MAE Macro II

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Contents

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Acknowledgements

Significant parts of these notes borrow from the following material:

- Recursive Methods in Economic Dynamics by Stokey, Lucas, with Prescott (SLP), Harvard University Press (1989);
- Recursive Macroeconomic Theory by Lars Ljungqvist and Thomas Sargent (LS), 3rd edition, MIT Press (2012)
- lecture notes of Matthias Kredler's lectures, redacted by Sergio Feijoo;
- lecture notes on Advanced Macroeconomic Theory by Nezih Guner;
- lecture notes on Macroeconomic Theory by Dirk Krueger;
- slides on continuous time macro and numerical methods by Benjamin Moll;
- lecture notes on continuous time macro by Pablo Kurlat;
- lecture notes on Macroeconomic Theory by Victor Rios-Rull, redacted by many of his students.

Course Information

Please refer to the syllabus in Aula Global for details on course evaluation, problem set grading, attendance and additional reference material.

1 Finite Horizon

Consider a single agent living for T periods and receiving an exogenous endowment w_t (think of wage).

The utility function is $u(c)$ such that $u' > 0, u'' < 0, \lim_{c \to 0} u'(c) = +\infty$ and u is twice differentiable. We will assume these things basically all the time.

The objective to be maximised is

$$\max_{\{c_t, a_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to the budget constraint

$$w_t + a_t = c_t + \frac{a_{t+1}}{R} \qquad \text{for } t = 0, 1, ..., T$$

and the finite-horizon version of the no-Ponzi-game (who was Ponzi? and who was Madoff?) condition that $a_{T+1} \geq 0$, which states that agents cannot die with debt (what would happen if they could?).

1.1 Sequential approach

Let's first look for the solution using the Lagrangian, which you probably already know.

The Lagrangian is

$$\mathcal{L}\left(\{c_t, a_{t+1}, \lambda_t\}_{t=0}^T, \mu_T\right) = \sum_{t=0}^{T} \beta^t \left\{ u(c_t) + \lambda_t \left[w_t + a_t - c_t - \frac{a_{t+1}}{R} \right] \right\} + \beta^T \mu_T (a_{T+1} - 0)$$

and represents the value of the agent's lifetime utility under a certain plan $\{c_t, a_{t+1}\}_{t=0}^T$ and given a certain value of the constraints (which will be zero by the complementary slackness condition).

First-order conditions (FOCs) (which are necessary but not sufficient for the solution to be an optimum of our problem)

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t \left[u'(c_t) - \lambda_t\right] = 0 && \text{for } t \leq T \\
\frac{\partial \mathcal{L}}{\partial a_{t+1}} &= -\beta^t \lambda_t \frac{1}{R} + \beta^{t+1} \lambda_{t+1} = 0 && \text{for } t \leq T-1 \\
\frac{\partial \mathcal{L}}{\partial a_{T+1}} &= -\beta^T \lambda_T \frac{1}{R} + \beta^T \mu_T = 0
\end{aligned}$$

Simplifying we get the following. First

$$u'(c_t) = \lambda_t$$

which states that, under the optimal consumption and savings path, the marginal utility (MU) of consumption must equal the shadow value of wealth. Why is λ_t the shadow value of wealth? Consider

$$\frac{\partial \mathcal{L}}{\partial w_t} = \beta^t \lambda_t$$

so λ_t represents the marginal change of the maximised value of our objective function $\mathcal{L}$ corresponding to a unitary increase in wealth (w_t in this case, but there are other way to increase wealth!).

Before we move to the other FOCs, consider the complementary slackness condition (also a necessary condition) for the no Ponzi game constraint

$$\mu_T(a_{T+1} - 0) = 0$$

which says that at least one of μ_T and a_{T+1} must equal zero. Since $\mu_T = \lambda_T$ and $\lambda_T = u'(c_T) > 0$, then it must be that $a_{T+1} = 0$. Note that this condition is not the same as the no Ponzi game constraint: the constraint says you can't die with negative wealth, this optimality condition says that the optimal thing to do is to die with zero wealth.

Combine consumption and savings FOCs to get the Euler equation for savings

$$u'(c_t) = \beta R u'(c_{t+1}) \qquad \text{for } t < T.$$

The left-hand side (LHS) is the marginal (opportunity) cost of saving a unit of your wealth today: you could instead buy a unit of consumption with it, which gives you some MU. The right-hand side (RHS) is the marginal benefit of saving a unit of wealth today: you get R units tomorrow, which you can consume and get some MU tomorrow, which you then discount to today with the discount factor β. The Euler equation says that in equilibrium these two things must be equal, otherwise the consumption path $\{c_t\}_{t=0}^T$ would not be optimal: the agent could improve upon it by saving more and consuming less or viceversa.

Another way to look at the Euler equation is to express it in a "micro" way

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = R \qquad\qquad \text{for } t < T$$

where the LHS is the marginal rate of substitution between two "goods", consumption today and consumption tomorrow (why is the β there?), and the RHS is the relative price of the two goods.

A solution of the problem is to find the optimal consumption path $\{c_t\}_{t=0}^T$, or equivalently the optimal savings path $\{a_{t+1}\}_{t=0}^T$ (can you derive one from the other?). Is what we have so far enough to find a solution? We have $T-1$ Euler equations and 1 terminal condition for a_{T+1}, and we need to find the value of the T variables $\{a_{t+1}\}_{t=0}^T$, so the answer is yes. It may however not be that easy to actually find these variables using our equilibrium conditions. Let's see some examples that help us with that.

Let's suppose that $\beta R = 1$. Then the Euler equation implies $u'(c_t) = u'(c_{t+1})$ for $t < T$, so c_t is constant for all t. We can use this! Let's derive first the present value budget constraint (PVBC). Start with the $t = 0$ and $t = 1$ BCs

$$\begin{aligned} a_0 &= c_0 - w_0 - a_1/R \\ a_1 &= c_1 - w_1 - a_2/R \end{aligned}$$

plug the second into the first

$$a_0 = c_0 - w_0 - \frac{1}{R}[c_1 - w_1 - a_2/R].$$

If you keep going all the way to T, you get

$$a_0 = \sum_{t=0}^{T} \frac{c_t - w_t}{R^t}.$$

Now use the fact that consumption is constant and rearrange

$$c = \frac{1-\beta}{1-\beta^{T+1}}\left[a_0 + \sum_{t=0}^{T}\frac{w_t}{R^t}\right] \tag{1}$$

where we used the properties of geometric sums and the fact that $\beta = 1/R$. Equation (1) is nice because it expresses the level of consumption as a function of the present value of wealth at $t = 0$ (the whole term in square brackets) and the discount factor. The term with the βs outside of the brackets is called marginal propensity to consume (MPC), and it represents the marginal change in consumption due to a unit change in the PV of wealth.

Suppose then that $\beta R \neq 1$ and that utility is CRRA: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Then the Euler equation becomes

$$c_t^{-\gamma} = \beta R c_{t+1}^{-\gamma}.$$

Taking logs of both sides we get

$$\log\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\gamma}[\log(\beta) + \log(R)]$$

which shows how the growth rate of consumption (left-hand side)[1] depends on the discount factor, the rate of return and the intertemporal elasticity of substitution (IES) given by $1/\gamma$. The latter is called like that because it represents the elasticity of consumption growth with respect to R.

1.2 Dynamic Programming approach

We now use an alternative approach, on which we will focus most of the course this term. We will do so because this approach is very useful in representing and solving dynamic problems, but we'll say more on this later.

Consider the problem sequentially, starting from the last period. Let $V_T(a)$ denote the (maximised) value (of lifetime utility) for the agent entering period T with a units of the asset.

[1]To see this note that

$$\log\left(\frac{c_{t+1}}{c_t}\right) = \log(1 + g^c_{t,t+1}) \approx g^c_{t,t+1}$$

where $g^c_{t,t+1}$ is the growth rate of consumption between periods t and $t+1$.

The stuff in parenthesis is usually omitted for brevity from the definition of the value function V, but it helps to understand what we're talking about. This value is given by

$$V_T(a) = \max_{a'} u(a + w_T - a'/R) = u(a + w_T) \tag{2}$$

where the last equality is due to the fact that we already know it is optimal to set $a_{T+1} = 0$. Note that

- We carry around income w_T like that because it's fully known and exogenous, so we treat it as if it was a parameter basically.
- We are not using a time subscript for a (which represents a_T) or for a' (which represents a_{T+1}): it's not necessary, as V_T is supposed to indicate the value of lifetime utility for an agent with *some* wealth a, whatever that value is, as we don't know that yet. And a' is *some* choice of future assets, and we can call it whatever we want as we already know we are in period T. We will refer to a as a *state* variable, because it is some sort of pre-determined initial condition in the problem at hand, and to a' as *control* variable, because it is what we can actually choose in the problem at hand.
- in equation (2), we substituted out c and replace it with the budget constraint: this approach may be useful sometimes, but it is by no means compulsory. You can also leave c there and put the budget constraint as the constraint it is. See an example of that in V_{T-2}.

Let's go back to $T - 1$

$$V_{T-1}(a) = \max_{a'} \left\{ u(a + w_{T-1} - a'/R) + \beta V_T(a') \right\}.$$

Here a represents initial wealth at $T - 1$, and a' represents the wealth choice at $T - 1$, but also initial wealth at T (as we saw in the previous step). Now V_{T-1} denotes the (maximised, as the agent is behaving optimally, that's why there's a max) value (of lifetime utility, as it includes all remaining future periods) for the agent entering period $T - 1$ with a units of the asset.

Let's find the optimal value of a': take the derivative of the RHS and set it to zero

$$-u'(a + w_{T-1} - a'/R)\frac{1}{R} + \beta \frac{\mathrm{d}}{\mathrm{d}a'} V_T(a') = 0. \tag{3}$$

We can easily compute the derivative in the last term because we know the form of V_T from above

$$\frac{\mathrm{d}}{\mathrm{d}a}V_T(a) = u'(a + w_T).$$

What we've just done is to derive a value function (the one for the last period) with respect to its state variable. We'll refer to this derivative as the *envelope condition*. Equation (3) then becomes

$$u'(a + w_{T-1} - a'/R)\frac{1}{R} = \beta u'(a' + w_T)$$

which is one equation in one unknown (a'). Its solution will give us the optimal savings choice at $T - 1$ when initial wealth is a. We'll denote this as a function with $a' = g_{T-1}(a)$ and call it *policy function*. Again, it doesn't matter if we use a and a' or a_{T-1} and a_T, it's just notation and it doesn't change the substance. It should be clear that what we just derive is equivalent to

$$u'(c) = \beta R u'(c')$$

which is identical to the Euler equation we got earlier.

We won't solve explicitly for g_{T-1}, but let's use it to write the maximised value of V_{T-1}

$$V_{T-1}(a) = u\left(a + w_{T-1} - \frac{g_{T-1}(a)}{R}\right) + \beta V_T(g_{T-1}(a)).$$

Let's do one more step and go back one more period. The problem is

$$\begin{aligned} V_{T-2}(a) = &\max_{c,a'}\{u(c) + \beta V_{T-1}(a')\} \\ \text{s.t.} \quad & c + \frac{a'}{R} = a + w_{T-2}. \end{aligned}$$

Here we represented the problem with its constraint. The way to take FOCs with this approach is to build something equivalent to the Lagrangian in the sequential approach, i.e.

$$V_{T-2}(a) = \max_{c,a'}\{u(c) + \beta V_{T-1}(a') + \lambda\left[a + w_{T-2} - c - \frac{a'}{R}\right]\} \tag{4}$$

where you can think of the objective function to be maximised (which now also includes the Lagrange multiplier and the budget contraint) as the Lagrangian

$$\mathcal{L}(a, c, a') = u(c) + \beta V_{T-1}(a') + \lambda\left[a + w_{T-2} - c - \frac{a'}{R}\right].$$

The FOCs for c and a' are

$$u'(c) = \lambda$$
$$\lambda = \beta \frac{\mathrm{d}}{\mathrm{d}a'} V_{T-1}(a').$$

We need the envelope condition too

$$\frac{\mathrm{d}}{\mathrm{d}a} V_{T-1}(a) = u'\left(a + w_{T-1} - \frac{g_{T-1}(a)}{R}\right).$$

The reason why we call that envelope condition is given by the *envelope theorem*, which is also the reason why deriving $V_{T-2}(a)$ with respect to a we did not worry about deriving $g_{T-1}(a)$ with respect to a. That is, if we did that, we'd have

$$\frac{\mathrm{d}}{\mathrm{d}a} V_{T-1}(a) = u'(c) - \frac{\mathrm{d}g_{T-1}(a)}{\mathrm{d}a}\left(\frac{u'(c)}{R} - \beta \frac{\mathrm{d}}{\mathrm{d}a'} V_T(a')\right).$$

You can see that the term that multiplies $\frac{\mathrm{d}g_{T-1}(a)}{\mathrm{d}a}$ must equal zero, because it is exactly equal to the FOC for the $T-1$ problem we took in equation (3). This is what the envelope theorem states, that you can ignore the response of a' to a because it's already optimal, and just focus on the effect of the state variable a on the value function $V_t(a)$. We will see this more formally later on in the course.

You can see that in all steps we found a policy function g_t and a value function V_t, and we could keep going. Once we finish going back, we'll have a whole sequence

$$\{V_t(a), g_t(a)\}_{t=0}^{T}$$

of value and policy functions, which is the solution to the problem in the same way as the sequences $\{c_t, a_{t+1}, \lambda_t\}_{t=0}^{T}$ were the solution to the sequential problem. The nice thing with the dynamic programming approach will be that in infinite horizon we'll get rid of the t subscripts and so the solution will just be two functions V and g, while the solution to the sequential problem will continue to be an infinite sequence of variables, which is a much less handy object to deal with.

1.3 A more general formulation

Sequential Problem. A general version of our initial problem in the sequential form is

$$\begin{aligned} V_0(x_0) = & \sup_{\{x_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t F_t(x_t, x_{t+1}) \\ \text{s.t.} \quad & x_{t+1} \in \Gamma(x_t) \qquad \text{for } t = 0, 1, ..., T. \\ & x_0 \text{ given.} \end{aligned} \tag{SP}$$

We'll call this SP for "Sequential Problem".

A bit of notation now. Let

- X denote the set of all possible values for x_{t+1}, so it must be that $x_{t+1} \in X$ $\forall t$.
- $\Gamma_t : X \to X$ denotes the correspondence (which is a one-to-many function) that maps the set of feasible actions x_{t+1} that can be taken in a given period, for a certain value of the variable x_t in that period.
- $F_t(x_t, x_{t+1}) : X \times X \to \mathbb{R}$ denote the period return function.

The ingredients of our problem are (X, F, β, Γ), where X, Γ are somehow related to the *technology* of the problem, while F, β are related to preferences. And $V_0(x_0)$ is, again, the maximised lifetime value for an agent starting with initial condition x_0.

Recursive Problem. As we did in our example before, we can rewrite (SP) as a Functional Equation (FE henceforth)[2]

$$V_t(x) = \sup_{x' \in \Gamma_t(x)} \{F_t(x, x') + \beta V_{t+1}(x')\} \qquad \text{for } t = 0, 1, ..., T. \tag{FE}$$

In this formulation, V_t is a value function, x and t are the state variables, x' is the control variable, Γ_t is the feasible set correspondence and $F_t(x, x')$ is the return function.

V_t is the value function at t: it denotes the maximised *residual* lifetime value for an agent that enters preiod t with state x.

What is a state? It's something *sufficient to summarise the problem* at any point in time.

[2]Once again, it's irrelevant whether you write variables with or without time subscripts, as here the only distinction that matters is current vs. future periods. So (x, x'), (x_t, x_{t+1}) or (x, x_+) are all examples of valid notation pairs.

It's the smallest set of variables at time t that allows to:

1. determine the feasible set of controls $\Gamma_t(x)$
2. determine the current-period return $F_t(x, x')$ given a choice of controls x'
3. determine the value tomorrow given a choice of controls

These are the main criteria to decide what variables need to belong to the set of state variables. What instead does not need to belong to such set, are the fixed objects of the problem, that do not really change with choices or shocks. Things like the discount factor and the utility function parameters are not state variables, they are just parameters that are fixed, so we do not need to carry them around in our value function arguments. With respect to things like endowments or wages, if they are deterministic, they also do not really need to be included in the set of states, because they are already fixed and known. But if you do include these last ones as state variables, it's not a mistake, so there is some flexibility.

The policy function associated to the problem will be

$$g_t(x) = \arg \sup_{x' \in \Gamma_t(x)} \{F_t(x, x') + \beta V_{t+1}(x')\}. \tag{5}$$

As we will see later on, (FE) and (SP) are equivalent!

Analogy with the Consumption-Savings problem. In the consumption-savings problem:

- the state variable is a (again, think of w_t as a parameter given it's fully exogenous and deterministic)
- the control variable is a'
- the return function is $F(a, a') = u(w_t + a - a'/R)$
- the feasible set correspondence is $\Gamma_t(a) = [-\infty, (w_t + a)R]$ for $t = 0, 1, ..., T - 1$ (since $c \geq 0$) and $\Gamma_T(a) = [0, (w_T + a)R]$ since the no Ponzi condition does not allow borrowing.
- the value function is

$$\begin{aligned} V_t(a) &= \max_{a' \in \Gamma_t(a)} \{u(w_t + a - a'/R) + \beta V_{t+1}(a')\} \quad \text{for } t = 0, 1, ..., T \\ V_{T+1}(a) &= 0. \end{aligned}$$

2 Infinite Horizon

We now move to the analysis of the neoclassical growth model (NGM) in infinite horizon.

