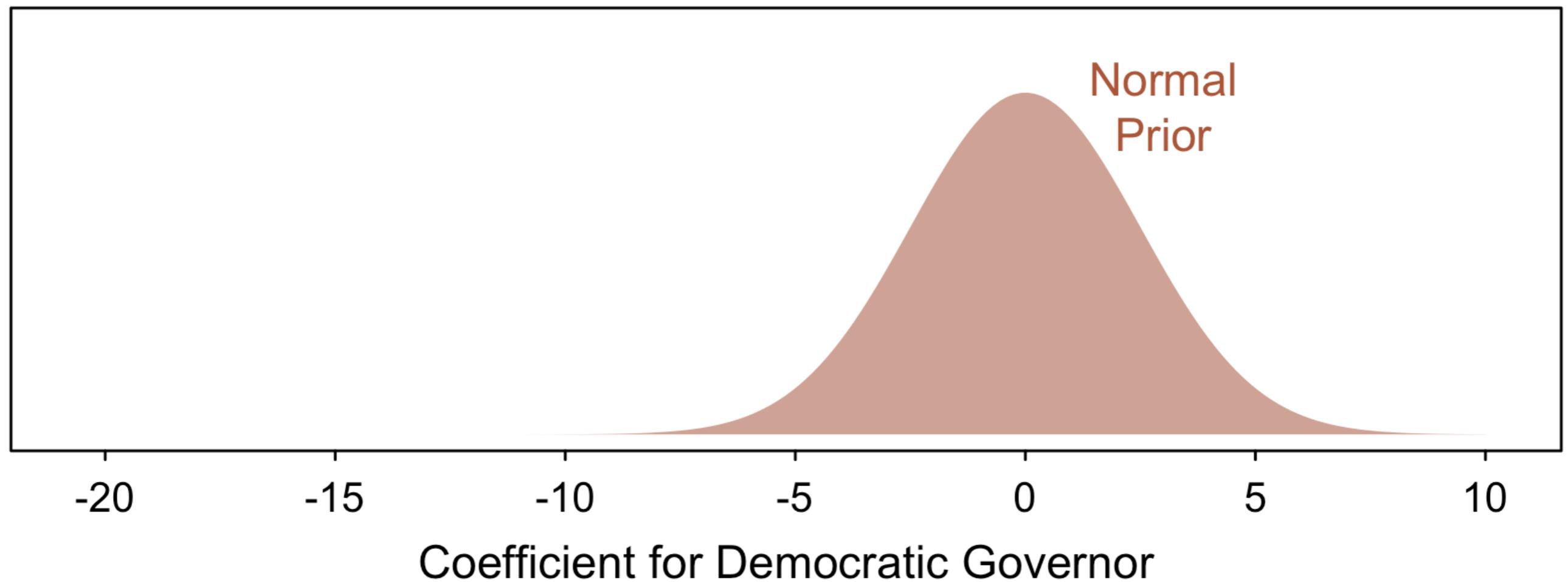
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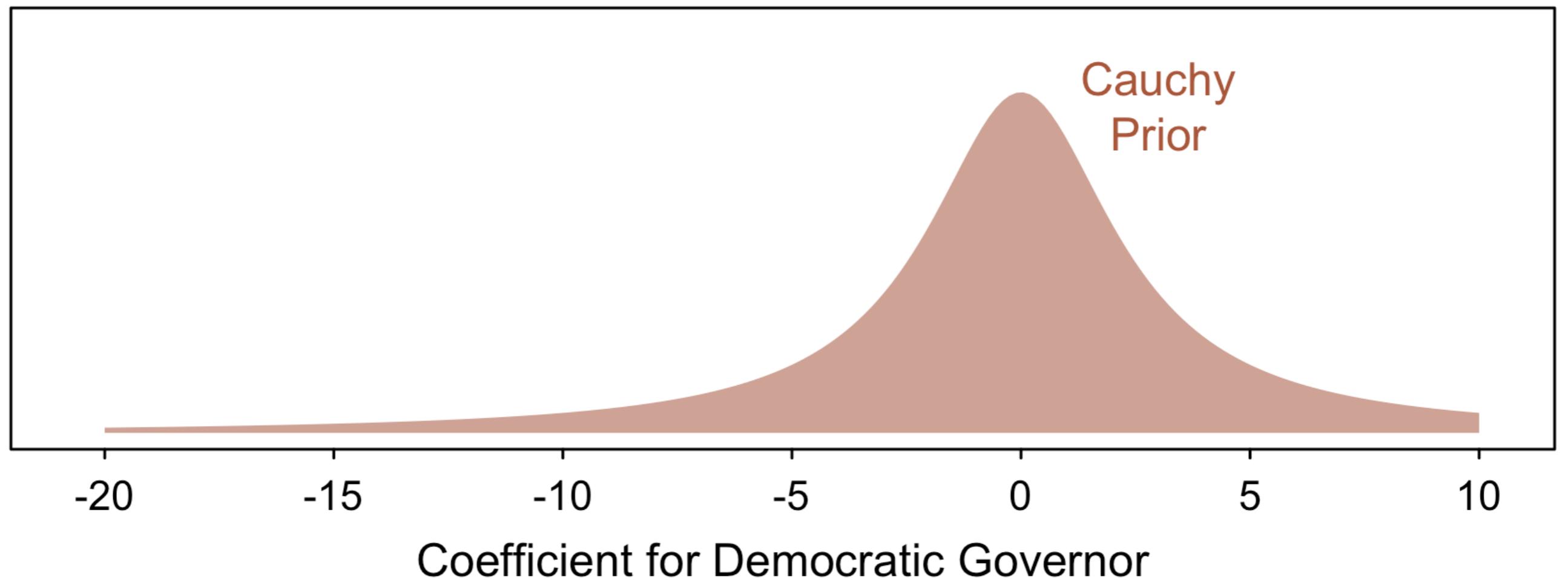
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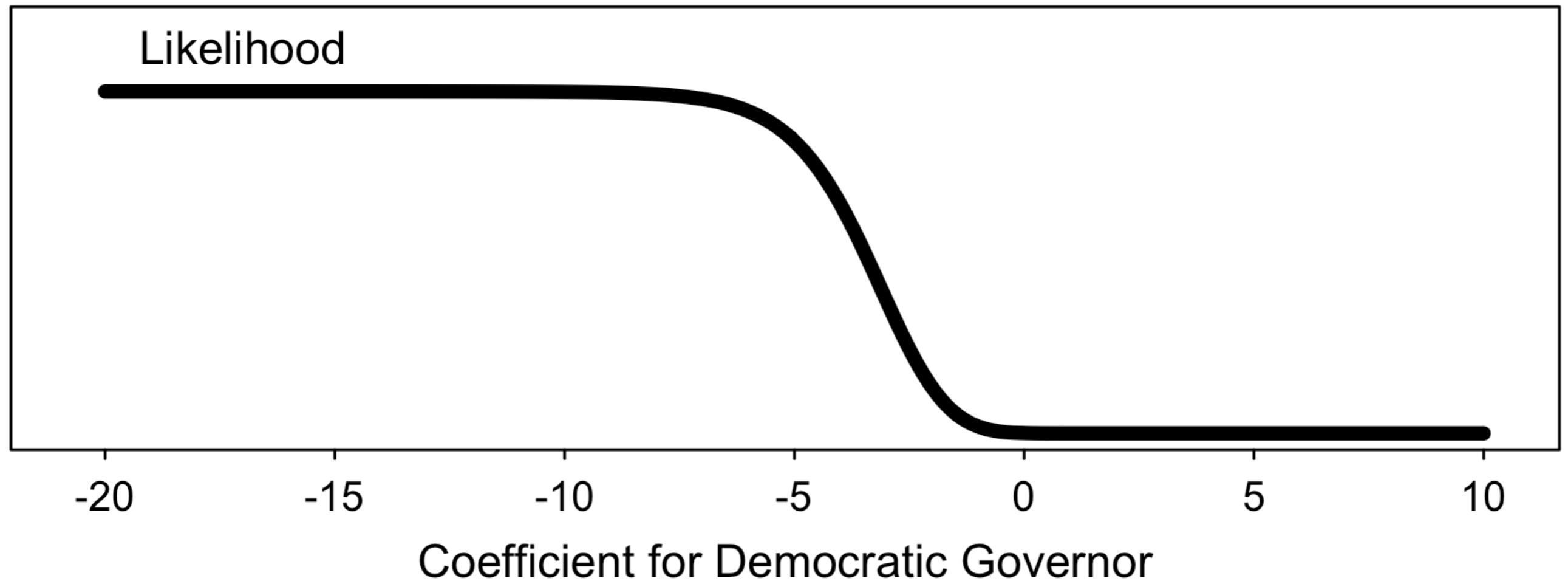
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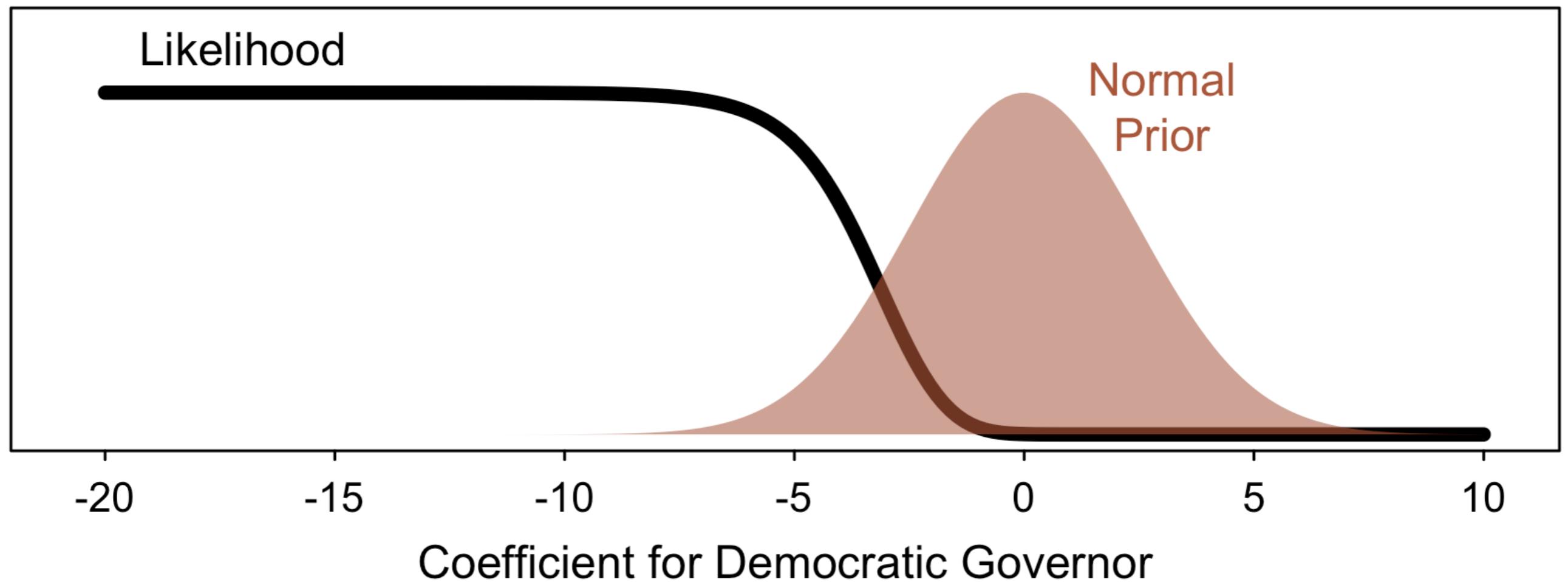
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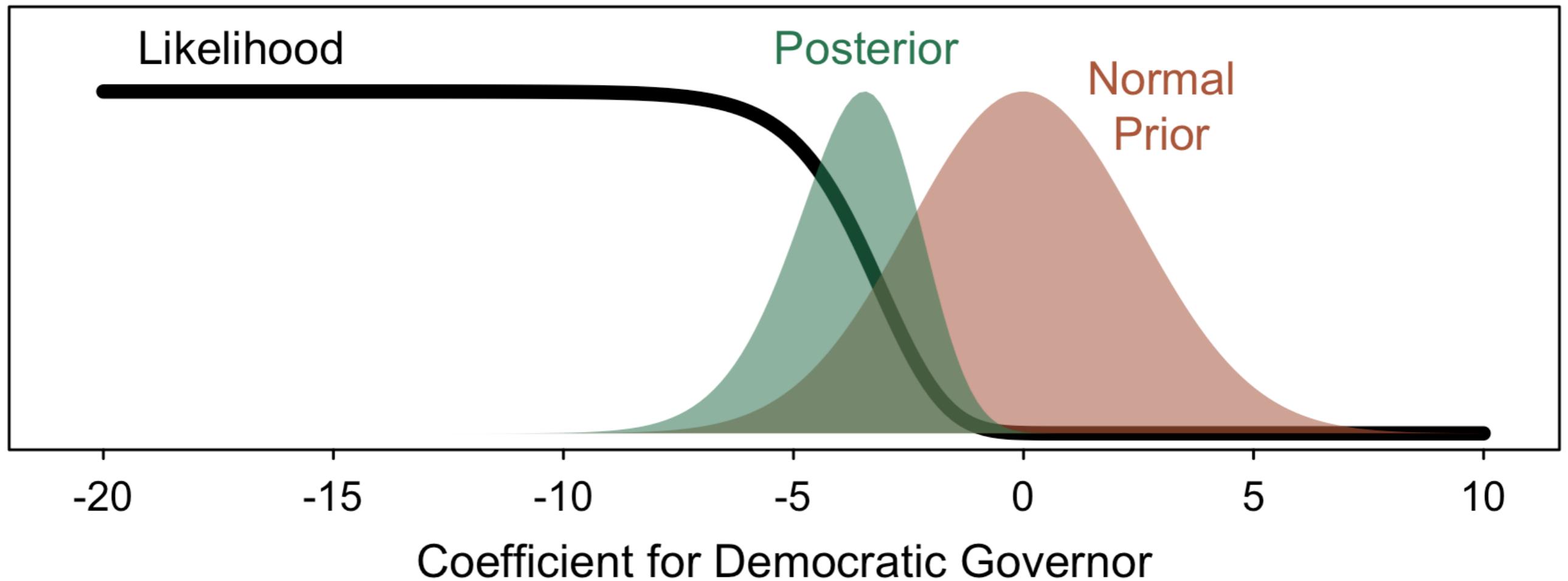
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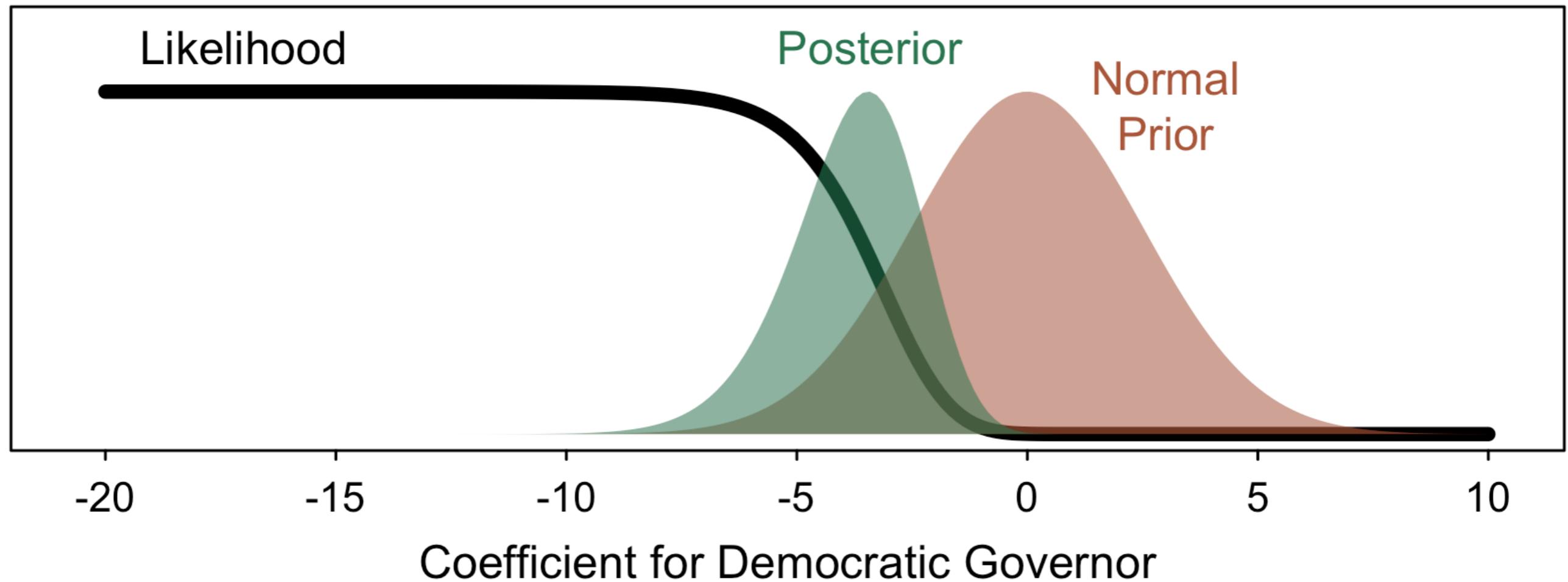
