Asset Purchases in Noisy Financial Markets with Fiscal-Monetary Interactions*

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Abstract

We study how Asset Purchase (AP) policies affect the *real* price of defaultable nominal bonds, accounting for the effect on inflation through the central bank balance sheet. In the context of noisy financial markets where investors have position limits and private information on default probabilities, APs twist the distribution of equilibrium prices from which investors learn and effectively reduce real returns. In the absence of fiscal backing from the treasury, APs however create inflation through their effect on the real value of the central bank balance sheet. We study the social efficiency of AP policies in a stylized heterogeneous agents model, where lower bond returns and higher inflation have offsetting effects on aggregate consumption and welfare. We find that a positive but finite amount of APs optimally balances this trade-off, when we restrict to simple, uncontingent AP policies. We then show that policies that target a specific asset price can reduce interest rates while minimizing inflation pressures, even when the central bank lacks fiscal backing or has the same information as the market.

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1 Introduction

The most significant change in monetary policy over the last decade is the persistent use of large-scale asset purchases (APs), also known as quantitative easing (QE). Central banks introduced this unconventional policy to overcome the zero lower bound on overnight rates and reduce long-term yields. APs also played a macroprudential role, stabilizing sovereign bond markets and mitigating large and sudden shocks such as Covid-19. In the Eurozone, they were used to fight financial fragmentation and speculative attacks. Despite their practical importance, economists continue to debate the effectiveness of QE. As Ben Bernanke famously remarked, "QE works in practice but not in theory".[1]

This paper provides a general equilibrium analysis of the social efficiency of APs, and the channels through which they work, explicitly accounting for the noisy nature of the financial markets in which it operates. We consider a setting where investors only have noisy information on the fundamental value of an asset, and take advantage of the information aggregated by the price emerging upon trade; for example, a price increase may convey the information that either other investors have positive private news about fundamentals, or that the unobservable part of the asset net supply is small. AP policies may affect such inference, even when publicly announced, in that they twist the mapping between market prices and asset supply, conditional on fundamentals.

Our contribution is twofold. First, we study this mechanism in the context of a financial market where nominal defaultable debt is traded. We characterize the impact of APs on the *real* price of bonds, i.e. accounting for the general equilibrium implications that APs have on inflation, via the balance sheet of the central bank. Second, we provide insights on the optimal AP policy relying on a stylized model of fiscal-monetary interactions where APs reduce inefficiently high interest rates, at the cost of generating socially harmful inflation.

We consider a two-period model where the players are a government, a central bank, and households. The government issues nominal, defaultable bonds in the short run (first period) to finance its stochastic spending needs, and eventually repays such debt in the long run (second period) by raising non-distortionary taxes, whose real value depends on bond interest rates as well as inflation. Repayment is a stochastic event that follows an exogenous lottery. The central bank issues money, whose long-run real value depends on that of its invested assets. The proceeds

[1] See Bernanke (2012).

of money issuance can be invested in a safe asset[2] or in government bonds.[3] The central bank may or may not receive fiscal backing of its balance sheet by the government. In particular, central bank and government interact in a regime of *monetary dominance* when fiscal transfers are set such that the value of the central bank's balance sheet is kept constant; in a regime of *fiscal dominance* instead, such transfers are absent, and a default event may affect the value of the central bank balance sheet depending on its AP policy.

The private sector consists of households holding an endowment that can be consumed in the first period, or saved and consumed in the second period. Agents are born of two types, only differing in which asset they can use in order to save: *savers* can only invest in money, whereas *investors* can invest in either bonds or the safe asset. Both types need to decide how much to consume and save in the first period before they learn any information. Once savings are set, investors receive private information on the default lottery, and decide how to allocate their portfolio between bonds and safe assets within a Bayesian trading game. In the trading game, investors face bounds on their short positions, and learn from the market-clearing bond price. Such price aggregates information about everyone's private signals, but also depends on some unobserved, stochastic supply. Investors thus face an inference problem, as they cannot tell apart whether, for example, the bond price is high because of low supply or high demand.

In the modelling of financial markets, we adopt a framework closely related to Hellwig et al. (2006), Albagli et al. (2021) and the sovereign debt application of Bassetto and Galli (2019), where the assumption of investors' risk neutrality and position limits allows considering nonlinear asset payoffs (such as that of defaultable debt). This class of models features an extensive margin mechanism, where the equilibrium price depends on the beliefs of the marginal agent, that is, the agent who is indifferent between buying government debt or investing in the alternative assets.[4]

If either heterogeneous information or position bounds are absent from the investors' portfolio allocation problem, we get a neutrality result as in Wallace (1981): there is no difference between money and bonds, the distinction between agent types is immaterial, and APs have no effect on bond prices, inflation or welfare. When instead both frictions are present, we show that APs have pervasive effects on bond prices, the information contained therein, and anything linked to such prices, specifically the central bank balance sheet, inflation, the consumption-saving decisions of

[2] This can be interpreted as a perfectly diversified portfolio, or a storage technology.

[3] Although we do not explicitly model the possibility that the central bank buys private assets, nothing prevents us from interpreting the government as the consolidated public sector.

[4] This contrasts with the key mechanism in the vast CARA-Normal noisy REE literature, where equilibrium prices depend on the risk premium priced by risk-averse investors that solve a risk-return trade-off problem. See for example Iovino and Sergeyev (2023) for an application of this framework to APs.

households, and welfare. We now briefly discuss these effects.

First, APs affect bond prices, and their mapping with fundamentals and supply shocks. When conducting APs, the central bank buys at the market price, crowding out the bond demand of a specific part of the investor distribution, that is, the least optimistic investors among those that would otherwise buy the bonds. This increases the market probability of repayment, by selecting a more optimistic investor as the marginal agent that is pricing the bond, which results in a lower bond interest rate (higher price).

Second, APs increase the precision of the information revealed by financial market prices. We show that this takes the form of a truncation in the distribution of investors' posterior beliefs on the fundamental. This result, for which we find general conditions, extends the analysis of noisy information aggregation in financial market by allowing for truncated belief distributions, being a theoretical contribution per se. This belief truncation is typically asymmetric, allowing investors to better detect default states. Intuitively, when APs are large and investors observe a high bond price, they cannot tell if that is because the government is indeed solvent, or prices are just inflated by central bank purchases. When instead APs are large and bond prices are low, investors infer that the government must be close to a default, since the price remained low even after the central bank intervened. This implies that APs render some prices fully informative of the underlying fundamental, thus eliminating all residual uncertainty in the corresponding states.

Third, APs reduce the ex ante expected value of the profits (excess returns of bonds over the safe asset) that investors make when saving and participating in the bond market. With dispersed information, agents expect to make positive profits, because they anticipate facing a call option in the bond market: if they receive a good private signal, they take on default risk and buy bonds, if not, they just save in the safe asset. This is a source of inefficiency, because it induces investors to save too much and consume too little in the first period. APs reduce these expected profits, stimulate consumption and increase welfare by reducing bond returns and revealing information in states where investors earn the most.

Fourth, in the presence of fiscal dominance, APs introduce correlation between the returns of bond and money. The long-run real value of central bank liabilities (i.e. money) depends on the real value of its investments. When the central bank invests in bonds, a default event implies a balance sheet loss that depresses the long-run real value of money and generates inflation; on the contrary, when government debt is repaid, the central bank makes a profit that generates long-run deflationary pressures. This simple result sheds light on the empirical observation that APs do not necessarily lead to inflationary pressure; on the contrary, deflation is an outcome that is perfectly consistent with the central bank investing in assets that increase the value of its liabilities. Under a simple "uncontingent" AP rule where the central bank always buys a fixed quantity of bonds, APs generate expected central bank losses and inflation. In our setting, inflation is costly for savers because it depresses the rate of return on their investment, money, below its efficient level. Hence, a trade-off emerges: on the one hand, APs reduce inefficiently high return for investors and increase their consumption; on the other hand, APs increase inflation, make the rate of return on money inefficiently low, and reduce savers' consumption. We show that the optimal uncontingent AP policy is to buy a positive but finite amount of bonds, trading off the welfare gains for investors with the losses for savers.

Finally, we study a more sophisticated yet tractable class of AP policies that target a specific bond interest rate. These policies are implemented as limit orders by the central bank to buy up to a certain quantity of bonds if the price is weakly smaller than a given target. Importantly, this class of polices does not require the central bank to know the fundamental shocks in the economy, or to have information that is superior to that of investors. We show that price-targeting policies have two important features: first, they are "beliefs-neutral", in the sense that they do not distort the information contained in the price, and actually correct a wedge that derives from the presence of information frictions; second, they are "budget-neutral", because we set a price target such that APs result in zero expected profits or losses for the central bank. This implies that, with this class of policies, the central bank can reduce interest rates and increase investors' consumption, while minimizing the drawback of creating costly inflation for savers.

Related Literature. Since Wallace (1981), the irrelevance of open market operations has been a benchmark theoretical result in rational expectation macroeconomic models. It states that, taking fiscal policy as given, any purchase of assets by public authorities is allocation-neutral insofar as taxes adjust to offset any gain or loss in public budgets. Thus, the composition of public liabilities does not matter, similarly to what Modigliani and Miller (1958) show for corporate liabilities.

Wallace's irrelevance result crucially obtains under complete information and frictionless financial markets. A literature questioning the complete information assumption focused on the role of APs to serve as a signal about uncertain central banks' objectives and fundamentals (see Mussa (1981)) or as a commitment device to future accommodative stance (Jeanne and Svensson (2007), Christensen and Rudebusch (2012) and Bhattarai et al. (2022)). Recent work by Iovino and Sergeyev (2023) focuses on the lack of rational expectations as a source of non-neutrality of APs in an otherwise frictionless model. A larger stream of literature has emphasized the importance of market segmentation for the workings of AP policies, in the vein of seminal papers like

Cúrdia and Woodford (2011) and Gertler and Karadi (2015).[5] Some papers have emphasized the role of asset purchases in incomplete markets economies with structurally heterogeneous agents in alleviating a lack of risk sharing or insurance on the side of firms or households.[6]

To the best of our knowledge, this is the first paper showing the role of dispersed information in economies where structurally homogeneous investors take bounded positions. In particular, we show that absent one of these two frictions – private uncertainty or bounded asset demand – the neutrality benchmark obtains. In our model, positions bounds prevent private demand to perfectly offset the heterogeneous crowding-out effect of APs, which has effects on the asset price and the information private agents extract from it.

Such absence of perfect private arbitrage echoes the assumption of market segmentation that is common in the finance literature on asset purchases. Market segmentation is essential for APs to induce "portfolio balance" effects, i.e. a relative price change across asset classes and maturities. These effects have been measured since the great recession of 2008-2009 in a flourishing empirical literature.[7] Some works support the view that asset purchases have mostly a "local" effect limited to the specific market targeted by the program.[8] Others have identified sizeable "global" portfolio rebalancing effects of asset purchases that pervade financial markets beyond those that are targeted directly by the program.[9]

On the theoretical front, the literature has developed models to account for the "local" vs "global" effects of asset purchases on financial markets, building on the seminal paper by Vayanos and Vila (2021) (see for example Hamilton and Wu (2012), Greenwood and Vayanos (2014), King (2013) and King (2019)). All these papers focus on the financial market impact of APs, abstracting from their general equilibrium implications on inflation and macroeconomic risks. An applied macro literature found robust evidence for expansionary general equilibrium effects of asset purchases.[10]

[5]See for example Chen et al. (2012), Del Negro et al. (2017), Wen (2014), Campbell et al. (2012), Harrison (2017) and Sims and Wu (2021).

[6]See for example Gornemann et al. (2016), Auclert (2019), Luetticke (2018), Ravn and Sterk (2021), Kaplan et al. (2018), Debortoli and Galí (2017), Hagedorn et al. (2019) and Cui and Sterk (2021).

[7]See Gagnon et al. (2011) for the US; Joyce et al. (2012) and Breedon et al. (2012) for the UK and more recently Altavilla et al. (2021) for the Eurozone, among others.

[8]See Krishnamurthy and Vissing-Jorgensen (2011), D'Amico and King (2013), and McLaren et al. (2014) on Fed LSAPs; Eser and Schwaab (2016) for the ECB Securities Markets Programme (SMP); Altavilla et al. (2016) for the ECB Outright Monetary Transactions (OMT); Krishnamurthy et al. (2017) for both SMP and OMT.

[9]E.g. Cahill et al. (2013), Li and Wei (2013), Gilchrist et al. (2015) and Rogers et al. (2018).

[10]See Bhattarai and Neely (2016) and Kim et al. (2020) for a survey of the literature.

2 Model

There are two periods $t \in \{1, 2\}$. In period 1 a continuum of agents of mass two chooses how much endowment to allocate for current consumption and how much to save in a financial asset. Agents differ in the saving assets to which they have access. *Savers* only save in money, *Investors* can choose between a safe asset and a defaultable nominal bond issued by the government, yielding a return in the third period. Once the consumption-saving choice had been made, investors receive a private signals on the likelihood of default. In period 2 agents consume.

The public sector consists in a government and a central bank. The central bank is financed via a uncontingent lump-sum tax in period 0. The government needs to consume in period 1, but it issues bonds as it cannot raise taxes in period 1. The central bank can eventually use resources to buy government bonds at market prices. In period 2, the default occurs as the outcome of an exogenous lottery, the government raises lump-sum taxes to repay the debt and rebate central banks profits to investors.

2.1 The Government and the Central Bank

The government issues a quantity of defaultable nominal bonds b to finance the realization of stochastic and wasteful public consumption needs in the first period, $\widetilde{S}$, which follows a Uniform$[0, 1]$ distribution. A unit of bonds is a promise by the government to pay R units of money in the second period, in exchange for one unit of money in the first period. For notation convenience we assume the price of consumption in the first period as a numeraire, $P_1 = 1$, so that the nominal amount of bonds issued by the government is equal to its real needs $\widetilde{S}$. The real return of bonds depends on the occurrence of default and inflation in the second period. Default is an exogenous event occurring stochastically according to the following lottery:

$$\theta = \begin{cases} \theta_H = 1 & \text{with probability } \; q, \\ \theta_L \in (0, 1) & \text{with probability } \; 1 - q. \end{cases}$$

where θ denotes the fraction of debt effectively repaid by the government. Inflation $\Pi := P_2/P_1$ occurs when there is variation in the price of consumption between the first and the second period. Finally, the government raises real resources by collecting lump-sum taxes T. The budget set of the government in periods 1 and 2 are respectively given by

$$\begin{aligned} b &= \widetilde{S} \\ T &= \frac{R\,\theta}{\Pi}\,\widetilde{S} + \tau \end{aligned} \tag{1}$$

where τ represents eventual transfers from the government to the central bank. The central bank has an initial endowment e_{cb} and issues a stock of money m to buy a nominal quantity of government bonds b_{cb} or invest in a safe asset s_{cb}, so that

$$e_{cb} + m = b_{cb} + s_{cb} \tag{2}$$

is the budget constraint of the central bank in the first period. In the second period, the central bank collects bond and safe asset revenues, reimburses money and transfers the endowment to the government. The budget constraint of the central bank in the second period is therefore:

$$\frac{R\,\theta}{\Pi}\,b_{cb} + s_{cb} + \tau = \frac{m}{\Pi}. \tag{3}$$

where, without loss of generality, we normalize the real return of the safe asset to one.[11] The transfer τ is critical for the determination of monetary fiscal interactions. We consider the following generic fiscal rule:

$$\tau = \left(1 - \frac{R\,\theta}{\Pi}\right)\kappa\,b_{cb} - e_{cb}, \tag{4}$$

where $\kappa \in [0, 1]$ describes the degree of fiscal backing by the government. Hence, the rate of return on money obtains as

$$\frac{1}{\Pi} = \left((1-\kappa)\,\frac{R\,\theta}{\Pi} + \kappa\right)\alpha + (1-\alpha), \tag{5}$$

with $\alpha := \frac{b_{cb}}{m}$ represents the fraction of money stock invested in asset purchases. We will refer to $\hat{\alpha} := (1-\kappa)\alpha$ as the degree of fiscal dominance: when $\hat{\alpha} = 0$, inflation is equal to 1 irrespective of the default realization; as $\hat{\alpha}$ increases, the correlation between the real ex post return on money (the inverse of inflation) and bonds increases. In other words, the rate of the return on money $1/\Pi$ is a weighted average, with weight $\hat{\alpha}$, of the rate of return on bonds $R\,\theta/\Pi$ and on the safe asset 1.

