UNIVERSITY
OF OSLO

Master’s thesis

My Master’s Thesis

With Subtitle

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60 ECTS study points

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May 7, 2024

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Chapter 1

Introduction

This is where I introduce the master's thesis.

Chapter 2

Quantum Field Theory

2.1 Perturbative Quantum Field Theory

PHANTOM PARAGRAPH: Here I want to introduce the Lagrangian formulation of quantum field theory, and the types of fields we will be working with. Furthermore, I want to introduce the perturbation series through the path integral, and talk about the nature of quantum effects. Also derive Feynman rules from path integral.

TODO:

- ☐ Introduce perturbative QFT.
- ☐ Talk about reading Feynman diagrams and special care to take with Majorana fermions.

In this thesis, I will use the Lagrangian framework to formulate QFT. Here I will introduce the basics of how to formulate a QFT in such a way using the path integral formalism. This leads to a perturbative formulation of scattering and computation of correlation functions, which is the basis for the calculations that will be made.

2.1.1 The Path Integral

I will start by introducing some useful shorthands that will be used throughout this section. Consider an action $S[\{\Phi\}]$ as a functional of some fields $\{\Phi\}$. Let $\phi_i \equiv \phi(x_i)$ for some arbitrary field $\phi \in \{\Phi\}$ evaluated at some point in space-time x_i—the path integral approach to quantum field theory is built on time-ordered correlation functions through the relation ⋮©⋮

$$\langle\Omega| T\left\{\hat{\phi}_1 \cdots \hat{\phi}_n\right\} |\Omega\rangle = \frac{\int \mathcal{D}\phi\, \phi_1 \cdots \phi_n e^{iS[\{\Phi\}]}}{\int \mathcal{D}\phi\, e^{iS[\{\Phi\}]}}, \tag{2.1}$$

where $T\{.\}$ denotes the time-ordering operation and $\mathcal{D}\phi$ is the measure denoting integration over all possible *field configurations*. A field configuration here is understood as a given set of values for the fields $\{\Phi\}$, one for each point in space-time. The time-ordering operation will put the fields in chronological order according to the time at which they are evaluated, with the "first" field being farthest to the right. The left-hand side of Eq. (2.1) the fields are understood as operators on the Hilbert space of states in our interacting theory (denoted by their hats), whereas on the right-hand side they are considered as classical fields. In this way quantum effects are encapsulated through the weighted sum of all classical *paths* through configuration space, rather than just whichever one minimises the action.

2.1.2 Feynman Rules

In interacting theories, correlation functions can be obtained through a *perturbation series* by expanding them around their coupling constant, here denoted λ. If the Lagrangian of the action can be written on the form $\mathcal{L} = \mathcal{L}_0 + \lambda\mathcal{L}_{\text{int}}$, then the exponentiation of the action can be written as

$$e^{i\int \mathrm{d}^4x\,(\mathcal{L}_0+\mathcal{L}_{\text{int}})} = e^{i\int \mathrm{d}^4x\,\mathcal{L}_0}\left(1+\lambda\int \mathrm{d}^4x_1\,\mathcal{L}_{\text{int}}(x_1)+\frac{\lambda^2}{2}\int \mathrm{d}^4x_1\int \mathrm{d}^4x_2\,\mathcal{L}_{\text{int}}(x_1)\mathcal{L}_{\text{int}}(x_2)+\dots\right). \tag{2.2}$$

Now say the interaction Lagrangian $\mathcal{L}_{\text{int}}$ is some monomial of degree p in the fields $\{\Phi\}$,[1] then the interacting correlation functions can be written in terms of free-theory correlation functions! To see this, we consider the interacting n-point function $D^n_{\text{int}}(1,\dots,n) = \langle\Omega|\,T\left\{\hat{\phi}_1\cdots\hat{\phi}_n\right\}|\Omega\rangle$, and write it out in terms of the free and interacting Lagrangians:

$$D^n_{\text{int}}(1,\dots,n) = \frac{1}{\mathcal{N}}\int\mathcal{D}\phi\,\phi_1\cdots\phi_n e^{i\int \mathrm{d}^4x\,\mathcal{L}_0}\left(1+i\lambda\int \mathrm{d}^4y\,\mathcal{L}_{\text{int}}+\mathcal{O}(\lambda^2)\right), \tag{2.3}$$

where the normalisation is given by $\mathcal{N} = \int\mathcal{D}\phi\, e^{i\int \mathrm{d}^4x\,\mathcal{L}}$. To relate this to the free-theory, let us take a moment to write this out. Given the free-field n-point correlator $D^n_0(1,\dots,n) \equiv \frac{1}{\mathcal{N}_0}\int\mathcal{D}\phi\,\phi_i\cdots\phi_n e^{i\int \mathrm{d}^4x\,\mathcal{L}_0}$ with normalisation $\mathcal{N}_0 = \int\mathcal{D}\phi\, e^{i\int \mathrm{d}^4x\,\mathcal{L}_0}$, expand the interacting normalisation as

$$\mathcal{N} = \mathcal{N}_0\left(1+i\lambda\int \mathrm{d}^4y\,D^p_0(\underbrace{y,\dots,y}_{p\text{ times}})-\frac{\lambda^2}{2}\int \mathrm{d}^4y\int \mathrm{d}^4z\,D^{2p}_0(\underbrace{y,\dots,y}_{p\text{ times}},\underbrace{z,\dots,z}_{p\text{ times}})+\mathcal{O}(\lambda^3)\right). \tag{2.4}$$

Inserting this into Eq. (2.3) and expanding around $\lambda = 0$ we get

$$\begin{aligned}D^n_{\text{int}}(1,\dots,n) &= \frac{D^n_0(1,\dots,n)+i\lambda\int \mathrm{d}^4y\,D^{n+p}_0(1,\dots,n,\overbrace{y,\dots,y}^{p\text{ times}})+\mathcal{O}(\lambda^2)}{1+i\lambda D^p_0(\underbrace{y,\dots,y}_{p\text{ times}})+\mathcal{O}(\lambda^2)}\\ &= D^n_0(1,\dots,n)+i\lambda\left(D^{n+p}_0(1,\dots,n,\underbrace{y,\dots,y}_{p\text{ times}})-D^n_0(1,\dots,n)D^p_0(\underbrace{y,\dots,y}_{p\text{ times}})\right)+\mathcal{O}(\lambda^2)\end{aligned} \tag{2.5}$$

$$\left(\Box_x - m^2\right)D^2_0(x,y) = -i\delta^4(x-y) \tag{2.6}$$

[1] In the more general case where it can be expressed with some polynomial in the fields, we can look at each monomial term separately, without loss of generality.

2.2 Renormalised Quantum Field Theory

PHANTOM PARAGRAPH: TALK ABOUT LOOP INTEGRALS, DIVERGENCES, REGULARISATION AND RENORMALISATION.

TODO: Mention Wick rotation and evaluation of loop integrals?

Divergences appear in perturbative correlation functions in QFT, and can be categorised into *ultraviolet* (UV) divergences and *infrared* (IR) divergences. They are so named after which region of momentum space they originate from — high momentum for UV and low momentum for IR. The two types of divergences are dealt with entirely differently, and here I will lay out how to deal with UV divergences through *renormalisation*.

Consider a loop like the one in Fig. 2.1: If we consider a massless particle in the loop, the loop integral will take the form

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 + i\epsilon)^2} = \frac{i}{(2\pi)^4} \int \mathrm{d}\Omega_4 \int_0^\infty \mathrm{d}q_E \, \frac{1}{q_E}, \tag{2.7}$$

which diverges for both low and high momenta. ⌜Mention what went into this integral, i.e. Wick rotation and $i\epsilon$.⌟ Had the particle been massive, the momentum would have a non-zero lower limit, and the IR divergence would disappear. However, the UV divergence must be handled differently.

2.2.1 Regularisation

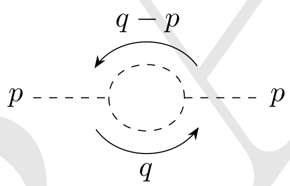


Figure 2.1: Simple example of a loop diagram in a scalar theory.

A first step to handle the divergences is to deform our theory in some way to make the loop integral formally finite, but recovering the divergence in the limit that the deformation disappears. An intuitive deformation would be to cap the momentum integral at some Λ, recovering our original theory in the limit $\Lambda \to \infty$. To illustrate the procedure of regularisation and subsequently renormalisation, it will be useful to have an example, for which I choose a scalar Lagrangian $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m_\phi^2 \phi^2 - \frac{\lambda}{3!}\phi^3$. ⌜Perhaps introduce this model earlier?⌟ Regularising the IR divergence in Eq. (2.7) by giving our scalar a mass m, and the UV divergence with a momentum cap Λ, we are left with

$$\begin{aligned} \int_{|q|<\Lambda} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{((q^2 - m^2) + i\epsilon)^2} &= \frac{i}{(2\pi)^4} \int \mathrm{d}\Omega_4 \int_0^\Lambda \mathrm{d}q_E \, \frac{q_E^3}{(q_E^2 - m^2)^2} \\ &= \frac{i}{16\pi^2} \left\{ \ln\left(1 + \frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2 + m^2} \right\}, \end{aligned} \tag{2.8}$$

where now evidently the divergences manifest as a logarithm.

Another popular choice of regularisation, which I will use in this thesis, is *dimensional regularisation*. It entails analytically continuing the number of space-time dimension from the ordinary 4 dimensions to $d = 4 - 2\epsilon$ dimensions for some small ϵ.[2] This removes much of the intuition for what we are doing, but turns out to be computationally very efficient. Our loop integral Eq. (2.7) will then turn into

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 + i\epsilon)^2} = \frac{i 2\pi^{d/2}}{(2\pi)^d} \frac{1}{\Gamma(d/2)} \int_0^\infty \mathrm{d}q \, q_E^{d-5}$$
$$= \frac{i 2\pi^{2-\epsilon}}{(2\pi)^{4-2\epsilon}} \frac{1}{\Gamma(2-\epsilon)} \left\{ \int_0^\mu \mathrm{d}q \, \frac{1}{q_E^{1+2\epsilon}} + \int_\mu^\infty \mathrm{d}q \, \frac{1}{q_E^{1+2\epsilon}} \right\} = \frac{i}{16\pi^2} \left(\frac{1}{\epsilon_{\mathrm{IR}}} - \frac{1}{\epsilon_{\mathrm{UV}}} \right) + O(\epsilon), \tag{2.9}$$

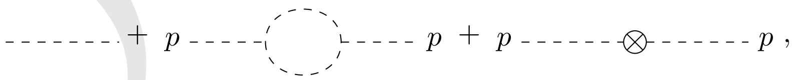
where in the second equality, the momentum integral is split into a low-energy and high-energy part with some scale μ. Here a trick was performed, as the low-energy part requires $\epsilon < 0$ to be convergent, whereas the high-energy part requires $\epsilon > 0$. The two different divergences thus require different deformations of the theory to be finite, and should be handled separately, hence the subscripts. In the end, the divergences when using dimensional regularisation come out as $\frac{1}{\epsilon}$-terms.

2.2.2 Counterterm Renormalisation

To take care of UV divergences, we note that there is freedom in how we define the contents of our Lagrangian. We should be able to rescale our fields like $\phi_0 = \sqrt{Z_\phi}\phi$, and rescale our couplings like $m_{\phi,0}^2 = Z_m m_\phi^2$ and $\lambda_0 = Z_\lambda \lambda$. Although suggestively naming terms such as *mass term* with mass m_ϕ^0 implies a connection to the mass of a particle, we have yet to define what that would mean experimentally. Thus, rescaling our parameters and fields parametrises the way in which we can tune our theory, allowing us freedom in choosing the way our theory connects to experiments.

This approach actually allows us to make a perturbative scheme for fixing our (re)normalisations of the fields and couplings. There are many choices for how to connect theory to experiment, but one common approach for field and mass renormalisation is to identify the pole of the two-point correlation function $\mathcal{G}(x, y)$ of a particle to the mass resonance measurable in experiment. This allows us to perturbatively calculate the two-point correlator, and then fix our normalisations accordingly, such that our imposed condition on it holds at every order in the perturbation series. We achieve this systematically with *counterterms*, which in essence are additional Feynman rules added to the theory. By expanding the renormalisation parameters as $Z = 1 + \delta$, the δ will carry the correction to the normalisation to any given order in a coupling constant. To one-loop order, the self-energy of our scalar theory© is diagrammatically given by

Is there a better explanation for this experiment than 'mass resonance'?



where the crossed dot represents an insertion of the δ into the LO amplitude. Since the δ carries corrections proportional to the NLO amplitudes, it should be seen as coming in at NLO.

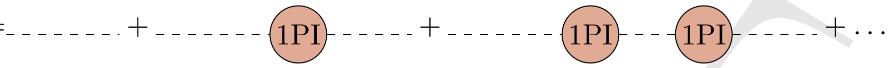
2.2.3 On-Shell Renormalisation

Categorising all higher order contributions that can arise to the LO self-energy of a massive particle, they come in the form of *one-particle-irreducible* (1PI) diagrams. These

[2] The reason for choosing 2ϵ is purely aesthetical, making some expressions neater.

are diagrams where all lines with loop momentum running through are connected. Other diagrams can be reconstructed as the sum of 1PI diagrams. Denoting the leading order correlator $\mathcal{G}_0(p)$ and the contribution from one insertion of all 1PI diagrams $i\Sigma(p)$, we get a series[3]

$i\mathcal{G}(p) =$



$$
\begin{aligned}
&= i\mathcal{G}_0(p) + i\mathcal{G}_0(p) i\Sigma(p) i\mathcal{G}_0(p) + i\mathcal{G}_0(p) i\Sigma(p) i\mathcal{G}_0(p) i\Sigma(p) i\mathcal{G}_0(p) + \dots \\
&= i\mathcal{G}_0(p) \left[i\Sigma(p) i\mathcal{G}_0(p) + (i\Sigma(p) i\mathcal{G}_0(p))^2 + \dots \right] \\
&= i\mathcal{G}_0(p) \frac{1}{1 + \Sigma(p)\mathcal{G}_0(p)} = \frac{i}{\mathcal{G}_0^{-1}(p) + \Sigma(p)}.
\end{aligned} \tag{2.10}
$$

So the computation of the two-point correlator to any order can be done simply by computing the sum of the 1PI diagrams to that order. These contributions will generally diverge, but then we can take into account the renormalisation parameters. Since this is a *bare* function, i.e. using the non-renormalised quantities, we can get the renormalised two-point correlator $\mathcal{G}^{\mathrm{R}}(p)$ through

$$
\mathcal{G}^{\mathrm{R}}(p) = \frac{1}{Z_\psi} \mathcal{G}^{\mathrm{bare}}(p) = \frac{1}{1+\delta_\psi} \mathcal{G}^{\mathrm{bare}}(p), \tag{2.11}
$$

for any field ψ, seeing as the two-point correlator is quadratic in ψ and thereby quadratic in $\sqrt{Z_\psi}$.

On-shell mass renormalisation seeks to identify the pole of the two-point correlator with the physical mass as observed in experiment. This is a generalisation the property of the free theory two-point function to the perturbative interacting two-point function at any order. It yields two conditions:

(I) $\left[(1+\delta_\psi) \left(\mathcal{G}_0^{\mathrm{bare}}(p) \right)^{-1} + \Sigma(p) \right] \Big|_{p^2 = m^2_{\mathrm{pole}}} = 0,$

(II) $\mathrm{Res} \left\{ \mathcal{G}^{\mathrm{R}}(p), p^2 = m^2_{\mathrm{pole}} \right\} = 1,$

where $\mathrm{Res}\{f(z), z = z_0\}$ is the residue of the function f at z_0.

For our scalar theory, where the leading order *bare* two-point correlator is $\mathcal{G}_0^{\mathrm{bare}}(p) = \frac{1}{p^2 - m_0^2}$, this means that we get the relations

(I) $\delta_m m_\phi^2 = \Sigma(m_\phi^2),$

(II) $\delta_\phi = -\frac{\mathrm{d}}{\mathrm{d}p^2} \Sigma(p^2) \Big|_{p^2 = m_\phi^2}.$

[←]■

TODO: Outline chiral mass renormalisation.

[3] A note on the argument p of these functions: The two-point-correlators in momentum space depend on the four-momentum p^μ in such a way that when it is put in between the external particle representations (i.e. 1 for scalars, spinors for fermions and polarisation vectors for vector bosons) the result will be Lorentz invariant. This means in principle that the correlator could carry Lorentz indices too, which will be suppressed here for simplicity.

2.2.4 Renormalised Parton Distribution Functions

2.3 Yang-Mills Theories

Gauge theory in QFT is based on imposing *internal symmetries* on the Lagrangian. Internal symmetries are symmetries separate from *external symmetries* in that they are not symmetries of coordinate transformations, but rather symmetries based on transformations of the fields. Typically, the field transformations under which the Lagrangian is invariant are Lie groups, and are referred to as the *gauge group*. A collection of fields that transform into each other under a particular representation[4] is called a *multiplet*.

Let us consider a complex scalar field theory to illustrate. Let ϕ_i be a multiplet of complex scalar fields, and let the gauge group be a general non-Abelian Lie group, locally defined by a set of hermitian generators T^a. Locally, the group elements can then be described using the exponential map as

$$g(\boldsymbol{\alpha}) = \exp\left(i\alpha^a T^a\right), \tag{2.12}$$

for a set of real parameters $\boldsymbol{\alpha}$.[5] This way of parametrising the group is convenient in that the inverse of the group elements are the hermitian conjugate, i.e. $g^{-1}(\boldsymbol{\alpha}) = g^\dagger(\boldsymbol{\alpha})$. The transformation law for $\Phi = (\phi_1, \ldots)^T$ is

$$\Phi \to g(\boldsymbol{\alpha})\Phi = \exp\left(i\alpha^a T^a\right)\Phi, \tag{2.13}$$

which for an infinitesimal set of parameters ϵ^a becomes

$$\Phi \to \left(1 + i\epsilon^a T^a\right)\Phi. \tag{2.14}$$

Now, we would like to categorise the Lagrangian terms that are invariant under such transformations. The ordinary free Klein-Gordon Lagrangian

$$\mathcal{L}_{\text{KG}} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - m^2 \Phi^\dagger \Phi \tag{2.15}$$

is invariant. However, if we promote our gauge symmetry to be a local symmetry, i.e. let the parameters become spacetime-dependent $\boldsymbol{\alpha} \to \boldsymbol{\alpha}(x)$, this is no longer the case. Since space-time coordinates are unchanged under gauge transformations, it follows that so too is the derivative ∂_μ. However, it will be useful to rewrite this in as somewhat convoluted way, letting it "transform" according to[6,7]

$$\partial_\mu \to \partial_\mu = g\partial_\mu g^{-1} + \left(\partial_\mu g\right) g^{-1}, \tag{2.16}$$

which in turn makes the field derivative transform to

$$\partial_\mu \Phi \to g\partial_\mu \Phi + \left(\partial_\mu g\right)\Phi, \tag{2.17}$$

which does *not* leave the kinetic term invariant. So we must rethink the kinetic term of the Lagrangian. To get the right transformation properties of the derivative term, we need a *covariant derivate* D_μ such that $D_\mu \Phi \to g D_\mu \Phi$. In order to create such a D_μ, it might be pru- to mention that is a real-valued somehow.

we must require that it transforms as $D_\mu \to g D_\mu g^{-1}$. This can be done by introducing the *gauge field* $\mathcal{A}_\mu(x) \equiv A^a_\mu(x) T^a$ which transforms according to

$$\mathcal{A}_\mu \to g \mathcal{A}_\mu g^{-1} - \frac{i}{q} \left(\partial_\mu g\right) g^{-1}. \tag{2.18}$$

The last term can compensate for the extra term in the "transformation" law of ∂_μ. We can then define the covariant derivative $D_\mu = \partial_\mu - iq\mathcal{A}_\mu$ to achieve this.

In summary, with a local gauge symmetry, a gauge field $\mathcal{A}_\mu$ must be introduced such that kinetic terms in the original Lagrangian can be invariant under the gauge transformation. In our case this amounts to adding the interaction term

$$\mathcal{L}_{\mathcal{A}\Phi\text{-int}} = -iq\left[(\partial^\mu \Phi^\dagger)\mathcal{A}_\mu \Phi - \Phi^\dagger \mathcal{A}^\mu (\partial_\mu \Phi)\right] + q^2 \Phi^\dagger \mathcal{A}^\mu \mathcal{A}_\mu \Phi \tag{2.19}$$

to the Klein-Gordon Lagrangian $\mathcal{L}_{KG}$.