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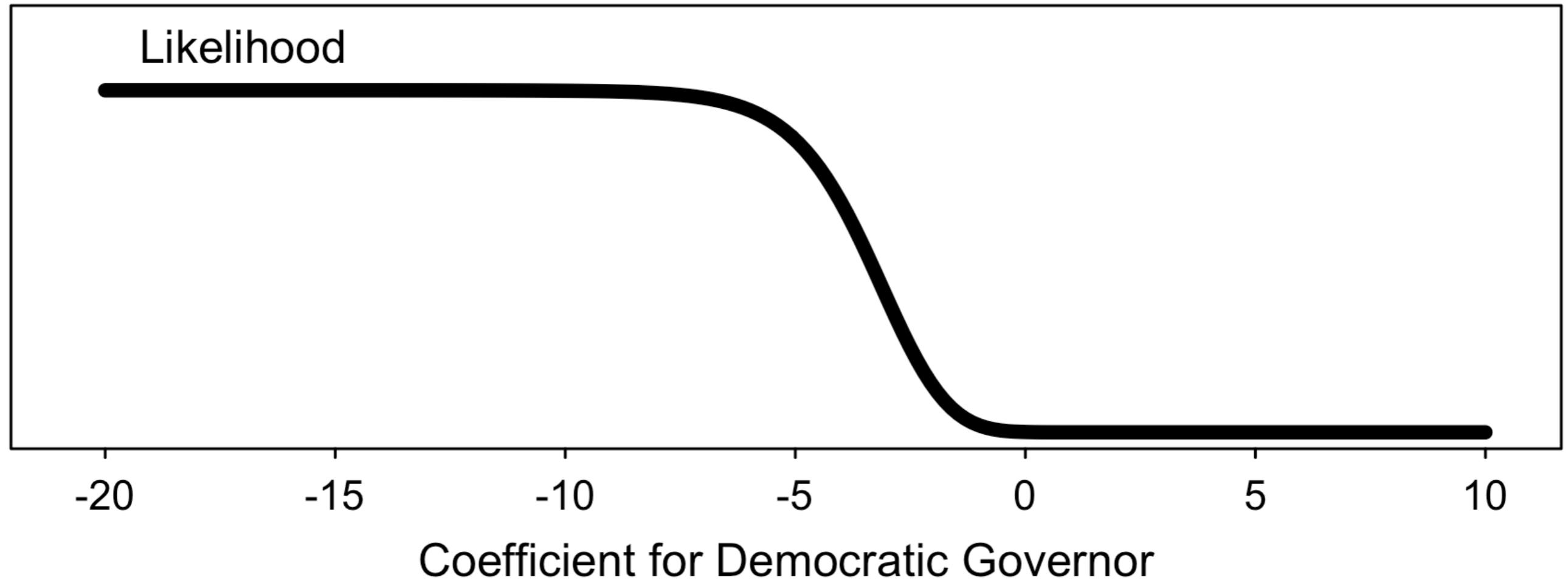
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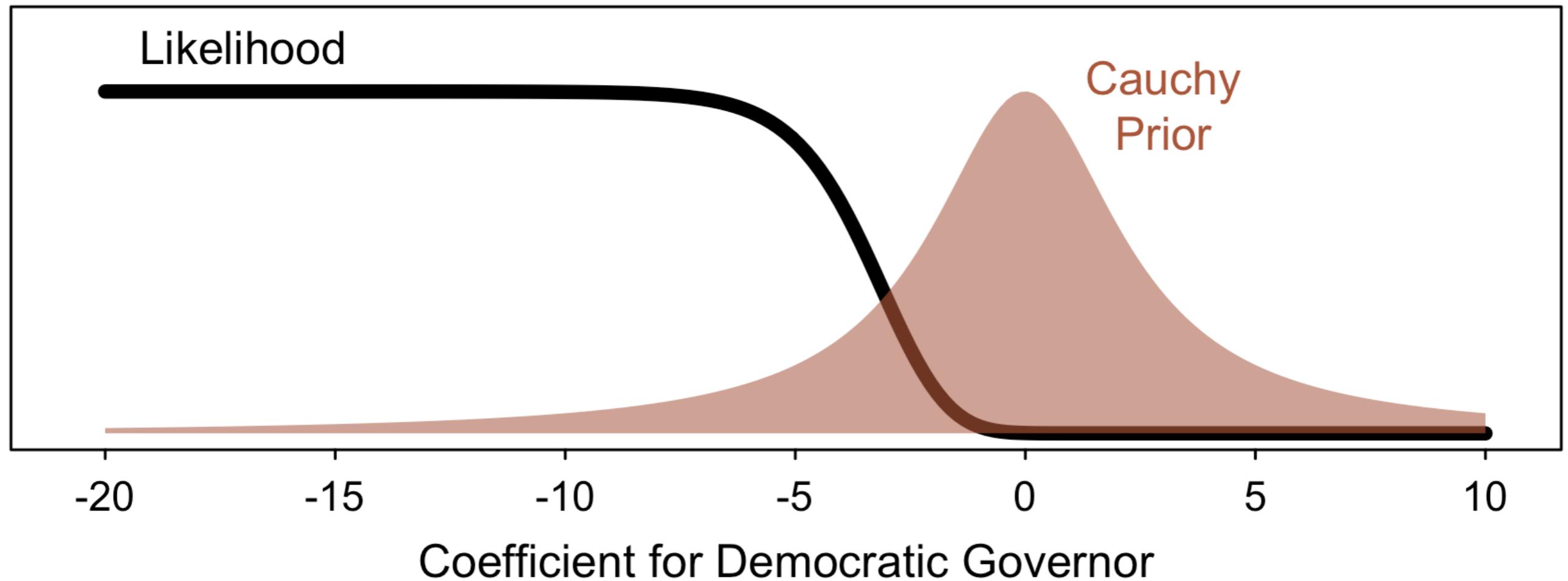
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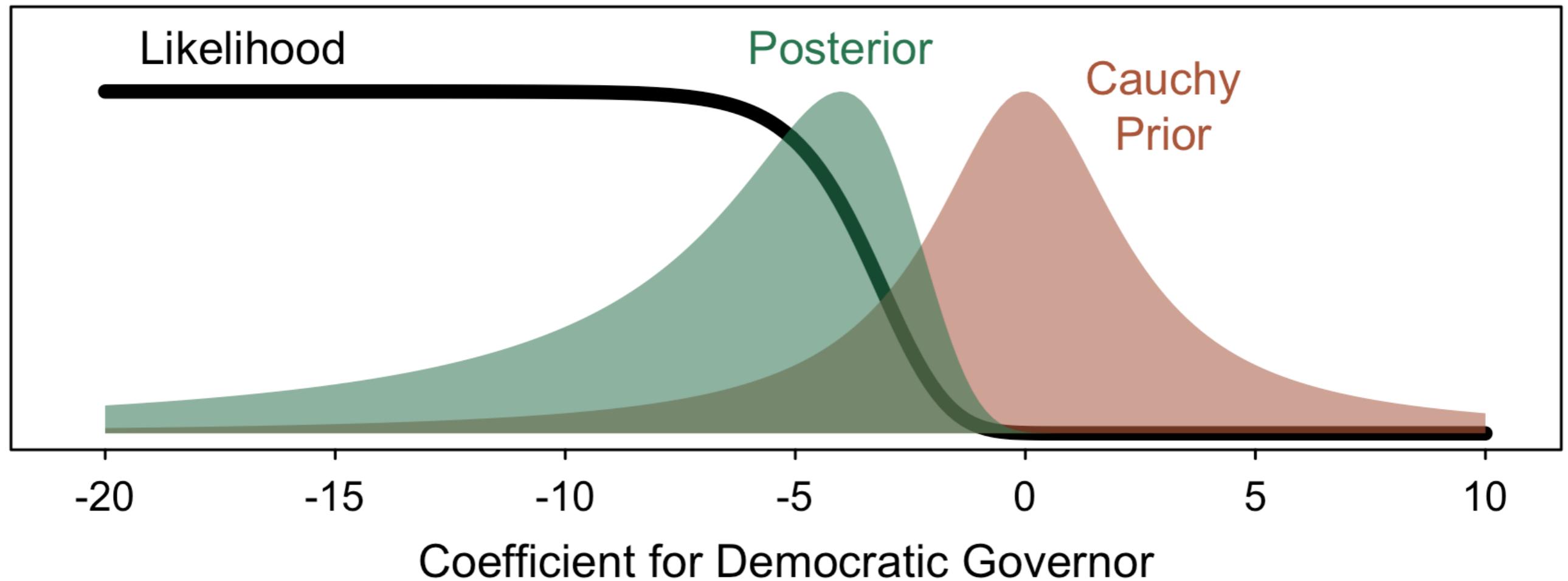
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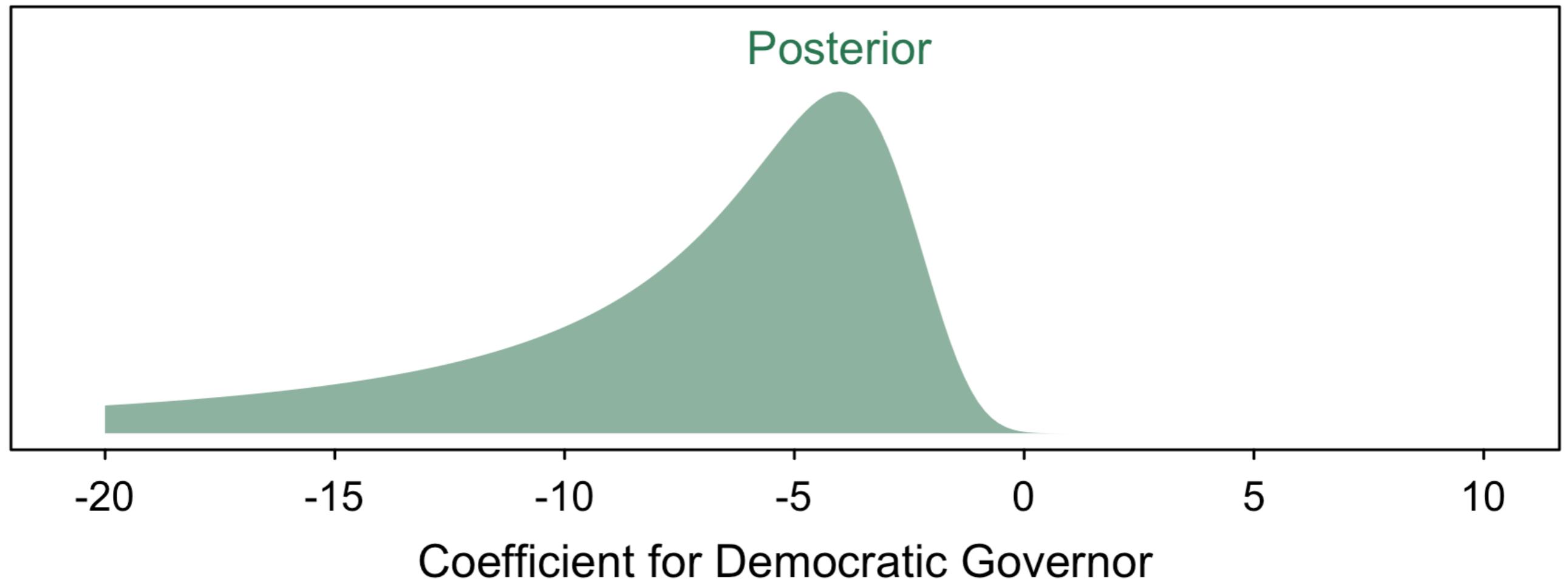
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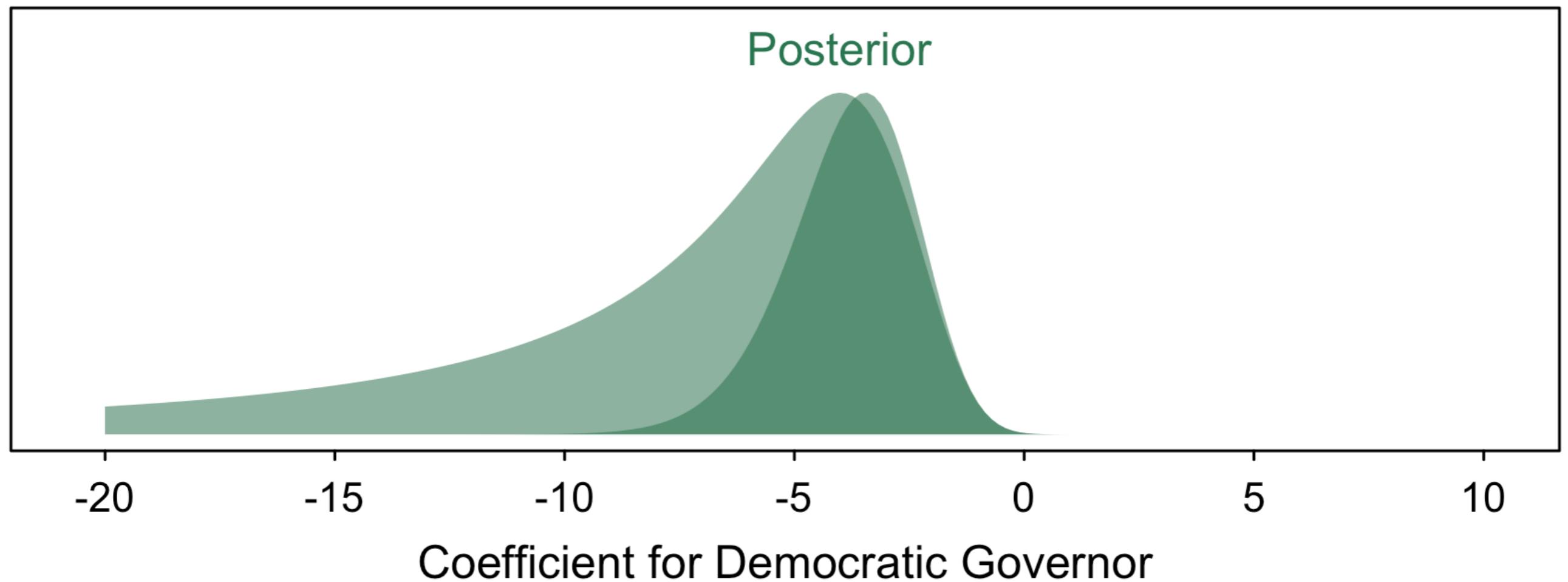
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The prior *determines*
crucial parts of the posterior.