There is a single, representative agent in a production economy. There is a single good y_t that can be either consumed c_t or transformed into capital used for production. The production technology is $y_t = H(k_t, n_t)$ where $k_t \geq 0$ is capital and $n_t \in [0,1]$ is labour. We'll assume that H is concave, continuously differentiable and features constant returns to scale.[3] The resource constraint of the economy is

$$c_t + i_t \leq y_t$$

the capital law of motion (LOM) is

$$i_t = k_{t+1} - (1-\delta)k_t$$

and initial capital is equal to k_0 and is exogenously given.

The agent has preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

which satisfies the usual assumptions. We assume labour is fixed ($n_t = 1$) so there is no disutility from labour and we can define

$$f(k_t) = H(k_t, 1) + (1-\delta)k_t$$

so that the resource constraint becomes

$$c_t + k_{t+1} = f(k_t).$$

[3] That is, it is homogeneous of degree one, which means that

$$H(zk_t, zn_t) = zH(k_t, n_t).$$

2.1 Sequential problem

Using the new resource constraint to substitute out consumption we can write the Sequential Problem

$$
\begin{aligned}
V_0(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\
\text{s.t.} \quad & k_{t+1} \in [0, f(k_t)] \\
& k_0 \quad \text{given.}
\end{aligned}
\tag{SP-NGM}
$$

Let's find the optimality conditions using the Lagrangian, and let's assume for a second that we're in a finite horizon setting where the last period is T (then we'll take the limit for $T \to \infty$, but this helps). The Lagrangian is

$$
\mathcal{L}\left(\{k_{t+1}, \mu_t\},\right) = \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}) + \mu_t(k_{t+1} - 0)
$$

where we ignored the constraint $c_t \geq 0$ (or equivalently $k_{t+1} \leq f(k_t)$) given that it will always be satisfied since marginal utility is infinity at $c \to 0$. The FOC for capital is

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &= -\beta^t u'(c_t) + \beta^{t+1} f'(k_{t+1}) u'(c_{t+1}) + \mu_t = 0 \quad \text{for} \quad t = 0, 1, ..., T-1 \\
\frac{\partial \mathcal{L}}{\partial k_{T+1}} &= -\beta^T u'(c_T) + \mu_T = 0.
\end{aligned}
$$

and the complementary slackness condition for the non-negativity constraint is $\mu_t k_{t+1} = 0$.

For all periods except the last one there is no danger that $k_{t+1} = 0$ because that would imply zero production and consumption in the following period, so $\mu_t = 0$ for $t < T$ and we get the usual Euler equation

$$
u'(c_t) = \beta f'(k_{t+1}) u'(c_{t+1}).
$$

In the last period, investment is a bad idea as there is no production tomorrow, and it reduces consumption today, so $k_{T+1} = 0$ and the non-negativity constraint is binding (if you could, you would set k_{T+1} negative, but you can't). You can also see that $\mu_T > 0$ because $u'(c_T)$ must be positive. Using the second FOC, the complementary slackness condition for the last period can be rewritten as

$$
\beta^T u'(c_T) k_{T+1} = 0.
$$

The intuition is simple: either the MU of consumption is zero, and then you don't care about having k_{T+1} positive because changes in consumption are irrelevant, or the MU of consumption is positive and then investing for no production tomorrow is a bad idea.

Now let's go back to infinite horizon: the equivalent of the complementary slackness condition becomes

$$\lim_{t\to\infty} \beta^t u'(c_t) f'(k_t) k_t = 0.$$

This is what is called *tranversality condition* (TVC henceforth) and is an important optimality condition in infinite horizon models. Using the Euler equation, we can rewrite is as

$$\lim_{t\to\infty} \beta^t u'(c_t) f'(k_t) k_t = 0.$$

This says that the NPV of "terminal" capital (in discounted marginal utility terms), where the value is computed using the capital price given by $f'(k_t)$ must be zero. This is the infinite-horizon equivalent of saying that it's not optimal to die with positive wealth in a finite horizon model.

The Euler equation and the TVC below

$$u'(c_t) = \beta f'(k_{t+1}) u'(c_{t+1})$$
$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0$$

are necessary *and sufficient* condition for an optimum in this model (you will have to prove the sufficiency part in one of the problem sets, where you will see why the TVC is needed and has the form it has).

Now the question is, how do we find a solution of this problem from these conditions? In Macro I, you may have seen that in the special case with $u(c) = \log c, \delta = 1$ and $f(k) = k^\alpha$, one can guess that $k_{t+1} = \gamma k_t^\alpha$ and using the Euler equation verify that $k_{t+1} = \alpha\beta k_t^\alpha$ and in turn $c_t = (1 - \alpha\beta) k_t^\alpha$. But it is very rare to be able to derive a solution by pen. When this is not possible, then solving the problem boils down to solving a second-order difference equation in k_t, k_{t+1}, k_{t+2} (which is the Euler equation) with a terminal condition given by the TVC, which is not an easy task. Dynamic Programming is meant to give us the tools to solve this problem in a simpler and faster way, by hand (also rare) or with a computer.

2.2 Recursive problem

Going back to our general formulation, we now have:

- return function $F(x_t, x_{t+1})$ given by $u(f(k_t) - k_{t+1})$
- feasible set correspondence given by $\Gamma(k_t) = [0, f(k_t)]$

First, let's see how to go from the sequential to the recursive formulation. The sequential problem was

$$V_0(k_0) = \max_{\substack{\{k_{t+1}\}_{t=0}^{\infty} \\ k_0 \text{ given}}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \tag{6}$$

using the properties of the max operator we can rewrite (note the changes in the time subscripts!)

$$V_0(k_0) = \max_{\substack{k_1 \in \Gamma(k_0) \\ k_0 \text{ given}}} \left\{ u(f(k_0) - k_1) + \beta \max_{\substack{\{k_{t+1}\}_{t=1}^{\infty} \\ k_1 \text{ given}}} \sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right\}$$

let's do a change of variable (let $s = t-1$ and use x_s instead of k_{t+1}) to make things even clearer

$$V_0(k_0) = \max_{\substack{k_1 \in \Gamma(k_0) \\ k_0 \text{ given}}} \left\{ u(f(k_0) - k_1) + \beta \max_{\substack{\{x_s\}_{s=0}^{\infty} \\ x_0 \text{ given}}} \sum_{s=0}^{\infty} \beta^{s} u(f(x_s) - x_{s+1}) \right\}$$

and now finally note that the last object in the equation is equivalent to what equation (6) would be for $V_0(x_0)$, which is the same as $V_0(k_0)$ if we change the variable name. Now let's get rid of the time subscript on V given that, as we just showed, the value function is *timeless*, because "time to death" is always infinity. In other words, at any point in time the problem is *only* defined by the state variable k and nothing else!

We can thus write

$$V(k_0) = \max_{k_1 \in \Gamma(k_0)} \{u(f(k_0) - k_1) + \beta V(k_1)\}$$

but here the time subscripts in k_0 and k_1 only really denote "current" vs "future" capital, so we can just use k, k' for that purpose. To conclude, we have shown that our value function is

$$V(k) = \max_{k' \in \Gamma(k)} \{u(f(k) - k') + \beta V(k')\} \tag{7}$$

and the associated policy function will have the form

$$g(k) = \arg \max_{k' \in \Gamma(k)} \left\{ u(f(k) - k') + \beta V(k') \right\}. \tag{8}$$

As said before, the DP (or recursive) approach boils down to finding the value and policy *functions*, i.e. just two functions which can thus be applied to any value of the state variable k to get the policy and the value of lifetime utility in that state, rather than finding the infinite sequence $\{k_{t+1}\}_{t=0}^{\infty}$ that solves the problem.

2.2.1 Solving the recursive problem.

Let us take FOCs and derive once more the Euler equation from the recursive formulation of the neoclassical growth model in infinite horizon. In (7) we have formulated the problem in reduced form, i.e. plugging the resource constraint instead of c. As we say with (4), there is an alternative, more flexible way to pose the problem which is to set up something similar to a Lagrangian, as we were doing in the sequential problem:

$$V(k) = \max_{k' \in \Gamma(k)} \left\{ u(c) + \beta V(k') + \lambda [f(k) - k'] \right\}$$

The FOCs are

$$u'(c) = \lambda$$
$$V'(k') = \lambda$$

and the envelope condition is $V'(k) = u'(c) f'(k)$, which can be rolled forward by one period and gives $V'(k') = u'(c') f(k')$. Putting everything together we get our usual Euler equation

$$u'(c) = \beta f'(k) u'(c').$$

We've properly specified the neoclassical growth model in recursive form and we've shown that it looks equivalent (we'll be more formal on this later) to the sequential one. Now, how do we find a solution though? How do we know there's only one solution, for example?

One option is to solve for the policy function in the Euler equation, but this is a similar idea to the way we solve the sequential problem, it's difficult and there is rarely an explicit solution.

Another option is to solve for function V first. Mathematically, (7) is a specific type (called *Bellman equation*) of what is known as a *functional equation*. Let T be an *operator* on function V, given by

$$(TV)(k) = \max_{k' \in \Gamma(k)} \{u(f(k) - k') + \beta V(k')\}. \tag{9}$$

What T does to function V (last object you see inside curly brackets) is the following (going from right to left in the right-hand side of equation (9)): it evaluates it at some point k', it discounts it with β, it adds $u(f(k) - k')$ to it, and then it looks for the k' that maximises this expression within some feasible set $\Gamma(k)$. This is the meaning, in words, of applying operator T to function V and evaluating it at k. Note that, since we are taking a max with respect to k', TV does not depend on k' but just on k. Then, looking for the function V that solves our Bellman equation actually means we are solving for a fixed point (but really it's a "fixed function") of the operator T, i.e. any function V such that, once you apply T to it, returns V itself. Mathematically, we're looking for any solution to $V = TV$.

We already know what is a fixed point: for example, function $f(x) = x^3$ has three fixed points, i.e. three solutions to $x = f(x)$, which are $x = \{-1, 0, 1\}$.

Let us now look at some functional equations and their solutions.

Example 1. *One example is*

$$f(x) = x - y + f(y).$$

The RHS of the equation is an operator that takes function f, evaluates it at some (any) point y, adds x and subtracts y to it, finally returning $(Tf)(x)$ as output. That is, we are applying operator T to function f for whatever value of y, evaluating it at point x: $(Tf)(x) = x - y + f(y)$. One solution (may not be the only one) to this functional equation is $f(x) = x + c$ for any $c \in \mathbb{R}$ (check it!).

Another example is

$$f(x)f(y) = f(x + y)$$

where one solution is $f(x) = e^x$ (check this too!).

A last example:

$$f(x)f(y) = f(xy).$$

One solution of this FE is $f(x) = x^n$. But note that also $f(x) = 0$ is a solution for $x = 0$, while also $f(x) = 1$ is a solution for any $x \neq 0$. This example is particularly instructive of the fact that the restrictions we impose on the solution, or on the domain and codomain of our variables, do matter.

So, how do we find a solution to our special functional equation, i.e. any dynamic macro problem expressed in a recursive way? The two most popular answers are Guess & Verify and Value Function Iteration. The former is a pretty quick method, but it is rarely applicable as it requires you to have a good idea of what shape the value function will have, which may not always be the case, and if you make the wrong guess you will not find a solution. The latter method instead is a slower but very robust method: you make some guess for the value function (any guess will work!) and then iterate on it until you converge to the solution. The powerful thing of this method is that it can easily be done with the computer and, under some conditions which we will see soon, it always yields the unique solution of the problem.

In general, this is how value function iteration (VFI) works (using the NGM as an example).

- Start with an initial guess $V_0(k)$ (here the subscript denotes the iteration count, not the time!). You can even guess that $V_0(k) = 0$ for all k, for example.
- Write down the first iteration

  $$V_1(k) = \max_{k' \in \Gamma(k)} u(f(k) - k') + \beta V_0(k')$$

  and since we know the shape of V_0, we can take the FOCs for k', find the optimal policy $g_0(k)$, and get

  $$V_1(k) = u(f(k) - g_0(k)) + \beta V_0(g_0(k)).$$

  This step consisted in applied the Bellman operator T once, i.e. $V_1 = TV_0$.
- You can keep going, and at the n-th step you will have

  $$V_{n+1}(k) = \max_{k' \in \Gamma(k)} u(f(k) - k') + \beta V_n(k).$$

  The goal is that as you continue to iterate, you will eventually converge to the true value function $V(k)$, that is $\lim_{n\to\infty} V_n(k) = V(k)$. The true V is in fact a fixed point of the Bellman operator, $V = TV$.

As mentioned above, the "magic" of dynamic programming is that, under some conditions, V_n will always converge to V as $n \to \infty$, and that there exists only one solution V to the Bellman equation.

Guess & Verify example. Let's take the NGM example with log utility and full depreciation. Our Bellman equation is

$$V(k) = \max_{k' \in [0, k^\alpha]} \{\log(k^\alpha - k') + \beta V(k')\}. \tag{10}$$

Let's guess that the value function has the shape $V(k) = A + D\log(k)$, where A and D are two constants that we must solve for. Our Bellman equation thus becomes

$$V(k) = \max_{k' \in [0, k^\alpha]} \{\log(k^\alpha - k') + \beta[A + D\log(k')]\}.$$

Taking the FOC for k' yields $k' = \frac{\beta D}{1+\beta D}k^\alpha$, which implies $c = \frac{1}{1+\beta D}k^\alpha$. So our Bellman equation is now

$$V(k) = \log(k^\alpha) - \log(1+\beta D) + \beta A + \beta D \log(k^\alpha) + \beta D \log\left(\frac{\beta D}{1+\beta D}\right)$$

where we have expressed the whole RHS as a function of k and parameters only. Let us now plug our guess into the LHS as well

$$A + D\log(k) = \alpha\log(k) - \log(1+\beta D) + \beta A + \beta D \alpha \log(k) + \beta D \log\left(\frac{\beta D}{1+\beta D}\right).$$

For this equality to hold, we need the constant terms on the LHS to equal those on the RHS, and the terms with $\log(k)$ on the LHS to equal those on the RHS. Let's start with the latter, that requires $D = \alpha(1+\beta D)$ so we get $D = \frac{\alpha}{1-\alpha\beta}$. Solving for the constant terms is a bit more tedious, and after a few lines of algebra we get

$$A = \frac{1}{1-\beta}\left[\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)\right].$$

We got value of A and D as a function of parameters only, so we're done! We can write down the policy function too, which is $g(k) = \frac{\alpha\beta}{1+\alpha\beta}k^\alpha$.

To conclude, let's check if the solution we found satisfies the transversality condition:

$$\begin{aligned}
&\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = \\
&= \lim_{t\to\infty} \beta^t \frac{1}{k_t^\alpha \left(1 - \frac{\alpha\beta}{1-\alpha\beta}\right)} \frac{\alpha\beta}{1-\alpha\beta} k_t^\alpha = \\
&= \lim_{t\to\infty} \beta^t \alpha\beta = 0
\end{aligned}$$

3 Maths Preliminaries

As anticipated, we will now study the properties of the Bellman equation (BE) and find out when VFI will yield a solution, when such solution is unique, and why that is so. Recall that the BE in its general form[4] is

$$v(x) = \sup_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta v(x') \right\}. \tag{BE}$$

In other words, the Bellman operator is

$$(Tv)(x) = \sup_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta v(x') \right\}. \tag{BO}$$

The questions we will ask are:

- When is (BE) well defined?
- When does (BE) have a solution?
- A solution to (BE) is a fixed point of the operator (BO): is Tv of the same "type" of v?
- When is the solution unique?
- How do we find the solution? Is it always the case that $T^n v \to v$?
- Are (BE) and the sequential problem (SP) the same? Does v coincide with the solution of (SP)?

First, let us introduce some maths concepts. What we will do here will be an application of what you have seen with Juan Pablo.

[4] We now use small v to denote the value function.

3.1 Metric spaces.

Definition 1 (Metric)**.** *Pick any set X. A metric (or distance) of set X is a function $d : X \times X \to \mathbb{R}_+$ such that for all $(x, y, z) \in X$*

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ *if and only if* $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$ *(this is known as triangle inequality)*

Definition 2. *A set-distance pair (X, d) is called a metric space.*

Examples of metric spaces:

- $X = \mathbb{R}$ and $d(x, y) = |x - y|$
- $X = \mathbb{R}^2$ and $d((x_1, y_1), (x_2, y_2)) = [(x_1 - x_2)^2 + (y_1 + y_2)^2]^{1/2}$ (this distance is called d_2 or 2-norm or Euclidean/Pythagorean distance)
- $X = \mathbb{R}^k$ and $d(\boldsymbol{x}, \boldsymbol{y}) = \left[\sum_{i=1}^{k} (x_i - y_i)^n\right]^{1/n}$ (this distance is called d_n or n-norm)
- $X = \mathbb{R}^k$ and $d(\boldsymbol{x}, \boldsymbol{y}) = \max\left(|x_1 - y_1|, ..., |x_k - y_k|\right)$ (this distance is called d_∞ or sup-norm)

Metric spaces need not just be sets of elements and metrics/distance between elements. One can also have spaces of functions and metrics between functions. Some examples of sets of functions are

- $\mathcal{C}(X)$ is the set of all continuos functions
- $\mathcal{B}(X)$ is the set of all bounded functions[5]
- $\{f : X \to \mathbb{R} \text{ s.t. } f \text{ bounded and continuous}\}$ is the set of all bounded and continuos functions.

