Transformations

1. Determine which (if any) of the following operators correspond to linear transformations on some arbitrary function $f(x)$:

   (a) $\hat{A} = \frac{d}{dx}$

   (b) $\hat{B} = \frac{d}{dx}x$

   (c) $\hat{C} = \int dx$

2. Consider an operator $\hat{O}$ that acts on a four-dimensional vector space with basis functions $|\phi_k\rangle$ for $k = 1, 2, 3,$ and 4. Somehow you know the operator acts on each of the basis functions according to

$$\hat{O}\,|\phi_1\rangle = i\,|\phi_2\rangle + 3\,|\phi_3\rangle \tag{1}$$

$$\hat{O}\,|\phi_2\rangle = -i\,|\phi_1\rangle - 2i\,|\phi_3\rangle \tag{2}$$

$$\hat{O}\,|\phi_3\rangle = 3\,|\phi_1\rangle + 2i\,|\phi_2\rangle + |\phi_3\rangle \tag{3}$$

$$\hat{O}\,|\phi_4\rangle = \frac{\pi}{2}\,|\phi_4\rangle \tag{4}$$

   (a) Assuming these four basis functions form an orthonormal set ($\langle\phi_i|\phi_j\rangle = \delta_{ij}$), write the matrix representation for the operator $\hat{O}$ in this basis.

   (b) Determine the action of this operator on the vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle - |\phi_3\rangle\right) \tag{5}$$

Classes of Operators

3. In the notes (Example 5.1) we found the following form for a matrix that rotates a vector by some angle θ about the y-axis:

$$\mathcal{R}(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}. \tag{6}$$

   Show that this matrix is a unitary matrix.

4. Let $|a\rangle$, $|b\rangle$, and $|c\rangle$ be vectors defined in terms of the Cartesian basis vectors according to

$$|a\rangle = \frac{1}{\sqrt{2}}\,|x\rangle + \frac{1}{\sqrt{2}}\,|y\rangle \tag{7}$$

$$|b\rangle = \frac{1}{\sqrt{2}}\,|x\rangle - \frac{1}{\sqrt{2}}\,|y\rangle \tag{8}$$

$$|c\rangle = |z\rangle\,. \tag{9}$$

Let the vector space V consist of the span of $|a\rangle$ and $|b\rangle$ and the vector space W of the span of $|a\rangle$ and $|c\rangle$.

(a) Write the operators for the projection onto each of these two vector spaces, V and W.

(b) Determine the matrix representation for each of these two operators.

(c) Using either the operator you obtained in (a) or the matrix you obtained in (b), obtain the projection of

$$|\omega\rangle = \frac{1}{\sqrt{3}} \left(|x\rangle + |y\rangle + |z\rangle\right)$$

onto each of the vector spaces V and W.