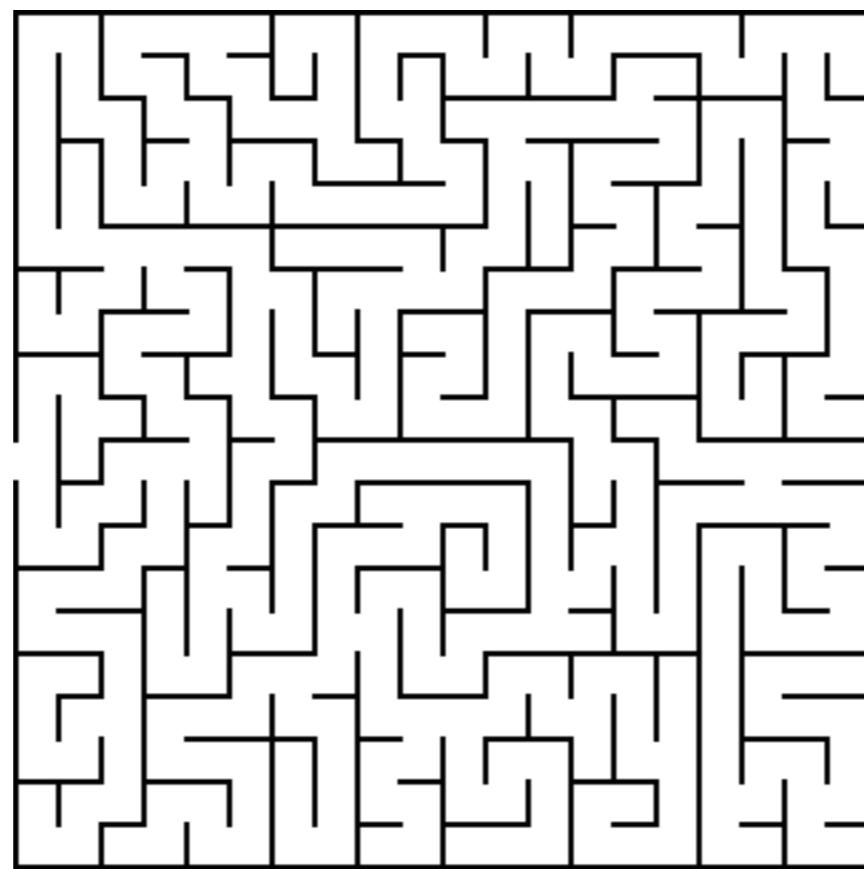
Key Concepts

for Choosing a Good Prior

$$Pr(y_i) = \Lambda(\beta_c + \beta_s s_i + \beta_1 x_{i1} + ... + \beta_k x_{ik})$$

Prior Predictive Distribution

$$p(y_{new}) = \int_{-\infty}^{\infty} p(y_{new} | \beta) p(\beta) d(\beta)$$



$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1k} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \dots & \sigma_{2k} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \dots & \sigma_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \sigma_{k3} & \dots & \sigma_{kk} \end{pmatrix}$$