Some examples of metrics on sets of functions are:

- $\mathrm{d}_n(f, g) = \left\{\int_X [f(x) - g(x)]^n \mathrm{d}x\right\}^{1/n}$ (when $n = 1$, this is the area between the graphs of the two functions)
- $\mathrm{d}_\infty(f, g) = \max_{x \in X} |f(x) - g(x)|$

[5] A function $f : X \to \mathbb{R}$ is bounded if there exists some $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in X$.

3.2 Completeness and convergence.

Definition 3 (Convergence). *A sequence $\{x_n\}_{n=0}^{\infty}$ in metric space X converges to limit $x \in X$ if for any $\epsilon > 0$, there exists a number N_ϵ such that $d(x_n, x) < \epsilon$ for all $n \geq N_\epsilon$.*

Definition 4 (Cauchy Sequence). *A sequence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence if for any $\epsilon > 0$, there exists a number N_ϵ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N_\epsilon$.*

Both of these two definitions basically ask that the elements of a sequence stay close to each other after a certain number of steps. A convergent sequence is always Cauchy, but a Cauchy sequence is not necessarily a convergent one. For example, consider $x_n = 1/n$ in the metric space $((0,1), |\cdot|)$. The sequence is Cauchy in $(0,1)$, but does not converge to any point inside the interval. This however is a somehow fine point, and henceforth we will focus on sequences where also the converse is true.

Definition 5 (Completeness). *A metric space (X, d) is complete if every Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n \in X \, \forall n$ converges to some $x \in X$.*

Some examples of metric spaces (X, d) that are (or not) complete, taken from exercise 3.6 of Stokey, Lucas with Prescott (SLP) and its solutions by Irigoyen, Rossi-Hansberg and Wright:

- X is the set of all integers, $d(x, y) = |x - y|$.

  The metric space is complete. Take a Cauchy sequence with $x_n \in X$ for all n. Choose $\epsilon \in (0, 1)$. Being the sequence Cauchy, there exists N_ϵ such that $|x_n - x_m| < \epsilon < 1$ for all $n, m \geq N_\epsilon$. Hence, $x_n = x_m = x \in X$ for all $n, m \geq N_\epsilon$.
- X is the set of integers, $d(x, y) = \mathbb{1}(x \neq y)$.

  The metric space is complete. The reasoning is the same as above, we can pick an $\epsilon \in (0, 1)$ such that $x_n = x_m = x \in X$ for all $n, m \geq N_\epsilon$.
- X is the set of functions that are $\mathcal{C}([a, b])$ and are strictly increasing, $d(f, g) = \max_{a \leq x \leq b} |f(x) - g(x)|$.

  The metric space is not complete. Consider $f_n(x) = 1 + x/n$. This is a Cauchy sequence because $f_n(x)$ and $f_m(x)$ get arbitrarily close as n, m grow. But $f_n(x) \to f(x)$ as $n \to \infty$, and $f(x) = 1$ which is not strictly increasing and thus $f(x) \notin X$.
- X is the set of functions that are $\mathcal{C}([a, b])$ and $d(f, g) = \int_a^b |f(x) - g(x)| dx$.

  The metric space is not complete. Take $f_n(x) = \left(\frac{x-a}{b-a}\right)^n$. The limit of this sequence is $f(x) = \begin{cases} 1 & \text{if} \quad x = b \\ 0 & \text{if} \quad x \in [a, b) \end{cases}$ which is clearly not a continuous function so $f(x) \notin X$.

- X is the set of functions that are $\mathcal{C}([a,b])$ and $d(f,g) = \sup_{a \leq x \leq b} |f(x) - g(x)|$.

  This metric is complete. The reason why the previous counterexample does not apply to this case too, is that sequences of the form $f_n(x) = x^n$ are not Cauchy under the sup-norm metric. Why? For the sequence to be Cauchy, we need that for all ϵ we can find a N_ϵ such that $\sup_x |x^n - x^m| < \epsilon$. With the sup-norm, N_ϵ depends on x and so the sequence is not Cauchy.

  The formal reasoning is the following. Suppose the N_ϵ that satisfies the definition exists. Then it must be that $\sup_x |x^N - x^m| < \epsilon$ for all $m \geq N$. Now note that x^N for a fixed N is continuous[6], so for any η we can find a δ such that (take the definition of continuity and let $x = 1$) $|f(1) - f(x')| = |1 - (x')^N| < \eta$ for $|1 - x'| < \delta$. In other words, if $x' > 1 - \delta$ then $(x')^N > 1 - \eta$. Now, note that there exists an m large enough that $(x')^m < \eta$. So at x' we have that

  $$|(x')^N - (x')^m| = (x')^N - (x')^m > 1 - 2\eta$$

  Let $2\eta = \epsilon$. We finally get

  $$\sup_x |x^N - x^m| \geq |(x')^N - (x')^m| \geq \epsilon$$

  which contradicts the definition of Cauchy sequence.

It will be useful to know that the sets $\mathbb{R}^n$ for any n always forms a complete metric spaces.

Definition 6. *Let (X,d) be a metric space.*

- *set $A \subset X$ is closed if $a_n \in A$ such that $a_n \to a$ implies that $a \in A$*
- *set $A \subset X$ is bounded if there exists a D such that $d(a,a') \leq D$ for all $(a,a') \in A$*
- *set $A \subset X$ is compact if it is closed and bounded.*

The following is a useful theorem that we will use going forward.

Theorem 1 (Completeness of continuous bounded functions (SLP Theorem 3.1))**.** *Let $X \in \mathbb{R}^n$, $\mathcal{C}(X)$ denote the set of bounded and continuous functions $f : X \to \mathbb{R}$ and $d(f,g) = \sup_{x \in X} |f(x) - g(x)|$. The metric space $(\mathcal{C}(X), d)$ is a complete metric space.*

[6] A function $f(x)$ is continuous in x if for any $\epsilon > 0$ we can find a $\delta > 0$ such that $|f(x) - f(x')| < \epsilon$ for $|x - x'| < \delta$.

To clarify, what's the difference between completeness and closedness? First, as we saw completeness depends on the metric, while closedness is just a property of a set. Second, completeness implies closedness: since a complete metric space is such that all Cauchy sequences converge, then all sequence within a set converge to that set.

3.3 Contractions and fixed points.

Definition 7 (Contraction Mapping). *Let (X, d) be a metric space. The operator $T : X \to X$ is a contraction mapping with contraction parameter (or modulus) β if and only if*

$$d(T(x), T(y)) \leq \beta d(x, y)$$

for any pair $(x, y) \in X$.

Note that this definition also applies to sets of functions.

Let us look at some examples.

- $T(x) = 0$ is a contraction, since $d(T(x), T(y)) = 0$ while $\beta d(x, y) \geq 0$ for any $(x, y) \in X$.
- Consider $T : [a, b] \to [a, b]$ where T is continuous, differentiable and with a slope that is uniformly less than β, i.e. $\sup_{x \in (a,b)} |T'(x)| \leq \beta < 1$. We have that T is a contraction. Why? Because of the mean-value theorem, there exists a $c \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$ if f is continuous and differentiable. Then

$$|T(x) - T(y)| \leq \sup_{z \in (a,b)} |T'(z)||x - y| \leq \beta|x - y|$$

  and T satisfies the definition of contraction.

Definition 8 (Fixed Point). *A fixed point of a mapping $T : X \to X$ is some element $x \in X$ such that $T(x) = x$.*

Again, X can be a set of functions, in which case x will be some function in the set. Some examples of fixed points of functional equations can be found in Section 2.2.1 where we gave some examples.

Example 2. *Let us look at an example now. Consider the functional equation*

$$v(x) = \sup_{y \in [0,2x]} \{y - 2x + \beta v(y)\}$$

where $\beta \in (0,1)$. Let us assume that v is differentiable, derive with respect to y and write down the FOC

$$1 + \beta v'(y) = 0.$$

To find $v'(x)$ we typically apply the envelope condition, but (as we will see formally later) the envelope theorem only applies to cases where the optimal policy is interior, which may not be the case here, so let us be careful and proceed in steps.

Suppose first that the optimal policy is $y = 0$ for any x, in which case $v'(x) = -2x$ and indeed the FOC for y confirms $y = 0$ is optimal. Then the functional equation becomes

$$v(x) = -2x + \beta v(0).$$

Consider $x = 0$ (we did not restrict the domain of the state variable x, so we assume $x \in \mathbb{R}$):

$$v(0) = \beta v(0)$$

which shows us that one solution of this functional equation must satisfy $v(0) = 0$. We now know that $v(0) = 0$ and that $v'(x) = -2$, so have enough to find one solution: $v(x) = -2x$.

Suppose now that the optimal policy is $y = 2x$ (the other corner solution) for all x. Then $v(x) = \beta v(2x)$, whose only solution is a constant function independent of x. Since at $x = 0$ we still have that $v(0) = \beta v(0)$, we have found another solution which is $v(x) = 0$ for all x.

The third possibility is that the optimal policy is $y \in [0, 2x]$ which is only possible if $v'(x) = -1/\beta$. Then let's guess $v(x) = A - x/\beta$ for some constant A and see if we can find a third solution. We have

$$A - x/\beta = y - 2x + \beta(A - y/\beta)$$

simplifying we get

$$A(1 - \beta) = x(1/\beta - 2)$$

which must hold for all values of x. This would be a solution if $A = 0$ and $\beta = 1/2$, which is however ruled out by our restriction on β.

This was an example of a functional equation with multiple fixed points. We will come back to this example to show why one of the two fixed points is "better" than the other.

Example 3. *One more example. Consider the operator*

$$Tv(x) = \sup_{x' \in \mathbb{R}} \frac{x^2}{2} - x' + \beta v(x').$$

Our functional equation is

$$v(x) = Tv(x) = \sup_{x' \in \mathbb{R}} \frac{x^2}{2} - x' + \beta v(x').$$

The FOC is $-1 + \beta v'(x') = 0$. Here the solution is interior, so we can write down the envelope condition is $v'(x) = x$, so the optimal policy is $x' = 1/\beta$ for any x. Plug it into our FE

$$v(x) = \frac{x^2}{2} - 1/\beta + \beta v(1/\beta).$$

This FE must hold for any value of x, so let us look at the case where $x = 1/\beta$:

$$v(1/\beta) = \frac{x^2}{2} - 1/\beta + \beta v(1/\beta)$$

which gives $v(1/\beta) = \frac{1/\beta\left(\frac{1}{2\beta}-1\right)}{1-\beta}$. We found the value of $v(1/\beta)$! But we still don't know the value of $v(x)$ when $x \neq 1/\beta$. It is

$$v(x) = \frac{x^2}{2} - 1/\beta + \beta \frac{1/\beta\left(\frac{1}{2\beta} - 1\right)}{1-\beta}.$$

We now have all the elements to state the contraction mapping theorem.

Theorem 2 (Contraction Mapping Theorem, SLP Theorem 3.2)**.** *If*

- *(X, d) is a complete metric space*
- *$T : X \to X$ is a contraction mapping with parameter β*

then

- *T has exactly one fixed point $v \in X$ (i.e. $v = Tv$) (this is the "existence & uniqueness" part of the theorem)*
- *for any $v_0 \in X$, we have that $d(T^n(v_0), v) \leq \beta^n d(v_0, v)$ for any $n = 0, 1, ...$ (this is the "convergence everywhere" part of the theorem)*

This is a central theorem in dynamic programming and recursive methods in general, so we will go through the proof.

Proof. First, we prove the existence of a fixed point. Let $\{v_n\}_{n=0}^{\infty}$ where $v_{n+1} = Tv_n$. We know

$$d(v_{n+1}, v_n) = d(Tv_n, Tv_{n-1}) \leq \beta d(v_n, v_{n-1}) = \beta d(Tv_{n-1}, Tv_{n-2})$$

where the inequality comes from the properties of a contraction. We can keep following the backwards iteration to get to

$$d(v_{n+1}, v_n) \leq \beta^n d(v_1, v_0) \quad \text{for } n = 0, 1, ...$$

Now let us verify that $d(v_m, v_n)$ for $m > n$ is a Cauchy sequence. First

$$d(v_m, v_n) \leq d(v_m, v_{m-1}) + d(v_{m-1}, v_n)$$

by the triangle inequality. We can keep going for all the numbers between n and m

$$d(v_m, v_n) \leq d(v_m, v_{m-1}) + d(v_{m-1}, v_{m-2}) + ... + d(v_{n+1}, v_n)$$

and using the results from above

$$\begin{aligned} d(v_m, v_n) &\leq \beta^m d(v_1, v_0) + \beta^{m-1} d(v_1, v_0) + ... + \beta^n d(v_1, v_0) \\ &= \beta^n \left(\beta^{m-n} + \beta^{m-n-1} + ... + \beta + 1\right) d(v_1, v_0) \\ &\leq \frac{\beta^n}{1-\beta} d(v_1, v_0) \end{aligned}$$

where the last line comes from the properties of geometric sums, and is such that $\frac{\beta^n}{1-\beta} d(v_1, v_0) \to 0$ as $n \to \infty$. So we have a Cauchy sequence, because we can always pick a n that makes $d(v_m, v_n)$ as small as we want. Given that (X, d) is a complete metric space by assumption, then every Cauchy sequence has a limit inside X, that is $\{v_n\}_{n=0}^{\infty} \to v \in X$. Now, let's show that the limit v is also a fixed point of operator T

$$d(Tv, v) \leq d(Tv, T^n v_0) + d(T^n v_0, v) \leq \beta d(v, T_{n-1} v_0) + d(T^n v_0, v) \to_{n\to\infty} 0.$$

The first inequality comes from the triangle inequality, the second comes from the properties of a contraction, the last limit is what we proved in the previous paragraph. We have thus proved the existence of a fixed point.

Second, let's prove that the fixed point of T is also unique. We do this by contradiction. Suppose $\exists \hat{v}$ such that $T\hat{v} = \hat{v}$ and $\hat{v} \neq v$. Then it must be that $d(\hat{v}, v) = a > 0$ for some a, and

$$a = d(\hat{v}, v) = d(T\hat{v}, Tv) \leq \beta d(\hat{v}, v) = \beta a$$

which is a contradiction. It must thus be that v is the unique fixed point of T.

Third, let's prove convergence everywhere. We do this by induction. For any v_0, the initial step is

$$d(T^0 v_0, v) = d(v_0, v) \leq \beta^0 d(v_0, v)$$

and is true by the definition of contraction. The n-th step is

$$d(T^n v_0, v) \leq \beta^n d(v_0, v)$$

and since $d(T^{n+1} v_0, v) \leq \beta d(T^n v_0, v)$ then

$$d(T^{n+1} v_0, v) \leq \beta^{n+1} d(v_0, v)$$

which completes our proof. ■

So to apply the Contraction Mapping Theorem (CMT) we need a complete metric space and a contraction operator. The former is typically the case, although not always, and we've seen that checking for it can be tough. The latter instead is something that must be verified case by case, since our Bellman operator will depend on the economic problem at hand. We can however find some conditions that are sufficient for an operator to be a contraction.

Theorem 3 (Blackwell Sufficient Conditions, SLP Theorem 3.3)**.** *Let*

- $X \subseteq \mathbb{R}^L$
- $\mathcal{B}(X)$ *denote the set of bounded function* $f : X \to X$
- $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$.

If $T : \mathcal{B}(X) \to \mathcal{B}(X)$ *is such that*

1. *for any* $f, g \in \mathcal{B}(X)$ *such that* $f(x) \leq g(x)$ *for all* $x \in X$*, it holds that* $(Tf)(x) \leq (Tg)(x)$ *for all* $x \in X$ *(monotonicity)*
2. *there exists a* $\beta \in (0, 1)$ *such that* $T(f + a)(x) \leq (Tf)(x) + \beta a$ *for any* $f \in \mathcal{B}(X)$*,* $a \geq 0$ *and* $x \in X$ *(discounting)*

then T is a contraction with parameter β.

Proof. See SLP (Theorem 3.3). ■

Let us check whether the Bellman operator for the NGM satisfies the Blackwell Conditions and is a contraction. Let's pick the required metric, composed by the set of bounded function on $X = [0, \infty)$ and the sup norm. Our operator is, once again,

$$(Tv)(k) = \max_{k' \in \Gamma(x)} \{u(f(k) - k') + \beta v(k')\}$$

where we assume that u is a bounded function. First, let's show T maps X into itself. Since we assume that both u and v are bounded, then Tv must be bounded as it is the maximum of the sum of two bounded functions. Second, consider some function $w(k) \geq v(k)$ for all k. Then

$$(Tw)(k) = \max_{k' \in \Gamma(x)} \{u(f(k) - k') + \beta w(k')\} \geq \max_{k' \in \Gamma(x)} \{u(f(k) - k') + \beta v(k')\} = (Tv)(k)$$

which proved monotonicity. Third, let's check discounting

$$\begin{aligned} T(v + a)(k) &= \max_{k' \in \Gamma(x)} \{u(f(k) - k') + \beta[v(k') + a]\} \\ &= \max_{k' \in \Gamma(x)} \{u(f(k) - k') + \beta v(k')\} + \beta a = (Tv)(k) + \beta a \end{aligned}$$

so the discounting property is satisfied. It follows that the Bellman operator associated to the NGM is a contraction and therefore there exists a unique solution which can be obtained by iterating on the operator from *any* initial guess. We'll see that this is true in the computational part of problem set 2.