Now, the Lagrangian is gauge invariant, but there still remains to add dynamics to the gauge field $\mathcal{A}_\mu$ through a kinetic term. To this end, we can make a field-strength tensor $\mathcal{F}_{\mu\nu} \equiv F^a_{\mu\nu} T^a$ that transforms as $\mathcal{F}_{\mu\nu} \to g \mathcal{F}_{\mu\nu} g^{-1}$. The covariant derivative already has this property, and so we can define $\mathcal{F}_{\mu\nu} = \frac{i}{q}[D_\mu, D_\nu]$, which will include derivative terms for the $\mathcal{A}_\mu$ gauge field and let us construct a gauge invariant kinetic term $\mathrm{Tr}\left\{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\right\}$. Antisymmetrising $D_\mu D_\nu \to [D_\mu, D_\nu]$ serves to get rid of the $\partial_\mu\partial_\nu$-term which would result in third derivatives of the gauge field. The kinetic term can be shown to be gauge invariant using the transformation law the field-strength tensor and the cyclic property of the trace

$$\mathrm{Tr}\{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\} \to \mathrm{Tr}\{g\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}g^{-1}\} = \mathrm{Tr}\{g^{-1}g\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\} = \mathrm{Tr}\{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\}. \tag{2.20}$$

This results in a kinetic term for the $\mathcal{A}_\mu$-field

$$\mathcal{L}_{\mathcal{A}\text{-kin}} = -\frac{1}{4T(R)}\,\mathrm{Tr}\{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu_\nu}\} = -\frac{1}{4}F^{a\,\mu\nu}F^a_{\mu\nu}, \tag{2.21}$$

where $T(R)$ is the Dynkin index of the representation R of the group defined by the relation $\mathrm{Tr}\{T^a T^b\} = T(R)\delta^{ab}$ when T^a are the generators of the group in that representation.

2.4 The Standard Model

PHANTOM PARAGRAPH: INTRODUCE THE *RELEVANT* FIELDS OF THE STANDARD MODEL AND ITS CONSTRUCTION. MAYBE MENTION THE HIGGS MECHANISM?

2.5 Loop Integrals and Regularisation

PHANTOM PARAGRAPH: INTRODUCE LOOP INTEGRALS, HOW TO CALCULATE THEM, WHERE DIVERGENCES APPEAR AND HOW TO REGULARISE THEM.

TODO:

[4]More on this later.

[5]I will use bold notation $\boldsymbol{\alpha}$ to refer to the collection of parameters α^a, of which there is one for each generator T^a.

[6]It can be shown to be equivalent to ∂_μ when applied to any field (whether they transform under the gauge transformations or not).

[7]In the following I suppress the argument so that $g = g\left(\boldsymbol{\alpha}(x)\right)$.

☐ Introduce $\overline{\mathrm{DR}}$ renormalisation scheme and talk about Yukawa counterterm in relation to SUSY breaking.

2.5.1 Dimensional Regularisation

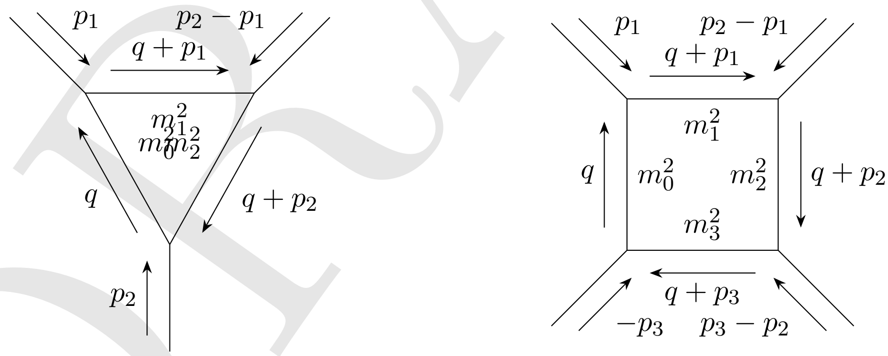
2.5.2 Passarino-Veltman Loop Integrals

By Lorentz invariance, there are a limited set of forms that loop integrals can take. Why is this?© These can be categorised according to the number of propagator terms they include, which corresponds to the number of externally connected points there are in the loop. A general scalar N-point loop integral takes the form

$$T_0^N\left(p_i^2,(p_i-p_j)^2;m_0^2,m_i^2\right)=\frac{(2\pi\mu)^{4-d}}{i\pi^2}\int \mathrm{d}^d q\,\mathcal{D}_0\prod_{i=1}^{N-1}\mathcal{D}_i, \tag{2.22}$$

where $\mathcal{D}_0=\left[q^2-m_0^2\right]^{-1}$ and $\mathcal{D}_i=\left[(q+p_i)^2-m_i^2\right]^{-1}$. The first 4 scalar loop integrals are named accordingly

$$T_0^1 \equiv A_0(m_0^2) \tag{2.23}$$
$$T_0^2 \equiv B_0(p_1^2;m_0^2,m_1^2) \tag{2.24}$$
$$T_0^3 \equiv C_0(p_1^2,p_2^2,(p_1-p_2)^2;m_0^2,m_1^2,m_2^2) \tag{2.25}$$
$$T_0^4 \equiv D_0(p_1^2,p_2^2,p_3^2,(p_1-p_2)^2,(p_1-p_3)^2,(p_2-p_3)^2;m_0^2,m_1^2,m_2^2) \tag{2.26}$$



(a) (b)

Figure 2.2: Illustration of the momentum conventions for loop diagrams used in the Passarino-Veltman functions.

More complicated Lorentz structure can be obtained in loop integrals, however, these can still be related to the scalar integrals by exploiting the possible tensorial structure they can have. Defining an arbitrary loop integral

$$T^N_{\mu_1\cdots\mu_P}\left(p_i^2,(p_i-p_j)^2;m_0^2,m_i^2\right)=\frac{(2\pi\mu)^{4-d}}{i\pi^2}\int \mathrm{d}^d q\,q_{\mu_1}\cdots q_{\mu_P}\mathcal{D}_0\prod_{i=1}^{N-1}\mathcal{D}_i. \tag{2.27}$$

These tensors can only depend on the metric $g^{\mu\nu}$ and the external momenta p_i. The possible structures up to four-point loops are as following:

$$
\begin{aligned}
B^{\mu} &= p_1^{\mu} B_1, && \text{(2.28a)}\\
B^{\mu\nu} &= g^{\mu\nu} B_{00} + p_1^{\mu} p_1^{\nu} B_{11}, && \text{(2.28b)}\\
C^{\mu} &= \sum_{i=1}^{2} p_i^{\mu} C_i, && \text{(2.28c)}\\
C^{\mu\nu} &= g^{\mu\nu} C_{00} + \sum_{i,j=1}^{2} p_i^{\mu} p_j^{\nu} C_{ij}, && \text{(2.28d)}\\
C^{\mu\nu\rho} &= \sum_{i=1}^{2} (g^{\mu\nu} p_i^{\rho} + g^{\mu\rho} p_i^{\nu} + g^{\nu\rho} p_i^{\mu}) C_{00i} + \sum_{i,j,k=1}^{2} p_i^{\mu} p_j^{\nu} p_k^{\rho} C_{ijk}, && \text{(2.28e)}\\
D^{\mu} &= \sum_{i=1}^{3} p_i^{\mu} D_i, && \text{(2.28f)}\\
D^{\mu\nu} &= g^{\mu\nu} D_{00} + \sum_{i,j=1}^{3} p_i^{\mu} p_j^{\nu} D_{ij}, && \text{(2.28g)}\\
D^{\mu\nu\rho} &= \sum_{i=1}^{3} (g^{\mu\nu} p_i^{\rho} + g^{\mu\rho} p_i^{\nu} + g^{\nu\rho} p_i^{\mu}) D_{00i} + \sum_{i,j,k=1}^{3} p_i^{\mu} p_j^{\nu} p_k^{\rho} D_{ijk}, && \text{(2.28h)}\\
D^{\mu\nu\rho\sigma} &= (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) D_{0000} \\
&\quad + \sum_{i,j=1}^{3} (g_{\mu\nu} p_i^{\rho} p_j^{\sigma} + g_{\mu\nu} p_i^{\sigma} p_j^{\rho} + g_{\mu\rho} p_i^{\nu} p_j^{\sigma} + g_{\mu\rho} p_i^{\sigma} p_j^{\nu} + g_{\mu\sigma} p_i^{\rho} p_j^{\nu} + g_{\mu\nu} p_i^{\nu} p_j^{\rho}) D_{00ij} \\
&\quad + \sum_{i,j,k,l=1}^{3} p_i^{\mu} p_j^{\nu} p_k^{\rho} p_l^{\sigma} D_{ijkl}, && \text{(2.28i)}
\end{aligned}
$$

where all coefficients must be completely symmetric in i, j, k, l.

Chapter 3

Supersymmetry

3.1 Introduction to Supersymmetry

In this chapter, I introduce the basic ideas behind supersymmetry, what it is, and how to construct field theories that are *supersymmetric*. I will discuss the Super-Poincaré group as an extension of the Poincaré group, and introduce superspace as a vessel for supersymmetric field theories. I go on to describe the Minimal Supersymmetric Standard Model (MSSM), the minimal (broken) supersymmetric QFT containing the Standard Model (SM) particles as a subset. The electroweakinos, the main focus of this thesis, are introduced, including general mixing of fields into mass eigenstates, and I go into some depth to derive the interaction Feynman rules of these particles from the MSSM *superlagrangian*.

This chapter makes extensive use of Weyl spinor notation and Grassmann calculus. For more details on this and the specific notation I use, I refer to Appendix A. Some background in group theory is necessary to follow certain parts of the chapter, but is not necessary to understand the broader ideas.

A Simple Supersymmetric Theory

To illustrate what supersymmetry looks like in practice, it can be helpful to look at a simple example. Take a Lagrangian for a massive complex scalar field ϕ and a massive Weyl spinor field ψ,

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) + i\psi\sigma^\mu\partial_\mu\psi^\dagger - |m_\phi|^2\,\phi\phi^* - \frac{1}{2}m_\psi(\psi\psi) - \frac{1}{2}m_\psi^*(\psi\psi)^\dagger. \tag{3.1}$$

To impose some symmetry between the bosonic and fermionic degrees of freedom, we want to examine a transformation of the scalar field through the spinor field and vice versa. Imposing Lorentz invariance a general, infinitesimal, such transformation can be parametrised by

$$\delta\phi = \epsilon a(\theta\psi), \tag{3.2a}$$

$$\delta\phi^* = \epsilon a^*(\theta\psi)^\dagger, \tag{3.2b}$$

$$\delta\psi_\alpha = \epsilon\left(c(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu\phi + F(\phi,\phi^*)\theta_\alpha\right), \tag{3.2c}$$

$$\delta{\psi^\dagger}_{\dot\alpha} = \epsilon\left(c^*(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^* + F^*(\phi,\phi^*)\theta^\dagger_{\dot\alpha}\right), \tag{3.2d}$$

where ϵ is some infinitesimal parameter for the transformation, θ is some Grassmann-valued Weyl spinor, a, c are complex coefficients of the transformation and $F(\phi,\phi^*)$ is some linear function of ϕ and ϕ^*. The resulting change in the scalar field part of the Lagrangian is

$$\delta\mathcal{L}_\phi/\epsilon = a(\theta\partial_\mu\psi)\,(\partial^\mu\phi^*) - a\,|m_\phi|^2\,(\theta\psi)\phi^* + \text{c. c.}, \tag{3.3}$$

and likewise for the spinor field part

$$\delta\mathcal{L}_\psi/\epsilon = -ic^*(\psi\sigma^\mu\bar{\sigma}^\nu\theta)\partial_\mu\partial_\nu\phi^* + i(\psi\sigma^\mu\theta^\dagger)\partial_\mu F^* + m_\psi\left[c(\psi\sigma^\mu\theta^\dagger)\partial_\mu\phi + (\psi\theta)F\right] + \text{c. c.} \tag{3.4}$$

The first term in Eq. (3.4) can be rewritten using the commutativity of partial derivatives and the identity $\left(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu\right)_\alpha{}^\beta = -2g^{\mu\nu}\delta_\alpha^\beta$ to get $ic^*\,(\theta\psi)\,\partial_\mu\partial^\mu\phi^*$. Up to a total derivative, we can then write the change in the spinor part as

$$\delta\mathcal{L}_\psi/\epsilon = -ic^*(\theta\partial_\mu\psi)\partial^\mu\phi^* + (\psi\sigma^\mu\theta^\dagger)\partial_\mu\left(iF^* + m_\psi c\phi\right) + m_\psi(\theta\psi)F + \text{c. c.}\,. \tag{3.5}$$

The change of the total Lagrangian (again up to a total derivate) can then be grouped as

$$\begin{aligned}\delta\mathcal{L}/\epsilon = &\,(a - ic^*)\,(\theta\partial_\mu\psi)\,(\partial^\mu\phi^*) + (\psi\sigma^\mu\theta^\dagger)\partial_\mu\left(iF^* + m_\psi c\phi\right)\\ &+ (\theta\psi)\left(a\,|m_\phi|^2\,\phi^* + m_\psi F\right) + \text{c. c.},\end{aligned} \tag{3.6}$$

giving us three different conditions for the action to be invariant:

$$a - ic^* = 0, \tag{3.7a}$$

$$iF^* + m_\psi c\phi = 0, \tag{3.7b}$$

$$a\,|m_\phi|^2\,\phi^* + m_\psi F = 0. \tag{3.7c}$$

This is fulfilled if

$$c = ia^*, \tag{3.8a}$$

$$F = -am_\psi^*\phi^*, \tag{3.8b}$$

$$a\,|m_\phi|^2 = a^*\,|m_\psi|^2\,. \tag{3.8c}$$

What is interesting is the last condition, because it requires a to be real, as both $|m_\phi|^2$ and $|m_\psi|^2$ are real, but also requires $|m_\phi|^2 = |m_\psi|^2$. For the theory to be supersymmetric in this sense, the masses of the boson and fermion must be the same! Since the phase of m_ϕ does not appear in the Lagrangian, we are free to set $m_\phi = m_\psi \equiv m$, suppressing any mass subscripts henceforth.

Revisiting F, it can be introduced as an auxiliary field to bookkeep the supersymmetry transformation. By including the non-dynamical term to the Lagrangian $\mathcal{L}_F = F^*F + mF\phi + m^*F^*\phi^*$, we make sure F takes the correct value in the transformation from its equation of motion $\frac{\partial\mathcal{L}}{\partial F} = F^* + m\phi \stackrel{!}{=} 0$. Inserting F back into the Lagrangian reproduces the mass term of the scalar field, allowing us to write the original Lagrangian as

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) + i\psi\sigma^\mu\partial_\mu\psi^\dagger + F^*F + mF\phi + m^*F^*\phi^* - \frac{1}{2}m(\psi\psi) - \frac{1}{2}m^*(\psi\psi)^\dagger, \tag{3.9}$$

with the *supersymmetry transformation* rules

$$\delta\phi = \epsilon(\theta\psi), \qquad \delta\phi^* = \epsilon(\theta\psi)^\dagger, \tag{3.10a}$$

$$\delta\psi_\alpha = \epsilon\left(-i(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu\phi + F\theta_\alpha\right), \qquad \delta\psi^\dagger_{\dot\alpha} = \epsilon\left(i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^* + F^*\theta^\dagger_{\dot\alpha}\right), \tag{3.10b}$$

$$\delta F = i\epsilon\left(\partial_\mu\psi\sigma^\mu\theta^\dagger\right), \qquad \delta F^* = -i\epsilon\left(\theta\sigma^\mu\partial_\mu\psi^\dagger\right), \tag{3.10c}$$

where I have set $a = 1$ without loss of generality,[1] and found the appropriate transformation law for F such that the Lagrangian is invariant up to total derivatives. The dynamics of this Lagrangian are the same as before, but the supersymmetry is now made manifest, i.e. the transformation is free of any dependence on the contents of the Lagrangian as we had in Eq. (3.8).

In fact, one can show that a general supersymmetric Lagrangian consisting of a scalar field and a fermion field can be written

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) + i\psi\sigma^\mu\partial_\mu\psi^\dagger + F^*F + \left\{ mF\phi - \frac{1}{2}m(\psi\psi) - \lambda\phi(\psi\psi) + \text{c. c.} \right\} \tag{3.11}$$

up to renormalisable interactions.

3.2 The Super-Poincaré Group

To introduce more involved supersymmetric QFTs than our simple example from Section 3.1, it will be useful to first introduce a framework that will manifestly carry the supersymmetry. This will alleviate the need to figure out the correct transformation laws, and the constraints they may carry to the parameters of the theory. To this end, I will outline a common way of introducing supersymmetric theories – extending our fields from representations of the Poincaré group of cooridinate transformations to the super-Poincaré group. This will hopefully give an algebraic geometrical understanding to *superfields* as the building blocks of a supersymmetric field theory.

3.2.1 The Poincaré and Super-Poincaré Algebras

As we have already seen in Section 2.3, sets of transformations for a symmetry can be described by a group. To introduce supersymmetry in this context, it will be clearer to study the *generators* of the algebra of the group, so I would like to take a moment to motivate this change of perspective, before describing the fundamental symmetries we will be using.

The group describing the basic set of *coordinate transformations* under which the fields theories we will consider are symmetric is called the *Poincaré group*, denoted P. Theories that are symmetric under this group will be manifestly relativistic, and will exhibit the ordinary freedom in choice of coordinate system. The Poincaré group consists of any transformation of space-time coordinates x^μ such that

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{3.12}$$

for a real, orthogonal 4×4 matrix Λ and real numbers a^μ. As a group it is the semi-direct product of Lorentz group $O(1,3)$ and group of 4D space-time translations $T(1,3)$

$$P \equiv O(1,3) \rtimes T(1,3). \tag{3.13}$$

For completeness, the semi-direct product is defined such that the product of two group elements $(\Lambda_1, p_1), (\Lambda_2, p_2) \in P$ where $\Lambda_1, \Lambda_2 \in O(1,3)$ and $p_1, p_2 \in T(1,3)$ is

$$(\Lambda_1, p_1) \circ (\Lambda_2, p_2) \equiv (\Lambda_1 \circ_O \Lambda_2, p_1 \circ_T \Lambda_1(p_2)), \tag{3.14}$$

[1] It can be absorbed by a redefinition of the parameter ϵ for instance.

where we understand $\circ_{O/T}$ as the group multiplication operations of $O(1,3)$ and $T(1,3)$ respectively.[2]

For our purposes, it will suffice to work simply with the local structure of the Poincaré group, and being Lie groups, this can be reproduced with the exponential map we have used already in Eq. (2.12) $\exp : \mathfrak{g} \to G$, where $\mathfrak{g}$ is the *Lie algebra* of the Lie group G. In this way, the algebra is said to *generate* the group, and a basis set $\{T^a\}$ of the algebra $\mathfrak{g}$ is said to be the *generators* of the group.[3] Accordingly, the local behaviour of the group can be inferred simply from the properties of the generators T^a. The generators of the Poincaré group can be structured by an antisymmetric Lorentz tensor $M^{\mu\nu}$, and a four-vector P^μ. The properties of the algebra these generators span can be inferred from their commutation relations

$$
\begin{aligned}
[P^\mu, P^\nu] =& 0, && (3.15a)\\
[M^{\mu\nu}, P^\rho] =& i\left(g^{\mu\sigma} P^\nu - g^{\nu\sigma} P^\mu\right), && (3.15b)\\
[M^{\mu\nu}, M^{\rho\sigma}] =& i\left(g^{\mu\rho} M^{\nu\sigma} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\rho} M^{\mu\sigma} + g^{\nu\sigma} M^{\nu\rho}\right). && (3.15c)
\end{aligned}
$$

To construct the super-Poincaré group, we can then just extend the algebra, and the rest of the group will follow. This is done by extending the Lie algebra to a *graded Lie superalgebra* by adding new generators. A *graded Lie superalgebra* is constructed from two vector spaces $\mathfrak{l}_0, \mathfrak{l}_1$ and is denoted $\mathfrak{l}_0 \oplus \mathfrak{l}_1$. It is itself a vector space with a bilinear operation such that for any elements $x_i \in \mathfrak{l}_i$ we have

$$
\begin{aligned}
& x_j \circ x_j \in \mathfrak{l}_{i+j \bmod 2}, && \text{(grading)}\\
& x_i \circ x_j = -(-1)^{i\cdot j} x_j \circ x_i, && \text{(supersymmetrisation)}\\
& x_i \circ (x_j \circ x_k)(-1)^{i\cdot k} + x_j \circ (x_k \circ x_i)(-1)^{j\cdot i} + x_k \circ (x_i \circ x_j)(-1)^{k\cdot j} = 0. && \text{(generalised Jacobi identity)}
\end{aligned}
$$

I note that in this case, $\mathfrak{l}_0$ acts as an ordinary Lie algebra where $\circ$ is the ordinary commutator, and $\mathfrak{l}_1$ gets anti-commutator relations rather than commutator relations.[4]

The *super-Poincaré algebra*, denoted $\mathfrak{sp}$, is the graded Lie superalgebra resulting from the Poincaré algebra $\mathfrak{p}$ and the vector space $\mathfrak{q}$. Here $\mathfrak{p}$ is the Lie algebra of the Poincaré group P and $\mathfrak{q}$ is the vector space spanned by the generators $Q_\alpha, Q^\dagger_{\dot\alpha}$ that form two Weyl spinors. In addition to the commutation relations Eqs. (3.15a) to (3.15c), the Poincaré superalgebra is specified by the (anti-)commutator relations

$$
\begin{aligned}
[Q_\alpha, P^\mu] = [Q^\dagger_{\dot\alpha}, P_\mu] =& 0 && (3.16a)\\
[Q_\alpha, M^{\mu\nu}] =& (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta && (3.16b)\\
\{Q_\alpha, Q_\beta\} = \{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} =& 0, && (3.16c)\\
\{Q_\alpha, Q^\dagger_{\dot\beta}\} =& 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu && (3.16d)
\end{aligned}
$$

where $\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu \bar\sigma^\nu - \sigma^\nu \bar\sigma^\mu\right)$, $\sigma^\mu = \left(\mathbb{I}, \sigma^i\right)$, $\bar\sigma^\mu = \left(\mathbb{I}, -\sigma^i\right)$ and σ^i are the Pauli matrices.