simplify

We Already Know Few Things

We Already Know Few Things

$$\beta_1 \approx \hat{\beta}_1^{mle}$$

$$\beta_2 \approx \hat{\beta}_2^{mle}$$

⋮

$$\beta_k \approx \hat{\beta}_k^{mle}$$

We Already Know Few Things

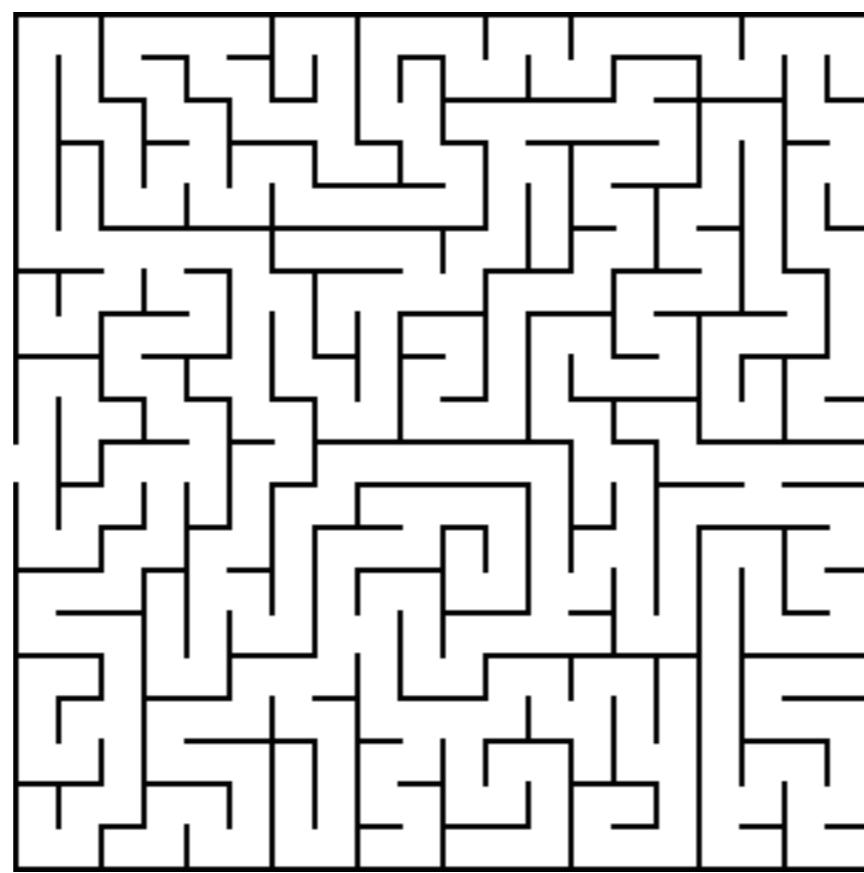
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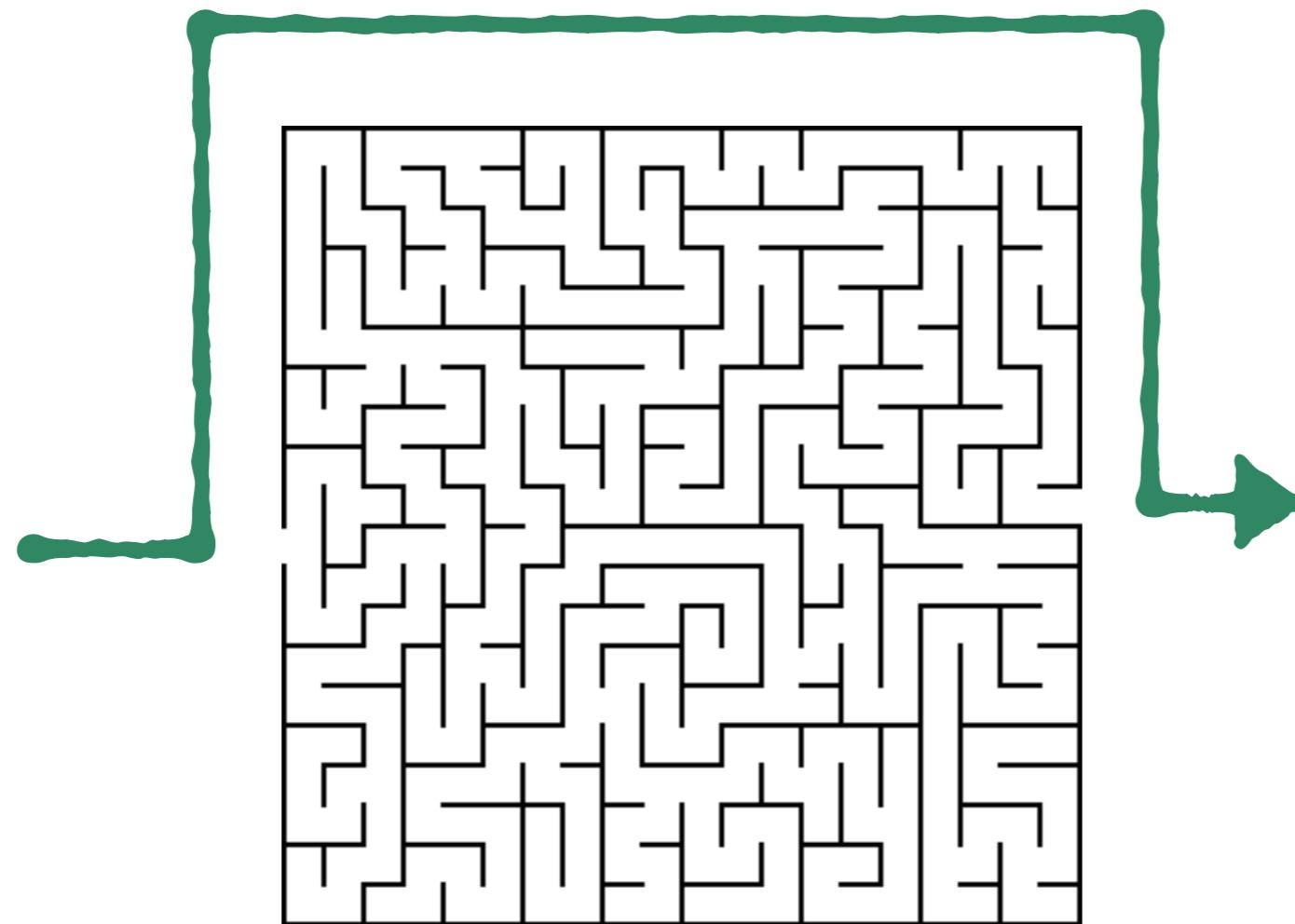
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$$\beta_s < 0$$

$$\beta_k \approx \hat{\beta}_k^{mle}$$



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Partial Prior Predictive Distribution

$$p^*(y_{new}) = \int_{-\infty}^0 p(y_{new} | \beta_s, \hat{\beta}_{-s}^{mle}) p(\beta_s | \beta_s \leq 0) d(\beta_s)$$

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1. Choose a prior distribution $p(\beta_s)$.
2. Estimate the model coefficients $\hat{\beta}^{mle}$.
3. For i in 1 to n_{sims} , do the following:
 - (a) Simulate $\tilde{\beta}_s^{[i]} \sim p(\beta_s)$.
 - (b) Replace $\hat{\beta}_s^{mle}$ in $\hat{\beta}^{mle}$ with $\tilde{\beta}_s^{[i]}$, yielding the vector $\tilde{\beta}^{[i]}$.
 - (c) Calculate and store the quantity of interest $\tilde{q}^{[i]} = q(\tilde{\beta}^{[i]})$.
4. Keep only the simulations in the direction of the separation.
5. Summarize the simulations \tilde{q} using quantiles, histograms, or density plots.
6. If the prior is inadequate, then update the prior distribution $p(\beta_s)$.

Example

Nuclear Weapons and War

Table I. The Ratio of the Probability of War in a Nonnuclear Dyad to That in a Nuclear Dyad According to (1) GEE and (2) Firth Logit, Including and Excluding the Kargil War, along with 95 Percent Confidence Intervals.^a

	(1) GEE		(2) Firth logit	
	Estimate	95 percent confidence interval	Estimate	95 percent confidence interval
Kargil excluded	2,717,000	[893,000, 8,531,000]	1.606	[0.088, 30.079]
Kargil included	0.693	[0.545, 9.563]	0.471	[0.077, 2.985]

Note: GEE = generalized estimating equation. All other covariates are held constant at their median values. An estimate of 1 indicates equal probability of war in nuclear and nonnuclear dyads. Ratios are simulated as recommended by King, Tomz, and Wittenberg (2000) and are based on 1,000 simulations.

^aAll models use the same battery of control variables used by Rauchhaus: contiguity, distance between states, state capabilities, presence of an alliance, a dummy for major powers, democracy, economic interdependence, and intergovernmental organization membership. Full regression tables for all models run are included in the online appendix.

