Investigating the Structure of an A-type Star

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ABSTRACT
This project models the structure of a 1.75 solar mass star at the zero-age main sequence. The results are then compared to a MESA simulation of the same type of star.

Key words: Stellar Structure

1 INTRODUCTION

In this project, the structure of a 1.75 solar mass, with a metallicity of 0.03, and at the zero-age main sequence is modeled. A star of this mass would correspond to a spectral type A star like Beta Pictoris. The star is taken to have a roughly solar composition. Since the star that was modeled was a ZAMS star, this means that the star has just finished contraction onto the main sequence. Before this, the star was contracting on a Kelvin Helmholtz time scale along the Hayashi trackKippenhahn et al. (2020). Since the star does not have a solar-like structure the core should be convective and the envelope should be radiativeHansen (2013).

The main sequence of stellar evolution is defined as when the primary energy generation of the star becomes nuclear fusion rather than gravitational contractionHansen (2013). Since the star was previously fully convective while it collapsed onto the main sequence, the interior composition is well-mixed. This allows us to have no compositional gradients which makes the modeling process easier. Energy transport in the star was assumed to be only by either radiation or convection at any point. This allows the code to simply decide which and calculate the luminosity at any point, instead of relying on more complicated mixing length theories. Additionally, the star was assumed to have no rotation which also simplifies the model. Finally, since the star has contracted to the main sequence and is young, the only source of energy was taken to be nuclear generation. Additional energy from further gravitational collapse or loss due to neutrinos was ignored.

2 METHODOLOGY

To model the structure of this star the four equations of stellar structure were integrated. There is a stellar structure equation for luminosity, pressure, radius, and temperature. These quantities were expressed as a function of mass, which means they were in Lagrangian form.

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2.1 Stellar Structure Equations

There is one stellar structure for luminosity, radius, pressure, and temperature. All are nonlinear first-order differential equations. They are as follows:

$$\frac{\partial l}{\partial m} = \epsilon(\rho, T),$$
$$\frac{\partial P}{\partial m} = \frac{-Gm}{4\pi r^4},$$
$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} l$$
$$\frac{\partial T}{\partial m} = \frac{-GmT}{4\pi r^4 P}\nabla$$

Kippenhahn et al. (2020). In the first luminosity equation, epsilon is the nuclear generation rate. In the pressure equation, G is the gravitational constant, and r is the radius as a function of mass. In the radius equation, ρ is the density at a given point. In the temperature equation, G is the gravitational constant, and P is the pressure. and ∇, is the gradient for radiation transport. These four quantities along with the density fully describe the structure of a non-rotating star.

2.2 Nuclear Energy Generation

The star is taken to have two sources of nuclear fusion. The first is the proton-proton chain and the other is the CNO cycle. There are analytical approximations of these equations in the book Stellar Interiors which this work used. Since the star modeled is 1.75 solar masses, the main source of energy generation should be the CNO cycleHansen (2013). However, there still is a contribution from the pp-chain that needs to be taken into account. The pp-chain is the process by which hydrogen is fused into helium through intermediate steps that involve deuterium. The first step of turning two protons into one deuteron is the rate-limiting step. The equation that describes this was found in the book Stellar Interiors Hansen (2013):

$$\epsilon_{pp} = 2.57 \cdot 10^4 \psi f_{11} g_{11} \rho X^2 T_9^{\frac{-2}{3}} e^{\frac{-3.381}{T_9^{\frac{1}{3}}}}$$

where rho is the density, T_9 is the temperature in units of billions of kelvin, ψ is a temperature-dependent shielding factor, and X is the hydrogen mass fraction. f_{11} and g_{11} are factors representing nuclear interactions between the different species involved in the reaction. Since the nuclear electric charges of the atoms are small and the temperatures are comparatively low weak screening approximations were used. The equations for these factors are:

$$f_{11} = e^{5.92\cdot 10^{-3}\sqrt{\frac{\rho}{T_7^3}}}$$

$g_{11} = 1 + 3.82T_9 + 1.51T_9^2 + 0.144T_9^3 - 0.0114T_9^4.$

The CNO cycle is the main source of energy for stars of masses greater than about 1.2 solar masses Hansen (2013). The equation that describes this more efficient fusion process is:

$\epsilon_{CNO} = 8.24 \cdot 10^{25} g_{14,1} ZX\rho t_9^{-2/3} e^{-15.231T_9^{-1/3} - (T_9/0.8)^2},$

where Z is the metallicity and the factor $G_{14,1}$ is given by the equation:

$g_{14,1} = 1 - 2.00T_9 + 3.41T_9^2 - 2.43T_9^3.$

The total energy generated is the sum of these two sources.

