Classification of Orbits in the Two-Body Problem with Post-Newtonian and Post-Minkowskian Corrections using Numerical Methods

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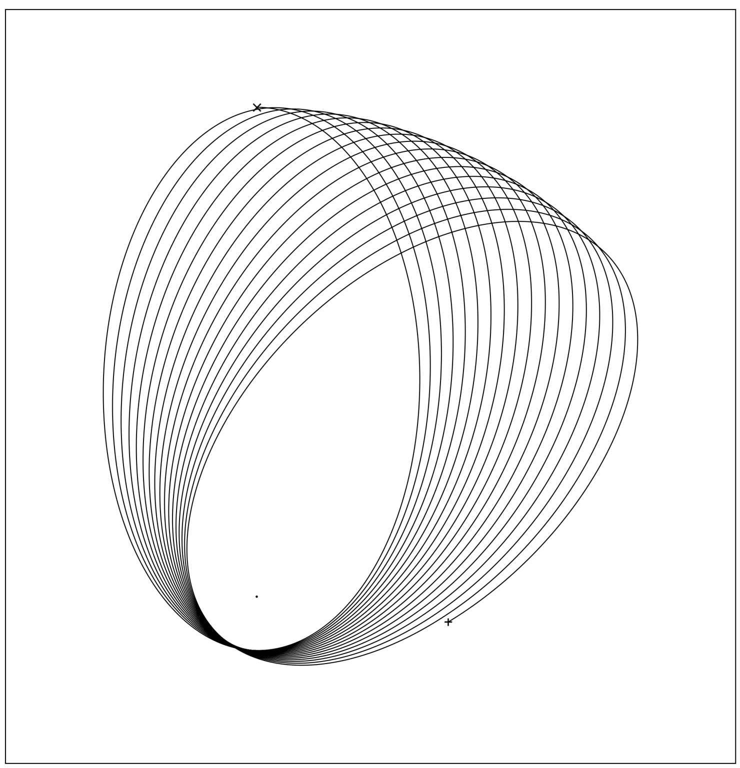

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Abstract

The post-Minkowskian expansion is a new (???) way of further correcting the motion of massive bodies around each other due to gravitational interaction. In this paper we will be classifying the different orbits achieved in the two-body problem using classical mechanics and both post-Newtonian and -Minkowskian correction methods. We will be looking at each methods significance on the orbits and their mutual discrepancies in describing the motion.

Introduction

Gravitational waves (GWs) were predicted quite early by Einstein in 1916 as presented in [9]. There he used a first order perturbative method in the weak field limit of gravity on the field equations of general relativity to linearize the equations and derive wave solutions describing fluctuations in the fabric of reality; spacetime. The existence of GWs was confirmed in 2015 by LIGO's observations of a binary black hole merger, made possible by modeling of such binary systems and their potential graviational waveforms. Due to the remarkably small amplitudes of GWs, modeling of this kind is essential, since it makes distinguishing the signal from background noise possible [1].

In order to describe binary neutron star and black hole systems accurately, it is necessary to make analytical approximations in general relativity (GR). Since the discovery of gravitational waves caused by such systems, by the LIGO and Virgo detectors, there have emerged different approaches to solving the two body problem in GR [4]. The most popular approximation has, by far, been the post-Newtonian (PN) expansion which is an expansion in the velocity, v/c, around zero. Despite the PN expansion being only formally valid in a low-velocity regime (compared to the speed of light) and weak gravity, it has had surprising success outside of those regimes [17]. This is not without its limits though and that is where the post-Minkowskian (PM) expansion comes into play. The PM expansion is in all orders of velocity but expanded in Gm/rc^2 around zero, thus formally valid in weak gravity but for any velocity and can thus be used to analyse ultra-relativistic scattering and other cases that fall outside of the scope of the PN expansion.

Add paragraph describing what we do in this paper

Chapter 1

The Two-Body Problem in Classical Mechanics

In this chapter the two-body problem and some of the hurdles that have to be overcome to solve it are discussed. The test-body limit of the two-body problem and how it reduces nicely to the one-body problem, i.e. movement of a single particle in an external static gravitational field, will also be discussed. Furthermore the solution to these two problems for two distinct potentials will be worked through using both the Newtonian and the Hamiltonian methods.

1.1 The Two-Body Problem in Newtonian Mechanics

The two-body problem is an isolated system of two interacting massive particles. One can picture gravitational fields emerging from the center of both particles (see Figure 1.1), which move along with the particles. This makes the problem tricky to solve, but there exists a way to simplify it, and make it solvable using Newtonian mechanics.

1.1.1 The Center of Mass Transformation

Let's name the particles particle 1 and 2 and give them masses m_1 and m_2, positions $\mathbf{r}_1$ and $\mathbf{r}_2$ and velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ respectively. Using the conservation of momentum one can find the position, $\mathbf{R}_{\text{CM}}$, and velocity, $\mathbf{v}_{\text{CM}}$, of the center of mass in the inertial frame, the latter of which is constant due to the system being isolated, i.e. no external force is acting upon it:

$$\mathbf{R}_{\text{CM}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \quad \dot{\mathbf{R}}_{\text{CM}} = \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{M} \quad \text{and} \quad \ddot{\mathbf{R}}_{\text{CM}} = \mathbf{0}, \tag{1.1}$$

where $M = m_1 + m_2$. Using the relative distance and velocity, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, one can rewrite:

$$\mathbf{r}_1 = \mathbf{R}_{\text{CM}} + \frac{m_2}{M}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R}_{\text{CM}} - \frac{m_1}{M}\mathbf{r}. \tag{1.2}$$

Now, using Newton's second and third law, $\mathbf{F}_{21} = m_1\ddot{\mathbf{r}}_1 = m_2\ddot{\mathbf{r}}_2 = -\mathbf{F}_{12}$,

$$m_2\mathbf{F}_{21} - m_1\mathbf{F}_{12} = m_1 m_2(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) \tag{1.3}$$

$$(m_2 + m_1)\mathbf{F}_{21} = m_1 m_2\left(\frac{m_2}{m_1 + m_2}\ddot{\mathbf{r}} + \frac{m_1}{m_1 + m_2}\ddot{\mathbf{r}}\right) \tag{1.4}$$

$$\mathbf{F}_{21} = \frac{m_1 m_2}{m_1 + m_2}\left(\frac{m_1 + m_2}{m_1 + m_2}\ddot{\mathbf{r}}\right) = \frac{m_1 m_2}{m_1 + m_2}\ddot{\mathbf{r}} \tag{1.5}$$

as well as knowing that for a central potential $\mathbf{F} = -\boldsymbol{\nabla} V$, one arrives at

$$\mu\ddot{\mathbf{r}} = -\boldsymbol{\nabla} V \tag{1.6}$$

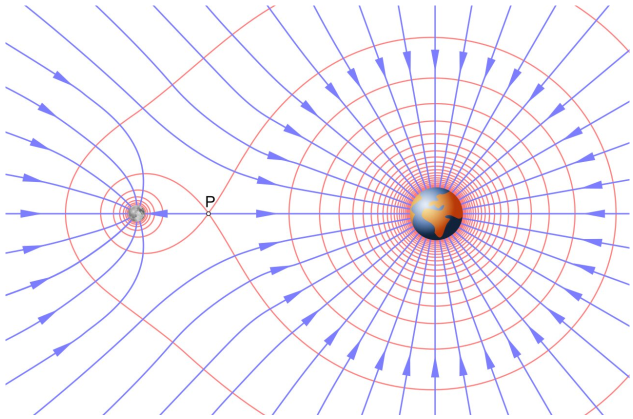

Figure 1.1: The Earth and the Moon interact through gravity. Here, their mutual gravitational field is shown in 2 dimensions with the point P experiencing no gravitational force. (Picture: Wikipedia)

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. [11]

Thus, the problem has been reduced from having to solve for the motion of both particles $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ separately, to solving for the relative motion of the, $\mathbf{r}(t)$, in the center of mass frame, and then using equations (1.2) to find the positions of each individual particle after solving (write here how we solve and include that it is important that the equations are in a CM frame of reference).

Example: The One-Body Problem

Let's look at the two-body problem (TBP) in the limit where $m_1 = m \ll M = m_2$, which is called the test-body limit of the TBP. Here, we see that

$$(1.1) \quad \rightarrow \quad \mathbf{R}_{\text{CM}} = \mathbf{r}_2, \quad \mathbf{v}_{\text{CM}} = \mathbf{v}_2 \tag{1.7}$$

$$(1.2) \quad \rightarrow \quad \mathbf{r}_1 = \mathbf{r}_2 + \mathbf{r} = \mathbf{r}_1, \quad \mathbf{r}_2 = \mathbf{r}_1 - \mathbf{r} = \mathbf{r}_2 \tag{1.8}$$

$$(1.6) \quad \rightarrow \quad m(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = m\left(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{R}}_{\text{CM}}\right) = m\ddot{\mathbf{r}}_1 \tag{1.9}$$

We see that the TBP completely reduces to the problem of one particle m subject to a conservative force due to an external potential $\mathbf{F} = -\boldsymbol{\nabla} V$ centered at $\mathbf{r}_2$. This is also called the one-body problem.

Let's take the potential to be that of a classical Newtonian gravitational potential $V(r) = -Gm_1m_2/r$, where $G = 6.6743 \cdot 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ is Newtons gravitational constant. The force on the particle m due to the gravitational field then becomes

$$m\ddot{\mathbf{r}} = \mathbf{F} = -\boldsymbol{\nabla} V(r) = -\frac{GMm}{r^2}\hat{\mathbf{r}}, \tag{1.10}$$

which leads to the equations of motion which, given an initial position and velocity, can be integrated[1] to give the position of the particle at any time t,

$$\mathbf{r}(t) = \iint \ddot{\mathbf{r}}(t)\, dt^2 = -GM \iint \frac{\hat{\mathbf{r}}}{r(t)^2}\, dt^2. \tag{1.11}$$

We have now solved the one-body problem for a traditional Newtonian gravitational potential, which we have shown to be equivalent to solving the two-body problem, although one needs to do the correct variable transformation.

[1] A derivation of the analytical solution of the orbit can be found in [10], but since we will be solving the equations numerically, this is not of that big concern here.

1.1.2 Two-Body Motion is Planar

In all of the equations above are actually hidden a multitude of additional equations, because of the fact that vector notation has been used. This means that one has to solve a system of equations representing the movement in all (three) spatial dimensions, which in Cartesian coordinates looks like

$$\mu\ddot{r}_x = -\frac{\partial V}{\partial x}, \quad \mu\ddot{r}_y = -\frac{\partial V}{\partial y} \quad \text{and} \quad \mu\ddot{r}_z = -\frac{\partial V}{\partial z}, \tag{1.12}$$

but again, one can use one of the fundamental laws of physics, namely the conservation of angular momentum, to further simplify the problem. If $\mathbf{r}$ is the relative position between two particles and $\mathbf{p}$ is their relative momentum, then the angular momentum $\mathbf{L}$ is described as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \mu\dot{\mathbf{r}} \tag{1.13}$$

The conservation of angular momentum then states that

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mu\dot{\mathbf{r}}) = \mathbf{0}, \tag{1.14}$$

Therefore, the displacement vector $\mathbf{r}$ and its velocity $\dot{\mathbf{r}}$ will be in the plane perpendicular to the constant vector $\mathbf{L}$ at all times t.[16] Thus, by a certain transformation of coordinates, more precisely the rotation that brings the $\mathbf{r}$ and $\dot{\mathbf{r}}$ vectors into the x, y-plane, one can reduce the amount of equations from 3 to only 2:

$$\mu\ddot{r}_x = -\frac{\partial V}{\partial x} \quad \text{and} \quad \mu\ddot{r}_y = -\frac{\partial V}{\partial y}. \tag{1.15}$$

since the last one is trivial $\mu\ddot{r}_z = -\,\partial V/\partial z = 0$.

