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My Master's Thesis

With Subtitle

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Chapter 1

Introduction

This is where I introduce the master's thesis.

Chapter 2

Quantum Field Theory

2.1 Lagrangian Formalism and the Path Integral

PHANTOM PARAGRAPH: Here I want to introduce the Lagrangian formulation of quantum field theory, and the types of fields we will be working with. Furthermore, I want to introduce the perturbation series through the path integral, and talk about the nature of quantum effects. Also derive Feynman rules from path integral.

TODO:

- ☐ Introduce perturbative QFT.
- ☐ Talk about reading Feynman diagrams and special care to take with Majorana fermions.

In this thesis, I will use the Lagrangian framework to formulate QFT. Here I will introduce the basics of how to formulate a QFT in such a way using the path integral formalism. This leads to a perturbative formulation of scattering and computation of correlation functions, which is the basis for the calculations that will be made.

2.2 Renormalised Quantum Field Theory

PHANTOM PARAGRAPH: Talk about loop integrals, divergences, regularisation and renormalisation.

TODO: Mention Wick rotation and evaluation of loop integrals?

Divergences appear in perturbative correlation functions in QFT, and can be categorised into *ultraviolet* (UV) divergences and *infrared* (IR) divergences. They are so named after which region of momentum space they originate from — high momentum for UV and low momentum for IR. The two types of divergences are dealt with entirely differently, and here I will lay out how to deal with UV divergences through *renormalisation*.

Consider a loop like the one in Fig. 2.1: If we consider a massless particle in the loop, the loop integral will take the form

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 + i\epsilon)^2} = \frac{i}{(2\pi)^4} \int \mathrm{d}\Omega_4 \int_0^\infty \mathrm{d}q_E \, \frac{1}{q_E}, \tag{2.1}$$

which diverges for both low and high momenta. ⌜Mention what went into this integral, i.e. Wick rotation and $i\epsilon$.⌟ Had the particle been massive, the momentum would have a non-zero lower limit, and the IR divergence would disappear. However, the UV divergence must be handled differently.

2.2.1 Regularisation

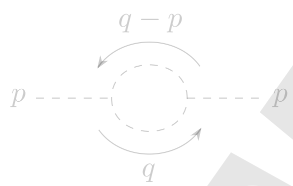


Figure 2.1: Simple example of a loop diagram in a scalar theory.

A first step to handle the divergences is to deform our theory in some way to make the loop integral formally finite, but recovering the divergence in the limit that the deformation disappears. An intuitive deformation would be to cap the momentum integral at some Λ, recovering our original theory in the limit $\Lambda \to \infty$. To illustrate the procedure of regularisation and subsequently renormalisation, it will be useful to have an example, for which I choose a scalar Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{\lambda}{3!}\phi^3$. ⌜Perhaps introduce this model earlier?⌟ Regularising the IR divergence in Eq. (2.1) by giving our scalar a mass m, and the UV divergence with a momentum cap Λ, we are left with

$$\begin{aligned}\int_{|q|<\Lambda} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{\left((q^2 - m^2) + i\epsilon\right)^2} &= \frac{i}{(2\pi)^4} \int \mathrm{d}\Omega_4 \int_0^\Lambda \mathrm{d}q_E \, \frac{q_E^3}{\left(q_E^2 - m^2\right)^2} \\ &= \frac{i}{16\pi^2} \left\{ \ln\left(1 + \frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2 + m^2} \right\}, \end{aligned} \tag{2.2}$$

where now evidently the divergences manifest as a logarithm.

Another popular choice of regularisation, which I will use in this thesis, is *dimensional regularisation*. It entails analytically continuing the number of space-time dimension from the ordinary 4 dimensions to $d = 4 - 2\epsilon$ dimensions for some small ϵ.[1] This removes much of the intuition for what we are doing, but turns out to be computationally very efficient. Our loop integral Eq. (2.1) will then turn into

$$\begin{aligned}\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{\left(q^2 + i\epsilon\right)^2} &= \frac{i2\pi^{d/2}}{(2\pi)^d} \frac{1}{\Gamma(d/2)} \int_0^\infty \mathrm{d}q \, q_E^{d-5} \\ = \frac{i2\pi^{2-\epsilon}}{(2\pi)^{4-2\epsilon}} \frac{1}{\Gamma(2-\epsilon)} \left\{ \int_0^\mu \mathrm{d}q \, \frac{1}{q_E^{1+2\epsilon}} + \int_\mu^\infty \mathrm{d}q \, \frac{1}{q_E^{1+2\epsilon}} \right\} &= \frac{i}{16\pi^2} \left(\frac{1}{\epsilon_{\mathrm{IR}}} - \frac{1}{\epsilon_{\mathrm{UV}}} \right) + O(\epsilon), \end{aligned} \tag{2.3}$$

where in the second equality, the momentum integral is split into a low-energy and high-energy part with some scale μ. Here a trick was performed, as the low-energy part requires $\epsilon < 0$ to be convergent, whereas the high-energy part requires $\epsilon > 0$. The two different divergences thus require different deformations of the theory to be finite, and should be handled separately, hence the subscripts. In the end, the divergences when using dimensional regularisation come out as $\frac{1}{\epsilon}$-terms.

[1] The reason for choosing 2ϵ is purely aesthetical, making some expressions neater.

2.2.2 Counterterm Renormalisation

To take care of UV divergences, we note that there is freedom in how we define the contents of our Lagrangian. We should be able to rescale our fields like $\phi_0 = \sqrt{Z_\phi}\phi$, and rescale our couplings like $m_{\phi,0}^2 = Z_m m_\phi^2$ and $\lambda_0 = Z_\lambda \lambda$. Although suggestively naming terms such as *mass term* with mass m_ϕ^0 implies a connection to the mass of a particle, we have yet to define what that would mean experimentally. Thus, rescaling our parameters and fields parametrises the way in which we can tune our theory, allowing us freedom in choosing the way our theory connects to experiments.

This approach actually allows us to make a perturbative scheme for fixing our (re)normalisations of the fields and couplings. There are many choices for how to connect theory to experiment, but one common approach for field and mass renormalisation is to identify the pole of the two-point correlation function $\mathcal{G}(x, y)$ of a particle to the mass resonance measurable in experiment. This allows us to perturbatively calculate the two-point correlator, and then fix our normalisations accordingly, such that our imposed condition on it holds at every order in the perturbation series. We achieve this systematically with *counterterms*, which in essence are additional Feynman rules added to the theory. By expanding the renormalisation parameters as $Z = 1 + \delta$, the δ will carry the correction to the normalisation to any given order in a coupling constant. To one-loop order, the self-energy of our scalar theory© is diagrammatically given by

Is there a better explanation for this experiment than 'mass resonance'?

where the crossed dot represents an insertion of the δ into the LO amplitude. ⌈Since the δ carries corrections proportional to the NLO amplitudes, it should be seen as coming in at NLO⌋.

2.2.3 On-Shell Renormalisation

Categorising all higher order contributions that can arise to the LO self-energy of a massive particle, they come in the form of *one-particle-irreducible* (1PI) diagrams. These are diagrams where all lines with loop momentum running through are connected. Other diagrams can be reconstructed as the sum of 1PI diagrams. Denoting the leading order correlator $\mathcal{G}_0(p)$ and the contribution from one insertion of all 1PI diagrams $i\Sigma(p)$, we get a series[2]

$i\mathcal{G}(p) =$


$$
\begin{aligned}
&= i\mathcal{G}_0(p) + i\mathcal{G}_0(p) i\Sigma(p) i\mathcal{G}_0(p) + i\mathcal{G}_0(p) i\Sigma(p) i\mathcal{G}_0(p) i\Sigma(p) i\mathcal{G}_0(p) + \dots \\
&= i\mathcal{G}_0(p) \left[i\Sigma(p) i\mathcal{G}_0(p) + (i\Sigma(p) i\mathcal{G}_0(p))^2 + \dots \right] \\
&= i\mathcal{G}_0(p) \frac{1}{1 + \Sigma(p)\mathcal{G}_0(p)} = \frac{i}{\mathcal{G}_0^{-1}(p) + \Sigma(p)}.
\end{aligned} \tag{2.4}
$$

So the computation of the two-point correlator to any order can be done simply by computing the sum of the 1PI diagrams to that order. These contributions will generally

[2] A note on the argument p of these functions: The two-point-correlators in momentum space depend on the four-momentum p^μ in such a way that when it is put in between the external particle representations (i.e. 1 for scalars, spinors for fermions and polarisation vectors for vector bosons) the result will be Lorentz invariant. This means in principle that the correlator could carry Lorentz indices too, which will be suppressed here for simplicity.

diverge, but then we can take into account the renormalisation parameters. Since this is a *bare* function, i.e. using the non-renormalised quantities, we can get the renormalised two-point correlator $\mathcal{G}^{\mathrm{R}}(p)$ through

$$\mathcal{G}^{\mathrm{R}}(p) = \frac{1}{Z_\psi}\mathcal{G}^{\mathrm{bare}}(p) = \frac{1}{1+\delta_\psi}\mathcal{G}^{\mathrm{bare}}(p), \tag{2.5}$$

for any field ψ, seeing as the two-point correlator is quadratic in ψ and thereby quadratic in $\sqrt{Z_\psi}$.

On-shell mass renormalisation seeks to identify the pole of the two-point correlator with the physical mass as observed in experiment. This is a generalisation the property of the LO two-point function to arbitrary order. It yields two conditions:

(I) $\left[\left(\mathcal{G}_0^{\mathrm{R}}(p)\right)^{-1} + \Sigma(p)\right]\Big|_{p^2=m^2_{\mathrm{pole}}} = 0,$

(II) $\mathrm{Res}\left\{\mathcal{G}^{\mathrm{R}}(p), p^2 = m^2_{\mathrm{pole}}\right\} = 1,$

where $\mathrm{Res}\left\{f(z), z = z_0\right\}$ is the residue of the function f at z_0.

For our scalar theory, where the leading order two-point correlator is $\mathcal{G}(p) = \frac{i}{p^2 - m_\phi^2}$, this means that we get the relations

(I) $\delta_m m_\phi^2 = \Sigma(m_\phi^2),$

(II) $\delta_\phi = -\frac{\mathrm{d}}{\mathrm{d}p^2}\Sigma(p^2)\Big|_{p^2=m_\phi^2}.$

[←]■

TODO: Outline chiral mass renormalisation.

2.3 Yang-Mills Theories

Gauge theory in QFT is based on imposing *internal symmetries* on the Lagrangian. Internal symmetries are symmetries separate from *external symmetries* in that they are not symmetries of coordinate transformations, but rather symmetries based on transformations of the fields. Typically, the field transformations under which the Lagrangian is invariant are Lie groups, and are referred to as the *gauge group*. A collection of fields that transform into each other under the fundamental representation[3] is called a *multiplet*.

Let us consider a complex scalar field theory to illustrate. Let ϕ_i be a multiplet of complex scalar fields, and let the gauge group be a general non-Abelian group, locally defined by a set of generators T^a. Locally, the group elements can then be described using the exponential map as

$$g(\boldsymbol{\alpha}) = \exp\left(i\alpha^a T^a\right), \tag{2.6}$$

[3]More on this later.

for a set of real parameters $\boldsymbol{\alpha}$.[4] The transformation law for $\Phi = (\phi_1, \ldots)^T$

$$\Phi \to g(\boldsymbol{\alpha})\Phi = \exp\left(i\alpha^a T^a\right)\Phi, \tag{2.7}$$

which for an infinitesimal set of parameters ϵ^a becomes

$$\Phi \to \left(1 + i\epsilon^a T^a\right)\Phi. \tag{2.8}$$

Now, we would like to categorise the Lagrangian terms that are invariant under such transformations. The ordinary free Klein-Gordon Lagrangian $\mathcal{L}_{\text{KG}} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - m^2 \Phi^\dagger \Phi$ is invariant. However, if we promote our gauge symmetry to be a local symmetry, i.e. let the parameters become spacetime-dependent $\boldsymbol{\alpha} \to \boldsymbol{\alpha}(x)$, this is no longer the case. Since space-time coordinates are unchanged under gauge transformations, it follows that so too is the derivative ∂_μ. However, it will be useful to rewrite this in as somewhat convoluted way, letting it "transform" according to[5][6]

$$\partial_\mu \to \partial_\mu = g\partial_\mu g^{-1} + \left(\partial_\mu g\right) g^{-1}, \tag{2.9}$$

which in turn makes the field derivative transform to

$$\partial_\mu \Phi \to g\partial_\mu \Phi + \left(\partial_\mu g\right)\Phi, \tag{2.10}$$

leaving the kinetic term variant. So we must rethink the kinetic term of Lagrangian. To get the right transformation properties of the derivative term, we need a *covariant derivate* D_μ such that $D_\mu \Phi \to g D_\mu \Phi$. In order to create such a D_μ, we must require that it transforms as $D_\mu \to g D_\mu g^{-1}$. This can be done by introducing the *gauge field* $\mathcal{A}_\mu(x) \equiv A^a_\mu(x) T^a$ which transforms according to

Here it might be prudent to mention that $\mathcal{A}_\mu$ is a real-valued field somehow.

$$\mathcal{A}_\mu \to g\mathcal{A}_\mu g^{-1} - \frac{i}{q}\left(\partial_\mu g\right) g^{-1}. \tag{2.11}$$

The last term can compensate for the extra term in the "transformation" law of ∂_μ. We can then define the covariant derivative $D_\mu = \partial_\mu - iq\mathcal{A}_\mu$ to achieve this.