2.3 Opacities

One of the variables in the stellar structure equations is the Rossaland mean opacity. This is the wavelength-averaged opacity of the matter in the star. These values were obtained from the OPAL project database. The values had to be interpolated. I had great difficulty getting the interpolation function to work until my colleague Koji Shukawa advised me on the code that worked for him. The logarithms of the opacity in a table are indexed by the temperature and density. This table had to be interpolated by means of rectangular grid interpolations so that intermediate values could be obtained.

2.4 Equation of State and Radiation Transport

to complete the system of equations that describe a star, an equation of state for the gas must be provided. This gives the pressure as a function of density and temperature at every point. The equation of state used is a combination of the ideal gas law and radiation pressure. The equation is:

$P = \frac{\rho kT}{m_h \mu} + \frac{1}{3} aT^4,$

where rho is the density, k is the Boltzmann constant, T is the temperature, m_h is the mass of a hydrogen atom, μ is the mean molecular weight, and a is the radiation constant. Since the composition is well mixed and the metallicity is small the following equation was used to approximate the mean-molecular weight:

$\mu = \frac{4}{3 + 5X},$

where X is the hydrogen mass fraction.

Energy transport within the star is either by convection or radiation. Both of these modes of transport there are radiative nablas that have the forms:

$$\nabla_{ad} = \frac{2(4 - 3\beta)}{32 - 24\beta - 3\beta^2} \tag{1}$$

$$\nabla_{rad} = \frac{3}{16\pi acG} \frac{Pl\kappa}{mT^4}, \tag{2}$$

where β is the ratio of gas pressure to total pressure. At every step of the integration, the values of the two gradients were compared, and if the radiative nabla was smaller than the adiabatic gradient the energy transport was by radiation else it was by adiabatic convection.

3 NUMERICAL COMPUTATION

To model the structure of this star the four equations of stellar structure were integrated. Since these are four coupled non-linear differential equations, there had to be corresponding boundary conditions provided. This is also a two-point boundary value problem with the conditions specified at the center and surface of the star. The equations were solved using the shooting point method where integrations from both boundaries were integrated until they converged on a common answer. The answer at the end of this process is taken to be the structure of the star.

3.1 method of computation

The method used in this calculation is the shooting point method. The four stellar structure equations have a singularity at the center of the star. This is where the mass coordinate is zero. For the very center, an expansion of the stellar structure equations is used to for the first step. Two functions respectively load the surface and center boundary conditions into the integration. Then a Runga-Kutta integration scheme is used to integrate the center boundary conditions and outward and the outer conditions inward until they reach a point at half the total stellar mass Press (2007). At this meeting point, the percent difference between the two solutions is calculated. This solution is then minimized using a root-solving algorithm to find the minimum boundary conditions. The root-solving algorithm used was the Levenberg-Marquardt algorithm which minimizes the vector quantity that was the percent difference between each variable in the inward and outward solutions. The tolerance used was $1e - 35$ Press (2007).

3.2 Boundary conditions

At the boundaries, there are four parameters each that need to be determined. The free parameters that need to be guessed are the central temperature and pressure and the surface luminosity and radius. The other boundary values can be determined from the other four that have been specified. The interior used boundary Taylor expanded versions of the stellar structure equations to begin the integration. These were:

$$dP = P_{center} - \frac{3G}{8\pi}(\frac{4\pi\rho}{3})^{\frac{4}{3}} dm$$

$$dr = (\frac{3dm}{4\pi\rho})^{\frac{1}{3}}$$

$$dl = \epsilon_{nuc} dm,$$

where dm is a small mass step. The temperature was kept as the specified center temperature since the mass step was very small.