---

[2] We see also that $O(1,3)$ must also be a map $T(1,3) \to T(1,3)$. We will later see that this means that the generators of translations are in a representation of the Lorentz group.

[3] The algebra of a Lie group can be shown to be a vector space, and as such there exists a basis set spanning the algebra.

[4] This can be seen from supersymmetrisation as for any $x_1, x_1' \in \mathfrak{l}_1$ we have that $x_1 \circ x_1' = x_1' \circ x_1$.

3.2.2 Superspace

The idea behind *superspace* is to create a coordinate system for which supersymmetry transformation manifest as coordinate transformations similarly to the way Poincaré transformations work on ordinary space-time coordinates. To this end, we can start by considering a general element of the super-Poincaré group $g \in SP$; it can be parametrised through the exponential map like this.

$$g = \exp\left(ix^\mu P_\mu + i(\theta Q) + i(\theta Q)^\dagger + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right), \tag{3.17}$$

where $x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha}, \omega_{\mu\nu}$ parametrise the group, and $P_\mu, Q_\alpha, Q^{\dagger\dot\alpha}, M^{\mu\nu}$ are the generators of the group as we have already seen. Since the parameters $x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha}$ live in irreps of the Lorentz algebra (four-vector and Weyl spinor representations respectively) generated by $M^{\mu\nu}$, the effect of the Lorentz part of the super-Poincaré group on the parameters can be determined easily. Likewise, the parameters $\omega_{\mu\nu}$ are in a trivial representation of the algebra generated by $P_\mu, Q_\alpha, Q^{\dagger\dot\alpha}$, and need not then be considered. It is therefore expedient to create a space with $x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha}$ as the coordinates, modding out the Lorentz algebra part.

We create superspace as a coordinate system with coordinates $z^\pi = (x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha})$, and look at how they transform under super-Poincaré group transformations. A function $F(z)$ on superspace can then be written using the generators $K_\pi = (P_\mu, Q_\alpha, Q^{\dagger\dot\alpha})$ as $F(z) = \exp(iz^\pi K_\pi)\, F(0)$. Applying a super-Poincaré group element without the Lorentz generators $\bar g(a,\eta) = \exp\left(ia^\mu P_\mu + i(\eta Q) + i(\eta Q)^\dagger\right)$ we have

$$F(z') = \exp(iz'^\pi K_\pi)\, F(0) = \exp\left(ia^\mu P_\mu + i(\eta Q) + i(\eta Q)^\dagger\right)\exp(iz^\pi K_\pi)\, F(0), \tag{3.18}$$

which by the Baker-Campbell-Hausdorff formula (BCH) gives to first order in the commutators

$$z'^\pi K_\pi = (x^\mu + a^\mu)P_\mu + (\theta^\alpha + \eta^\alpha)Q_\alpha + (\theta^\dagger_{\dot\alpha} + \eta^\dagger_{\dot\alpha})Q^{\dagger\dot\alpha} + \frac{i}{2}\left[a^\mu P_\mu + (\eta Q) + (\eta Q)^\dagger, z^\pi K_\pi\right] + \dots \tag{3.19}$$

Now, P_μ commutes with all of K_π, and Q_α ($Q^{\dagger\dot\alpha}$) anti-commute with themselves, for every combination of different α ($\dot\alpha$), so the only relevant part of the commutator is

$$[(\eta Q), (\theta Q)^\dagger] + [(\eta Q)^\dagger, (\theta Q)] = -\eta^\alpha\{Q_\alpha, Q^\dagger_{\dot\alpha}\}\theta^{\dagger\dot\alpha} + (\eta \leftrightarrow \theta) = -2(\eta\sigma^\mu\theta^\dagger)P_\mu + (\eta \leftrightarrow \theta). \tag{3.20}$$

Since this commutator is proportional to P_μ which in turn commutes with everything, all higher order commutators of BCH vanish, and we can conclude that the transformed coordinates z'^π are given by

$$z'^\pi = \left(x^\mu + a^\mu + i(\theta\sigma^\mu\eta^\dagger) - i(\eta\sigma^\mu\theta^\dagger), \theta^\alpha + \eta^\alpha, \theta^\dagger_{\dot\alpha} + \eta^\dagger_{\dot\alpha}\right). \tag{3.21}$$

This gives us a differential representation of the K_π generators as

$$P_\mu = -i\partial_\mu, \tag{3.22a}$$

$$Q_\alpha = -(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu - i\partial_\alpha, \tag{3.22b}$$

$$Q^\dagger_{\dot\alpha} = -(\theta\bar\sigma^\mu)_{\dot\alpha}\partial_\mu - i\partial_{\dot\alpha}. \tag{3.22c}$$

Now, to see what the these functions of superspace look like, we can expand $F(z)$ in terms of the coordinates $\theta^\alpha, \theta^\dagger_{\dot\alpha}$, as these expansions are finite due to the fact that none of these coordinates can appear more than once per term. Demanding that the function $F(z)$ be invariant under Lorentz transformations, the x^μ-dependent coefficients of the expansion must transform such that each term is a scalar (or fully contracted Lorentz structure). This limits a general such function of superspace to be written as

$$\begin{aligned}F(z) =& f(x) + \theta^\alpha\phi_\alpha(x) + \theta^\dagger_{\dot\alpha}\chi^{\dagger\dot\alpha}(x) + (\theta\theta)m(x) + (\theta\theta)^\dagger n(x)\\ &+ (\theta\sigma^\mu\theta^\dagger)V_\mu(x) + (\theta\theta)\theta^\dagger_{\dot\alpha}\lambda^{\dagger\dot\alpha}(x) + (\theta\theta)^\dagger\theta^\alpha\psi_\alpha(x) + (\theta\theta)(\theta\theta)^\dagger d(x).\end{aligned} \tag{3.23}$$

Temporary

For future reference, a supersymmetry transformation $\bar g(0,\epsilon)$ from some infinitesimal Weyl spinor ϵ_α transforms the component fields of the general superfield $F(z)$ according to[5] ⋮©⋮

$$\delta f(x) =(\epsilon\phi(x)) + (\epsilon\chi(x))^\dagger \tag{3.24a}$$
$$\delta\phi_\alpha(x) =2\epsilon_\alpha m(x) - (\sigma^\mu\epsilon^\dagger)_\alpha(V_\mu(x) + i\partial_\mu f(x)) \tag{3.24b}$$
$$\delta\chi^{\dagger\dot\alpha}(x) =2\epsilon^{\dagger\dot\alpha}n(x) + (\bar\sigma^\mu\epsilon)^{\dagger\dot\alpha}(V_\mu(x) - i\partial_\mu f(x)) \tag{3.24c}$$
$$\delta m(x) =(\epsilon\lambda(x))^\dagger - \frac{i}{2}(\epsilon^\dagger\bar\sigma^\mu\partial_\mu\phi(x)) \tag{3.24d}$$
$$\delta n(x) =(\epsilon\psi(x)) - \frac{i}{2}(\epsilon\sigma^\mu\partial_\mu\chi^\dagger(x)) \tag{3.24e}$$
$$\delta V_\mu(x) =(\epsilon\sigma_\mu\lambda^\dagger(x)) - (\epsilon^\dagger\bar\sigma_\mu\psi(x)) - \frac{i}{2}(\epsilon\sigma^\nu\bar\sigma_\mu\partial_\nu\phi(x)) + \frac{i}{2}(\epsilon^\dagger\bar\sigma^\nu\sigma_\mu\partial_\nu\chi^\dagger(x)) \tag{3.24f}$$
$$\delta\psi_\alpha =2\epsilon_\alpha d(x) - i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu n(x) - \frac{i}{2}(\sigma^\nu\bar\sigma^\mu\epsilon)_\alpha\partial_\mu V_\nu(x) \tag{3.24g}$$
$$\delta\lambda^{\dagger\dot\alpha} =2\epsilon^{\dagger\dot\alpha}d(x) - i(\bar\sigma^\mu\epsilon)^{\dagger\dot\alpha}\partial_\mu m(x) + \frac{i}{2}(\bar\sigma^\nu\sigma^\mu\epsilon^\dagger)^{\dot\alpha}\partial_\mu V_\nu(x) \tag{3.24h}$$
$$\delta d(x) =-\frac{i}{2}(\epsilon^\dagger\bar\sigma^\mu\partial_\mu\psi(x)) - \frac{i}{2}(\epsilon\partial^\mu\partial_\mu\lambda^\dagger(x)) \tag{3.24i}$$

3.2.3 Superfields

To construct a manifestly supersymmetric theory, it will be useful to start with finding representations of the super-Poincaré group. This is exactly what we have already done; the functions on superspace find themselves in the representation space of a differential representation of the K_π generators of the super-Poincaré group, and a scalar representation of the remaining Lorentz generators (i.e. the Lorentz generators leave the superspace functions unchanged). Inside the general function on superspace Eq. (3.23), we find many component functions in different representation spaces of the Lorentz group. Furthermore, supersymmetry transformations transform these fields into one another. This seems like an ideal vessel for constructing supersymmetric fields theories.

We define the *superfield* Φ as an operator-valued function on superspace.[6] The general superfield from Eq. (3.23) is in a reducible representation space of the super-Poincaré group, so we define three *irreducible* representations that will be useful going

[5]The transformations parameters a^μ are set to zero, which we can do without loss of generality, as the P_μ generators commute with the rest of the generators.

[6]For our purposes, it suffices to look at them simply as complex valued functions, but strictly speaking, they are operator-valued in a quantised field theory.

forward:[7]

$$\textit{Left-handed scalar superfield:} \qquad \bar{D}_{\dot{\alpha}}\Phi = 0, \tag{3.25}$$
$$\textit{Right-handed scalar superfield:} \qquad D_{\alpha}\Phi^{\dagger} = 0, \tag{3.26}$$
$$\textit{Vector superfield:} \qquad \Phi^{\dagger} = \Phi. \tag{3.27}$$

Here the dagger operation refers to complex conjugation, and the differential operators $D_\alpha, \bar{D}_{\dot{\alpha}}$ are defined as

$$D_\alpha = \partial_\alpha + i(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu, \tag{3.28a}$$
$$\bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu. \tag{3.28b}$$

These differential operators are covariant differentials in the sense that the commute with supersymmetry transformations, i.e. $D_\alpha F(z) \to D'_\alpha(\bar{g}F(z)) = \bar{g}\,(D_\alpha F(z))$ Collectively, the left- and right-handed scalar superfields are referred to as *chiral superfields*.

For future reference, the general forms of a left-handed scalar superfield Φ, a right-handed scalar superfield $\Phi^\dagger$ and a vector superfield V_{WZ} in the so-called Wess-Zumino gauge is [1]:

$$\begin{aligned}\Phi(x,\theta,\theta^\dagger) =& A(x) + i(\theta\sigma^\mu\theta^\dagger)\partial_\mu A(x) - \frac{1}{4}(\theta\theta)(\theta\theta)^\dagger\,\Box\,A(x)\\ &+ \sqrt{2}(\theta\psi(x)) - \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi(x)\sigma^\mu\theta^\dagger) + (\theta\theta)F(x),\end{aligned} \tag{3.29a}$$

$$\begin{aligned}\Phi^\dagger(x,\theta,\theta^\dagger) =& A^*(x) - i(\theta\sigma^\mu\theta^\dagger)\partial_\mu A^*(x) - \frac{1}{4}(\theta\theta)(\theta\theta)^\dagger\,\Box\,A^*(x)\\ &+ \sqrt{2}(\theta\psi(x))^\dagger + \frac{i}{\sqrt{2}}(\theta\theta)^\dagger(\theta\sigma^\mu\partial_\mu\psi^\dagger(x)) + (\theta\theta)^\dagger F^*(x),\end{aligned} \tag{3.29b}$$

$$V_{\mathrm{WZ}}(x,\theta,\theta^\dagger) = (\theta\sigma^\mu\theta^\dagger)V_\mu\,(x) + (\theta\theta)(\theta\lambda(x))^\dagger + (\theta\theta)^\dagger(\theta\lambda(x)) + \frac{1}{2}(\theta\theta)(\theta\theta)^\dagger D(x). \tag{3.29c}$$

3.2.4 Superlagrangian

We are now ready to define the action of a supersymmetric quantum field theory on superspace. Given a set of superfields $\{\Phi_i\}$, we want to define an action through a Lagrangian density comprised of the component fields in Φ_i. A function of the superfields will still be a superfield, and will therefore take the form from Eq. (3.23). To get a supersymmetry invariant Lagrangian density, we can therefore look to extract some part of such a superspace function that at most transforms as a total derivative under a supersymmetry transformation according to Eq. (3.19). It can be show that the $d(x)$ component field of Eq. (3.23) transforms in such a way, and likewise for the $F(x)$ component field of a chiral superfield Eqs. (3.29a) and (3.29b), so projecting out these would constitute a valid Lagrangian density for a supersymmetry invariant action. Keeping in mind that the Lagrangian density must be real, we can then get a supersymmetry invariant action through a Lagrangian density on the form[8]

$$\mathcal{L} = \mathrm{proj}_D(V[\Phi_i]) + \mathrm{proj}_F(W[\Phi_i]) + \mathrm{proj}_{F^\dagger}(W^\dagger[\Phi_i]), \tag{3.30}$$

where $V[\Phi_i]$ is a vector superfield and $W[\Phi_i]$ ($W^\dagger[\Phi_i]$) is some left-handed (right-handed) scalar superfield.

[7] I will not prove here that these in fact are irreducible representations.

[8] To clarify potential confusion on the capitalisation of the D-projection here – for a vector superfield Eq. (3.29c) the $d(x)$ component field is the $D(x)$ auxiliary component field.

The projection operators can be realised using Grassmann integration:[9]

$$\operatorname{proj}_D V[\Phi_i] = \int \mathrm{d}^4\theta\, V[\Phi_i], \tag{3.31a}$$

$$\operatorname{proj}_F W[\Phi_i] = \int \mathrm{d}^4\theta\, (\theta\theta)^\dagger W[\Phi_i], \tag{3.31b}$$

$$\operatorname{proj}_{F^\dagger} W^\dagger[\Phi_i] = \int \mathrm{d}^4\theta\, (\theta\theta) W^\dagger[\Phi_i]. \tag{3.31c}$$

Accordingly, we can write down the general supersymmetry invariant action, letting $\left\{\bar{\Phi}_i\right\}$ be the subset of chiral superfields in $\{\Phi_i\}$ using a Lagrangian density on the form

$$\mathcal{L} = \int \mathrm{d}^4\theta\, \left\{V[\Phi_i] + (\theta\theta)^\dagger W[\bar{\Phi}_i] + (\theta\theta) W[\bar{\Phi}_i^\dagger]\right\}, \tag{3.32}$$

where we restrict W to be holonomic function of its argument superfields called the *superpotential*. W being holonomic in this context simply means that $W[\bar{\Phi}_i]$ will be a left-handed scalar superfield and $W[\bar{\Phi}_i^\dagger]$ a right-handed scalar superfield. This leads to defining the *superlagrangian* $\tilde{\mathcal{L}}$ as a Lagrangian density analogue on superspace, where we can recognise

$$\tilde{\mathcal{L}} = V[\Phi_i] + (\theta\theta)^\dagger W[\bar{\Phi}_i] + (\theta\theta) W[\bar{\Phi}_i^\dagger], \tag{3.33}$$

and subsequently the action as

$$S\left[\{\Phi_i\}\right] = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \tilde{\mathcal{L}}\left(\{\Phi_i\}, \left\{\frac{\partial \Phi_i}{\partial z^\pi}\right\}, z\right). \tag{3.34}$$

Renormalisability puts severe restrictions on the form of the superlagrangian by imposing that any parameter of the theory cannot have a negative mass dimension. Recognising that $1 = \int \mathrm{d}^4\theta\, (\theta\theta)(\theta\theta)^\dagger$, we must have that $[\int \mathrm{d}^4\theta\,] = M^2$ for some mass reference scale M. Consequently, for the ordinary Lagrangian density to have mass dimension four, we must have that $[\tilde{\mathcal{L}}] = M^2$. From Eq. (3.29a) we recognise that the scalar superfield contains a scalar field term, and consequently has mass dimension $[\Phi] = M^1$. Thus, the general form of the superpotential is

$$W[\Phi_i] = \sum_i \lambda_i \Phi_i + \sum_{ij} m_{ij} \Phi_i \Phi_j + \sum_{ijk} y_{ijk} \Phi_i \Phi_j \Phi_k, \tag{3.35}$$

and the only possible form of $V[\Phi_i]$ only containing scalar superfields is

$$V[\Phi_i] = \sum_i \Phi_i \Phi_i^\dagger, \tag{3.36}$$

where the prefactor of the terms are set to 1, which can be done without loss of generality by rescaling the fields.

3.2.5 Revisiting our Simple Supersymmetric Theory

Now that we have developed a structure for creating manifestly supersymmetric theories using superfields, we can take a moment to revisit our simple theory from Eq. (3.1) to see what it would look like within the superspace framework. We can use a left-handed

[9]As a reminder, I detail how the calculus of Grassmann coordinates is defined in Appendix A.

scalar superfield Φ as the vessel for our scalar field ϕ, fermionic field ψ and auxiliary field F:

$$\begin{aligned}\Phi(\theta, \theta^\dagger, x) =& \phi(x) + i(\theta\sigma^\mu\theta^\dagger)\partial_\mu\phi(x) - \frac{1}{4}(\theta\theta)(\theta\theta)^\dagger \Box \phi(x) \\ &+ \sqrt{2}(\theta\psi(x)) - \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi(x)\sigma^\mu\theta^\dagger) + (\theta\theta)F(x). \end{aligned} \tag{3.37}$$

The kinetic terms are reproduced through the first term in Eq. (3.32):

$$\begin{aligned}\mathcal{L}_{\text{kin}} = \int \mathrm{d}^4\theta\, \Phi^\dagger\Phi = \int \mathrm{d}^4\theta \Big\{ &- \frac{1}{4}\left(\phi^* \Box \phi + \phi \Box \phi^*\right) + (\theta\sigma^\mu\theta^\dagger)(\theta\sigma^\nu\theta^\dagger)\partial_\mu\phi^*\partial_\nu\phi \\ &- i\left[(\theta\psi)^\dagger(\theta\theta)(\partial_\mu\psi\sigma^\mu\theta^\dagger) - (\theta\theta)^\dagger(\theta\sigma^\mu\partial_\mu\psi^\dagger)(\theta\psi)\right] + (\theta\theta)(\theta\theta)^\dagger F^* F\Big\} \\ =& (\partial_\mu\phi)(\partial^\mu\phi^*) + i(\psi\sigma^\mu\partial_\mu\psi^\dagger) + F^*F. \end{aligned} \tag{3.38}$$

The remaining mass term can be recreated by the superlagrangian term $\frac{m}{2}(\theta\theta)^\dagger\Phi\Phi + \frac{m^*}{2}(\theta\theta)\Phi^\dagger\Phi^\dagger$, equivalent to a superpotential $W[\Phi] = \frac{m}{2}\Phi\Phi$, yielding

$$\begin{aligned}\mathcal{L}_{\text{mass}} = \int \mathrm{d}^4\theta \left\{\frac{m}{2}(\theta\theta)^\dagger\Phi\Phi + \text{c. c.}\right\} &= \int \mathrm{d}^4\theta \left\{\frac{m}{2}(\theta\theta)^\dagger \left(2\phi(\theta\theta)F + 2(\theta\psi)(\theta\psi)\right) + \text{c. c.}\right\} \\ =& m\phi F + m^*\phi^*F^* + \frac{m}{2}(\psi\psi) + \frac{m^*}{2}(\psi\psi)^\dagger. \end{aligned} \tag{3.39}$$

So our simple supersymmetric theory is encapsulated simply by the superlagrangian

$$\tilde{\mathcal{L}} = \Phi\Phi^\dagger + \frac{m}{2}(\theta\theta)^\dagger\Phi\Phi + \frac{m^*}{2}(\theta\theta)\Phi^\dagger\Phi^\dagger, \tag{3.40}$$

showing how superspace simplifies the model building considerably.