In summary, with a local gauge symmetry, a gauge field $\mathcal{A}_\mu$ must be introduced such that kinetic terms in the original Lagrangian can be invariant under the gauge transformation. In our case this amounts to adding the interaction term

$$\mathcal{L}_{\mathcal{A}\Phi\text{-int}} = -iq\left[(\partial^\mu \Phi^\dagger)\mathcal{A}_\mu \Phi - \Phi^\dagger \mathcal{A}^\mu (\partial_\mu \Phi)\right] + q^2 \Phi^\dagger \mathcal{A}^\mu \mathcal{A}_\mu \Phi \tag{2.12}$$

to the Klein-Gordon Lagrangian $\mathcal{L}_{KG}$. Now, the Lagrangian is gauge invariant, but there still remains to add dynamics to the gauge field $\mathcal{A}_\mu$. To this end, we can make field-strength tensor $\mathcal{F}_{\mu\nu} \equiv F^a_{\mu\nu} T^a$ that transforms as $\mathcal{F}_{\mu\nu} \to g\mathcal{F}_{\mu\nu} g^{-1}$. The covariant derivative already has this property, and so we can define $\mathcal{F}_{\mu\nu} = \frac{i}{q}\left[D_\mu, D_\nu\right]$, which would include derivative terms for the $\mathcal{A}_\mu$ gauge field and let us construct a gauge invariant term $\operatorname{Tr}\left\{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\right\}$. Antisymmetrising $D_\mu D_\nu \to [D_\mu, D_\nu]$ serves to get rid of the $\partial_\mu\partial_\nu$-term which would result in third derivatives of the gauge field. The kinetic term can be

[4] I will use bold notation $\boldsymbol{\alpha}$ will refer to the collection of parameters α^a, of which there is one for each generator T^a.

[5] It can be shown to be equivalent to ∂_μ when applied to any field (whether they transform under the gauge transformations or not).

[6] In the following I suppress the argument such that $g = g\left(\boldsymbol{\alpha}(x)\right)$

shown to be gauge invariant using the transformation law the field-strength tensor and the cyclic property of the trace

$$\mathrm{Tr}\left\{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\right\} \to \mathrm{Tr}\left\{g\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}g^{-1}\right\} = \mathrm{Tr}\left\{g^{-1}g\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\right\} = \mathrm{Tr}\left\{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\right\}. \tag{2.13}$$

This results in a kinetic term for the $\mathcal{A}_\mu$-field

$$\mathcal{L}_{\mathcal{A}\text{-kin}} = -\frac{1}{4T(R)}\mathrm{Tr}\left\{\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\right\} = -\frac{1}{4}F^{a\,\mu\nu}F^{a}_{\mu\nu}, \tag{2.14}$$

where $T(R)$ is the Dynkin index of the representation R of the group defined by the relation $\mathrm{Tr}\left\{T^aT^b\right\} = T(R)\delta^{ab}$ when T^a are the generators of the group in that representation.

2.4 The Standard Model

PHANTOM PARAGRAPH: INTRODUCE THE *RELEVANT* FIELDS OF THE STANDARD MODEL AND ITS CONSTRUCTION. MAYBE MENTION THE HIGGS MECHANISM?

2.5 Loop Integrals and Regularisation

PHANTOM PARAGRAPH: INTRODUCE LOOP INTEGRALS, HOW TO CALCULATE THEM, WHERE DIVERGENCES APPEAR AND HOW TO REGULARISE THEM.

TODO:

- ☐ Introduce $\overline{\mathrm{DR}}$ renormalisation scheme and talk about Yukawa counterterm in relation to SUSY breaking.

2.5.1 Dimensional Regularisation

2.5.2 Passarino-Veltman Loop Integrals

By Lorentz invariance, there are a limited set of forms that loop integrals can take.Why is this?© These can be categorised according to the number of propagator terms they include, which corresponds to the number of externally connected points there are in the loop. A general scalar N-point loop integral takes the form

$$T_0^N\left(p_i^2, (p_i - p_j)^2; m_0^2, m_i^2\right) = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int \mathrm{d}^d q\, \mathcal{D}_0 \prod_{i=1}^{N-1}\mathcal{D}_i, \tag{2.15}$$

where $\mathcal{D}_0 = \left[q^2 - m_0^2\right]^{-1}$ and $\mathcal{D}_i = \left[(q+p_i)^2 - m_i^2\right]^{-1}$. The first 4 scalar loop integrals are named accordingly

$$T_0^1 \equiv A_0(m_0^2) \tag{2.16}$$
$$T_0^2 \equiv B_0(p_1^2; m_0^2, m_1^2) \tag{2.17}$$
$$T_0^3 \equiv C_0(p_1^2, p_2^2, (p_1 - p_2)^2; m_0^2, m_1^2, m_2^2) \tag{2.18}$$
$$T_0^4 \equiv D_0(p_1^2, p_2^2, p_3^2, (p_1 - p_2)^2, (p_1 - p_3)^2, (p_2 - p_3)^2; m_0^2, m_1^2, m_2^2) \tag{2.19}$$

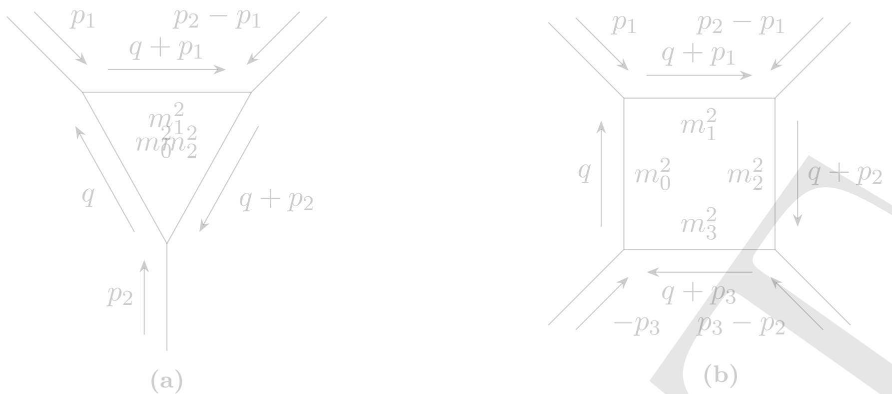


Figure 2.2: Illustration of the momentum conventions for loop diagrams used in the Passarino-Veltman functions.

More complicated Lorentz structure can be obtained in loop integrals, however, these can still be related to the scalar integrals by exploiting the possible tensorial structure they can have. Defining an arbitrary loop integral

$$T^N_{\mu_1\cdots\mu_P}\left(p_i^2,(p_i-p_j)^2;m_0^2,m_i^2\right) = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int \mathrm{d}^d q\, q_{\mu_1}\cdots q_{\mu_P}\,\mathcal{D}_0\prod_{i=1}^{N-1}\mathcal{D}_i. \tag{2.20}$$

These tensors can only depend on the metric $g^{\mu\nu}$ and the external momenta p_i. The possible structures up to four-point loops are as following:

$$B^\mu = p_1^\mu B_1, \tag{2.21a}$$
$$B^{\mu\nu} = g^{\mu\nu}B_{00} + p_1^\mu p_1^\nu B_{11}, \tag{2.21b}$$
$$C^\mu = \sum_{i=1}^{2} p_i^\mu C_i, \tag{2.21c}$$
$$C^{\mu\nu} = g^{\mu\nu}C_{00} + \sum_{i,j=1}^{2} p_i^\mu p_j^\nu C_{ij}, \tag{2.21d}$$
$$C^{\mu\nu\rho} = \sum_{i=1}^{2}(g^{\mu\nu}p_i^\rho + g^{\mu\rho}p_i^\nu + g^{\nu\rho}p_i^\mu)C_{00i} + \sum_{i,j,k=1}^{2} p_i^\mu p_j^\nu p_k^\rho C_{ijk}, \tag{2.21e}$$
$$D^\mu = \sum_{i=1}^{3} p_i^\mu D_i, \tag{2.21f}$$
$$D^{\mu\nu} = g^{\mu\nu}D_{00} + \sum_{i,j=1}^{3} p_i^\mu p_j^\nu D_{ij}, \tag{2.21g}$$
$$D^{\mu\nu\rho} = \sum_{i=1}^{3}(g^{\mu\nu}p_i^\rho + g^{\mu\rho}p_i^\nu + g^{\nu\rho}p_i^\mu)D_{00i} + \sum_{i,j,k=1}^{3} p_i^\mu p_j^\nu p_k^\rho D_{ijk}, \tag{2.21h}$$
$$\begin{aligned} D^{\mu\nu\rho\sigma} &= (g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})D_{0000} \\ &\quad + \sum_{i,j=1}^{3}(g_{\mu\nu}p_i^\rho p_j^\sigma + g_{\mu\nu}p_i^\sigma p_j^\rho + g_{\mu\rho}p_i^\nu p_j^\sigma + g_{\mu\rho}p_i^\sigma p_j^\nu + g_{\mu\sigma}p_i^\rho p_j^\nu + g_{\mu\nu}p_i^\nu p_j^\rho)D_{00ij} \\ &\quad + \sum_{i,j,k,l=1}^{3} p_i^\mu p_j^\nu p_k^\rho p_l^\sigma D_{ijkl}, \end{aligned} \tag{2.21i}$$

where all coefficients must be completely symmetric in i, j, k, l.

Chapter 3

Supersymmetry

3.1 Introduction to Supersymmetry

PHANTOM PARAGRAPH: INTRODUCE SUPERSYMMETRY, WHAT THE SYMMETRY IS AND HOW IT TRANSFORMS FERMIONIC AND BOSONIC FIELDS THROUGH EACH OTHER. INTRODUCE SUPERSPACE, GRASSMANN CALCULUS, SUPERFIELDS AND SUPERLAGRANGIANS.

Introductory passage...

In this section, I will make extensive use of Weyl spinor notation and Grassmann calculcus for which I go in depth in Appendix ⋮©⋮ .