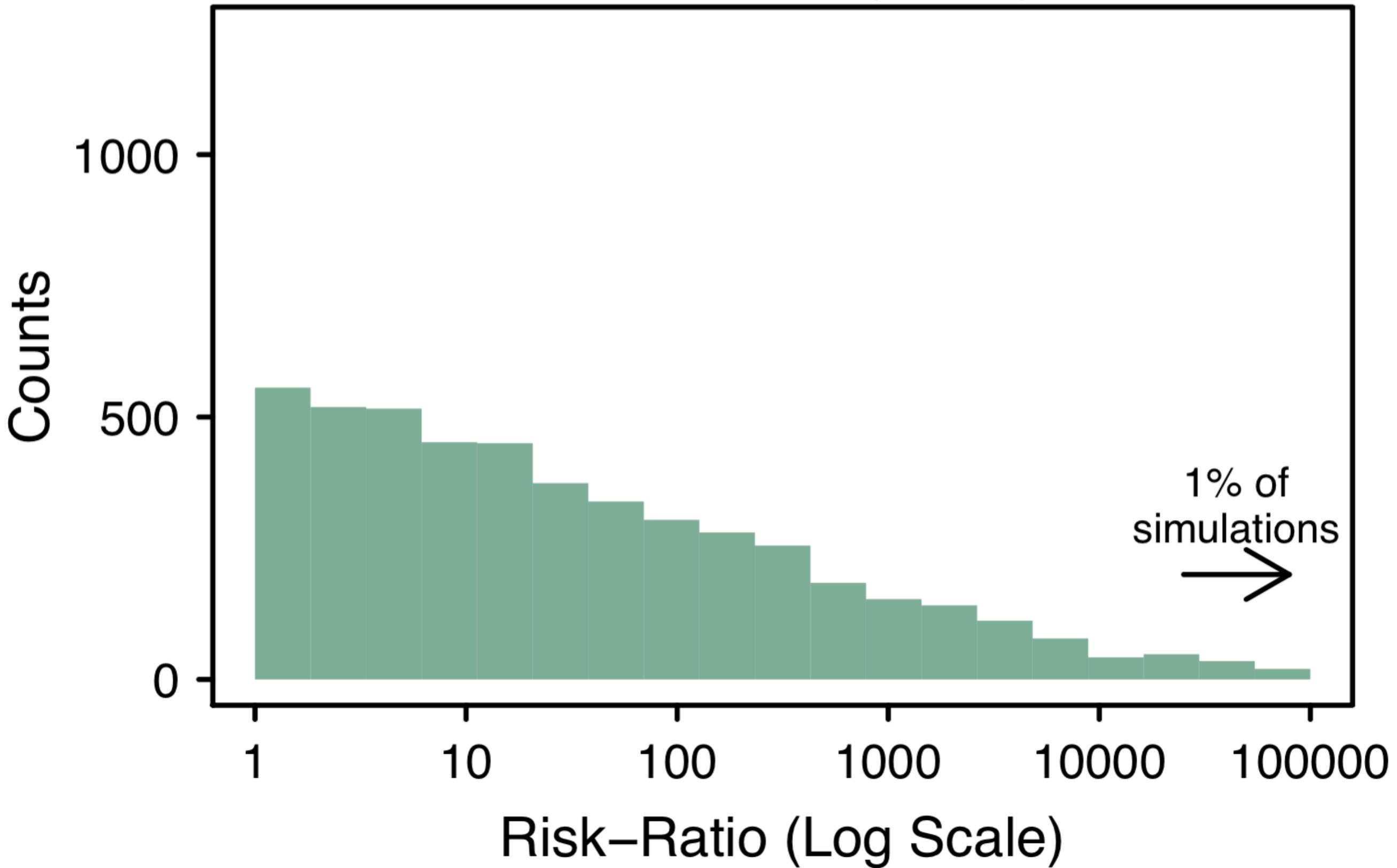
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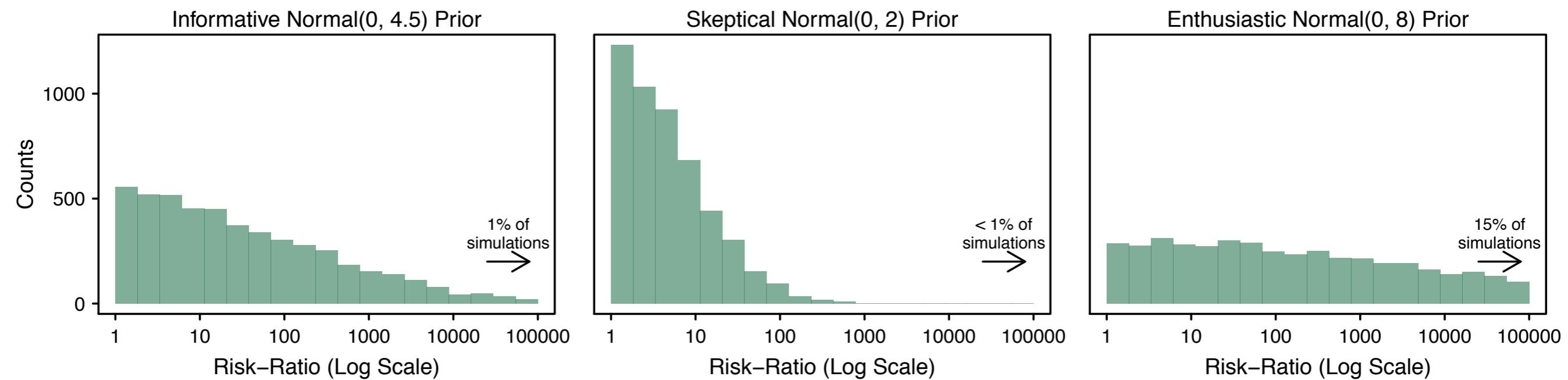
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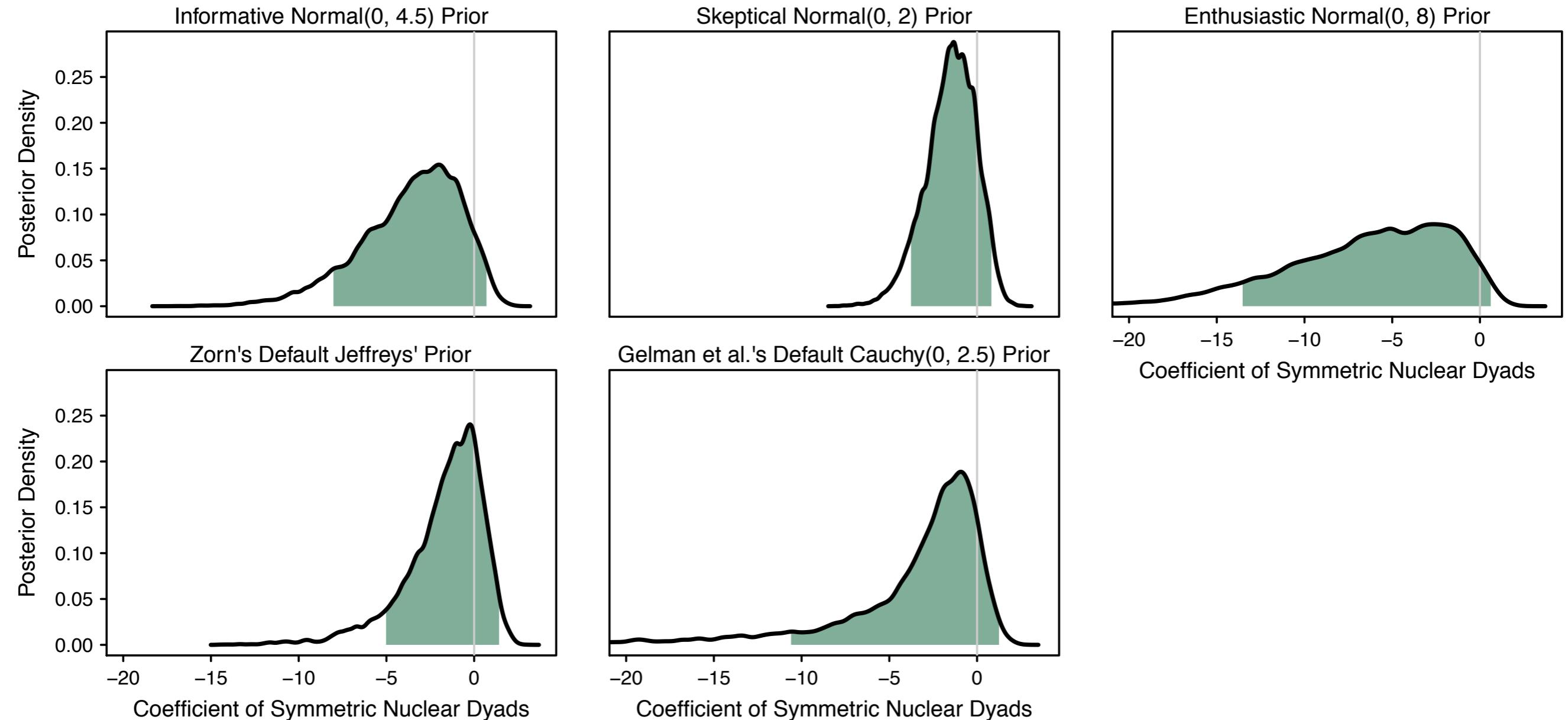
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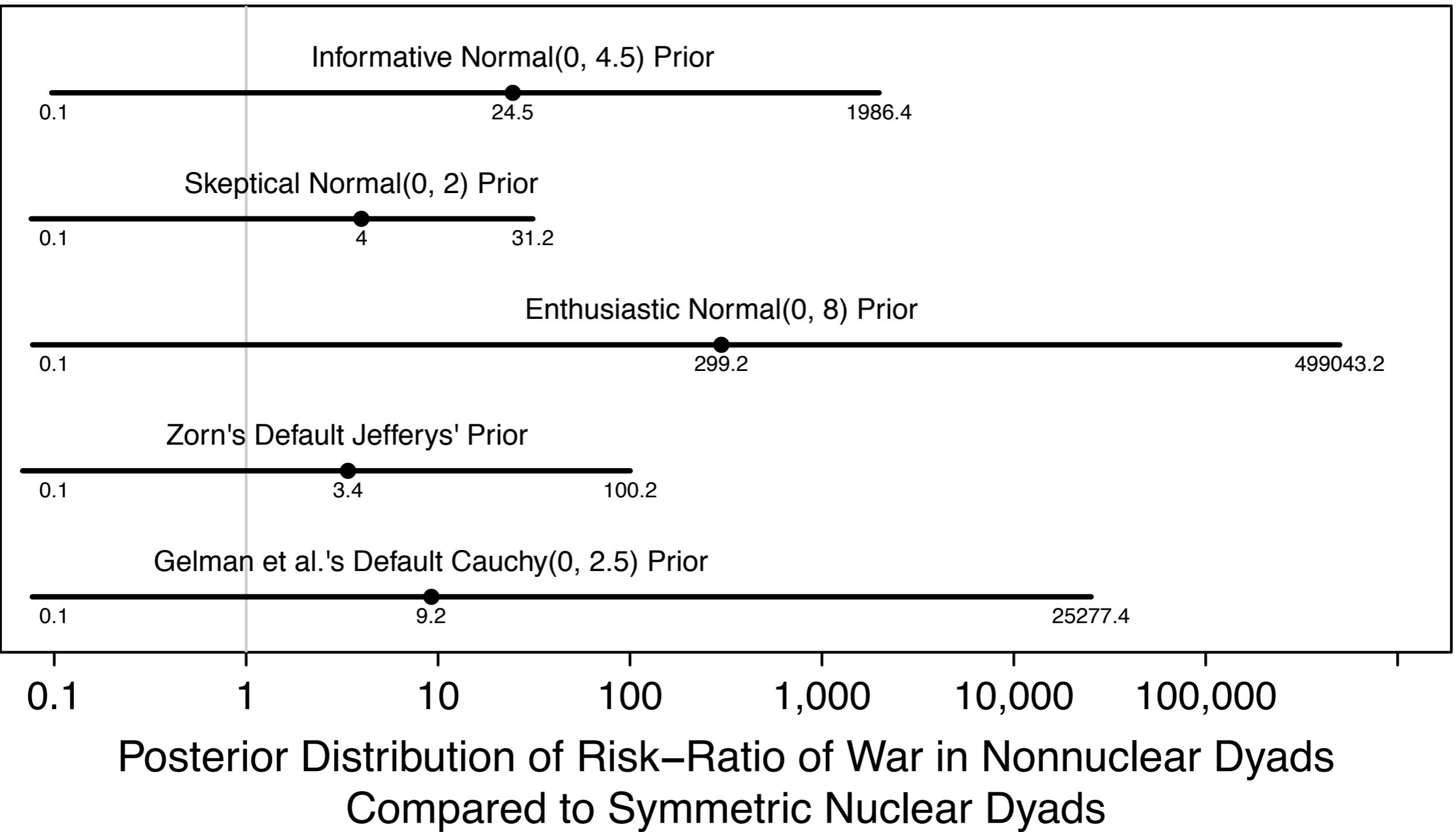
Informative Normal($0, 4.5$) Prior



The prior matters,
so *robustness checks*
are critical.







Software for Choosing a Good Prior

separation
(on GitHub)

crain.co/example

```
# install packages
devtools::install_github("carlislerainey/compactr")
devtools::install_github("carlislerainey/separation")

# load packages
library(separation)
library(arm) # for rescale()

# load and recode data
data(politics_and_need)
d <- politics_and_need
d$dem_governor <- 1 - d$gop_governor
d$st_percent_uninsured <- rescale(d$percent_uninsured)

# formula to use throughout
f <- oppose_expansion ~ dem_governor +
  percent_favorable_aca + gop_leg +
  st_percent_uninsured + bal2012 +
  multiplier + percent_nonwhite +
  percent_metro
```

Workflow

Workflow

1. Calculate the PPPD: `calc_pppd()`

Workflow

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2. Simulate from the posterior: `sim_post_*`

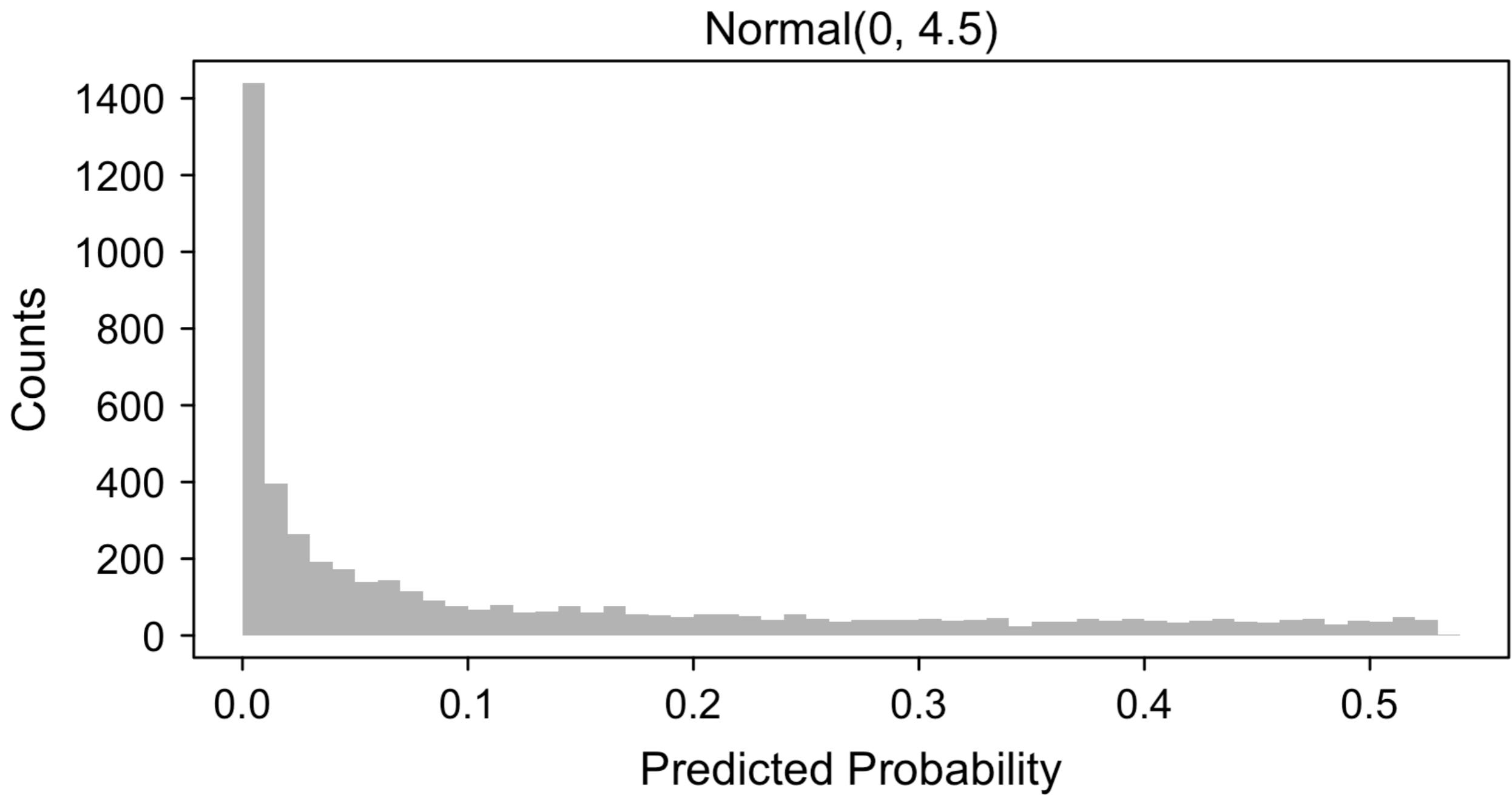
Workflow

1. Calculate the PPPD: `calc_pppd()`
2. Simulate from the posterior: `sim_post_*`()
3. Calculate quantities of interest: `calc_qi()`