Example 4 (Exercise 137 of Guner's notes.)**.** *Let $X = (1, \infty)$, $d = |x - y|$ and $f : X \to \mathbb{R}$ be a function given by*

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

Show that f is a contraction for $a \in (1, 3)$.

1. *First, let's show f maps X into itself. The function first derivative $f'(x) = \frac{1}{2}\left(1 + \frac{a}{x^2}\right)$, which has a stationary point $f'(x^*) = 0$ at $x^* = \sqrt{a}$. Since $f''(x) = \frac{a}{x^3} > 0$, x^* is the unique global minimum and $f(x^*) = \sqrt{a}$. Given that $a > 1$, we have proved that $f(x) \in (1, \infty)$ when $x \in (1, \infty)$.*

2. *Second, let's check that f is a contraction using the definition itself: we need to show that*

$$d(f(x), f(y)) \leq \beta d(x, y).$$

Consider the LHS

$$\left| \frac{1}{2}\left(x + \frac{a}{x}\right) - \frac{1}{2}\left(y + \frac{a}{y}\right) \right| = ... = \left| \frac{1}{2}(x - y)\left(1 - \frac{a}{xy}\right) \right| = \left| \frac{1}{2}\left(1 - \frac{a}{xy}\right) \right| |x - y|$$

so going back to the inequality

$$\left| \frac{1}{2}\left(1 - \frac{a}{xy}\right) \right| |x - y| \leq \beta |x - y|$$

implies

$$\left| \frac{1}{2}\left(1 - \frac{a}{xy}\right) \right| \leq \beta.$$

Since the LHS is decreasing in a, its largest possible value is $\left| \frac{1}{2}\left(1 - \frac{3}{xy}\right) \right| = \left| -\frac{2}{2} \right| = 1$, which proves that for all $a \in (1, 3)$ we have that $d(f(x), f(y)) \leq \beta d(x, y)$.

3.4 Theorem of the Maximum

We will now ask what can we say about the properties of the value function v and its associated policy function g, being as general as possible. We will focus on a particular type of operator which is the Bellman operator, in general form

$$v(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}.$$

Going forward, let us define

$$f(x, y) := F(x, y) + \beta v(y). \tag{11}$$

To describe the feasible set, we use correspondences: $\Gamma(x)$ maps each element of set X into some set Y of feasible choices. First, we want to show when T is a self-mapping, e.g. that Tf is continuous when f is continuous. Let's define some properties of Γ which we will need later. A correspondence $\Gamma : X \to Y$ is

- compact-valued if $\Gamma(x)$ is a compact subset of Y for all $x \in X$
- closed-valued if $\Gamma(x)$ is a closed subset of Y for all $x \in X$

- convex-valued if $\Gamma(x)$ is a convex subset of Y for all $x \in X$.

Note that convex sets can be open, closed sets can be non-convex, and compact sets are closed and bounded.

We denote with *correspondence graph* the set $A = \{(x, y) : y \in \Gamma(x)\}$, in words the set of state-choice pairs such that choice y is feasible given state x.

We will need correspondences to satisfy some notion of continuity. We consider two types of continuity for correspondences.

Definition 9. *A correspondence $\Gamma : X \to Y$ is lower hemi-continuous (LHC) at x if*

- *$\Gamma(x)$ is non-empty*
- *for every $y \in \Gamma(x)$ and every sequence $x_n \to x$, there exist a number N and a sequence $\{y_n\}_{n=N}^{\infty}$ such that $y_n \to y$ and $y_n \in \Gamma(x_n)$ for all $n \geq N$.*

In words, a correspondence is LHC if any feasible choice $y \in \Gamma(x)$ can be reached from within the set by some sequence $y_n \in \Gamma(x_n)$. LHC fails if the limit y belongs to the correspondence but the convergent sequence does not.

Definition 10. *A correspondence $\Gamma : X \to Y$ is upper hemi-continuous (UHC) at x if*

- *$\Gamma(x)$ is non-empty*
- *for every sequence $x_n \to x$, every sequence $y_n \in \Gamma(x_n)$ has limit $y_n \to y \in \Gamma(x)$.*

In words, a correspondence is UHC if any sequence $y_n \in \Gamma(x_n)$ converges to $y \in \Gamma(x)$. UHC fails if a sequence belongs to the correspondence but the limit does not.

Definition 11. *A correspondence is continuous if it is both UHC and LHC.*

Please refer to your maths notes for some examples of LHC and UHC. Guner's notes contain two graphs (Fig. 20, 21) that provide some intuition.

We can now go back to our Bellman operator. When does the max of $f(x, y)$ exist? If f is continuous in y and Γ is non-empty and compact-valued, then $\max_{y \in \Gamma(x)} f(x, y)$ exists and we need not use the sup notation. It follows that a general version of the policy function can be written as

$$G(x) = \arg \max_{y \in \Gamma(x)} f(x, y) = \{y \in \Gamma(x) : f(x, y) = v(x)\}.$$

We can now say more about the properties of v and G.

Theorem 4 (Theorem of the Maximum (SLP Theorem 3.6))**.** *Let*

- *$X \in \mathbb{R}^L, Y \in \mathbb{R}^M$*
- *$f : X \times Y \to \mathbb{R}$ be a continuous function*
- *$\Gamma : X \rightrightarrows Y$ be a compact-valued and continuous correspondence*

Then

- *$v : X \to \mathbb{R}$ is a continuous function*
- *$G : X \rightrightarrows Y$ is a non-empty, compact-valued and UHC correspondence.*

Proof. See SLP. ■

This theorem has such a name because it essentially states that the maximum of the function $f(x, y)$ (defined in (11)), i.e. $\max_{y \in \Gamma(x)} f(x, y)$, both exists and is a continuous function of the state variable x. We can say this because we are maximising a continuous function on a compact set.

Corollary 1 (Convex Corollary)**.** *If also*

- *f is strictly concave in y for all x*
- *Γ is convex-valued*

then $G : X \rightrightarrows Y$ is a single-valued and continuous function.

Let's look at our NGM example:

- $f(x, y) = u(f(k) - y) + \beta v(y)$ has u and $k^\alpha + (1 - \delta)k - y$ as continuous functions, so if v is continuous then Tv must also be so.
- $\Gamma(x) = [0, f(k)]$ where both 0 and $f(k)$ are continuous, which implies that Γ is a continuous, non-empty and compact correspondence.

4 Dynamic Programming

4.1 Principle of Optimality

The last part of our maths work concerns the equivalence between the sequential and recursive formulations of the problem. The sequential problem in general form is

$$\begin{aligned} V^*(x_0) = \sup_{\{x_{t+1}\}_{x=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ \text{s.t.} \quad & x_{t+1} \in \Gamma(x_t) \\ & x_0 \text{ given.} \end{aligned} \tag{SP}$$

We will refer to V^* as the supremum function.

Our recursive problem instead is given by our functional equation

$$v(x) = \sup_{x' \in \Gamma(x)} \{f(x, x') + \beta v(x')\}. \tag{FE}$$

We want to show the following:

- V^* solves (FE) ($SP \Rightarrow FE$)
- v evaluated at x_0 solves (SP) ($FE \Rightarrow SP$)
- the solution to (FE) exists (proved it already)
- the sequence $\{x^*_{t+1}\}_{t=0}^{\infty}$ attains the maximum in (SP) if it satisfies

$$v(x^*_t) = f(x^*_t, x^*_{t+1}) + \beta v(x^*_{t+1}).$$

Some notation:

- $\tilde{x} := \{x_{t+1}\}_{t=0}^{\infty}$ denotes a plan, which will be feasible if $x_{t+1} \in \Gamma(x_t)$ for all $t \geq 0$;
- $\Pi(x_0) = \{\{x_{t+1}\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t) \forall t \geq 0\}$ is the set of all feasible plans/sequences;
- for some feasible plan $\tilde{x}$, we define

$$u_n(\tilde{x}) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$
$$u(\tilde{x}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = \lim_{n \to \infty} u_n(\tilde{x}).$$

Assumptions we'll make

- **(A1)**: $\Gamma(x)$ is non-empty for all $x \in X$
- **(A2)**: $F : A \to R$ is bounded, where $A = \{(x, x') : x \in X, x' \in \Gamma(x)\}$

We will sometimes make A2 a bit looser by only asking that $u(\tilde{x}) = \lim_{n \to \infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists, even though it may be plus or minus infinity. In other words, we just need F to be bounded from one side, above or below.

These assumptions imply that the set of feasible plans is non-empty and that $u(\tilde{x})$ is well defined, and so is the (SP) problem. Specifically, $V^*(x_0) = \sup_{\tilde{x} \in \Pi(x_0)} u(\tilde{x})$ will be uniquely defined, although there may exist multiple $\tilde{x}$ that attain it.

Our interest is in connecting the supremum function V^* and the solution(s) v to (FE). It is important to remember that while V^* is always uniquely defined, we know much less about the solutions to (FE), which may very well be zero, one or many.

Before we move to the theorem and its proof, it will be useful to be more precise on the meaning of the statements "to satisfy" (FE) and (SP). We refer the reader to SLP for alternative conditions for the case where V^* or v are not bounded.

- $V^*(x_0)$ is the supremum function if it satisfies

$$V^*(x_0) \geq u(\tilde{x}) \text{ for all } \tilde{x} \in \Pi(x_0) \tag{SP1}$$

$$V^*(x_0) \leq u(\tilde{x}) + \epsilon \text{ for some } \tilde{x} \in \Pi(x_0), \text{ any } \epsilon > 0. \tag{SP2}$$

  These conditions do sound quite obvious and will be needed for the proof of the Principle of Optimality. To see what they mean in words, think of $V^*(x_0)$ as some value which is not necessarily related to $u(\tilde{x})$. Such value is the supremum if it is weakly larger than the value of the best possible sequence (SP1)), and if it is actually attained by some sequence (SP2).

- v satisfies (FE) if it satisfies

$$v(x_0) \geq F(x_0, y) + \beta v(y) \text{ for all } y \in \Gamma(x_0) \tag{FE1}$$

  and if for all $\epsilon > 0$

$$v(x_0) \leq F(x_0, y) + \beta v(y) + \epsilon \text{ for } \textit{some } y \in \Gamma(x_0), \text{ any } \epsilon > 0. \tag{FE2}$$

  The intuitive explanation is that same as that of the previous bullet point.

Theorem 5 (Principle of optimality (SLP Theorems 4.2-4.3))**.** *Under assumptions (A1), (A2), the following statements hold*

- *($FE \Leftarrow SP$) the supremum function $V^*(x_0) = \sup_{\tilde{x} \in \Pi(x_0)} u(\tilde{x})$ satisfies (FE)*
- *($FE \Rightarrow SP$) if v satisfies (FE) and*

$$\lim_{n \to \infty} \beta^n v(x_n) = 0 \textit{ for all } x_0 \in X \textit{ and } \tilde{x} \in \Pi(x_0) \tag{12}$$

  then $v = V^*$.

Proof. The direction ($FE \Leftarrow SP$) is not our primary objective so please refer to SLP Theorem 4.2 for its proof.

Let us prove the second statement, i.e. direction ($FE \Rightarrow SP$), in two parts.

(FE1)⇒(SP1). If v satisfies (FE), then (FE1) holds. That implies

$$\begin{aligned} v(x_0) &\geq F(x_0, x_1) + \beta v(x_1) \quad \text{ for all } x_1 \in \Gamma(x_0) \\ &\geq F(x_0, x_1) + +\beta F(x_1, x_2) + \beta^2 v(x_2) \quad \text{ for all } x_1 \in \Gamma(x_0), x_2 \in \Gamma(x_1) \\ &\vdots \\ &\geq u_n(\tilde{x}) + \beta^n v(x_{n+1}) \quad \text{ for any } \tilde{x} \text{ and for all } n \geq 1. \end{aligned}$$

Taking the limit for $n \to \infty$ and using (12) we get

$$v(x_0) \geq u(\tilde{x})$$

which is condition (SP1).

(FE2)⇒(SP2). If v satisfies (FE), then (FE2) holds. That implies

$$v(x_t) \leq F(x_t, x_{t+1}) + \beta v(x_{t+1}) + \epsilon_t \quad \forall t \geq 0.$$

Starting from $t = 0$ and iterating forward for some plan $\tilde{x}$

$$\begin{aligned} v(x_0) &\leq \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} v(x_{n+1}) + \sum_{t=0}^{\infty} \epsilon_t \\ &\leq u_n(\tilde{x}) + \beta^{n+1} v(x_{n+1}) + \bar{\epsilon} \end{aligned}$$

and since the second-last term is equal to zero by (12), taking the limit for $n \to \infty$ we get

$$v(x_0) \leq u(\tilde{x}) + \bar{\epsilon}$$

which is condition (SP2). ■

Recall that when we solved the (SP) using the Lagrangian, the necessary and sufficient conditions for an optimum were the Euler equation and the TVC. In some sense, condition (12) is to the functional equation what the TVC is for the sequential problem.

As said before, we have characterised under what conditions the (unique) solution to the SP problem is equivalent to one particular solution to the FE. Note that in this theorem we have not said anything about the uniqueness of the solution to FE: there may be many, and

Theorem 5 identifies which of the possibly many solutions is related to the sequential problem. More precisely, the theorem implies that (FE) has *at most* one solution satisfying (12). Also note that condition (12) is a *sufficient* condition, but it is not necessary. We now present an example that combines all these observations.

Example 5. *Consider a simple problem where the rate of return on storage is equal to the discount factor and utility is linear. The supremum function is*

$$\begin{aligned} V^*(x_0) = \sum_{t=0}^{\infty} \beta^t c_t & \\ s.t. \quad & c_t + \beta x_{t+1} \leq x_t \\ & c_t \geq 0. \end{aligned}$$

Note we are not imposing any no-Ponzi game condition here, so the budget set is such that $x_{t+1} \in \Gamma(x_t) = (-\infty, x_t/\beta]$. *Given that, pick a sequence where* $x_1 = -\infty$ *and* $x_t = x_{t+1}/\beta$ *for all* $t \geq 2$. *The constraints are satisfied in all periods, so this is a feasible sequence and* $V^*(x_0) = +\infty$.

Now consider the corresponding functional equation

$$\begin{aligned} v(x) = \sup_{x' \in \Gamma(x)} x - \beta x' + \beta v(x') & \\ s.t. \quad & \Gamma(x) = (-\infty, x/\beta]. \end{aligned}$$

This has two solutions. First, $v(x) = +\infty$ *is a valid solution here*[7], *and it obviously coincides with the supremum function, which we know because* $v(x_0) = V^*(x_0)$. *Note that this solution for* v *does not satisfy (12), but that's fine because that condition is sufficient but not necessary, as we said before.*

Second, $\tilde{v}(x) = x$ *is also a solution associated with policy function* $g(x) = y$ *for any* $y \in \Gamma(x)$, *i.e. with any feasible sequence (check it!). This second solution also does not satisfy condition (12), and on top of that it is clearly different to the supremum function. To see why condition*

[7] To see that, plug it in as a continuation value, find that the optimal policy is always $x' = -\infty$, which implies $v(x) = +\infty$.

(12) does not hold here, take sequence $\tilde{x} = \{x_t = x_{t-1}/\beta = \beta^{-t}x_0\}_{t=0}^{\infty}$. Then

$$\lim_{n\to\infty} \beta^n v(x_n) = \lim_{n\to\infty} \beta^n x_n = \lim_{n\to\infty} \beta^n \beta^{-n} x_0 = x_0 \neq 0.$$

So this is an example of a case where there exist multiple solutions to the functional equation, and some of them are not the supremum function V^*.

Example 6. *Consider again an example from earlier where the FE is*

$$v(x) = \sup_{y\in[0,2x]} \{y - 2x + \beta v(y)\}.$$

We know that there are two solutions, $v(x) = 0$ and $v(x) = -2x$. If we can prove that one of the two satisfies (12), then we know that will be the one that coincides with the supremum function. Clearly $v(x) = 0$ satisfies the condition since $\lim_{n\to\infty} \beta^n v(x_n) = 0$ for any feasible sequence. We cannot say the same for the second solution: pick sequence $x_t = 2x_{t-1}$, which is feasible. Then

$$\lim_{n\to\infty} \beta^n v(x_n) = \lim_{n\to\infty} \beta^n(-22^n x_0) = \infty$$

given that $\beta \in (1/2, 1)$.

We have thus seen two examples where (FE) admits multiple solutions, and additional criteria are useful to identify which of the solutions is the one that coincides with the supremum function. Clearly, when the Contraction Mapping Theorem applies, this machinery is not necessary because you know that we converge always to the unique solution to (FE), which must thus coincide with the supremum function. The Principle of Optimality gives us additional tools we can use when the CMT cannot be used.