A Simple Supersymmetric Theory

To illustrate what supersymmetry looks like in practice, it can be helpful to look at a simple example. Take a Lagrangian for a massive complex scalar field ϕ and a massive Weyl spinor field ψ

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) + i\psi\sigma^\mu\partial_\mu\psi^\dagger - |m_B|^2\,\phi\phi^* - \frac{1}{2}m_F(\psi\psi) - \frac{1}{2}m_F^*(\psi\psi)^\dagger. \tag{3.1}$$

To impose some symmetry between the bosonic and fermionic degrees of freedom, we want to examine a transformation of the scalar field through the spinor field and vice versa. Imposing Lorentz invariance a general, infinitesimal such transformation can be parametrised by

$$\delta\phi = \epsilon a(\theta\psi), \tag{3.2a}$$
$$\delta\phi^* = \epsilon a^*(\theta\psi)^\dagger, \tag{3.2b}$$
$$\delta\psi_\alpha = \epsilon\left(c(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu\phi + F(\phi,\phi^*)\theta_\alpha\right), \tag{3.2c}$$
$$\delta\psi^\dagger{}_{\dot\alpha} = \epsilon\left(c^*(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^* + F^*(\phi,\phi^*)\theta^\dagger_{\dot\alpha}\right), \tag{3.2d}$$

where ϵ is some infinitesimal parameter for the transformation, θ is some Grassmann-valued Weyl spinor, a, c are complex coefficients of the transformation and $F(\phi,\phi^*)$ is some linear function of ϕ and ϕ^*. The change in the scalar field part of the Lagrangian is

$$\delta\mathcal{L}_\phi/\epsilon = a(\theta\partial_\mu\psi)\,(\partial^\mu\phi^*) - a\,|m_B|^2\,(\theta\psi)\phi^* + \mathrm{c.\,c.}, \tag{3.3}$$

and likewise for the spinor field part

$$\delta\mathcal{L}_\psi/\epsilon = -ic^*(\psi\sigma^\mu\bar{\sigma}^\nu\theta)\partial_\mu\partial_\nu\phi^* + i(\psi\sigma^\mu\theta^\dagger)\partial_\mu F^* + m_F\left[c(\psi\sigma^\mu\theta^\dagger)\partial_\mu\phi + (\psi\theta)F\right] + \text{c. c.} \tag{3.4}$$

The first term in Eq. (3.4) can be rewritten using the commutativity of partial derivatives and the identity $(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta = -2g^{\mu\nu}\delta_\alpha^\beta$ to get $ic^*(\theta\psi)\partial_\mu\partial^\mu\phi^*$. Up to a total derivative, we can then write the change in the spinor part as

$$\delta\mathcal{L}_\psi/\epsilon = -ic^*(\theta\partial_\mu\psi)\partial^\mu\phi^* + (\psi\sigma^\mu\theta^\dagger)\partial_\mu\left(iF^* + m_F c\phi\right) + m_F(\theta\psi)F + \text{c. c.}. \tag{3.5}$$

The change of the total Lagrangian (again up to a total derivate) can then be grouped as

$$\begin{aligned}\delta\mathcal{L}/\epsilon =& \left(a - ic^*\right)\left(\theta\partial_\mu\psi\right)\left(\partial^\mu\phi^*\right) + (\psi\sigma^\mu\theta^\dagger)\partial_\mu\left(iF^* + m_F c\phi\right) \\ &+ (\theta\psi)\left(a\left|m_B\right|^2\phi^* + m_F F\right) + \text{c. c.},\end{aligned} \tag{3.6}$$

giving us three different conditions for the action to be invariant:

$$a - ic^* = 0, \tag{3.7a}$$

$$iF^* + m_F c\phi = 0, \tag{3.7b}$$

$$a\left|m_B\right|^2\phi^* + m_F F = 0. \tag{3.7c}$$

This is fulfilled if

$$c = ia^*, \tag{3.8a}$$

$$F = -am_F^*\phi^*, \tag{3.8b}$$

$$a\left|m_B\right|^2 = a^*\left|m_F\right|^2. \tag{3.8c}$$

What is interesting is the last condition, because it requires a to be real, as both $\left|m_B\right|^2$ and $\left|m_F\right|^2$ are real, but also requires $\left|m_B\right|^2 = \left|m_F\right|^2$. For the theory to be supersymmetric in this sense, the masses of the boson and fermion must be the same!

Revisiting F, it can be introduced as an auxiliary field to bookkeep the supersymmetry transformation. By including the non-dynamical term to the Lagrangian $\mathcal{L}_F = F^*F + mF\phi + m^*F^*\phi^*$, we make sure F takes the correct value in the transformation from its equation of motion $\frac{\partial\mathcal{L}}{\partial F} = F^* + m\phi \overset{!}{=} 0$. Inserting F back into the Lagrangian reproduces the mass term of the scalar field, allowing us to write the original Lagrangian as

$$\mathcal{L} =(\partial_\mu\phi)(\partial^\mu\phi^*) + i\psi\sigma^\mu\partial_\mu\psi^\dagger + F^*F + mF\phi + m^*F^*\phi^* - \frac{1}{2}m(\psi\psi) - \frac{1}{2}m^*(\psi\psi)^\dagger, \tag{3.9}$$

with the *supersymmetry transformation* rules

$$\delta\phi = \epsilon(\theta\psi), \qquad \delta\phi^* = \epsilon(\theta\psi)^\dagger, \tag{3.10a}$$

$$\delta\psi_\alpha = \epsilon\left(-i(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu\phi + F\theta_\alpha\right), \qquad \delta\psi^\dagger_{\dot\alpha} = \epsilon\left(i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^* + F^*\theta^\dagger_{\dot\alpha}\right), \tag{3.10b}$$

$$\delta F = i\epsilon\left(\partial_\mu\psi\sigma^\mu\theta^\dagger\right), \qquad \delta F^* = -i\epsilon\left(\theta\sigma^\mu\partial_\mu\psi^\dagger\right), \tag{3.10c}$$

where I have set $a = 1$ without loss of generality, and found the appropriate transformation law for F such that the Lagrangian is invariant up to total derivatives.

The dynamics of this Lagrangian are the same as before, but the supersymmetry is now made manifest, i.e. the transformation is free of any dependence on the contents of the Lagrangian.

In fact, one can show that a general supersymmetric Lagrangian consisting of a scalar field and a fermion field can be written

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) + i\psi\sigma^\mu\partial_\mu\psi^\dagger + F^*F + \left\{ mF\phi - \frac{1}{2}m(\psi\psi) - \lambda\phi(\psi\psi) + \text{c. c.} \right\} \tag{3.11}$$

up to renormalisable interactions.

3.2 The Super-Poincaré Group

PHANTOM PARAGRAPH: Introduce the super-Poincaré algebra, and superspace as a vessel for manifestly supersymmetric theories. Lead into superfields, and general supersymmetric superlagrangians.

To introduce more involved supersymmetric QFTs than our simple example from before, it will be useful to first introduce a framework that will manifestly carry the supersymmetry. This will alleviate the need to figure out the correct transformation laws, and the constraints they may carry to the parameters of the theory. To this end, I will outline a common way of introducing supersymmetric theories – extending our fields from representations of the Poincaré group of transformations to the super-Poincaré group. This will hopefully give an algebraic geometrical understanding to *superfields* as the building blocks of a supersymmetric field theory.

3.2.1 The Poincaré and Super-Poincaré Algebras

As we have already seen in Section 2.3, sets of transformations can be described by a group. To introduce supersymmetry in this context, it will be clearer to study the *generators* of the algebra of the group, so I would like to take a moment to motivate this change of perspective, before describing the fundamental symmetries we will be using in this context.

The group describing the basic set of *coordinate transformations* under which the fields theories we will consider are symmetric is called the *Poincaré group*, denoted P. Theories that are symmetric under this group will be manifestly relativistic, and will exhibit the ordinary freedom in choice of coordinate system. The Poincaré group consists of any transformation of space-time coordinates x^μ such that

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{3.12}$$

for a real 4×4 matrix Λ and real numbers a^μ. As a group it is the semi-direct product of Lorentz group $O(1,3)$ and group of 4D space-time translations $T(1,3)$

$$P \equiv O(1,3) \rtimes T(1,3). \tag{3.13}$$

For completeness, the semi-direct product is defined such that the product of two group elements $(\Lambda_1, p_1), (\Lambda_2, p_2) \in P$ where $\Lambda_1, \Lambda_2 \in O(1,3)$ and $p_1, p_2 \in T(1,3)$ is

$$(\Lambda_1, p_1) \circ (\Lambda_2, p_2) \equiv (\Lambda_1 \circ_O \Lambda_2, p_1 \circ_T \Lambda_1(p_2)), \tag{3.14}$$

where we understand $\circ_{O/T}$ as the group multiplication operations of $O(1,3)$ and $T(1,3)$ respectively.[1]

For our purposes, it will suffice to work simply with the local structure of the Poincaré group, and being Lie groups© , this can be reproduced with the exponential map we have used earlier $\exp : \mathfrak{g} \to G$, where $\mathfrak{g}$ is the *Lie algebra* of the Lie group G. In this way, the algebra is said to *generate* the group, and a basis set $\{T^a\}$ of the algebra $\mathfrak{g}$ is said to be the *generators* of the group.[2] Accordingly, the local behaviour of the group can be inferred simply from the properties of the generators T^a. The generators of the Poincaré group can be structured by an antisymmetric Lorentz tensor $M^{\mu\nu}$, and a four-vector P^μ. Their properties of the algebra these generators span can be inferred from their commutation relations

$$[P^\mu, P^\nu] = 0 \tag{3.15a}$$
$$[M^{\mu\nu}, P^\rho] = i\left(g^{\mu\sigma}P^\nu - g^{\nu\sigma}P^\mu\right) \tag{3.15b}$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\nu\rho}\right) \tag{3.15c}$$

To construct the super-Poincaré group, we can then just extend the algebra, and the rest of the group will follow. This is done in two ways: First, the Lie algebra is extended to a *graded Lie superalgebra*, and second, by adding four new generators $Q_\alpha, Q^\dagger_{\dot\alpha}$ where $\alpha, \dot\alpha \in \{1,2\}$. A *graded Lie superalgebra* is constructed from two vector spaces $\mathfrak{l}_0, \mathfrak{l}_1$ and is denoted $\mathfrak{l}_0 \oplus \mathfrak{l}_1$. It is itself a vector space with a bilinear operation such that for any elements $x_i \in \mathfrak{l}_i$ we have

$$x_j \circ x_j \in \mathfrak{l}_{i+j \bmod 2} \tag{grading}$$
$$x_i \circ x_j = -(-1)^{i\cdot j} x_j \circ x_i \tag{supersymmetrisation}$$
$$x_i \circ (x_j \circ x_k)(-1)^{i\cdot k} + x_j \circ (x_k \circ x_i)(-1)^{j\cdot i} + x_k \circ (x_i \circ x_j)(-1)^{k\cdot j} = 0 \tag{generalised Jacobi identity}$$

I note that in this case, $\mathfrak{l}_0$ acts as an ordinary Lie algebra, and $\mathfrak{l}_1$ gets anti-commutator relations rather than commutator relations.[3]

The *super-Poincaré algebra*, denoted $\mathfrak{sp}$, is the graded Lie superalgebra resulting from Poincaré algebra $\mathfrak{p}$ and the vector space $\mathfrak{q}$. Here $\mathfrak{p}$ is the Lie algebra of the Poincaré group P and $\mathfrak{q}$ is the vector space spanned by the generators $Q_\alpha, Q^\dagger_{\dot\alpha}$. In addition to the commutation relations Eq. (3.15a) and **????**, the Poincaré superalgebra is specified by the (anti-)commutator relations

$$[Q_\alpha, P^\mu] = \left[Q^\dagger_{\dot\alpha}, P_\mu\right] = 0 \tag{3.16a}$$
$$[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \tag{3.16b}$$
$$\left\{Q_\alpha, Q_\beta\right\} = \left\{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\right\} = 0, \tag{3.16c}$$
$$\left\{Q_\alpha, Q^\dagger_{\dot\beta}\right\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \tag{3.16d}$$

where $\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right)$, $\sigma^\mu = \left(\mathbb{I}, \sigma^i\right)$, $\bar\sigma^\mu = \left(\mathbb{I}, -\sigma^i\right)$ and σ^i are the Pauli matrices.

[1] We see also that $O(1,3)$ must also be a map $T(1,3) \to T(1,3)$. We will later see that this means that the generators of translations are in a representation of the Lorentz group.

[2] The algebra of a Lie group can be shown to be a vector space, and as such there exists a basis set spanning the algebra.

[3] This can be seen from supersymmetrisation as for any $x_1, x_1' \in \mathfrak{l}_1$ we have that $x_1 \circ x_1' = x_1' \circ x_1$.