[11] Note that the inclusion of an initial endowment of the central bank and the reimbursement of money are artefacts of the finite nature of time considered here.

In particular, we denote the regime where $\kappa = 1$ as *monetary dominance*. In this regime, the budget constraint of the central bank holds with $\Pi = 1$ for any values of (θ, R, m, b_{cb}). This implies that the value of money is stable irrespective of fiscal variables, since losses or gains from central bank APs are perfectly offset by transfers to or from the fiscal authority. When instead $\kappa = 0$, we will say that taxes are determined under the *fiscal dominance* regime: transfers to the central bank do not vary with profits (or losses) from APs, and inflation Π must adjust for (3) to hold. The value of money must thus fluctuate with profits and losses from APs, inflating or deflating the value of central bank liabilities (money), in order to balance the budget. Given $\kappa = 0$, the higher the α, the higher the share of the central bank assets that are invested in bonds, the closer the real return of money is to the one on bonds.

2.2 Households: Savers and Investors

A household, denoted by j, has a quasi-linear utility function $U_j = u(c_{j,1}) + c_{j,2}$. First-period utility $u(c) := \frac{c^{1-\gamma}}{1-\gamma}$ is concave in first-period consumption $c_{j,1}$, with $\gamma > 0, \gamma \neq 1$ measuring the degree of concavity of consumption utility in the first period, and linear in second-period consumption $c_{j,2}$.

Each agent has a productive endowment $e_j = 1 + c^*$. Agents' problem consists in deciding in the first period how much $\bar{b}_j \in [0, e_j]$ to save for second-period consumption. In doing so, they anticipate that the per-unit return on saving $\mathcal{R}_j(\bar{b}_j)$ will depend on portfolio choice b_j, that is made at a later stage within the first period, after receiving pay-off relevant information Ω_j. Formally, agent j solves

$$\max_{\bar{b}_j \in [0, e_j]} \mathbb{E}\left[\frac{(e_j - \bar{b}_j)^{1-\gamma}}{1-\gamma} + \max_{b_j} \mathbb{E}\left[\mathcal{R}_j(b_j) |\, \Omega_j\,\right] \bar{b}_j - \frac{T}{2}\right], \tag{6}$$

where T is a lump-sum tax. It follows that the amount of savings $\bar{b}_j$ that an agent j decides to hold is given by

$$(e_j - \bar{b}_j)^{-\gamma} = E\left[\max_{b_j \in [\,\underline{b}, \bar{b}_j]} \mathbb{E}\left[\mathcal{R}_j(b_j) \,|\, \Omega_j\right]\right], \tag{7}$$

i.e., the marginal utility in consumption must equate the expected real return on saving. We fix $c^* = 1$, so that $\bar{b}_j = 1$ is the amount of saving equating marginal utility to the inverse of the discount factor, assumed equal to one for simplicity. The optimality condition on the saving choice, entails the traditional intertemporal substitution motive at the core of workhorse models of the business cycle: an expected real return above (resp. below) the natural level 1 induces a negative (resp. a positive) gap of current consumption $c_{1,j}$ with the natural level c^*.

Let us now describe the portfolio choice. A structural heterogeneity exists in that agents differ in the type of financial assets they have access to. We distinguish between two types of agents, each of unitary mass. We refer to the first type as *savers*, denoted by $j = s \in [0,1]$. These agents can only save in the form of money, i.e. $b_s = \bar{b}_s$, which means

$$\mathcal{R}_s(b_s) = \frac{1}{\Pi}\,, \tag{8}$$

i.e. what savers do not use for consumption in the first period can only be saved in money, yielding a real return equal to the inverse of inflation. Moreover, savers do not have additional information other than their prior beliefs, so $\Omega_s = \emptyset$.

The second type of agents, denoted by $j = i \in [0,1]$, are *investors*, who do have access to the financial market. In the financial market, defaultable debt competes with a safe asset (e.g., a fully diversified portfolio) available in infinitely elastic supply, and with a risk-free real rate of return of unity. Given their choice $\bar{b}_i$, each investor i chooses a quantity of government bond purchases, denoted by $b_i \in [-\underline{b}, \bar{b}_i]$, in order to maximize her expected per-unit rate of return:

$$\mathcal{R}_i(b_i) := Q(b_i) = \frac{b_i}{\bar{b}_i}\,\frac{R\,\theta}{\Pi} + \left(1 - \frac{b_i}{\bar{b}_i}\right), \tag{9}$$

where $\underline{b}$ and $\bar{b}_i$ denote bounds on investors' short and long asset positions respectively. Note that, while the long position bound $\bar{b}_i$ is an endogenous object, the short position bound $\underline{b}$ is an exogenous parameter. Because of risk neutrality, agent i with information set Ω_i chooses

$$b_i = \begin{cases} \bar{b}_i & \text{if and only if} \quad \mathbb{E}\left[\,\frac{R\theta}{\Pi}\,|\,\Omega_i\,\right] > 1, \\ -\underline{b} & \text{if and only if} \quad \mathbb{E}\left[\,\frac{R\theta}{\Pi}\,|\,\Omega_i\,\right] < 1, \end{cases} \tag{10}$$

and she is indifferent otherwise. Investors submit demand schedules contingent on the market clearing interest rate R (i.e., the inverse of the bond price) and other available information summarised by Ω_i, to be specified later.

The market clearing condition for bonds is

$$\int_0^1 b_i \,\mathrm{d}i + b_{cb} = \widetilde{S}, \tag{11}$$

which states that investors' bond demand must equal gross bond supply by the government net of central bank asset purchases. This market clearing condition determines the equilibrium interest rate R, that will generally depend on the repayment state θ, gross supply $\widetilde{S}$, asset purchases b_{cb}, and position bounds $\underline{b}, \overline{b}_i$. We define $\mathbf{b} := \{b_{cb}, \overline{b}_i\}$ as the variable collecting central bank APs and investors' savings or long position bounds.

2.3 Equilibrium

We are now ready to give a general definition of an equilibrium in this economy.

Definition 1. *An equilibrium consists of central bank policy (m, s_{cb}, b_{cb}), government policy (b, τ, T), allocations for savers $(c_{s,1}, c_{2,s}, \overline{b}_s)$ and investors $(c_{i,1}, c_{i,2}, \overline{b}_i, b_i)$, investors' information set Ω_i, investors' bidding strategy $b_i(\Omega_i)$ and posterior beliefs $p_i(\Omega_i)$, bond price function $R(x_m, \mathbf{b})$ and inflation function $\Pi(\theta, \widetilde{S}, b_{cb}/m)$, such that*

1. *savers' and investors' allocations solve their respective $t = 1$ problem, given the unconditional distributions of $Q(b_i)$ and Π.*
2. *investors' bidding strategies b_i are optimal given posterior beliefs p_i*
3. *investors' posterior beliefs p_i satisfy Bayes' law,*
4. *all markets clear (money, bonds, goods),*
5. *the government and central bank budget constraints are satisfied.*

2.4 First Best and Pecuniary Externalities

Social welfare in the economy is equal to the sum of utilities of agents and government, i.e. is equal to total utility from consumption. Using (1)-(11), aggregate welfare is given by

$$\mathcal{W} := \mathbb{E}[U_j] = u(e_s - m) + u(e_i - \overline{b}_i) + m + \overline{b}_i + e_{cb}, \tag{12}$$

which means that choices in the financial market are irrelevant for total consumption, as the market for bonds accounts only for a liquidity friction on the side of the government and do not generate any new production. Nevertheless, the safe asset provides a lower bound to the rate of return in the economy. The first best allocation, entailing maximum welfare $\mathcal{W}^*$, is the one characterized by $\mathcal{W}^* = 2\, u(c^*) + 2 + e_{cb}$, which obtains when $\overline{b}_i = 1$ and $Q(b_i) = 1$.

The market solution, however, may not necessarily implement the first best allocation, for two key reasons. First, investors do not internalize that taxes in the second period vary with the market price, which ultimately depends on investors' actions as a whole. Second, given taxes, there are strictly positive gains to be done in the financial market exploiting private information since the presence of bounds to asset positions prevents investors' full arbitrage. In fact, it is worth noting that the market solution implements the first best under complete or homogeneous information. In case of complete information, $R\,\theta = 1$ at any $(\theta, \widetilde{S}, \mathbf{b})$ state and as a consequence $Q(b_i) = 1$ for any b_i. However, complete information also necessarily occurs with homogeneous information sets. Absent information frictions, investors are indifferent in their portfolio choices, and bond demand is uniform and such that $b_i = \widetilde{S} - b_{cb}$ for any i. Since b_{cb} is observable, we conclude that $\widetilde{S}$ would be revealed and θ inferred by observing the prevailing equilibrium price R and bank's purchases.

3 Equilibrium in the Financial Market

3.1 Equilibrium Price Given Marginal Investor Beliefs

In this section, we derive a generic characterisation of the equilibrium bond price in the financial market, as a function of the beliefs of the *marginal investor*, which represent a sufficient statistic for a given distribution of investors' posterior beliefs. This is useful because the mapping between the marginal investor's beliefs and the equilibrium in both the financial market and the macroeconomic model are independent of the way in which such beliefs are formed. In the Section 4 we instead make specific assumptions on the AP rule and the information structure, and derive the *rational expectation equilibrium (REE) distribution* of investors' beliefs.

Let us denote the repayment probability held by investor i by $p_i := Prob(\theta = \theta_H \,|\, \Omega_i) \in [0, 1]$, which is distributed according to a generic distribution G. It follows that the expected value of the fundamental θ for investor i is given by $\mathbb{E}[\theta \,|\, \Omega_i] = \theta_L + p_i(\theta_H - \theta_L)$, which is strictly increasing in p_i. Using equation (5) and provided that $R\,\theta < 1/\alpha$, we get that the real bond payoff is

$$\frac{\theta}{\Pi} = \frac{1 - \hat{\alpha}}{\frac{1}{\theta} - \hat{\alpha} R}, \tag{13}$$

where $\hat{\alpha} = (1 - \kappa)\alpha$, so that for $\kappa = 1$ or $\alpha = 0$ it is easy to verify that $\Pi = 1$. It is also easy to check that monotonicity in beliefs on repayment maps into monotonicity in beliefs about ex-post real returns for a given R, that is:

$$p_i \geq p_j \quad \Leftrightarrow \quad \mathbb{E}\left[\theta \,|\, \Omega_i\right] \geq \mathbb{E}\left[\theta \,|\, \Omega_i\right] \quad \Leftrightarrow \quad R\,\mathbb{E}\left[\frac{\theta}{\Pi} \,|\, \Omega_i\right] \geq R\,\mathbb{E}\left[\frac{\theta}{\Pi} \,|\, \Omega_i\right], \tag{14}$$

for any degree of fiscal dominance $\hat{\alpha} \in [0,1)$ and pair of agents $(i,j) \in [0,1]^2$. We introduce then the following definition.[12]

Definition 2 (Marginal agent and market clearing price)**.** *For a given net supply, $\widetilde{S} - b_{cb}$, the marginal agent $m \in [0,1]$ is the agent who holds posterior beliefs $p_m \in [0,1]$ such that the market clears with*

$$\bar{b}\,(1 - G(p_m)) - \underline{b}\,G(p_m) = \widetilde{S} - b_{cb} \tag{15}$$

where $G(p_m)$ is the mass of investors who are more pessimistic than the marginal investor, and sell bonds short on the market. The market clearing price R is then determined by the marginal agent's beliefs according to

$$R\,\mathbb{E}\left[\frac{\theta}{\Pi}\,\Big|\,\Omega_m\right] = R\left[p_m\,\theta_H \frac{1-\hat{\alpha}}{1-\hat{\alpha}R\theta_H} + (1-p_m)\,\theta_L \frac{1-\hat{\alpha}}{1-\hat{\alpha}R\theta_L}\right] = 1, \tag{16}$$

as the price that makes the marginal investor indifferent between buying or short selling.

Proposition 1 (Equilibrium price function)**.** *Provided $R\theta < 1/\alpha$ for any $\theta \in \{\theta_L, \theta_H\}$, then, for a given belief of the marginal agent p_m, the equilibrium interest rate $R : p_m \to [\theta_H^{-1}, \theta_L^{-1}]$ is given by*

$$R\,(p_m) = \frac{(1-\hat{\alpha})\mathbb{E}[\theta \,|\, \Omega_m] + (\theta_H + \theta_L)\hat{\alpha} - \sqrt{\left((1-\hat{\alpha})\mathbb{E}[\theta \,|\, \Omega_m] + \hat{\alpha}(\theta_H + \theta_L)\right)^2 - 4\hat{\alpha}\theta_H\theta_L}}{2\hat{\alpha}\theta_H\theta_L}. \tag{17}$$

The equilibrium price function $R(p_m)$ has the following properties:

i. it is monotonically decreasing in the posterior of the marginal agent p_m with:

$$\lim_{p_m \to 0} R(p_m) = \frac{1}{\theta_L} > \lim_{p_m \to 1} R(p_m) = \frac{1}{\theta_H}, \quad \textit{with} \quad \frac{\partial R(p_m)}{\partial p_m} < 0;$$

ii. it is monotonically decreasing in the degree of fiscal dominance $\hat{\alpha}$ with:

$$\lim_{\hat{\alpha} \to 0} R(p_m) = \frac{1}{\mathbb{E}[\theta|\Omega_m]}, \quad \lim_{\hat{\alpha} \to 0} \frac{\partial R(p_m)}{\partial \hat{\alpha}} = 0, \quad \frac{\partial R(p_m)}{\partial \hat{\alpha}}\Big|_{\alpha \neq 0} < 0.$$

[12] In all of this section focusing on the financial market, we omit the subscript i in investors' total savings (and long position bounds) $\bar{b}_i$ to lighten up notation.

Proof. Postponed to Proof 1 in the appendix. ■

The proposition characterises the one-to-one mapping $R : p_m \rightarrow [\theta_H^{-1}, \theta_L^{-1}]$ between the beliefs of the marginal agent and the equilibrium price or interest rate. The first part states that the equilibrium nominal return is equal to its minimum $1/\theta_H$ when the marginal agent believes repayment occurs with certainty, and it increases as the marginal agent considers default more likely, until it reaches its maximum $1/\theta_L$, where the marginal agent believes default occurs with certainty. Intuitively, a higher probability of repayment by the marginal agent maps into a lower equilibrium return because a smaller remuneration is sufficient to clear the market when the distribution of beliefs in the population is more optimistic.