3.3 Minimal Supersymmetric Standard Model

Up to this point, the building blocks for the MSSM have been introduced, and I will now shift focus how these are put together to create the minimal supersymmetric extension of the SM. I will also outline the process of spontaneous symmetry breaking, and state a general parametrisation of how this is done in the MSSM.

3.3.1 Supersymmetric Yang-Mills Theory

Before getting into the MSSM content, we must introduce what Yang-Mills theory looks like at a superlagangian level. We define a *supergauge transformation* of a left-handed scalar superfield multiplet Φ analogously to the ordinary case Eq. (2.13)

$$\Phi \to \exp\left(i\Lambda\right)\Phi, \tag{3.41}$$

where $\Lambda \equiv \Lambda^a T^a$, Λ^a are the parameters of the transformation and T^a are again the generators of the gauge group. To get a sense of what these parameters are, we can require the transformed superfield to be left-handed

$$\begin{aligned}D^\dagger_{\dot\alpha}\exp\left(i\Lambda\right)\Phi =& i\left(D^\dagger_{\dot\alpha}\Lambda^a\right)T^a\exp\left(i\Lambda^aT^a\right)\Phi + \exp\left(i\Lambda^aT^a\right)D^\dagger_{\dot\alpha}\Phi \\ =& i\left(D^\dagger_{\dot\alpha}\Lambda^a\right)T^a\exp\left(i\Lambda^aT^a\right)\Phi \stackrel{!}{=} 0, \end{aligned}$$

which means that we must require $D^\dagger_{\dot\alpha}\Lambda^a = 0$, meaning that the parameters are themselves left-handed scalar superfields. Examining how the kinetic term $\Phi^\dagger\Phi$ does under this transformation we can see that[10]

$$\Phi^\dagger\Phi \to \Phi^\dagger e^{-i\Lambda^\dagger} e^{i\Lambda}\Phi = \Phi^\dagger e^{i\left(\Lambda-\Lambda^\dagger\right)-\frac{1}{2}[\Lambda,\Lambda^\dagger]+\cdots}\Phi, \tag{3.42}$$

which is not invariant. To remedy this, we will introduce a term to compensate for this change, like before. For this we define a *supergauge field* $\mathcal{V} \equiv V^a T^a$ which transforms according to[11]

$$e^{2q\mathcal{V}} \to e^{i\Lambda^\dagger} e^{2q\mathcal{V}} e^{-i\Lambda} \tag{3.43}$$

or infinitesimally

$$\mathcal{V} \to \mathcal{V} - \frac{i}{2q}\left(\Lambda - \Lambda^\dagger\right) + \frac{i}{2}[\Lambda + \Lambda^\dagger, \mathcal{V}]. \tag{3.44}$$

Changing the kinetic term to $\Phi^\dagger e^{2q\mathcal{V}}\Phi$ will then yield it invariant under supergauge transformations. Since we require the superlagrangian term to be real, we must require $\mathcal{V}^\dagger = \mathcal{V}$, meaning it must be a vector superfield according to Eq. (3.29c).

As before, we would also like to add dynamics to the (super)gauge field $\mathcal{V}$. To this end, we introduce the supersymmetric field strength $\mathcal{W}_\alpha \equiv W^a_\alpha T^a$ for which we require the transformation law

$$\mathcal{W}_\alpha \to e^{i\Lambda}\mathcal{W}_\alpha e^{-i\Lambda}. \tag{3.45}$$

It can be shown that the left-handed chiral superfield construction

$$\mathcal{W}_\alpha = -\frac{1}{4}(\bar D\bar D)\left(e^{-2\mathcal{V}} D_\alpha e^{2\mathcal{V}}\right) \tag{3.46}$$

transforms this way, and recreates field-strength tensor earlier in Section 2.3.[2] The gauge invariant superlagrangian kinetic term for the supergauge field becomes

$$\mathcal{L}_{\mathcal{V}\text{-kin}} = \frac{1}{4T(R)}\operatorname{Tr}\{\mathcal{W}^\alpha\mathcal{W}_\alpha\} \tag{3.47}$$

analogously to Eq. (2.21).

3.3.2 Field Content

Here I give a very brief overview of the field content and naming scheme of the MSSM – for a more comprehensive introduction I will refer to ⋮©⋮ . The basic idea is to embed every SM fermion into a chiral superfield, and the vector bosons into the vector superfields arising from local gauge invariance. Since the SM fermions are Dirac fermions, they require two different Weyl spinors, which means that two superfields are required to provide each fermion.

Consider an SM Dirac fermion

$$f_D = \begin{pmatrix} f \\ \bar f^\dagger \end{pmatrix}, \tag{3.48}$$

where f and $\bar f^\dagger$ are two *different* left-handed and right-handed Weyl spinors respectively. The left-handed Weyl spinor part f is embedded into a superfield f wherein it receives

[10]Using the Baker-Campell-Hausdorff formula (BCH) to combine the exponentials.

[11]The factor of 2 in the exponential here seems arbitrary at first, and is just a matter of choice. It is chosen to be 2 here such that the transformation of law for $\mathcal{V}$ is proportional to Λ without any numerical prefactors.

a scalar *superpartner* $\tilde{f}_L$.[12] The superfield and Weyl spinor have the exact same name, which might seem needlessly confusing. However, it does lead to less cluttered notation, and context should clarify which is meant. The right-handed Weyl spinor part $\bar{f}^\dagger$ is likewise embedded into a right-handed scalar superfield $\bar{F}^\dagger$, with a scalar superfield partner $\tilde{f}_R$. Furthermore, the left-handed scalar superfield f is part of an $SU(2)_L$ doublet of superfields F, matching the uppercase naming of the right-handed superfield $\bar{F}^\dagger$. The bar on superfields and right-handed Weyl spinors signify that they are $SU(2)_L$ singlets, i.e. they do not transform under such symmetry transformations, and makes it clear that the two Weyl spinors f and $\bar{f}^\dagger$ are separate variables belonging to the same SM fermion field. Collectively, the scalar superpartners to the SM fermions are referred to as *sfermions*.

The gauge groups of the MSSM are the same as in the SM, but the gauge fields are replaced by vector superfield gauge fields as detailed in Section 3.3.1. This way, an SM vector boson V^μ is embedded in a vector superfield V where it receives a left-handed Weyl spinor superpartner $\tilde{V}$ with its right-handed compliment $\tilde{V}^\dagger$.

Lastly, and perhaps the most intricate, is the extension of the Higgs sector in the MSSM. As it turns out, the MSSM requires two Higgs doublets for anomaly cancellation within the $U(1)_Y$ gauge group sector, and to construct the Yukawa terms giving mass to particles with both positive and negative weak isospin.[13] This means that there are two scalar Higgs doublets H_u, H_d before electroweak symmetry breaking (EWSB), giving mass to fermions in the upper/lower part of $SU(2)_L$ fermion doublets respectively. For the anomaly cancellation to work out, we must require hypercharge $+1/2$ for H_u and $-1/2$ for H_d. These scalar Higgs field doublets are embedded in left-handed chiral superfields $H_{u/d}$ together with fermion superpartners. The superfield doublet components are named according to $H_u = (H_u^+, H_u^0)^T$ and $H_d = (H_d^0, H_d^-)^T$, where the superscript indicates the post EWSB electric charge of the superfields. The fermion partners to both the vector bosons and the Higgs bosons are called *bosinos* collectively. For reference all the superfields in the MSSM, their symbols and their component fields are tabulated in Table 3.1.

3.3.3 Superlagrangian and Supersymmetry Breaking

Now that we have defined the field content of the MSSM, we need to define the interaction between them through the superlagrangian. As has already been noted, the gauge groups of the MSSM are the same as for the SM, and all kinetic terms are defined according to the super Yang-Mills theory of Section 3.3.1. A summary of the gauge numbers of the scalar superfields is given in Table 3.2. This results in the kinetic part of the MSSM superlagrangian being

$$
\begin{aligned}
\mathcal{L}_{\text{kin}}^{\text{MSSM}} =& H_u^\dagger e^{g'B+2g\mathcal{W}} H_u + H_d^\dagger e^{-g'B+2g\mathcal{W}} H_d + L_i^\dagger e^{-g'B+2g\mathcal{W}} L_i + \bar{E}_i^\dagger e^{2g'B} \bar{E}_i \\
&+ Q_i^\dagger e^{\frac{1}{3}g'+2g\mathcal{W}+2g_s\mathcal{C}} Q_i + \bar{U}_i^\dagger e^{-\frac{4}{3}g'+2g_s\mathcal{C}} \bar{D}_i + e^{\frac{2}{3}g'+2g_s\mathcal{C}} \bar{D}_i \\
& \frac{1}{4} B^\alpha B_\alpha + \frac{1}{2}\operatorname{Tr}\{\mathcal{W}^\alpha \mathcal{W}_\alpha\} + \frac{1}{2}\operatorname{Tr}\{\mathcal{C}^\alpha \mathcal{C}_\alpha\},
\end{aligned}
\tag{3.49}
$$

where $B^\alpha, \mathcal{W}^\alpha, \mathcal{C}^\alpha$ are the supersymmetric field strengths of the gauge superfields B, $\mathcal{W} = W^k \frac{1}{2}\sigma^k$ and $\mathcal{C} = C^a \frac{1}{2}\lambda^a$ respectively. The matrices λ^a are the Gell-Mann matrices — the generators of $SU(3)$.

[12] The subscript L on the scalar fields carries no indication of any chirality, but rather alludes to the origin of the field as a superpartner to the left-handed chiral part of the fermion field f_D.

[13] For a more detailed explanation, I will refer the reader to [2].

	Superfield		Boson field	Fermion field	Auxiliary field
Higgs	H_u	H_u^+	H_u^+	$\tilde{H}_u^+$	$F_{H_u^+}$
		H_u^0	H_u^0	$\tilde{H}_u^0$	$F_{H_u^0}$
	H_d	H_d^0	H_d^0	$\tilde{H}_d^0$	$F_{H_d^0}$
		H_d^-	H_d^-	$\tilde{H}_d^-$	$F_{H_d^-}$
Leptons	L_i	ν_i	$\tilde{\nu}_{iL}$	ν_i	F_{ν_i}
		l_i	$\tilde{l}_{iL}$	l_i	F_{l_i}
	-	$\bar{E}_i$	$\tilde{l}_{iR}^*$	$\bar{e}_i$	$F_{\bar{E}_i}^*$
Quarks	Q_i	u_i	$\tilde{u}_{iL}$	u_i	F_{u_i}
		d_i	$\tilde{d}_{iL}$	d_i	F_{d_i}
	-	$\bar{U}_i$	$\tilde{u}_{iR}^*$	$\bar{u}_i$	$F_{\bar{U}_i}^*$
	-	$\bar{D}_i$	$\tilde{d}_{iR}^*$	$\bar{d}_i$	$F_{\bar{D}_i}^*$
Bosons	-	B^0	B_μ^0	$\tilde{B}^0$	D_{B^0}
	W^k	W^0	W_μ^0	$\tilde{W}^0$	D_{W^0}
		$W^\pm$	$W_\mu^\pm$	$\tilde{W}^\pm$	$D_{W^\pm}$
	-	C^a	C_μ^a	$\tilde{g}$	D_C

Table 3.1: Table of superfields of the MSSM, and their component field names. Note that the fermion fields are left-handed Weyl spinors, in spite of any L or R in the boson field subscript. The conjugate superfields changes these to right-handed Weyl spinors. The indices i enumerate the three generations of leptons/quarks, k the three $SU(2)_L$ gauge fields and a the eight $SU(3)_C$ gauge fields.

The superpotential up to gauge invariant and R-parity conserving terms is given by

$$W_{\text{MSSM}} = \mu H_u^T i\sigma_2 H_d + y_{ij}^e (L_i^T i\sigma_2 H_d)\bar{E}_j + y_{ij}^u (Q_i^T i\sigma_2 H_u)\bar{U}_j + y_{ij}^d (Q_i^T i\sigma_2 H_d)\bar{D}_j, \quad (3.50)$$

where μ is some complex, massive parameters and $y_{ij}^{e/u/d}$ are the ordinary SM Yukawa couplings. This leaves two new degrees of freedom in the MSSM superpotential beyond what is in the SM – the phase and magnitude of μ.

Seeing as we have not discovered any particles with the same mass but opposite spin-statistics to the SM particles we know, we must conclude that supersymmetry is broken at low energy. A mechanism for spontaneous symmetry breaking of supersymmetry would therefore be necessary. Constructing such a mechanism in a way as to not reintroduce the hierarchy problem leads to what we call *soft* breaking of supersymmetry [2]. Disregarding the high-energy completion of the theory, we can parametrise the terms that can arise in the low-energy MSSM superlagrangian up to gauge invariant and R-parity conserving terms, as

$$
\begin{aligned}
\mathcal{L}_{\text{soft}}^{\text{MSSM}} =&(\theta\theta)(\theta\theta)^\dagger\Big\{-\frac{1}{4}M_1 B^\alpha B_\alpha - \frac{1}{2}M_2\,\mathrm{Tr}\{\mathcal{W}^\alpha\mathcal{W}_\alpha\} - \frac{1}{2}M_3\,\mathrm{Tr}\{\mathcal{C}^\alpha\mathcal{C}_\alpha\} + \text{c. c.}\\
&- \frac{1}{6}a_{ij}^e L_i^T i\sigma_2 H_d \bar{E}_j - \frac{1}{6}a_{ij}^u Q_i^T i\sigma_2 H_u \bar{U}_j - \frac{1}{6}a_{ij}^d Q_i^T i\sigma_2 H_d \bar{D}_j + \text{c. c.}\\
&- \frac{1}{2}b H_u^T i\sigma_2 H_d + \text{c. c.}\\
&- (m_{ij}^L)^2 L_i^\dagger L_j - (m_{ij}^e)^2 \bar{E}_i^\dagger \bar{E}_j - (m_{ij}^Q)^2 Q_i^\dagger Q_j - (m_{ij}^u)^2 \bar{U}_i^\dagger \bar{U}_j - (m_{ij}^d)^2 \bar{D}_i^\dagger \bar{D}_j\\
&- m_{H_u}^2 H_u^\dagger H_u - m_{H_d}^2 H_d^\dagger H_d\Big\}.
\end{aligned}
\tag{3.51}
$$

All the parameters are potentially complex numbers, although all the mass terms m_{ij}^2 must be hermitian in the that $m_{ij}^2 = (m_{ji}^2)^*$, which leads to m_{ii}^2 having to be real. This is the source of the great many parameters of the MSSM, as these soft-breaking parameters alone amount to 109 degrees of freedom![14] For this reason, most searches of the MSSM focus on various simplified models ⋮ © ⋮ . These can be based on simplifications like assuming all parameters to be real or assuming no flavour-violation, or by making theoretical assumptions on the specific mechanism for symmetry breaking, as in Constrained MSSM© or mSUGRA© , and combinations of these. In this thesis, I will not make any such assumptions and work with the general form of the MSSM, unless otherwise stated. The full MSSM superlagrangian is then

$$
\mathcal{L}_{\text{MSSM}} = \mathcal{L}_{\text{kin}}^{\text{MSSM}} + (\theta\theta)^\dagger W_{\text{MSSM}} + (\theta\theta) W_{\text{MSSM}}^\dagger + \mathcal{L}_{\text{soft}}^{\text{MSSM}}. \tag{3.52}
$$

3.4 Electroweakinos

The focus in this thesis we be on a particular set of superpartners, namely the *electroweakinos*. These are fermion superpartners to the electroweak bosons, i.e. the photon, Z and W bosons and the Higgs bosons. These are subdivided into the vector boson superpartners, the *gauginos*, and the Higgs boson partners, the *higgsinos*. Before EWSB, the gauge fields naturally occurring in the Lagrangian are the B- and W^k-fields, and it is customary to work with the fermion superpartners of these fields. These are naturally called the *binos* and *winos* respectively.

3.4.1 Mass mixing

After EWSB, we get two oppositely charged winos, and two mixed bino/wino states, mirroring the electroweak gauge bosons. However, the higgsinos come in an oppositely charged pair and two neutral ones, so the gauginos and higgsinos can further mix. So the general electroweak fermionic sector in the MSSM includes two particle-antiparticle pairs of charged Dirac fermions, and four neutral Majorana fermions, respectively referred to as *charginos* and *neutralinos*. The two chargino fields are denoted with the Weyl spinors $\tilde{\chi}_{i=1,2}^{\pm}$, and the four neutralinos are denoted with the Weyl spinors $\tilde{\chi}_{i=1,2,3,4}^{0}$. The indices i are numbered according to the mass hierarchy, with 1 being the lightest chargino/neutralino and 2/4 being the heaviest.

[14] A few of these can be eliminated through field redefinitions, however.

	Superfield		Hypercharge Y	Isospin I^3	Electric Charge Q_e	Colour
Higgs	H_u	H_u^+	$+1/2$	$+1/2$	$+1$	-
		H_u^0	$+1/2$	$-1/2$	0	-
	H_d	H_d^0	$-1/2$	$+1/2$	0	-
		H_d^-	$-1/2$	$-1/2$	-1	-
Leptons	L_i	ν_i	$-1/2$	$+1/2$	0	-
		l_i	$-1/2$	$-1/2$	-1	-
	-	$\bar{E}_i$	$+1$	-	$+1$	-
Quarks	Q_i	u_i	$+1/6$	$+1/2$	$+2/3$	yes
		d_i	$+1/6$	$-1/2$	$-1/3$	yes
	-	$\bar{U}_i$	$-2/3$	-	$-2/3$	yes
	-	$\bar{D}_i$	$+1/3$	-	$+1/3$	yes

Table 3.2: Summary of quantum numbers for the MSSM scalar superfields. The charges of barred fields $\bar{F}$ supplying the right-handed part of SM fermions are defined such that the charge of $\bar{F}^\dagger$ matches that of its left-handed compliment. I note that the convention for the hypercharge differs from some sources, seeing as I use 1 as the generator of $U(1)_Y$ instead of $\frac{1}{2}$ used elsewhere. This amounts to shuffling some factors of $\frac{1}{2}$ around. The indices i enumerate the three generations of leptons/quarks.