3.2.2 Superspace

The idea behind *superspace* is to create a coordinate system for which supersymmetry transformation manifest as coordinate transformations similarly to the way Poincaré transformations work on ordinary space-time coordinates. To this end, we can start by considering a general element of the super-Poincaré group $g \in SP$; it can be parametrised through the exponential map like this.

$$g = \exp\left(ix^\mu P_\mu + i(\theta Q) + i(\theta Q)^\dagger + \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} \right), \tag{3.17}$$

where $x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha}, \omega_{\mu\nu}$ parametrise the group, and $P_\mu, Q_\alpha, Q^{\dagger\dot\alpha}, M^{\mu\nu}$ are the generators of the group as we have already seen. Since the parameters $x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha}$ live in irreps of the Lorentz algebra (four-vectors and Weyl spinors respectively) generated by $M^{\mu\nu}$, the effect of this part of the super-Poincaré group on the parameters can be determined easily. Likewise, the parameters $\omega_{\mu\nu}$ are in a trivial representation of the algebra generated by $P_\mu, Q_\alpha, Q^{\dagger\dot\alpha}$, and need not then be considered. It is therefore poignant to create a space with $x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha}$ as the coordinates, modding out the Lorentz algebra part. We create superspace as a coordinate system with coordinates $z^\pi = (x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha})$, and look at how they transform under super-Poincaré group transformations. A function $F(z)$ on superspace can then be written using the generators $K_\pi = (P_\mu, Q_\alpha, Q^{\dagger\dot\alpha})$ as $F(z) = \exp\left(iz^\pi K_\pi\right) F(0)$. Applying a super-Poincaré group element without the Lorentz generators $\bar g(a,\eta) = \exp\left(ia^\mu P_\mu + i(\eta Q) + i(\eta Q)^\dagger \right)$ we have

$$F(z') = \exp\left(iz'^\pi K_\pi\right) F(0) = \exp\left(ia^\mu P_\mu + i(\eta Q) + i(\eta Q)^\dagger \right) \exp\left(iz^\pi K_\pi\right) F(0), \tag{3.18}$$

which by the Baker-Campbell-Hausdorff© formula gives

$$z'^\pi K_\pi = (x^\mu + a^\mu) P_\mu + (\theta^\alpha + \eta^\alpha) Q_\alpha + (\theta^\dagger_{\dot\alpha} + \eta^\dagger_{\dot\alpha}) Q^{\dagger\dot\alpha} + \frac{i}{2}\left[a^\mu P_\mu + (\eta Q) + (\eta Q)^\dagger, z^\pi K_\pi \right] + \ldots \tag{3.19}$$

Now, P_μ commutes with all of K_π, and Q_α ($Q^{\dagger\dot\alpha}$) anti-commute with themselves, for every combination of different α ($\dot\alpha$), so the only relevant part of the commutator is

$$\left[(\eta Q), (\theta Q)^\dagger\right] + \left[(\eta Q)^\dagger, (\theta Q)\right] = -\eta^\alpha \left\{ Q_\alpha, Q^\dagger_{\dot\alpha} \right\} \theta^{\dagger\dot\alpha} + (\eta \leftrightarrow \theta) = -2(\eta\sigma^\mu\theta^\dagger) P_\mu + (\eta \leftrightarrow \theta). \tag{3.20}$$

Since this commutator is proportional to P_μ which in turn commutes with everything, all higher order commutators vanish, and we can conclude that the transformed coordinates z'^π are given by

$$z'^\pi = \left(x^\mu + a^\mu + i(\theta\sigma^\mu\eta^\dagger) - i(\eta\sigma^\mu\theta^\dagger), \theta^\alpha + \eta^\alpha, \theta^\dagger_{\dot\alpha} + \eta^\dagger_{\dot\alpha} \right). \tag{3.21}$$

This gives us a differential representation of the K_π generators as

$$P_\mu = -i\partial_\mu, \tag{3.22a}$$
$$Q_\alpha = -(\sigma^\mu\theta^\dagger)_\alpha \partial_\mu - i\partial_\alpha, \tag{3.22b}$$
$$Q^\dagger_{\dot\alpha} = -(\theta\bar\sigma^\mu)_{\dot\alpha}\partial_\mu - i\partial_{\dot\alpha}. \tag{3.22c}$$

Now, to look into what the these functions of superspace look like, we can expand $F(z)$ in terms of the coordinates $\theta^\alpha, \theta^\dagger_{\dot\alpha}$, as these expansions are finite due to the fact that none of these coordinates can appear more than once. Demanding that the function $F(z)$ be invariant under Lorentz transformations, the x^μ-dependendt coefficients of the expansion must transform such that each term is a scalar (or fully contracted Lorentz structure). This limits a general such function of superspace to be written as

$$\begin{aligned} F(z) =& f(x) + \theta^\alpha \phi_\alpha(x) + \theta^\dagger_{\dot\alpha} \chi^{\dagger\dot\alpha}(x) + (\theta\theta)m(x) + (\theta\theta)^\dagger n(x) \\ &+ (\theta\sigma^\mu\theta^\dagger)V_\mu(x) + (\theta\theta)\theta^\dagger_{\dot\alpha}\lambda^{\dagger\dot\alpha}(x) + (\theta\theta)^\dagger\theta^\alpha\psi_\alpha(x) + (\theta\theta)(\theta\theta)^\dagger d(x). \end{aligned} \tag{3.23}$$

3.2.3 Superfields

To construct a manifestly supersymmetric theory, it will then be useful to start with finding representations of the super-Poincaré group. This is exactly what we have already done; the functions on superspace find themselves in a representation space of a differential representation of the K_π generators of the super-Poincaré group, and a scalar representation of the remaining Lorentz generators (i.e. the Lorentz generators leave the superspace functions unchanged). Inside the general function on superspace Eq. (3.23), we find many component functions in different representation spaces of the Lorentz group. Furthermore, supersymmetry transformations transform these fields into one another. This seems like an ideal vessel for constructing supersymmetric fields theories.

We define the *superfield* Φ as an operator-valued function on superspace.[4] The general one Eq. (3.23) is in a reducible representation space of the super-Poincaré group, so we define three *irreducible* representations that will be useful going forward:[5]

$$\text{\textit{Left-handed scalar superfield:}} \qquad \bar{D}_{\dot\alpha}\Phi = 0 \tag{3.24}$$

$$\text{\textit{Right-handed scalar superfield:}} \qquad D_\alpha\Phi^\dagger = 0 \tag{3.25}$$

$$\text{\textit{Vector superfield:}} \qquad \Phi^\dagger = \Phi \tag{3.26}$$

Here the dagger operation refers to complex conjugation, and the differential operators $D_\alpha, \bar{D}_{\dot\alpha}$ are defined as

$$D_\alpha = \partial_\alpha + i(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu, \tag{3.27a}$$

$$\bar{D}_{\dot\alpha} = -\partial_{\dot\alpha} - i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu. \tag{3.27b}$$

These differential operators are covariant differentials in the sense that the commute with supersymmetry transformations, i.e. $D_\alpha F(z) \to D'_\alpha(\bar{g}F(z)) = \bar{g}\,(D_\alpha F(z))$

3.2.4 Superlagrangian

We are now ready to define the action of a quantum field theory on superspace. Letting the superlagrangian $\mathcal{L}$ be a function superfields $\{\Phi\}$, their derivatives and of superspace coordinates z^π to the reals, the action becomes the functional

$$S\left[\{\Phi_i\}\right] = \int \mathrm{d}^4x\mathrm{d}^4\theta\, \mathcal{L}\left(\{\Phi\}, \left\{\frac{\partial\Phi}{\partial z^\pi}\right\}, z\right). \tag{3.28}$$

The ordinary Lagrangian density as a function of the component fields ϕ of the superfields is recovered simply by integrating over the Grassmann coordinates:

$$\mathcal{L}_{\text{ordinary}}\left(\{\phi\}, \left\{\frac{\partial\phi}{\partial x^\mu}\right\}, x\right) = \int \mathrm{d}^4\theta\, \mathcal{L}\left(\{\Phi\}, \left\{\frac{\partial\Phi}{\partial z^\pi}\right\}, x, \theta, \theta^\dagger\right) \tag{3.29}$$

[4] For our purposes, it suffices to look at them simply as complex valued functions, but strictly speaking, they are operator-valued in a quantised field theory.

[5] I will not prove that these in fact are irreducible representations.

For details on how the calculus of Grassmann coordinates is defined, I refer to Appendix ⋮©⋮ .
⌜Perhaps this is the place for a superspace calculus interlude?⌟

3.3 Minimal Supersymmetric Standard Model

PHANTOM PARAGRAPH: INTRODUCE THE FIELD CONTENT OF THE MSSM AND EXPLAIN CONVENTIONS AND NAMES. TALK ABOUT THE RELEVANT PARTS OF THE SUPERLAGRANGIAN.

TODO:

- ☐ Talk about Wess-Zumino gauge.

3.3.1 Superymmetric Yang-Mills Theory

Before getting into the MSSM content, we must introduce what Yang-Mills theory looks like at a superlagangian level. We define a *supergauge transformation* of a left-handed scalar superfield multiplet Φ analogously to the ordinary case Eq. (2.7)

$$\Phi \to \exp\left(i\Lambda\right)\Phi, \tag{3.30}$$

where $\Lambda \equiv \Lambda^a T^a$, Λ^a are the parameters of the transformation and T^a are again the generators of the gauge group. To get a sense of what these parameters are, we can require the transformed superfield to be left-handed

$$\begin{aligned} D^\dagger_{\dot\alpha} \exp\left(i\Lambda\right)\Phi =& i\left(D^\dagger_{\dot\alpha}\Lambda^a\right)T^a \exp\left(i\Lambda^a T^a\right)\Phi + \exp\left(i\Lambda^a T^a\right)D^\dagger_{\dot\alpha}\Phi \\ =& i\left(D^\dagger_{\dot\alpha}\Lambda^a\right)T^a \exp\left(i\Lambda^a T^a\right)\Phi \overset{!}{=} 0, \end{aligned}$$

which means that we must require $D^\dagger_{\dot\alpha}\Lambda^a = 0$, meaning that the parameters are themselves left-handed scalar superfields. Examining how the kinetic term $\Phi^\dagger\Phi$ does under this transformation we can see that[6]

$$\Phi^\dagger\Phi \to \Phi^\dagger e^{-i\Lambda^\dagger} e^{i\Lambda}\Phi = \Phi^\dagger e^{i\left(\Lambda-\Lambda^\dagger\right)-\frac{1}{2}\left[\Lambda,\Lambda^\dagger\right]+\cdots}\Phi, \tag{3.31}$$

which is not invariant. To remedy this, we will introduce a term to compensate for this change, like before. For this we define a *supergauge field* $\mathcal{V} \equiv V^a T^a$ which transforms according to[7]

$$e^{2q\mathcal{V}} \to e^{i\Lambda^\dagger} e^{2q\mathcal{V}} e^{-i\Lambda} \tag{3.32}$$

or infinitesimally

$$\mathcal{V} \to \mathcal{V} - \frac{i}{2q}\left(\Lambda - \Lambda^\dagger\right) + \frac{i}{2}\left[\Lambda + \Lambda^\dagger, \mathcal{V}\right]. \tag{3.33}$$

Changing the kinetic term to $\Phi^\dagger e^{2q\mathcal{V}}\Phi$ will then yield it invariant under supergauge transformations. Since we require the superlagrangian term to be real, we must require $\mathcal{V}^\dagger = \mathcal{V}$, meaning it must be a vector superfield according to Eq. (3.26).

[6] Using the Baker-Campell-Hausdorff formula© to combine the exponentials.
[7] The factor of 2 in the exponential here seems arbitrary at first, and is just a matter of choice. It is chosen to be 2 here such that the tansformation of law for $\mathcal{V}$ is proportional to Λ without any numerical factors.

As before, we would also like to add dynamics to the (super)gauge field $\mathcal{V}$. To this end, we introduce the supersymmetric field strength $\mathcal{W}_\alpha \equiv W_\alpha^a T^a$ for which we require the transformation law

$$\mathcal{W}_\alpha \to e^{i\Lambda}\mathcal{W}_\alpha e^{-i\Lambda}. \tag{3.34}$$

It can be shown that the left-handed chiral superfield construction

$$\mathcal{W}_\alpha = -\frac{1}{4}(\bar{D}\bar{D})\left(e^{-2\mathcal{V}}D_\alpha e^{2\mathcal{V}}\right) \tag{3.35}$$

transforms this way, and recreates field-strength tensor earlier in Section 2.3.[1] The gauge invariant superlagrangian kinetic term for the supergauge field becomes

$$\mathcal{L}_{\mathcal{V}\text{-kin}} = \frac{1}{4T(R)}\operatorname{Tr}\{\mathcal{W}^\alpha\mathcal{W}_\alpha\} \tag{3.36}$$

analogously to Eq. (2.14).
⌜Why is it that the coupling q sometimes is included in the exponentials ?? to be cancelled Eq. (3.36)?⌟

3.4 Electroweakinos

PHANTOM PARAGRAPH: Introduce the electroweakinos and how they are derived from the various fermions partners of electroweak bosons. Talk about mass matrices and mass eigenstates

3.4.1 Mass mixing

3.5 Feynman Rules of Neutralinos

PHANTOM PARAGRAPH: Derive the ordinary Lagrangian for neutralinos from the superlagrangian.