The surface boundary had a luminosity and radius specified. The surface temperature was calculated using the equation:

$L = 4\pi\sigma R^2 T^4.$

The surface pressure was more complicated to calculate since the pressure could not just be set to zero. The method used on the advice of my colleague was the Eddington grey atmosphere model. This model uses the Rossaland mean opacity to calculate the pressure where the atmosphere becomes optically thick. The atmosphere becomes optically thick when the optical depth,τ becomes two-thirds. The temperature at which this happens is given by:

$T = (\frac{3L}{16\pi\sigma R^2})(\tau + \frac{2}{3})^{1/4}$

[1]	Center T	1.85e7 K
	Center P	$1.7e17dyne/cm^2$
	Surface L	$1e33ergs/s$
	Surface R	1e12 cm

Table 1. This table contains the starting parameters used in the model

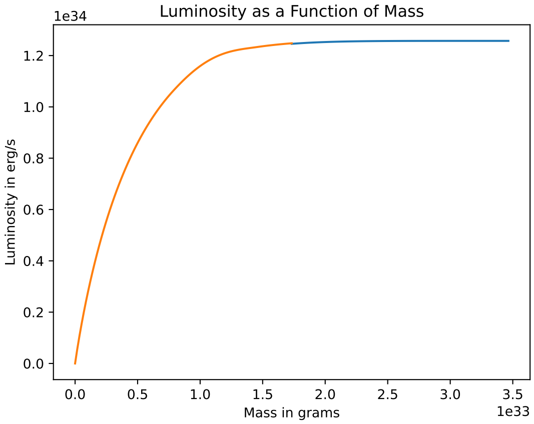

Figure 1. This figure shows the luminosity as a function of mass that resulted from the program converging.

Using this temperature the opacity at that point was calculated. This opacity was then used to find how pressure varied with optical depth by:

$$\frac{dp}{d\tau} = \frac{GM}{R^2\kappa}$$

This equation was integrated until $\tau = 2/3$ at which the pressure was taken to be the surface pressure.

4 RESULTS

4.1 Results from Calculation

The code integrating the stellar equations with the provided equation of state and boundary conditions was run. The results are a graph of luminosity, radius, pressure, and temperature each as a function of mass. These graphs represent the structure of the star.

4.2 Comparison with MESA Results

In order to compare to the results of the code I wrote. A simulation for a star with 1.75 solar masses and a metallicity of 0.03 was run with mesa. The results where plotted together with the output of my program.

5 CONCLUSION

Table two is a comparison of the values generated by this project and the MESA code. The results are considerably different. In the case of the luminosity, it is different by four orders of magnitude. The reason for this may be the limits of the opacity table. When I attempted to change the initial parameters of the simulation to be closer to the mesa values, I received overflow errors from the exponential and logarithm functions in my code. Despite my best efforts, I could not

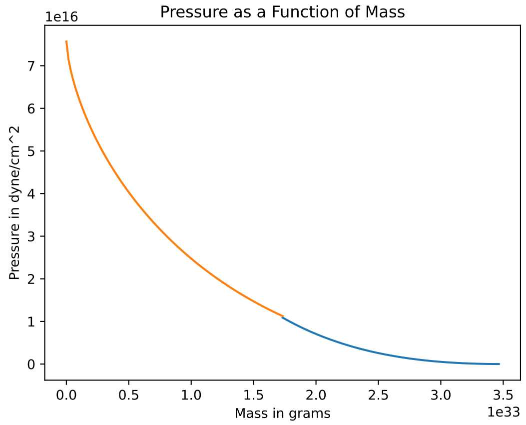

Figure 2. This figure shows the pressure as a function of mass that resulted from the program converging.

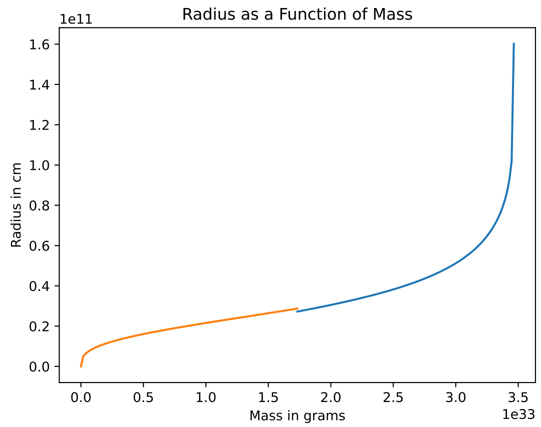

Figure 3. This figure shows the radius as a function of mass that resulted from the program converging.

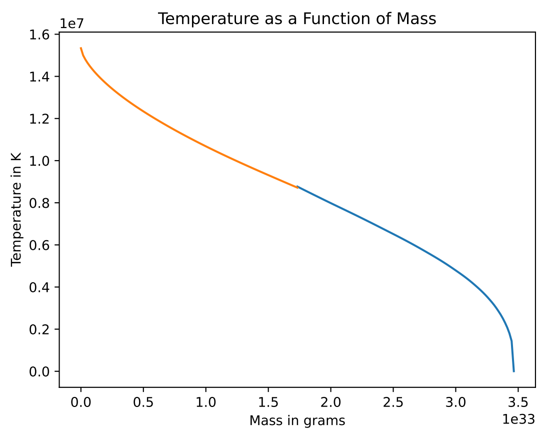

Figure 4. This figure shows the temperature as a function of mass that resulted from the program converging.