The second part of the proposition states that a higher degree of fiscal dominance $\hat{\alpha}$ decreases the equilibrium nominal interest rate. To see why, recall first that fiscal dominance makes inflation comove with the fundamental: in repayment states, central bank profits from APs generate deflation, which increases real bond returns; in default states instead, central bank losses generate inflation, which reduces real bond returns. In the appendix we show that the former effect is always stronger than the latter, so R increases with $\hat{\alpha}$ for any marginal agent belief p_m.

3.2 Information

3.2.1 Exogenous Information: APs and Private Signals

Uncontingent AP rule. Since the mass of buyers must belong to the $[0, 1]$ interval, bonds purchased by the central bank must be such that net supply is non-negative. Accounting for this constraint, we assume that asset purchases follow the rule

$$b_{cb} = \min\left\{\widetilde{S} + \underline{b}, \overline{b}_{cb}\right\} \quad \text{and} \quad \overline{b}_{cb} \in [0, 1 + \underline{b}] \tag{18}$$

for any $(\theta, \widetilde{S})$. That is, the central bank commits to buying $\overline{b}_{cb}$ at the prevailing market price, whenever gross bond supply $(\widetilde{S}+\underline{b})$ is high enough to allow it, or to buy the whole gross supply at the risk-free interest rate $1/\theta_H$,[13] when such supply is below $\overline{b}_{cb}$. With a little abuse of language, we refer to this policy as *uncontingent* because purchases depend on $\widetilde{S}$ only for feasibility reasons, since we do not allow for negative asset purchases. As we discuss later, this is a conservative assumption in evaluating central bank losses. We denote with $P_0 := \overline{b}_{cb} - \underline{b}$ the probability that

[13] This is the price at which no investor is willing to buy.

the central bank purchases the whole bond supply, that is, no supply is available to buyers in the market. We refer to this scenario by saying that the market is *passive*. It follows that with probability $1 - P_0$ there is a non-zero mass of investors buying bonds in the market, in which case we say the market is *active*.

It is important to note that, to carry out APs according to rule (18), the central bank does not need to know gross bond supply $\widetilde{S}$. We assume that the central bank submits a limit order in the bond market to buy up to $\bar{b}_{cb}$ bonds at an interest rate weakly larger than $1/\theta_H$. If bond supply is below $\bar{b}_{cb}$, the central bank buys all the supply at its reservation price. If instead bond supply is above $\bar{b}_{cb}$, then the central bank buys $\bar{b}_{cb}$ at the prevailing market interest rate. We think that this is a realistic feature of our setting: if the central bank did observe $\widetilde{S}$ (or θ, for that matter), it could reveal its value directly, which would allow investors to infer the value of the fundamental, thus resolving all uncertainty and achieving the efficient level of savings and consumption.

Private Information on Default. In period 1 bond investors do not observe the realization of θ, but they may receive information about it: we denote the information set of investor i in stage 1 with Ω_i. We assume that each agent has a private noisy signal on θ given by

$$x_i = \theta + \sigma_x \, \xi_i, \tag{19}$$

where $\xi_i \sim N(0,1)$ for each i are mutually orthogonal white noise shocks. We denote the unconditional distribution of x_i with $\mathcal{N}_x$.[14]

3.2.2 Endogenous Information: the Price Signal

The Distribution of the Marginal Investor. We can now derive explicitly how the mass of investors taking long and short bond positions is determined in equation (15). First, whenever the market price does not fully reveal the value of θ, posterior beliefs are increasing in the private signal x_i in the sense of first-order stochastic dominance. Second, investors' expected payoff are an increasing function of beliefs, as shown in (14). This implies that agents follow monotone

[14] The unconditional distribution of x_i is given by

$$f_{x_i}(x) = \sum_{j \in \{L,H\}} q_j f_{x_i}(x|\theta_j) = \frac{1}{\sigma_x} \left[q \, \phi\left(\frac{\theta_H - x}{\sigma_x}\right) + (1-q)\, \phi\left(\frac{\theta_L - x}{\sigma_x}\right) \right]$$

where $q_H = q$ and $q_L = 1 - q$.

threshold strategies, and we can rewrite equation (10) as

$$b_i(x_m) = \begin{cases} \bar{b} & \text{if} \quad x_i \geq x_m, \\ -\underline{b} & \text{if} \quad x_i < x_m \end{cases} \tag{20}$$

where $b_i(x_m)$ is the bond position taken by agent i when the private signal threshold is x_m, which is endogenous to the equilibrium. We assume that a law of large numbers across investors applies as in Judd (1985): for a given value of the fundamental, the mass of investors buying bonds is given by the share of agents with a private signal larger than x_m, that is

$$1 - G(p_m) = \text{Prob}(x_i \geq x_m \,|\, \theta) = \Phi\left(\frac{\theta - x_m}{\sigma_x}\right),$$

where Φ denotes the standard normal cumulative distribution function. The cutoff private signal x_m identifies the marginal agent on the market, i.e., the investor whose private signal is such that she is indifferent between buying the bonds or investing in the safe asset. Rearranging equation (15) we can express the private signal of the marginal agent as

$$x_m(\theta, S) = \theta + \sigma_x \Phi^{-1}(1 - S). \tag{21}$$

where

$$S(\widetilde{S}, \mathbf{b}) := \frac{\widetilde{S} + \underline{b} - b_{cb}}{\bar{b} + \underline{b}} \sim \text{Uniform}\,[S_{min},\, S_{max}] \tag{22}$$

defines *net bond supply per individual exposure*, or simply *net supply*, as the supply available to each buyer, in units of individual exposure $\Delta := (\bar{b} + \underline{b})^{-1}$, given position bounds and APs.

Equation (21) states that the marginal agent's private signal x_m must be equal, in equilibrium, to a function that is linear in the fundamental shock θ, and nonlinear in the gross supply shock $\widetilde{S}$, the central bank's AP policy, and the bond position bounds. Henceforth, we will focus on equilibria where x_m and R convey the same information, in which case conditioning beliefs on the marginal agent signal x_m is equivalent to conditioning them on the endogenous price R. We thus refer to x_m as the price, or market, signal, which is a public signal that is endogenous to the equilibrium.

We use $\mathcal{M}(\mathbf{b})$ to denote the unconditional distribution of x_m. Note that this distribution is not necessarily the same as the private signal distribution $\mathcal{N}_x$, because of market clearing restrictions. That is, the support of net supply S depends on the central bank AP policy, as well as the bond position bounds. In particular, $\mathcal{M}(\mathbf{b}) = \mathcal{N}_x$ if and only if $[-\underline{b}, \bar{b}] = [0, 1]$ and

$b_{cb} = 0$ for any $(\theta, \widetilde{S})$.[15] More generally, the support of the marginal investor distribution ranges from $S_{min} = (\underline{b} - b_{cb})\,\Delta > 0$ to $S_{max} = (1 + \underline{b} - b_{cb})\,\Delta < 1$. This implies that the support of x_m conditional on θ may differ from the support of x_i, and ranges from $x_{min} := x_m(\theta_L, S_{max})$ to $x_{max} := x_m(\theta_H, S_{min})$.

It is important to note that, since the support of x_m depends on θ and may have finite bounds, there may exist an upper interval of price signals $[x_+, x_{max}]$ that realise only if $\theta = \theta_H$, and a lower interval $[x_{min}, x_+]$ that realise only if $\theta = \theta_L$. We define S_- and S_+ as the values of net supply that correspond to (x_-, x_+):

$$\begin{aligned} S_- \;&:\; x_+ := x_m(\theta_L, S_{min}) = x_m(\theta_H, S_-), \\ S_+ \;&:\; x_- := x_m(\theta_H, S_{max}) = x_m(\theta_L, S_+). \end{aligned} \tag{23}$$

In practice, S_- (resp. S_+) is the value of net supply at which, when θ is high (resp. low), the marginal investor receives the same private signal that the most (resp. least) optimistic marginal investor would receive when θ is instead low (resp. high). This means that observing any $x_m \in (x_+, x_{max}] \cup [x_{min}, x_-)$ is revealing of the underlying value of θ. On the contrary, observing any $x_m \in [x_+, x_-]$ is compatible with both values of θ and leaves uncertainty on which θ has realised.

The conditional p.d.f. of x_m conditional on θ is given by

$$f_{x_m|\theta}(y \mid \theta) = \begin{cases} \max\{\overline{b}_{cb} - \underline{b}, 0\} & \text{for } y = x_{max} \\ \frac{\overline{b}+\underline{b}}{\sigma_x}\phi\left(\frac{\theta-y}{\sigma_x}\right) & \text{for } y \in (x_{min}, x_{max}). \end{cases} \tag{24}$$

Note that we are treating the market signal as a random variable on the *extended* real line, that is, including the infinity elements as actual numbers. This is useful to deal with the particular AP policy we are assuming: when $\overline{b}_{cb} > \underline{b}$, the conditional density has a mass point at $x_m = x_{max} = +\infty$, because there is a non-empty set of states in which net supply is zero and the market is passive.[16] When instead $\overline{b}_{cb} < \underline{b}$, the conditional density of x_m becomes a standard truncated normal density with no mass points.

[15] This can be shown using the fact $\Phi(S) \sim N(0,1)$ if $S \sim$ Uniform$[0,1]$.

[16] It is easy to verify that

$$\int_{\theta_L - \sigma_x \Phi^{-1}\left(\frac{1+\underline{b}-\overline{b}_{cb}}{\overline{b}+\underline{b}}\right)}^{+\infty} \frac{\overline{b}+\underline{b}}{\sigma_x}\phi\left(\frac{\theta-y}{\sigma_x}\right) \mathrm{d}y + (\overline{b}_{cb} - \underline{b}) = 1$$

An illustration. Figure 1 illustrates the mapping from fundamentals (θ, S) to the equilibrium marginal investor and price signal entailed by (21). The figure plots the realisation of x_m on the y-axis as a function of the net supply shock S on the x-axis, and the fundamental shock θ: the solid and dashed lines refer to the case where θ is equal to θ_H and θ_L respectively. Each panel illustrate different combinations of $(\underline{b}, \overline{b}, b_{cb})$.

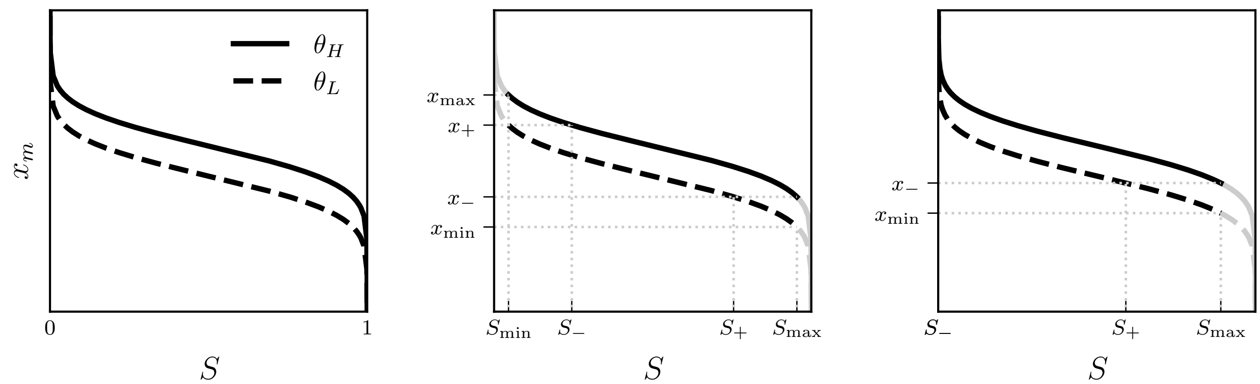

Figure 1: In the right panel, the top (resp. bottom) line plots the realisation of the price signal in case θ^H (resp. θ_L) as a function of *net supply per individual exposure* S. In the left panel we plot on the x-axis the probability density function of the price signal realisation, which is measured on the y-axis. The left panel plots the benchmark case of unit position bounds and no APs $(\underline{b}, \overline{b}, \overline{b}_{cb}) = (0, 1, 0)$. The centre panel plots the generic case with short-selling and no APs $(\underline{b}, \overline{b}, \overline{b}_{cb}) = (-0.05, 1.05, 0)$. The left panel plots a case with APs $(\underline{b}, \overline{b}, \overline{b}_{cb}) = (-0.05, 1.05, 0.07)$.

The left panel, illustrates our benchmark case, $\{\underline{b}, \overline{b}, b_{cb}\} = \{0, 1, 0\}$. In this case, $\widetilde{S} = S$ and $\mathcal{M} = \mathcal{N}_x$. When $S = 1$, the whole population of investors is needed to clear the market, so the most pessimistic investor, i.e. the one with the lowest private signal ($x_i \to -\infty$), is marginal. When instead $S = 0$, there is an infinitesimal amount of supply, such that only the most optimistic investor ($x_i \to \infty$) buys bonds and is marginal. For a given value of $S \in (0, 1)$, the mass of investors required to clear the market and the marginal investor's identity (or position in the distribution) do not change with θ, but her signal will be more optimistic when θ is high, because that shifts the mean of the private signal distribution. In other words, $\theta = \theta_H$ as $x_m(\theta_H, S) > x_m(\theta_L, S)$ always.

The central panel illustrates a case with position bounds outside unity, that is, short-selling

and thus the integral of the p.d.f. on the extended real line is equal to 1. When instead $\overline{b}_{cb} < \underline{b}$

$$\int_{\theta_L - \sigma_x \Phi^{-1}\left(\frac{1+\underline{b}-\overline{b}_{cb}}{\overline{b}+\underline{b}}\right)}^{\theta_H - \sigma_x \Phi^{-1}\left(\frac{\underline{b}-\overline{b}_{cb}}{\overline{b}+\underline{b}}\right)} \frac{\overline{b}+\underline{b}}{\sigma_x} \phi\left(\frac{\theta - y}{\sigma_x}\right) \mathrm{d}y = 1.$$

is possible and long positions can be larger than one: $-\underline{b} < 0, \overline{b} > 1, b_{cb} = 0$. When short and long position bounds are outside unity, the marginal investor distribution has truncated tails as $S_{min} > 0$ and $S_{max} < 1$. Some very optimistic investors are never marginal, as there is always enough supply from short sellers to satisfy their demand; on the other extreme, some very pessimistic buyers are never marginal either, as long position bounds are such that more optimistic investors are always enough to meet supply.

Finally, the right panel shows the effect of central bank uncontingent asset purchases, keeping the same position bounds of the central panel: $-\underline{b} < 0, \overline{b} > 0, \overline{b}_{cb} > \underline{b} > 0$. We assume $\overline{b}_{cb} > \underline{b}$ to highlight a difference from the central panel of the figure: sufficiently large APs have the effect of absorbing all the short selling, which implies that we are back in the case where $S_{min} = 0$, zero net supply is a possibility, and the most optimistic investor in the whole population can be marginal. Similarly to the central panel, in the left tail of the marginal agent distribution, there is an interval of investors which are never marginal, as the intervention of the central bank always crowds their purchases out. The presence of a left tail truncation in the marginal agent distribution gives rise to an interval $[x_{min}, x_{-})$ of price signals whose observation is uniquely associated with the realization of θ_L.

It is worth noting that any combination of $(\underline{b}, \overline{b}, \overline{b}_{cb})$ that delivers the same (S_{min}, S_{max}) pairs generates identical truncations of the marginal agent distribution. In this sense,[17] implementing uncontingent asset purchases is analogous to an expansion of the long position limit, since it effectively increases the amount of bonds purchased at any market price, as if investors could absorb a larger stock of assets.

3.3 Marginal Investor Beliefs

In the analysis that follows, we often condition on the market being active, because we are interested in characterising the equilibrium price when it is determined by the market, rather than the central bank. This implies that we condition on $x_m \in (x_{min}, x_{max})$, and not on the case where $x_m = x_{max}$ whenever that is a possibility.