TODO: Make sure the component form of the superfields is introduced somewhere

Temporary

As a reminder of the form of the superfields we will use, I list them on component form here:

$$\Phi = A + i(\theta\sigma^\mu\theta^\dagger)\partial_\mu A - \frac{1}{4}(\theta\theta)(\theta\theta)^\dagger\Box A + \sqrt{2}(\theta\psi) - \frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi\sigma^\mu\theta^\dagger) + (\theta\theta)F, \tag{3.37a}$$

$$\Phi^\dagger = A^* - i(\theta\sigma^\mu\theta^\dagger)\partial_\mu A^* - \frac{1}{4}(\theta\theta)(\theta\theta)^\dagger\Box A^* + \sqrt{2}(\theta\psi)^\dagger + \frac{i}{\sqrt{2}}(\theta\theta)^\dagger(\theta\sigma^\mu\partial_\mu\psi^\dagger) + (\theta\theta)^\dagger F^*, \tag{3.37b}$$

$$V_{\text{WZ}} = (\theta\sigma^\mu\theta^\dagger)V_\mu + (\theta\theta)(\theta\lambda)^\dagger + (\theta\theta)^\dagger(\theta\lambda) + \frac{1}{2}(\theta\theta)(\theta\theta)^\dagger D, \tag{3.37c}$$

3.5.1 Fermion Interactions from Supersymmetric Yang-Mills theory

⌜This subsection might be best suited for an appendix?⌟

Considering an superlagrangian kinetic term $\mathcal{L} = \Phi_i^\dagger \left(e^{2q\mathcal{V}}\right)_{ij} \Phi_j$, I will extract the interaction terms containing either the fermion fields multiplets ψ from Φ and the fermion fields $\lambda \equiv \lambda^a T^a$ from $\mathcal{V} \equiv V^a T^a$. Up to terms with the appropriate amount of θs, we have

$$\begin{aligned} \mathcal{L} &\overset{\psi,\psi^\dagger}{\supset} 2q\left\{A_i^*(\theta\theta)^\dagger(\theta\lambda_{ij})\sqrt{2}(\theta\psi_j) + \sqrt{2}(\theta\psi_i)^\dagger(\theta\sigma^\mu\theta^\dagger)\left(\mathcal{V}_\mu\right)_{ij}\sqrt{2}(\theta\psi_j) + \sqrt{2}(\theta\psi_i)^\dagger(\theta\theta)(\theta\lambda_{ij})^\dagger A_j\right\} \\ &= q(\theta\theta)(\theta\theta)^\dagger\left\{-\sqrt{2}A^*(\lambda\psi) + (\psi\sigma^\mu\mathcal{V}_\mu\psi^\dagger) - \sqrt{2}(\psi\lambda)^\dagger A\right\}, \end{aligned} \tag{3.38}$$

where I have used Weyl spinor relations© . ⌜Perhaps it should be clarified that these are all the ψ- *and* λ-interactions.⌟

There are also Yukawa terms coming from the superpotential of the form $\mathcal{L} = y_{ij}(\theta\theta)^\dagger\Phi_i\Phi\Phi_j + \text{c. c.}$ Extracting the interaction terms of fermion field ψ from Φ, we find

$$\begin{aligned} \mathcal{L} &\overset{\psi,\psi^\dagger}{\supset} y_{ij}(\theta\theta)^\dagger\sqrt{2}(\theta\psi)\left\{A_i\sqrt{2}(\theta\psi_i) + \sqrt{2}(\theta\psi_j)A_j\right\} + \text{c. c.} \\ &= -y_{ij}(\theta\theta)(\theta\theta)^\dagger\left\{A_i(\psi\psi_j) + (\psi_i\psi)A_j + \text{c. c.}\right\} \end{aligned} \tag{3.39}$$

3.5.2 Wino and Bino Interactions

First, I will look at the bino and wino interactions. Writing out the W^a vector superfields in the basis $W^\pm, W^0$, we are only interested in the electrically neutral W^0 bit. The interactions will come from kinetic terms of scalar superfields Φ, whose relevant part can be written as

$$\mathcal{L} = \Phi^\dagger e^{2g\left\{Y t_W B^0\left(+\frac{1}{2}\sigma_3 W^0\right)\right\}}\Phi, \tag{3.40}$$

where $t_W \equiv \tan\theta_W$ is the tangent of the Weinberg angle and Y is the hypercharge of Φ. To generalise this, I will use the isospin I^3, which is $+\frac{1}{2}$ for fields in the upper part of an SU(2) doublet, $-\frac{1}{2}$ for fields in the lower part and 0 for SU(2) singlet fields. Then the kinetic term can compactly be written as

$$\mathcal{L} = \Phi^\dagger e^{2g\left\{(Q_e - I^3)t_W B^0 + I^3 W^0\right\}}\Phi, \tag{3.41}$$

where Q_e is the electric charge of Φ.

Extracting the interactions of the fermion fields $\widetilde{B}^0, \widetilde{W}^0$ in B^0, W^0 using Eq. (3.38), we are left with (up to approprate θs)

$$\mathcal{L} \overset{\widetilde{B}^0,\widetilde{W}^0}{\supset} -\sqrt{2}g(\theta\theta)(\theta\theta)^\dagger\Big\{(Q_e - I^3)t_W(\widetilde{B}^0\psi)A^* + I^3(\widetilde{W}^0\psi)A^* + \text{c. c.}\Big\}. \tag{3.42}$$

Temporary

$q, \tilde{q}_L \in Q$, $\bar{q}, \tilde{q}_R^* \in \bar{Q}$

Considering an SM quark derive from the superfields Q and $\bar{Q}$, with electric charge Q_e and isospins I^3 and 0 respectively, we can write out the interaction as

$$\mathcal{L} = -\sqrt{2}g\Big\{(Q_e - I^3)t_W(\widetilde{B}^0 q)\tilde{q}_L^* + I^3(\widetilde{W}^0 q)\tilde{q}_L^* + Q_e t_W(\widetilde{B}^0\bar{q})\tilde{q}_R^* + \text{c. c.}\Big\}. \tag{3.43}$$

Changing to the $\widetilde{\chi}^0$-basis, we have that $\widetilde{B}^0 = \sum_i N^*_{i1}\widetilde{\chi}^0_i$, $\widetilde{W}^0 = \sum_i N^*_{i2}\widetilde{\chi}^0_i$, which together with writing out the Weyl products on Dirac spinor form yields

$$\mathcal{L}_{\widetilde{\chi}^0\tilde{q}q} = -\sqrt{2}g\sum_i \bar{\widetilde{\chi}}^0_i\Big\{\underbrace{\big[\left(Q - I^3\right)t_W N^*_{i1} + I^3 N^*_{i2}\big]}_{\equiv C^{L*}_{\widetilde{\chi}^0_i\tilde{q}q}}\widetilde{q}^*_L P_L \underbrace{-Q_f t_W N_{i1}}_{\equiv C^{R*}_{\widetilde{\chi}^0_i\tilde{q}q}}\widetilde{q}^*_R P_R\Big\}q_D + \text{c. c.} \tag{3.44}$$

Generalising this further to include squark mixing between the left- and right-handed squarks in a generation i, we have

$$\tilde{q}_A = R^{\tilde{q}_i}_{A1}\tilde{q}_L + R^{\tilde{q}_i}_{A2}\tilde{q}_R, \tag{3.45}$$

where $R^{\tilde{q}_i}$ is a 2×2 unitary matrix transforming the quarks of type $q = u, d$ in generation i to their mass eigenstates. As such, we can write $\tilde{q}_L = \left(R^{\tilde{q}_i}_{A1}\right)^*\tilde{q}_A$, $\tilde{q}_R = \left(R^{\tilde{q}_i}_{A2}\right)^*\tilde{q}_A$ to get

$$\mathcal{L}_{\widetilde{\chi}^0\tilde{q}q} = -\sqrt{2}g\sum_i \bar{\widetilde{\chi}}^0_i\Big\{\underbrace{\left(R^{\tilde{q}_i}_{A1}\right)^* C^{L*}_{\widetilde{\chi}^0_i\tilde{q}q}}_{\equiv C^{L*}_{\widetilde{\chi}^0_i\tilde{q}_A q}} P_L + \underbrace{\left(R^{\tilde{q}_i}_{A2}\right)^* C^{R*}_{\widetilde{\chi}^0_i\tilde{q}q}}_{\equiv C^{R*}_{\widetilde{\chi}^0_i\tilde{q}_A q}} P_R\Big\}\widetilde{q}^*_A q_D + \text{c. c.} \tag{3.46}$$

TODO: Maybe comment on the extension to flavour violation.

3.5.3 Higgsino Interactions

The Higgsino interaction with the (s)quarks comes from the Yukawa terms of the superpotential, but seeing that this interaction is proportional to the quark mass, it will be ignored at the centre-of-mass energies we are interested in.

The relevant interaction that remains is that with the Z-boson. This interaction again comes from the kinetic term, but this time of the neutral Higgs superfields in the superfield multiplets $H_u = (H^+_u, H^0_u)^T$, $H_d = \left(H^0_d, H^-_d\right)^T$. The Lagrangian is of the form

$$\mathcal{L} = \left(H^0_{u/d}\right)^\dagger e^{\mp g\left(W^0 - t_W B^0\right)} H^0_{u/d}. \tag{3.47}$$

Integrating over the Grassman variables and using equation Eq. (3.38) we get

$$\int \mathrm{d}^4\theta\,\mathcal{L} \overset{\tilde{H}^0_{u/d}, W^0_\mu, B^0_\mu}{=} \mp\frac{g}{2}\left(\tilde{H}^0_{u/d}\sigma^\mu\left(\tilde{H}^0_{u/d}\right)^\dagger\right)\left(W^0_\mu - t_W B^0_\mu\right). \tag{3.48}$$

Mention Weyl/Dirac identities necessary for this.

Switching to Dirac spinors, the mass eigenbasis for the neutralinos and the Z boson $Z_\mu = c_W W^0_\mu - s_W B^0_\mu$, we end up with

$$\begin{aligned}\mathcal{L}_{Z\tilde{\chi}^0} &= \frac{g}{2c_W}Z_\mu\sum_{ij}\left(N_{i4}N^*_{j4} - N_{i3}N^*_{j3}\right)\bar{\widetilde{\chi}}^0_i\gamma^\mu P_L\widetilde{\chi}^0_j \\ &= \frac{g}{2}Z_\mu\sum_{ij}\bar{\widetilde{\chi}}^0_i\gamma^\mu\Big[\underbrace{\frac{1}{2c_W}\left(N_{i4}N^*_{j4} - N_{i3}N^*_{j3}\right)}_{\equiv O''^L_{ij}}P_L \underbrace{-\frac{1}{2c_W}\left(N^*_{i4}N_{j4} - N^*_{i3}N_{j3}\right)}_{\equiv O''^R_{ij}}P_R\Big]\widetilde{\chi}^0_j\end{aligned} \tag{3.49}$$

Chapter 4

Neutralino Pair Production at Parton Level

TODO:

- ☐ Formulate a section on the dipole formalism used in Debove et al. and make a comparison.

4.1 Kinematics

To start off, it will be useful to introduce some procedure for going forward in the phase space of an inclusive $2 \to 2(+1)$ cross-section process. The phase space of 2-body and 3-body final states are quite different as there are more degrees of freedom in the 3-body final state. In the end, these extra degrees of freedom will be need to be integrated over to make an additive comparison between the 2-body and 3-body processes, however, exactly how we choose to parametrise and subsequently integrate over the extra degrees of freedom can matter quite a bit. To start out, let us count the degrees of freedom of a

scattering problem involving N four-momenta $p_{i=1,\ldots,N}$. Assuming our end result to be Lorentz invariant, there are $N(N+1)/2$ different scalar products that can be produced using N different four-momenta. Momentum conservation allows us to eliminate one momentum, such that we have $N(N-1)/2$ possible scalar products. Denoting the scalar products by $m_{ij}^2 \equiv (p_i + p_j)^2$ for $j \neq i$, and $m_i^2 \equiv p_i^2$, we can find a relation between scalar products by using momentum conservation.

$$
\begin{aligned}
m_{ij}^2 &= \left(p_i - \sum_{k\neq j} p_k\right)^2 = \left(\sum_{k\neq i,j} p_k\right)^2 = \sum_{k\neq i,j}\sum_{l\neq i,j} p_k \cdot p_l \\
&= \sum_{k\neq i,j}\sum_{l\neq i,j,k} \frac{m_{kl}^2 - m_k^2 - m_l^2}{2} + \sum_{k\neq i,j} m_k^2 \\
&= \sum_{k\neq i,j}\sum_{\substack{l\neq i,j\\ l>k}} m_{kl}^2 - \frac{1}{2}\sum_{k\neq i,j}(N-3)m_k^2 - \frac{1}{2}\sum_{l\neq i,j}(N-3)m_l^2 + \sum_{k\neq i,j} m_k^2 \\
&= \sum_{k\neq i,j}\sum_{\substack{l\neq i,j\\ l>k}} m_{kl}^2 - (N-4)\sum_{k\neq i,j} m_k^2. \qquad (4.1)
\end{aligned}
$$

⌜This little generalised relation might not be immediately necessary...⌟ Furthermore, we assign the N scalar products m_i^2 to the invariant masses of the incoming and outgoing particles, thus not counting them as degrees of freedom, leaving us with $n_{\text{dof}} = \frac{N(N-3)}{2}$ degrees of freedom.[1] This means that in a $2 \to 2$ process, we have 2 degrees of freedom, and in a $2 \to 3$ process we have 5.