Ignoring higher order corrections, the mass terms for the gauginos and higgsinos in the MSSM Lagrangian can be structured as

$$\mathcal{L}_{\tilde{\chi}\text{-mass}} = -\frac{1}{2}(\psi^0)^T M_{\tilde{\chi}^0}\psi^0 - \frac{1}{2}\psi^{\pm T} M_{\tilde{\chi}^\pm}\psi^\pm + \text{c. c.}, \tag{3.53}$$

where $\psi^0 = \left(\tilde{B}^0, \tilde{W}^0, \tilde{H}_d^0, \tilde{H}_u^0\right)^T$, $\psi^\pm = (\psi^+, \psi^-)^T = \left(\tilde{W}^+, \tilde{H}_u^+, \tilde{W}^-, \tilde{H}_d^-\right)^T$ and $M_{\tilde{\chi}^0}$, $M_{\tilde{\chi}^\pm}$ are the neutralino and chargino mass matrices respectively. They are given by

$$M_{\tilde{\chi}^0} = \begin{bmatrix} M_1 & 0 & -m_Z c_\beta s_W & m_Z s_\beta s_W \\ 0 & M_2 & m_Z c_\beta c_W & -m_Z s_\beta c_W \\ -m_Z c_\beta s_W & m_Z c_\beta c_W & 0 & -\mu \\ m_Z s_\beta s_W & -m_Z s_\beta c_W & -\mu & 0 \end{bmatrix}, \tag{3.54}$$

$$M_{\tilde{\chi}^\pm} = \begin{bmatrix} 0 & 0 & M_2 & \sqrt{2}c_\beta m_W \\ 0 & 0 & \sqrt{2}s_\beta m_W & \mu \\ M_2 & \sqrt{2}s_\beta m_W & 0 & 0 \\ \sqrt{2}c_\beta m_W & \mu & 0 & 0 \end{bmatrix}. \tag{3.55}$$

These mass matrices can be diagonalised to get the mass eigenstate *neutralinos* $\tilde{\chi}_i^0$ and *charginos* $\tilde{\chi}_i^\pm$, respectively. Both the matrices are symmetric, but we will diagonalise them slightly differently. The neutralino mass matrix can be diagonalised by a unitary matrix N such that

$$\begin{aligned} \mathcal{L}_{\tilde{\chi}^0\text{-mass}} &= -\frac{1}{2}(\psi^0)^T M_{\tilde{\chi}^0}\psi^0 + \text{c. c.} = -\frac{1}{2}\underbrace{(\psi^0)^T N^T}_{\equiv(\tilde{\chi}^0)^T}\ \underbrace{N^* M_{\tilde{\chi}^0} N^\dagger}_{=\text{diag}(m_{\tilde{\chi}_1^0},\ldots,m_{\tilde{\chi}_4^0})}\ \underbrace{N\psi^0}_{\equiv\tilde{\chi}^0} + \text{c. c.} \\ &= -\frac{1}{2}(\tilde{\chi}^0)^T \,\text{diag}(m_{\tilde{\chi}_1^0}, \ldots, m_{\tilde{\chi}_4^0})\tilde{\chi}^0 + \text{c. c.} \end{aligned} \tag{3.56}$$

The chargino mass matrix is handled slightly differently, seeing as it has the structure $M_{\tilde{\chi}^\pm} = \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix}$. Using singular value decomposition, we can write $X = U^T D V$ for two unitary matrices U, V and a diagonal matrix of positive singular values $D = \mathrm{diag}(m_{\tilde{\chi}_1^\pm}, m_{\tilde{\chi}_2^\pm})$. This results in

$$
\begin{aligned}
\mathcal{L}_{\tilde{\chi}^\pm\text{-mass}} &= -\frac{1}{2}\psi^{\pm T} M_{\tilde{\chi}^\pm}\psi^\pm + \text{c. c.} = -\frac{1}{2}\begin{pmatrix}\psi^+ \\ \psi^-\end{pmatrix}^T \begin{bmatrix} 0 & U^T D V \\ V^T D U & 0 \end{bmatrix}\begin{pmatrix}\psi^+ \\ \psi^-\end{pmatrix} + \text{c. c.} \\
&= -\frac{1}{2}\underbrace{(\psi^+)^T U^T}_{\equiv(\tilde{\chi}^+)^T} D \underbrace{V\psi^-}_{\equiv\tilde{\chi}^-} - \frac{1}{2}\underbrace{(\psi^-)^T V^T}_{\equiv(\tilde{\chi}^-)^T} D \underbrace{U\psi^+}_{\equiv\tilde{\chi}^+} + \text{c. c.} \\
&= -\left(\tilde{\chi}^+\right)^T \mathrm{diag}(m_{\tilde{\chi}_1^\pm}, m_{\tilde{\chi}_2^\pm})\tilde{\chi}^- + \text{c. c.}
\end{aligned}
\tag{3.57}
$$

This tells us that there are two doubly degenerate mass eigenvalues of the chargino mass matrix, constituting two massive Dirac fermion particle-antiparticle pairs.

Proof of Takagi factorisation and an algorithm for realising it are done in Appendix B.

3.5 Feynman Rules for Neutralinos

To calculate the cross-section for electroweakino production later on, we will need the Feynman rules of the relevant particle interactions. I will not explicitly derive the Feynman rules for all the electroweakinos, but rather exemplify how they can be derived by deriving all the relevant neutralino interactions from the MSSM superlagrangian. The relevant Feynman rules for the remaining electroweakino processes follow in much the same manner, and are listed in the end.

3.5.1 Fermion Interactions in Super Yang-Mills and Yukawa Theory

I will start by deriving the interactions of fermions in the chiral superfields and vector superfields of a supersymmetric Yang-Mills superlagrangian. As a reminder, the super Yang-Mills superlagrangian kinetic term is $\Phi_i^\dagger \left(e^{2q\mathcal{V}}\right)_{ij}\Phi_j$. Extracting the interaction terms containing either the fermion field multiplets ψ ($\psi^\dagger$) from the left-handed (right-handed) scalar superfield multiplets Φ ($\Phi^\dagger$), and the fermion fields $\lambda \equiv \lambda^a T^a$ from vector superfields $\mathcal{V} \equiv V^a T^a$ from terms with the appropriate amount of θ's to survive the projection of Eq. (3.30), we have

$$
\begin{aligned}
\mathcal{L} \overset{\psi,\psi^\dagger,\lambda}{\supset} 2q\sum_{ij}\Big\{ & A_i^*(\theta\theta)^\dagger(\theta\lambda_{ij})\sqrt{2}(\theta\psi_j) + \sqrt{2}(\theta\psi_i)^\dagger(\theta\sigma^\mu\theta^\dagger)\left(\mathcal{V}_\mu\right)_{ij}\sqrt{2}(\theta\psi_j) \\
& + \sqrt{2}(\theta\psi_i)^\dagger(\theta\theta)(\theta\lambda_{ij})^\dagger A_j\Big\} \\
= q(\theta\theta)(\theta\theta)^\dagger & \sum_{ij}\left\{-\sqrt{2}A_i^*(\lambda_{ij}\psi_j) + \left(\psi_i\sigma^\mu\left(\mathcal{V}_\mu\right)_{ij}\psi_j^\dagger\right) - \sqrt{2}(\psi_i\lambda_{ij})^\dagger A_j\right\},
\end{aligned}
\tag{3.58}
$$

where I have used Weyl spinor relations© .

There are also Yukawa terms coming from the superpotential terms of the form $y_{ij}(\theta\theta)^\dagger\Phi_i\Phi\Phi_j + \text{c. c.}$ Here Φ will later represent one of the Higgs superfields. Extracting the interaction terms of fermion field ψ from Φ, we find

$$\begin{aligned}\mathcal{L} &\overset{\psi,\psi^\dagger}{\supset} y_{ij}(\theta\theta)^\dagger\sqrt{2}(\theta\psi)\left\{A_i\sqrt{2}(\theta\psi_i)+\sqrt{2}(\theta\psi_j)A_j\right\}+\text{c. c.}\\ &= -y_{ij}(\theta\theta)(\theta\theta)^\dagger\left\{A_i(\psi\psi_j)+(\psi_i\psi)A_j+\text{c. c.}\right\}\end{aligned} \tag{3.59}$$

3.5.2 Wino and Bino Interactions

First, I will look at the bino and wino interactions coming from the kinetic terms. Writing out the W^a vector superfields in the basis $W^\pm, W^0$, we are now only interested in the electrically neutral W^0 bit. The interactions will come from kinetic terms of scalar superfields Φ, whose relevant part can be written as

$$\mathcal{L} = \Phi^\dagger e^{2g\left\{Y t_W B^0\left(+\frac{1}{2}\sigma_3 W^0\right)\right\}}\Phi, \tag{3.60}$$

where $t_W \equiv \tan\theta_W$ is the tangent of the Weinberg angle, Y is the hypercharge of Φ and the term in parentheses only appears for fields in $SU(2)_L$ superfield doublets. To generalise this, I will use the isospin I^3, which is $+\frac{1}{2}$ for fields in the upper part of an SU(2) doublet, $-\frac{1}{2}$ for fields in the lower part and 0 for SU(2) singlet fields. Then the kinetic term can be written compactly as

$$\mathcal{L} = \Phi^\dagger e^{2g\left\{(Q_e-I^3)t_W B^0+I^3W^0\right\}}\Phi, \tag{3.61}$$

where Q_e is the electric charge of Φ.

Extracting the interactions of the fermion fields $\tilde{B}^0, \tilde{W}^0$ in B^0, W^0 using Eq. (3.58), we are left with (up to appropriate θ's)

$$\mathcal{L} \overset{\tilde{B}^0,\tilde{W}^0}{\supset} -\sqrt{2}g(\theta\theta)(\theta\theta)^\dagger\Big\{(Q_e-I^3)t_W(\tilde{B}^0\psi)A^*+I^3(\tilde{W}^0\psi)A^*+\text{c. c.}\Big\}. \tag{3.62}$$

Consider an SM quark q_g of generation g, from the scalar superfield components A and ψ contained in the superfields Q_g and $\bar{Q}_g$, with electric charge Q_e and weak isospins I^3 and 0 respectively, we can write out the interaction as

$$\mathcal{L} = -\sqrt{2}g\Big\{(Q_e-I^3)t_W(\tilde{B}^0q_g)\tilde{q}^*_{gL}+I^3(\tilde{W}^0q_g)\tilde{q}^*_{gL}+Q_et_W(\tilde{B}^0\bar{q}_g)\tilde{q}^*_{gR}+\text{c. c.}\Big\}. \tag{3.63}$$

Changing to the $\tilde{\chi}^0$-basis, we have that $\tilde{B}^0=\sum_i N^*_{i1}\tilde{\chi}^0_i$, $\tilde{W}^0=\sum_i N^*_{i2}\tilde{\chi}^0_i$, which together with writing out the Weyl products on Dirac spinor form yields

$$\mathcal{L}_{\tilde{q}q\tilde{\chi}^0} = -\sqrt{2}g\sum_i\bar{\tilde{\chi}}^0_i\Big\{\big[\underbrace{\left(Q_e-I^3\right)t_WN^*_{i1}+I^3N^*_{i2}}_{\equiv\left(C^L_{\tilde{q}q_g\tilde{\chi}^0_i}\right)^*}\big]\tilde{q}^*_{gL}P_L\underbrace{-Q_ft_WN_{i1}}_{\equiv\left(C^R_{\tilde{q}q_g\tilde{\chi}^0_i}\right)^*}\tilde{q}^*_{gR}P_R\Big\}q_g+\text{c. c.}, \tag{3.64}$$

where we understand $\tilde{\chi}^0_i$ and q_g as Dirac spinors.

Generalising this further to include squark mixing between the left- and right-handed squarks of the same flavour in a generation g, we have

Mention that this is typical in the third generation and perhaps because of the larger Yukawa couplings.

$$\tilde{q}_{gA} = R^{\tilde{q}_g}_{A1}\tilde{q}_{gL}+R^{\tilde{q}_g}_{A2}\tilde{q}_{gR}, \tag{3.65}$$

where $R^{\tilde{q}_g}$ is a 2×2 unitary matrix transforming the squarks of the same flavour and generation g to their mass eigenstates. As such, we can write $\tilde{q}_{gL} = \sum_A \left(R^{\tilde{q}_g}_{A1}\right)^* \tilde{q}_{gA}$, $\tilde{q}_{gR} = \sum_A \left(R^{\tilde{q}_g}_{A2}\right)^* \tilde{q}_{gA}$ to get

$$\mathcal{L}_{\tilde{q}q_g\tilde{\chi}^0} = -\sqrt{2}g \sum_i \sum_A \bar{\tilde{\chi}}^0_i \Big\{ \underbrace{R^{\tilde{q}_g}_{A1} \left(C^L_{\tilde{q}q_g\tilde{\chi}^0_i}\right)^*}_{\equiv\left(C^L_{\tilde{q}_A q_g \tilde{\chi}^0_i}\right)^*} P_L + \underbrace{R^{\tilde{q}_g}_{A2} \left(C^R_{\tilde{q}q_g\tilde{\chi}^0_i}\right)^*}_{\equiv\left(C^R_{\tilde{q}_A q_g \tilde{\chi}^0_i}\right)^*} P_R \Big\} \tilde{q}^*_{gA} q_{gD} + \text{c. c.} \tag{3.66}$$

Flavour-Violating Squark Sector

The previous derivation was done under the assumption that squarks do not mix between fermion generations. However, this can happen if there are non-zero supersymmetry-breaking parameters coupling squarks between generations or if loop corrections are added to the squark sector. The generalisation is fairly straight forward: Instead of one unitary, 2×2 mixing matrix $R^{\tilde{q}_g}$ for each of the six quark flavours, there is one 6×6 mixing matrix $R^{\tilde{q}}$ for each of the two quark types (either up or down). These mixing matrices can be defined using different conventions, but in this thesis I will follow the SLHA2 standard [3]

$$\begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \\ \tilde{q}_4 \\ \tilde{q}_5 \\ \tilde{q}_6 \end{pmatrix} = R^{\tilde{q}} \begin{pmatrix} \tilde{q}_{1L} \\ \tilde{q}_{2L} \\ \tilde{q}_{3L} \\ \tilde{q}_{1R} \\ \tilde{q}_{2R} \\ \tilde{q}_{3R}. \end{pmatrix} \tag{3.67}$$

This means that the chiral squarks in generation $g = 1, 2, 3$ will rather be given by

$$\tilde{q}_{gL} = \sum_A (R^{\tilde{q}}_{A,g})^* \tilde{q}_A, \tag{3.68a}$$

$$\tilde{q}_{gR} = \sum_A (R^{\tilde{q}}_{A,g+3})^* \tilde{q}_A. \tag{3.68b}$$

What this means for the interaction Lagrangian in **??** is that the sum over A changes to go from 1 to 6 and the definition of the coupling parameter changes slightly to

$$C^L_{\tilde{q}_A q_g \tilde{\chi}^0_i} = \left(R^{\tilde{q}}_{A,g}\right)^* \left[\left(Q_e - I^3\right) t_W N_{i1} + I^3 N_{i2}\right], \tag{3.69a}$$

$$C^R_{\tilde{q}_A q_g \tilde{\chi}^0_i} = -\left(R^{\tilde{q}}_{A,g+3}\right)^* Q_e t_W N^*_{i1}. \tag{3.69b}$$

3.5.3 Higgsino Interactions

The higgsino interaction with the (s)quarks comes from the Yukawa terms of the superpotential, but seeing that this interaction is proportional to the quark mass, it is safe to ignore them at the centre-of-mass energies we are interested in. ⌜Talk about sbottom and stop squarks with large Yukawas.⌟

The relevant interaction that remains is that with the Z-boson. This interaction again comes from the kinetic term, but this time for the neutral Higgs superfields in the superfield multiplets $H_u = (H^+_u, H^0_u)^T$, $H_d = \left(H^0_d, H^-_d\right)^T$. The Lagrangian is of the form

$$\mathcal{L} = \left(H^0_{u/d}\right)^\dagger e^{\mp g\left(W^0 - t_W B^0\right)} H^0_{u/d}. \tag{3.70}$$

Integrating over the Grassmann variables and using equation Eq. (3.58) we get

$$\int \mathrm{d}^4\theta\, \mathcal{L} \overset{\tilde{H}^0_{u/d}, W^0_\mu, B^0_\mu}{=} \mp\frac{g}{2}(\tilde{H}^0_{u/d}\sigma^\mu(\tilde{H}^0_{u/d})^\dagger)(W^0_\mu - t_W B^0_\mu). \tag{3.71}$$

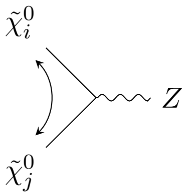
Mention Weyl/Dirac identities necessary for this.

Switching to Dirac spinors, the mass eigenbasis for the neutralinos and the Z boson $Z_\mu = c_W W^0_\mu - s_W B^0_\mu$, we end up with

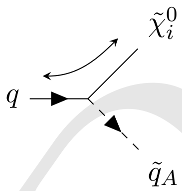
$$\mathcal{L}_{Z\tilde{\chi}^0_i\tilde{\chi}^0_j} = \frac{g}{2c_W} Z_\mu \sum_{ij}\left(-N_{i4}N^*_{j4} + N_{i3}N^*_{j3}\right)\bar{\tilde{\chi}}^0_i\gamma^\mu P_L \tilde{\chi}^0_j$$
$$= -\frac{g}{2} Z_\mu \sum_{ij} \bar{\tilde{\chi}}^0_i \gamma^\mu \Bigg[\underbrace{\frac{1}{2c_W}\left(N_{i4}N^*_{j4} - N_{i3}N^*_{j3}\right)}_{\equiv O''^L_{ij}} P_L - \underbrace{\frac{1}{2c_W}\left(N^*_{i4}N_{j4} - N^*_{i3}N_{j3}\right)}_{\equiv O''^R_{ij}} P_R\Bigg]\tilde{\chi}^0_j \tag{3.72}$$

3.5.4 Summary of Coupling Definitions

In summary, the Feynman rules for the interactions of neutralinos with the electroweak bosons and (s)quarks are given by the interaction Lagrangians in Eqs. (3.66) and (3.72) as



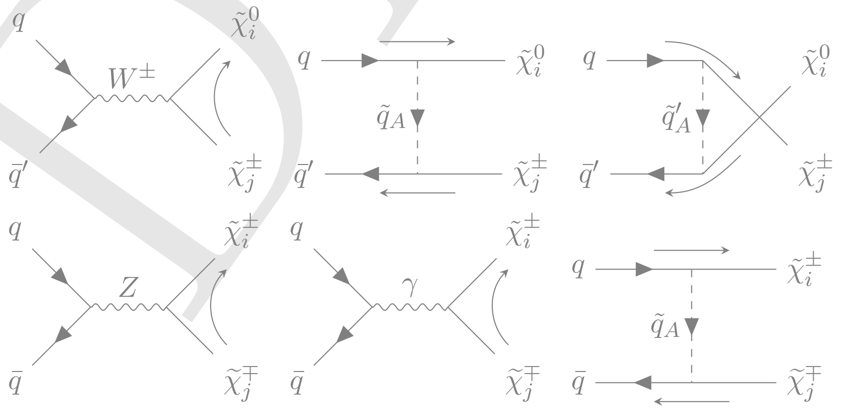
$$= -ig\gamma^\mu\left[O''^L_{ij}P_L + O''^R_{ij}P_R\right], \tag{3.73a}$$



$$= -i\sqrt{2}g\left[\left(C^L_{\tilde{q}_A q_g \tilde{\chi}^0_i}\right)^* P_L + \left(C^R_{\tilde{q}_A q_g \tilde{\chi}^0_i}\right)^* P_R\right]. \tag{3.73b}$$

In fact, the interactions of all electroweakinos with W/Z-bosons and (s)quarks take the same form, and we can generalise by replacing O''^X_{ij} or $C^X_{\tilde{q}_A q_g \tilde{\chi}^0_i}$ with the appropriate definitions in Table 3.3. ⌜Add citation for this.⌟

Temporary



Interaction	Coupling	Definition
$\tilde{q}q\tilde{\chi}^0$	$C^L_{\tilde{q}_A q_g \tilde{\chi}^0_i}$	$\left(R^{\tilde{q}}_{A,g}\right)^* \left[\left(Q_e - I^3_q\right) t_W N_{i1} + I^3_q N_{i2}\right]$
	$C^R_{\tilde{q}_A q_g \tilde{\chi}^0_i}$	$-\left(R^{\tilde{q}}_{A,g+3}\right)^* Q_e t_W N^*_{i1}$
$W\tilde{\chi}^0\tilde{\chi}^\pm$	O^L_{ij}	$\frac{1}{\sqrt{2}} N_{i4} V^*_{j2} - N_{i2} V^*_{j1}$
	O^R_{ij}	$-\frac{1}{\sqrt{2}} N^*_{i3} U_{j2} - N^*_{i2} U_{j1}$
$Z\tilde{\chi}^\pm\tilde{\chi}^\mp$	O'^L_{ij}	$\frac{1}{c_W}\left(V_{i1} V^*_{j1} + \frac{1}{2} V_{i2} V^*_{j2} - \delta_{ij} s^2_W\right)$
	O'^R_{ij}	$\frac{1}{c_W}\left(U_{i1} U^*_{j1} + \frac{1}{2} U_{i2} U^*_{j2} - \delta_{ij} s^2_W\right)$
$Z\tilde{\chi}^0\tilde{\chi}^0$	O''^L_{ij}	$\frac{1}{2c_W}\left(N_{i4} N^*_{j4} - N_{i3} N^*_{j3}\right)$
	O''^R_{ij}	$-\frac{1}{2c_W}\left(N^*_{i4} N_{j4} - N^*_{i3} N_{j3}\right)$
$\tilde{q}q'\tilde{\chi}^\pm$	$C^L_{\tilde{d}_A u_g \tilde{\chi}^\pm_i}$	$\frac{1}{\sqrt{2}} U_{i1} \left(R^{\tilde{d}}_{A,g}\right)^* V^{\mathrm{CKM}}_{u_g d_g}$
	$C^L_{\tilde{u}_A d_g \tilde{\chi}^\pm_i}$	$\frac{1}{\sqrt{2}} V_{i1} \left(R^{\tilde{u}}_{A,g}\right)^* \left(V^{\mathrm{CKM}}_{u_g d_g}\right)^*$
	$C^R_{\tilde{q}_A q'_g \tilde{\chi}^\pm_i}$	0
qqZ	C^L_{qqZ}	$-\frac{I^3_q - Q_e s^2_W}{c_W}$
	C^R_{qqZ}	$\frac{Q_e s^2_W}{c_W}$
$qq'W$	$C^L_{qq'W}$	$-\frac{V^{\mathrm{CKM}}_{qq'}}{c_W}$
	$C^R_{qq'W}$	0
$\tilde{q}\tilde{q}Z$	$C^L_{\tilde{q}_A \tilde{q}_B Z}$	$-\frac{I^3_q - Q_e s^2_W}{c_W} R^{\tilde{q}}_{A,g} \left(R^{\tilde{q}}_{B,g}\right)^*$
	$C^R_{\tilde{q}_A \tilde{q}_B Z}$	$\frac{Q_e s^2_W}{c_W} R^{\tilde{q}}_{A,g+3} \left(R^{\tilde{q}}_{B,g+3}\right)^*$
$\tilde{q}\tilde{q}W$	$C^L_{\tilde{q}_A \tilde{q}'_B W}$	$-\frac{V^{\mathrm{CKM}}_{qq'}}{c_W} R^{\tilde{q}}_{A,g} \left(R^{\tilde{q}'}_{B,g}\right)^*$
	$C^R_{\tilde{q}_A \tilde{q}'_B W}$	0

Table 3.3: A summary of the variables used in the derived Feynman rules and their definitions.

