Technical Appendix for "Dealing with Separation in Logistic Regression Models"

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Proof of Theorem 1

Recall Theorem 1:

> For a monotonic likelihood $p(y|\beta)$ increasing [decreasing] in β_s, proper prior distribution $p(\beta|\sigma)$, and large positive [negative] β_s, the posterior distribution of β_s is proportional to the prior distribution for β_s, so that $p(\beta_s|y) \propto p(\beta_s|\sigma)$. More formally, $\lim\limits_{\substack{\beta_s \to \infty \\ [-\infty]}} \dfrac{p(\beta_s|y)}{p(\beta_s|\sigma)} = k$, for postive constant k.

Proof. Due to separation, $p(y|\beta)$ is monotonic increasing in β_s to a limit $\underline{\mathscr{L}}$, so that $\lim\limits_{\beta_s \to \infty} p(y|\beta_s) = \underline{\mathscr{L}}$. By Bayes' rule,

$$p(\beta|y) = \frac{p(y|\beta)p(\beta|\sigma)}{\int\limits_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta} = \frac{p(y|\beta)p(\beta|\sigma)}{\underbrace{p(y|\sigma)}_{\text{constant w.r.t. } \beta}}.$$

Integrating out the other parameters $\beta_{-s} = \langle \beta_{cons}, \beta_1, \beta_2, ..., \beta_k \rangle$ to obtain the posterior distribution of β_s,

$$p(\beta_s|y) = \frac{\int\limits_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta_{-s}}{p(y|\sigma)}, \tag{1}$$

and the prior distribution of β_s,

$$p(\beta_s|\sigma) = \int\limits_{-\infty}^{\infty} p(\beta|\sigma)d\beta_{-s}.$$

Notice that $p(\beta_s|y) \propto p(\beta_s|\sigma)$ iff $\dfrac{p(\beta_s|y)}{p(\beta_s|\sigma)} = k$, where the constant $k \neq 0$. Thus, Theorem **??** implies that

$$\lim_{\beta_s \to \infty} \frac{p(\beta_s|y)}{p(\beta_s|\sigma)} = k$$

Substituting in Equation 1,

$$\lim_{\beta_s \to \infty} \frac{\frac{\int\limits_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta_{-s}}{p(y|\sigma)}}{p(\beta_s|\sigma)} = k.$$

Multiplying both sides by $p(y|\sigma)$, which is constant with respect to β,

$$\lim_{\beta_s \to \infty} \frac{\int\limits_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta_{-s}}{p(\beta_s|\sigma)} = kp(y|\sigma).$$

Setting $\int_{-\infty}^{\infty} p(y|\beta)p(\beta|\sigma)d\beta_{-s} = p(y|\beta_s)p(\beta_s|\sigma)$,

$$\lim_{\beta_s \to \infty} \frac{p(y|\beta_s)p(\beta_s|\sigma)}{p(\beta_s|\sigma)} = kp(y|\sigma).$$

Canceling $p(\beta_s|\sigma)$ in the numerator and denominator,

$$\lim_{\beta_s \to \infty} p(y|\beta_s) = kp(y|\sigma).$$

Recalling that $\lim_{\beta_s \to \infty} p(y|\beta) = \underline{\mathscr{L}}$ and substituting,

$$\underline{\mathscr{L}} = kp(y|\sigma),$$

which implies that $k = \frac{p(y|\sigma)}{\underline{\mathscr{L}}}$, which is a positive constant. $\square$