[17] But not in general, as different combinations of $(\underline{b}, \overline{b}, \overline{b}_{cb})$ imply different mappings between gross and net supply.

3.3.1 "Cursed" Posterior Beliefs: Only Private Signals

It is instructive to first consider the simplest case where agents are "cursed" in the terminology of Eyster et al. (2019), i.e. they condition their beliefs on the private signal (and APs rule) but neglect the information content of the price they observe. In this case, the posterior probability of $\theta = \theta_H$ conditional only on private and prior information is given by

$$p_i^{cur} := P(\theta = \theta_H \,|\, x_i \sim \mathcal{N}_x) = \frac{q\,\phi\left(\frac{\theta_H - x_i}{\sigma_x}\right)}{q\,\phi\left(\frac{\theta_H - x_i}{\sigma_x}\right) + (1-q)\,\phi\left(\frac{\theta_L - x_i}{\sigma_x}\right)},$$

i.e. it is the probability that a certain net supply realisation consistent with the observation of x_i occurred conditional to θ_H rather than θ_L. Therefore $R(p_m^{cur})$, with m defined as the identity of the investor observing the threshold signal $x_i = x_m$ with x_m given by (21), would be the equilibrium return prevailing in a market with cursed investors. We will use the "cursed" case as our simplest benchmark of beliefs formation and market price to isolate the effect of learning from prices.

3.3.2 "Public" Posterior Beliefs: Only Price Signals

As an intermediate step, it is instructive to derive the posterior belief conditional on public information only, i.e. the probability of θ_H conditional on the observation of the realisation of the marginal agent (21), equivalent to the observation of the market price. As we highlighted previously, depending on $\{\underline{b}, \overline{b}, \overline{b}_{cb}\}$ there may be partitions of the price signal support in which the value of the fundamental is fully revealed conditional on observing $x_m \in [x_{min}, x_-) \cup (x_+, x_{max}]$. In this case, the posterior probability of $\theta = \theta_H$ conditional only on public and prior information is given by

$$p_m^{pub} := P(\theta_H \,|\, x_m \sim \mathcal{M}(\mathbf{b})) = \begin{cases} 1 & \text{if} \quad x_m \in (x_+, x_{max}], \\ \dfrac{q\,\phi\left(\frac{\theta_H - x_m}{\sigma_x}\right)}{q\phi\left(\frac{\theta_H - x_m}{\sigma_x}\right) + (1-q)\phi\left(\frac{\theta_L - x_m}{\sigma_x}\right)} & \text{otherwise} \\ 0 & \text{if} \quad x_m \in [x_{min}, x_-), \end{cases}$$

where note $p_m^{pub} = p_m^{cur}$ within the non-revealing region. In fact, in this region, the information given by the price coincides with the information contained in the marginal agent's private signal. In fully revealing regions, however, conditioning on public information will take advantage of the fact that a particular realisation of the price signal x_m is consistent with only a particular realisation of θ. $R(p_m^{ext})$ is thus a "publicly-evaluated" interest rate, i.e. it represents the price that would make an external observer using only public information indifferent between buying bonds and selling them short.

3.3.3 Equilibrium Posterior Beliefs: Private Signals and Learning from Prices

We can now characterise the equilibrium posterior beliefs of investor i conditional on her whole information set, which includes: *i)* the AP rule and exogenous prior distributions, *ii)* the private signal x_i, drawn from its distribution $\mathcal{N}_x$; and *iii)* the endogenous public signal given by the market price R, which is equivalent to that contained in x_m. In this case, the posterior probability of $\theta = \theta_H$ conditional on $\Omega_i = \{x_i \sim \mathcal{N}_x, x_m \sim \mathcal{M}(\mathbf{b})\}$ is given by

$$
\begin{aligned}
p_{i,m} :=& P(\theta_H \,|\, x_i \sim \mathcal{N}_x, x_m \sim \mathcal{M}(\mathbf{b})) = \\
&= \begin{cases} 1 & \text{if} \quad x_m \in (x_+, x_{max}], \\ \dfrac{q\,\phi\left(\frac{\theta_H - \frac{x_i + x_m}{2}}{\sigma_x/\sqrt{2}}\right)}{q\,\phi\left(\frac{\theta_H - \frac{x_i + x_m}{2}}{\sigma_x/\sqrt{2}}\right) + (1-q)\,\phi\left(\frac{\theta_L - \frac{x_i + x_m}{2}}{\sigma_x/\sqrt{2}}\right)} & \text{otherwise} \\ 0 & \text{if} \quad x_m \in [x_{min}, x_-), \end{cases}
\end{aligned} \tag{25}
$$

where note $p_{i,m} = p_m^{pub}$ within the fully revealing regions, whereas it is different otherwise. It is worth noting that, whenever the price signal is not fully revealing, the precision of the posterior beliefs of an investor observing private and public information is exactly double that of both a cursed investor not updating public information, and an external observer not holding private information. This follows from the fact that, in the non-revealing region, both private and price signal have the same precision, and observing both doubles the precision of posterior beliefs. Importantly, $R(p_{m,m})$ defines the equilibrium return in our model, which obtains by evaluating $p_{i,m}$ with $x_i = x_m$:

$$
R\,\mathbb{E}\left[\frac{\theta}{\Pi}\,|\, x_m \sim \mathcal{N}_x,\, x_m \sim \mathcal{M}(\mathbf{b})\right] = R\,\left[p_{m,m}(x_m)\theta_H \frac{1-\hat{\alpha}}{1-\hat{\alpha}R\theta_H} + (1-p_{m,m}(x_m))\theta_L \frac{1-\hat{\alpha}}{1-\hat{\alpha}R\theta_L}\right] = 1. \tag{26}
$$

3.4 A Characterisation of Equilibrium Returns

To characterise the equilibrium price, it is useful to establish the following statement.

Lemma 1. *When prices are not fully revealing, the equilibrium posterior of the marginal agent is larger than the public posterior of the marginal agent, i.e.* $p_{m,m} > p_m^{cur} = p_m^{pub}$*, if and only if*

$$x_m > x_* := \frac{\theta_H + \theta_L}{2},$$

with $x_m \in (x_-, x_+)$*. In particular,* $p_{*,*} = p_*^{cur} = p_*^{pub} = q$ *where* $p_{*,*}$*,* p_*^{pub} *and* p_*^{cur} *represent the equilibrium, public and cursed posterior of the marginal agent, respectively, computed conditional to* x_**.*

Proof. Postponed to appendix 2 ∎

The lemma states that the equilibrium posterior of the marginal agent is above (below) the public one when the price signal is not fully-revealing and higher (lower) than the uninformative value x_*.

To gain intuition, consider first the effect of exogenous public news (e.g. investors observing an exogenous public signal on θ) on the equilibrium. A public signal above the prior mean of θ makes investors' beliefs shift up, and the equilibrium interest rate shift down, with equal elasticity, without affecting the relative mass of buyers.[18] In this case, investors do not learn anything from the price change in itself, because it is entirely due to the variation in public news. Consider now the effect of a change in the equilibrium price (hence x_m) due to a shock to the fundamental θ. This will shift up the distribution of investors' private signals as well as the equilibrium price, without affecting the mass of buyers. The crucial difference with the previous example is that x_m is an *endogenous* signal that aggregates private information: when investors see the equilibrium price go up, they revise their beliefs up again. This update triggers a further shift up in the market price, in a loop of amplification.

Figure 2 illustrates Lemma 1, plotting cursed (with circles), public (with a dashed line) and equilibrium (with a solid line) posterior beliefs of the marginal agent as a function of the realisation of the price (or marginal agent) signal, in the same three specifications of Figure 1. In the first panel, unit position bounds and no APs imply that the private signal and price signal distributions $\mathcal{N}_x$ and $\mathcal{M}$ coincide. The market signal does not generate fully-revealing regions, and the cursed and public posteriors both lie on the same curve $p_m^{curs} = p_m^{pub}$. Cursed, public

[18] This is evident from the market clearing condition that only depends on the dispersion of investors' beliefs.

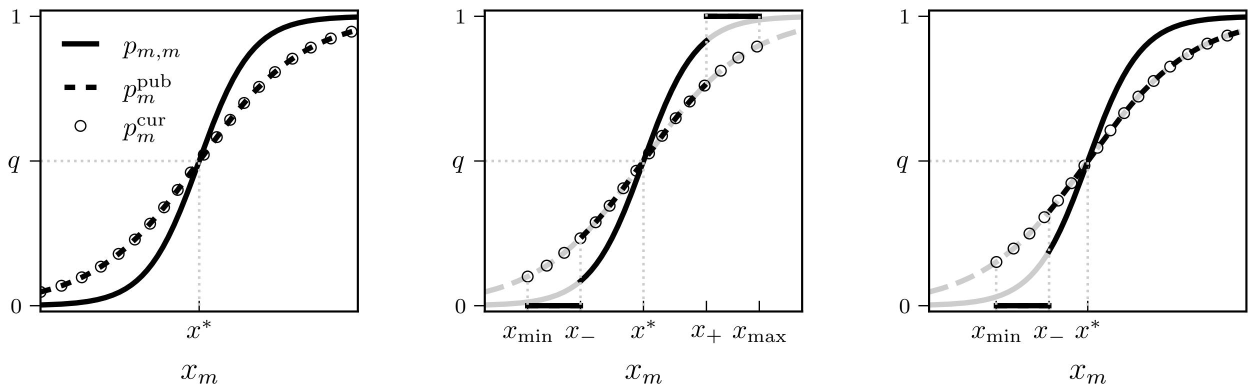

Figure 2: The figure illustrate the relation among cursed, public and equilibrium marginal posteriors as a function of the market signal x_m. Panel *(a)* plots the benchmark case of unit position bounds and no APs $(\underline{b}, \overline{b}, \overline{b}_{cb}) = (0, 1, 0)$. Panel *(b)* plots the generic case with short-selling and no APs $(\underline{b}, \overline{b}, \overline{b}_{cb}) = (-0.05, 1.05, 0)$. Panel *(c)* plots a case with APs $(\underline{b}, \overline{b}, \overline{b}_{cb}) = (-0.05, 1.05, 0.07)$.

and equilibrium posteriors all take value q at x_*, and are strictly monotonically increasing in x_m, ranging from 0 at $x_m \to -\infty$ to 1 at $x_m \to \infty$. At x_*, cursed and public marginal posteriors are flatter than equilibrium ones, meaning that the latter react to market news more than the former.

In analogy to Figure 1, the central panel illustrates a case with position bounds outside $[0, 1]$ and without APs. The presence of truncations in the support of the market signals implies $-\infty < x_{min} < x_{max} < +\infty$, which narrows the range of possible marginal agents. Cursed and public posteriors now differ in the regions of full revelation. Full revelation occurs only for the external observer who takes advantage of the observation of the realisation x_m vis a vis its distribution $\mathcal{M}$, whereas cursed agents do not. In full revelation regions, equilibrium marginal posteriors are equal to public ones, since x_m becomes an infinitely precise signal on θ and all uncertainty is resolved.

Finally, the right panel illustrates the effect of APs, which is essentially to shift the support of x_m towards the right. On the one hand, larger APs make it possible that relatively more optimistic investors can become marginal, as the central bank absorbs all the short sales, and states with zero net bond supply can realise. On the other hand, larger APs imply that relatively more pessimistic investors can never become marginal, as states with large net supply states cannot realise due to APs. All in all, the effect of APs is to shift right towards the right all the bounds $x_{min}, x_-, x_+, x_{max}$, making full revelation more likely to occur for bad states (θ_L) than good ones (θ_H).

Finally, it is instructive to note that, in the non-revealing region (x_-, x_+), the curves of the centre and right panels perfectly overlap with the curves in the left panel (in light gray): this is a property of truncated normal distributions for which the ratio of probability of two outcomes from the same distribution does not change with the size of the truncation, as far as these probabilities do not take degenerate values. In other words, position bounds and APs only affect the support of x_m and its partitions and distribution, and not the posterior distribution of θ conditional on x_m when uncertainty remains.

Average Ex-Post Returns. A consequence of learning from prices is that the equilibrium interest rate does not reflect a fair evaluation of the asset. To appreciate this statement, it is useful to write down the expression for the average ex-post rate of return for bonds

$$\mathbb{E}[R(p_{m,m})\,\theta] = \mathbb{E}[\mathbb{E}[R(p_{m,m})\,\theta\,|\,x_m \sim \mathcal{M}(\mathbf{b})]] = \mathbb{E}\left[\frac{\mathbb{E}[\theta\,|\,x_m \sim \mathcal{M}(\mathbf{b})]}{\mathbb{E}[\theta\,|\,x_m \sim \mathcal{N}_x\,,\,x_m \sim \mathcal{M}(\mathbf{b})]}\right] = \mathbb{E}\left[\frac{R(p_{m,m}(x_m))}{R(p_m^{pub}(x_m))}\right],$$

where, by exploiting the law of iterated expectation, we show that the average ex-post return on bonds is equal to the average ratio between the equilibrium interest rate and its analogous evaluated according to public posterior beliefs. It is immediate to see that, in states where $R(p_m, m) = R(p_m^{pub})$, the ratio is exactly equal to the safe rate of one. These states correspond to fully revealing regions, while the equivalence is generally not true in the whole range of x_m. We can then state the following Lemma.

Lemma 2. *In equilibrium, sufficiently large APs decrease the average ex-post return* $\mathbb{E}[\,R(p_{m,m})\,\theta]$ *strictly below the safe rate of one. In particular,*

$$\frac{\partial \mathbb{E}[\,R(p_{m,m})\,\theta]}{\partial b_{cb}}\Big|_{b_{cb}=\underline{b}} < 0, \qquad and \qquad \frac{\partial \mathbb{E}[\,R(p_{m,m})\,\theta]}{\partial b_{cb}}\Big|_{b_{cb}=\underline{b}+\bar{b}} > 0,$$

that is, the amount of uncontingent APs that minimizes $\mathbb{E}[\,R(p_{m,m})\,\theta]$ *is interior in* $(\underline{b}, \underline{b}+\bar{b})$.

Proof. Postponed in Appendix 3. ■

The Lemma provides a characterization of the impact of asset purchases on the average ex-post market return. It highlights how asset purchases prevent the realization of states where relatively more pessimistic investors become marginal, therefore taking out states in which the market rate is lower than the public one.

4 Monetary-Fiscal Interactions and Welfare

4.1 Individual Rates of Return

Our final goal is to write down savers' and investors' unconditional expectation of the total return on their savings, $\mathbb{E}[\mathcal{R}_s(\bar{b}_s)]$ and $\mathbb{E}[\mathcal{R}_i(\bar{b}_i)]$ respectively. We are interested in the unconditional version of these expectations for two related reasons. First, because they determine the first-period demand for consumption and savings by savers and investors via Euler equation (7). Second, because these expectations depend on agents' savings decisions and on central bank APs. In fact, an equilibrium of the macroeconomic model is a fixed point of this two-way relationship between prices and allocations.

Savers. In the case of savers, the expected return on money depends on the unconditional distribution of inflation, which in turn depends on the interaction between the treasury and the central bank, and the profits and losses of the latter. We postpone this discussion to Subsections 4.2 and 4.3, where we consider the cases of monetary and fiscal dominance respectively.