Chapter 4

Neutralino Pair Production at Parton Level

TODO:

- ☐ Formulate a section on the dipole formalism used in Debove et al. and make a comparison.

4.1 Phase Space and Kinematics in Scattering Processes

To start off, it will be useful to introduce some procedure for going forward in the phase space of an inclusive $2 \to 2(+1)$ scattering process. The phase space of 2-body and 3-body final states are quite different as there are more degrees of freedom in the 3-body final state. In the end, these extra degrees of freedom will be need to be integrated over to make an additive comparison between the 2-body and 3-body processes, however, exactly how we choose to parametrise and subsequently integrate over the extra degrees of freedom can matter quite a bit.

To start out, let us count the degrees of freedom of a scattering problem involving N four-momenta $p_{i=1,\ldots,N}$. Assuming our end result to be Lorentz invariant, there are $N(N+1)/2$ different scalar products that can be produced using N different four-momenta. Momentum conservation allows us to eliminate one momentum, such that we have $N(N-1)/2$ possible scalar products. Denoting the scalar products by $m_{ij}^2 \equiv (p_i + p_j)^2$ for $j \neq i$, and $m_i^2 \equiv p_i^2$, we can find a relation between scalar products by using momentum conservation.

$$
\begin{aligned}
m_{ij}^2 &= \left(p_i - \sum_{k\neq j} p_k \right)^2 = \left(\sum_{k\neq i,j} p_k \right)^2 = \sum_{k\neq i,j} \sum_{l\neq i,j} p_k \cdot p_l \\
&= \sum_{k\neq i,j} \sum_{l\neq i,j,k} \frac{m_{kl}^2 - m_k^2 - m_l^2}{2} + \sum_{k\neq i,j} m_k^2 \\
&= \sum_{k\neq i,j} \sum_{\substack{l\neq i,j \\ l>k}} m_{kl}^2 - \frac{1}{2} \sum_{k\neq i,j} (N-3) m_k^2 - \frac{1}{2} \sum_{l\neq i,j} (N-3) m_l^2 + \sum_{k\neq i,j} m_k^2 \\
&= \sum_{k\neq i,j} \sum_{\substack{l\neq i,j \\ l>k}} m_{kl}^2 - (N-4) \sum_{k\neq i,j} m_k^2.
\end{aligned} \tag{4.1}
$$

⌜This little generalised relation might not be immediately necessary. . . ⌟

To count the degrees of freedom in an N-body final state, we need to classify how many scalar products need to be specified for every scalar product to be defined. We assign the N scalar products m_i^2 to the invariant masses of the incoming and outgoing particles, thereby not counting them as kinematic degrees of freedom, which leaves us with $n_{\text{dof}} = \frac{N(N-3)}{2}$ degrees of freedom.[1] This means that in a $2 \to 2$ process, we must specify 2 kinematic variables, and in a $2 \to 3$ process we must specify 5. For instance, in the $2 \to 2$ case, the canonical Mandelstam variables s, t, u can be used together with the restriction that $s + t + u = \sum_i m_i^2$.

4.1.1 2-body Phase Space

The Lorentz invariant phase space differential for a 2-body final state with four-momenta p_i, p_j in d dimensions is

$$\mathrm{d}\Pi_{2\to 2} = (2\pi)^d\, \delta^d\left(P - p_i - p_j\right) \frac{\mathrm{d}^{d-1}\boldsymbol{p}_i}{(2\pi)^{d-1}} \frac{1}{2E_i} \frac{\mathrm{d}^{d-1}\boldsymbol{p}_j}{(2\pi)^{d-1}} \frac{1}{2E_j}. \tag{4.2}$$

Going to the centre-of-mass frame of the incoming partons, we have $P^\mu = (\sqrt{s}, 0, 0, 0)$, where $s \equiv P^2$. This allows us to integrate over the spatial part of Dirac delta-function to arrive at

$$\mathrm{d}\Pi_{2\to 2} = \frac{1}{(2\pi)^{d-2}} \mathrm{d}^{d-1}\boldsymbol{p} \frac{1}{4E_iE_j} \delta\left(\sqrt{s} - E(p, m_i) - E(p, m_j)\right), \tag{4.3}$$

where the $E(p, m) = \sqrt{p^2 + m^2}$. We can write out the differential of the spatial component of p_i in spherical coordinates as $\mathrm{d}^{d-1}\boldsymbol{p} = \mathrm{d}\Omega_{d-1} \mathrm{d}p\, p^{d-2} = \mathrm{d}\Omega_{d-2} \sin^{d-3}\theta\, \mathrm{d}\theta\, \mathrm{d}p\, p^{d-2}$. As a $2 \to 2$ process is restricted to planar motion, we can always go to a frame of reference such that any amplitude we calculate will not be dependent on the spatial angles $\mathrm{d}\Omega_{d-2}$, allowing us to integrate over them using that $\int \mathrm{d}\Omega_{d-2} = 2\pi^{\frac{d-2}{2}} \frac{1}{\Gamma\left(\frac{d-2}{2}\right)}$ to get

$$\mathrm{d}\Pi_{2\to 2} = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \frac{1}{\Gamma\left(\frac{d-2}{2}\right)} \frac{p^{d-3}}{2\sqrt{s}} \sin^{d-3}\theta\, \mathrm{d}\theta, \tag{4.4}$$

where we understand the momentum to be given by $p = \frac{\sqrt{\lambda(s, m_i^2, m_j^2)}}{2\sqrt{s}}$. The λ function is known as the Källén function and is defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{4.5}$$

In $d = 4$ dimensions, it is often convenient to change to the Mandelstam variable t, which for massless initial state particles becomes $t = \frac{1}{2}\left(-s + m_i^2 + m_j^2 + \sqrt{\lambda(s, m_i^2, m_j^2)} \cos\theta\right)$. Making the change of variable, the differential phase space reduces to

$$\mathrm{d}\Pi_{2\to 2}\big|_{d=4} = \frac{1}{8\pi s} \mathrm{d}t \tag{4.6}$$

[1] I note that we often consider the invariant mass of the incoming bodies to be fixed, which would reduce our degrees of freedom by one.

4.1.2 3-body Phase Space

TODO: Fill out an introduction here.

The differential Lorentz invariant phase space for a 3-body final state with four-momenta p_i, p_j, k, where $k^2 = 0$ in d dimensions is

$$\mathrm{d}\Pi_{2\to 3} = (2\pi)^d \delta^d(P - p_i - p_j - k) \frac{\mathrm{d}^{d-1}\boldsymbol{p}_i}{(2\pi)^{d-1}} \frac{1}{2E_i} \frac{\mathrm{d}^{d-1}\boldsymbol{p}_j}{(2\pi)^{d-1}} \frac{1}{2E_j} \frac{\mathrm{d}^{d-1}\boldsymbol{k}}{(2\pi)^{d-1}} \frac{1}{2\omega}. \tag{4.7}$$

First, it will be useful to write out the differential in $\boldsymbol{k}$ in spherical coordinates where it reads $\mathrm{d}^{d-1}\boldsymbol{k} = \omega^{d-2}\mathrm{d}\Omega_{d-1}\mathrm{d}\omega$. The differentials in $\boldsymbol{p}_{i/j}$ together with the delta-function are easier to compute in the centre-of-mass frame of the neutralinos where we have $P - k = (Q, 0, 0, 0)$. This leaves

$$\mathrm{d}\Pi_{2\to 3} = \frac{1}{8}\frac{1}{(2\pi)^{2d-3}} \delta(Q - E_i^\star - E_j^\star)\delta^{d-1}(\boldsymbol{p}_i^\star + \boldsymbol{p}_j^\star)\frac{\omega^{d-3}}{E_i^\star E_j^\star} \mathrm{d}^{d-1}\boldsymbol{p}_i^\star \, \mathrm{d}^{d-1}\boldsymbol{p}_j^\star \, \mathrm{d}\Omega_{d-1} \, \mathrm{d}\omega, \tag{4.8}$$

where the stars denote quantities calculated in the aforementioned reference frame. Integrating trivially over $\boldsymbol{p}_j^\star$ using the delta-function, and using polar coordinates $\mathrm{d}^{d-1}\boldsymbol{p}_i = \mathrm{d}\Omega_{d-1}^\star \, \mathrm{d}|\boldsymbol{p}_i^\star| \, |\boldsymbol{p}_i^\star|^{d-2}$ to integrate over $\delta\left(Q - E_i^\star - E_j^\star\right)$, we get

$$\mathrm{d}\Pi_{2\to 3} = \frac{1}{(2\pi)^{2d-3}} \frac{\omega^{d-3}\,|\boldsymbol{p}_i^\star|^{d-3}}{8Q} \mathrm{d}\Omega_{d-1}^\star \, \mathrm{d}\Omega_{d-1} \, \mathrm{d}\omega. \tag{4.9}$$

Here, we understand the magnitude of the three-momenta to be given by $|\boldsymbol{p}_i^\star| = \frac{\sqrt{\lambda(Q^2, m_i^2, m_j^2)}}{2Q}$ and $\omega = \frac{s - Q^2}{2\sqrt{s}}$. It will also be useful to make a change of integration variable to Q^2, leaving us finally with

$$\mathrm{d}\Pi_{2\to 3} = \frac{1}{(2\pi)^{2d-3}} \frac{\omega^{d-3}\,|\boldsymbol{p}_i^\star|^{d-3}}{16Q\sqrt{s}} \mathrm{d}\Omega_{d-1}^\star \, \mathrm{d}\Omega_{d-1} \, \mathrm{d}Q^2. \tag{4.10}$$

TODO: Comment on integration boundaries.

Temporary

$$(m_i + m_j)^2 \le Q^2 \le s$$

With two initial state momenta, the amplitude will be independent of the azimuthal angle in the centre-of-mass frame of the initial partons. This lets us integrate over it for a factor of 2π.

$$\mathrm{d}\Pi_{2\to 3} = \frac{1}{(2\pi)^{2d-3}} \frac{1}{\Gamma\left(\frac{d-2}{2}\right)} \frac{1}{2^d \pi^{\frac{3d-4}{2}}} \frac{\lambda^{\frac{d-3}{2}}(Q^2, m_i^2, m_j^2)}{s} \frac{(1-z)^{\frac{d-3}{2}}}{z^{\frac{d-2}{2}}} \left(y(1-y)\right)^{\frac{d-4}{2}} \mathrm{d}y \, \mathrm{d}\Omega_{d-1}^\star \, \mathrm{d}Q^2. \tag{4.11}$$

Parametrising the free variables in a $2 \to 3$ process can be tricky. I will define some natural variables in two different frames of reference, and rediscover the Lorentz transformation between them to parametrise all scalar products in terms of the variables in these reference frames. First, we will consider the lab frame, or the centre-of-mass frame of the incoming partons with momenta $\boldsymbol{k}_{i,j}$. We can reduce this to an ordinary $2 \to 2$ scattering by considering the outgoing neutralinos with momenta $\boldsymbol{p}_{i,j}$ as a single system. This lets us write the momenta as

$$k_i^\mu = \frac{\sqrt{s}}{2}\left(1,0,0,1\right), \tag{4.12a}$$

$$k_j^\mu = \frac{\sqrt{s}}{2}\left(1,0,0,-1\right), \tag{4.12b}$$

$$k^\mu = \frac{\sqrt{s}}{2}(1-z)\left(1,\sin\theta,0,\cos\theta\right), \tag{4.12c}$$

$$(p_i+p_j)^\mu = \frac{\sqrt{s}}{2}\left((1+z),-(1-z)\sin\theta,0,-(1-z)\cos\theta\right). \tag{4.12d}$$

The centre-of-mass frame of the neutralinos is defined by $(p_i^*+p_k^*)^\mu = (\sqrt{zs},0,0,0)$.[2] We find the transformation to this frame then by making appropriate boosts and rotations of this four-vector. Let us start by rotating the 3-momentum to lie along the positive z-direction. As the y-component is already zero in the lab-frame, we only require a rotation around the y-axis, we can be parametrised by the following matrix

$$\mathrm{Rot}_y(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{pmatrix}. \tag{4.13}$$

Using $\alpha = -\theta - \pi$ we get that $\mathrm{Rot}_y(-\theta-\pi)(p_i+p_j)^\mu = \frac{\sqrt{s}}{2}\left((1+z),0,0,(1-z)\right)$. We can subsequently boost along the z-axis to eliminate the z-component. Such a boost can be parametrised by

$$\mathrm{Boost}_z(\beta) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}, \tag{4.14}$$

where $\gamma = \left(1-\beta^2\right)^{-1/2}$. The z-component is eliminated using $\beta = -\frac{1-z}{1+z}$, such that we end up with

$$(p_i^*+p_j^*)^\mu \equiv \mathrm{Boost}_z\left(-\frac{1-z}{1+z}\right)\mathrm{Rot}_y\left(-\theta-\pi\right)\left(p_i+p_j\right)^\mu = \left(\sqrt{zs},0,0,0\right)$$

as we expected.

Now we can parametrise ${p_{i,j}^*}^\mu$ in this frame using two angular variables θ^*, ϕ^*, knowing that $\boldsymbol{p}_i + \boldsymbol{p}_j = 0$,

$$p_i^{*\mu} = \left(E_i, p\sin\theta^*\cos\phi^*, p\sin\theta^*\sin\phi^*, p\cos\theta^*\right), \tag{4.15a}$$

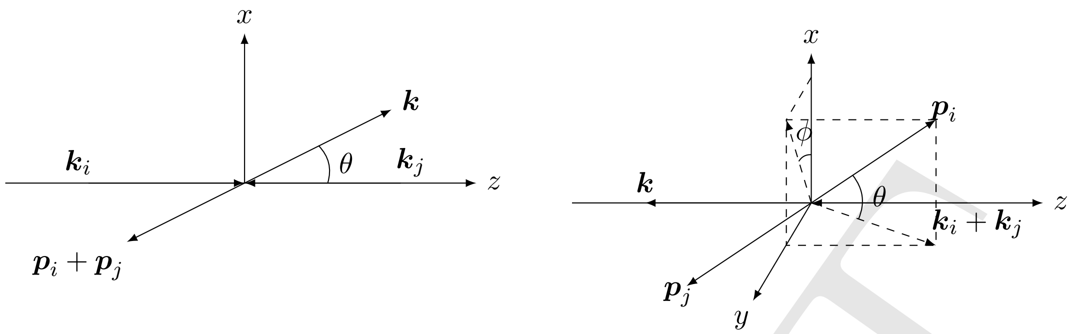
$$p_j^{*\mu} = \left(E_j, -p\sin\theta^*\cos\phi^*, -p\sin\theta^*\sin\phi^*, -p\cos\theta^*\right). \tag{4.15b}$$

To find what $E_{i,j}$ and p need to be, we can transform k^μ and $k_{i,j}^\mu$ to this frame of reference, finding

$$k^{*\mu} = \frac{\sqrt{s}}{2}\frac{1-z}{\sqrt{z}}\left(1,0,0,-1\right), \tag{4.16a}$$

$$\left(k_i^*+k_j^*\right)^\mu = \frac{s}{2\sqrt{z}}\left(1+z,0,0,-(1-z)\right), \tag{4.16b}$$

[2] I will from now on always put a star on quantities pertaining to the centre-of-mass frame of the neutralinos.



(a) Angular definition in the centre-of-mass frame of the initial particles with momenta $\boldsymbol{k}_{i,j}$.

(b) Angular definitions in the centre-of-mass frame of the outgoing particles with momenta $\boldsymbol{p}_{i,j}$.

Figure 4.1

and use conservation of momentum and the fact that ${p^*_{i,j}}^2 = m^2_{i,j}$ to get that

$$E_{i,j}(z) = \frac{zs + m^2_{i,j} - m^2_{j,i}}{2\sqrt{zs}}, \tag{4.17a}$$

$$p(z) = \frac{\sqrt{\lambda\left(zs, m_i^2, m_j^2\right)}}{2\sqrt{zs}}. \tag{4.17b}$$

Now to get all momenta in the lab frame, we can apply the reverse transformations on ${p^*_{i,j}}^\mu$ using that $\mathrm{Rot}_y^{-1}(\alpha) = \mathrm{Rot}_y(-\alpha)$ and $\mathrm{Boost}_z^{-1}(\beta) = \mathrm{Boost}_z(-\beta)$:

$$p^\mu_{i,j} = \mathrm{Rot}_y\left(\theta + \pi\right) \mathrm{Boost}_z\left(\frac{1-z}{1+z}\right) {p^*_{i,j}}^\mu. \tag{4.18}$$

4.1.3 Differential Cross-Section

$$\mathrm{d}\sigma = \frac{1}{2\pi} \left|\mathcal{M}\right|^2 \mathrm{d}\Pi \tag{4.19}$$

$$\mathrm{d}\hat{\sigma}^d = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \frac{1}{\Gamma\left(\frac{d-2}{2}\right)} \frac{p^{d-3}}{4\hat{s}\sqrt{\hat{s}}} \left|\mathcal{M}\right|^2 \sin^{d-3}\theta \,\mathrm{d}\theta \tag{4.20}$$

$$\mathrm{d}\hat{\sigma} = \frac{1}{16\pi} \frac{1}{\hat{s}^2} \left|\mathcal{M}\right|^2 \mathrm{d}\hat{t} \tag{4.21}$$

Averaged over spin and colour, and taking account of symmetry if the particles are identical, the differential cross-section in $d = 4$ dimensions is.

$$\mathrm{d}\hat{\sigma} = \left(\frac{1}{2}\right)^{\delta_{ij}} \frac{1}{64 N_C^2 \pi} \frac{1}{\hat{s}^2} \sum_{\substack{\text{spin}\\\text{colour}}} \left|\mathcal{M}\right|^2 \mathrm{d}\hat{t} \tag{4.22}$$

4.2 Leading Order Cross-Section

TODO:

☐ Comment on reason for using Breit-Wigner approximation.

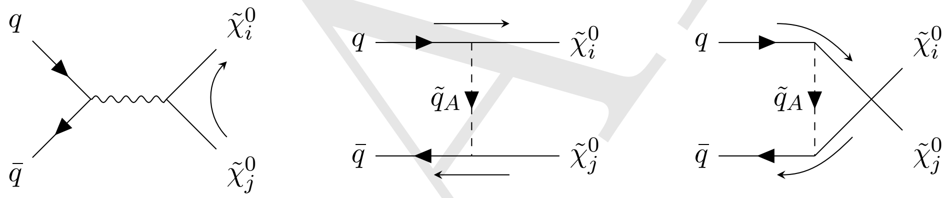
4.2.1 Kinematic Definitions

Before getting into the details of the calculation, it will be helpful to present some definitions of the variables we will need. I will make use of the shorthand notation for the spinors $w_{i/j} = w(p_{i/j}), w_{1,2} = w(k_{i/j})$ where w is either u or v. Furthermore, I will use the shorthand $m_{i,j}$ for the neutralino masses $p^2_{i,j}$. We will also need to define an appropriate set of kinematic variables. Seeing as the inclusive scattering cross-section is only a $2 \to 2$ process to leading order, I will make use of the Mandelstam variables, which in this case will be defined as

$$\hat{s} \equiv (k_i + k_j)^2 = (p_i + p_j)^2, \tag{4.23a}$$
$$\hat{t} \equiv (k_i - p_i)^2 = (k_j - p_j)^2, \tag{4.23b}$$
$$\hat{u} \equiv (k_i - p_j)^2 = (k_j - p_i)^2, \tag{4.23c}$$

which by Eq. (4.1) is constrained by $\hat{s} + \hat{t} + \hat{u} = m_i^2 + m_j^2$. For clarity later on when we will be working with hadron-level kinematics, I will put a hat on variables that are defined at parton level that have an unhatted hadron-level counterpart. This includes the Mandelstam variables above and cross-sections.