1.2 Hamiltonian Mechanics

Now it has been shown how the two-body problem can be solved using Newtonian Mechanics by correctly describing the position, velocity and force vectors of the problem, doing the right variable transformation and integrating. This can be quite tedious though, and for more complicated interactions, unfeasible.

To bypass this hurdle, one has to introduce another way to solve the TBP, namely Hamiltonian Mechanics. This approach rests on entirely different principles than Newtonian mechanics, but has been shown to be equivalent (i.e. leading to the same equations of motion and hence the same physics) [14].

1.2.1 Generalized Coordinates

The power of Hamiltonian Mechanics comes from the fact that it is coordinate independent generally. If one wishes to describe a system of N particles in motion in 3-dimensional space, then this has $n = 3N$ degrees of freedom, and can be described by the coordinates

$$x_i \quad \text{for} \quad i = (1, 2, 3),\ (4, 5, 6),\ \ldots, (n-2, n-1, n), \tag{1.16}$$

where the first triplet refers to the first particle, the second triplet refers to the second particle and so on [10]. Suppose there exists k functions

$$f_j(x_1, \ldots, x_n) = c_j \quad \text{for} \quad j = 1, \ldots, k, \tag{1.17}$$

that describe k holonomic constraints of the system, then the Hamiltonian can in general be described by $2m = (n - k)$ generalized coordinates q_σ and momenta p_σ (for $\sigma = 1, \ldots, n$) and maybe even time t. An example of such a constraint is the length L of the arm of a pendulum constraining radial motion (see Figure 1.2).

$$H = H(q_1, \ldots, q_n;\, p_1, \ldots, p_n;\, t). \tag{1.18}$$

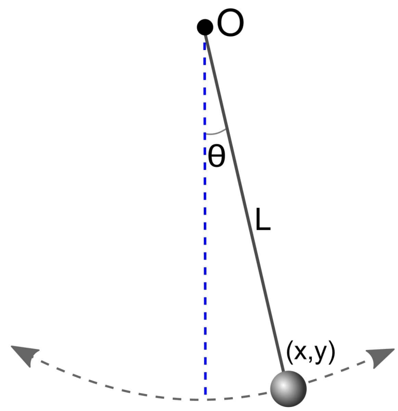

Figure 1.2: The length L of the arm of a pendulum is a holonomic constraint which can be expressed like $f(x, y) = x^2 + y^2 - L^2 = 0$. (Picture: Wikipedia)

This, coupled with the Hamiltonian equations of motion

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}, \quad \text{and} \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \tag{1.19}$$

gives $2n$ coupled first order differential equations in the time, which then determines the subsequent motion of the particles [10].

Note that we can choose any generalized coordinates, as long they describe the system and it's constraints fully, and thus Hamiltonian mechanics is invariant to different coordinate changes, making it a powerful tool to solve a variety of different mechanical problems.

1.2.2 Conservative Dynamics

For a conservative system the Hamiltonian describes the total conserved energy of the system

$$H = T + V = E = \text{const.} \tag{1.20}$$

and the Hamiltonian equations (1.19) then results in the same equations of motion as Newton's laws for the conserved system in question. [10].

Example: Solving the Two-Body Problem

Imagine a 2 particles with masses m_1 and m_2 gravitationally bound to each other particle through a conservative potential V(r). The Hamiltonian for the system can be written as

$$H(\mathbf{p}_1, \mathbf{p}_2, \mathbf{r}_1, \mathbf{r}_2) = T + V = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(r), \tag{1.21}$$

where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta of the two particles, and $r = \|\mathbf{r}\| = \|\mathbf{r}_1 - \mathbf{r}_2\|$ is the length of their mutual displacement vector $\mathbf{r}$.

Moving to the center of mass (COM) frame

$$\mathbf{p}_1 \;\rightarrow\; \mathbf{p} \qquad \text{and} \qquad \mathbf{p}_2 \;\rightarrow\; -\mathbf{p}, \tag{1.22}$$

and taking $V(r)$ to be the Newtonian potential, the Hamiltonian can now be written as

$$H(\mathbf{p}, \mathbf{r}) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{p}^2 - \frac{Gm_1m_2}{r} \tag{1.23}$$

Using the fact that the orbital motion of these two particles will be confined to a plane, we ensure that our coordinate system is situated such that the Hamiltonian only be a function of the radial coordinate r and the angular coordinate ϕ.[2] Identifying our generalized variables as $\mathbf{r} = (r, \phi)$ and $\mathbf{p} = (p_r, p_\phi)$ the Hamiltonian equations of motion in vector form are then

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} \tag{1.24}$$

$$\text{and} \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}} = -\frac{\partial H}{\partial r}\frac{\partial r}{\partial \mathbf{r}} = \frac{GMm}{r^2}\hat{\mathbf{r}}, \tag{1.25}$$

from which we can obtain

$$\ddot{\mathbf{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{p}}{m}\right) = \frac{\dot{\mathbf{p}}}{m} = \mathbf{a}(t) \tag{1.26}$$

Specifying the initial conditions $\mathbf{r}_0$ and $\mathbf{p}_0$ is then sufficient to determine the subsequent motion at all times t. We can now use the results from Section 1.1.1 to change from our center of mass motion $\mathbf{r}(t)$ to the motion of the two individual masses

$$(1.2) \quad \rightarrow \quad \mathbf{r}_1(t) = \mathbf{R}_{\mathrm{CM}} + \frac{m_2}{M}\mathbf{r}(t) \qquad \text{and} \qquad \mathbf{r}_2(t) = \mathbf{R}_{\mathrm{CM}} - \frac{m_1}{M}\mathbf{r}(t), \tag{1.27}$$

and thus we have solved the two-body problem for the Newtonian potential generically using Hamiltonian mechanics.

[2] Equivalently one could have used the Cartesian coordinates x and y.

Chapter 2

General Relativity

"The theory of General Relativity is arguably the most beautiful theory of physics. [...]
Once one understands the underlying mathematics, its formulation is simple and natural."
- Troels Harmark

In 1915 Albert Einstein published his theory of General Relativity (GR) which, in a sense, is an extension of Newtons theory of gravity to regimes where the force of gravity, or the relative velocities of bodies, is very large. One of the main points of General Relativity is that gravity is a manifestation of the geometry of the 4-dimensional space-time that makes up the universe. This geometry is expressed mathematically with the space-time metric $g_{\mu\nu}$.

2.1 Special Relativity

2.1.1 Minkowski Space and the Proper Time

In Einsteins theory of Special Relativity space and time are related and inseparable and this medium of the universe is called space-time. An *event* is a point in the 4-dimensional space-time described by the 4 coordinates

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \tag{2.1}$$

where c is the speed of light. In the absence of gravity one can find special coordinate systems where objects not subject to external forces will either stand still or move in a straight line called *Inertial Systems*. When in one inertial system x^μ, one can move into another one, $\tilde{x}^\mu$ by doing a *Lorentz Transformation*, which includes rotations, translations and boosts [12].

Describing the 4 space-time coordinates using the expression x^μ, $\mu = 0, 1, 2, 3$, and defining the infinitesimal distance between two events x^μ and $x^\mu + dx^\mu$ as dx^μ, then one can describe the *line-element* in flat space-time as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{2.2}$$

where the *Einstein summation convention* is used to sum over the repeated indices μ and ν. With this definition of the line-element one can define the *proper time* between two events in Minkowski space in the case where $ds^2 < 0$ as

$$d\tau^2 = -ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu, \tag{2.3}$$

measuring the time elapsed between two events as seen by an observer moving on a straight path (through space-time) between the events [5]. According to the theory of Special Relativity, the proper time $d\tau^2$ is the same in all Inertial Systems, making it an invariant under Lorentz transformations [12]. This can be written mathematically as

$$-\eta_{\mu\nu} dx^\mu dx^\nu = d\tau^2 = -\eta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu, \tag{2.4}$$

where x^μ and $\tilde{x}^\mu$ are the coordinates of the two inertial systems. In flat space-time, a particle not subject to external forces moves along a straight line, and it can be shown that the straight line is the curve that maximzes the proper time, i.e. no curve has a longer proper time than the one followed by non-accelerating particles [12]. In the expressions above, the Minkowksi metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.5}$$

is used to describe flat space-time, and thus this is also called *Minkowski Space*. The theory of Special Relativity is the study of Minkowski space.

2.2 General Space-Times

2.2.1 The Space-Time Metric

Since the General Theory of Relativity is a theory of geometry, and the metric is the object that manifests this geometry, it is of fundamental importance. In the evaluation of the proper time in flat space-time, the Minkowski metric is used, although, in general, one could have used any metric $g_{\mu\nu}$ thus describing a curved geometry of space-time. The space-time metric is a symmetric tensor $g_{\mu\nu} = g_{\nu\mu}$, and it is often useful to write it as a 4×4-matrix, as so

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{10} & g_{20} & g_{30} \\ g_{01} & g_{11} & g_{21} & g_{31} \\ g_{02} & g_{12} & g_{22} & g_{32} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}. \tag{2.6}$$

The metric is what determines the geometry of, and thus, distances in the space-time, and therefore the expressions for the proper time and the line-element in a general space-time is actually

$$d\tau^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu \tag{2.7}$$

To give a sense of the all-important role that the metric plays in GR, here's a list from [5] to illustrate: (1) The metric supplies a notion of "past" and "future"; (2) The metric allows the computation of path length and proper time; (3) The metric determines the "shortest distance" between two points, and therefore the motion of test particles; (4) The metric replaces the Newtonian gravitational field ϕ; (5) The metric provides a notion of locally inertial frames and therefore a sense of "no rotation"; (6) The metric determines causality, by defining the speed of light faster than which no signal can travel; (7) The metric replaces the traditional Euclidean three-dimensional dot product of Newtonian mechanics.

2.2.2 Geodesics

As mentioned above, in the flat Minkowski space of Special Relativity a particle not subject to external forces will follow the curve that maximizes its proper time, which happens to also be the straight line through the space-time. For more general space-times, this fact still applies, but now the curvature of space-time bends the straight lines, and the apparent trajectories aren't perceived as straight lines anymore.

To better understand this concept to people on the equator started walking straight north. At first their trajectories would be parallel, but in due time, as they get further and further north their trajectories slowly converge until reaching the north pole where they bump into each other. Their trajectories started out parallel, but still ended up crossing, a feat which is impossible in normal Euclidean geometry. (See Figure 2.1).

Another example is the one about straight flight paths seeming distorted on 2-dimensional maps and can be see in Figure 2.2.