4.1.1 2-body Phase Space

The Lorentz invariant phase space differential for a 2-body final state with four-momenta p_i, p_j in d dimensions is

$$\mathrm{d}\Pi_{2\to 2} = (2\pi)^d\, \delta^d\left(P - p_i - p_j\right) \frac{\mathrm{d}^{d-1}\boldsymbol{p}_i}{(2\pi)^{d-1}} \frac{1}{2E_i} \frac{\mathrm{d}^{d-1}\boldsymbol{p}_j}{(2\pi)^{d-1}} \frac{1}{2E_j}. \tag{4.2}$$

Going to the centre-of-mass frame of the incoming partons, we have $P^\mu = (\sqrt{s}, 0, 0, 0)$, allowing us to integrate over the spatial part of Dirac delta-function to arrive at

$$\mathrm{d}\Pi_{2\to 2} = \frac{1}{(2\pi)^{d-2}} \mathrm{d}^{d-1}\boldsymbol{p} \frac{1}{4E_i E_j} \delta\left(\sqrt{s} - E(p, m_i) - E(p, m_j)\right), \tag{4.3}$$

where the $E(p,m) = \sqrt{p^2 + m^2}$. We can write out the differential of the spatial component of p_i in spherical coordinates as $\mathrm{d}^{d-1}\boldsymbol{p} = \mathrm{d}\Omega_{d-1} \mathrm{d}p\, p^{d-2} = \mathrm{d}\Omega_{d-2} \sin^{d-3}\theta\, \mathrm{d}\theta\, \mathrm{d}p\, p^{d-2}$. As a $2 \to 2$ process is restricted to planar motion, we can always go to a frame of reference such that any amplitude we calculate will not be dependent on the spatial angles $\mathrm{d}\Omega_{d-2}$, allowing us to integrate over them using that $\int \mathrm{d}\Omega_{d-2} = 2\pi^{\frac{d-2}{2}} \frac{1}{\Gamma\left(\frac{d-2}{2}\right)}$ to get

$$\mathrm{d}\Pi_{2\to 2} = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \frac{1}{\Gamma\left(\frac{d-2}{2}\right)} \frac{p^{d-3}}{2\sqrt{s}} \sin^{d-3}\theta\, \mathrm{d}\theta, \tag{4.4}$$

where we understand the momentum to be given by $p = \frac{\sqrt{\lambda(s, m_i^2, m_j^2)}}{2\sqrt{s}}$. In $d = 4$ dimensions, it is often convenient to change to the Mandelstam variable t, which for massless initial state particles becomes $t = \frac{1}{2}\left(-s + m_i^2 + m_j^2 + \sqrt{\lambda(s, m_i^2, m_j^2)} \cos\theta\right)$. Making the change of variable, the differential phase space reduces to

$$\mathrm{d}\Pi_{2\to 2}\big|_{d=4} = \frac{1}{8\pi s}\, \mathrm{d}t \tag{4.5}$$

4.1.2 3-body Phase Space

TODO: Fill out an introduction here.

The differential Lorentz invariant phase space for a 3-body final state with four-momenta p_i, p_j, k, where $k^2 = 0$ in d dimensions is

$$\mathrm{d}\Pi_{2\to 3} = (2\pi)^d \delta^d(P - p_i - p_j - k) \frac{\mathrm{d}^{d-1}\boldsymbol{p}_i}{(2\pi)^{d-1}} \frac{1}{2E_i} \frac{\mathrm{d}^{d-1}\boldsymbol{p}_j}{(2\pi)^{d-1}} \frac{1}{2E_j} \frac{\mathrm{d}^{d-1}\boldsymbol{k}}{(2\pi)^{d-1}} \frac{1}{2\omega}. \tag{4.6}$$

First, it will be useful to write out the differential in $\boldsymbol{k}$ in spherical coordinates where it reads $\mathrm{d}^{d-1}\boldsymbol{k} = \omega^{d-2} \mathrm{d}\Omega_{d-1} \mathrm{d}\omega$. The differentials in $\boldsymbol{p}_{i/j}$ together with the delta-function

[1] I note that we often consider the invariant mass of the incoming bodies to be fixed, which would reduce our degrees of freedom by one.

are easier to compute in the centre-of-mass frame of the neutralinos where we have $P - k = (Q, 0, 0, 0)$. This leaves

$$\mathrm{d}\Pi_{2\to 3} = \frac{1}{8}\frac{1}{(2\pi)^{2d-3}}\delta(Q - E_i^\star - E_j^\star)\delta^{d-1}(\boldsymbol{p}_i^\star + \boldsymbol{p}_j^\star)\frac{\omega^{d-3}}{E_i^\star E_j^\star}\mathrm{d}^{d-1}\boldsymbol{p}_i^\star\,\mathrm{d}^{d-1}\boldsymbol{p}_j^\star\,\mathrm{d}\Omega_{d-1}\,\mathrm{d}\omega, \tag{4.7}$$

where the stars denote quantities calculated in the aforementioned reference frame. Integrating trivially over $\boldsymbol{p}_j^\star$ using the delta-function, and writing using polar coordinates $\mathrm{d}^{d-1}\boldsymbol{p}_i = \mathrm{d}\Omega_{d-1}^\star\,\mathrm{d}|\boldsymbol{p}_i^\star|\,|\boldsymbol{p}_i^\star|^{d-2}$ to integrate over $\delta\left(Q - E_i^\star - E_j^\star\right)$, we get

$$\mathrm{d}\Pi_{2\to 3} = \frac{1}{(2\pi)^{2d-3}}\frac{\omega^{d-3}\,|\boldsymbol{p}_i^\star|^{d-3}}{8Q}\mathrm{d}\Omega_{d-1}^\star\,\mathrm{d}\Omega_{d-1}\,\mathrm{d}\omega. \tag{4.8}$$

Here, we understand the magnitude of the three-momenta to be given by $|\boldsymbol{p}_i^\star| = \frac{\sqrt{\lambda(Q^2, m_i^2, m_j^2)}}{2Q}$ and $\omega = \frac{s - Q^2}{2\sqrt{s}}$. It will also be useful to make a change of integration variable to Q^2, leaving us finally with

$$\mathrm{d}\Pi_{2\to 3} = \frac{1}{(2\pi)^{2d-3}}\frac{\omega^{d-3}\,|\boldsymbol{p}_i^\star|^{d-3}}{16Q\sqrt{s}}\mathrm{d}\Omega_{d-1}^\star\,\mathrm{d}\Omega_{d-1}\,\mathrm{d}Q^2. \tag{4.9}$$

TODO: Comment on integration boundaries.

Temporary

$$(m_i + m_j)^2 \leq Q^2 \leq s$$

With two initial state momenta, the amplitude will be independent of the azimuthal angle in the centre-of-mass frame of the initial partons. This lets us integrate over it for a factor of 2π.

$$\mathrm{d}\Pi_{2\to 3} = \frac{1}{(2\pi)^{2d-3}}\frac{1}{\Gamma\left(\frac{d-2}{2}\right)}\frac{1}{2^d\pi^{\frac{3d-4}{2}}}\frac{\lambda^{\frac{d-3}{2}}\left(Q^2, m_i^2, m_j^2\right)}{s}\frac{(1-z)^{\frac{d-3}{2}}}{z^{\frac{d-2}{2}}}\left(y(1-y)\right)^{\frac{d-4}{2}}\,\mathrm{d}y\,\mathrm{d}\Omega_{d-1}^\star\,\mathrm{d}Q^2. \tag{4.10}$$

Parametrising the free variables in a $2 \to 3$ process can be tricky. I will define some natural variables in two different frames of reference, and rediscover the Lorentz transformation between them to parametrise all scalar products in terms of the variables in these reference frames. First, we will consider the lab frame, or the centre-of-mass frame of the incoming partons with momenta $\boldsymbol{k}_{i,j}$. We can reduce this to an ordinary $2 \to 2$ scattering by considering the outgoing neutralinos with momenta $\boldsymbol{p}_{i,j}$ as a single system. This lets us write the momenta as

$$k_i^\mu = \frac{\sqrt{s}}{2}\left(1, 0, 0, 1\right), \tag{4.11a}$$

$$k_j^\mu = \frac{\sqrt{s}}{2}\left(1, 0, 0, -1\right), \tag{4.11b}$$

$$k^\mu = \frac{\sqrt{s}}{2}\left(1 - z\right)\left(1, \sin\theta, 0, \cos\theta\right), \tag{4.11c}$$

$$(p_i + p_j)^\mu = \frac{\sqrt{s}}{2}\left((1+z), -(1-z)\sin\theta, 0, -(1-z)\cos\theta\right). \tag{4.11d}$$

The centre-of-mass frame of the neutralinos is defined by $(p_i^* + p_k^*)^\mu = (\sqrt{zs}, 0, 0, 0)$.[2] We find the transformation to this frame then by making appropriate boosts and rotations of this four-vector. Let us start by rotating the 3-momentum to lie along the positive z-direction. As the y-component is already zero in the lab-frame, we only require a rotation around the y-axis, we can be parametrised by the following matrix

$$\mathrm{Rot}_y(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{pmatrix}. \tag{4.12}$$

Using $\alpha = -\theta - \pi$ we get that $\mathrm{Rot}_y(-\theta - \pi)(p_i + p_j)^\mu = \frac{\sqrt{s}}{2}\left((1+z), 0, 0, (1-z)\right)$. We can subsequently boost along the z-axis to eliminate the z-component. Such a boost can be parametrised by

$$\mathrm{Boost}_z(\beta) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}, \tag{4.13}$$

where $\gamma = \left(1 - \beta^2\right)^{-1/2}$. The z-component is eliminated using $\beta = -\frac{1-z}{1+z}$, such that we end up with

$$(p_i^* + p_j^*)^\mu \equiv \mathrm{Boost}_z\left(-\frac{1-z}{1+z}\right)\mathrm{Rot}_y\left(-\theta - \pi\right)\left(p_i + p_j\right)^\mu = (\sqrt{zs}, 0, 0, 0)$$

as we expected.

Now we can parametrise ${p_{i,j}^*}^\mu$ in this frame using two angular variables θ^*, ϕ^*, knowing that $\boldsymbol{p}_i + \boldsymbol{p}_j = 0$,

$$p_i^{*\mu} = \left(E_i, p\sin\theta^*\cos\phi^*, p\sin\theta^*\sin\phi^*, p\cos\theta^*\right), \tag{4.14a}$$
$$p_j^{*\mu} = \left(E_j, -p\sin\theta^*\cos\phi^*, -p\sin\theta^*\sin\phi^*, -p\cos\theta^*\right). \tag{4.14b}$$

To find what $E_{i,j}$ and p need to be, we can transform k^μ and $k_{i,j}^\mu$ to this frame of reference, finding

$$k^{*\mu} = \frac{\sqrt{s}}{2}\frac{1-z}{\sqrt{z}}\left(1, 0, 0, -1\right), \tag{4.15a}$$
$$\left(k_i^* + k_j^*\right)^\mu = \frac{s}{2\sqrt{z}}\left(1+z, 0, 0, -(1-z)\right), \tag{4.15b}$$

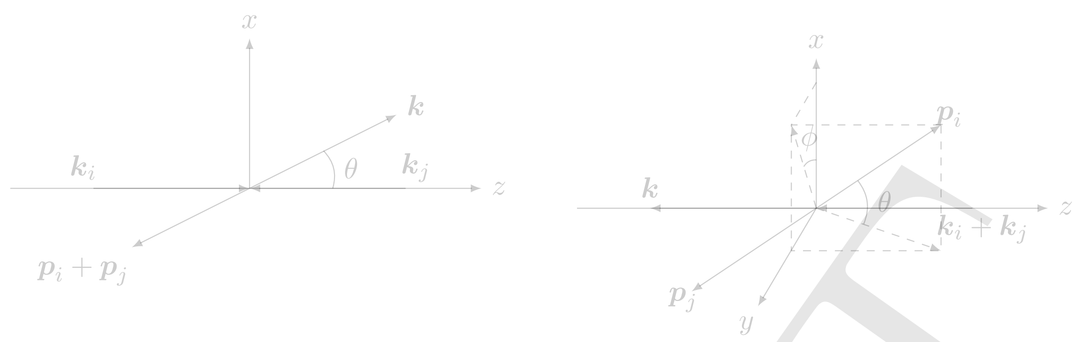
and use conservation of momentum and the fact that ${p_{i,j}^*}^2 = m_{i,j}^2$ to get that

$$E_{i,j}(z) = \frac{zs + m_{i,j}^2 - m_{j,i}^2}{2\sqrt{zs}}, \tag{4.16a}$$
$$p(z) = \frac{\sqrt{\lambda\left(zs, m_i^2, m_j^2\right)}}{2\sqrt{zs}}. \tag{4.16b}$$

Now to get all momenta in the lab frame, we can apply the reverse transformations on ${p_{i,j}^*}^\mu$ using that $\mathrm{Rot}_y^{-1}(\alpha) = \mathrm{Rot}_y(-\alpha)$ and $\mathrm{Boost}_z^{-1}(\beta) = \mathrm{Boost}_z(-\beta)$:

$$p_{i,j}^\mu = \mathrm{Rot}_y\left(\theta + \pi\right)\mathrm{Boost}_z\left(\frac{1-z}{1+z}\right){p_{i,j}^*}^\mu. \tag{4.17}$$

[2] I will from now on always put a star on quantities pertaining to the centre-of-mass frame of the neutralinos.