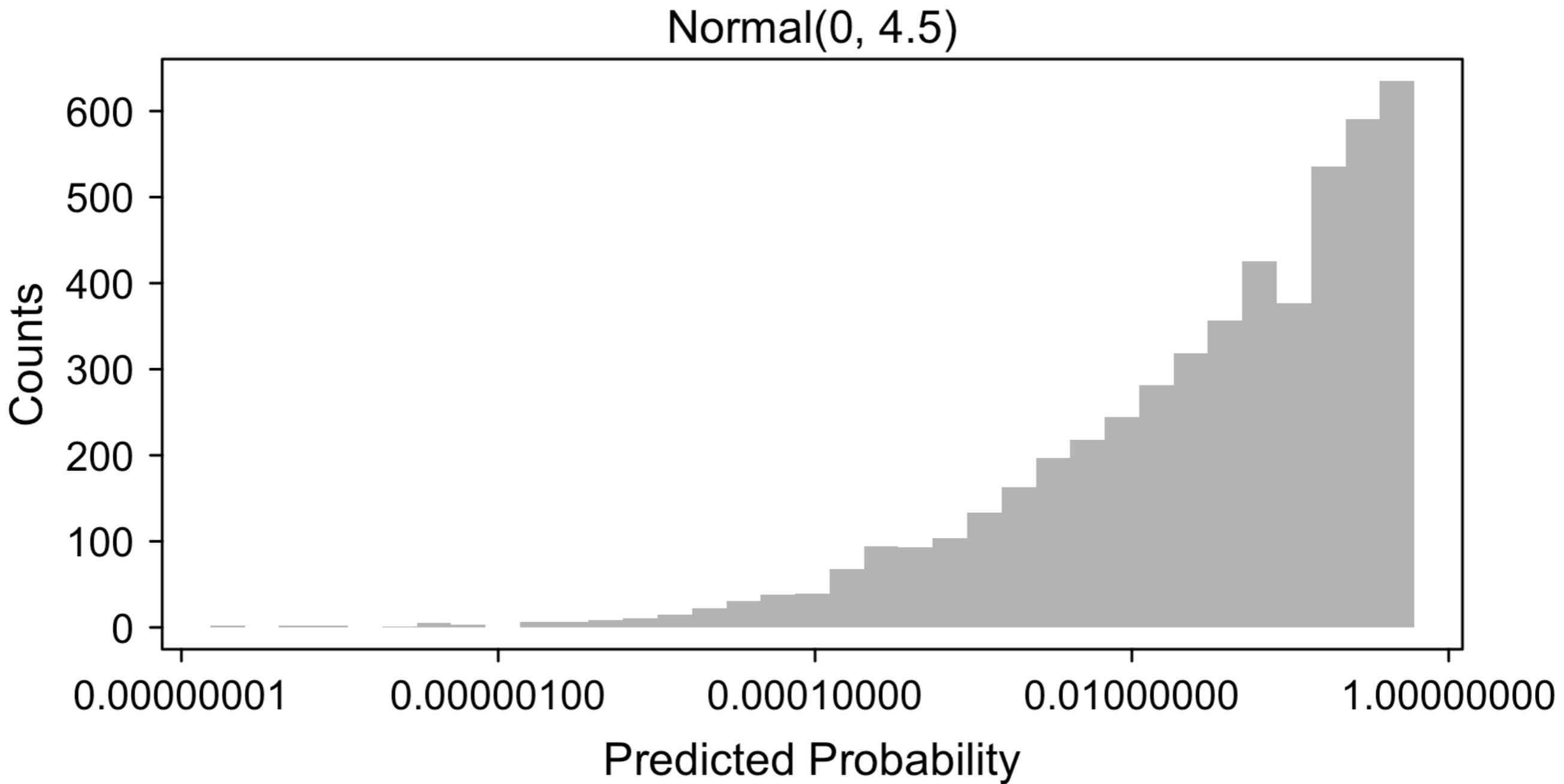
`calc_ppd()`

```
# informative prior
prior_sims_4.5 <- rnorm(10000, 0, 4.5)
pppd <- calc_pppd(formula = f,
                     data = d,
                     prior_sims = prior_sims_4.5,
                     sep_var_name = "dem_governor",
                     prior_label = "Normal(0, 4.5)")
```

```
plot(pppd)
```



```
plot(pppd, log_scale = TRUE)
```



`sim_post_normal()`

`sim_post_gelman()`

`sim_post_jeffreys()`

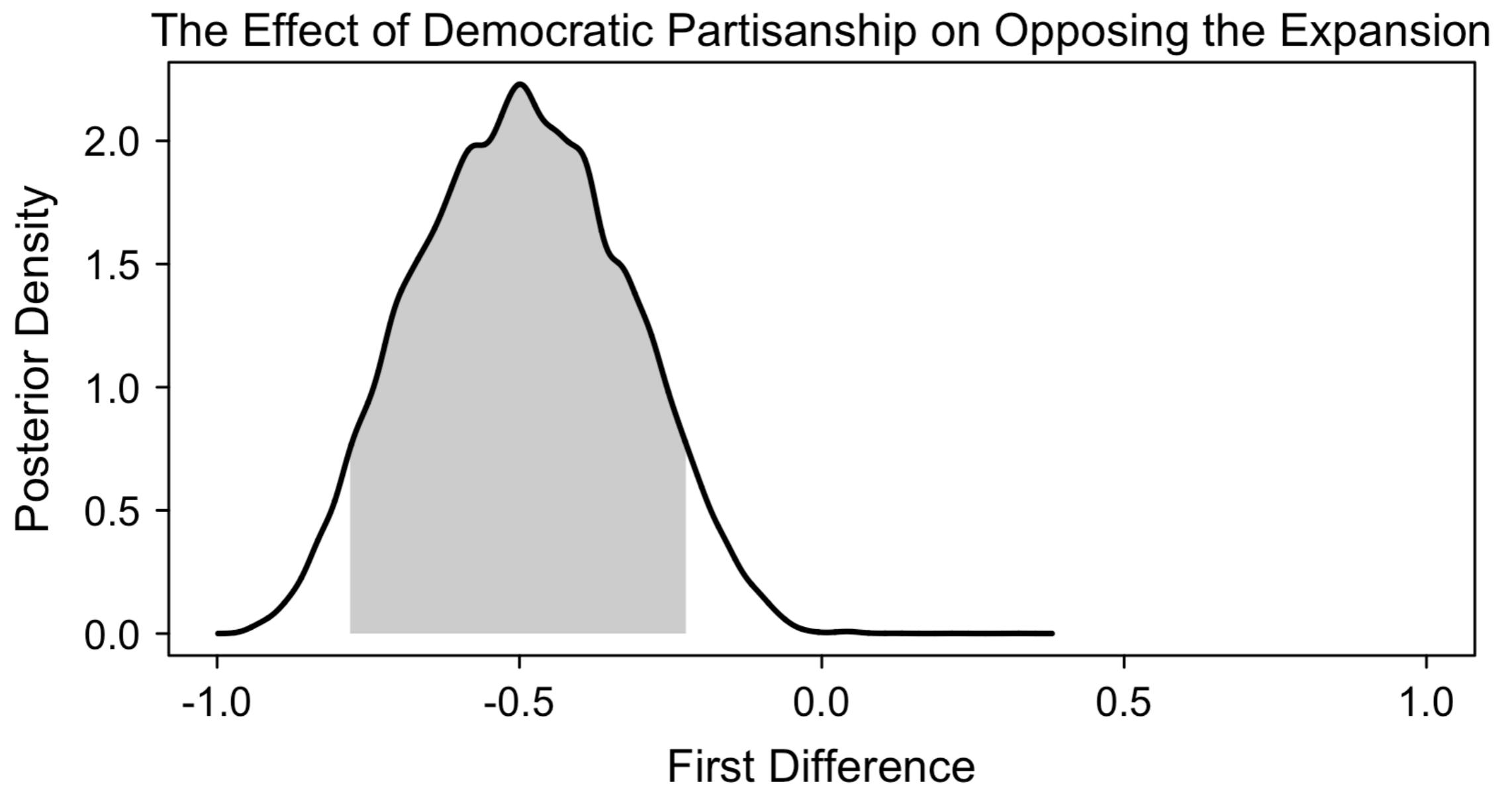
```
# mcmc estimation
post <- sim_post_normal(f, d, sep_var = "dem_governor",
                        sd = 4.5,
                        n_sims = 10000,
                        n_burnin = 1000,
                        n_chains = 4)
```

calc_qi()

```
# compute quantities of interest

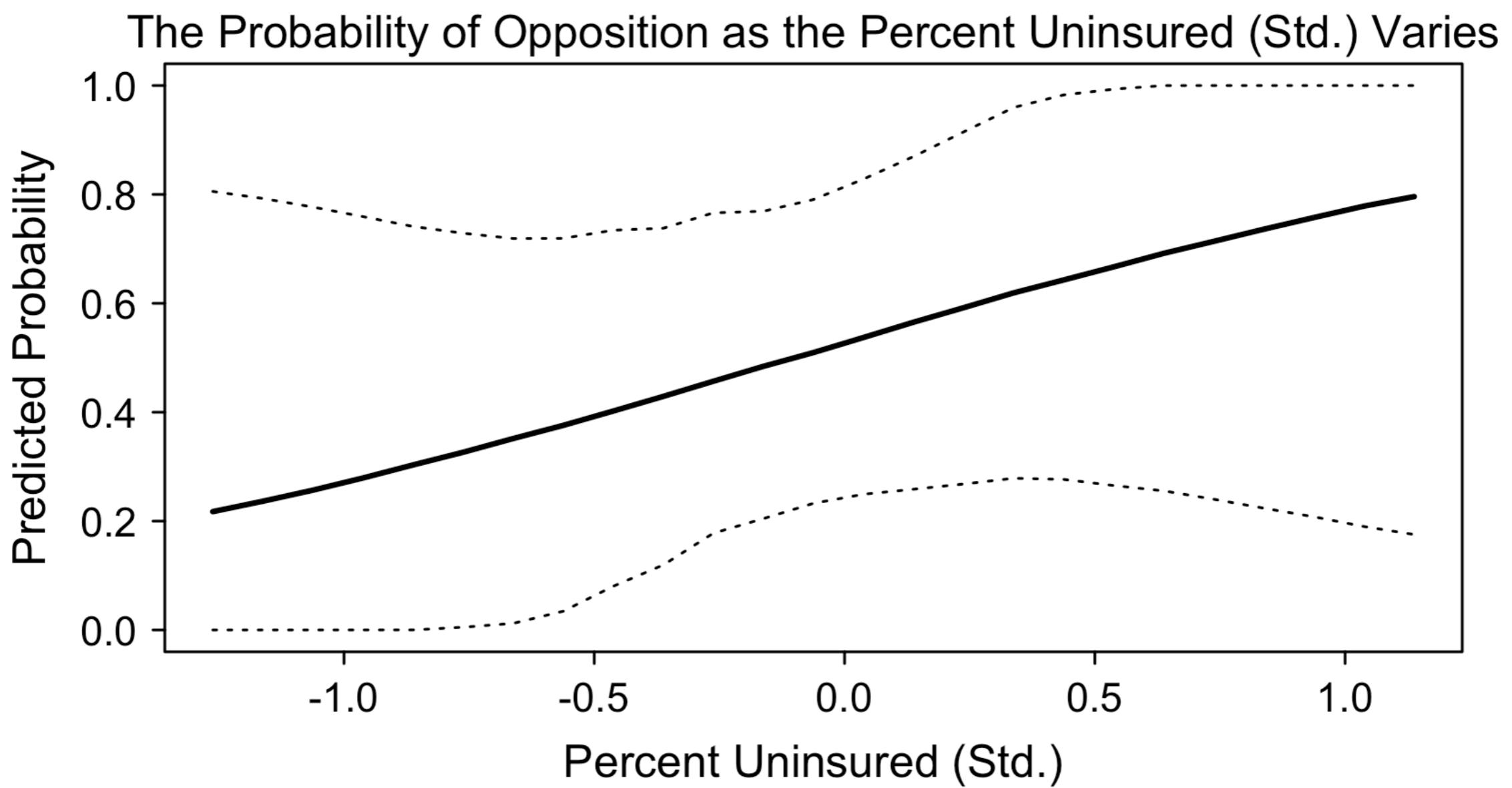
## dem_governor
x_pred_list <- set_at_median(f, d)
x <- c(0, 1)
x_pred_list$dem_governor <- x
qi <- calc_qi(post, x_pred_list, qi_name = "fd")
```

```
plot(qi, xlim = c(-1, 1),  
     xlab = "First Difference",  
     ylab = "Posterior Density",  
     main = "The Effect of Democratic Partisanship on  
Opposing the Expansion")
```



```
## st_percent_uninsured
x_pred_list <- set_at_median(f, d)
x <- seq(min(d$st_percent_uninsured),
          max(d$st_percent_uninsured),
          by = 0.1)
x_pred_list$st_percent_uninsured <- x
qi <- calc_qi(post, x_pred_list, qi_name = "pr")
```

```
plot(qi, x,
      xlab = "Percent Uninsured (Std.)",
      ylab = "Predicted Probability",
      main = "The Probability of Opposition as the
Percent Uninsured (Std.) Varies")
```



15 lines

Conclusion

The prior matters a lot,
so choose a good one.

The prior matters
in practice.

The prior matters
in theory.

The **partial prior predictive distribution**
simplifies the choice of prior.