4.2.2 The Matrix Elements



(a) s-channel **(b)** t-channel **(c)** u-channel

Figure 4.2: The leading order diagrams contributing to neutralino pair production at parton-level.

At leading order the contributing diagrams to the parton-level process are shown in Fig. 4.2. The resulting amplitudes, using the Feynman rules in Feynman rules section©
, are then

$$\begin{aligned}\mathcal{M}_{\hat{s}} = -g^2 D_Z(\hat{s}) &\left[\bar{u}_i \gamma^\mu \left(O''^L_{ij} P_L + O''^R_{ij} P_R\right) v_j\right] \\ &\times \left[\bar{v}_2 \gamma_\mu \left(C^L_{Zqq} P_L + C^R_{Zqq} P_R\right) u_1\right],\end{aligned} \tag{4.24a}$$

$$\begin{aligned}\mathcal{M}_{\hat{t}} = -\sum_A 2g^2 D_{\tilde{q}_A}(\hat{t}) &\left[\bar{u}_i \left(\left(C^L_{\tilde{\chi}^0_i \tilde{q}_A q_g}\right)^* P_L + \left(C^R_{\tilde{\chi}^0_i \tilde{q}_A q_g}\right)^* P_R\right) u_1\right] \\ &\times \left[\bar{v}_2 \left(C^R_{\tilde{\chi}^0_i \tilde{q}_A q_g} P_L + C^L_{\tilde{\chi}^0_i \tilde{q}_A q_g} P_R\right) v_j\right],\end{aligned} \tag{4.24b}$$

$$\begin{aligned}\mathcal{M}_{\hat{u}} = (-1) - \sum_A 2g^2 D_{\tilde{q}_A}(\hat{u}) &\left[\bar{u}_j \left(\left(C^L_{\tilde{\chi}^0_i \tilde{q}_A q_g}\right)^* P_L + \left(C^R_{\tilde{\chi}^0_i \tilde{q}_A q_g}\right)^* P_R\right) u_1\right] \\ &\times \left[\bar{v}_2 \left(C^R_{\tilde{\chi}^0_i \tilde{q}_A q_g} P_L + C^L_{\tilde{\chi}^0_i \tilde{q}_A q_g} P_R\right) v_i\right],\end{aligned} \tag{4.24c}$$

where $D_p(q^2) = \frac{1}{q^2 - m_p^2 + i\Gamma_p m_p}$ is the Breit-Wigner propagator© of a particle with mass m_p and decay width Γ_p and the extra minus sign in $\mathcal{M}_{\hat{u}}$ comes from it being an odd permutation of the external spinors to the other two amplitudes. TODO: Cite/reference

this.
TODO: Discuss lack of Breit-Wigner for squarks and perhaps touch on non-gauge invariance with BW Z-boson.

These matrix elements can be expanded using the *effective charges* defined by

$$Z^{XY} = C^{X}_{qqZ} O''^{Y}_{ij}\,, \tag{4.25a}$$
$$Q^{XY}_{A} = C^{X}_{\tilde{\chi}^0_i \tilde{q}_A q} \left(C^{Y}_{\tilde{\chi}^0_j \tilde{q}_A q}\right)^{*}, \tag{4.25b}$$

and the Dirac bilinears

$$b_{L/R}(w_a, w_b) = \bar{w}_a P_{L/R} w_b, \tag{4.26a}$$
$$b^{\mu}_{L/R}(w_a, w_b) = \bar{w}_a \gamma^{\mu} P_{L/R} w_b, \tag{4.26b}$$

to arrive at

$$\begin{aligned}\mathcal{M}_{\hat{s}} = -g^2 D_Z(\hat{s}) \Big[& Z^{LL} b^{\mu}_{L}(u_i, v_j) b_{L\mu}(v_2, u_1) - \left(Z^{RR}\right)^{*} b^{\mu}_{L}(u_i, v_j) b_{R\mu}(v_2, u_1) \\ & - \left(Z^{LL}\right)^{*} b^{\mu}_{R}(u_i, v_j) b_{L\mu}(v_2, u_1) + Z^{RR} b^{\mu}_{R}(u_i, v_j) b_{R\mu}(v_2, u_1) \Big],\end{aligned} \tag{4.27a}$$

$$\begin{aligned}\mathcal{M}_{\hat{t}} = -\sum_{A} 2g^2 D_{\tilde{q}_A}(\hat{t}) \Big[& \left(Q^{LR}_{A}\right)^{*} b_L(u_i, u_1) b_L(v_2, v_j) + \left(Q^{LL}_{A}\right)^{*} b_L(u_i, u_1) b_R(v_2, v_j) \\ & + \left(Q^{RR}_{A}\right)^{*} b_R(u_i, u_1) b_L(v_2, v_j) + \left(Q^{RL}_{A}\right)^{*} b_R(u_i, u_1) b_R(v_2, v_j) \Big],\end{aligned} \tag{4.27b}$$

$$\begin{aligned}\mathcal{M}_{\hat{u}} = \sum_{A} 2g^2 D_{\tilde{q}_A}(\hat{u}) \Big[& Q^{RL}_{A} b_L(v_2, v_i) b_L(u_j, u_1) + Q^{RR}_{A} b_L(v_2, v_i) b_R(u_j, u_1) \\ & + Q^{LL}_{A} b_R(v_2, v_i) b_L(u_j, u_1) + Q^{LR}_{A} b_R(v_2, v_i) b_R(u_j, u_1) \Big].\end{aligned} \tag{4.27c}$$

To square the amplitudes we will need to use that the complex conjugate of the Dirac bilinears is

$$\left(b_{L/R}(w_a, w_b)\right)^{\dagger} = b_{R/L}(w_b, w_a), \tag{4.28a}$$
$$\left(b^{\mu}_{L/R}(w_a, w_b)\right)^{\dagger} = b^{\mu}_{L/R}(w_b, w_a). \tag{4.28b}$$

Furthermore, when summing over the spins of the various spinors in the bilinears, they have the sum identities

$$\sum_{\text{spins}} b_X(w_a, w_b) b_Y(w_b, w_a) = 2\Big[(1 - \delta_{XY})(p_a \cdot p_b) + \operatorname{rsgn} \delta_{XY} m_a m_b\Big], \tag{4.29}$$

$$\begin{aligned}\sum_{\text{spins}} b^{\mu}_{X}(w_a, w_b) b^{\nu}_{Y}(w_b, w_a) = 2\Big[& \delta_{XY} \left(p^{\mu}_a p^{\nu}_b - g^{\mu\nu}(p_a \cdot p_b) + p^{\nu}_a p^{\mu}_b + (-1)^{\delta_{XL}} i \epsilon^{\mu\nu\alpha\beta} (p_a)_{\alpha} (p_b)_{\beta}\right) \\ & + (1 - \delta_{XY}) \operatorname{rsgn} m_a m_b g^{\mu\nu}\Big],\end{aligned} \tag{4.30}$$

where rsgn is 1 if w_a, w_b are spinors of the same type, e.g. both are u-spinors, and -1 otherwise.

$$
\begin{aligned}
\mathcal{M}^{\dagger}_{\hat{s}} = -g^2 D^*_Z(\hat{s}) \Big[\left(Z^L\right)^* b^{\mu}_L(v_j, u_i) b_{L\mu}(u_1, v_2) - Z^R b^{\mu}_L(v_j, u_i) b_{R\mu}(u_1, v_2) \\
- Z^L b^{\mu}_R(v_j, u_i) b_{L\mu}(u_1, v_2) + \left(Z^R\right)^* b^{\mu}_R(v_j, u_i) b_{R\mu}(u_1, v_2) \Big] \quad (4.31a) \\
\mathcal{M}^{\dagger}_{\hat{t}} = -\sum_A 2g^2 D^*_{\tilde{q}_A}(\hat{t}) \Big[Q^{RL}_A b_L(u_1, u_i) b_L(v_j, v_2) + Q^{RR}_A b_L(u_1, u_i) b_R(v_j, v_2) \\
+ Q^{LL}_A b_R(u_1, u_i) b_L(v_j, v_2) + Q^{LR}_A b_R(u_1, u_i) b_R(v_j, v_2) \Big] \quad (4.31b) \\
\mathcal{M}^{\dagger}_{\hat{u}} = -\sum_B 2g^2 D^*_{\tilde{q}_B}(\hat{u}) \Big[\left(Q^{LR}_A\right)^* b_L(v_i, v_2) b_L(u_1, u_j) + \left(Q^{LL}_A\right)^* b_L(v_i, v_2) b_R(u_1, u_j) \\
+ \left(Q^{RR}_A\right)^* b_R(v_i, v_2) b_L(u_1, u_j) + \left(Q^{RL}_A\right)^* b_R(v_i, v_2) b_R(u_1, u_j) \Big]. \quad (4.31c)
\end{aligned}
$$

4.2.3 Differential Result

Now, using Eq. (4.22), the partonic cross-section differential in $\hat{t}$ can be shown to be

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\sigma}^0}{\mathrm{d}\hat{t}} = & \frac{\pi \alpha_W^2}{N_C \hat{s}^2} \left(\frac{1}{2}\right)^{\delta_{ij}} \Bigg\{ \sum_{X,Y} \left[\left| C^{XY}_{\hat{t}} \right|^2 \left(\hat{t} - m_i^2\right) \left(\hat{t} - m_j^2\right) + \left| C^{XY}_{\hat{u}} \right|^2 \left(\hat{u} - m_i^2\right) \left(\hat{u} - m_j^2\right) \right] \\
& - \sum_X \left[2 \,\mathrm{Re} \left\{ \left(C^{XX}_u\right)^* C^{XX}_t \right\} m_i m_j \hat{s} - 2 \,\mathrm{Re} \left\{ \left(C^{XX'}_u\right)^* C^{XX'}_t \right\} \left(\hat{t}\hat{u} - m_i^2 m_j^2\right) \right] \Bigg\},
\end{aligned}
\tag{4.32}
$$

where I have defined

$$
C^{XY}_{\hat{t}} = \delta^{XY} \frac{Z^{XX}}{\hat{s} - \Delta^*_Z} + \sum_A \frac{Q^{XY}_A}{t - m_A^2}, \tag{4.33a}
$$

$$
C^{XY}_{\hat{u}} = \delta^{XY} \frac{(Z^{XX})^*}{\hat{s} - \Delta^*_Z} + \sum_A \frac{(Q^{XY}_A)^*}{t - m_A^2}. \tag{4.33b}
$$

(4.33c)

Adding another layer of abstraction in the effective charges on the cross-section might make things a little less clear now, but it will become useful when generalising the cross-section to other electroweakino processes later on.

4.2.4 Phase Space Integral

To get the full cross-section, we will need to integrate over the $\hat{t}$-variable. To do this, we can classify the types of integrals that will arise. All the integrals take the form

$$
T^p(\Delta_1, \Delta_2) \equiv \int_{t_-}^{t_+} \mathrm{d}\hat{t} \, \frac{\hat{t}^p}{(\hat{t} - \Delta_1)(\hat{t} - \Delta_2)} \tag{4.34}
$$

for some $\Delta_{1,2}$ dependent on $\hat{s}$, the neutralino masses and the squark masses, and p is some non-negative integer.

Using the integral limits are $t_\pm = -\frac{\hat{s}-m_i^2-m_j^2}{2} \pm p\sqrt{\hat{s}}$, we get that the possible integrals evaluate to

$$T^2(0,0) = 2p\sqrt{\hat{s}}, \tag{4.35a}$$

$$T^3(0,0) = -p\sqrt{\hat{s}}\left(\hat{s} - m_i^2 - m_j^2\right), \tag{4.35b}$$

$$T^4(0,0) = p\sqrt{\hat{s}}\left(\frac{8}{3}\hat{s}p^2 + 2m_i^2 m_j^2\right), \tag{4.35c}$$

$$T^1(\Delta,0) = -L(\Delta), \tag{4.35d}$$

$$T^2(\Delta,0) = 2p\sqrt{\hat{s}} - \Delta L(\Delta), \tag{4.35e}$$

$$T^3(\Delta,0) = p\sqrt{\hat{s}}\left(2\Delta - (\hat{s} - m_i^2 - m_j^2)\right) - \Delta^2 L(\Delta), \tag{4.35f}$$

$$T^0(\Delta_1,\Delta_2) = \begin{cases} \frac{1}{\Delta_2-\Delta_1}\left\{L(\Delta_1) - L(\Delta_2)\right\} & \text{if } \Delta_1 \neq \Delta_2 \\ \frac{2p\sqrt{\hat{s}}}{\Delta^2+\Delta(\hat{s}-m_i^2-m_j^2)+m_i^2m_j^2} & \text{if } \Delta_1 = \Delta_2 \equiv \Delta \end{cases}, \tag{4.35g}$$

$$T^1(\Delta_1,\Delta_2) = \begin{cases} \frac{1}{\Delta_2-\Delta_1}\left\{\Delta_1 L(\Delta_1) - \Delta_2 L(\Delta_2)\right\} & \text{if } \Delta_1 \neq \Delta_2 \\ \frac{2\Delta p\sqrt{\hat{s}}}{\Delta^2+\Delta(\hat{s}-m_i^2-m_j^2)+m_i^2m_j^2} - L(\Delta) & \text{if } \Delta_1 = \Delta_2 \equiv \Delta \end{cases}, \tag{4.35h}$$

$$T^2(\Delta_1,\Delta_2) = \begin{cases} 2p\sqrt{\hat{s}} + \frac{1}{\Delta_2-\Delta_1}\left\{\Delta_1^2 L(\Delta_1) - \Delta_2^2 L(\Delta_2)\right\} & \text{if } \Delta_1 \neq \Delta_2 \\ \frac{2(2\Delta^2+\Delta(\hat{s}-m_i^2-m_j^2)+m_i^2m_j^2)p\sqrt{\hat{s}}}{\Delta^2+\Delta(\hat{s}-m_i^2-m_j^2)+m_i^2m_j^2} - 2\Delta L(\Delta) & \text{if } \Delta_1 = \Delta_2 \equiv \Delta \end{cases},$$

where I have defined $L(\Delta) = \log \frac{\Delta + \frac{1}{2}(\hat{s}-m_i^2-m_j^2)+p\sqrt{\hat{s}}}{\Delta + \frac{1}{2}(\hat{s}-m_i^2-m_j^2)-p\sqrt{\hat{s}}}$. The two non-zero arguments to these functions that will arise are $\Delta_A^{\hat{t}} = m_A^2$ and $\Delta_A^{\hat{u}} = -(\hat{s} - m_i^2 - m_j^2) - m_A^2$, and I note that $L(\Delta_A^{\hat{u}}) = -L(\Delta_A^{\hat{t}})$. This lets us write the total cross-section

$$\hat{\sigma}^0 = \frac{4\pi p \alpha_W^2}{\hat{s}^{3/2} N_C}\left(F_{\tilde{q}} + F_Z + F_{\tilde{q}Z}\right), \tag{4.36}$$

with effective couplings defined as

$$\begin{aligned} F_{\tilde{q}} = \sum_{A,B,X,Y} \Big\{ & \mathrm{Re}\left\{Q_A^{XY}(Q_B^{XY})^*\right\}\left(1 + K_2^{AB}\right) + \delta_{XY}\,\mathrm{Re}\left\{Q_A^{XX}Q_B^{XX}\right\}K_1^{AB} \\ & + \delta_{XY}\,\mathrm{Re}\left\{Q_A^{XX'}Q_B^{X'X}\right\}\left(1 + K_3^{AB}\right)\Big\}, \end{aligned} \tag{4.37}$$

$$F_Z = \sum_X \left\{\frac{1}{6}\frac{2\hat{s}(\hat{s} - m_i^2 - m_j^2) + m_i^4 + m_j^4}{|\hat{s} - \Delta_Z|^2}|Z^{XX}|^2 - \frac{\hat{s}m_i m_j}{|\hat{s} - \Delta_Z|^2}\mathrm{Re}\left\{(Z^{XX})^2\right\}\right\}, \tag{4.38}$$

and

$$\begin{aligned} F_{\tilde{q}Z} = \sum_{A,X} \mathrm{Re}\left\{\frac{1}{\hat{s} - \Delta_Z}\right\}\Big\{\Big[& \hat{s}m_i m_j\,\mathrm{Re}\left\{Q_A^{XX}Z^{XX}\right\} \\ & - (m_A^2 - m_i^2)(m_A^2 - m_j^2)\,\mathrm{Re}\left\{Q_A^{XX}(Z^{XX})^*\right\}\Big]\frac{L(m_A^2)}{p\sqrt{\hat{s}}} \\ & - (\hat{s} + m_i^2 + m_j^2 - 2m_A^2)\,\mathrm{Re}\left\{Q_A^{XX}(Z^{XX})^*\right\}\Big\}, \end{aligned} \tag{4.39}$$

where $X, Y \in L, R$, $L'/R' = R/L$ and I have defined the shorthands for some functions of the kinematics

$$K_1^{AB} = \frac{\hat{s} m_i m_j}{m_A^2 + m_B^2 + \hat{s} - m_i^2 - m_j^2} \frac{L(m_A^2)}{p\sqrt{\hat{s}}}, \tag{4.40a}$$

$$K_2^{AB} = \begin{cases} \frac{(m_A^2 - m_i^2)(m_A^2 - m_j^2)}{m_A^2 - m_B^2} \frac{L(m_A^2)}{p\sqrt{\hat{s}}} & \text{if } A \neq B \\ \frac{1}{2}(2m_A^2 - m_i^2 - m_j^2) \frac{L(m_A^2)}{p\sqrt{\hat{s}}} & \text{if } A = B \end{cases}, \tag{4.40b}$$

$$K_3^{AB} = \frac{m_A^4 + m_A^2(\hat{s} - m_i^2 - m_j^2) + m_i^4 + m_j^4}{m_A^2 + m_B^2 + \hat{s} + m_i^2 - m_j^2} \frac{L(m_A^2)}{p\sqrt{\hat{s}}}. \tag{4.40c}$$

The sums over the squark mass eigenstates with indices A, B go to two for the non-flavour violating case, and from one to 6 in the flavour violating SLHA2 case.

4.3 NLO Corrections

TODO:

- ☐ Describe factorisation and derive the factorised expression for the cross-section
- ☐ Discuss supersymmetry breaking in dimensional regularisation and its effect on the cross-section.

4.3.1 Factorisation

As we will only look at NLO contributions to the s-channel contribution through a Z-boson, we can do a trick to simplify the process and its corrections. This trick is factorisation, which involves splitting the total cross-section into the two separate processes of the production of an off-shell Z-boson, and its subsequent decay into two neutralinos. Seeing as we are calculating the inclusive cross-section, I include the potential emission of another particle (gluon or quark) along with the Z-boson production.

To start off, we can factorise the d-dimensional differential $2 \to 3$ phase space into two processes by adding an intermediate momentum q with 'mass' squared Q^2. We end up with

$$\begin{aligned} \mathrm{d}q \delta^d (k + q - P) \, \mathrm{d}Q^2 \delta(q^2 - Q^2) \mathrm{d}\Pi_{2\to 3} &= \frac{1}{(2\pi)^{2d-3}} \mathrm{d}^{d-1}p_i \, \mathrm{d}^{d-1}p_j \, \mathrm{d}^{d-1}k \, \mathrm{d}^{d-1}q \, \mathrm{d}Q^2 \\ &\quad \times \frac{1}{16 E_i E_j \omega q^0} \delta^d (q + k - k_i - k_j) \, \delta^d (p_i + p_j + k - k_i - k_j) \\ &\equiv \frac{1}{2\pi} \mathrm{d}\Pi_H \, \mathrm{d}\Pi_N \, \mathrm{d}Q^2, \end{aligned} \tag{4.41}$$

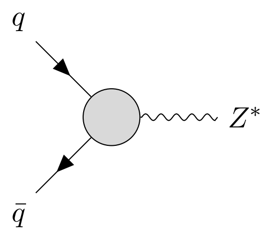
where

$$\mathrm{d}\Pi_H = \frac{\mathrm{d}^{d-1}k \, \mathrm{d}^{d-1}q}{(2\pi)^{d-2}} \frac{1}{4\omega q^0} \delta^d(q + k - k_i - k_j), \tag{4.42a}$$

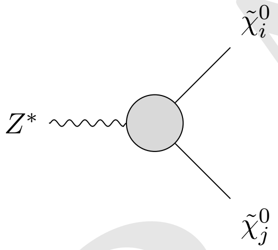
$$\mathrm{d}\Pi_N = \frac{\mathrm{d}^{d-1}p_i \, \mathrm{d}^{d-1}p_j}{(2\pi)^{d-2}} \frac{1}{4 E_i E_j} \delta^d(p_i + p_j - q), \tag{4.42b}$$

which are recognisable as differential phase spaces for a $2 \to 2$ processes going from momenta $k_i + k_j \to q + k$ and a $1 \to 2$ phase space going from $q \to p_i + p_j$. The total phase space integrates over all possible off-shell masses Q^2 for the intermediate momentum q.