Investors. With respect to investors, we can make some progress in characterising the way in which APs affect their expected return from savings, without making specific assumptions about fiscal-monetary interactions. Joining equations (9) and (10), the investors' expected return per unit of savings, conditional on her whole information set, is

$$\mathbb{E}\left[Q(b_i)\,|\,\Omega_i\right] = w(x_i, x_m, \mathbf{b})\mathbb{E}\left[\frac{R\,\theta}{\Pi}\,|\,\Omega_i\right] + \left(1 - w(x_i, x_m, \mathbf{b})\right)1 \tag{27}$$

where $\Omega_i = \{x_i \sim \mathcal{N}_x, x_m \sim \mathcal{M}(\mathbf{b})\}$, and

$$w(x_i, x_m, \mathbf{b}) = \mathbb{1}[x_i \geq x_m] - \frac{b}{\bar{b}_i}\mathbb{1}[x_i < x_m]$$

is a weighting function that says that optimistic agents put all their savings into bonds, while pessimistic agents sell bonds short and put all their savings (plus revenues from short sales) into the safe asset.

To derive investors' unconditional expected return from savings, let us first integrate (31) over the private signal distribution, while continuing to condition on public information $x_m \sim \mathcal{M}(\mathbf{b})$:

$$
\begin{aligned}
&\mathbb{E}\left[Q(b_i)\,|\,x_m \sim \mathcal{M}(\mathbf{b})\right] = \\
&= \int_{-\infty}^{+\infty} \left(w(x_i, x_m, \mathbf{b})\, \mathbb{E}\left[\frac{R\,\theta}{\Pi}\,|\,\Omega_i \right] + (1 - w(x_i, x_m, \mathbf{b}))1 \right) \mathrm{d}\, F_{x_i|x_m}(x_i|x_m, \mathbf{b}) = \\
&= 1 + \int_{-\infty}^{+\infty} w(x_i, x_m, \mathbf{b}) \left(\frac{\mathbb{E}[\theta/\Pi\,|\,x_i \sim \mathcal{N}_x, x_m \sim \mathcal{M}(\mathbf{b})]}{\mathbb{E}[\theta/\Pi\,|\,x_m \sim \mathcal{N}_x, x_m \sim \mathcal{M}(\mathbf{b})]} - 1 \right) \mathrm{d}\, F_{x_i|x_m}(x_i|x_m, \mathbf{b})
\end{aligned} \tag{28}
$$

where we used the equilibrium price equation (26) to substitute out R, and $F_{x_i|x_m}(x_i)$ is the c.d.f. of the private signal conditional on the market signal, and its p.d.f. is given by

$$
f(x_i|x_m, \mathbf{b}) = \frac{1}{\sigma_x} \phi\left(\frac{x_i - x_m}{\sigma_x \sqrt{2}} \right) \frac{\sum_{\theta \in \Theta(x_m, \boldsymbol{b})} q(\theta) \phi\left(\frac{\frac{x_i + x_m}{2} - \theta}{\frac{\sigma_x}{\sqrt{2}}} \right)}{\sum_{\theta \in \Theta(x_m, \boldsymbol{b})} q(\theta) \phi\left(\frac{x_m - \theta}{\sigma_x} \right)} \tag{29}
$$

where Θ is the set of values of θ that have positive probability conditional on $(x_m, \mathbf{b})$.[19] The last equality of equation (28) highlights two important facts. First, for any x_m we condition upon, the expected return on savings for an investor is bounded below by 1. This is due to the fact that the term

$$
\frac{\mathbb{E}[\theta/\Pi\,|\,x_i \sim \mathcal{N}_x, x_m \sim \mathcal{M}(\mathbf{b})]}{\mathbb{E}[\theta/\Pi\,|\,x_m \sim \mathcal{N}_x, x_m \sim \mathcal{M}(\mathbf{b})]} - 1
$$

inside the integral represents the *expected excess return of bonds over the safe asset*, conditional on investor i's private information, and must be non-negative: if $x_i \geq x_m$, the expected excess return is positive, and the investors puts all her savings in bonds (i.e. she takes a long position equal to 1); if instead $x_i < x_m$, the expected excess return is negative and the investor takes a short position in bonds equal to $\underline{b}/\overline{b}_i < 0$, and puts her savings plus the revenues from short sales into the safe asset. Second, whenever the price signal is fully revealing, all uncertainty is resolved, bonds and the safe option are equivalent assets with a deterministic payoff, bond excess returns are zero, and the whole integrand in equation (28) cancels out.

Integrating (28) once more with respect to the price signal distribution, we get the unconditional expectation of the return on savings as

$$
\mathbb{E}[Q(b_i)\,|\,\mathbf{b}] = \int_{\mathrm{Supp}(x_m)} \mathbb{E}\left[Q(b_i)\,|\,x_m \sim \mathcal{M}(\mathbf{b})\right] \mathrm{d}F_{x_m}(x_m, \mathbf{b})
$$

[19]The derivation can be found in Proof 4 in the appendix.

where $F_{x_m}(x_m, \mathbf{b})$ is the marginal c.d.f. of the market signal, whose p.d.f. is given by

$$f_{x_m}(x_m, \mathbf{b}) = \begin{cases} \max\{\bar{b}_{cb} - \underline{b}, 0\} & \text{for } x_m = x_{max} \\ \frac{\bar{b}+\underline{b}}{\sigma_x} \sum_{\Theta(x_m,\mathbf{b})} q(\theta)\phi\left(\frac{x_m-\theta}{\sigma_x}\right) & \text{for } x_m \in (x_{min}, x_{max}). \end{cases} \tag{30}$$

We can split the support of the market signal x_m in two partitions: the interval $[x_-, x_+]$ where uncertainty is never resolved, and the region $(x_{min}, x_-) \cup (x_+, x_{max})$ where the price signal is fully revealing. As we have discussed in the previous paragraph, in the latter region investors get zero excess returns from bonds. We thus integrate expected excess returns only in the former region, where uncertainty remains:

$$\mathbb{E}[Q(b_i)\,|\,\mathbf{b}] - 1 = \int_{x_-}^{x_+} \int_{-\infty}^{+\infty} w(x_i, x_m) \left(\frac{\mathbb{E}[\theta/\Pi\,|\,x_i, x_m]}{\mathbb{E}[\theta/\Pi\,|\,x_m, x_m]} - 1 \right) \mathrm{d}\, F_{x_i|x_m}(x_i|x_m, \mathbf{b})\, \mathrm{d}F_{x_m}(x_m, \mathbf{b}) \tag{31}$$

A few things are worth noting. First, the unconditional expected excess return over the safe asset is non-negative for the reasons illustrated in the previous paragraph. Second, expected returns depend on APs and position bounds (i.e. $\mathbf{b}$) through *(i)* the interval of non-revealing price signals $[x_-, x_+]$, *(ii)* the size of long and short positions investors can take, $w(x_i, x_m, \mathbf{b})$, *(iii)* the marginal distribution of the price signal, F_{x_m}, and *(iv)* the amount by which inflation depends on bond returns when in the fiscal dominance regime, α. Note that, conditional on x_m being in the non-revealing region, the ratio of conditional expectations and the conditional distribution of x_i given x_m do not depend on $\mathbf{b}$.

The next subsections make specific assumptions about fiscal-monetary interactions and the behaviour of inflation, and describe in detail the effect of APs on asset prices and expected returns, savings, consumption and welfare.

4.2 Monetary Dominance

We now assume that the transfer policy between the government and the central bank is such that the gross inflation rate is constant and equal to 1. That is, we assume a transfer policy such that all profits and losses the central bank makes as a result of engaging in APs are fully rebated to the government. In the wording and notation of Section 2.1, we assume full fiscal backing, $\kappa = 1$, and allow α to vary with $\bar{b}_{cb}$.

As a result, the behaviour of savers becomes uninteresting, because the rate of return on their savings is now equal to the first best level, and their savings and consumption decisions are efficient. We thus focus solely on the effect of uncontingent APs on investors' expected rate of return.

In the case of investors, the inflation term disappears from the ratio of conditional expectations in equation (31). In Figure 3, the bottom row represents the inner integral of that equation, i.e. the expected return from savings conditional on x_m. The top row plots the marginal distribution of the market signal x_m. Each column represents a different configuration of position bounds and asset purchases. Computing the unconditional expected return from savings amounts to integrating the product of these two objects over the whole support of the market signal x_m.

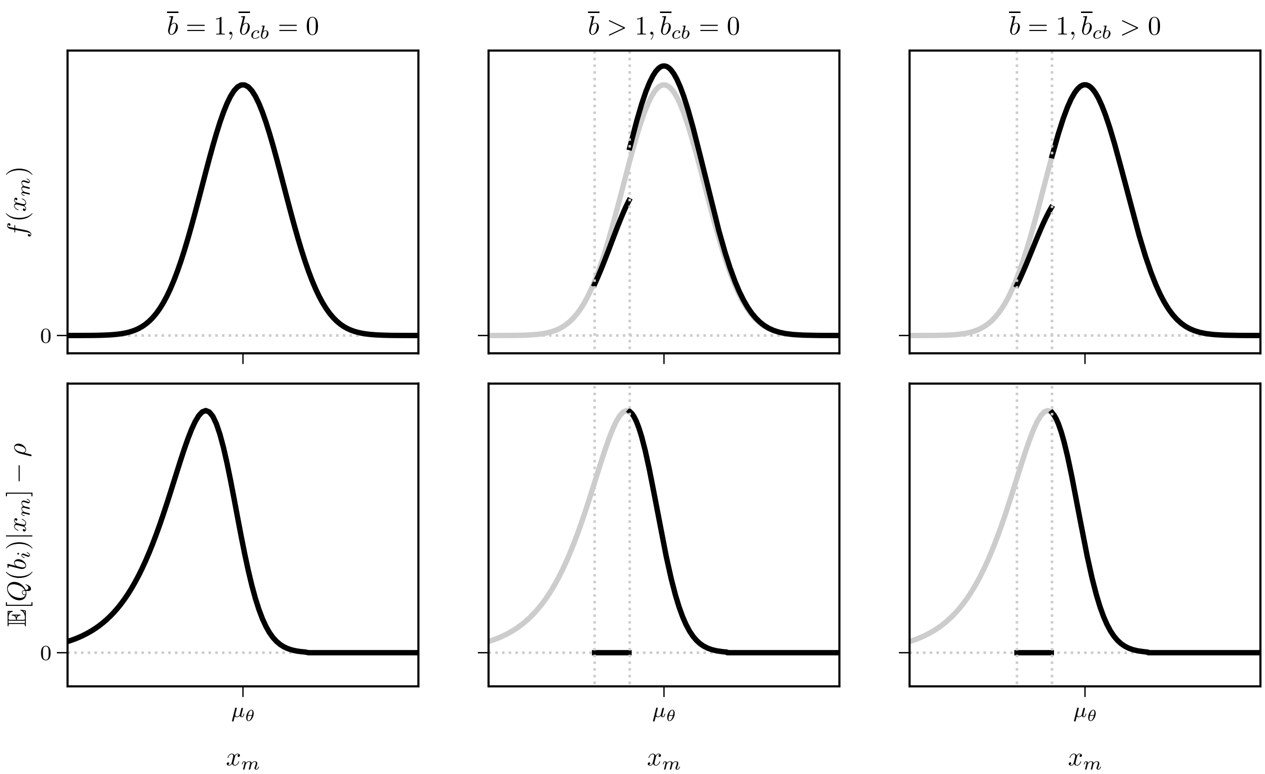

Figure 3: Marginal distribution of the market signal (top row) and expected excess return from saving in bonds (bottom row) as a function of the market signal x_m. The x axis displays values of x_m within 5 standard deviations of its marginal distribution. The light gray lines in the middle and right columns represent the case illustrated in the left column, with unit bounds and no APs.

In the first column we look at the baseline case with unit position bounds and no APs. The marginal distribution of x_m is symmetric around the prior mean and has support given by the entire real line, since in this case there are no instances of full revelation and support trunca-

tions. Expected excess returns are weakly positive, and have an asymmetric distribution that is skewed towards the left. Understanding its shape is not a straightforward exercise, so we now discuss one by one the different mechanisms that drive it. First, the p.d.f. of the private signal conditional on the market signal (see equation (29) moves in the same direction of, but less than, the market signal itself. This implies that the probability investor i is optimistic and buys bond (i.e. $x_i \geq x_m$) is decreasing in x_m. Second, investor i's excess returns $\mathbb{E}[\theta|x_i, x_m]/\mathbb{E}[\theta|x_m, x_m] - 1$ belong to the range
$[0,\ \theta_H/\mathbb{E}[\theta|x_m, x_m] - 1]$ as x_i moves inside the $[x_m, +\infty)$ interval. These two observations combined help us explain the behaviour of expected excess returns condition on x_m. When $x_m \to \infty$, excess returns tend to zero: it's less likely the investor will be more optimistic than the marginal agent, and even if she is, there is little upside to be made by buying the bonds if the price is close to its highest value θ_H. As x_m decreases, the probability that investor i receives a signal $x_i \geq x_m$ grows, and the range of excess returns that may realise in such cases becomes wider and populated with higher values: as the equilibrium price drops, there is more upside to be made when receiving a high private signal. But as x_m decreases, so does the density of $x_i|x_m$. When x_m becomes low enough, the probability mass over signals that correspond to high excess returns shrinks towards zero. As $x_m \to -\infty$, expected excess returns converge to zero, as the investor makes positive returns that are significantly above zero only when she receives private signals that are very large and have low probability.

The second column of Figure 3 plots the case where investor savings (i.e. the long position bound) are larger than unity. Larger savings imply the net supply per investor decreases. This shrinks the support of the market signal: all values of x_m to the left of the first vertical dotted line fall out of the support, the values in between the vertical lines are now only compatible with θ_L, and the remaining values are compatible with both values of θ. This explains why the density of the market signal has a discontinuous jump, and why expected excess bond returns fall to zero in the region where information is fully revealed. As is clear from equation (30), the reduction in the density of x_m inside the fully revealing region is balanced by an increase in probability mass inside the non-revealing region. While the changes in the support and revealing regions of x_m clearly point to a reduction in the expected excess return from savings, the increase in probability mass in the non-revealing partition of x_m goes in the opposite direction.

The third column of Figure 3 plots the case where the central bank is doing non-contingent APs. The effect is very similar to that of an increase in investors' savings, with one important difference: there is no adjustment in the probability density function of the market signal in the non-revealing region. This happens because APs reduce the probability that the market will be active, so rather than observing a "redistribution" of probability mass from the left to the right, this mass simply disappears and goes into the unconditional probability that the market is passive, which is $1 - P_0 = 1 - \bar{b}_{cb}$. This implies that the effect of APs on the expected return from savings is unambiguously negative.

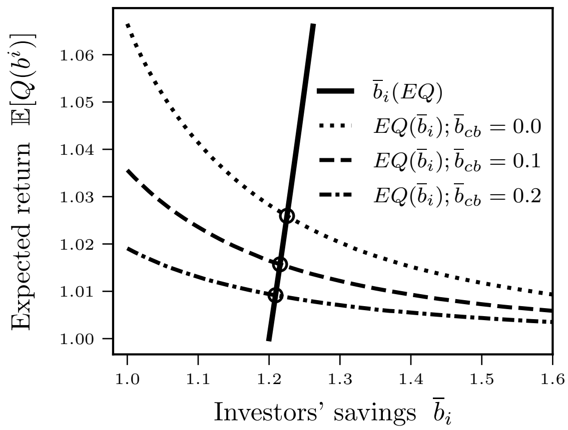

Figure 4: tbc

Equilibrium savings and returns. Having computed $\mathbb{E}[Q(b_i) \mid \mathbf{b}]$, we can look for an equilibrium of the economy, which we illustrate in Figure 4. An equilibrium is given by the intersection between two curves and represented by round markers. The first curve, depicted in black, represents investors' supply of savings $\bar{b}_i$ as a function of their expected return $\mathbb{E}[Q(b_i) \mid \mathbf{b}]$, which can be derived from Euler equation (7), and is increasing and concave. The second set of curves (dotted, dashed and dash-dotted line), represent the expected return from savings $\mathbb{E}[Q(b_i) \mid \mathbf{b}]$ as a function of their level $\bar{b}_i$ and of the central bank asset purchase policy $\bar{b}_{cb}$. We can show that this relationship is decreasing. The smaller the net supply of the bonds, because of higher long position bounds or larger APs, the smaller the excess returns investors expect to make. Importantly, expected returns are decreasing in both position bounds $\bar{b}_i$ and APs $\bar{b}_{cb}$, which implies that both the equilibrium level and the expected return of savings are decreasing in the central bank asset purchase policy. That is, the higher $\bar{b}_{cb}$, the closer to 1 the returns, and the larger first-period consumption.