Figure 2.1: The longitudinal lines on a globe are straight lines on the surface, but defy properties of straight lines in Euclidean geometry, for example, they seem parallel at the equator, but converge at the poles. (Picture: Wikipedia)

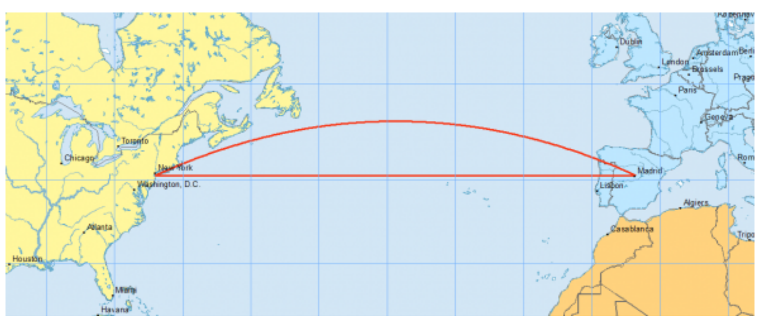

Figure 2.2: A straight line on a 2-dimensional map of the world is not shortest path on the actual curved surface of the earth but is percieved that way because of the distortion caused by the projection in the manufacturing of the map. maybe put a border around this one? (Picture: https://gisgeography.com/great-circle-geodesic-line-shortest-flight-path/)

There are infinitely many trajectories through different space-times but only a certain subset of trajectories follow straight curves through that space-time and these curves are called *geodesics*, and are defined precisely as the curves through space-time which maximize the proper time.

Imagine a curve through space-time between two events $x^\mu_{(1)}$ and $x^\mu_{(2)}$ parameterized by the parameter λ, then for the infinitesimal piece of the curve from $x^\mu(\lambda)$ to $x^\mu(\lambda)+x^\mu(\lambda+d\lambda)$ the proper time can be written as

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}d\lambda^2, \tag{2.8}$$

and then the proper time for the whole curve will be

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2}\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda. \tag{2.9}$$

Maximizing this integral leads to the *geodesic equation*[1]

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{2.10}$$

where $u^\mu = dx^\mu/d\tau$ and $a^\mu = d^2x^\mu/d\tau^2$ are the relativistic velocity and acceleration respectively and the Christoffel Symbol $\Gamma^\mu_{\nu\rho}$ is a function of the metric, and embodies the curvature of the space-time and is 0 for no curvature. In the latter case we see that the geodesic equation reduces to

$$\frac{d^2x^\mu}{d\tau^2} = a^\mu = 0, \tag{2.11}$$

precisely the motion with constant velocity along a straight line.

Newton Limit of the Geodesic Equation

Taking the Newton limit means assuming (1) weak gravity; (2) static gravity and (3) small velocities. The first assumption the assumption that we can write the metric as the Minkowskian metric plus a small perturbation. The second assumption is the assumption that the metric does not change over time, and the third assumption is that the velocities are small compared to the speed of light. They are all expressed mathematically as

$$(1) \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{where} \quad |h_{\mu\nu}| \ll 1, \qquad\qquad (2) \qquad \frac{\partial g_{\mu\nu}}{\partial t} = 0, \tag{2.12 a,b}$$

$$(3) \qquad \left|\frac{dx^\mu}{dt}\right| \ll c \quad \text{for} \quad i = 1,2,3. \tag{2.12 c}$$

In this limit the geodesic equation reduces to[2]

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i} \qquad \text{for} \quad \mu = i = 1,2,3, \tag{2.13}$$

and 0 in the case where $\mu = 0$. This can be rewritten as

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{1}{2}\boldsymbol{\nabla}h_{00}. \tag{2.14}$$

Comparing to Newtonian gravity which is written as

$$\frac{d^2\mathbf{x}}{dt^2} = -\boldsymbol{\nabla}\phi, \tag{2.15}$$

we can see that

$$h_{00} = -2\phi. \tag{2.16}$$

This means that the small perturbation to the metric is exactly 2 times the Newtonian gravitational potential ϕ confirming the fact that gravity is the same as the geometry of space-time in the theory of General Relativity.

[1] An excellent and concise derivation can be found in [12], but is apparent in virtually any book on General Relativity.
[2] Again we refer to [12] for the derivation.

2.3 The Einstein Field Equations

The Einstein field equations (EFEs) are a set of non-linear partial differential equations that are the general relativistic generalization to Poissons equation $\nabla^2\phi = 4\pi G\rho_m$, exactly as the metric is to the gravitational potential ϕ. The EFEs can be written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{2.17}$$

where $T_{\mu\nu}$ is the stress-energy tensor, $R_{\mu\nu}$ is the Ricci curvature tensor and R is its trace, called the Ricci scalar. The Ricci curvature tensor is a symmetric tensor that depends on $g_{\mu\nu}$ and its first and second derivatives, along with the Ricci scalar, describe the curvature of a spacetime. Equations (2.17) then relate said curvature to the energy, momentum and stress of that spacetime. [5]

Due to the non-linearity of the EFEs, and the fact that they are a set of 10 equations, it is cumbersome to find exact solutions, and in most cases even impossible. There do, however, exist exact solutions, such as the Schwarzschild and Kerr metrics, that have proven to be exceptionally useful for understanding black holes but which are limited in application due to their derivations assuming high levels of symmetry or simplicity, not representative of real world phenomena. For more practical applications of GR it is, therefore, useful and necessary to find approximate solutions using expansions in certain parameters. The most common way to accomplish this is by starting with a flat Minkowskian spacetime and then adding correction terms [17]. Two such methods will be explored here.

2.4 Post-Newtonian expansion

The post-Newtonian (PN) expansion is an expansion in the parameter $v^2/c^2 \sim GM/rc^2$, where v is the relative velocity, r is the relative distance and M is the total mass of the binary system [3]. This means that for an nth-order expansion one gets:

$$\sum_{i=0}^{n} v^{2i}G^{(n+1)-i} \tag{2.18}$$

as can be seen in Table 2.1.

	0PN		1PN		2PN		3PN		4PN		5PN		6PN		7PN			
1PM	(1	+	v^2	+	v^4	+	v^6	+	v^8	+	v^{10}	+	v^{12}	+	v^{14}	+	...)	G^1
2PM			(1	+	v^2	+	v^4	+	v^6	+	v^8	+	v^{10}	+	v^{12}	+	...)	G^2
3PM					(1	+	v^2	+	v^4	+	v^6	+	v^8	+	v^{10}	+	...)	G^3
4PM							(1	+	v^2	+	v^4	+	v^6	+	v^8	+	...)	G^4
5PM									(1	+	v^2	+	v^4	+	v^6	+	...)	G^5
6PM											(1	+	v^2	+	v^4	+	...)	G^6

Table 2.1: Comparison table of powers used for PN and PM approximations in the case of two non-rotating bodies. 0PN corresponds to the case of Newton's theory of gravitation. 0PM (not shown) corresponds to the Minkowsky flat space. (Source: Wikipedia [15])

The PN expansion has been one of the most important methods for finding approximate solutions to problems in GR, the two-body problem in particular. Since the expansion is made in the velocity, the PN expansion is formally valid at low velocities, compared to the speed of light, as well as in a weak gravitational field [4].

2.4.1 Einstein-Infeld-Hoffmann equations

Equations of motion for the relativistic two-body problem in first order Post-Newtonian expansion were first derived by Einstein, Infeld and Hoffmann (EIH), hence the name. This first order approximation is valid at low velocities, weak gravity and for arbitrary masses. Given the Hamiltonian from [2]

$$H_{\mathrm{EIH}} = \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)p^2 - \frac{1}{8}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\frac{p^4}{c^2} - \frac{G}{r}\left[m_1m_2 + \frac{1}{2}\left(1+3\frac{(m_1+m_2)^2}{m_1m_2}\right)\frac{p^2}{c^2} + \frac{1}{2}\frac{(\mathbf{p}\cdot\mathbf{r})^2}{r^2c^2}\right] + \frac{G^2}{r^2}\frac{m_1m_2(m_1+m_2)}{2c^2} \tag{2.19}$$

the Hamiltonian equations of motion are as follows:

$$\dot{\mathbf{r}} = \frac{\partial H_{\mathrm{EIH}}}{\partial \mathbf{p}} = \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\mathbf{p} - \frac{1}{2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)p^2\mathbf{p} - \frac{G}{r}\left[\left(1+3\frac{(m_1+m_2)^2}{m_1m_2}\right)\frac{\mathbf{p}}{c^2} + \frac{(\mathbf{p}\cdot\mathbf{r})}{r^2c^2}\mathbf{r}\right] \tag{2.20}$$

$$\dot{\mathbf{p}} = -\frac{\partial H_{\mathrm{EIH}}}{\partial \mathbf{r}} = -\frac{G}{r^3}\left[\left(m_1m_2 + \frac{1}{2}\left(1+3\frac{(m_1+m_2)^2}{m_1m_2}\right)\frac{p^2}{c^2} + \frac{3}{2}\frac{(\mathbf{p}\cdot\mathbf{r})^2}{r^2c^2}\right)\mathbf{r} - \frac{\mathbf{p}\cdot\mathbf{r}}{c^2}\mathbf{p}\right] + \frac{G^2}{r^4}\frac{m_1m_2(m_1+m_2)}{c^2}\mathbf{r} \tag{2.21}$$

where $\mathbf{r}$ refers to the relative distance between the particles and $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$ are the momenta of the particles in a center of mass frame. Lastly, the acceleration is found by taking the time derivative of $\dot{\mathbf{r}}$

$$\begin{aligned}\ddot{\mathbf{r}} = \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\dot{\mathbf{p}} - \frac{3}{2c^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)p^2\dot{\mathbf{p}} &- \frac{G}{r}\Bigg\{-\frac{\dot{r}}{r}\left[\left(1+3\frac{(m_1+m_2)^2}{m_1m_2}\right)\frac{\mathbf{p}}{c^2} + \frac{(\mathbf{p}\cdot\mathbf{r})^2}{r^2c^2}\mathbf{r}\right] \\ &+ \left(1+3\frac{(m_1+m_2)^2}{m_1m_2}\right)\frac{\dot{\mathbf{p}}}{c^2} \\ &+ \frac{(\dot{\mathbf{p}}\cdot\mathbf{r}+\mathbf{p}\cdot\dot{\mathbf{r}})\mathbf{r} + (\mathbf{p}\cdot\mathbf{r})\dot{\mathbf{r}}}{r^2c^2} - 2\frac{(\mathbf{p}\cdot\mathbf{r})\dot{r}}{r^3c^2}\mathbf{r}\Bigg\}\end{aligned} \tag{2.22}$$

The Nonrelativistic Limit **To include or not to include?**

We see that in the nonrelativistic limit, that is, $c \to \infty$ the EIH equations reduce to

$$\lim_{c\to\infty} H_{\mathrm{EIH}} = \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)p^2 - \frac{Gm_1m_2}{r} = H_N \tag{2.23}$$

$$\lim_{c\to\infty} \dot{\mathbf{p}} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} = \dot{\mathbf{p}}_N \tag{2.24}$$

$$\lim_{c\to\infty} \ddot{\mathbf{r}} = \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\dot{\mathbf{p}} = \ddot{\mathbf{r}}_N, \tag{2.25}$$

which are exactly the equations of classical Newtonian Mechanics.