We now consider a version of the Principle of Optimality that concerns policy rather than value functions.

Theorem 6 (Optimal policy (SLP Theorems 4.4-4.5))**.** *Under assumptions (A1), (A2), we have that*

1. *if $\bar{x} \in \Pi(x_0)$ solves (SP), then*

$$V^*(\bar{x}_t) = F(\bar{x}_t, \bar{x}_{t+1}) + \beta V^*(\bar{x}_{t+1}) \quad \text{for all} \quad t \geq 0.$$

2. if $\hat{x} \in \Pi(x_0)$ satisfies the functional equation

$$V^*(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta V^*(\hat{x}_{t+1})$$

and also is such that

$$\lim_{t\to\infty} \sup \beta^t V^*(\hat{x}_t) \leq 0 \tag{13}$$

then $\hat{x}$ solves (SP).

The first part of the theorem says that any optimal plan from the sequential problem is optimal in the recursive problem, that is, the plan satisfies the functional equation without the max operator when the supremum function is the value function. The second part says that, given the "right" value function that solves the functional equation, the associated policy function generates a plan that solves the sequential problem as long as it satisfies the additional limit condition.

Again, we'll focus on the second part of the theorem because we are more interested in that direction. This theorem may seem redundant: if we have a value function that satisfies (12), condition (13) automatically holds because (12) must hold for any feasible plan and so it does also for the optimal plan. The last theorem is however not redundant because the converse may not be true! That is, there may be cases where (13) holds but (12) does not.

Example 7. *Consider the following modification of a previous example*

$$\begin{aligned} V^*(x_0) = \sum_{t=0}^{\infty} &\beta^t (x_t - \beta x_{t+1}) \\ s.t. \quad & 0 \leq x_{t+1} \leq x_t/\beta \\ & x_0 \text{ given.} \end{aligned}$$

The associated FE is

$$v(x) = \max_{0 \leq x' \leq x/\beta} \{x - \beta x' + \beta v(x')\}.$$

To find the supremum function, iterate the objective function forward and use the fact that *now $x_t \geq 0$ for any t*[8]

$$V^*(x_0) = x_0 - \beta x_1 + \beta(x_1 - \beta x_2) + \beta^2(x_2 - \beta x_3) + ... = x_0.$$

So we know that $V^*(x_0) = x_0$ is the supremum function. As the first part of Theorem 5 implies, we can see that $V^*(x_0) = x_0$ solves the functional equation since $v(x) = x$ is a solution of the FE. By the way, note that this solution does not satisfy condition (12), because we can find a feasible plan ($x_t = x_0/\beta^t$) where that condition fails. So, if we did not know the supremum function directly, we couldn't use the second part of Theorem 5 to conclude that $v = V^*$ here.

We can however now apply the second part of Theorem 6 to find that certain plans are optimal, while some other plans are not. Consider the plan $\{\widehat{x}_t\}$ defined by $\widehat{x}_0 = x_0$ and $\widehat{x}_t = 0$ for all $t > 0$ (i.e. save everything in period 0 and consume everything in period 1). This plan is feasible, it satisfies (FE)[9] *and condition (13) since*

$$\lim_{t\to\infty} \beta^t V^*(\widehat{x}_t) = \lim_{t\to\infty} \beta^t V^*(0) = 0.$$

We can thus conclude that by Theorem 6 the plan $\widehat{x}_t$ solves (SP).

There are many other plans that are also optimal (which suggests that there is no unique solution here!): in fact, all plans such that all of x_0 is consumed in finite time. But not all plans are optimal, and condition (13) is useful to detect them. Pick again $\widehat{x}_t = x_0/\beta^t$, which is a plan where x_0 is all saved and never consumed: it does satisfy (FE) but it does not satisfy (13) since

$$\lim_{t\to\infty} \beta^t V^*(\widehat{x}_t) = \lim \sup_{t\to\infty} \beta^t \beta^{-t} x_0 = x_0.$$

Hence we cannot claim that $\widehat{x}_t$ is optimal.

4.2 Bounded returns

We now consider the subset of problems where the return function $F(x, y)$ is bounded. This will allow us to say more about the value function, the optimal policy and the Bellman operator.

[8] Note that without the no Ponzi game condition, we cannot do this derivation because $\lim_{t\to\infty} \beta^t x_t$ does not necessarily go to zero.

[9] To prove it, look at the case $t = 0$ and at the case $t > 0$.

We'll make the following additional assumptions

- **(B1)**: X is a convex subset of $\mathbb{R}^L$ and $\Gamma : X \rightrightarrows X$ is non-empty, compact-valued and continuous
- **(B2)**: $F(x, y) : A \to \mathbb{R}$ (where again $A = \{(x, x') : x \in X, x' \in \Gamma(x)\}$) is bounded and continuous.

First, these assumptions imply that assumptions (A1) and (A2) are satisfied. Second, they suggest that we will be looking for v in the space of bounded and continuous functions on X (let us call that $\mathcal{C}^{\mathcal{B}}(X)$) and under the sup-norm metric. Let T be the Bellman operator on $\mathcal{C}^{\mathcal{B}}(X)$ defined by

$$Tv(x) = \max_{x' \in \Gamma(x)} F(x, x') + \beta v(x')$$

and recall that the policy correspondence is defined as

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\}.$$

We have the following theorem.

Theorem 7 (Bounded returns (SLP Theorem 4.6)). *Under assumptions (B1) and (B2) the operator T is such that*

1. *it maps $\mathcal{C}^{\mathcal{B}}(X)$ into itself*
2. *it has a unique fixed point, i.e. $!\exists v \in \mathcal{C}^{\mathcal{B}}(X) : Tv = v$*
3. *for all $v_0 \in \mathcal{C}^{\mathcal{B}}(X)$ we have that $T(T^n v_0, v) \leq \beta^n d(v_0, v)$*
4. *the policy correspondence G associated with v is non-empty, compact-valued and UHC.*

Proof (Sketch). The proof uses a lot of the results we have used so far. To prove 1), we must show that T maps into itself because it preserves boundedness and continuity: the former is straightforward, the latter follows from the Theorem of the Maximum. To prove 2) and 3), we have to show that $\mathcal{C}^{\mathcal{B}}(X)$ with the sup-norm is a complete metric space, and that T satisfies the Blackwell sufficient conditions. It then follows that T is a contraction, and so the fixed point $v = Tv$ exists, is unique and we converge to it from any $v_0 \in \mathcal{C}^{\mathcal{B}}(X)$. To prove 4), we use again the Theorem of the Maximum. ■

Let us once again use our NGM

$$v(k) = \max_{k' \in [0, f(k)]} \{u(f(k) - k') + \beta v(k')\}$$

to check whether the assumptions of the theorem are satisfied:

- Let $\bar{k}$ define the highest maintainable capital stock, i.e. the level of capital at which, even setting $c = 0$, it is not possible to increase its stock: $\bar{k} = f(\bar{k})$. If we let $X = [0, \bar{k}]$, then X is a convex subset of R.
- $\Gamma(k) = [0, k^\alpha + (1-\delta)k]$ is compact-valued and continuous.
- For any continuous utility function, $u(c)$ is bounded because $u(0) \leq u(f(k) - k') \leq u(f(\bar{k}))$.
- $\beta \in (0, 1)$.

So both assumptions (B1) and (B2) are satisfied and Theorem 7 applies.

We now move on to discuss concavity and the single-valuedness of the policy function. Under the following additional assumptions

- **(B5)**: $F(x, y)$ is strictly concave, i.e.

$$F(\theta x + (1-\theta)x', \theta y + (1-\theta)y') \leq \theta F(x, y) + (1-\theta)F(x', y')$$

  for all $(x, y), (x', y') \in A$ where A was defined earlier (see Assumption (B2)).
- **(B6)**: Γ is convex, i.e. if $y, y' \in \Gamma(x)$ then $\theta y + (1-\theta)y' \in \Gamma(x)$.

we can state the following theorem

Theorem 8 ((SLP Theorem 4.10)). *Under assumptions (B1), (B2), (B5), (B6) we have that*

- *v is strictly concave*
- *G is a single-valued and continuous function.*

We now move on to discuss a very important property: value function differentiability.

Theorem 9 (Benveniste and Scheinkman (SLP Theorem 4.10)). *Let*

- *$X \in \mathbb{R}^L$ be a convex set*
- *$v : X \to \mathbb{R}$ be a concave function*
- *$x_0 \in int(X)$*[10]
- *D be a neighbourhood of x_0.*

If there exists a function $W : D \to \mathbb{R}$ that is concave, differentiable and such that $W(x_0) = v(x_0)$ and $W(x) \leq v(x)$ for all $x \in D$ then v is differentiable at x_0 and

$$\frac{\partial v(x_0)}{\partial x_i} = \frac{\partial W(x_0)}{\partial x_i}$$

[10]The notation $\text{int}(X)$ refers to the interior of set X.

for all $i = 1, \ldots, L$.

Adding one last additional assumption we can then apply this to dynamic programming

- **(B7)**: F is continuously differentiable inside set A.

We can now state the following theorem.

Theorem 10 (Differentiability of v (SLP Theorem 4.11))**.** *Under assumptions (B1), (B2), (B5), (B6), (B7) we have that, if* $x_0 \in int(X)$ *and* $g(x_0) \in int(\Gamma(X))$*, then*

- *v is continuously differentiable at* x_0
- *and*

$$v'(x_0) = F_1(x_0, g(x_0)).$$

What this theorem says in practice is that, under some conditions, we can disregard the response of the control variable when we differentiate the value function with respect to the state variable. Take our usual one-dimension FE

$$v(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}$$

and suppose you want to compute $v'(x_0)$, i.e., the derivative of the value function with respect to the state variable, evaluated at x_0. By hypothesis, x_0 and the optimal policy $g(x_0)$ are interior points of X and $\Gamma(X)$ respectively. Pick a function

$$W(x) = F(x, g(x_0)) + \beta v(g(x_0))$$

that is, a function where you choose for any state the policy that is optimal when the state is x_0. Clearly such value function will be lower than $v(x)$ when $x \neq x_0$ and will be equal to it when $x = x_0$. Also, W is concave and differentiable since it's a sum of a concave and differentiable function (F) and a constant ($v(g(x_0))$). We can then compute

$$W'(x) = F_1(x, g(x_0))$$

and once we evaluate it at x_0, we know it must be such that

$$W'(x_0) = F_1(x, g(x_0)) = v'(x_0).$$

4.3 Unbounded returns

We now move on to consider the subset of dynamic programming problems where the return function is not bounded. We need the following assumptions

- **(A1)**: $\Gamma(x) \neq \emptyset$ for all $x \in X$
- **(A2)**$'$: for all initial conditions $x_0 \in X$ and all feasible plans $\tilde{x} \in \Pi(x_0)$, $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$ exists (but need not be finite).

Assumption (A1) has not changed, while assumption (A2)$'$ is now a looser version of the assumption we made in Section 4.1.

We now present a theorem that is useful when the supremum function V^* satisfies the FE (first part of Theorem 5) but the boundedness hypothesis (second part of Theorem 5) does not hold.

Theorem 11 (Principle of optimality when returns are unbounded (SLP Theorem 4.14))**.** *Under assumptions (A1) and (A2)$'$, if*

1. *there exists a function $\hat{v} : X \to \mathbb{R}$ such that*
   (a) $T\hat{v} \leq \hat{v}$
   (b) $\lim_{n\to\infty} \beta^n \hat{v}(x_n) \leq 0$ *for all* $x_0 \in X$ *and all* $\tilde{x} \in \Pi(x_0)$
   (c) $u(\tilde{x}) \leq \hat{v}(x_0)$ *for all* $x_0 \in X$ *and all* $\tilde{x} \in \Pi(x_0)$
2. *the function v defined as*
$$v(x) = \lim_{n\to\infty} T^n \hat{v}(x)$$
   is a fixed point of T

then v is the supremum function (i.e. $v = V^*$).

Essentially, this theorem replaces condition (12). Let us look at an example[11] with the NGM with an unbounded return function. Consider

$$\begin{aligned} V^*(x_0) = & \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(k_t^\alpha - k_{t+1}) \\ \text{s.t.} \quad & k_{t+1} \in [0, k_t^\alpha] \end{aligned}$$

[11]If you want to see further examples, please refer to the quadratic example in SLP Section 4.4.

for $X = (0, \infty)$. Note that the choice of the return function and the feasible set implies that returns are unbounded from below and from above (although we can prove that capital can never go above 1 here).

Let us check whether we can use the latest theorem. Assumption (A1) is satisfied as the feasible set correspondence is never empty. To check Assumption (A2)$'$, it is sufficient to show that the PV of lifetime utility is bounded from either above or below[12]. Consider a (unfeasible) plan where we both consume *and* save all available resources: $c_t = k_{t+1} = k_t^{\alpha}$. Then capital will be such that $\log(k_{t+s}) = \alpha^s \log(k_t)$, and consumption such that $\log(c_{t+s}) = \log(k_{t+s}^{\alpha}) = \alpha \log(k_{t+s}) = \alpha^{s+1} \log(k_t)$. So the PV of lifetime utility under this plan is

$$\sum_{t=0}^{\infty} \beta^t \log(c_t) = \sum_{t=0}^{\infty} \beta^t \alpha^{t+1} \log(k_0) = \frac{\alpha \log(k_0)}{1 - \alpha\beta}.$$

Clearly this will be an upper bound for the PV of lifetime utility of any *feasible* plan, so (A2)$'$ holds.

Now, let us look for our $\hat{v}$ function. Let $\hat{v}(k_0) = \frac{\alpha \log(k_0)}{1-\alpha\beta}$. This function satisfies requirements (1a) to (1c) of the theorem: $T\hat{v} \leq \hat{v}$, because the resource constraint does not allow to consume *and* save everything; $\lim_{n\to\infty} \beta^n \hat{v}(x_n) = 0$ for any feasible plan, since $\hat{v}$ is bounded even when we follow the unfeasible plan proposed above; $\hat{v}(k_0) \geq u(\tilde{x})$ for any feasible plan, because $\hat{v}$ is the PV of lifetime utility of an infeasible plan. Finally, we know that $\lim_{n\to\infty} T^n \hat{v}(x) = v(x)$ because we know v is a contraction mapping and so we converge to it from any initial guess, including $\hat{v}$. Hence we proved that $v = V^*$.

5 Stochastic Environments

We now consider recursive problems in infinite horizon and stochastic environments.

Consider a vector of shocks $\boldsymbol{z}_t \in \mathbb{R}^n$ for $t = 0, 1, ...$ Let us introduce some notation:

- superscripts denote histories $\boldsymbol{z}^t = \{\boldsymbol{z}_0, \boldsymbol{z}_1, ..., \boldsymbol{z}_t\}$, so $\boldsymbol{z}^t$ is a $(t+1) \times n$ vector;
- the probability of any given history is given by $\pi(\boldsymbol{z}^t)$;
- most things will depend on such histories, e.g. $c_t(\boldsymbol{z}^t)$ is consumption at period t when history $\boldsymbol{z}^t$ has realised.

[12] Check discussion in Section 4.1 of SLP for more details.

We will consider iid processes such that $\mathrm{P}(\boldsymbol{z}_{t+1}|\boldsymbol{z}^t) = \mathrm{P}(\boldsymbol{z}_{t+1})$, and Markov process such that $\mathrm{P}(\boldsymbol{z}_{t+1}|\boldsymbol{z}^t) = \mathrm{P}(\boldsymbol{z}_{t+1}|\boldsymbol{z}_t)$. The latter are the protagonist in a large share of modern dynamic stochastic macro.

Before going directly to consider how this new stochastic environment affects our dynamic programming problem, let us review a bit of properties of Markov processes[13].

5.1 Markov chains

A stochastic process has the Markov property is $\mathrm{P}(\boldsymbol{z}_{t+1}|\boldsymbol{z}^t) = \mathrm{P}(\boldsymbol{z}_{t+1}|\boldsymbol{z}_t)$. A Markov chain is a triplet

$$(\{\boldsymbol{e}_i\}_{i=1}^n, \underset{=}{\mathrm{s}}P, \boldsymbol{\pi}_0)$$

where

- n is the number of possible event realisations;
- $\boldsymbol{e}_i$ is a "selector" vector that is composed by a 1 in the i-th position if even i has realised, and all zeros elsewhere;
- $\boldsymbol{P}$ is a $n \times n$ transition matrix;
- $\boldsymbol{\pi}_0$ is a vector that specified the probabilities of the initial realisation of the process.

Any Markov chain must satisfy the following assumptions

- $\boldsymbol{P}$ is a stochastic matrix, i.e. $\sum_{j=1}^n \boldsymbol{P}_{i,j} = 1$ for all i. In words, all rows of $\boldsymbol{P}$ sum up to 1.
- $\sum_{i=1}^n \boldsymbol{\pi}_{0,i} = 1$, i.e. the vector of initial probabilities must also sum up to 1.

Conditional and unconditional probabilities. Elements of $\boldsymbol{P}$ denote *conditional* probabilities. Each element of $\boldsymbol{P}$ indicates the probability of observing event j tomorrow conditional on observing event i today: $\boldsymbol{P}_{i,j} = \boldsymbol{P}(\boldsymbol{x}_{t+1} = \boldsymbol{e}_j|\boldsymbol{x}_t = \boldsymbol{e}_i)$.