In this setting, the optimal AP policy is to set $\bar{b}_{cb}$ as high as possible, to bring returns down to their efficient level of 1, and consumption up to its efficient level of one. This crucially relies on the fact that central bank profits or losses are rebated back to the government, and in turn to the household via lump-sum taxes. We illustrate this aspect in detail in the next paragraph, and then move to consider a setting where central bank losses create a welfare loss, creating a non-trivial trade-off for the central bank.

Central Bank Profits and Losses Like investors, the central bank makes profits or losses from asset purchases. Let us characterise the expected excess return (over the risk-free rate) of central bank asset purchases. Such return is illustrated in Figure 5 and is formally given by

$$\begin{aligned}\mathbb{E}[b_{cb}\,(R\,\theta-1)\,|\,\mathbf{b}] &= \left(\frac{\mathbb{E}[\theta]}{\theta_H}-1\right)\int_0^{\bar{b}_{cb}}\widetilde{S}\,\mathrm{d}\widetilde{S}\;+\;\bar{b}_{cb}\int_{x_-}^{x_+}\left(\mathbb{E}[R\,\theta\,|\,x_m\sim\mathcal{M}(\mathbf{b})]-1\right)\mathrm{d}F_{x_m}(x_m,\mathbf{b})\\ &= \left(\frac{\mathbb{E}[\theta]}{\theta_H}-1\right)\frac{\bar{b}_{cb}^2}{2}\;+\;\bar{b}_{cb}\int_{x_-}^{x_+}\left(\frac{\mathbb{E}[\theta\,|\,x_m\sim\mathcal{M}(\mathbf{b})]}{\mathbb{E}[\theta\,|\,x_m\sim\mathcal{N}_x,x_m\sim\mathcal{M}(\mathbf{b})]}-1\right)\mathrm{d}F_{x_m}(x_m,\mathbf{b}).\end{aligned} \tag{32}$$

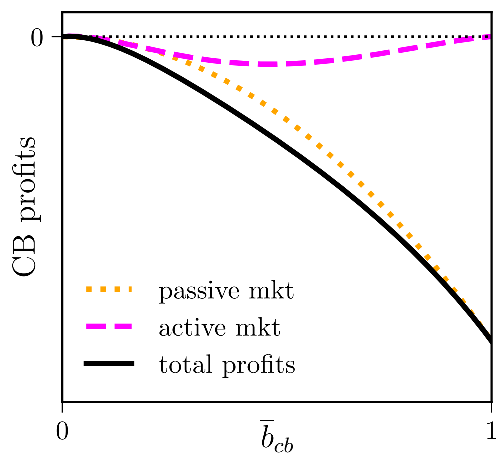

Figure 5: Central bank profits and losses as a function of uncontingent APs $\bar{b}_{cb}$.

The first term in equation (32) is illustrated by the yellow dotted line in the figure, and represents the states where the market is passive, that is, no investor buys (or sells short) any amount of the bonds, because the central bank absorbs the whole bond supply. In order to do so, the central bank must purchase the bonds at the highest possible price θ_H, which is the reservation price at which the most optimistic investor is indifferent between bonds and the safe asset. The states where this happens are those where gross supply is below the AP bound, $\widetilde{S}\in[0,\bar{b}_{cb}]$. In these states, the central bank earns an expected return equal to the prior mean of θ, which is strictly lower than the price it pays. It follows that this term is strictly negative, and decreasing and concave in $\bar{b}_{cb}$.

The second term in equation (32) is represented by the dashed purple line in the figure, and depends instead on the equilibrium bond price that arises in the states where the market is active, i.e. when gross bond supply exceeds the $\bar{b}_{cb}$, which happens with probability $1 - P_0$. In these states, the quantity of APs is constant, so profits and losses are solely determined by the behaviour of the expected excess return of bonds over the safe asset. As we show in equation (3.4) and the paragraph on ex-post returns, the expression

$$\mathbb{E}[R\,\theta \,|\, x_m \sim \mathcal{M}(\mathbf{b})] - 1 = \frac{\mathbb{E}[\theta \,|\, x_m \sim \mathcal{M}(\mathbf{b})]}{\mathbb{E}[\theta \,|\, x_m \sim \mathcal{N}_x, x_m \sim \mathcal{M}(\mathbf{b})]} - 1 = \frac{R}{R_p} - 1$$

is generally different from zero because of the different conditioning sets. In Figure 2 we show how the posterior probabilities behind these prices differ.[20] From there, we can see that APs create both a truncation and a fully revealing region in the left tail of the support of x_m, where $R/R_p > 1$ and bonds are under-priced. By taking away these states, APs reduce the expected excess return of bonds, so it follows that the central bank makes losses (and the dashed purple line is below zero) even when the market is active.

4.3 Fiscal Dominance

So far, we have developed the analysis under the assumption of monetary dominance, i.e. of a setting of full fiscal backing when central bank profits and losses are fully rebated to the government, and inflation is constant. We now make the opposite assumption and consider a situation without any fiscal backing ($\kappa = 0$). As equation (5) shows, this implies that the rate of return on money (inverse of the gross inflation rate) will be a weighted average of the real return on bonds and on the safe asset, with weight $\frac{b_{cb}}{m}$. Savers' money demand $\bar{b}_s$ will now depend, through inflation, on central bank APs, the equilibrium bond price, and default. To find the equilibrium of the two-period model we thus need to solve a two-dimensional fixed point problem: savers' and investors' unconditional expectation of the total return on their savings, $\mathbb{E}[\mathcal{R}_s(\bar{b}_s)]$ and $\mathbb{E}[\mathcal{R}_i(\bar{b}_i)]$, jointly depend on $\bar{b}_i, \bar{b}_s$ and on APs b_{cb}; at the same time, both savings decisions depend on the joint behaviour of the rate of return on bonds and money.

As we saw in the previous subsection, uncontingent APs generate expected losses for the central bank, which are increasing in the size of the purchases. Under fiscal dominance, these losses translate in expected inflation, which is also increasing in the size of the AP program.

[20] As we discuss at length in Section 3, the posterior probability of $\theta = \theta_H$ for the marginal agent with some information set is a sufficient statistic for the equilibrium price or interest rate, so we discuss the former rather than the latter.

This reduction in the rate of return of money has a welfare cost, because it decreases savers' incentive to hold money, increasing their first-period consumption above the efficient level. At the same time, APs retain their beneficial effect on the expected rate of return of bonds and on investors' first-period consumption. There is thus a trade-off between increasing investors' welfare and reducing savers', which under some conditions admits an interior solution, implying that the optimal amount of APs is positive and finite.

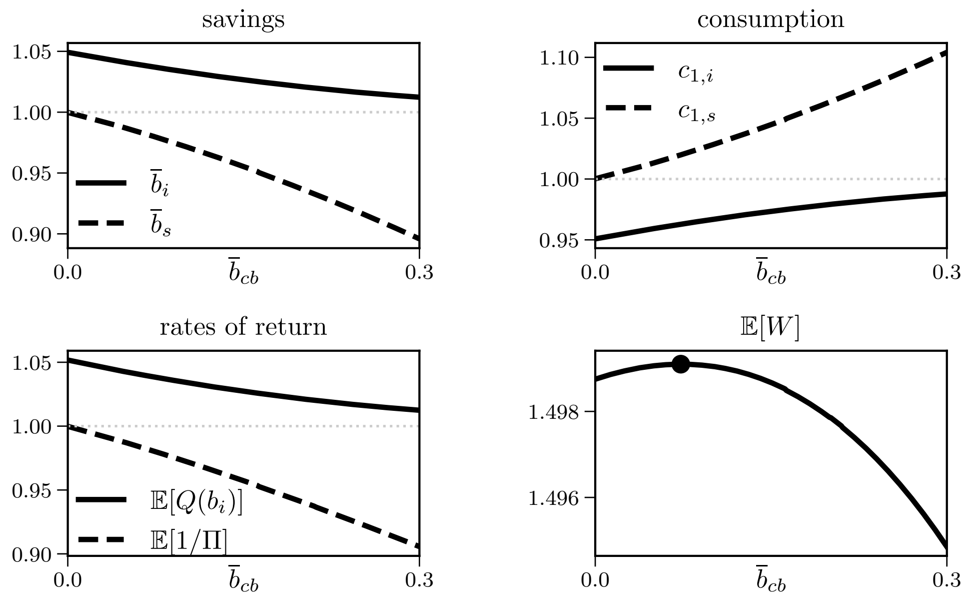

Figure 6: Equilibrium variables under fiscal dominance, as a function of uncontingent APs $\overline{b}_{cb}$. Variables related to investors and savers are represented by solid and dashed lines respectively. Parametric assumptions: $c^* = 1, \gamma = 1$ (log utility), $q = 1/2, \sigma_x = 1/2, \theta_L = 1/2, \underline{b} = 0$.

Figure 6 illustrates how equilibrium variables depend on the size of APs. The top-left panel plots investors' demand for bonds (which becomes their long position bound in the financial market) and savers' demand for money (which becomes the denominator of $\hat{\alpha}$). The consumption of each set of agents is shown in the top right panel. The bottom-left panel plots the unconditional expectation of the rate of return of bonds and money. As explained, higher APs affect saving demand, rates of return and first-period consumption of investors and savers by bringing each of these variables respectively towards and away from their efficient levels. The optimal uncontingent AP program thus trades off distortions for savers and investors, with the result that aggregate welfare (plotted in the bottom right panel) has an interior maximum at a level $\overline{b}^*_{cb} > 0$.

5 Optimal Price-Targeting Asset Purchases Policy

So far, we have focused the analysis on uncontingent asset purchases, which are a particularly simple type of program that allowed us to explain as clearly as possible what are the effects of such policy on consumption, savings, asset prices and welfare.

We have seen that, under fiscal dominance, uncontingent APs cannot eliminate all distortions. We now move on to ask whether a more sophisticated AP policy can do better than the optimal uncontingent policy studied in the previous subsection. We have shown that the main channels through which APs have effects are *(i)* the identity of the marginal agent, *(ii)* the information conveyed by the bond price, and *(iii)* the balance sheet of the central bank. We will consider a policy that targets x^* for aspect *(i)* and is neutral in aspects *(ii)*-*(iii)*.

We consider a class of asset purchase rules that specify an interval of feasible AP quantities $\mathcal{B} := [0, \overline{b}_{cb}]$ and a target equilibrium interest rate $R^\dagger$. When bidding in the bond market, the central bank submits a limit order to buy up to $\overline{b}_{cb}$ as long at an interest rate larger or equal to a target $R^\dagger$. The target we choose will be the interest rate the corresponds to the prior distribution, $R(q)$.

To characterise this policy, we follow three steps: *(i)* understand how an AP policy can target a certain marginal agent x_n; *(ii)* derive the public signal associated with observing a market signal equal to the target x_n; *(iii)* build the mapping between marginal agent x_n and market prices R_n conditional on the given policy.

Implementation and feasibility. Targeting price R_n is equivalent to targeting some marginal agent x_n. Later, we will derive the equilibrium mapping between these two variables. Consider the set of states where the central bank is buying (or is "active") and $R = R_n, x_m = x_n$. Using the market clearing equation (21), we can see that targeting x_n is equivalent to targeting net supply $S_{x_n}(\theta) := \Phi\left(\frac{\theta - x_n}{\sigma_x}\right)$, and we can back out the amount of bonds the central bank is buying

$$b_{x_n}(\theta, \widetilde{S}) = \widetilde{S} + \underline{b} - (\overline{b} + \underline{b}) S_{x_n}(\theta). \tag{33}$$

It follows that the equilibrium price is at the target whenever

$$b_{cb} \in [0, \overline{b}_{cb}] \quad \Leftrightarrow \quad \widetilde{S} \in \left[(\overline{b} + \underline{b})\, S_{x_n}(\theta, x_n) - \underline{b},\, \overline{b}_{cb} + (\overline{b} + \underline{b})\, S_{x_n}(\theta, x_n) - \underline{b}\right] =: \widetilde{\mathcal{S}}(\theta, x_n) \tag{34}$$

where $\widetilde{\mathcal{S}}(\theta, x_n)$ denotes the set of gross supply realisations such that APs are feasible and the price is at the target. This set has two important features. First, we assume that x_n, R_n are such that $\widetilde{\mathcal{S}}(\theta, x_n) \subseteq [0,1]$, so the set is always contained in the support of $\widetilde{S}$;[21] second, its length is equal to $\bar{b}_{cb}$ and is independent of θ. Since gross supply is uniformly distributed, the unconditional probability of observing $R = R_n$ and an active central bank is independent of θ and given by

$$\begin{aligned} P(R = R_n,\, b_{cb} \in [0, \bar{b}_{cb}]) &= P(S = S_{x_n}(\theta_H, x_n)) = P(\widetilde{S} \in \widetilde{\mathcal{S}}(\theta_H, x_n)) \\ &= P(S = S_{x_n}(\theta_L, x_n)) = P(\widetilde{S} \in \widetilde{\mathcal{S}}(\theta_L, x_n)). \end{aligned} \tag{35}$$

Information conveyed by the target price. Using the result from (35), it is straightforward to show that the joint observation of a price equal to target and non-zero asset purchases does not convey any information, and the posterior distribution of θ conditional on public information alone is equal to the prior. Formally[22]

$$P(\theta \,|\, R_n, b_{cb}) = \frac{P(b_{cb} \,|\, \theta_H, x_n) P(x_n \,|\, \theta_H) P(\theta_H)}{\sum_{j \in \{H,L\}} P(b_{cb} \,|\, \theta_j, x_n) P(x_n \,|\, \theta_j) P(\theta_j)} = q. \tag{36}$$

This implies that the beliefs of investor i conditional on her private information and on price signal $R = R_n$ are given by

$$E[\theta \,|\, x_i \sim \mathcal{N}_x,\, x_n \sim \mathcal{M}(b_{x_n})] = E[\theta \,|\, x_i \sim \mathcal{N}_x].$$

Intuitively, this happens because in these states the marginal bond buyer is always the central bank, who is not constrained by its position limits and can absorb any variation in bond supply, making the price is inelastic to it. As we will see below, price become informative when the marginal bond buyer goes back to being an investor again.

[21] This is true if

$$S_{x_n}(\theta, x_n) \in \left[\frac{\underline{b}}{\bar{b} + \underline{b}},\, \frac{1 + \underline{b} - \bar{b}_{cb}}{\bar{b} + \underline{b}}\right] \;\forall \theta \quad \Leftrightarrow \quad x_n \in \left[\theta_H - \sigma_x \Phi^{-1}\left(\frac{\underline{b}}{\bar{b} + \underline{b}}\right),\, \theta_L - \sigma_x \Phi^{-1}\left(\frac{1 - \bar{b}_{cb} + \underline{b}}{\bar{b} + \underline{b}}\right)\right].$$

[22] The complete derivation can be found in Proof 6 in the appendix.