2.5 Post-Minkowskian expansion

Another method of finding approximate solutions in GR is the post-Minkowskian (PM) expansion. It is found by approaching the problem from an effect field theory of quantum gravity, using relativistic scattering amplitudes [13]. This results in the PM expansion being only expanded in the parameter Gm/rc^2 but including all orders of v, unlike the PN expansion, this is due to scattering amplitudes appearing as series in powers of G [3]. The PM expansion is thus valid in a highly relativistic regime and weak gravity, and is therefore very useful in modeling ultra-relativistic scattering [4].

2.5.1 First order post-Minkowskian expansion

Given the post-Minkowskian Hamiltonian to first order from [6],

$$H_{1\mathrm{PM}}(p,r) = E_1 + E_2 + V_{1\mathrm{PM}}(p,r) = E_1 + E_2 + \frac{1}{E_1E_2}\frac{Gc_1}{r}, \tag{2.26}$$

where

$$c_1 = m_1^2m_2^2c^4 - 2\left(\frac{E_1E_2}{c^2} + p^2\right)^2, \quad E_i = \sqrt{m_i^2c^4 + p^2c^2} \tag{2.27}$$

the Hamiltonian equations of motion are:

$$\dot{\mathbf{r}} = \frac{\partial H_{1\mathrm{PM}}}{\partial \mathbf{p}} = \frac{\partial E_1}{\partial \mathbf{p}} + \frac{\partial E_2}{\partial \mathbf{p}} + \frac{G}{E_1^2E_2^2r}\left[E_1E_2\frac{\partial c_1}{\partial \mathbf{p}} - c_1\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right)\right] \tag{2.28}$$

$$\dot{\mathbf{p}} = -\frac{\partial H_{1\mathrm{PM}}}{\partial \mathbf{r}} = \frac{Gc_1}{E_1E_2r^3}\mathbf{r}, \tag{2.29}$$

where the equations are, again, formulated in a center of mass frame as described for the EIH equations of motion in section 2.4.1.

The acceleration can then be obtained: move to appendix?

$$\ddot{\mathbf{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}} \tag{2.30}$$

$$\begin{aligned}&+ \frac{G}{E_1^2E_2^2r}\Bigg\{-\left(\frac{\dot{r}}{r} + 2\frac{\dot{E}_1}{E_1} + 2\frac{\dot{E}_2}{E_2}\right)\left[E_1E_2\frac{\partial c_1}{\partial \mathbf{p}} - c_1\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right)\right] \\ &+ \Bigg[\left(\dot{E}_1E_2 + E_1\dot{E}_2\right)\frac{\partial c_1}{\partial \mathbf{p}} + E_1E_2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial c_1}{\partial \mathbf{p}} - \dot{c}_1\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right) \\ &- c_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}}E_2 + \frac{\partial E_1}{\partial \mathbf{p}}\dot{E}_2 + \dot{E}_1\frac{\partial E_2}{\partial \mathbf{p}} + E_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}}\right)\Bigg]\Bigg\}\end{aligned} \tag{2.31}$$

where

$$\frac{\partial c_1}{\partial \mathbf{p}} = -4\left(\frac{E_1E_2}{c^2} + p^2\right)\left[\frac{1}{c^2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right) + 2\mathbf{p}\right] \tag{2.32}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial c_1}{\partial \mathbf{p}} = -4\Bigg(&\left[\frac{1}{c^2}\left(\dot{E}_1E_2 + E_1\dot{E}_2\right) + 2\mathbf{p}\cdot\dot{\mathbf{p}}\right]\left[\frac{1}{c^2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right) + 2\mathbf{p}\right] \\ &+ \left(\frac{E_1E_2}{c^2} + p^2\right)\left[\frac{1}{c^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}}E_2 + \frac{\partial E_1}{\partial \mathbf{p}}\dot{E}_2 + \dot{E}_1\frac{\partial E_2}{\partial \mathbf{p}} + E_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}}\right) + 2\dot{\mathbf{p}}\right]\Bigg)\end{aligned} \tag{2.33}$$

$$\dot{c}_1 = -4\left(\frac{E_1E_2}{c^2} + p^2\right)\left[\frac{1}{c^2}\left(\dot{E}_1E_2 + E_1\dot{E}_2\right) + 2\mathbf{p}\cdot\dot{\mathbf{p}}\right] \tag{2.34}$$

and

$$\frac{\partial E_i}{\partial \mathbf{p}} = \frac{c^2}{E_i}\mathbf{p} \tag{2.35}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_i}{\partial \mathbf{p}} = \frac{c^2}{E_i}\dot{\mathbf{p}} - \frac{c^2}{E_i^2}\dot{E}_i\mathbf{p} \tag{2.36}$$

$$\dot{E}_i = \frac{\partial E_i}{\partial \mathbf{p}}\cdot\dot{\mathbf{p}}. \tag{2.37}$$

The Nonrelativistic Limit **To include or not to include?**

We see that in the nonrelativistic limit, the 1PM equations reduce to

$$\lim_{c\to\infty} H_{1\mathrm{PM}} = m_1 + m_2 - \frac{Gm_1m_2}{r} \tag{2.38}$$

$$\lim_{c\to\infty} \dot{\mathbf{p}} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \tag{2.39}$$

$$\lim_{c\to\infty} \ddot{\mathbf{r}} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\dot{\mathbf{p}} + \frac{G}{m_1^2m_2^2r}\Bigg\{ -\frac{\dot{r}}{r}\left[-8m_1^2m_2^2\mathbf{p} + m_1^2m_2^2\left(\frac{m_2}{m_1} + \frac{m_1}{m_2}\right)\mathbf{p}\right] \tag{2.40}$$

$$- 8m_1m_2(2(\mathbf{p}\cdot\dot{\mathbf{p}})\mathbf{p} + \dot{\mathbf{p}}) + 8\big(m_1^2 + m_2^2\big)(\mathbf{p}\cdot\dot{\mathbf{p}})\mathbf{p} \tag{2.41}$$

$$+ m_1^2m_2^2\left(\frac{m_2}{m_1} + \frac{m_1}{m_2}\right)\dot{\mathbf{p}}\Bigg\} \tag{2.42}$$

2.5.2 Second order post-Minkowskian expansion

Now adding the second order PM correction term the post-Minkowskian Hamiltonian becomes:

$$H_{2\mathrm{PM}}(p,r) = E_1 + E_2 + V_{1\mathrm{PM}}(p,r) + V_{2\mathrm{PM}}(p,r) \tag{2.43}$$

where

$$V_{2\mathrm{PM}}(p,r) = \frac{G^2}{E_1E_2r^2}\Bigg[\frac{1}{4c^2}\left(\frac{c_\triangleright}{m_1} + \frac{c_\triangleleft}{m_2}\right) + \frac{c_1(E_1+E_2)}{E_1E_2}\left(\frac{c_1}{2}\left(\frac{1}{(E_1+E_2)^2} - \frac{1}{E_1E_2}\right) - \frac{4}{c^2}\left(\frac{E_1E_2}{c^2} + p^2\right)^2\right)\Bigg]. \tag{2.44}$$

The equations of motions from the first three terms of the Hamiltonian in 2.43 have already been calculated in section 2.5.1. This section will, therefore, only show the derivations of the $V_{2\mathrm{PM}}$ term using the same definitions as in equations 2.32 - 2.37 along with:

$$c_\triangleright = 3m_1^2\left(m_1^2m_2^2c^4 - 5\left(\frac{E_1E_2}{c^2} + p^2\right)^2\right) \tag{2.45}$$

$$c_\triangleleft = 3m_2^2\left(m_1^2m_2^2c^4 - 5\left(\frac{E_1E_2}{c^2} + p^2\right)^2\right) \tag{2.46}$$

$$\frac{\partial c_\triangleright}{\partial \mathbf{p}} = -30m_1^2\left(\frac{E_1E_2}{c^2} + p^2\right)\left(\frac{1}{c^2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right) + 2\mathbf{p}\right) \tag{2.47}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial c_\triangleright}{\partial \mathbf{p}} = -30m_1^2\Bigg[\left(\frac{\dot{E}_1E_2 + E_1\dot{E}_2}{c^2} + 2\mathbf{p}\cdot\dot{\mathbf{p}}\right)\left(\frac{1}{c^2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right) + 2\mathbf{p}\right) + \left(\frac{E_1E_2}{c^2} + p^2\right)\left(\frac{1}{c^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}}E_2 + \frac{\partial E_1}{\partial \mathbf{p}}\dot{E}_2 + \dot{E}_1\frac{\partial E_2}{\partial \mathbf{p}} + E_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}}\right) + 2\dot{\mathbf{p}}\right)\Bigg] \tag{2.48}$$

$$\dot{c}_\triangleright = -30m_1^2\left(\frac{E_1E_2}{c^2} + p^2\right)\left(\frac{\dot{E}_1E_2 + E_1\dot{E}_2}{c^2} + 2\mathbf{p}\cdot\dot{\mathbf{p}}\right) \tag{2.49}$$

where $c_\triangleleft = {}^{m_2^2}/_{m_1^2}c_\triangleright$ has been used for simplification.

Thus: move to appendix?

$$-\frac{\partial V_{2\mathrm{PM}}}{\partial \mathbf{r}} = \frac{G^2}{r^4}\frac{2\mathbf{r}}{E_1E_2}\left[\frac{1}{4c^2}\left(\frac{c_\triangleright}{m_1}+\frac{c_\triangleleft}{m_2}\right)\right.$$
$$\left.+\frac{c_1(E_1+E_2)}{E_1E_2}\left(\frac{c_1}{2}\left(\frac{1}{(E_1+E_2)^2}-\frac{1}{E_1E_2}\right)-\frac{4}{c^2}\left(\frac{E_1E_2}{c^2}+p^2\right)^2\right)\right] \tag{2.50}$$

