Takagi Factorisation

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Abstract

This document describes the diagonalisation procedure by Takagi for diagonalising symmetric, complex-valued matrices.

1 Schur decomposition and singular-value decomposition

Schur decomposition tells us that any (potentially complex) matrix A can be written as

$$A = U^\dagger \Delta U,$$

where U is a unitary matrix, and Δ is an upper triangular matrix. It follows then that if A as a symmetric matrix ($A^T = A$), then

$$\left(U^\dagger \Delta U\right) = \left(U^\dagger \Delta U\right)^T = U^T \Delta^T U^*$$

2 Takagi factorisation

Assume $A = A^T$ is a symmetric, complex-valued, $n \times n$ matrix. Takagi factorisation[1] tells us that there exists a unitary matrix U, and a real, non-negative diagonal matrix D such that

$$A = U^T D U. \tag{1}$$

2.1 Factorisation algorithm

The algorithm is will be based on finding vector $\boldsymbol{v} \in \mathbb{C}^n$ that satisfy $A\boldsymbol{v}^* = \sigma\boldsymbol{v}$, for some real, non-negative σ. This vector will be called a *Takagi vector* for future reference. Existence of these vectors for any matrix A such that AA^* only has real, non-negative eigenvalues is detailed later.

To find U, I propose here an algorithm based on the proof for Takagi factorisation in.[2] Given a Takagi vector $\boldsymbol{v} \in \mathbb{C}^n$ of A,[3] and an orthonormal basis $\{\boldsymbol{v}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ of $\mathbb{C}^n$, it is possible to write A as

$$A = V \begin{bmatrix} \sigma & \mathbf{0} \\ \mathbf{0} & A_2 \end{bmatrix} V^T,$$

[1] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1990. ISBN: 0521386322. URL: `http://www.amazon.com/Matrix-Analysis-Roger-Horn/dp/0521386322%3FSubscriptionId%3D192BW6DQ43CK9FN0ZGG2%26tag%3Dws%26linkCode%3Dxm2%26camp%3D2025%26creative%3D165953%26creativeASIN%3D0521386322`.

[2] Horn and Johnson, *Matrix Analysis*.

[3] A proof that this can be found is detailed elsewhere.

where A_2 is a symmetric $(n-1)\times(n-1)$ matrix and V is a unitary matrix with the aforementioned orthonormal basis as its columns. This process can be repeated with A_2 and so on until you have

$$A = V_1 \cdots V_n \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V_n^T \cdots V_1^T,$$

where

$$V_p = \begin{bmatrix} \mathbb{I}_{(p-1)\times(p-1)} & \mathbf{0} \\ \mathbf{0} & \tilde{V}_p, \end{bmatrix}$$

and $\tilde{V}_p$ is the unitary matrix that makes a diagonalisation step on A_p. Comparing to Eq. (1), we find that

$$U = V_n^T \cdots V_1^T, \tag{2a}$$
$$D = \mathrm{diag}\left(\sigma_1, \ldots, \sigma_n\right). \tag{2b}$$

It is easy to show that U is unitary, as promised. Furthermore, by assumption, all the values σ_p are real and positive. Now the values on the diagonal of D can be permuted to any order using a permutation matrix P, such that we get

$$A = U_P^T D_P U_P,$$

where $U_P = PU$ and $D_P = PDP^T$. U_P is still unitary, and D_P diagonal.

3 Proofs

The Takagi vector. For any $A \in M_n(\mathbb{C})$ such that AA^* only has real, non-negative eigenvalues, there exists a non-zero vector $\boldsymbol{v} \in \mathbb{C}^n$ such that $A\boldsymbol{v}^* = \sigma\boldsymbol{v}$, where σ is a real, non-negative number.
Proof. Consider a vector $\boldsymbol{x} \neq \mathbf{0} \in \mathbb{C}^n$ that is an eigenvector of AA^* with corresponding eigenvalue λ. There are two cases:

(a) $A\boldsymbol{x}^*$ and $\boldsymbol{x}$ are linearly dependent.

(b) $A\boldsymbol{x}^*$ and $\boldsymbol{x}$ are linearly independent.