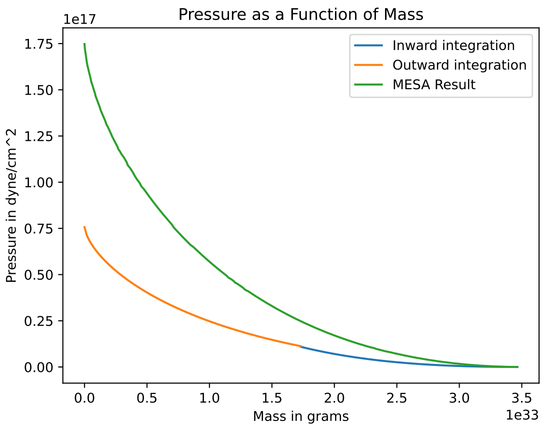

Figure 5. This figure shows the pressure as a function of mass that resulted from the program converging plotted with the MESA result. There is disagreement.

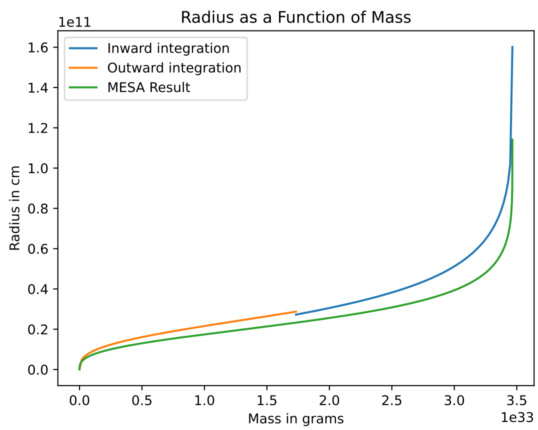

Figure 6. This figure shows the radius as a function of mass that resulted from the program converging plotted with the MESA result. There is disagreement.

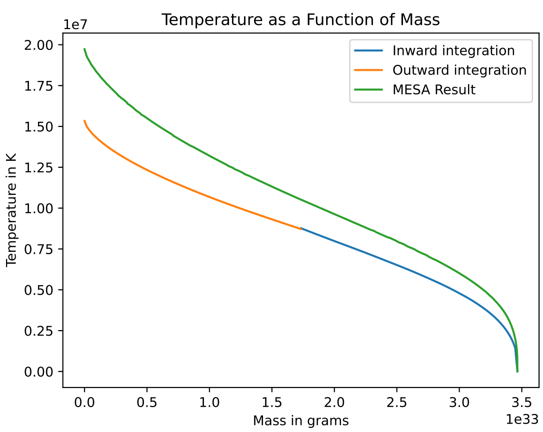

Figure 7. This figure shows the temperature as a function of mass that resulted from the program converging plotted with the MESA result. There is disagreement.

[2]

Result Name	Project value	MESA value	Percent Difference
Luminosity	$1.26e34 ergs/s$	$5.84e38 ergs/s$	0.999
Temperature	$5146K$	$7728K$	0.33
Surface gravity	$9007 gcm/s^2$	$17700 gcm/s^2$	0.49
Radius	$1.6e11 cm$	$1.14e11 cm$	0.29

Table 2. This table contains the results of the integration, the MESA result, and the percent difference.

find out why the true parameters for this star from the MESA code caused these errors in my own code. Eventually, I found parameters that allowed my code to converge but to values much different than that of MESA. This has inspired even greater respect for the brilliance of the MESA developers in overcoming these challenges and making a stellar evolution model.

All materials and code for this project can be found in the linked GitHub repository. https://github.com/carljingebretsen/Stellar_Project_2023/tree/main

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This research made use of NumPy (Harris et al. 2020) This research made use of matplotlib, a Python library for publication-quality graphics (Hunter 2007) This research made use of SciPy (Virtanen et al. 2020) This research made use of Astropy, a community-developed core Python package for Astronomy (Astropy Collaboration et al. 2018, 2013) This project made use of the MESA code for stellar evolution. Paxton et al. (2011, 2013, 2015, 2018, 2019); Jermyn et al. (2023)

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