$$\frac{\partial V_{2\mathrm{PM}}}{\partial \mathbf{p}} = \frac{G^2}{E_1E_2r^2}\left\{-\frac{\frac{\partial E_1}{\partial \mathbf{p}}E_2+E_1\frac{\partial E_2}{\partial \mathbf{p}}}{E_1E_2}\right.$$
$$\cdot\left[\frac{1}{4c^2}\left(\frac{c_\triangleright}{m_1}+\frac{c_\triangleleft}{m_2}\right)+\frac{c_1(E_1+E_2)}{E_1E_2}\left(\frac{c_1}{2}\left(\frac{1}{(E_1+E_2)^2}-\frac{1}{E_1E_2}\right)-\frac{4}{c^2}\left(\frac{E_1E_2}{c^2}+p^2\right)^2\right)\right]$$
$$+\frac{1}{4c^2}\left(\frac{1}{m_1}+\frac{m_2}{m_1^2}\right)\frac{\partial c_\triangleright}{\partial \mathbf{p}}$$
$$+\left[\frac{E_1+E_2}{E_1E_2}\frac{\partial c_1}{\partial \mathbf{p}}+\frac{c_1}{E_1E_2}\left(\frac{\partial E_1}{\partial \mathbf{p}}+\frac{\partial E_1}{\partial \mathbf{p}}-\frac{E_1+E_2}{E_1E_2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2+E_1\frac{\partial E_2}{\partial \mathbf{p}}\right)\right)\right]$$
$$\cdot\left(\frac{c_1}{2}\left(\frac{1}{(E_1+E_2)^2}-\frac{1}{E_1E_2}\right)-\frac{4}{c^2}\left(\frac{E_1E_2}{c^2}+p^2\right)^2\right)$$
$$+\frac{c_1(E_1+E_2)}{E_1E_2}\left[\frac{1}{2}\frac{\partial c_1}{\partial \mathbf{p}}\left(\frac{1}{(E_1+E_2)^2}-\frac{1}{E_1E_2}\right)+\frac{c_1}{2}\left(-2\frac{\frac{\partial E_1}{\partial \mathbf{p}}+\frac{\partial E_2}{\partial \mathbf{p}}}{(E_1+E_2)^3}+\frac{\frac{\partial E_1}{\partial \mathbf{p}}E_2+E_1\frac{\partial E_2}{\partial \mathbf{p}}}{E_1^2E_2^2}\right)\right.$$
$$\left.\left.-\frac{8}{c^2}\left(\frac{E_1E_2}{c^2}+p^2\right)\left(\frac{1}{c^2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2+E_1\frac{\partial E_2}{\partial \mathbf{p}}\right)+2\mathbf{p}\right)\right]\right\} \tag{2.51}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V_{2\mathrm{PM}}}{\partial \mathbf{p}} = &-\left(2\frac{\dot{r}}{r} + \frac{\dot{E}_1 E_2 + E_1\dot{E}_2}{E_1 E_2}\right)\frac{\partial V_{2\mathrm{PM}}}{\partial \mathbf{p}} \\
&+ \frac{G^2}{E_1E_2r^2}\Bigg\{ -\Bigg[\frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}}E_2 + \frac{\partial E_1}{\partial \mathbf{p}}\dot{E}_2 + \dot{E}_1\frac{\partial E_2}{\partial \mathbf{p}} + E_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}}}{E_1E_2} - \frac{\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}}{E_1^2E_2^2}\Big(\dot{E}_1E_2 + E_1\dot{E}_2\Big)\Bigg] \\
&\qquad\qquad \cdot\Bigg[\frac{1}{4c^2}\left(\frac{c_\triangleright}{m_1} + \frac{c_\triangleleft}{m_2}\right) + \frac{c_1(E_1+E_2)}{E_1E_2}\Bigg(\frac{c_1}{2}\left(\frac{1}{(E_1+E_2)^2} - \frac{1}{E_1E_2}\right) - \frac{4}{c^2}\left(\frac{E_1E_2}{c^2} + p^2\right)^2\Bigg)\Bigg] \\
&- \frac{\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}}{E_1E_2}\Bigg[\frac{\dot{c}_\triangleright}{4c^2}\left(\frac{1}{m_1} + \frac{m_2}{m_1^2}\right) + \Bigg(\frac{\dot{c}_1(E_1+E_2) + c_1(\dot{E}_1+\dot{E}_2)}{E_1E_2} - \frac{c_1(E_1+E_2)\Big(\dot{E}_1E_2 + E_1\dot{E}_2\Big)}{E_1^2E_2^2}\Bigg) \\
&\qquad\qquad \cdot\Bigg(\frac{c_1}{2}\left(\frac{1}{(E_1+E_2)^2} - \frac{1}{E_1E_2}\right) - \frac{4}{c^2}\left(\frac{E_1E_2}{c^2} + p^2\right)^2\Bigg) \\
&\qquad + \frac{c_1(E_1+E_2)}{E_1E_2}\Bigg(\frac{\dot{c}_1}{2}\left(\frac{1}{(E_1+E_2)^2} - \frac{1}{E_1E_2}\right) + \frac{c_1}{2}\Bigg(-2\frac{\dot{E}_1+\dot{E}_2}{(E_1+E_2)^3} + \frac{\dot{E}_1E_2 + E_1\dot{E}_2}{E_1^2E_2^2}\Bigg) \\
&\qquad\qquad - \frac{8}{c^2}\left(\frac{E_1E_2}{c^2} + p^2\right)\Bigg(\frac{\dot{E}_1E_2 + E_1\dot{E}_2}{c^2} + 2\mathbf{p}\cdot\dot{\mathbf{p}}\Bigg)\Bigg)\Bigg] \\
&+ \frac{1}{4c^2}\left(\frac{1}{m_1} + \frac{m_2}{m_1^2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial c_\triangleright}{\partial \mathbf{p}} \\
&+ \Bigg[\Bigg(\frac{\dot{E}_1+\dot{E}_2}{E_1E_2} - \frac{(E_1+E_2)(\dot{E}_1E_2 + E_1\dot{E}_2)}{E_1^2E_2^2}\Bigg)\frac{\partial c_1}{\partial \mathbf{p}} + \frac{E_1+E_2}{E_1E_2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial c_1}{\partial \mathbf{p}} \\
&\quad + \Bigg(\frac{\dot{c}_1}{E_1E_2} - \frac{c_1(\dot{E}_1E_2 + E_1\dot{E}_2)}{E_1^2E_2^2}\Bigg)\Bigg(\frac{\partial E_1}{\partial \mathbf{p}} + \frac{\partial E_1}{\partial \mathbf{p}} - \frac{E_1+E_2}{E_1E_2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right)\Bigg) \\
&\quad + \frac{c_1}{E_1E_2}\Bigg(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}} - \Bigg(\frac{\dot{E}_1+\dot{E}_2}{E_1E_2} - \frac{(E_1+E_2)(\dot{E}_1E_2 + E_1\dot{E}_2)}{E_1^2E_2^2}\Bigg)\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right) \\
&\qquad\qquad - \frac{E_1+E_2}{E_1E_2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}}E_2 + \frac{\partial E_1}{\partial \mathbf{p}}\dot{E}_2 + \dot{E}_1\frac{\partial E_2}{\partial \mathbf{p}} + E_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}}\right)\Bigg)\Bigg] \\
&\quad \cdot\Bigg(\frac{c_1}{2}\left(\frac{1}{(E_1+E_2)^2} - \frac{1}{E_1E_2}\right) - \frac{4}{c^2}\left(\frac{E_1E_2}{c^2} + p^2\right)^2\Bigg) \\
&+ \Bigg[\frac{E_1+E_2}{E_1E_2}\frac{\partial c_1}{\partial \mathbf{p}} + \frac{c_1}{E_1E_2}\Bigg(\frac{\partial E_1}{\partial \mathbf{p}} + \frac{\partial E_1}{\partial \mathbf{p}} - \frac{E_1+E_2}{E_1E_2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right)\Bigg)\Bigg] \\
&\quad \cdot\Bigg(\frac{\dot{c}_1}{2}\left(\frac{1}{(E_1+E_2)^2} - \frac{1}{E_1E_2}\right) + \frac{c_1}{2}\Bigg(-2\frac{\dot{E}_1+\dot{E}_2}{(E_1+E_2)^3} + \frac{\dot{E}_1E_2 + E_1\dot{E}_2}{E_1^2E_2^2}\Bigg) \\
&\qquad\qquad - \frac{8}{c^2}\left(\frac{E_1E_2}{c^2} + p^2\right)\Bigg(\frac{\dot{E}_1E_2 + E_1\dot{E}_2}{c^2} + 2\mathbf{p}\cdot\dot{\mathbf{p}}\Bigg)\Bigg) \\
&+ \Bigg(\frac{\dot{c}_1(E_1+E_2) + c_1(\dot{E}_1+\dot{E}_2)}{E_1E_2} - \frac{c_1(E_1+E_2)\Big(\dot{E}_1E_2 + E_1\dot{E}_2\Big)}{E_1^2E_2^2}\Bigg) \\
&\quad \cdot\Bigg[\frac{1}{2}\frac{\partial c_1}{\partial \mathbf{p}}\left(\frac{1}{(E_1+E_2)^2} - \frac{1}{E_1E_2}\right) + \frac{c_1}{2}\Bigg(-2\frac{\frac{\partial E_1}{\partial \mathbf{p}} + \frac{\partial E_2}{\partial \mathbf{p}}}{(E_1+E_2)^3} + \frac{\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}}{E_1^2E_2^2}\Bigg) \\
&\qquad\qquad - \frac{8}{c^2}\left(\frac{E_1E_2}{c^2} + p^2\right)\Bigg(\frac{1}{c^2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2 + E_1\frac{\partial E_2}{\partial \mathbf{p}}\right) + 2\mathbf{p}\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c_1(E_1+E_2)}{E_1E_2}\Bigg[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial c_1}{\partial \mathbf{p}}\left(\frac{1}{(E_1+E_2)^2}-\frac{1}{E_1E_2}\right)+\frac{1}{2}\frac{\partial c_1}{\partial \mathbf{p}}\left(-2\frac{\dot{E}_1+\dot{E}_2}{(E_1+E_2)^3}+\frac{\dot{E}_1E_2+E_1\dot{E}_2}{E_1^2E_2^2}\right)\\
&\qquad+\frac{\dot{c}_1}{2}\left(-2\frac{\frac{\partial E_1}{\partial \mathbf{p}}+\frac{\partial E_2}{\partial \mathbf{p}}}{(E_1+E_2)^3}+\frac{\frac{\partial E_1}{\partial \mathbf{p}}E_2+E_1\frac{\partial E_2}{\partial \mathbf{p}}}{E_1^2E_2^2}\right)\\
&\qquad+\frac{c_1}{2}\Bigg(-\frac{2}{(E_1+E_2)^3}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}}+\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2+E_1\frac{\partial E_2}{\partial \mathbf{p}}\right)\left(\dot{E}_1E_2+E_1\dot{E}_2\right)\right)\\
&\qquad\qquad+6\frac{\frac{\partial E_1}{\partial \mathbf{p}}+\frac{\partial E_2}{\partial \mathbf{p}}}{(E_1+E_2)^3}\left(\dot{E}_1+\dot{E}_2\right)+\frac{1}{E_1^2E_2^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}}E_2+\frac{\partial E_1}{\partial \mathbf{p}}\dot{E}_2+\dot{E}_1\frac{\partial E_2}{\partial \mathbf{p}}+E_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}}\right)\Bigg)\\
&\qquad-\frac{8}{c^2}\Bigg(\left(\frac{\dot{E}_1E_2+E_1\dot{E}_2}{c^2}+2\mathbf{p}\cdot\dot{\mathbf{p}}\right)\left(\frac{1}{c^2}\left(\frac{\partial E_1}{\partial \mathbf{p}}E_2+E_1\frac{\partial E_2}{\partial \mathbf{p}}\right)+2\mathbf{p}\right)\\
&\qquad\qquad+\left(\frac{E_1E_2}{c^2}+p^2\right)\left(\frac{1}{c^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_1}{\partial \mathbf{p}}E_2+\frac{\partial E_1}{\partial \mathbf{p}}\dot{E}_2+\dot{E}_1\frac{\partial E_2}{\partial \mathbf{p}}+E_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E_2}{\partial \mathbf{p}}\right)+2\dot{\mathbf{p}}\right)\Bigg)\Bigg]\Bigg\}
\end{aligned}
\tag{2.52}
$$

2.6 Relation between PN and PM expansions

Due to the PM expansion including all orders of v for each order of G, as previously mentioned, it is possible to recover the PN from the PM expansions. This is considered to be another of the strengths of the PM expansion. Looking at table 2.1, one can see that Taylor expanding the velocity term of the 1PM expansion to first order in v^2 and to zeroth order in 2PM, one recovers the 1PN expansion. expand on this

Chapter 3

Numerical Methods

As seen in the previous chapter, the equations of motion for higher orders of the PN- and PM-expansions get very large, and thus, doing the calculations by hand becomes quite unfeasible, and also, it is very difficult to find analytical solutions to the integrals of the equations of motion. SOURCE Therefore many scientists turn to the power of computers to do the simulation of complex problems, as they can do the hard work and calculation to any given order of approximation, costing only time. In this section the numerical implementation in Python of a two-body problem simulator is presented.