We can compute conditional probabilities of events further away in time too. For example

$$P(\boldsymbol{x}_{t+2} = \boldsymbol{e}_j|\boldsymbol{x}_t = \boldsymbol{e}_i) = \sum_{h=1}^{n} P(\boldsymbol{x}_{t+2} = \boldsymbol{e}_j|\boldsymbol{x}_{t+1} = \boldsymbol{e}_h)P(\boldsymbol{x}_{t+1} = \boldsymbol{e}_h|\boldsymbol{x}_t = \boldsymbol{e}_i) = \sum_{h=1}^{n} \boldsymbol{P}_{i,h}\boldsymbol{P}_{h,j} = \boldsymbol{P}^2_{i,j}$$

where the second-last step represents the dot product of two rows of $\boldsymbol{P}$. The example shows that the probability of observing event j conditional on observing event i k periods before is

[13]Most of this section follows LS chapter 2.

given by

$$P(\boldsymbol{x}_{t+k} = \boldsymbol{e}_j | \boldsymbol{x}_t = \boldsymbol{e}_i) = \boldsymbol{P}^k_{i,j}.$$

What we're really interested in are *unconditional* distributions. I.e., if I have a stochastic model, what states are visited more often than others? Let $\boldsymbol{\pi}'_1$ be the $1 \times n$ vector denoting the unconditional probability distribution at $t = 1$. It is computed as

$$\boldsymbol{\pi}'_0 \boldsymbol{P} = \boldsymbol{\pi}'_1$$

so for example

$$\boldsymbol{\pi}_{1,k} = \sum_{j=1}^{n} \boldsymbol{\pi}_{0,j} \boldsymbol{P}_{j,k}$$

that is, the probability of being in state k at $t = 1$ is given by the probability of being in j at $t = 0$ times the probability of going from j to k at $t = 1$, summed over all possible j's. It follows that

$$\boldsymbol{\pi}'_s = \boldsymbol{\pi}'_0 \boldsymbol{P}^s = \boldsymbol{\pi}'_{s-1} \boldsymbol{P}.$$

Stationarity. An unconditional distribution $\boldsymbol{\pi}_t$ is stationary if it is such that

$$\boldsymbol{\pi}' = \boldsymbol{\pi}' \boldsymbol{P} \tag{14}$$

that is, if it is constant over time. Rearranging equation (14) we get

$$\boldsymbol{\pi}'(\boldsymbol{I} - \boldsymbol{P}) = 0$$

where I is the identity matrix. This says that the stationary distribution is the left eigenvector[14] associated with the unit eigenvalue of matrix $\boldsymbol{P}$[15].

[14] Alternatively, we can write

$$(\boldsymbol{P}' - \boldsymbol{I})\boldsymbol{\pi} = 0$$

and say that $\boldsymbol{\pi}$ is the right eigenvector associated with the unit eigenvalue of $\boldsymbol{P}'$.

[15] Recall that, given matrix $\boldsymbol{A}$ and

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} \quad \Leftrightarrow \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = 0$$

we say that $\boldsymbol{v}$ is the right eigenvector associated with the λ eigenvalue of matrix $\boldsymbol{A}$.

Example 8. *Consider the stochastic matrix*

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ .2 & .5 & .3 \\ 0 & 0 & 1 \end{bmatrix}.$$

What are the stationary distributions associated with it? Is there more than one? We can look for the eigenvectors associated with the unit eigenvalue, solving

$$\boldsymbol{\pi}'(\boldsymbol{I} - \boldsymbol{P}) = 0.$$

In this example,

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -.2 & .5 & -.3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -.2\pi_2 & .5\pi_2 & -.3\pi_2 \end{bmatrix}.$$

It follows that any probability distribution $\begin{bmatrix} x & 0 & 1-x \end{bmatrix}$ *is stationary: the transition matrix is such that if you start in position 1 or 3, you stay there forever, so any distribution that gives positive probability only to those states will be stationary.*

This example is somewhat extreme in that, depending on the starting point of the stochastic process, convergence happens immediately. What is more common is that some stochastic system converges only *eventually* to a stationary distribution. In other words, is there a distribution $\boldsymbol{\pi}_\infty$ such that $\lim_{t\to\infty} \boldsymbol{\pi}_t = \boldsymbol{\pi}_\infty$? We will now consider the convergence properties of Markov chains.

We first state the conditions for existence and uniqueness of such limiting stationary distribution

Theorem 12 ((LS2 Theorem 1))**.** *If* $\boldsymbol{P}_{i,j} > 0$ *for all* i, j*, then there exists a unique* $\boldsymbol{\pi}_\infty$ *such that* $\lim_{t\to\infty} \boldsymbol{\pi}_t = \boldsymbol{\pi}_\infty$*.*

A looser version of this theorem is the following

Theorem 13 ((LS2 Theorem 2))**.** *If for some* n *we have that* $\boldsymbol{P}^n_{i,j} > 0$ *for all* i, j*, then there exists a unique* $\boldsymbol{\pi}_\infty$ *such that* $\lim_{t\to\infty} \boldsymbol{\pi}_t = \boldsymbol{\pi}_\infty$*.*

Expectations. Consider now a stochastic variable $\boldsymbol{x}_t$ that is a selector vector distributed according to a Markov chain, i.e. takes the value $\boldsymbol{e}_i$ for some i (we defined $\boldsymbol{e}_i$ before) according to some distribution $\boldsymbol{\pi}$ and some transition matrix $\boldsymbol{P}$. Then consider a random variable $y_t = \boldsymbol{y}'\boldsymbol{x}_t$ where $\boldsymbol{y} = \begin{bmatrix} y_1 \; y_2 \; ... \; y_n \end{bmatrix}$ is a vector that contains all possible realisations of some variable (e.g. GDP, productivity, and so on). When $\boldsymbol{x}_t = \boldsymbol{e}_j$, then y_t takes on the j-th value of vector $\boldsymbol{y}$ ($y_t = \boldsymbol{y}_j$).

The *conditional* expectations of y_{t+1} is given by

$$\mathbb{E}[y_{t+1}|\boldsymbol{x}_t = \boldsymbol{e}_i] = \sum_{j=1}^{n} \boldsymbol{P}_{i,j}\boldsymbol{y}_j = (\boldsymbol{P}\boldsymbol{y})_i$$

which is the i-th row of the $(\boldsymbol{P}\boldsymbol{y})$ matrix.

The *unconditional* expectation of y_t, on the other hand, is given by

$$\mathbb{E}[y_t] = \boldsymbol{\pi}_t'\boldsymbol{y} = \boldsymbol{\pi}_0'\boldsymbol{P}^t\boldsymbol{y}.$$

Invariance and ergodicity. Consider now a stationary Markov chain, that is a triplet $(\{\boldsymbol{e}_i\}_{i=1}^n, \boldsymbol{P}, \boldsymbol{\pi})$ such that $\boldsymbol{\pi}'P = \boldsymbol{\pi}$, and again a random variable $y_t = \boldsymbol{y}'\boldsymbol{x}_t$ where $\boldsymbol{x}_t$ is a selector vector[16]. We consider first a version of the law of large numbers (LLN): for any stationary Markov chain,

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} y_t = \mathbb{E}[y_\infty|\boldsymbol{x}_0]$$

with probability one.

This is a somewhat weak version of the LLN, because the time average of y_t only converges to the unconditional expectation of the limit of y_t, not that of any y_t. We will now get to a stronger version of the LLN, after having defined some additional concepts.

We say that random variable $y_t = \boldsymbol{y}'\boldsymbol{x}_t$ is invariant if $y_t = y_0$ for all t and all $\boldsymbol{x}_t$. That is, if y_t is constant over time.

[16]Henceforth we will refer to stationary Markov chains as $(\boldsymbol{P}, \boldsymbol{\pi})$ pairs, omitting the selector vector simply for brevity.

Theorem 14 ((LS2 Theorem 2.2.2))**.** *If a stationary Markov chain is such that*

$$\mathbb{E}[y_{t+1}|x_t] = y_t \tag{15}$$

then $y_t = \boldsymbol{y}'\boldsymbol{x}_t$ *is invariant.*

Any stochastic process satisfying (15) is defined as a *martingale*. The theorem is saying that random variables which depend on discrete and finite Markov chains must actually be constant over time, which is a special case of the Martingale convergence theorem. Equation (15) can actually be rewritten as

$$(\boldsymbol{P} - \boldsymbol{I})\boldsymbol{y} = 0 \tag{16}$$

which says that any invariant (random variable that is a) function (y_t) of the state (x_t) has a support $\boldsymbol{y}$ that is a right eigenvector associated to the unit eigenvalue of P.

Ergodicity.

Definition 12. *A stationary Markov chain* $(\boldsymbol{P}, \boldsymbol{\pi})$ *is ergodic if all invariant functions are constant with probability one, i.e. if* $\boldsymbol{y}_i = \boldsymbol{y}_j$ *for all* i, j *such that* $\boldsymbol{\pi}_i > 0, \boldsymbol{\pi}_j > 0$.

That is, if all vectors $\boldsymbol{y}$ that satisfy (16) are such that all positive probability elements are identical, then $(\boldsymbol{P}, \boldsymbol{\pi})$ is an ergodic Markov chain.

We now get to why this is useful.

Theorem 15 ((LS2 Theorem 2.2.3))**.** *If* y_t *is a random variable on an ergodic Markov chain* $(\boldsymbol{P}, \boldsymbol{\pi})$*, then*

$$\frac{1}{T}\sum_{t=1}^{T} y_t \to \mathbb{E}[y_0]$$

with probability 1*.*

Consider the following examples.

Example 9. *Let*

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The associated stationary distribution is any vector $\boldsymbol{\pi}$ *that solves*

$$\boldsymbol{\pi}'(\boldsymbol{P} - \boldsymbol{I}) = 0$$

which yields $\boldsymbol{\pi} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. *The invariant functions of the state satisfy*

$$(\boldsymbol{P} - \boldsymbol{I})\boldsymbol{y} = 0$$

which yields $\boldsymbol{y} = \begin{bmatrix} x \\ x \end{bmatrix}$ *for any value of* x. *So since all invariant functions are constant, the stationary Markov chain* $(\boldsymbol{P}, \boldsymbol{\pi})$ *is ergodic.*

Example 10. *Let*

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The associated stationary distribution is $\boldsymbol{\pi} = \begin{bmatrix} p \\ 1-p \end{bmatrix}$ *for any* $p \in [0, 1]$. *The invariant functions are* $\boldsymbol{y} = \begin{bmatrix} a \\ b \end{bmatrix}$ *for any* (a, b) *pair. It follows that when* $p \in (0, 1)$, *the stationary Markov chain* $(\boldsymbol{P}, \boldsymbol{\pi})$ *is not ergodic because its invariant functions are not constant. The intuition is that the random variable* $y_t = \boldsymbol{y}'\boldsymbol{x}_t$ *takes either value with some positive probability at* $t = 0$, *and then never moves away from that value. Since the other state is never visited, the sample average will converge to the value of* y_0 *that realises, and not to the unconditional expectation of* y_t *which is given by* $\boldsymbol{\pi}'\boldsymbol{y}$ *and is different from either* a *or* b.

When instead $p = 0$ *or* $p = 1$, *the stationary Markov chain* $(\boldsymbol{P}, \boldsymbol{\pi})$ *is ergodic because its invariant functions are constant in their positive-probability elements. The sample average will equal the first realisation of* y_t, *but since the initial distribution is degenerate, the sample average will coincide with the unconditional expectation of* y_t.

Continuous Markov chains. Let us now cover quickly the possibility that our random variable of interest follows a continuous Markov chain, i.e. that it takes on a continuum of values (while we continue to assume that time is discrete). Let s denote the random variable and S denote its domain. Let $\pi(s'|s)$ denote its transition density, which must be such that $\int_S \pi(s'|s)ds' = 1$, and let $\pi_0(s)$ denote the initial probability distribution, which must be such that $\int_S \pi_0(s)ds = 1$.

The unconditional distribution of s at period 1 is given by

$$\pi_1(s_1) = \int_S \pi(s_1|s_0)\pi_0(s_0)\mathrm{d}s_0.$$

The unconditional distribution of s at period t is given by

$$\pi_t(s_t) = \int_S \pi(s_t|s_{t-1})\pi_{t-1}(s_{t-1})\mathrm{d}s_{t-1}.$$

A stationary distribution is defined as

$$\pi_\infty(s') = \int_S \pi(s'|s)\pi_\infty(s)\mathrm{d}s.$$

An random variable $y(s)$ is invariant if

$$\int_S y(s')\pi(s'|s)\mathrm{d}s' = y(s).$$

A Markov chain is ergodic if all invariant functions $y(s)$ are constant with probability 1 according to π_∞, i.e. $y(s) = y(s')$ for all s, s' such that $\pi_\infty(s) > 0, \pi_\infty(s') > 0$.

Finally, if $y(s)$ is a random variable on a stationary and ergodic Markov chain $\pi(s'|s), \pi(s)$ and $\mathbb{E}[|y|] < \infty$, then

$$\frac{1}{T}\sum_{t=1}^{T} y_t \to \mathbb{E}[y] = \int_S y(s)\pi(s)\mathrm{d}s$$

with probability 1.

5.2 Stochastic Dynamic Programming

Consider a general formulation of a Bellman equation. Let x denote the endogenous state variable, and z denote the exogenous and stochastic state variable that follows a Markov chain with transition probabilities $P(z'|z)$. Let $\Gamma(x, z)$ define the feasible set correspondence and let $a \in \Gamma(x, z)$ denote the control variable. Let $F(x, z, a)$ denote the return function, and

$x' = q(x, z, a)$ denote the law of motion for the endogenous state variable.

$$v(x,z) = \max_{\substack{a \in \Gamma(x,z) \\ x' = q(x,z,a)}} F(x,z,a) + \beta \mathbb{E}[V(x',z')|z] \tag{17}$$

where the expectation can be either $\mathbb{E}[V(x',z')|z] = \sum_{z'} P(z'|z)V(x',z')$ or $\mathbb{E}[V(x',z')|z] = \int v(x',z')f(z'|z)\mathrm{d}z'$

Example 11. *Let's apply this new general notation to the NGM. One option is to continue to substitute out consumption by using the resource constraint. In this case, the endogenous state is k, the exogenous stochastic state is productivity z, the control variable is k', the feasible set correspondence is $\Gamma(k,z) = [0, zk^\alpha + (1-\delta)k]$, the return function is $F(k,z,k') = u(zk^\alpha + (1-\delta)k - k')$, the law of motion for capital is $k' = k'$ (trivial, says how the future state depends on the current control variable, which here happen to be same...) and for productivity is given by $\pi(z'|z)$, the Bellman equation is*

$$v(k,z) = \max_{k' \in \Gamma(k,z)} \left\{ F(k,z,k') + \beta \int_Z V(k',z')f(z'|z)dz' \right\}$$

Another option is to leave consumption in the problem and keep the resource constraint. Then we have that the endogenous state is k, the exogenous stochastic state is productivity z, the control variable is c, the feasible set correspondence is $\Gamma(k,z) = [0, zk^\alpha + (1-\delta)k]$, the return function is $F(c) = u(c)$, the law of motion for capital is $k' = q(k,z,c) = zk^\alpha + (1-\delta)k - c$ and for productivity is given by $\pi(z'|z)$, the Bellman equation is

$$v(k,z) = \max_{\substack{c \in \Gamma(k,z) \\ k' = q(k,z,c)}} \left\{ F(c) + \beta \int_Z V(k',z')f(z'|z)dz' \right\}.$$

Example 12. *Consider another example: a consumption-savings model with an AR(2) earnings process given by $w_t = \rho_1 w_{t-1} + \rho_2 w_{t-2} + \epsilon_t$ where $\epsilon_t \sim N(0, \sigma^2)$. The budget constraint is standard and given by*

$$c_t + a_{t+1}/R \leq a_t + w_t$$

and the no-borrowing constraint $a_{t+1} \geq 0$. The elements of our dynamic programming problem *are the following:*

- *the endogenous state is a, the exogenous states are w, w_- (the latter denotes income in the previous period)*
- *the control variable is a'*
- *the feasible set correspondence in $\Gamma(a, w) = [0, (a + w)R]$*
- *the return function is $F(w, a, a') = u(a + w - a'/R)$*
- *the laws of motion for the states are given by $a' = a'$ for the endogenous state, $w' = \pi(w'|w, w_-)$ and $w = w$ for the exogenous stochastic states*
- *the Bellman equation is given by*

$$v(a, w, w_-) = \max_{a' \in \Gamma(a,w)} \left\{ F(a, w, a') + \beta \int v(a', w', w) \pi(w'|w, w_-) dw' \right\}.$$

Let's derive the Euler equation to see how it looks like in this stochastic environment. The FOC for a' is given by

$$F_{a'}(w, a, a') + \beta \int v_{a'}(a', w', w) \pi(w'|w, w_-) dw'$$

where $F_{a'}(w, a, a') = -u'(c)\frac{1}{R}$. The envelope condition is given by

$$v_a(a, w, w_-) = F_a(w, a, a')$$

where $F_a(w, a, a') = u'(c)$. Putting things together we get the Euler equation

$$-F_{a'}(w, a, a') = \beta \int F_{a'}(w', a', a'') \pi(w'|w, w_-) dw'$$

which becomes the well-known

$$u'(c) = \beta R \int u'(c') \pi(w'|w, w_-) dw'.$$

5.3 The McCall job search model

This is an example of a finite horizon stochastic problem, which we'll use to get more acquainted with stochastic dynamic programming.