So, we have factorised the differential phase space of the differential cross-section Eq. (4.19), but it remains to factorise the amplitude part $|\mathcal{M}|^2$ into part only dependent on either q, k or p_i, p_j. Looking at the tree-level amplitudes Eqs. (4.24a) to (4.24c) that this happens neatly with the s-channel contribution Eq. (4.24a). It has the Lorentz structure $\mathcal{M}_s = D_Z(\hat{s}) g_{\mu\nu} \left[\bar{v}(k_j)\Gamma^{\mu}_{Zqq} u(k_i)\right] \left[\bar{u}(p_i)\Gamma^{\nu}_{Z\tilde{\chi}^0_i\tilde{\chi}^0_j} v(p_j)\right]$. The two terms in brackets are individually only dependent on couplings and the momenta of either the initial partons or the final neutralinos. In fact, they individually take the form of the processes



$$= \left[\bar{v}(k_j) i\Gamma^{\mu}_{Zqq} u(k_i)\right] \epsilon^*_{\mu}(q) \tag{4.43a}$$



$$= \left[\bar{u}(p_i) i\Gamma^{\mu}_{Z\tilde{\chi}^0_i\tilde{\chi}^0_j} v(p_j)\right] \epsilon_{\mu}(q) \tag{4.43b}$$

Squaring it, we can write the differential cross-section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{1}{4\pi\hat{s}} \left|D_Z(\hat{s})\right|^2 H^{\mu\nu} N_{\mu\nu}, \tag{4.44}$$

where

$$\epsilon_{\mu}(q)\epsilon^*_{\nu}(q) H^{\mu\nu} = \int \mathrm{d}\Pi_H \, \left|\mathcal{M}(q\bar{q} \to Z^*)\right|^2, \tag{4.45a}$$

$$\epsilon_{\mu}(q)\epsilon^*_{\nu}(q) N^{\mu\nu} = \int \mathrm{d}\Pi_N \, \left|\mathcal{M}(Z^* \to \tilde{\chi}^0_i \tilde{\chi}^0_j)\right|^2. \tag{4.45b}$$

4.3.2 Self-Energy Contributions

4.3.3 Vertex Corrections

4.3.4 Box Diagrams

4.3.5 Real Emission

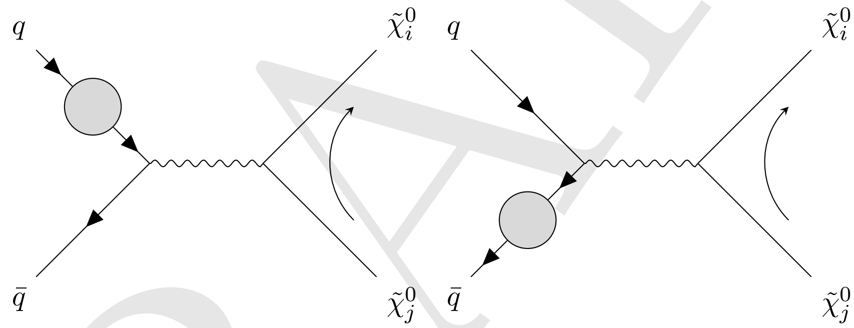

q
$\tilde{\chi}_i^0$
$\bar{q}$
$\tilde{\chi}_j^0$
q
$\tilde{\chi}_i^0$
$\bar{q}$
$\tilde{\chi}_j^0$

Figure 4.3

Chapter 5

Proton–Proton Electroweakino Pair Production

TODO:

- ☐ Describe parton model.

5.1 The Parton Model and pdf's

So far, we have worked with the parton level cross-section, figuring out the contribution of the individual constituents of a proton to the cross-section for our final state. Now these do not individually result in any observable, as the partons are confined to the proton, and can therefore not be singled out in an experiment. To get an observable quantity comparable to experiment, we must sew the individual contributions together. This is done with the *parton model*, where scattering interactions with the proton is modelled with the interaction of free constituent particles inside. The parton model builds on the concept of *factorisation* which, owing to the weakening of the QCD coupling at high-energies, divides interactions with colour-neutral particles into a high-energy and a low-energy regime that are treated separately. The low-energy regime dictates that partons each carry a fraction of the total momentum of the proton, and the probability of encountering a given parton with said momentum fraction.

5.1.1 Hadronic kinematics

Consider the scattering of two protons with momenta P_1^μ and P_2^μ respectively into a set of final state particles χ, χ', X where X is some collection of unlabelled particles. Table 5.1 lists the definitions of kinematic variables at the hadronic level and their relation of the partonic kinematic variables defined in Chapter 4. I define the centre-of-mass energy $S \equiv (P_1 + P_2)^2$. The cross-section for a given process is then given in terms of the cross-section of two partons i, j with momenta $k_i = x_1 P_1$ and $k_j = x_2 P_2$ where $x_1, x_2 \in [0, 1]$ are the respective fractions of the proton momenta the partons carry. The *hadronic cross-section* differential in the squared mass Q^2 of two final state particles χ and χ' is then given by

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2}(PP \to \chi\chi' + X) &= \sum_{ij} \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \, \theta\left(\hat{s} - Q^2\right) f_i(x_1) f_j(x_2) \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}Q^2}(ij \to \chi\chi' + X) \\
&= \sum_{ij} \int_\tau^1 \mathrm{d}x_1 \int_{\tau/x_1}^1 \mathrm{d}x_2 \, f_i(x_1) f_j(x_2) \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}Q^2}(ij \to \chi\chi' + X).
\end{aligned}
\tag{5.1}
$$

The Heaviside function $\theta\left(\hat{s} - Q^2\right) = \theta\left(x_1 x_2 - \tau\right)$ ensures that there is enough energy between the scattering partons to produce the final state $\chi\chi'$-pair with centre-of-mass energy $Q^2 = \tau S$.

Partonic variable	Definition in terms of hadronic variables
k_i^μ	$x_1 P_1^\mu$
k_j^μ	$x_2 P_2^\mu$
$\hat{s}$	$x_1 x_2 S$
z	$\frac{\tau}{x_1 x_2}$

Table 5.1: List of relations between hadronic and partonic kinematic variables.

5.1.2 Integration over pdf's

Practically, the two-dimensional integration over the parton momentum fractions x_1, x_2 can be alleviated by the fact that partonic cross-section contains terms proportional to either $\delta(1-z)$ or plus distributions $f_+(z)$ as we have seen in Chapter 4. Let us consider these types of integrals in some generality. Let $g(x_1, x_2)$ be some function of x_1, x_2, consider the integral

$$
\int_{\tau/x_1}^1 \frac{\mathrm{d}x_2}{x_2} g(x_1, x_2) \delta(1-z). \tag{5.2}
$$

Switching variables to $z = \frac{\tau}{x_1 x_2}$ and keeping x_1 constant yields

$$
\int_{\tau/x_1}^1 \frac{\mathrm{d}z}{z} g(x_1, \frac{\tau}{x_1 z}) \delta(1-z) = g(x_1, \frac{\tau}{x_1}). \tag{5.3}
$$

The plus-distributions are somewhat more complicated. Keeping in mind their definition

$$
\int_0^1 \mathrm{d}z \, g(z) f_+(z) = \int_0^1 \mathrm{d}z \, (g(z) - g(1)) f(z), \tag{5.4}
$$

we have that

$$
\begin{aligned}
\int_{\tau/x_1}^1 \frac{\mathrm{d}x_2}{x_2} g(x_1, x_2) f_+(z) &= \int_{\tau/x_1}^1 \frac{\mathrm{d}z}{z} g(x_1, \frac{\tau}{x_1 z}) f_+(z) \\
&= \int_0^1 \frac{\mathrm{d}z}{z} g(x_1, \frac{\tau}{x_1 z}) f_+(z) - \int_0^{\tau/x_1} \frac{\mathrm{d}z}{z} g(x_1, \frac{\tau}{x_1 z}) f_+(z) \\
&= \int_0^1 \mathrm{d}z \left(\frac{1}{z} g(x_1, \frac{\tau}{x_1 z}) - g(x_1, \frac{\tau}{x_1}) \right) f(z) - \int_0^{\tau/x_1} \frac{\mathrm{d}z}{z} g(x_1, \frac{\tau}{x_1 z}) f(z) \\
&= \int_{\tau/x_1}^1 \frac{\mathrm{d}z}{z} \left(g(x_1, \frac{\tau}{x_1 z}) - z g(x_1, \frac{\tau}{x_1}) \right) f(z) - g(x_1, \frac{\tau}{x_1}) \int_0^{\tau/x_1} \mathrm{d}z \, f(z),
\end{aligned}
\tag{5.5}
$$

where in the third line we have used that $f_+(z) = f(z)$ for $z < 1$. Now, the only plus distribution that have cropped up thus far have been $\lfloor\frac{1}{1-z}\rfloor_+$ and $\lfloor\frac{\ln(1-z)}{1-z}\rfloor_+$, so the last integral in Eq. (5.5) can be done analytically.

Temporary

Defining

$$F(x_1) \equiv \int_0^{\tau/x_1} \mathrm{d}z\, f(z) = \begin{cases} -\ln(1-\frac{\tau}{x_1}) & \text{if } f(z) = \frac{1}{1-z} \\ -\frac{1}{2}\ln^2(1-\frac{\tau}{x_1}) & \text{if } f(z) = \frac{\ln(1-z)}{1-z} \end{cases} \tag{5.6}$$

Should I define this like Tore?

Together, this reduces the integration over the parton momentum fractions into a 1-dimensional and a 2-dimensional integral, easing on the computational power necessary to compute it numerically. Writing the parton level differential cross-sections as

$$\frac{\mathrm{d}\hat{\sigma}_{ij}}{\mathrm{d}Q^2}(x_1,x_2) = \frac{1}{x_1x_2}\left\{w_{ij}^{\mathrm{rad}}(z) + w_{ij}^{\mathrm{soft}}(z)\delta(1-z) + \sum_f w_{ij}^{f_+}(z)f_+(z)\right\}, \tag{5.7}$$

we can factor the x_1, x_2 integrals into

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2}(Q^2) = \int_\tau^1 \frac{\mathrm{d}x_1}{x_1} f_i(x_1)\Bigg\{ f_j(\frac{\tau}{x_1})w_{ij}^{\mathrm{soft}}(1) + \sum_f w_{ij}^{f_+}(1)F(x_1)$$
$$+ \int_{\tau/x_1}^1 \frac{\mathrm{d}x_2}{x_2}\left\{ f_j(x_2)w_{ij}^{\mathrm{rad}}(\frac{\tau}{x_1x_2}) + \sum_f \left(f_j(x_2)w_{ij}^{f_+}(\frac{\tau}{x_1x_2}) - f_j(\frac{\tau}{x_1})\frac{\tau}{x_1x_2}w_{ij}^{f_+}(1)\right) f(\frac{\tau}{x_1x_2})\right\}\Bigg\} \tag{5.8}$$

Chapter 6

Numerical Results

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Appendix A
Weyl Spinors & Grassmann Calculus

Appendix B

Takagi Factorisation Algorithm

B.1 Schur decomposition and singular-value decomposition

Schur decomposition tells us that any (potentially complex) matrix A can be written as

$$A = U^\dagger \Delta U,$$

where U is a unitary matrix, and Δ is an upper triangular matrix. It follows then that if A is a symmetric matrix ($A^T = A$), then

$$\left(U^\dagger \Delta U\right) = \left(U^\dagger \Delta U\right)^T = U^T \Delta^T U^*$$

B.2 Takagi factorisation

Assume $A = A^T$ is a symmetric, complex-valued, $n \times n$ matrix. Takagi factorisation [4] tells us that there exists a unitary matrix U, and a real, non-negative diagonal matrix D such that

$$A = U^T D U. \tag{B.1}$$

B.2.1 Factorisation algorithm

The algorithm is will be based on finding vector $\boldsymbol{v} \in \mathbb{C}^n$ that satisfy $A\boldsymbol{v}^* = \sigma \boldsymbol{v}$, for some real, non-negative σ. This vector will be called a *Takagi vector* for future reference. Existence of these vectors for any matrix A such that AA^* only has real, non-negative eigenvalues is detailed later.

To find U, I propose here an algorithm based on the proof for Takagi factorisation in [4]. Given a Takagi vector $\boldsymbol{v} \in \mathbb{C}^n$ of A,[1] and an orthonormal basis $\{\boldsymbol{v}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ of $\mathbb{C}^n$, it is possible to write A as

$$A = V \begin{bmatrix} \sigma & \mathbf{0} \\ \mathbf{0} & A_2 \end{bmatrix} V^T,$$

where A_2 is a symmetric $(n-1)\times(n-1)$ matrix and V is a unitary matrix with the aforementioned orthonormal basis as its columns. This process can be repeated with A_2

[1] A proof that this can be found is detailed elsewhere.

and so on until you have

$$A = V_1 \cdots V_n \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} V_n^T \cdots V_1^T,$$

where

$$V_p = \begin{bmatrix} \mathbb{I}_{(p-1)\times(p-1)} & \mathbf{0} \\ \mathbf{0} & \tilde{V}_p, \end{bmatrix}$$

and $\tilde{V}_p$ is the unitary matrix that makes a diagonalisation step on A_p. Comparing to Eq. (B.1), we find that

$$U = V_n^T \cdots V_1^T, \tag{B.2a}$$
$$D = \operatorname{diag}(\sigma_1, \ldots, \sigma_n). \tag{B.2b}$$

It is easy to show that U is unitary, as promised. Furthermore, by assumption, all the values σ_p are real and positive. Now the values on the diagonal of D can be permuted to any order using a permutation matrix P, such that we get

$$A = U_P^T D_P U_P,$$

where $U_P = PU$ and $D_P = PDP^T$. U_P is still unitary, and D_P diagonal.

B.3 Proofs

The Takagi vector. For any $A \in M_n(\mathbb{C})$ such that AA^* only has real, non-negative eigenvalues, there exists a non-zero vector $\boldsymbol{v} \in \mathbb{C}^n$ such that $A\boldsymbol{v}^* = \sigma\boldsymbol{v}$, where σ is a real, non-negative number.
Proof. Consider a vector $\boldsymbol{x} \neq \mathbf{0} \in \mathbb{C}^n$ that is an eigenvector of AA^* with corresponding eigenvalue λ. There are two cases:

(a) $A\boldsymbol{x}^*$ and $\boldsymbol{x}$ are linearly dependent.

(b) $A\boldsymbol{x}^*$ and $\boldsymbol{x}$ are linearly independent.

In case (a), we must have that $A\boldsymbol{x}^* = \mu\boldsymbol{x}$ for some $\mu \in \mathbb{C}$, since they are linearly dependent. Then $AA^*\boldsymbol{x} = A\mu^*\boldsymbol{x}^* = |\mu|^2\,\boldsymbol{x} \equiv \lambda\boldsymbol{x}$, which is non-negative by definition.
In case (b), the vector $\boldsymbol{y} = A\boldsymbol{x}^* + \mu\boldsymbol{x}$ is non-zero for any $\mu \in \mathbb{C}$, since $A\boldsymbol{x}^*$ and $\boldsymbol{x}$ are linearly independent. Then we can choose μ such that $|\mu|^2 = \lambda$ to get that $A\boldsymbol{y}^* = A\left(A^*\boldsymbol{x} + \mu^*\boldsymbol{x}^*\right) = \lambda\boldsymbol{x} + \mu^* A\boldsymbol{x}^* = \mu\mu^*\boldsymbol{x} + \mu^* A\boldsymbol{x}^* = \mu^*\left(A\boldsymbol{x}^* + \mu\boldsymbol{x}\right) = \mu^*\boldsymbol{y}$.
As such, we can always find a vector $\tilde{\boldsymbol{v}} \in \mathbb{C}^n$ such that $A\tilde{\boldsymbol{v}}^* = \mu\tilde{\boldsymbol{v}}$ for some $\mu \in \mathbb{C}^n$. Furthermore, we can define a vector $\boldsymbol{v} = e^{i\theta}\tilde{\boldsymbol{v}}$ for a $\theta \in \mathbb{R}$ to get $A\boldsymbol{v}^* = A\left(e^{i\theta}\tilde{\boldsymbol{v}}\right)^* = e^{-i\theta}A\tilde{\boldsymbol{v}}^* = e^{-i\theta}\mu\tilde{\boldsymbol{v}} = e^{-2i\theta}\mu e^{i\theta}\tilde{\boldsymbol{v}} = e^{-2i\theta}\mu\boldsymbol{v} \equiv \sigma\boldsymbol{v}$. This allows us to choose the phase of $\sigma = e^{-2i\theta}\mu$ to be such that σ is real and non-negative.

Eigenvalues of AA^* for symmetric A. Given an $N \times N$ complex matrix A, the eigenvalues of AA^* are always real and non-negative.
Proof. Consider $\boldsymbol{x} \neq \boldsymbol{0}$ an eigenvector of AA^* with corresponding eigenvalue λ. Then we must have that

$$\lambda \boldsymbol{x}^\dagger \boldsymbol{x} = \boldsymbol{x}^\dagger AA^* \boldsymbol{x} = \left(A^\dagger \boldsymbol{x}\right)^\dagger (A^* \boldsymbol{x}) = (A^* \boldsymbol{x})^\dagger (A^* \boldsymbol{x}),$$

where we have used that $A^\dagger = \left(A^T\right)^* = A^*$. This means that $\lambda \geq 0$, since for any vector $\boldsymbol{v} \in \mathbb{C}^n$ we have that $\boldsymbol{v}^\dagger \boldsymbol{v} \geq 0$. As this holds for all eigenvectors $\boldsymbol{x}$ of AA^*, all its eigenvalues must be non-negative.

Diagonalisation step of a symmetric matrix A. For any symmetric matrix $A \in M_n(\mathbb{C})$, there exist a unitary matrix $V \in M_n(\mathbb{C})$ such that

$$V^\dagger A V^* = \begin{pmatrix} \sigma & \boldsymbol{0} \\ \boldsymbol{0} & A_2 \end{pmatrix},$$

where σ is a real, non-negative number and $A_2 \in M_{n-1}(\mathbb{C})$ is also a symmetric matrix.
Proof. Consider a normalised Takagi vector $\boldsymbol{v} \neq \boldsymbol{0}$ of A such that $A\boldsymbol{v}^* = \sigma \boldsymbol{v}$ for some real, non-negative σ and $\boldsymbol{v}^\dagger \boldsymbol{v} = 1$. We can then complete a basis for $\mathbb{C}^n$ with unit vectors $\boldsymbol{v}_i$ where $i \in 1, \ldots, n$, where we define $\boldsymbol{v}_1 \equiv \boldsymbol{v}$. Defining a unitary matrix $V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$, the first column of the product

$$\left(V^\dagger A V^*\right)_{i1} = \boldsymbol{v}_i^\dagger A \boldsymbol{v}^* = \boldsymbol{v}_i^\dagger \sigma \boldsymbol{v} = \sigma \delta_{i1},$$

where δ_{ij} is the Kronecker delta symbol, and we have used the Takagi property of $\boldsymbol{v}$ and the orthonormality of $\boldsymbol{v}_i^\dagger \boldsymbol{v}_j$. This means only the first component of the first column of $V^\dagger A V^*$ is non-zero, and has value σ. Now since A is symmetric, we have that $\left(V^\dagger A V^*\right)^T = V^\dagger A^T V^* = V^\dagger A V^*$ must also be symmetric, and thus must have the form

$$V^\dagger A V^* = \begin{pmatrix} \sigma & \boldsymbol{0} \\ \boldsymbol{0} & A_2 \end{pmatrix},$$

for a symmetric $A_2 \in M_{n-1}(\mathbb{C})$.