3.1 Units

When simulating any physical problem numerically, it is important to be consistent with ones system of units. The computer does not know kilometers from Coulombs, so it is important to include the right conversion factors etc.

Imagine simulating the Earth orbiting the Sun. Their masses in different units are

$$
\begin{aligned}
M_E &= 5.965 \times 10^{24}\,\mathrm{kg}, & M_\odot &= 1.988 \times 10^{30}\,\mathrm{kg}, && (3.1)\\
&= 3.146 \times 10^{-3}\,M_{\mathrm{Jup}}, & &= 1047\,M_{\mathrm{Jup}}, && (3.2)\\
&= 3.003 \times 10^{-6}\,M_\odot, & &= M_\odot, && (3.3)\\
&= 3.597 \times 10^{51}\,m_{\mathrm{p}}, & &= 1.197 \times 10^{57}\,m_{\mathrm{p}}. && (3.4)
\end{aligned}
$$

The choice of units is an arbitrary one, but it makes sense to have units that reflect well the scales of the problem. Another thing to take into account is that Pythons floating point numbers[1] have a limited range of around $\sim (10^{-308},\ 10^{308})$,[2] which makes exceedingly large or small numbers a problem in equations. Thus we want values that are representative of our problem and which don't stray too far from 0.

Simulating gravity using classical physics and relativity on a solar system type scale an appropriate choice of units could be

$$
\begin{aligned}
[\mathrm{mass}] &= 1 \cdot M_\odot = 1.989 \times 10^{30}\,\mathrm{kg} && (3.5)\\
[\mathrm{distance}] &= 1 \cdot GM_\odot/c^2\ = 1477\,\mathrm{m} && (3.6)\\
[\mathrm{time}] &= 1 \cdot GM_\odot/c^3 = 4.925 \times 10^{-6}\,\mathrm{s}, && (3.7)
\end{aligned}
$$

with the rest of the units being expressed as a linear combination of these.[3] Now giving the values of the problem to the computer, these units are used to *non-dimensionalize*, and then to *re-dimensionalize*, so to speak.

[1] More can be read on floating points here: https://www.geeksforgeeks.org/python-float-type-and-its-methods/ (accessed: 02/06/2022).

[2] Found at https://note.nkmk.me/en/python-sys-float-info-max-min/ (accessed: 01/06/2022).

[3] Except for things like Ampere or Coulomb, but these are not relevant in this context.

As an example imagine the Earth orbiting the Sun with parameters[4] in our non-dimensionalized units:

$$m_1 = 5.972 \times 10^{24}\,\mathrm{kg} \xrightarrow{\cdot M_\odot^{-1}} 3.003 \times 10^{-6}, \tag{3.8}$$

$$m_2 = 1.989 \times 10^{30}\,\mathrm{kg} \xrightarrow{\cdot M_\odot^{-1}} 1, \tag{3.9}$$

$$r_{\text{init}} = 1.496 \times 10^{8}\,\mathrm{km} \xrightarrow{\cdot c^2/GM_\odot} 1.013 \times 10^{8}, \tag{3.10}$$

$$v_{\text{init}} = 2.978 \times 10^{4}\,\mathrm{m/s} \xrightarrow{\cdot c^{-1}} 10^{-4}, \tag{3.11}$$

where r_{init} and v_{init} is the initial relative distance and tangential speed. The program takes in these, simulates, and then returns dimensionless values which are then converted back into SI with the chosen units. An example could be

$$T_{\text{period}} = 6.411 \times 10^{12} \xrightarrow{\cdot GM_\odot/c^3} 365.3\,\text{days}, \tag{3.12}$$

$$v_{\text{mean}} = 9.934 \times 10^{-5} \xrightarrow{\cdot c} 29.78\,\mathrm{km/s}, \tag{3.13}$$

$$r_{\text{perihelion}} = 1.030 \times 10^{8} \xrightarrow{\cdot GM_\odot/c^2} 152.1 \times 10^{6}\,\mathrm{km}. \tag{3.14}$$

In this way one can turn many different physical problems into something digestible for a computer, for it to crunch the numbers and give results unobtainable with conventional calculation.

3.2 Numerically Solving the Two-Body Problem

To solve the problem, first define the initial state of two particles. Here their positions $\mathbf{p}_i$, velocities $\mathbf{v}_i$ and masses m_i are relevant.[5]

$$s_0 = [\mathbf{r}_1,\ \mathbf{v}_1,\ m_1,\ \mathbf{r}_2,\ \mathbf{v}_2,\ m_2,\ t = 0], \tag{3.15}$$

Then choose the appropriate model Hamiltonian in the center of mass frame and derive the equations of motion in that frame

$$H(\mathbf{p},\mathbf{r}) \longrightarrow \begin{aligned} -\frac{\partial H}{\partial \mathbf{r}} &= \dot{\mathbf{p}}\,(\mathbf{p},\ \mathbf{r}), && (3.16)\\ \frac{\partial H}{\partial \mathbf{p}} &= \dot{\mathbf{r}}\,(\mathbf{p},\ \mathbf{r}), && (3.17)\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial \mathbf{p}} &= \ddot{\mathbf{r}}\,(\mathbf{p},\ \mathbf{r},\ \dot{\mathbf{p}}), && (3.18) \end{aligned}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the relative vector between the particles and $\mathbf{p} = \mathbf{p}_1^{\text{CM}} = -\mathbf{p}_2^{\text{CM}}$ is their COM momentum (opposite but equal for the particles). From this one gets a set of coupled differential equations

$$(i) \quad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \dot{\mathbf{r}} = \mathbf{v}, \qquad (ii) \quad \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \ddot{\mathbf{r}} = \mathbf{a}. \tag{3.19}$$

Using the results from Section 1.1.1 one can transform the initial conditions to the center of mass frame

$$s_0 = [\mathbf{r},\ \mathbf{p},\ m_1,\ m_2,\ t = 0], \tag{3.20}$$

and calculate the center of mass acceleration, which can then be iteratively and discretely integrated using an integrator of choice. elaborate with examples (in appendix?).

[4] Taken from https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html (accessed: 01/06/2022)
[5] Note that here we haven't specified the coordinates in which these variables are given.

3.2.1 Integration

The integration will occur within an iterative function $\mathcal{I}$ that takes the states of the system at time t, $s(t)$ and returns the state $s(t+dt)$,

$$\mathcal{I}(s(t)) = s(t+dt), \tag{3.21}$$

whereby the integrator of choice will be that of a symplectic Euler integration

$$\mathbf{a}(t) = \mathbf{a}(t) \tag{3.22}$$
$$\mathbf{v}(t+dt) = \mathbf{v}(t) + \mathbf{a}(t)\,dt \tag{3.23}$$
$$\mathbf{r}(t+dt) = \mathbf{r}(t) + \mathbf{v}(t+dt)\,dt \tag{3.24}$$

chosen due to its simplicity as well as ability to give quite precise results. Built-in integrators in Python had issues with not being energy-conserving and higher order symplectic integration did not give significantly better results than first order symplectic (Euler) integration.

As a final step for updating the state in each iteration, the newly found velocity has to be transformed to a COM momentum (since the acceleration is a function of momentum):

$$\mathbf{v}(t+dt) \quad \longrightarrow \quad \mathbf{p} = \gamma\mu\mathbf{v} \tag{3.25}$$

where $\gamma = 1/\sqrt{1-v^2/c^2}$ and μ is the reduced mass. Running the iterative function starting at $t=0$ and up to some final time t_f, will ultimately result in an array of relative positions $[\mathbf{r}]$, that can then be transformed back to arrays of $[\mathbf{r}_1]$ and $[\mathbf{r}_2]$, using equations (1.2), that will then describe the motion of each particle.

Chapter 4

Results

Having derived the equations of motion for a binary system in various expansions, one can now simulate and analyze the system. This will mainly consist of analyzing parameters in two states for all expansions, that of bound and scattering states.

4.1 Scattering

4.1.1 Analytical expressions

In order to properly analyze the simulated scattering, one has to be able to compare to some analytically derived values for the scattering. For the Newtonian case one has the well known expression, here taken from [10]:

$$\Delta\theta_{\mathrm{N}} = 2\arctan\left(\frac{G(m_1+m_2)}{v_\infty^2 b}\right) \tag{4.1}$$

where v_∞ is the initial velocity of the scattering particle "infinitely" far away and b is the impact parameter, i.e. the closest distance between the two particles if the scattering particle would keep traveling in a straight line.

For the 2PM case one has the following expression, taken from [6]:

$$\Delta\theta_{\mathrm{2PM}} = \frac{Gf_1}{p_0 L} + \frac{G^2 f_2 \pi}{2L^2} \tag{4.2}$$

where L is the angular momentum,

$$f_1 = -\frac{2c_1}{(E_1+E_2)/c}, \quad f_2 = -\frac{1}{2(E_1+E_2)/c}\left(\frac{c_\triangleright}{m_1} + \frac{c_\triangleleft}{m_2}\right), \quad p_0 = \sqrt{\frac{\left(E_1E_2/c^2 - p^2\right)^2 - m_1^2 m_2^2 c^4}{(E_1+E_2)^2/c^2}}, \tag{4.3}$$

c_1 and $c_\triangleright$ and $c_\triangleleft$ are as defined in equations (2.27) and (2.45) - (2.46). The 1PM scattering angle is then found analytically by dropping the second term in equation (4.2).

4.1.2 Scattering simulations

As mentioned before, one of the PM expansion's main strengths lies within the regime of ultra-relativistic scattering, $v \lesssim c$, with small deviation angle [4]. This can be seen in figure 4.1, where the EIH simulation completely breaks down at $v = 0.9c$, even showing repulsive behavior and scattering off in the wrong direction,

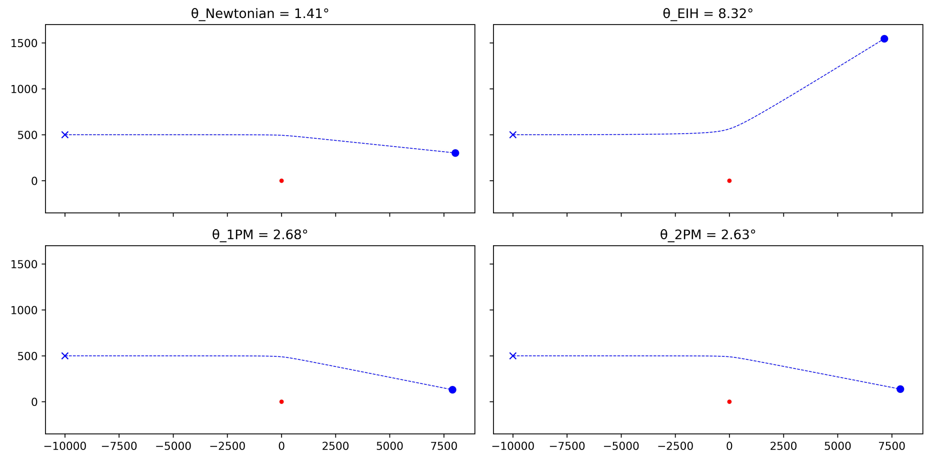

Figure 4.1: Ultra-relativistic scattering in eccentric test-body limit at $v = 0.9c$, Newton compared with three different expansions; EIH, 1PM, 2PM

meanwhile the 1PM and 2PM persist. Interestingly, the Newtonian simulation does not break down either but it does, however, return a considerably lower value than the PM simulations.