Mapping between x_n and R_n. We now have all the elements to define the mapping between the marginal investor x_n and the target bond price R_n, which must satisfy

$$R_n = \frac{1}{\mathbb{E}[\theta \,|\, x_n \sim \mathcal{N}_x,\, x_n \sim \mathcal{M}(b_{x_n})]} = \frac{1}{\mathbb{E}[\theta \,|\, x_n \sim \mathcal{N}_x]}. \tag{37}$$

As usual, the marginal investor is defined as the agent who is indifferent between trading bonds or the safe asset. The last equality uses our finding from the previous step, showing that when the price equals the target and the central bank is active, agents only condition on their private signals. It follows that R_n is defined as the expected bond payoff according to the *public* beliefs of the marginal agent who receives the private signal x_n.

Information and prices away from target. We now consider cases where the price target is not achieved. A direct implication of (34) is that when $\widetilde{S} \notin \mathcal{S}(\theta, x_n)$, then $\bar{b}_{cb} \in \{0, \bar{b}_{cb}\}$ and $R \neq R_n$. In this set of states, the price differs from target R_n, and the marginal agent is not x_n: either $R < R_n$ and the central bank limit order is not executed ($b_{cb} = 0$), or $R > R_n$ because APs are constrained by their upper bound ($b_{cb} = \bar{b}_{cb}$). In both cases, the marginal bond buyer is an investor rather than the central bank, the price is elastic to gross supply, agents use the information contained in the price, and the equilibrium price and marginal agent are determined by the following equation, as in the setup with uncontingent APs:

$$R = \frac{1}{\mathbb{E}[\theta \,|\, x_m \sim \mathcal{N}_x,\, x_m \sim \mathcal{M}]} \tag{38}$$

where x_m is given by $x_m(\theta, S(\widetilde{S}, 0))$ as per (21) and (22). That is, the price in this case is defined by the *market* beliefs of the marginal agent with private signal x_m. Note that, exactly as in the case with uncontingent APs, there may be values of x_m which correspond to only one value of the fundamental θ, and thus imply there is full information revelation.

Optimal price target. As illustrated in Figure 2 and the accompanying analysis, the equilibrium prices in (37) and (38) are generated by different belief distributions, and therefore differ from one another in the way described by Lemma 1. When supply is low, $R < R_n$ and the central bank does not intervene. As soon as R reaches and goes above the target, the central bank limit order starts being executed. This implies that the equilibrium price function may jump, because it is not determined by $p_{m,m}$ any more, but by p_m^{pub} instead. If $R_n = R(q)$, then R reaches the target when $x_m = x^*$, at which point $p_m^{pub} = p_{m,m}$. If instead $R_n < (>) R(q)$, then b_{cb}

would have to jump up (down) to a positive (negative) level needed to affect net supply, since posterior probabilities for the public are below (above) those for the market in such a region of the state space. This is one reason why we consider $R(q)$ as a target, the other being its budget neutrality which we analyse next.

Illustration. Figure 7 shows an example of a price-targeting policy where $\bar{b}_{cb}$ is large enough that it never binds. In all panels, dashed and solid line represent θ_L and θ_H respectively, gray lines plot the case without any APs, and gross supply $\widetilde{S}$ is on the x-axis. The left panel plots how much of the central bank limit order is executed; the central panel shows the marginal agent signal; and the right panel illustrates the equilibrium interest rate. The vertical gray dotted lines highlight the intervals of gross supply realisations for which the policy is active and the price is at the target. Outside of this set, either APs are zero and $R < R_n$, or APs are at the upper bound and $R > R_n$. In this particular case, the upper bound on APs is defined by the highest possible purchase the central bank is making when $\theta = \theta_H$ and $\widetilde{S} = 1$. This creates a bound an APs, because if the central bank purchased a larger quantity when $\theta = \theta_L$, that would perfectly reveal the fundamental. As a result, when there is a default and gross supply is above its feasible set, there is full information revelation and R jumps to its highest value.

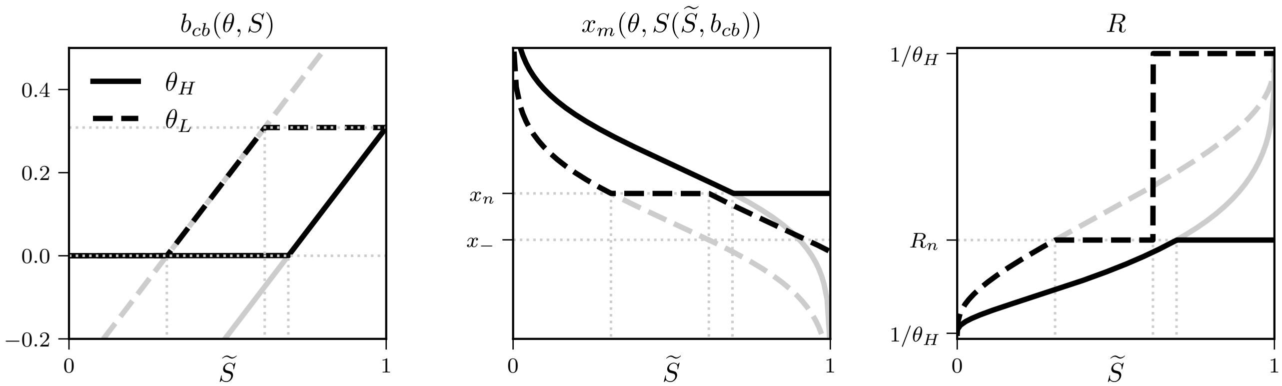

Figure 7:

Central bank profits. We now have all the elements to ask what happens to the central bank balance sheet, inflation, investors' and savers' rates of return, and welfare. When $R_n = R(q)$, the central bank makes profits or losses $\theta - R(q)$ ex post, but in expectation these are always zero: if $\widetilde{S} \in \mathcal{S}(\theta, x_n)$, the central bank buys a varying amount at the actuarially fair price, so expected profits are $\mathbb{E}[R\,\theta] - 1 = 0$; if APs are at the upper bound, there is full revelation, in which case again expected profits must be zero because there is no uncertainty; in the remaining states, APs are zero. We have thus proved that

1. price-targeting policies are belief-neutral,
2. a policy with target $R_n = R(q)$ and upper bound $\bar{b}_{cb} \geq b_{x^*}(\theta_H, 1)$ is central-bank-budget-neutral.

The consequence for inflation is that, while some inflation or deflation will happen ex-post, the ex-ante value of expected gross inflation will be one, which means that savers' consumption-savings decision are efficient.

6 Conclusion

TBC

Appendix

A Irrelevance benchmark

In subsection we show that when asset purchases are publicly observable, but investors are unconstrained in their asset positions or they have homogeneous information, asset purchases are irrelevant as in Wallace (1981).

Unconstrained asset positions. We first show here that when asset positions are unconstrained – i.e. they can take any value on the real line – asset purchases are neutral in the sense of not moving equilibrium asset prices. This occurs because with unconstrained asset positions, agents can always reduce the amount of demanded asset so that their overall exposure to the risky bond, including the one implicit in taxation, remain optimal.

To see this consider now that agents have an arbitrary utility function $u(c_i)$ with $u'(c) \geq 0$ and $u''(c) \leq 0$. suppose that, without central bank asset purchases $b_{cb} = 0$ and for a given information sets $\{\Omega_i\}_{i\in(0,1)}$, a certain profile of individual demand $b_i^\star \in \mathbb{R}$ exists such that

$$b_i^\star(0) = \text{argmax}\{E[u(c_i(b_i))|\Omega_i]\big|_{\alpha=0}\},$$

for each $i \in (0,1)$. Let us denote with $R^\star(\Lambda)$ the price such that the market clears for given $\Lambda := (\theta, \widetilde{S}, \{\Omega_i\}_{i\in(0,1)})$ realisation.

Now consider, a positive value for central bank purchases, i.e. $b_{cb} > 0$. By substituting the values of τ_1 and τ_2, we can rewrite the budget constraint of the agents as:

$$c_i = (b_i + b_{cb})(R\theta - 1) + e - \widetilde{S}R\theta,$$

where we it is easy to note that there always exists a $b_i^\star(b_{cb}) := b_i^\star(0) - b_{cb}$ such that, for any agent i, consumption is not affected by b_{cb} as the net individual exposure to bonds is always optimally equal to $b_i^\star(0)$. This also implies that $\int b_i^\star(b_{cb})di + b_{cb} = \int b_i^\star(0)di = S$ and so that $R^\star(\Lambda)$ is the equilibrium asset price for any b_{cb}.

B Nonlinear Asset Purchase Rules

Any AP rule such that each (b_{cb}, R) pair is compatible with both values of θ is non-revealing and may generate a one-to-one relationship between b_{cb} and R. As a generic example, consider a rule such that[23]

$$b_{cb}(\theta_H, \widetilde{S}) = b_{cb}(\theta_L, \widetilde{S}') = b \quad \text{and} \quad \theta_L - \sigma_x\Phi^{-1}(\widetilde{S}' - b) = \theta_H - \sigma_x\Phi^{-1}(\widetilde{S} - b) \qquad \text{for } (S, S') \in [0,1]^2. \tag{39}$$

This implies that the same marginal agent is selected when the shocks are (θ_H, S) and (θ_L, S'). The figure at this link shows an example, assuming that (in the notation of equation (39)) $S' = \kappa S$ (i.e. the quantity b_{cb} that equalises the marginal agent identity when supply conditional on θ_L is a fraction κ of supply conditional on θ_H.):

Remark 1 (Belief formation). If b_{cb} is one-to-one with R, then APs should not change posterior belief formation at all. If b_{cb} is one-to-many with R, then beliefs should take that into account and I have not worked that out yet, but I think the intuition given by the example still stands.

Remark 2 (Generality). I have a very loose conjecture that "our" policies (R target or b_{cb} target) are special cases of this policy. It all boils down to how you specify the pairs (S, S') in which we want b_{cb} to *(i)* be positive, and *(ii)* solve equation (39).

[23] For simplicity, let $\overline{b} = 1$ and $\underline{b} = 0$.

C Proofs

Proof 1 (Lemma 1)**.** *Given probability p of $\theta = \theta_H$ of the marginal agent then market clearing price is given by the following fixed point equation*

$$\left(p\,\frac{(1-\hat{\alpha})}{\frac{1}{\theta_H} - \hat{\alpha}R} + (1-p)\,\frac{(1-\hat{\alpha})}{\frac{1}{\theta_L} - \hat{\alpha}R} \right) R = 1,$$

where $\hat{\alpha} := (1-\kappa)\alpha$. The fix point equation can be rewritten as

$$\frac{(R\hat{\alpha}\theta_H - 1)\,(R\hat{\alpha}\theta_L - 1)}{\theta_L + p\theta_H - p\theta_L - R\hat{\alpha}\theta_H\theta_L} - R\,(1-\hat{\alpha}) = 0$$

or, provided $\theta^e := \theta_L + p\theta_H - p\theta_L \neq 0$ and $\theta^p := \theta_H\theta_L \neq 0$,

$$(-\hat{\alpha}\theta^p)\,R^2 + ((1-\hat{\alpha})\theta^e + \hat{\alpha}\theta^s)\,R - 1 = 0,$$

with $\theta^s := \theta_L + \theta_H$. The only solution positive and smaller than $1/\theta_L$ is

$$R\,(p) = \frac{(1-\hat{\alpha})\theta^e + \theta^s\hat{\alpha} - \sqrt{((1-\hat{\alpha})\theta^e + \hat{\alpha}\theta^s)^2 - 4\hat{\alpha}\theta^p}}{2\hat{\alpha}\theta^p}$$

where we can show $R\,(1) = 1/\theta_H$, $R\,(0) = 1/\theta_L$. In fact, one can verify that

$$\begin{aligned}
\frac{(1-\alpha)\theta^e + \theta^s\alpha - \sqrt{((1-\alpha)\theta^e + \alpha\theta^s)^2 - 4\alpha\theta^p}}{2\alpha\theta^p} &< \frac{1}{\alpha} \\
(1-\alpha)\theta^e + \theta^s\alpha - 2\theta^p &< \sqrt{((1-\alpha)\theta^e + \alpha\theta^s)^2 - 4\alpha\theta^p} \\
((1-\alpha)\theta^e + \theta^s\alpha - 2\theta^p)^2 - ((1-\alpha)\theta^e + \alpha\theta^s)^2 + 4\alpha\theta^p &< 0 \\
4\theta^p\,(\alpha + \theta^p - \theta^s\alpha + \theta^e\alpha - \theta^e) &< 0 \\
\alpha - \theta^e + \theta^p + \theta^e\alpha - \theta^s\alpha &< 0 \\
\alpha - (\theta_L + p\,(\theta_H - \theta_L)) + \theta_H\theta_L + (\theta_L + p\,(\theta_H - \theta_L))\alpha - (\theta_H + \theta_L)\,\alpha &< 0 \\
\alpha + p\theta_L - \theta_L + p\alpha\theta_H - p\theta_H - \alpha\theta_H - p\alpha\theta_L + \theta_H\theta_L &< 0 \\
\alpha + p\theta_L - \theta_L + p\alpha - p - \alpha - p\alpha\theta_L + \theta_L &< 0 \\
-p\,(1-\alpha)\,(1-\theta_L) &< 0.
\end{aligned}$$

The derivative of the equilibrium return with respect to p is given by

$$\frac{\partial R}{\partial p} = -\frac{\left(\theta_H - \theta_L\right)\left(1-\hat{\alpha}\right)}{\sqrt{\left((1-\hat{\alpha})\theta^e + \hat{\alpha}\theta^s\right)^2 - 4\hat{\alpha}\theta^p}} R < 0.$$

By using L'Hopital rule at $\hat{\alpha} = 0$ one can show $\lim_{\hat{\alpha}\to 0} R = \frac{1}{\theta^e}$. Moreover, we have

$$\frac{\partial R}{\partial \hat{\alpha}} = \frac{1}{\hat{\alpha}} \frac{\theta^e R - 1}{\sqrt{\left((1-\hat{\alpha})\theta^e + \hat{\alpha}\theta^s\right)^2 - 4\hat{\alpha}\theta^p}} < 0,$$

since

$$\frac{(1-\alpha)\theta^e + \theta^s\alpha - \sqrt{\left((1-\alpha)\theta^e + \alpha\theta^s\right)^2 - 4\alpha\theta^p}}{2\alpha\theta^p} < \frac{1}{\theta^e},$$

holds if and only if

$$\begin{aligned}
(1-\alpha)\theta^{2e} + \theta^e\theta^s\alpha - \theta^e\sqrt{\left((1-\alpha)\theta^e + \alpha\theta^s\right)^2 - 4\alpha\theta^p} &< 2\alpha\theta^p \\
(1-\alpha)\,\theta^{2e} + \theta^e\theta^s\alpha - 2\alpha\theta^p &< \theta^e\sqrt{\left((1-\alpha)\theta^e + \alpha\theta^s\right)^2 - 4\alpha\theta^p} \\
\left((1-\alpha)\,\theta^{2e} + \theta^e\theta^s\alpha - 2\alpha\theta^p\right)^2 &< \theta^{2e}\left(\left((1-\alpha)\theta^e + \alpha\theta^s\right)^2 - 4\alpha\theta^p\right) \\
\left((1-\alpha)\,\theta^{2e} + \theta^e\theta^s\alpha - 2\alpha\theta^p\right)^2 - \theta^{2e}\left((1-\alpha)\theta^e + \alpha\theta^s\right)^2 + 4\alpha\theta^p\theta^{2e} &< 0 \\
4\theta^p\alpha^2\left(\theta^p + \theta^{2e} - \theta^{s+e}\right) &< 0 \\
\theta^p + \theta^{2e} - \theta^{s+e} &< 0, \\
\theta_H\theta_L + \left(\theta_L + p\left(\theta_H - \theta_L\right)\right)^2 - \left(\theta_L + p\left(\theta_H - \theta_L\right)\right)\left(\theta_H + \theta_L\right) &< 0,
\end{aligned}$$

or

$$-p\left(\theta_H - \theta_L\right)^2\left(1-p\right) < 0,$$

where notice $Var_p\left(\theta\right) = p\left(\theta_H - \theta_L\right)^2\left(1-p\right) > 0$ always.