Consider the problem of a single agent, a worker. She can be either employed or unemployed. If she's unemployed, she receives a random wage offer $w_t \in [\underline{w}, \overline{w}]$, where w_t is distributed according to some distribution F. The agent consumes her income y_t and has linear utility.

The worker can accept the offer, in which case she gets income equal to the wage w_t for the rest of her life, or reject the offer and receive unemployment benefit b that are some constant inside the $[\underline{w}, \overline{w}]$ interval. The ingredients of the problem of the unemployed worker are:

- the state variable is the wage offer w
- the choice variable is the discrete accept/reject choice $c \in \{a, r\}$
- the return function is $F(w, c)$ which is equal to w if $c = a$ and to b if $c = r$.

Two-period case. Consider the first a worker that only lives two periods $t = 0, 1$. Clearly, we could also look at the problem in sequential form. The objective function would be $\mathbb{E}_0(y_0 + \beta y_1)$, and we could derive the optimal worker's policy and the value of her maximised utility. We'll use the dynamic programming approach to get used to that, but you can try to solve the problem in sequential form as an exercise and check that you do get the same results.

Let's solve the problem backwards. Let V_1 denote the maximised value for the unemployed worker of drawing offer w at $t = 1$

$$V_1(w) = \max_{\{a,r\}} \{V_1^a(w), V_1^r(w)\}$$

where

$$\begin{aligned} V_1^a(w) &= w \\ V_1^r(w) &= b. \end{aligned}$$

The optimal policy of the worker clearly is

$$g_1(w) = \begin{cases} a & \text{if} \quad w \geq b \\ r & \text{if} \quad w < b \end{cases}$$

that is, the worker accepts if the offer is above a threshold which we call reservation wage and we denote with $\hat{w}_1 = b$.

In $t = 0$, let V_0 denote the maximised value for the unemployed worker of drawing offer w at $t = 0$

$$V_0(w) = \max_{\{a,r\}} \{V_0^a(w), V_0^r(w)\}$$

where

$$V_0^a(w) = w + \beta w$$
$$V_0^r(w) = b + \beta \mathbb{E}_1[V_1(w')] = b + \beta \left[\int_{\underline{w}}^{\overline{w}} V_1(w') \mathrm{d}F(w') \right] = b + \beta \left[\int_{\underline{w}}^{\hat{w}_0} b \, \mathrm{d}F(w') + \int_{\hat{w}_0}^{\overline{w}} w' \, \mathrm{d}F(w') \right].$$

Here too the value of accepting is increasing in w and the value of rejecting is constant and independent of w, so the worker will accept if and only if the wage offer is above a threshold which is the reservation wage at $t = 0$ $\hat{w}_0$. What can we say about $\hat{w}_0$? How does it change with b? Is it larger or smaller than $\hat{w}_1$?

By construction, $\hat{w}_0$ must be the wage offer at which the worker is indifferent between a and r. So let's write this with math

$$\begin{aligned}
V_0^a(\hat{w}_0) &= V_0^r(\hat{w}_0) \\
\hat{w}_0(1+\beta) &= b + \beta \left[bF(b) + \int_b^{\overline{w}} w' \, \mathrm{d}F(w') \right] \\
\hat{w}_0(1+\beta) &= b(1+\beta) + \beta \int_b^{\overline{w}} (w' - b) \, \mathrm{d}F(w') \\
\hat{w}_0 &= b + \frac{\beta}{1+\beta} \int_b^{\overline{w}} (w' - b) \, \mathrm{d}F(w').
\end{aligned}$$

This tells us a few things. First, $\hat{w}_0 > b$, because in this model there is an *option value* of waiting: if you reject the offer at $t = 0$, you consume b today but you have a chance of getting a very high draw tomorrow, so you're willing to risk it. Second, we can show that $\frac{\mathrm{d}\hat{w}_0}{\mathrm{d}b} > 0$ (try to show that at home!), because obviously you're less willing to accept a certain offer if the alternative to that is to get a higher unemployment benefit.

Infinite horizon. Given the assumptions that employed workers stay on the job forever, the infinite horizon version of this problem is actually extremely similar to the $t = 0$ version of the previous problem, with the only different that now the future lasts an infinite number of periods rather than only one period.

The optimised value of lifetime utility for an employed agent with wage w is

$$V^e(w) = \frac{w}{1-\beta}.$$

The optimised value of lifetime utility for an unemployed agent with a wage offer w is

$$V^u(w) = \max_{a\in\{0,1\}} \{aV^e(w) + (1-a)[b + \beta\mathbb{E}[V^u(w')]]\}.$$

As in the finite-horizon environment, the optimal policy of the unemployed agent is to accept the wage offer if it is higher than some reservation wage, which can be derived as we did previously.

Extension: receive offer with probability ϕ. Consider the case where unemployed agents only receive an employment offer with probability ϕ. The value of being employed is unchanged. The value of being unemployed with no offer is

$$V^n = b + \beta(1-\phi)V^n + \beta\phi\mathbb{E}[V^o(w')].$$

The value of being unemployed with an offer w is

$$V^o(w) = \max_{a,r}\left\{\frac{w}{1-\beta}, V^n\right\}.$$

We can keep going by first solving for V^n and then writing it as a weighted average

$$V^n = a(\phi)\frac{b}{1-\beta} + [1-a(\phi)]\mathbb{E}[V^o(w')]$$

where $a(\phi) = \frac{1-\beta}{1-\beta(1-\phi)}$. With this we can for example compute the impact of changes in ϕ on the reservation wage.

Extension: receive multiple offers. Consider the case where an unemployed worker received two wage offers in each period. Then the wage offer to be considered is $y = \max\{w_1, w_2\}$. One can show that

$$P(y < y') = P(\max\{w_1, w_2\} < y') = F(y')^2.$$

So the value of being unemployed is the same as in the one-offer case, just that now

$$\mathbb{E}[V^u(w')] = \int V(w')dF(y')^2.$$

6 Recursive Competitive Equilibrium

So far we have seen either planner's problems (the NGM), where there are no prices or markets to clear, or partial equilibrium problems where, where pricesare taken as given and market clearing is ignored. We will now move on to consider competitive equilibria (CE). So far you have seen Arrow-Debreu time-0 equilibria as well as equilibria with sequential markets. Now we will consider recursive competitive equilibria (RCE).

Let us get started by considering once again the NGM example. Let K, N respectively denote aggregate capital and labour. Consider a continuum of measure 1 of households indexed by $i \in [0,1]$. Each household faces a budget constraint given by

$$c_t^i + k_{t+1}^i \leq w_t n_t^i + (1 - \delta + r_t) k_t^i \tag{18}$$

for all $i \in [0,1]$ and all t. Agents now rent their capital and labour in a competitive market respectively at rental rate r_t and wage w_t. Note that these are equilibrium prices so they are not agent-specific. We look for a symmetric equilibrium, since all agents have the same utility function, face the same budget constraint and are therefore identical. Since there is a continuum of households, each of them is infinitesimally small and takes prices as given[17]. That is, this is different from a setting where agents have market power (think of industrial organisation examples where firms choose quantities knowing that revenues are given by $Q \cdot P(Q)$) and therefore internalise the effect of their decisions on equilibrium prices.

Aggregate variables are determined by individual variables through the following

$$K_t = \int_0^1 k_t^i \mathrm{d}i$$
$$N_t = \int_0^1 n_t^i \mathrm{d}i.$$

Given agents are all identical, we will have that $K = k^i$ and $N = n^i$ for all i. Below we will

[17] To gain intuition, it is useful to see the continuum assumption as the limit of a case where there is a finite number N of agents, each with weight $\lambda_i = 1/N$ such that $\sum_{i=1}^N \lambda_i = 1$. Taking the limit for $N \to \infty$ the weight of each individual goes to zero. Aggregate variables are given by $X = \sum_{i=1}^N \frac{1}{N} x_i$, so in the limit for $N \to \infty$ it follows that $\frac{\partial X}{\partial x_i} \to 0$. By the way, this feature is what gives rise to externalities and coordination problems (but we won't cover these here).

sometimes drop the i superscript to make notation lighter, but we will always use small-caps notation to denote individual variables and capital letters to denote aggregate variables.

Let $X = (Z, K)$ denote the aggregate state vector, where Z denotes aggregate productivity, which is exogenous and follows some (for now) undefined stochastic process.

Before considering the household individual problem, let us analyse the firms' problem. There is also a continuum of measure 1 of firms, that are perfectly competitive and subject to free entry, so they make zero profits. Like households, firms also take prices as given, and firm f faces the following *static* profit-maximisation problem

$$\max_{k^f, n^f} ZF(k^f, n^f) - wn^f - rk^f$$

where $F(k^f, n^f)$ is a standard Cobb-Douglas production function and k^f, n^f denote capital and labour demanded by firm f. The first-order conditions are

$$\begin{aligned} ZF_k(k^f, n^f) - r &= 0 \\ ZF_n(k^f, n^f) - w &= 0. \end{aligned}$$

Aggregation in the firms' sector works in the same way described above, so $K = \int_0^1 k^f \mathrm{d}f$ and $N = \int_0^1 n^f \mathrm{d}f$. It follows that, since all firms are identical, aggregate capital and labour in the firms' sector is given by $K = k^f$ and $N = n^f$ for any f, and in a symmetric CE the prices r and w are functions of the aggregate state $X = (Z, K)$ only, via a time-invariant mapping which is a crucial feature of a recursive equilibrium. It follows that firms' capital and labour demand will also be a time-invariant function of the aggregate states.

We now go back to look at the households. They face an intertemporal consumption-savings problem which is affected, for example, by the future rental rate of capital $r(X')$. To derive optimal individual behaviour we thus need to specify the law of motion of prices. Since prices only depend on the aggregate states, we need the law of motion for the aggregate states. Let us assume for now that agents have a *perceived* (i.e., for now this can be arbitrary) law of motion for capital given by $K' = G(K)$: given current aggregate capital, agents postulate what future capital will be. The Bellman equation for the household problem is

$$\begin{aligned}
v(X;k) = \max_{c,k',n} & \{u(c,1-n) + \beta\mathbb{E}\left[v(X';k')\right]\} \\
\text{s.t.} \quad & c + k' \leq w(X)n + (1-\delta + r(X))k \\
& K' = G(X) \\
& c \geq 0.
\end{aligned}$$

Solving the household problem will give us the usual Euler equation

$$u_c(c,1-n) = \beta\mathbb{E}\left[u_c(c',1-n')\right](1-\delta + r(X')) .$$

Current and future prices depend on the current and future aggregate states, and the expected future aggregate state depends on the perceived law of motion G. Hence the optimal individual policy of any agent will be a function of the individual state as well as all these factors: let us denote the policy functions for consumption, labour supply and investment with $c(X;k,G)$, $n(X;k,G)$ and $g(X;k,G)$.

Competitive equilibrium definition.

Definition 13. *A Recursive Competitive Equilibrium (RCE) with arbitrary expectations G is*

- *a set of functions (v,c,n,g) for the individuals, (w,r) for prices and (k^f,n^f) for firms*
- *perceived and actual laws of motion (G,H)*

such that

- *given price functions (r,w) and perceived law of motion G, the individual value and policy functions (v,c,n,g) solve the household problem (i.e. the Bellman equation);*
- *the actual law of motion for aggregate capital is given by H and is such that*

$$K' = H(X) = g(X,k=K,G);$$

- *given price functions (r,w), the firm policy functions (k^f,n^f) solve the firm problem;*
- *(prices are such that) the markets for labour, capital and consumption (i.e. the resource*

constraint) clear

$$\int n^f(X)\mathrm{d}f = \int n(X,k,G)\,\mathrm{d}i$$
$$\int k^f(X)\mathrm{d}f = \int k\,\mathrm{d}i$$
$$c(X;K,G) + g(X;K,G) = ZF(K,L) + (1-\delta)K.$$

A few things are worth noting. First, we use small-cap and big-cap notation here because in a symmetric equilibrium individual and aggregate variables coincide. Second, we defined the actual law of motion of capital by taking the individual investment policy function ($g(X;k,G)$) and plugging in aggregate capital ($g(X;K,G)$), which is sometime referred to as *representative agent* condition.

Rational expectations. We have not however asked that the resulting law of motion, which we labelled $H(X)$, is consistent with G. When that is the case, we have a RCE with rational expectations, and we have the extra requirement that

$$H(X) = G(X) = g(X;K,G)$$

i.e. that the realised law of motion of aggregate capital is consistent with the law of motion perceived by the individual agents.

Example 13. *To see the meaning of the rational expectations condition, consider the following toy example. Suppose that agents optimally choose some action a' as a linear function of their individual state a and the future value of the aggregate state A', that is*

$$a' = g_0 + g_1 a + g_2 A'$$

where g_0, g_1 and g_2 are functions of the model parameters. Next suppose agents believe A' follows some function $\eta(A) = \eta_0 + \eta_1 A$. It follows that for any individual

$$a' = g_0 + g_1 a + g_2(\eta_0 + \eta_1 A) = g(A;a,\eta).$$

Impose the representative agent condition ($a = A$) and obtain the actual law of motion of A

which is given by individual actions and beliefs according to

$$A' = g(A; A, \eta) = (g_0 + g_2\eta_0) + (g_1 + g_2\eta_1)A.$$

The rational expectation condition requires that agents' beliefs are consistent with the actual LOM, i.e. that

$$\begin{cases} \eta_0 = g_0 + g_2\eta_0 \\ \eta_1 = g_1 + g_2\eta_1 \end{cases} \Rightarrow \begin{cases} \eta_0 = \frac{g_0}{1-g_2} \\ \eta_1 = \frac{g_1}{1-g_2} \end{cases}.$$

Note that (as you probably already know) the resource constraint can be obtained by combining the household budget constraint with the firm optimality conditions

$$c + k' = ZF_n(K, N)n + [ZF_k(K, N)k + (1 - \delta)]k$$

and then replacing individual with aggregate variables and using the fact that the production function is assumed to be homogenous of degree 1[18], which allows to go from this

$$C + K' = Z[F_n(K, N)N + F_k(K, N)K] + (1 - \delta)K$$

to this

$$C + K' = ZF(K, N) + (1 - \delta)K.$$

6.1 RCE with Government

We now consider a variation of the previous problem which includes a government. The government taxes labour income using a proportional tax τ, and uses the proceeds of such tax to buy "medals" M.

Wasteful spending and inelastic labour supply. For now, think of M as wasteful government spending, which we'll assume to follow some exogenously given policy function or rule.

[18]Recall that a function F is homogeneous of degree n if, for a vector of inputs $\boldsymbol{x}$ and a scalar α, we have that

$$F(\alpha\boldsymbol{x}) = \alpha^n F(\boldsymbol{x}).$$

For simplicity we consider a deterministic setting without productivity shocks and with labour supply exogenously fixed at 1 and no utility from leisure

The household problem is

$$
\begin{aligned}
v(K;k) = \max_{c,k'} & \{u(c) + \beta v(K';k')\} \\
\text{s.t.} \quad & c + k' \leq w(K)(1-\tau) + (1-\delta+r(K))k \\
& K' = G(X).
\end{aligned}
$$

For brevity we do not solve the firm problem explicitly and just assume that wages and rental rates are function of the aggregate state, but we know that's true here.

The government budget constraint is given by

$$\tau w(K) = M$$

where we don't specify what medal policy is. But for any given medal policy (that is not a linear function of $w(K)$), we will have that labour taxes must be a function of K as well, for the government budget to be balanced.

Solving the household problem yields our usual Euler equation

$$u'(c) = \beta[1-\delta+r(K)]u'(c').$$

Will the equilibrium be efficient? To answer that let's look at the Planner's problem

$$
\begin{aligned}
v(K) = \max_{C,M,K'} & \{u(C) + \beta v(K')\} \\
\text{s.t.} \quad & C + K' + M \leq F(K,1) + (1-\delta)K.
\end{aligned}
$$

The Euler equation for this problem is

$$u'(C) = \beta[1-\delta+F_K(K',1)]u'(C')$$

which is identical to the one we derived just above and thus implies that the RCE in this particular problem will be efficient. Let us define such RCE explicitly

Definition 14. *A Recursive Competitive Equilibrium (RCE) with rational expectations (RE) is*

- *a set of functions (v, c, g) for the individuals, (w, r) for prices and (τ, M) for the government*
- *capital law of motion G*

such that

- *given price functions (r, w), capital law of motion G and government policy (τ, M), the individual value and policy functions (v, c, g) solve the household problem (i.e. the Bellman equation);*
- *the law of motion for aggregate capital is consistent with individuals' policy function*

$$K' = G(K) = g(K, k = K, G);$$

- *the factor price functions (r, w) equal the factor marginal product (equivalently, firms behave optimally and the labour and capital market clear);*
- *government policy is feasible (i.e. the government budget constraint is satisfied)*
- *(prices are such that) the market for consumption (i.e. the resource constraint) clears*

$$c(K; K, G) + g(K; K, G) + M(K) = F(K, 1) + (1 - \delta)K.$$

Note that the last bullet point here is redundant, because the market for consumption clears by Walras' law when that all other markets also do so, which is what we have ensured in the previous points.