To calculate the scattering angle from our simulations we do the following:

$$\Delta\theta_{\text{simulation}} = \pi - \arccos\left(\frac{(\mathbf{x}_i - \mathbf{b}) \cdot (\mathbf{x}_f - \mathbf{b})}{\|\mathbf{x}_i - \mathbf{b}\| \|\mathbf{x}_f - \mathbf{b}\|}\right) \tag{4.4}$$

where $\mathbf{x}_i$ and $\mathbf{x}_f$ are the initial and final positions of the test body, and $\mathbf{b}$ is the vector with the magnitude of the impact parameter b pointing in the direction perpendicular to the initial momentum of the test body $\mathbf{p}$, which we subtract from $\mathbf{x}_i$ and $\mathbf{x}_f$ in order to move the point of closest approach to the origo.

4.1.3 Scattering angle vs angular momentum

The relationship between scattering angle and angular momentum at velocities between $0.3c \leq v \leq 0.6$ can be seen in figure 4.2. Figure 4.3 shows the same relationship at ultra-relativistic velocities $0.6c \leq v \leq 0.9$ where the EIH case has been omitted, due to the EIH simulation failing miserably[1] at high velocities, as well as the Newtonian not being relevant. Both of the figures show the same sort of behavior for the 1PM and 2PM cases, the 1PM scattering angles being ever so slightly higher than the 2PM in the simulation while being noticeably smaller in the analytical calculations. This could be indicative of either some failure in deriving the equations of motion for the 1PM and 2PM, failure in the code or the analytical expressions not being the right ones to use for this situation. When plotting the simulated values along with the analytically calculated results it is clear, however, that the differences are in general not very large as can be seen in figure 4.4. (Put in reflection/conclusion)

This relationship is of significance... why is it of significance and in what context? Talk to Johan maybe.

4.2 Bound

Bound state systems are systems where the two masses orbit their mutual center of mass, and both orbits are then characterized by their elliptical (in the Newtonian case) or quasi-elliptical forms. An important parameter of a bound state system is it's eccentricity

something about bound states being like the sun and earth and are the most familiar.

[1] To quote our supervisor...

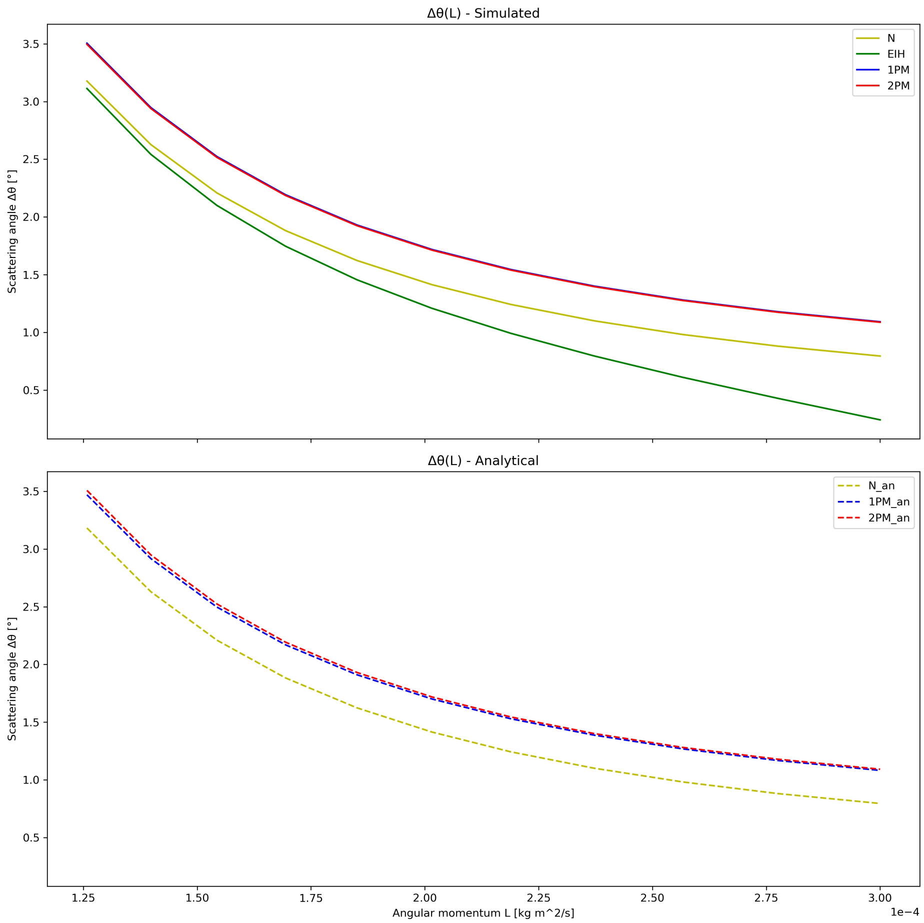

Figure 4.2: Scattering angle as a function of angular momentum simulated above and analytical results below, Newton compared with three different expansions; EIH, 1PM, 2PM

4.2.1 Eccentricity

The eccentricity of an orbit is defined as

$$e = \sqrt{1 - \frac{b^2}{a^2}}, \tag{4.5}$$

where a and b are the semi-major and -minor axes, respectively, of the first annual ellipse[2] of the orbit. (See Figure 4.5)

[2] That is, the (quasi-)ellipse that one mass forms after one "year" of it's orbit.

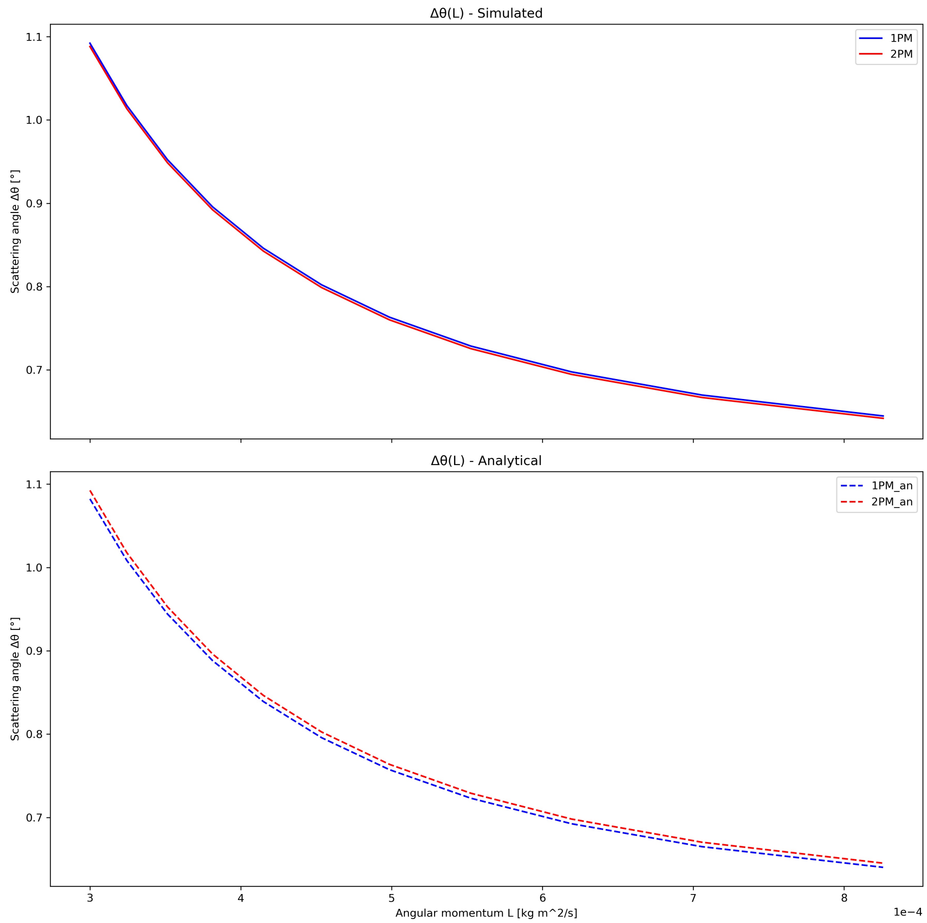

Figure 4.3: Scattering angle as a function of angular momentum at ultra-relativistic velocities simulated above and analytical results below, for 1PM and 2PM

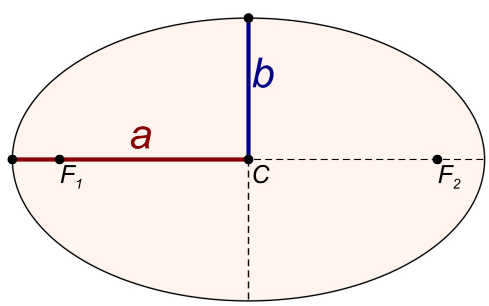

Figure 4.5: An ellipse with clear definitions of it's semi-major (a) and semi-minor (b) axes. It's foci F_1 and F_2 and it's center C can also be seen.

4.2.2 Precession of the orbits

One of the greatest confirmations of the validity of General Relativity is its ability to explain the precession of the orbit of the planet Mercury [12]. This section will compare how much the various expansions contribute to this precession in N? different cases, the test-body limit and similar mass binary system, both with eccentric orbits. (We have to explain what eccentricity is)

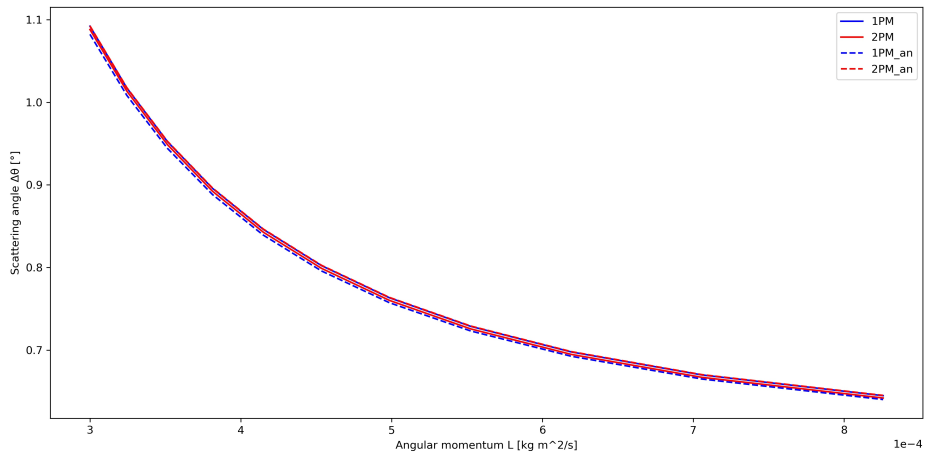

Figure 4.4: Scattering angle as a function of angular momentum at ultra-relativistic velocities, simulations and analytical results in one plot, for 1PM and 2PM

Analytical expressions

In the Newtonian case physicists have obtained analytical solutions to the orbits in the form of ellipses, parabolas and hyperbolas [10], showing that there will be no precession.

$$\Delta\phi = 0. \tag{4.6}$$

In the EIH and 1PM cases we use Robertson's formula (which is to order 1 in G taken from [8], although one has to be careful because these regimes both include special relativistic effects, while the formula doesn't. (This needs a citation!)

$$\Delta\phi = \frac{6\pi G(m_1 + m_2)}{c^2 a(1 - e^2)}, \tag{4.7}$$

where a is the semi-major axis of the (quasi-Newtonian) relative orbit.