Proof 2 (Lemma 1)**.** *The repayment probability held by agent i holding an information set Ω_i is*

$$P(\theta^H|x_i,\, x_m) = \frac{q\,\phi\left(\frac{\theta^H - \left(\frac{\sigma_m^2}{\sigma_x^2}x_i + \frac{\sigma_m^2}{\sigma_x^2}x_m + \frac{\sigma_m^2}{\sigma_y^2}y\right)}{\sigma_m}\right)}{q\,\phi\left(\frac{\theta^H - \left(\frac{\sigma_m^2}{\sigma_x^2}x_i + \frac{\sigma_m^2}{\sigma_x^2}x_m + \frac{\sigma_m^2}{\sigma_y^2}y\right)}{\sigma_m}\right) + (1-q)\,\phi\left(\frac{\theta^L - \left(\frac{\sigma_m^2}{\sigma_x^2}x_i + \frac{\sigma_m^2}{\sigma_x^2}x_m + \frac{\sigma_m^2}{\sigma_y^2}y\right)}{\sigma_m}\right)} \tag{40}$$

whereas the one held by a public observer is

$$P(\theta^H|\, x_m) = \frac{q\,\phi\left(\frac{\theta^H - \left(\frac{\sigma_p^2}{\sigma_x^2}x_m + \frac{\sigma_p^2}{\sigma_y^2}y\right)}{\sigma_p}\right)}{q\,\phi\left(\frac{\theta^H - \left(\frac{\sigma_p^2}{\sigma_x^2}x_m + \frac{\sigma_p^2}{\sigma_y^2}y\right)}{\sigma_p}\right) + (1-q)\,\phi\left(\frac{\theta^L - \left(\frac{\sigma_p^2}{\sigma_x^2}x_m + \frac{\sigma_p^2}{\sigma_y^2}y\right)}{\sigma_p}\right)} \tag{41}$$

where

$$\sigma_m^2 := \frac{1}{2\,\sigma_x^{-2} + \sigma_y^{-2}} \quad \text{and} \quad \sigma_p^2 := \frac{1}{\sigma_x^{-2} + \sigma_y^{-2}}$$

denote the conditional standard deviation of investors and public observers respectively.

We note that

$$P(\theta^H|x_i = x_m,\, x_m) = \frac{q}{q + (1-q)\,\kappa_m}$$

where

$$\kappa_m := \frac{\phi\left(\frac{\theta^L - \left(2\frac{\sigma_m^2}{\sigma_x^2}x_m + \frac{\sigma_m^2}{\sigma_y^2}y\right)}{\sigma_m}\right)}{\phi\left(\frac{\theta^H - \left(2\frac{\sigma_m^2}{\sigma_x^2}x_m + \frac{\sigma_m^2}{\sigma_y^2}y\right)}{\sigma_m}\right)} = e^{-\frac{1}{2\sigma_m^2}\left(\left(\theta^L - \left(2\frac{\sigma_m^2}{\sigma_x^2}x_m + \frac{\sigma_m^2}{\sigma_y^2}y\right)\right)^2 - \left(\theta^H - \left(2\frac{\sigma_m^2}{\sigma_x^2}x_m + \frac{\sigma_m^2}{\sigma_y^2}y\right)\right)^2\right)} =$$

$$= e^{\frac{1}{2\sigma_m^2}(\theta_H - \theta_L)\left(\theta_H + \theta_L - 2\left(2\frac{\sigma_m^2}{\sigma_x^2}x_m + \frac{\sigma_m^2}{\sigma_y^2}y\right)\right)}.$$

Performing analogous computations, we can write:

$$P(\theta^H|\, x_m) \;=\; \frac{q}{q + (1-q)\,\kappa_p},$$

where

$$\kappa_p = e^{\frac{1}{2\sigma_p^2}(\theta_H - \theta_L)\left(\theta_H + \theta_L - 2\left(\frac{\sigma_p^2}{\sigma_x^2}x_m + \frac{\sigma_p^2}{\sigma_y^2}y\right)\right)}$$

from which the limit statements can be easily proved. Then we verify that x^* is the solution to

$$\frac{1}{2\sigma_m^2}(\theta_H - \theta_L)\left(\theta_H + \theta_L - 2\left(2\frac{\sigma_m^2}{\sigma_x^2}x_m + \frac{\sigma_m^2}{\sigma_y^2}y\right)\right) = \frac{1}{2\sigma_p^2}(\theta_H - \theta_L)\left(\theta_H + \theta_L - 2\left(\frac{\sigma_p^2}{\sigma_x^2}x_m + \frac{\sigma_p^2}{\sigma_y^2}y\right)\right)$$

which happens when $x^* = (\theta_H + \theta_L)/2$ at which

$$\kappa_p = \kappa_m = e^{\frac{1}{\sigma_y^2}(\theta_H - \theta_H)\left(\frac{\theta_H+\theta_L}{2} - y\right)}.$$

As a result,

$$P(\theta^H | x_i = x_m^*, x^*, y) = P(\theta^H | x^*, y) = \frac{q}{q + (1-q)e^{(\theta_H - \theta_L)\left(\frac{\theta_H+\theta_L}{2} - y\right)\frac{1}{\sigma_y^2}}},$$

which are equal to the prior q in the limit $\sigma_y^2 \to \infty$.

For $x_m < x^*$ it is easy to check that the right hand side is larger than the left hand side, which proves the inequality statement.

Finally, the maximal distance between the two posteriors obtains when

$$\frac{\partial P(\theta^H | x_i = x_m,\, x_m)}{\partial x_m} = \frac{\partial P(\theta^H |\, x_m)}{\partial x_m} \quad \Rightarrow \quad \frac{-q(1-q)\frac{\partial \kappa_m}{\partial x_m}}{(q + (1-q)\kappa_m)^2} = \frac{-q(1-q)\frac{\partial \kappa_p}{\partial x_m}}{(q + (1-q)\kappa_p)^2}$$

and since

$$\frac{\partial \kappa_m}{\partial x_m} = -\frac{2}{\sigma_x^2}(\theta_H - \theta_L)\kappa_m \quad \textit{and} \quad \frac{\partial \kappa_p}{\partial x_m} = -\frac{1}{\sigma_x^2}(\theta_H - \theta_L)\kappa_p$$

we get the two solutions x^+ and x^- as solution to

...

Proof 3 (Lemma 2)**.** *We can rewrite the average ex-post equilibrium return as*

$$\mathbb{E}[\, R(p_{m,m})\, \theta\,] \;\; = \;\; \Big[\, P + (1-P) \,\Big(\int \underbrace{\int_{S_*(\theta)}^{S_+(\theta)} \frac{R(p_m(S,\theta))}{R(p_{m,m}(S,\theta))}\, dS}_{>1} + \underbrace{\int_{S_-(\theta)}^{S_*(\theta)} \frac{R(p_m(S,\theta))}{R(p_{m,m}(S,\theta))}\, dS}_{<1} \Big) d\theta \,\Big]$$

where P denotes the cumulative probability that $x_m(\theta, S)$ is fully revealing of θ.

By construction, both $S_-(\theta)$ and $S_+(\theta)$ strictly decrease (i.e. $x_-(\theta)$ and $x_+(\theta)$) with b_{cb} for any θ. The proof obtains since at $b_{cb} = \underline{b}$, for given θ, the upper boundary of the non revealing region S_- (resp. x_+) reaches its minimum $S_- = 0$ (resp. its maximum $x_+ \to +\infty$), so that marginal increases in b_{cb} at $\underline{b}$ will only shrink the range of S where $R(p_m) > R(p_{m,m})$ without affecting the one where $R(p_m) < R(p_{m,m})$. The second part of the prof obtains since, for given θ, at the limit $S_- \to S_+ \to 0$ (resp. $x_- \to x_+ \to +\infty$), which we get as $b_{cb} \to \underline{b} + \overline{b}$ it is $\mathbb{E}[\, R(p_{m,m})\, \theta\,] \to 1$, *and marginal decreases in b_{cb} at $\underline{b}$ will enlarge the range of S where $R(p_m) < R(p_{m,m})$ pushing $\mathbb{E}[\,R(p_{m,m})\,\theta\,]$ below the natural rate of one.*

Proof 4 (Derivation of the p.d.f. of x_i conditional on x_m in equation (29))**.**

$$
\begin{aligned}
f(x_i|x_m) &= \sum_j q_j f(x_i|x_m,\theta_j) = \sum_j q_j f(x_i|\theta_j) f(x_m|\theta_j)\frac{1}{f(x_m)} = \\
&= \sum_j q_j \frac{\overline{b}+\underline{b}}{\sigma_x}\phi\left(\frac{\theta_j - x_i}{\sigma_x}\right)\frac{1}{\sigma_x}\phi\left(\frac{\theta_j - x_m}{\sigma_x}\right)\frac{1}{f(x_m)} \\
&= \sum_j q_j \frac{\overline{b}+\underline{b}}{\sigma_x/\sqrt{2}}\phi\left(\frac{\theta_j - \frac{x_i+x_m}{2}}{\sigma_x/\sqrt{2}}\right)\frac{1}{\sigma_x\sqrt{2}}\phi\left(\frac{x_i - x_m}{\sigma_x\sqrt{2}}\right)\frac{1}{f(x_m)} \\
&= \frac{1}{\sigma_x\sqrt{2}}\phi\left(\frac{x_i - x_m}{\sigma_x\sqrt{2}}\right)\sum_j q_j \frac{\overline{b}+\underline{b}}{\sigma_x/\sqrt{2}}\phi\left(\frac{\theta_j - \frac{x_i+x_m}{2}}{\sigma_x/\sqrt{2}}\right)\frac{1}{\sum_j q_j \frac{\overline{b}+\underline{b}}{\sigma_x}\phi\left(\frac{x_m-\theta_j}{\sigma_x}\right)} \\
&= \frac{1}{\sigma_x}\phi\left(\frac{x_i - x_m}{\sigma_x\sqrt{2}}\right)\sum_j q_j \phi\left(\frac{\theta_j - \frac{x_i+x_m}{2}}{\sigma_x/\sqrt{2}}\right)\frac{1}{\sum_j q_j \phi\left(\frac{x_m-\theta_j}{\sigma_x}\right)}
\end{aligned}
$$

Proof 5 (Derivations behind equation (31))**.**

$$
\begin{aligned}
\mathbb{E}[Q(b_i)] = &\int_{x_m \in [x_{min}, x_-]\cup[x_+, x_{max}]} f(x_m)dx_m + \\
&+ \int_{x_-}^{x_+}\int_{-\infty}^{+\infty}\left[w(x_i,x_m)\,\mathbb{E}[R\,\theta\,|\,\Omega_i] + (1-w(x_i,x_m))1\right]\; dF(x_i|x_m)\; dF(x_m) = \\
&= \left[P_H + P_L + \int_{x_-}^{x_+}\int_{-\infty}^{+\infty} f(x_i|x_m)\,dx_i dF_{x_m}(x_m)\right] + \\
&+ \int_{x_-}^{x_+}\int_{-\infty}^{x_m}\frac{\underline{b}}{\overline{b}_i}\left(\mathbb{E}[R\,\theta\,|\,\Omega_i] - 1\right) dF_{x_i|x_m}(x_i)\, dF_{x_m}(x_m) + \\
&+ \int_{x_-}^{x_+}\int_{x_m}^{+\infty}\left(\mathbb{E}[R\,\theta\,|\,\Omega_i] - 1\right) dF_{x_i|x_m}(x_i)\, dF_{x_m}(x_m). \\
&= 1 + \int_{x_-}^{x_+}\int_{-\infty}^{+\infty}\left(\mathbb{1}[x_i \geq x_m]1 + \mathbb{1}[x_i < x_m]\frac{\underline{b}}{\overline{b}_i}\right)\left(\mathbb{E}[R\,\theta\,|\,\Omega_i] - 1\right) dF_{x_i|x_m}(x_i)\, dF_{x_m}(x_m). \\
&+ \int_{x_-}^{x_+}\int_{-\infty}^{x_m}\frac{\underline{b}}{\overline{b}_i}\left(\mathbb{E}[R\,\theta\,|\,\Omega_i] - 1\right) dF_{x_i|x_m}(x_i)\, dF_{x_m}(x_m) + \\
&+ \int_{x_-}^{x_+}\int_{x_m}^{+\infty}\left(\mathbb{E}[R\,\theta\,|\,\Omega_i] - 1\right) dF_{x_i|x_m}(x_i)\, dF_{x_m}(x_m).
\end{aligned}
$$

Proof 6 (Derivation of equation (36))**.** *We use Bayes' law to write*

$$\begin{aligned}
P(\theta_H|x_n, b_{cb}) &= \frac{P(b_{cb}|\theta_H, x_n)P(x_n|\theta_H)P(\theta_H)}{\sum_{j\in\{H,L\}} P(b_{cb}|\theta_j, x_n)P(x_n|\theta_j)P(\theta_j)} \\
&= \frac{P\left(\widetilde{S} \in \widetilde{\mathcal{S}}(\theta_H, x_n) : b_{x_n}(\theta_H, \widetilde{S}) = b_{cb}\right) P\left(S = S_{x_n}(\theta_H, x_n)\right) P(\theta_H)}{\sum_{j\in\{H,L\}} P\left(\widetilde{S} \in \widetilde{\mathcal{S}}(\theta_j, x_n) : b_{x_n}(\theta_j, \widetilde{S}) = b_{cb}\right) P(S = S_{x_n}(\theta_j, x_n))\, P(\theta_j)} \\
&= \frac{P(\widetilde{S} = b_{cb} + (\bar{b} + \underline{b})S_{x_n}(\theta_H, x_n))\, P\left(S = S_{x_n}(\theta_H, x_n)\right) P(\theta_H)}{\sum_{j\in\{H,L\}} P(\widetilde{S} = b_{cb} + (\bar{b} + \underline{b})S_{x_n}(\theta_j, x_n))\, P\left(S = S_{x_n}(\theta_j, x_n)\right) P(\theta_j)} \\
&= \frac{P(\widetilde{S} = b_{cb} + (\bar{b} + \underline{b})S_{x_n}(\theta_H, x_n))\, P\left(S = S_{x_n}(\theta_H, x_n)\right) P(\theta_H)}{P(\widetilde{S} = b_{cb} + (\bar{b} + \underline{b})S_{x_n}(\theta_H, x_n))\, P\left(S = S_{x_n}(\theta_H, x_n)\right) \sum_{j\in\{H,L\}} P(\theta_j)} = q.
\end{aligned}$$

The last step exploits the result in (35), and implies most terms cancel out. In the previous steps, we simply use the definition of b_{x_n} from (33) and that of $\mathcal{S}(\theta, x_n)$.

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