Software makes choosing a prior,
estimating the model, and
interpreting the estimates easy.

What should *you* do?

What should *you* do?

1. Notice the problem and do something.

What should *you* do?

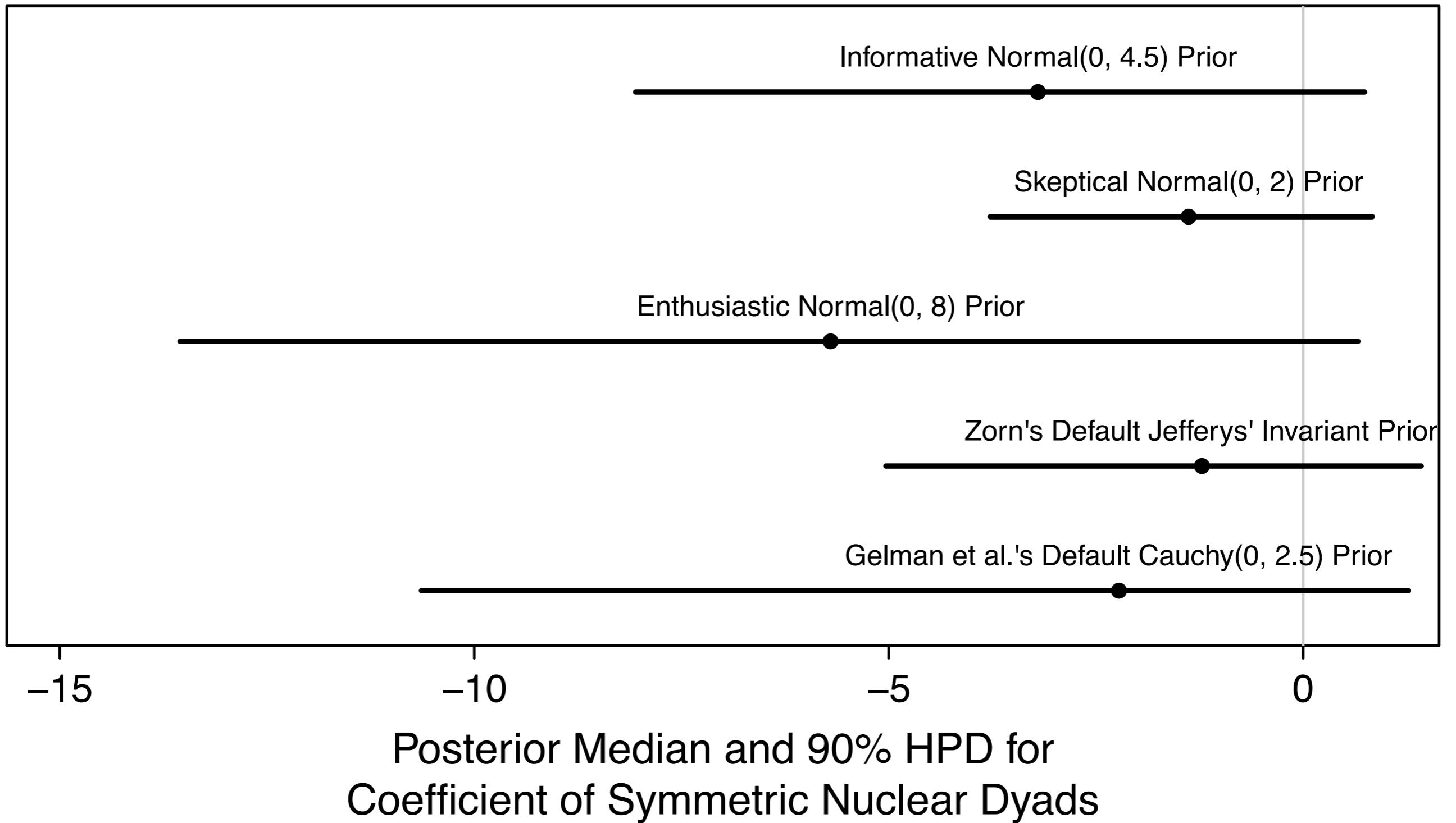
1. Notice the problem and do something.
2. Recognize the prior affects the inferences and choose a good one.

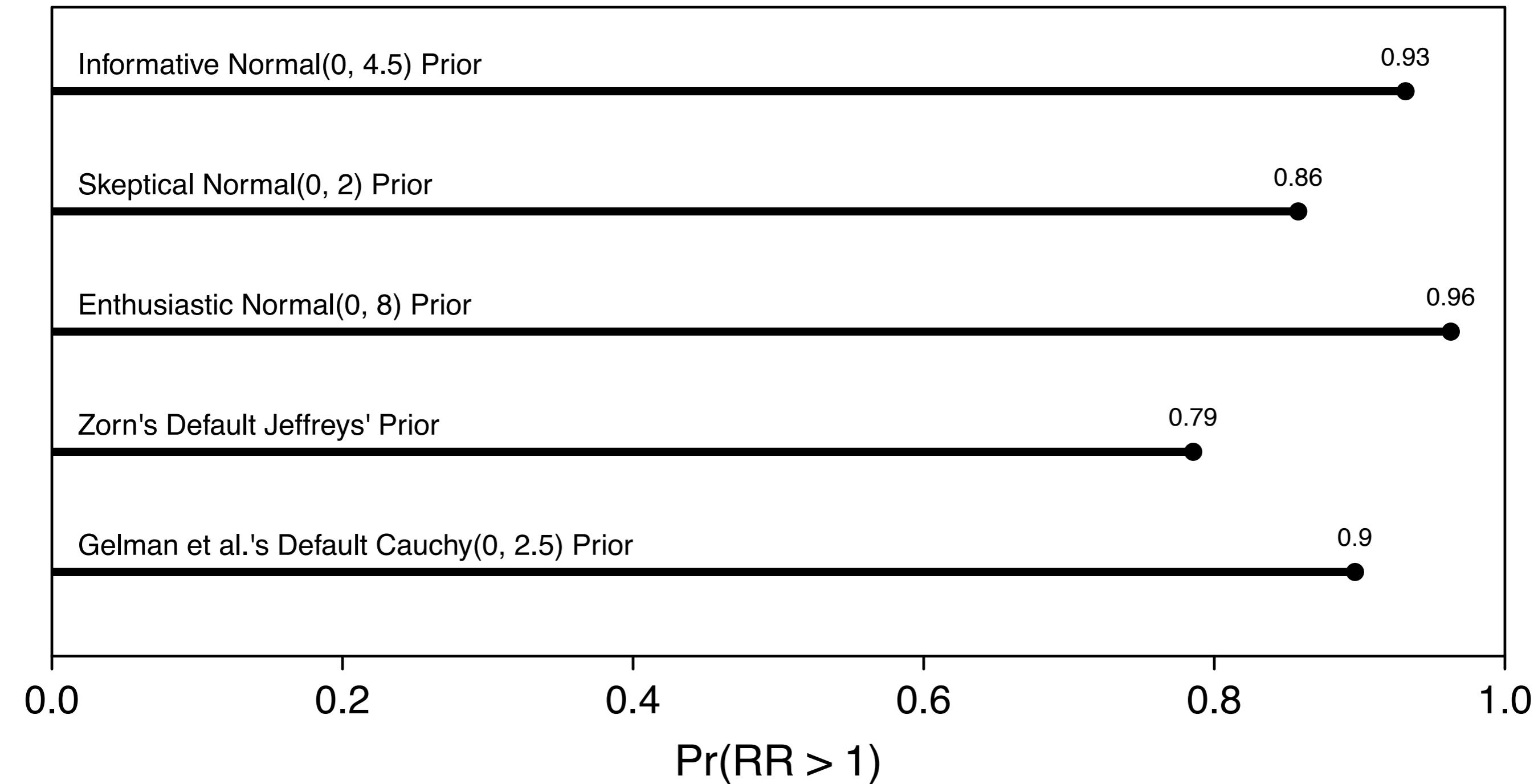
What should *you* do?

1. Notice the problem and do something.
2. Recognize the the prior affects the inferences and choose a good one.
3. Assess the robustness of your conclusions to a range of prior distributions.

Questions?