In the 2PM case we use the following formula taken from [7]

$$\Delta\phi = \frac{3\pi}{2}\left(\frac{Gm_1m_2}{L}\right)^2\left(\frac{E}{M}\right)(5\gamma^2 - 1), \tag{4.8}$$

where $L = \|\mathbf{r} \times \mathbf{p}\|$ is the angular momentum $E = M\sqrt{1 + 2\nu(\gamma - 1)}$ is the total energy with $M = m_1 + m_2$ and $\nu = m_1m_2/M^2$.

Precession in eccentric test-body limit

This case of an eccentric orbit of a small mass around a much larger mass is comparable to, for example, Mercury's orbit around the sun, although Mercury's precession is very small, so we've chosen another, more eccentric system to better illustrate the shift.

In Figure 4.6 it can be seen how almost no precession (the small amount that is measured is due to uncertainties in the way the precession angle is calculated) occurs when using Newtonian equations of motion and some precession appearing with the EIH. The biggest precession appears with the 1PM expansion and a bit less with the 2PM. why is that?

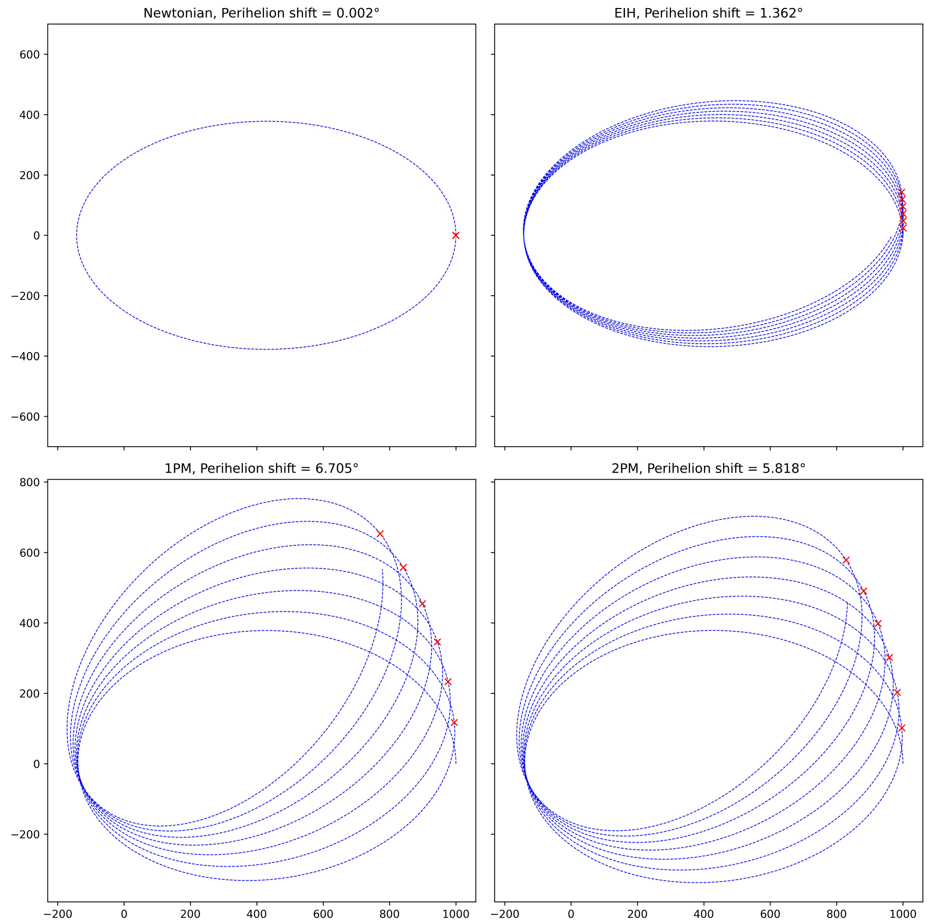

Figure 4.6: Precession in eccentric test-body limit, Newton compared with three different expansions; EIH, 1PM, 2PM

Precession in eccentric similar mass binary system

This case of an eccentric orbit of two similarly massive objects can be compared with that of two black holes or massive stars orbiting each other. In figure 4.7 one can see similar behaviour as for the precession in the test-body limit, albeit a bit different here. There is negligible precession in the Newtonian case and then increasing precession with EIH and 1PM. Here, the precession for the 2PM case is, however, less than the EIH again, why is that?. Only the trajectory and precession of one of the particles is shown since they are the same for both particles (only mirrored) due to the masses being the same.

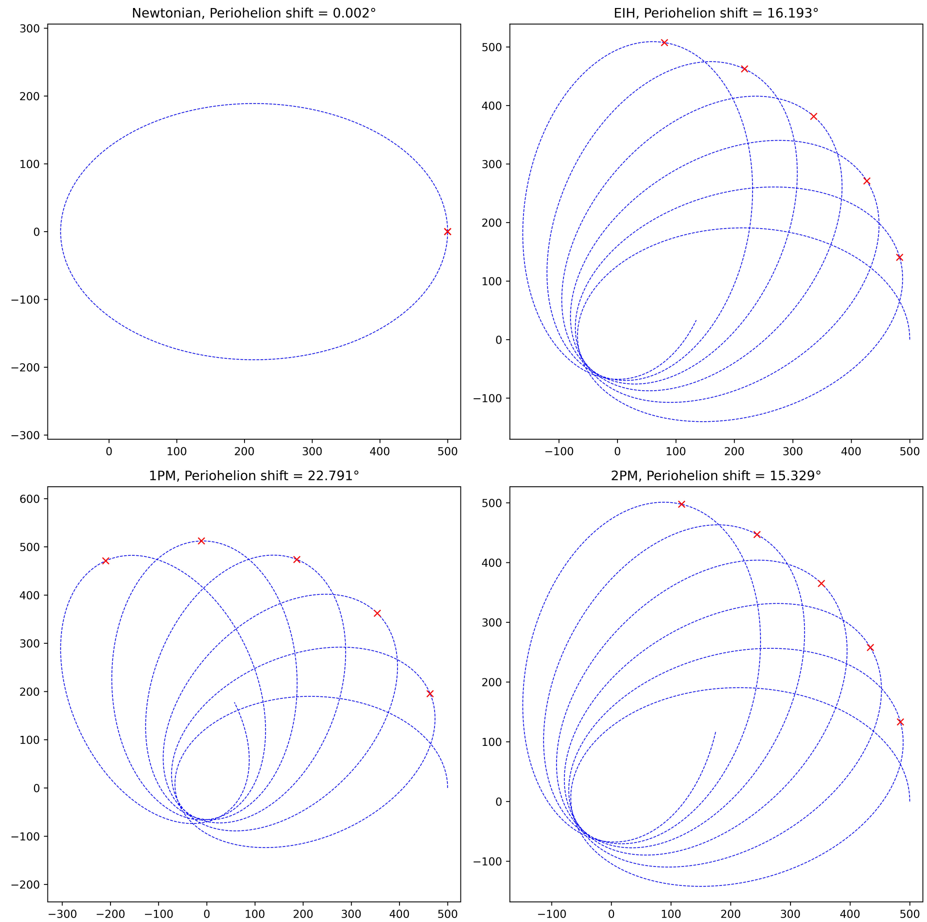

Figure 4.7: Precession of one particle in eccentric similar mass binary system, Newton compared with three different expansions; EIH, 1PM, 2PM Check the precession angle in 2PM case!

4.3 Other cool stuff (loopity-loop?)

4.4 Similarities and discrepancies between formalisms

Chapter 5

Conclusions

Chapter 6

Further work

6.1 Gravitational Waves

6.2 Including more terms

6.3 Other expansions

6.4 Coordinate dependencies

6.5 Optimizing the code to vary dt with relative velocity or relative distance

6.6 N-body problem

Bibliography

[1] Benjamin P Abbott et al. "Observation of gravitational waves from a binary black hole merger". In: *Physical review letters* 116.6 (2016), p. 061102.

[2] Bruce M. Barker and Robert F O'Connell. "Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments". In: *Physical Review D* 12.2 (1975), p. 329.

[3] Zvi Bern et al. "Black hole binary dynamics from the double copy and effective theory". In: *Journal of High Energy Physics* 2019.10 (2019), pp. 1–135.

[4] Luc Blanchet. "Analytic approximations in GR and gravitational waves". In: *International Journal of Modern Physics D* 28.06 (2019), p. 1930011.

[5] Sean M. Carroll. *Spacetime and geometry.* Pearson Education Limited, 2014.

[6] Andrea Cristofoli et al. "Post-Minkowskian Hamiltonians in general relativity". In: *Physical Review D* 100.8 (2019), p. 084040.

[7] Poul H Damgaard and Pierre Vanhove. "Remodeling the effective one-body formalism in post-Minkowskian gravity". In: *Physical Review D* 104.10 (2021), p. 104029.

[8] Thibault Damour and G Schäeer. "Higher-order relativistic periastron advances and binary pulsars". In: *Il Nuovo Cimento B (1971-1996)* 101.2 (1988), pp. 127–176.

[9] Albert Einstein. "Approximative integration of the field equations of gravitation". In: *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* 1916.688-696 (1916), p. 1.

[10] Alexander L. Fetter and John Dirk Walecka. *Theoretical Mechanics of Particles and Continua.* Dover Publications, Inc., 2003. ISBN: 9780486432618.

[11] Keith Fratus. *The Two-Body Problem.* URL: `https://web.physics.ucsb.edu/~fratus/phys103/LN/TBP.pdf`. (accessed: 06/05/2022).

[12] Troels Harmark. *General Relativity and Cosmology.* 2021.

[13] Yoichi Iwasaki. "Quantum theory of gravitation vs. classical theory: fourth-order potential". In: *Progress of Theoretical Physics* 46.5 (1971), pp. 1587–1609.

[14] David Morin. *The Hamiltonian Method.* URL: `https://scholar.harvard.edu/files/david-morin/files/cmchap15.pdf`. (accessed: 24/05/2022).

[15] *Post-Minkowskian expansion.* URL: `https://en.wikipedia.org/wiki/Post-Minkowskian_expansion`. (accessed: 01/06/2022).

[16] *The Two-body problem.* URL: `https://en.wikipedia.org/wiki/Two-body_problem`. (accessed: 25/05/2022).

[17] Clifford M. Will. "On the unreasonable effectiveness of the post-Newtonian approximation in gravitational physics". In: *Proceedings of the National Academy of Sciences* 108.15 (2011), pp. 5938–5945.