(a) Angular definition in the centre-of-mass frame of the initial particles with momenta $\boldsymbol{k}_{i,j}$.

(b) Angular definitions in the centre-of-mass frame of the outgoing particles with momenta $\boldsymbol{p}_{i,j}$.

Figure 4.1

4.1.3 Differential Cross-Section

$$\mathrm{d}\sigma = \frac{1}{2\pi} |\mathcal{M}|^2 \, \mathrm{d}\Pi \tag{4.18}$$

$$\mathrm{d}\hat{\sigma}^d = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \frac{1}{\Gamma\left(\frac{d-2}{2}\right)} \frac{p^{d-3}}{4\hat{s}\sqrt{\hat{s}}} |\mathcal{M}|^2 \sin^{d-3}\theta \, \mathrm{d}\theta \tag{4.19}$$

$$\mathrm{d}\hat{\sigma} = \frac{1}{16\pi} \frac{1}{\hat{s}^2} |\mathcal{M}|^2 \, \mathrm{d}\hat{t} \tag{4.20}$$

Averaged over spin and colour, and taking account of symmetry if the particles are identical, the differential cross-section in $d = 4$ dimensions is.

$$\mathrm{d}\hat{\sigma} = \left(\frac{1}{2}\right)^{\delta_{ij}} \frac{1}{64 N_C^2 \pi} \frac{1}{\hat{s}^2} \sum_{\substack{\text{spin}\\ \text{colour}}} |\mathcal{M}|^2 \, \mathrm{d}\hat{t} \tag{4.21}$$

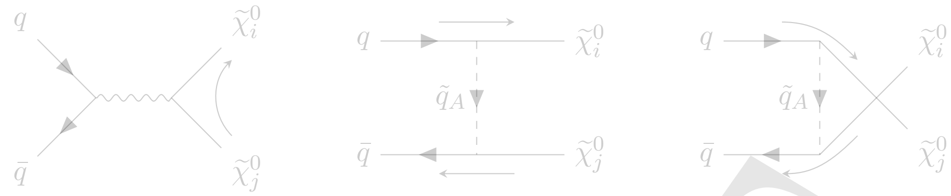
4.2 Leading Order Cross-Section

TODO:

- ☐ Comment on reason for using Breit-Wigner approximation.

4.2.1 The Matrix Elements

At leading order the contributing diagrams to the parton-level process are shown in Fig. 4.2. In the following, I will make use of the shorthand notation for the spinors $w_{i/j} = w(p_{i/j}), w_{1,2} = w(k_{i/j})$ where w is either u or v. The resulting amplitudes, using



(a) s-channel **(b)** t-channel **(c)** u-channel

Figure 4.2: The leading order diagrams contributing to neutralino pair production at parton-level.

the Feynman rules in Feynman rules section© , are then

$$\begin{aligned}
\mathcal{M}_{\hat{s}} = -\frac{g^2}{2} D_Z(\hat{s}) \left[\bar{u}_i \gamma^\mu \left(O''^L_{ij} P_L + O''^R_{ij} P_R\right) v_j\right] & \\
\times \left[\bar{v}_2 \gamma_\mu \left(C^L_{Zqq} P_L + C^R_{Zqq} P_R\right) u_1\right], & \qquad (4.22a) \\
\mathcal{M}_{\hat{t}} = -\sum_A 2g^2 D_{\tilde{q}_A}(\hat{t}) \left[\bar{u}_i \left(C^{L*}_{\widetilde{\chi}_i^0 \tilde{q}_A q} P_L + C^{R*}_{\widetilde{\chi}_i^0 \tilde{q}_A q} P_R\right) u_1\right] & \\
\times \left[\bar{v}_2 \left(C^R_{\widetilde{\chi}_j^0 \tilde{q}_A q} P_L + C^L_{\widetilde{\chi}_j^0 \tilde{q}_A q} P_R\right) v_j\right], & \qquad (4.22b) \\
\mathcal{M}_{\hat{u}} = -\sum_B 2g^2 D_{\tilde{q}_B}(\hat{u}) \left[\bar{u}_j \left(C^{L*}_{\widetilde{\chi}_j^0 \tilde{q}_B q} P_L + C^{R*}_{\widetilde{\chi}_j^0 \tilde{q}_B q} P_R\right) u_1\right] & \\
\times \left[\bar{v}_2 \left(C^R_{\widetilde{\chi}_i^0 \tilde{q}_B q} P_L + C^L_{\widetilde{\chi}_i^0 \tilde{q}_B q} P_R\right) v_i\right], & \qquad (4.22c)
\end{aligned}$$

where $D_p(q^2) = \frac{1}{q^2 - m_p^2 + i\Gamma_p m_p}$ is the Breit-Wigner propagator© of a particle with mass m_p and decay width Γ_p.

These matrix elements can be expanded using the *supercharges*

$$Z^X = C^X_{qqZ} O''^X_{ij}, \qquad (4.23a)$$

$$Q^{XY}_A = C^X_{\widetilde{\chi}_i^0 \tilde{q}_A q} \left(C^Y_{\widetilde{\chi}_j^0 \tilde{q}_A q}\right)^*, \qquad (4.23b)$$

and the Dirac bilinears

$$b_{L/R}(w_a, w_b) = \bar{w}_a P_{L/R} w_b, \qquad (4.24a)$$

$$b^\mu_{L/R}(w_a, w_b) = \bar{w}_a \gamma^\mu P_{L/R} w_b, \qquad (4.24b)$$

to arrive at

$$\begin{aligned}\mathcal{M}_{\hat{s}} = -\frac{g^2}{2} D_Z(\hat{s}) \Big[& Z^L b_L^\mu(u_i, v_j) b_{L_\mu}(v_2, u_1) - \left(Z^R\right)^* b_L^\mu(u_i, v_j) b_{R_\mu}(v_2, u_1) \\ & - \left(Z^L\right)^* b_R^\mu(u_i, v_j) b_{L_\mu}(v_2, u_1) + Z^R b_R^\mu(u_i, v_j) b_{R_\mu}(v_2, u_1) \Big]\end{aligned} \tag{4.25a}$$

$$\begin{aligned}\mathcal{M}_{\hat{t}} = -\sum_A 2g^2 D_{\tilde{q}_A}(\hat{t}) \Big[& \left(Q_A^{LR}\right)^* b_L(u_i, u_1) b_L(v_2, v_j) + \left(Q_A^{LL}\right)^* b_L(u_i, u_1) b_R(v_2, v_j) \\ & + \left(Q_A^{RR}\right)^* b_R(u_i, u_1) b_L(v_2, v_j) + \left(Q_A^{RL}\right)^* b_R(u_i, u_1) b_R(v_2, v_j) \Big]\end{aligned} \tag{4.25b}$$

$$\begin{aligned}\mathcal{M}_{\hat{u}} = -\sum_A 2g^2 D_{\tilde{q}_A}(\hat{u}) \Big[& Q_A^{RL} b_L(v_2, v_i) b_L(u_j, u_1) + Q_A^{RR} b_L(v_2, v_i) b_R(u_j, u_1) \\ & + Q_A^{LL} b_R(v_2, v_i) b_L(u_j, u_1) + Q_A^{LR} b_R(v_2, v_i) b_R(u_j, u_1) \Big].\end{aligned} \tag{4.25c}$$

To get the matrix element squared we can use that the complex conjugate of the Dirac bilinears is given by

$$\left(b_{L/R}(w_a, w_b)\right)^\dagger = b_{R/L}(w_b, w_a), \tag{4.26a}$$

$$\left(b^\mu_{L/R}(w_a, w_b)\right)^\dagger = b^\mu_{L/R}(w_b, w_a). \tag{4.26b}$$

Furthermore, when summing over the spins of the various spinors in the bilinears, they have the sum identities

$$\sum_{\text{spins}} b_X(w_a, w_b) b_Y(w_b, w_a) = 2\Big[(1 - \delta_{XY})(p_a \cdot p_b) + \operatorname{rsgn} \delta_{XY} m_a m_b\Big], \tag{4.27}$$

$$\begin{aligned}\sum_{\text{spins}} b_X^\mu(w_a, w_b) b_Y^\nu(w_b, w_a) = 2\Big[& \delta_{XY} \left(p_a^\mu p_b^\nu - g^{\mu\nu}(p_a \cdot p_b) + p_a^\nu p_b^\mu + (-1)^{\delta_{XY}} i\epsilon^{\mu\nu\alpha\beta}(p_a)_\alpha (p_b)_\beta\right) \\ & + (1 - \delta_{XY}) \operatorname{rsgn} m_a m_b g^{\mu\nu}\Big],\end{aligned} \tag{4.28}$$

where rsgn is 1 if w_a, w_b are spinors of the same type, e.g. both are u-spinors, and -1 otherwise.

Temporary

$$\begin{aligned}\mathcal{M}^\dagger_{\hat{s}} = -\frac{g^2}{2} D^*_Z(\hat{s}) \Big[& \left(Z^L\right)^* b_L^\mu(v_j, u_i) b_{L_\mu}(u_1, v_2) - Z^R b_L^\mu(v_j, u_i) b_{R_\mu}(u_1, v_2) \\ & - Z^L b_R^\mu(v_j, u_i) b_{L_\mu}(u_1, v_2) + \left(Z^R\right)^* b_R^\mu(v_j, u_i) b_{R_\mu}(u_1, v_2) \Big]\end{aligned} \tag{4.29a}$$

$$\begin{aligned}\mathcal{M}^\dagger_{\hat{t}} = -\sum_A 2g^2 D^*_{\tilde{q}_A}(\hat{t}) \Big[& Q_A^{RL} b_L(u_1, u_i) b_L(v_j, v_2) + Q_A^{RR} b_L(u_1, u_i) b_R(v_j, v_2) \\ & + Q_A^{LL} b_R(u_1, u_i) b_L(v_j, v_2) + Q_A^{LR} b_R(u_1, u_i) b_R(v_j, v_2) \Big]\end{aligned} \tag{4.29b}$$

$$\begin{aligned}\mathcal{M}^\dagger_{\hat{u}} = -\sum_B 2g^2 D^*_{\tilde{q}_B}(\hat{u}) \Big[& \left(Q_A^{LR}\right)^* b_L(v_i, v_2) b_L(u_1, u_j) + \left(Q_A^{LL}\right)^* b_L(v_i, v_2) b_R(u_1, u_j) \\ & + \left(Q_A^{RR}\right)^* b_R(v_i, v_2) b_L(u_1, u_j) + \left(Q_A^{RL}\right)^* b_R(v_i, v_2) b_R(u_1, u_j) \Big].\end{aligned} \tag{4.29c}$$

4.2.2 Differential Result

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\sigma}_0}{\mathrm{d}\hat{t}} =& \frac{\pi\alpha_W^2}{N_C\hat{s}^2}\left(\frac{1}{2}\right)^{\delta_{ij}}\left\{\sum_{X,Y}\left[\left|Q_{\hat{t}}^{XY}\right|^2\left(\hat{t}-m_i^2\right)\left(\hat{t}-m_j^2\right)+\left|Q_{\hat{u}}^{XY}\right|^2\left(\hat{u}-m_i^2\right)\left(\hat{u}-m_j^2\right)\right]\right. \\
&\left. -\sum_{X}\left[2\,\mathrm{Re}\left\{\left(Q_u^{XX}\right)^* Q_t^{XX}\right\}m_i m_j\hat{s}-2\,\mathrm{Re}\left\{\left(Q_u^{XX'}\right)^* Q_t^{XX'}\right\}\left(\hat{t}\hat{u}-m_i^2m_j^2\right)\right]\right\}
\end{aligned}
\tag{4.30}
$$

4.2.3 Phase Space Integral

To integrate over the variable $\hat{t}$, we can classify the types of integrals that will arise. All the integrals take the form

$$
T^p(\Delta_1,\Delta_2) \equiv \int_{t_-}^{t_+}\mathrm{d}\hat{t}\,\frac{\hat{t}^p}{(\hat{t}-\Delta_1)(\hat{t}-\Delta_2^*)} \tag{4.31}
$$

for some $\Delta_{1,2}$ dependent on $\hat{s}$, the neutralino masses and the squark masses and decay rates, and p is some integer.