In case (a), we must have that $A\boldsymbol{x}^* = \mu\boldsymbol{x}$ for some $\mu \in \mathbb{C}$, since they are linearly dependent. Then $AA^*\boldsymbol{x} = A\mu^*\boldsymbol{x}^* = |\mu|^2 \boldsymbol{x} \equiv \lambda\boldsymbol{x}$, which is non-negative by definition. In case (b), the vector $\boldsymbol{y} = A\boldsymbol{x}^* + \mu\boldsymbol{x}$ is non-zero for any $\mu \in \mathbb{C}$, since $A\boldsymbol{x}^*$ and $\boldsymbol{x}$ are linearly independent. Then we can choose μ such that $|\mu|^2 = \lambda$ to get that $A\boldsymbol{y}^* = A\left(A^*\boldsymbol{x} + \mu^*\boldsymbol{x}^*\right) = \lambda\boldsymbol{x} + \mu^* A\boldsymbol{x}^* = \mu\mu^*\boldsymbol{x} + \mu^* A\boldsymbol{x}^* = \mu^*\left(A\boldsymbol{x}^* + \mu\boldsymbol{x}\right) = \mu^*\boldsymbol{y}$. As such, we can always find a vector $\tilde{\boldsymbol{v}} \in \mathbb{C}^n$ such that $A\tilde{\boldsymbol{v}}^* = \mu\tilde{\boldsymbol{v}}$ for some $\mu \in \mathbb{C}^n$. Furthermore, we can define a vector $\boldsymbol{v} = e^{i\theta}\tilde{\boldsymbol{v}}$ for a $\theta \in \mathbb{R}$ to get $A\boldsymbol{v}^* = A\left(e^{i\theta}\tilde{\boldsymbol{v}}\right)^* = e^{-i\theta}A\tilde{\boldsymbol{v}}^* = e^{-i\theta}\mu\tilde{\boldsymbol{v}} = e^{-2i\theta}\mu e^{i\theta}\tilde{\boldsymbol{v}} = e^{-2i\theta}\mu\boldsymbol{v} \equiv \sigma\boldsymbol{v}$. This allows us to choose the phase of $\sigma = e^{-2i\theta}\mu$ to be such that σ is real and non-negative.

Eigenvalues of AA^* for symmetric A. Given an $N\times N$ complex matrix A, the eigenvalues of AA^* are always real and non-negative.
Proof. Consider $\boldsymbol{x} \neq \mathbf{0}$ an eigenvector of AA^* with corresponding eigenvalue λ. Then we must have that

$$\lambda\boldsymbol{x}^\dagger\boldsymbol{x} = \boldsymbol{x}^\dagger AA^*\boldsymbol{x} = \left(A^\dagger\boldsymbol{x}\right)^\dagger\left(A^*\boldsymbol{x}\right) = \left(A^*\boldsymbol{x}\right)^\dagger\left(A^*\boldsymbol{x}\right),$$

where we have used that $A^\dagger = \left(A^T\right)^* = A^*$. This means that $\lambda \geqslant 0$, since for any vector $\boldsymbol{v} \in \mathbb{C}^n$ we have that $\boldsymbol{v}^\dagger \boldsymbol{v} \geqslant 0$. As this holds for all eigenvectors $\boldsymbol{x}$ of AA^*, all its eigenvalues must be non-negative.

Diagonalisation step of a symmetric matrix A**.** For any symmetric matrix $A \in M_n(\mathbb{C})$, there exist a unitary matrix $V \in M_n(\mathbb{C})$ such that

$$V^\dagger A V^* = \begin{pmatrix} \sigma & \mathbf{0} \\ \mathbf{0} & A_2 \end{pmatrix},$$

where σ is a real, non-negative number and $A_2 \in M_{n-1}(\mathbb{C})$ is also a symmetric matrix. *Proof.* Consider a normalised Takagi vector $\boldsymbol{v} \neq \mathbf{0}$ of A such that $A\boldsymbol{v}^* = \sigma \boldsymbol{v}$ for some real, non-negative σ and $\boldsymbol{v}^\dagger \boldsymbol{v} = 1$. We can then complete a basis for $\mathbb{C}^n$ with unit vectors $\boldsymbol{v}_i$ where $i \in 1, \ldots, n$, where we define $\boldsymbol{v}_1 \equiv \boldsymbol{v}$. Defining a unitary matrix $V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$, the first column of the product

$$\left(V^\dagger A V^*\right)_{i1} = \boldsymbol{v}_i^\dagger A \boldsymbol{v}^* = \boldsymbol{v}_i^\dagger \sigma \boldsymbol{v} = \sigma \delta_{i1},$$

where δ_{ij} is the Kronecker delta symbol, and we have used the Takagi property of $\boldsymbol{v}$ and the orthonormality of $\boldsymbol{v}_i^\dagger \boldsymbol{v}_j$. This means only the first component of the first column of $V^\dagger A V^*$ is non-zero, and has value σ. Now since A is symmetric, we have that $\left(V^\dagger A V^*\right)^T = V^\dagger A^T V^* = V^\dagger A V^*$ must also be symmetric, and thus must have the form

$$V^\dagger A V^* = \begin{pmatrix} \sigma & \mathbf{0} \\ \mathbf{0} & A_2 \end{pmatrix},$$

for a symmetric $A_2 \in M_{n-1}(\mathbb{C})$.

References

Horn, Roger A. and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1990. ISBN: 0521386322. URL: http://www.amazon.com/Matrix-Analysis-Roger-Horn/dp/0521386322%3FSubscriptionId%3D192BW6DQ43CK9FN0ZGG2%26tag%3Dws%26linkCode%3Dxm2%26camp%3D2025%26creative%3D165953%26creativeASIN%3D0521386322.