Appendix

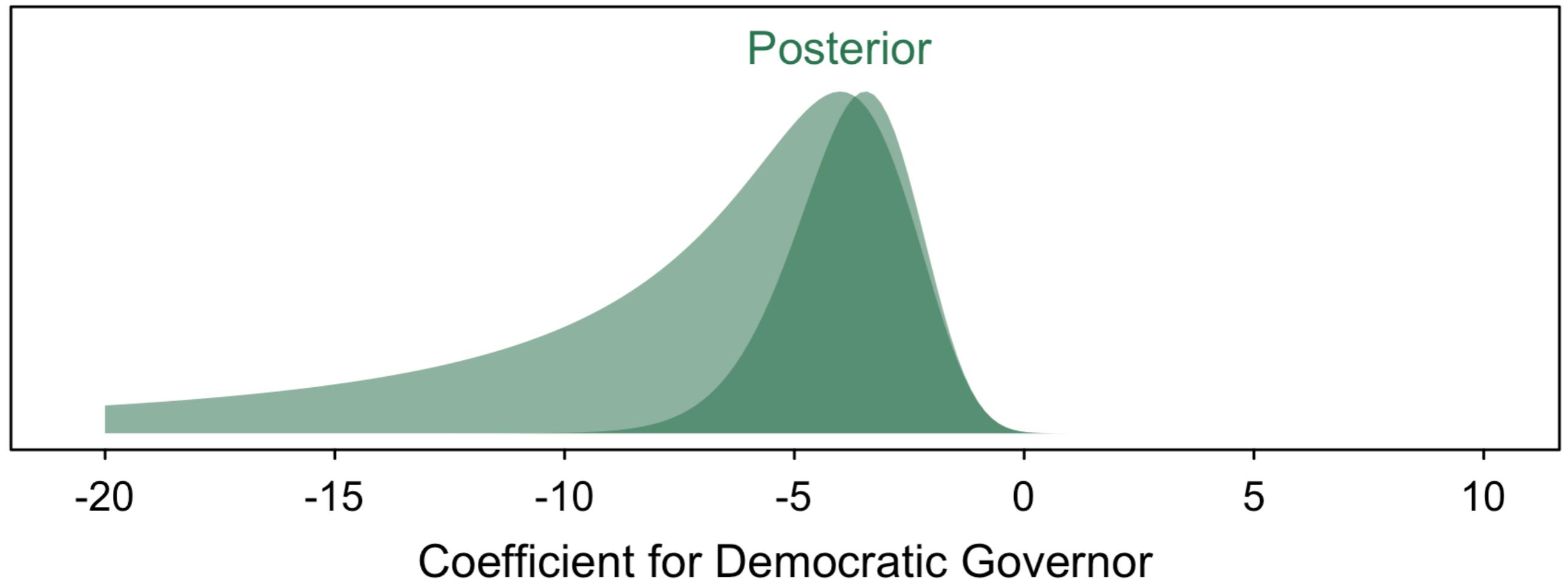


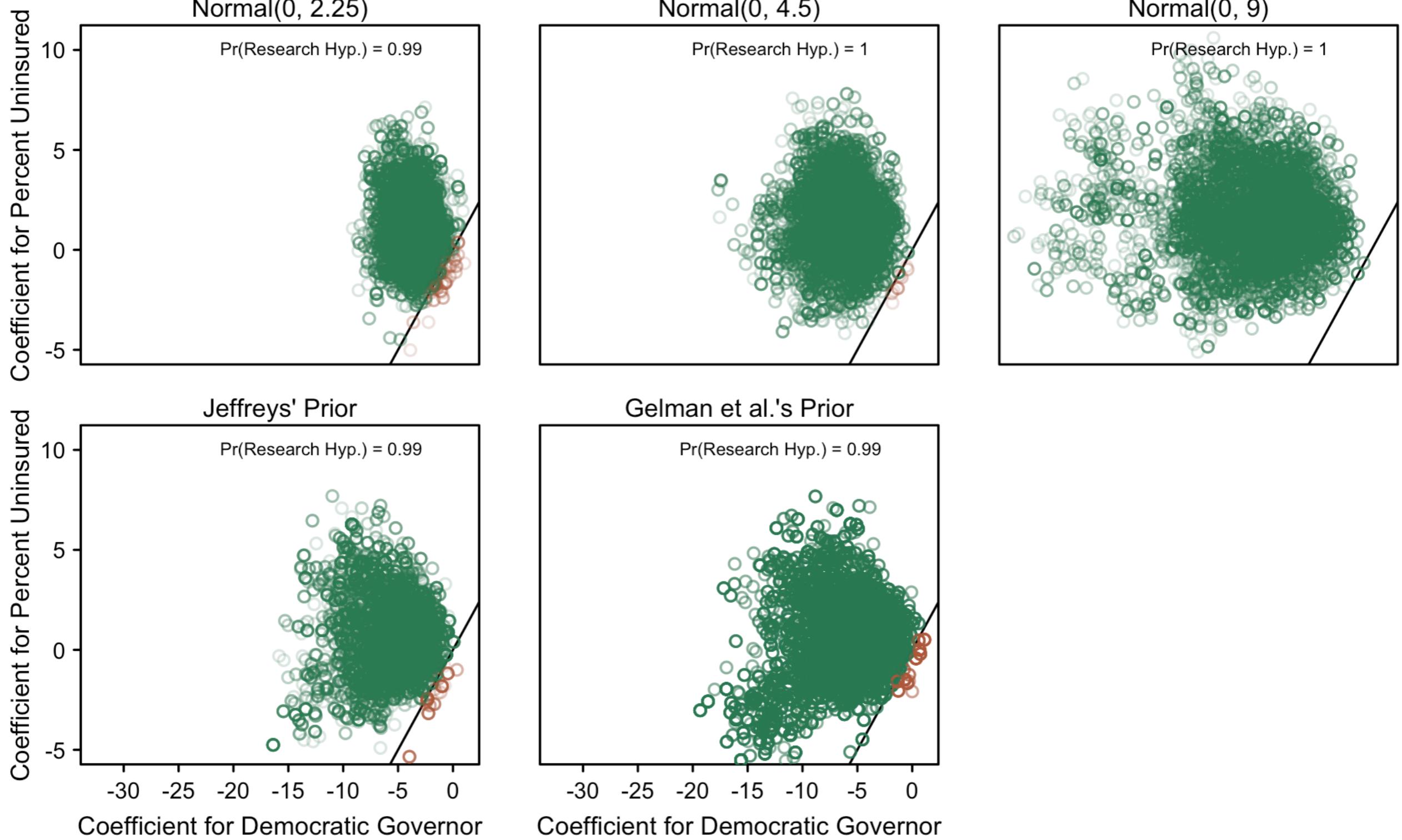


For

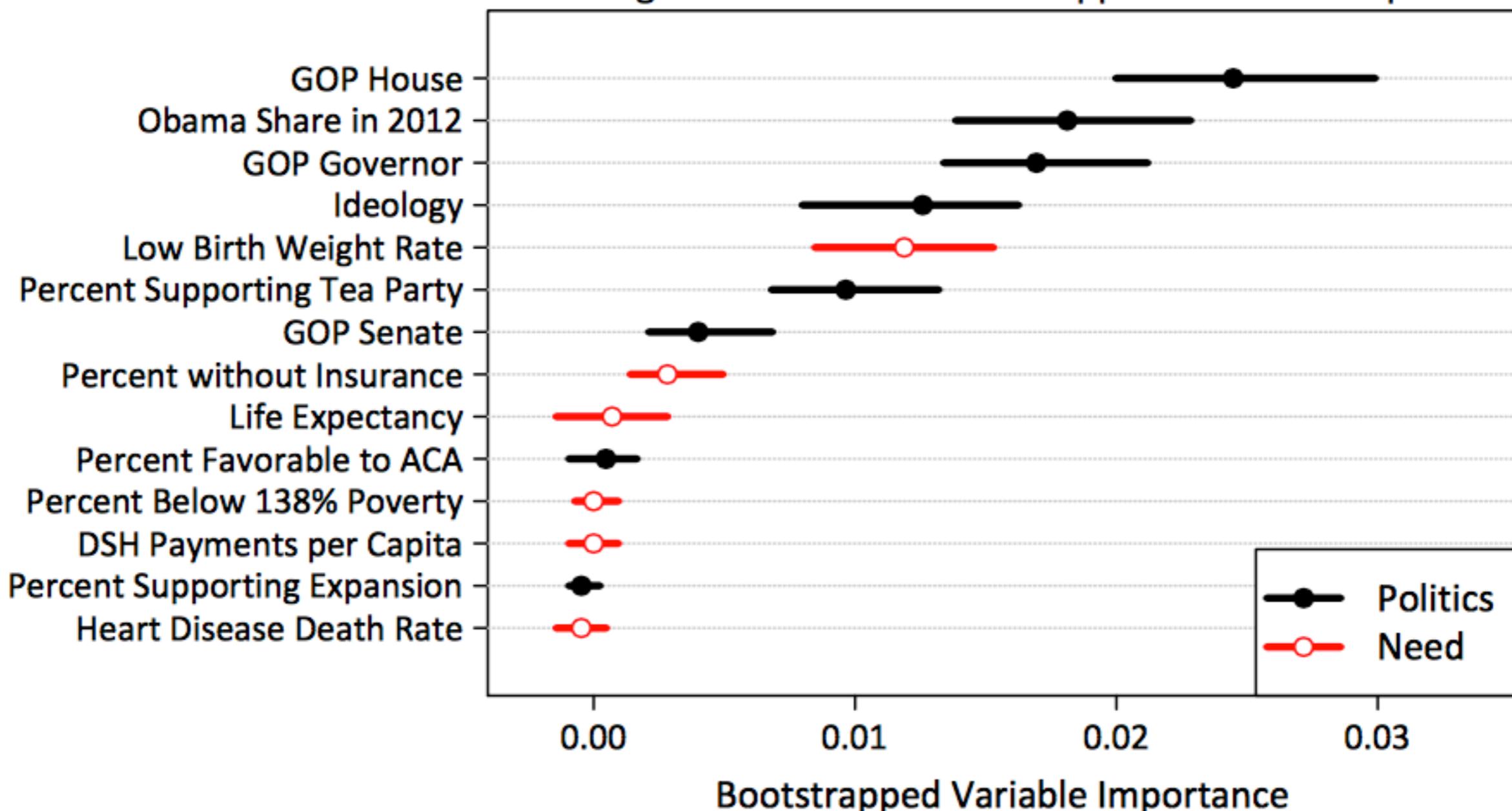
1. a monotonic likelihood $p(y|\beta)$ decreasing in β_s ,
2. a proper prior distribution $p(\beta|\sigma)$, and
3. a large, negative β_s ,

the posterior distribution of β_s is proportional to the prior distribution for β_s , so that $p(\beta_s|y) \propto p(\beta_s|\sigma)$.





Predicting Governor's Decision to Oppose Medicaid Expansion



Hypothesis	$Pr(H_r \text{data})$	Evidence
Gubernatorial Partisanship	> 0.99	Strong Evidence in Favor
Public Opinion	0.57	Ambiguous Evidence
Legislative Partisanship	0.97	Strong Evidence in Favor
Need	0.25	Ambiguous Evidence

Hypothesis	$Pr(H_r \text{data})$	Evidence
The effect of gubernatorial partisanship is larger than the effect of need.	> 0.99	Strong Evidence in Favor
The effect of public opinion is larger than the effect of need.	0.75	Ambiguous Evidence
The effect of legislative partisanship is larger than the effect of need.	0.96	Strong Evidence in Favor

Hypothesis	Variable	Expectation	Estimate	$Pr(H_r data)$	Evidence
Public Opinion	Percent Favorable to ACA	-	-0.21	0.57	Ambiguous Evidence
	Obama's 2012 Vote Share	-	-0.82	0.76	Ambiguous Evidence
	Obama Victory in 2012	-	-1.85	0.94	Moderate Evidence in Favor
	State Ideology	+	2.33	0.92	Moderate Evidence in Favor
	Percent Supporting Medicaid Expansion	-	1.01	0.17	Ambiguous Evidence
	Percent Supporting Tea Party	+	1.09	0.82	Ambiguous Evidence
Legislative Composition	GOP Controls Both House and Senate	+	2.31	0.97	Strong Evidence in Favor
	GOP House	+	6.48	1.00	Strong Evidence in Favor
	GOP Senate	+	2.17	0.97	Strong Evidence in Favor
	GOP House	+	6.44	0.98	Strong Evidence in Favor
	GOP Senate (as separate variables in the model)	+	0.21	0.55	Ambiguous Evidence
Need	Percent Without Health Insurance	-	0.91	0.25	Ambiguous Evidence
	DSH Payments per Capita	-	1.21	0.10	Moderate Evidence Against
	Percent Below 138% Poverty	-	0.53	0.35	Ambiguous Evidence
	Low Birth Weight	-	2.76	0.02	Strong Evidence Against
	Heart Disease Death Rate	-	1.23	0.09	Moderate Evidence Against
	Life Expectancy	+	-1.50	0.09	Moderate Evidence Against

Theorem 1. *For a monotonic likelihood $p(y|\beta)$ increasing [decreasing] in β_s , proper prior distribution $p(\beta|\sigma)$, and large positive [negative] β_s , the posterior distribution of β_s is proportional to the prior distribution for β_s , so that $p(\beta_s|y) \propto p(\beta_s|\sigma)$.*

Proof. Due to separation, $p(y|\beta)$ is monotonic increasing in β_s to a limit $\underline{\mathcal{L}}$, so that $\lim_{\beta_s \rightarrow \infty} p(y|\beta_s) = \underline{\mathcal{L}}$. By Bayes' rule,

$$p(\beta|y) = \frac{p(y|\beta)p(\beta|\sigma)}{\int\limits_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta} = \underbrace{\frac{p(y|\beta)p(\beta|\sigma)}{p(y|\sigma)}}_{\text{constant w.r.t. } \beta}.$$

Integrating out the other parameters $\beta_{-s} = \langle \beta_{cons}, \beta_1, \beta_2, \dots, \beta_k \rangle$ to obtain the posterior distribution of β_s ,

$$p(\beta_s|y) = \frac{\int\limits_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta_{-s}}{p(y|\sigma)}, \quad (1)$$

and the prior distribution of β_s ,

$$p(\beta_s|\sigma) = \int\limits_{-\infty}^{\infty} p(\beta|\sigma)d\beta_{-s}.$$

Notice that $p(\beta_s|y) \propto p(\beta_s|\sigma)$ iff $\frac{p(\beta_s|y)}{p(\beta_s|\sigma)} = k$, where the constant $k \neq 0$. Thus,

$$p(\beta_s | \sigma) = \int_{-\infty}^{\infty} p(\beta | \sigma) d\beta_{-s}.$$

Notice that $p(\beta_s | y) \propto p(\beta_s | \sigma)$ iff $\frac{p(\beta_s | y)}{p(\beta_s | \sigma)} = k$, where the constant $k \neq 0$. Thus, Theorem 1 implies that

$$\lim_{\beta_s \rightarrow \infty} \frac{p(\beta_s | y)}{p(\beta_s | \sigma)} = k$$

Substituting in Equation 1,

$$\lim_{\beta_s \rightarrow \infty} \frac{\frac{\int_{-\infty}^{\infty} p(y | \beta) p(\beta | \sigma) d\beta_{-s}}{p(y | \sigma)}}{p(\beta_s | \sigma)} = k.$$

Multiplying both sides by $p(y | \sigma)$, which is constant with respect to β ,

$$\lim_{\beta_s \rightarrow \infty} \frac{\int_{-\infty}^{\infty} p(y | \beta) p(\beta | \sigma) d\beta_{-s}}{p(\beta_s | \sigma)} = kp(y | \sigma).$$

Setting $\int_{-\infty}^{\infty} p(y | \beta) p(\beta | \sigma) d\beta_{-s} = p(y | \beta_s) p(\beta_s | \sigma)$,

$$\beta_s \rightarrow \infty p(\beta_s | \sigma)$$

Substituting in Equation 1,

$$\lim_{\beta_s \rightarrow \infty} \frac{\int_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta_{-s}}{p(\beta_s|\sigma)} = k.$$

Multiplying both sides by $p(y|\sigma)$, which is constant with respect to β ,

$$\lim_{\beta_s \rightarrow \infty} \frac{\int_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta_{-s}}{p(\beta_s|\sigma)} = kp(y|\sigma).$$

Setting $\int_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta_{-s} = p(y|\beta_s)p(\beta_s|\sigma)$,

$$\lim_{\beta_s \rightarrow \infty} \frac{p(y|\beta_s)p(\beta_s|\sigma)}{p(\beta_s|\sigma)} = kp(y|\sigma).$$

Cancelling $p(\beta_s|\sigma)$ in the numerator and denominator,

$$\lim_{\beta_s \rightarrow \infty} p(y|\beta_s) = kp(y|\sigma).$$