TODO: Talk about complex logarithms and define log-difference function.
TODO: Maybe substitute in p.

Using the the integral limits are $t_\pm = \frac{\hat{s}-m_i^2-m_j^2}{2} \pm p\sqrt{\hat{s}}$, we get that the possible integrals evaluate to

$$
\begin{aligned}
T^2(0,0) =& 2p\sqrt{\hat{s}} && (4.32a)\\
T^3(0,0) =& -p\sqrt{\hat{s}}\left(\hat{s}-m_i^2-m_j^2\right) && (4.32b)\\
T^4(0,0) =& p\sqrt{\hat{s}}\left(\frac{8}{3}\hat{s}p^2+2m_i^2m_j^2\right) && (4.32c)\\
T^1\left(m^2,0\right) =& -L(m^2) && (4.32d)\\
T^2\left(m^2,0\right) =& 2p\sqrt{\hat{s}}-m^2L(m^2) && (4.32e)\\
T^3\left(m^2,0\right) =& -p\sqrt{\hat{s}}\left(\hat{s}-m_i^2-m_j^2\right)+2m^2p\sqrt{\hat{s}}-m^4L(m^2) && (4.32f)\\
T^0\left(m_1^2,m_2^2\right) =& \frac{1}{m_2^2-m_1^2}\left\{L(m_1^2)-L(m_2^2)\right\} && (4.32g)\\
T^1\left(m_1^2,m_2^2\right) =& \frac{1}{m_2^2-m_1^2}\left\{m_1^2L(m_1^2)-m_2^2L(m_2^2)\right\} && (4.32h)\\
T^2\left(m_1^2,m_2^2\right) =& 2p\sqrt{\hat{s}}+\frac{1}{m_2^2-m_1^2}\left\{m_1^4L(m_1^2)-m_2^4L(m_2^2)\right\}
\end{aligned}
$$

$L(m^2) = \log\frac{m^2+\frac{1}{2}(s-m_i^2-m_j^2)+p\sqrt{s}}{m^2+\frac{1}{2}(s-m_i^2-m_j^2)-p\sqrt{s}}$

In here, I have defined $\mathrm{dLog}(z,w)$ to be the log-difference between to complex number z,w such that

$$
\mathrm{dLog}(z,w) \equiv \ln\left|\frac{z}{w}\right| + i\left(\arctan\frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}} - \arctan\frac{\mathrm{Im}\{w\}}{\mathrm{Re}\{w\}}\right). \tag{4.33}
$$

The only non-zero arguments that will appear in these integrals are $\Delta_A^{\hat{t}} = m_{\tilde{q}_A}^2 - i\Gamma_{\tilde{q}_A}m_{\tilde{q}_A}$ and $\Delta_A^{\hat{u}} = m_i^2+m_j^2-\hat{s}-m_{\tilde{q}_A}^2+i\Gamma_{\tilde{q}_A}m_{\tilde{q}_A}$.

Temporary

$$\hat{\sigma}^0 = \frac{\pi\alpha^2}{\hat{s}^2 c_W^4 N_C} \left(\frac{1}{2}\right)^{\delta_{ij}} \tag{4.34}$$

$$\times \left\{ n \frac{2p\sqrt{\hat{s}} \left(\left|Z^L\right|^2 + \left|Z^R\right|^2 \right) \left(\frac{32}{3}\hat{s}p^2 - (m_i^2 - m_j^2)^2 + 8m_i^2 m_j^2 + \hat{s}(m_i^2 + m_j^2) \right)}{\left|\hat{s} - \Delta_Z\right|^2} \right. \tag{4.35}$$

$$- \frac{4p\sqrt{\hat{s}}\,\mathrm{Re}\left\{(Z^L)^2 + (Z^R)^2\right\} \hat{s} m_i m_j}{\left|s - \Delta_Z\right|^2} \tag{4.36}$$

$$\sum_{X,Y} 4\,\mathrm{Re}\left\{ \mathrm{dLog}(t_+ - \Delta_A, t_- - \Delta_A) \left[\frac{\delta_{XY}\,\mathrm{Re}\left\{Q_A^{XX} Q_B^{XX}\right\} \hat{s} m_i m_j}{\Delta_A + \Delta_B^* + \hat{s} - m_i^2 - m_j^2} \right.\right. \tag{4.37}$$

$$\frac{(1 - \delta_{XY})\,\mathrm{Re}\left\{Q_A^{XY} Q_B^{YX}\right\} \left(\Delta_A(\Delta_A - \hat{s} + m_i^2 + m_j^2) + m_i^2 m_j^2 \right)}{\Delta_A + \Delta_B^* + \hat{s} - m_i^2 - m_j^2} \tag{4.38}$$

$$\left.\left.\left. + \frac{\mathrm{Re}\left\{Q_A^{XY} \left(Q_B^{XY}\right)^*\right\} (\Delta_A - m_i^2)(\Delta_A - m_j^2)}{\Delta_A - \Delta_B^*} \right] \right\} \right\} \tag{4.39}$$

4.3 NLO Corrections

TODO:

- ☐ Describe factorisation and derive the factorised expression for the cross-section
- ☐ Discuss supersymmetry breaking in dimensional regularisation and its effect on the cross-section.

4.3.1 Factorisation

As we will only look at NLO contributions to the s-channel contribution through a Z-boson, we can do a trick to simplify the process and its corrections. This trick is factorisation, which involves splitting the total cross-section into the two separate processes of the production of an off-shell Z-boson, and its subsequent decay into two neutralinos. Seeing as we are calculating the inclusive cross-section, I include the potential emission of another particle (gluon or quark) along with the Z-boson production.

To start off, we can factorise the d-dimensional differential $2 \to 3$ phase space into two processes by adding an intermediate momentum q with 'mass' squared Q^2. We end up with

$$\begin{aligned}\mathrm{d}q\delta^d\left(k+q-P\right)\mathrm{d}Q^2\delta(q^2-Q^2)\mathrm{d}\Pi_{2\to 3} &= \frac{1}{(2\pi)^{2d-3}}\mathrm{d}^{d-1}p_i\,\mathrm{d}^{d-1}p_j\,\mathrm{d}^{d-1}k\,\mathrm{d}^{d-1}q\;\mathrm{d}Q^2\\ &\quad\times\frac{1}{16E_iE_j\omega q^0}\delta^d\left(q+k-k_i-k_j\right)\delta^d\left(p_i+p_j+k-k_i-k_j\right)\\ &\equiv\frac{1}{2\pi}\mathrm{d}\Pi_H\,\mathrm{d}\Pi_N\,\mathrm{d}Q^2,\end{aligned} \tag{4.40}$$

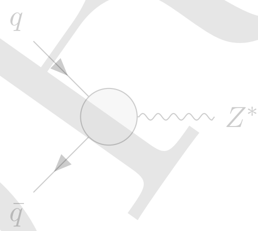
where

$$\mathrm{d}\Pi_H = \frac{\mathrm{d}^{d-1}k\,\mathrm{d}^{d-1}q}{(2\pi)^{d-2}}\frac{1}{4\omega q^0}\delta^d(q+k-k_i-k_j), \tag{4.41a}$$

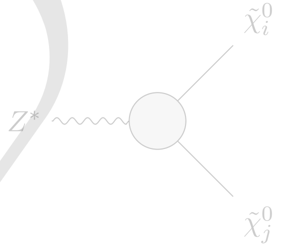
$$\mathrm{d}\Pi_N = \frac{\mathrm{d}^{d-1}p_i\,\mathrm{d}^{d-1}p_j}{(2\pi)^{d-2}}\frac{1}{4E_iE_j}\delta^d(p_i+p_j-q), \tag{4.41b}$$

which are recognisable as differential phase spaces for a $2\to 2$ processes going from momenta $k_i+k_j\to q+k$ and a $1\to 2$ phase space going from $q\ \to p_i+p_j$. The total phase space integrates over all possible off-shell masses Q^2 for the intermediate momentum q.

So, we have factorised the differential phase space of the differential cross-section Eq. (4.18), but it remains to factorise the amplitude part $|\mathcal{M}|^2$ into part only dependent on either q,k or p_i,p_j. Looking at the tree-level amplitudes Eqs. (4.22a) to (4.22c) that this happens neatly with the s-channel contribution Eq. (4.22a). It has the Lorentz structure $\mathcal{M}_s = D_Z(\hat{s})g_{\mu\nu}\left[\bar{v}(k_j)\Gamma^{\mu}_{Zqq}u(k_i)\right]\left[\bar{u}(p_i)\Gamma^{\nu}_{Z\tilde{\chi}^0_i\tilde{\chi}^0_j}v(p_j)\right]$. The two terms in brackets are individually only dependent on couplings and the momenta of either the initial partons or the final neutralinos. In fact, they individually take the form of the processes



$$= \left[\bar{v}(k_j)i\Gamma^{\mu}_{Zqq}u(k_i)\right]\epsilon^*_{\mu}(q) \tag{4.42a}$$



$$= \left[\bar{u}(p_i)i\Gamma^{\mu}_{Z\tilde{\chi}^0_i\tilde{\chi}^0_j}v(p_j)\right]\epsilon_{\mu}(q) \tag{4.42b}$$

Squaring it, we can write the differential cross-section as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2} = \frac{1}{4\pi\hat{s}}\left|D_Z(\hat{s})\right|^2 H^{\mu\nu}N_{\mu\nu}, \tag{4.43}$$

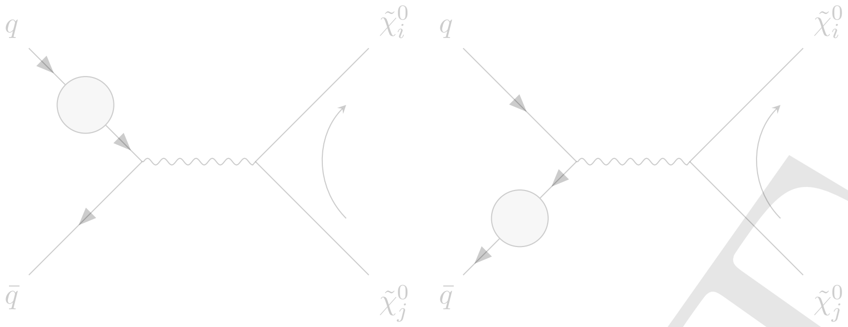


Figure 4.3

where

$$\epsilon_\mu(q)\epsilon_\nu^*(q)H^{\mu\nu} = \int \mathrm{d}\Pi_H \left|\mathcal{M}(q\bar{q} \to Z^*)\right|^2, \tag{4.44a}$$

$$\epsilon_\mu(q)\epsilon_\nu^*(q)N^{\mu\nu} = \int \mathrm{d}\Pi_N \left|\mathcal{M}(Z^* \to \tilde{\chi}_i^0\tilde{\chi}_j^0)\right|^2. \tag{4.44b}$$

4.3.2 Self-Energy Contributions

4.3.3 Vertex Corrections

4.3.4 Box Diagrams

4.3.5 Real Emission

Chapter 5

Proton—Proton Neutralino Pair Production

Bibliography

[1] S. P. Martin, *A Supersymmetry primer*, *Adv. Ser. Direct. High Energy Phys.* **18** (1998) 1–98, [hep-ph/9709356].