Useful spending. Now let's consider the case where households get utility from government spending (medals) according to $u(c, M)$. Since government policy is taken as given by households, the Euler equation will not change much neither for the households

$$u_c(c, M) = \beta[1 - \delta + r(K)]u_c(c', M')$$

nor for the government

$$u_c(C, M) = \beta[F_K(K, 1) + (1 - \delta)K]u_c(C', M').$$

However, the planner problem will include one more condition, namely

$$u_c(C, M) = u_m(C, M) \tag{19}$$

i.e. the MRS between consumption of the private and the public good must equal the relative price, which is 1 here.

It follows that an RCE with useful spending will be efficient if and only if government policy is such that condition (19) is always satisfied.

Endogenous labour supply. Lastly, consider the case where households get utility from leisure and have a labour/leisure choice. The household problem is

$$
\begin{aligned}
v(K;k) = \max_{c,n,k'} & \{u(c,n) + \beta v(K';k')\} \\
\text{s.t.} \quad & c + k' \leq w(K)n(1-\tau) + (1-\delta+r(K))k \\
& K' = G(X)
\end{aligned}
$$

and the government budget constraint is given by

$$\tau w(K)n = M.$$

What's new here is that households will have a consumption-leisure optimality condition, typically defined as the labour supply equation, given by

$$u_c(c,n) = (1-\tau)w(K)u_n(c,n).$$

In the planner's problem, the equivalent of this condition is given by

$$u_c(C,N) = F_n(K,N)u_n(C,N)$$

and since $w(K) = F_n(K,N)$ we can clearly see that the RCE in this economy will not be efficient because labour taxes create a distortion. The only case where the welfare theorems hold is that where $M = 0$ always, which implies taxes are also zero and thus there is no distortion.

6.2 RCE with Heterogeneity

We now consider something different from the perfectly symmetric world we have analysed so far.

Suppose there are two groups of agents, with a possibly different endowment of initial wealth or capital. Aggregate capital in this economy is given by

$$K = \mu K_1 + (1-\mu)K_2$$

where μ is the measure of the first group, and K_i denotes the amount of capital held by group i.

There is still a continuum of identical firms, so factor prices remain a function $r(K), w(K)$ of aggregate capital only.

The Bellman equation for an individual in group i is given by

$$\begin{aligned} v(K_1, K_2; k) = \max_{c,k'} &\{u(c) + \beta v(K_1', K_2'; k')\} \\ \text{s.t.} \quad & c + k' \leq w(K) + (1-\delta + r(K))k \\ & K' = G(K_1, K_2). \end{aligned}$$

Note that the aggregate state is now given by the capital of each group. The reason is that wages and capital rental rates are a function of aggregate capital K, and future values of K' may depend on how K is distributed across groups. To see that, consider the Euler equation for an individual

$$u'(c) = \beta[1-\delta + r(K')]u'(c')$$

whose solution will yield a policy function $k' = g(k, K)$. Applying the representative agent condition we get that

$$\begin{aligned} K_1' &= g(K_1, K) \\ K_2' &= g(K_2, K) \end{aligned}$$

and therefore

$$K' = \mu\, g(K_1, K) + (1-\mu)\, g(K_2, K)$$

which will be different from $g(K_1 + K_2, K)$, unless g is linear in its first argument.

What can we say about the steady state? The Euler equation implies that aggregate capital is pinned down by the usual condition

$$1 = \beta[1-\delta + F_k(K^{ss}, 1)]$$

which is the same for any group i. How do we recover the group-specific levels of consumption and investment? The household budget constraint

$$c_i^{ss} = w(K^{ss}) + k_i^{ss}[F_k(K^{ss}, 1) - \delta]$$

pins down the combination of c_i^{ss} and k_i^{ss}, but not each of them separately. It follows that we know steady-state aggregate consumption and capital, but the individual levels are undetermined.

7 Ordinary Differential Equations Review

First order differential equations are of the form

$$F(t, y, \dot{y}) = 0$$

and $y = f(t)$ is said to be a solution if

$$F(t, f(t), f'(t)) = 0 \quad \forall t.$$

An ODEs is **homogeneous** if it has the (manageable) form

$$\dot{y} = h(y, t).$$

A first-order ODE is **separable** if it has the form

$$\dot{y} = f(t)g(y).$$

In such case, we can attempt to solve it as

$$\int \frac{1}{g(y)} \mathrm{d}y = \int f(t) \mathrm{d}t.$$

7.1 Homogeneous, Separable, Linear.

Consider

$$\dot{y} + p(t)y = 0.$$

It is separable, and both y and $\dot{y}$ appear at the first power. The solution is

$$\int \frac{1}{y}\mathrm{d}y = \int -p(t)\mathrm{d}t$$

which yields

$$\log y = \int -p(t)\mathrm{d}t + C$$

and

$$y = Ae^{\int -p(t)\mathrm{d}t}$$

where $A = e^C$.

Given an initial condition $y(t_0) = y_0$ we can easily recover the value of A .

7.2 Non-Homogeneous.

It is of the form

$$\dot{y} + p(t)y = q(t)$$
$$y(t_0) = y_0.$$

First, you find the general solution to the "associated homogeneous equation" $\dot{y} + p(t)y = 0$, which is $y(t) = Ae^{P(t)}$ where $P(t) = \int -p(t)dt$.

Then, we make the guess that $A = a(t)$ and therefore $y(t) = a(t)h(t)$ where $h(t) = e^{P(t)}$. Then the left-hand side of the non-homogenous equation is given by

$$\begin{gathered}\dot{y} + p(t)y = \\ = \dot{a}(t)h(t) + a(t)\dot{h}(t) + p(t)a(t)h(t) = \\ = a(t)[\dot{h}(t) + p(t)h(t)] + \dot{a}(t)h(t) = \\ = \dot{a}(t)h(t).\end{gathered}$$

which means we must solve

$$\dot{a}(t)h(t) = q(t)$$

to get a particular solution of the non-homogenous equation. That requires finding the antiderivative of $\dot{a}(t) = \frac{q(t)}{h(t)}$. Since q and h are known functions of time, we'll get

$$a(t) = \int \frac{q(t)}{h(t)}\mathrm{d}t.$$

There could be a constant, but we can set it to zero when solving for a particular solution of the non-homogeneous equation.

The final solution is given by

$$y(t) = A\underbrace{e^{P(t)}}_{h(t)} + a(t)\underbrace{e^{P(t)}}_{h(t)} = Ah(t) + \int \frac{q(t)}{h(t)}\mathrm{d}t\ h(t)$$

and A can be obtained by solving $y(t_0) = y_0$.

When $q(t) = \kappa$ and $p(t) = \phi$ are constant and $y(0) = y_0$, the solution is given by

$$y(t) = y_0 e^{-\phi t} + \frac{\kappa}{\phi}(1 - e^{\phi t})$$

which is a weighted average (with exponentially decaying weights) between the initial condition y_0 and the steady state κ/ϕ (which can be obtained setting $\dot{y} = 0$).

8 Dynamic Optimisation in Continuous Time

8.1 Finite horizon

Our typical problem has the form

$$V(x_0, 0) = \max_{x(t),a(t)} \int_0^T r(t, x(t), a(t))\mathrm{d}t$$

subject to

$$\begin{aligned}
&\dot{x}(t) := \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x(t), a(t)) \\
&x(0) = x_0 \text{ given} \\
&g(x(T)) \geq 0.
\end{aligned}$$

where x, a are time functions that map from $[0, T]$ into a set $\mathcal{A}$ to be specified, $x(t)$ is the state variable, $a(t)$ is the control variable, r is the return function and g specifies the terminal condition on the state variable (e.g. the no Ponzi game condition).

The continuous time equivalent of the Lagrangian is the Hamiltonian[19]

$$H(t, x, a, \lambda) = r(t, x, a) + \lambda f(t, x, a).$$

The necessary equilibrium conditions are the following:

$$\begin{aligned} H_a &= 0 && \text{(maximum principle)} \\ \dot{\lambda}(t) &= -H_x && \text{(adjoint equation)} \\ \dot{x}(t) &= f(t, x, a) && \text{(dynamics)} \\ \lambda(T) &= \zeta\ g'(x(T)) && \text{(transversality condition)} \end{aligned}$$

We now formally derive these conditions using the so-called "variational" approach.

Proof. Consider function $\hat{a}(t)$ that achieves the optimum of our problem. Consider a function that deviates from $\hat{a}$ in the following way

$$a(t, \epsilon) = \hat{a}(t) + \epsilon\eta(t)$$

where η is an arbitrary continuous function. Using $a(t, \epsilon)$ as our solution for the control variable we'd get associated dynamics for the state variable of the form

$$\dot{x}(t, \epsilon) = f(t, x(t, \epsilon), a(t, \epsilon)).$$

Let

$$\Phi(\epsilon) = \int_0^T r(t, x(t, \epsilon), a(t, \epsilon))\mathrm{d}t.$$

If $\hat{a}$ is the solution, then we should have that $\Phi(0) \geq \Phi(\epsilon)$ for any ϵ. Take the dynamics, multiply by the co-state and integrate, and do the same (but without integration) for the terminal condition

$$\int_0^T \lambda(t)[f(t, x(t, \epsilon), a(t, \epsilon)) - \dot{x}(t, \epsilon)]\mathrm{d}t + \zeta\ g(x(T, \epsilon)) = 0$$

[19] We now start omitting the dependence on time to lighten up notation, but keep in mind that x, a, λ are all functions of time, not constants.

which holds by construction. The co-state ζ is just a number since the constraint is only there for period T. Since this whole expression is zero, we can add this to our previous equation

$$\Phi(\epsilon) = \int_0^T \Big[r(t, x(t,\epsilon), a(t,\epsilon)) + \lambda(t)\Big(f(t, x(t,\epsilon), a(t,\epsilon)) - \dot{x}(t,\epsilon)\Big)\Big] \, \mathrm{d}t + \zeta \; g(x(T,\epsilon)).$$

Integrate the term $\lambda(t)\dot{x}(t)$ by parts

$$\int_0^T \lambda(t)\dot{x}(t)\mathrm{d}t = \lambda(T)x(T,\epsilon) - \lambda(0)x_0 - \int_0^T \dot{\lambda}(t)x(t,\epsilon)\mathrm{d}t.$$

Plug it back into our main equation and drop some function dependences for simplicity (we keep time dependence for the terminal condition to remember things are a function of T there)

$$\Phi(\epsilon) = \int_0^T \Big[r(t,x,a) + \lambda(t)f(t,x,a) + \dot{\lambda}(t)x\Big] \, \mathrm{d}t - \lambda(T)x(T,\epsilon) + \lambda(0)x_0 + \zeta \; g(x(T,\epsilon)).$$

Now differentiate with respect to ϵ

$$\Phi'(\epsilon) = \int_0^T \Big[(r_x + \lambda f_x + \dot{\lambda})x_\epsilon + (r_a + \lambda f_a)a_\epsilon\Big] \, \mathrm{d}t - [\lambda(T) - \zeta \; g'(x(T,\epsilon))] \, x_\epsilon(T,\epsilon).$$

Evaluate this derivative at $\epsilon = 0$, where recall $\hat{a}(t) = a(t,0)$

$$\Phi'(0) = \int_0^T \Big[(r_x(t,\hat{x},\hat{a}) + \lambda f_x(t,\hat{x},\hat{a}) + \dot{\lambda})x_\epsilon + (r_a(t,\hat{x},\hat{a}) + \lambda f_a(t,\hat{x},\hat{a}))\eta\Big] \, \mathrm{d}t - [\lambda(T) - \zeta \; g'(\hat{x}(T))] \, x_\epsilon(T,0).$$

Now let's suppose that we picked the adjoint function $\lambda(t)$ such that the term multiplying x_ϵ goes away, i.e.

$$\dot{\lambda} = -r_x(t,\hat{x},\hat{a}) - \lambda f_x(t,\hat{x},\hat{a}).$$

Then as long as the following conditions hold

$$\begin{aligned} &r_a(t,\hat{x},\hat{a}) + \lambda f_a(t,\hat{x},\hat{a}) = 0 \\ &\lambda(T) = \zeta \; g'(\hat{x}(T)) \end{aligned}$$

we are sure that $\Phi'(0) = 0$ for any arbitrary "deviation function" η.

The last three equations we have derived correspond to the optimality conditions we spelled out earlier, together with the dynamics equation. ■

Remarks. First, the transversality condition (TVC), which is an optimality condition, relates the multiplier of the terminal condition (which is instead a constraint) to the co-state λ evaluated at the last period. Joining that condition with the complementary slackness condition of the terminal constraint itself will yield a more familiar version of the TVC, as we will see in the consumption-savings example.

Second, conditions (maximum principle)-(transversality condition) are necessary conditions for an optimum. They are also sufficient if we make additional concavity assumptions which we have not made here so far.

Third, the adjoint (or costate) variable $\lambda(t)$ is a time function that has the same role of the Lagrange multiplier in discrete time. Here is represents the "flow" value of relaxing the dynamics equation (also called flow constraint) by one unit, or the marginal value of incrementing the state variable by one unit at time t in an optimal plan.

8.2 Infinite horizon

We now assume exponential discounting, return functions and state dynamics that do not depend explicitly on time, and terminal conditions instead of terminal values.

The problem is

$$V(x_0, 0) = \max_{x(t),a(t)} \int_0^\infty e^{-\rho t} r(x(t), a(t)) \mathrm{d}t$$

subject to

$$\begin{aligned}
&\dot{x}(t) = f(x(t), a(t)) \\
&x(0) = x_0 \text{ given} \\
&\lim_{t\to\infty} b(t)x(t) = 0
\end{aligned}$$

where $b(t)$ is some exogenously defined function. We can also ignore the time argument inside the value function, since the time subscript of the initial condition coincides with it.

We define the *present-value* Hamiltonian as

$$H(t, x, a, \lambda) = e^{-\rho t} r(x, a) + \lambda f(x, a).$$

The equilibrium conditions are still the maximum principle, the state dynamics, the adjoint equation and the TVC

$$
\begin{aligned}
&H_a = e^{-\rho t} r_a + \lambda f_a = 0 \\
&\dot{x} = f(x, a) \\
&\dot{\lambda} = -H_x \\
&\lim_{t \to \infty} \lambda(t) x(t) = 0.
\end{aligned}
$$

The intuitive (but informal) way to derive the TVC is to assume finite horizon first. We get

$$\zeta\, b(T) = \lambda(T).$$

Since the complementary slackness condition is $\zeta\, b(T)x(T) = 0$, combine this to get $\lambda(T)a(T)$ and then take the limit for $T \to \infty$ to get the infinite horizon TVC.

Now that we have discounting, the adjoint becomes

$$\dot{\lambda} = -H_x = -e^{-\rho t} r_x + \lambda f_x.$$

We can also use a different version of the Hamiltonian, called *current-value* because we discount "forward" the costate. Define $\mu(t) = e^{\rho t}\lambda(t)$. We have a Hamiltonian which is not an explicit function of time anymore, i.e. where there is no discounting

$$H^{cv}(x, a, \mu) = r(x, a) + \mu f(x, a).$$

The equilibrium conditions become

$$
\begin{aligned}
&H^{cv}_a = r_a + \mu f_a = 0 \\
&\dot{x} = f(x, a) \\
&\dot{\mu} - \rho\mu = -H^{cv}_x = -r_x - \mu f_x \\
&\lim_{t \to \infty} e^{-\rho t} \mu(t) x(t) = 0.
\end{aligned}
$$

The third equation follows from the fact that, since $\lambda(t) = e^{-\rho t}\mu(t)$, we get that $e^{\rho t}\lambda(t) = \dot{\mu}(t) - \rho\mu(t)$.

8.3 Consumption-savings model

The problem is

$$V(a_0) = \max_{a(t),c(t)} \int_0^\infty e^{-\rho t} u(c(t)) \mathrm{d}t$$

such that

$$\begin{aligned}
\dot{a}(t) &= a(t)r - c(t) \\
a(0) &= a_0 \text{ given} \\
\lim_{t\to\infty} e^{-rt} a(t) &\geq 0.
\end{aligned}$$

To see how to derive the flow budget constraint from its discrete time equivalent, start with it

$$a_{t+1} = a_t(1+r) - c_t$$

then evaluate it not at $t+1$ but at $t+\Delta$ where Δ is an arbitrary time interval

$$a_{t+\Delta} = a_t(1 + r\Delta) - c_t\Delta$$

where we put a Δ in front of the flow variables (consumption and interest) but not the stock variables (the stock of savings). Divide through by Δ and then take the limit

$$\lim_{\Delta\to 0} \frac{a_{t+\Delta} - a_t}{\Delta} = a_t r - c_t$$

which finally yields

$$\dot{a}(t) = a(t)r - c(t).$$

The current-value Hamiltonian is[20]

$$H^{cv}(a, c, \mu) = u(c) + \mu[ar - c]$$

[20] The present-value Hamiltonian is

$$H^{pv}(t, x, a, \mu) = e^{-\rho t} u(c) + \lambda[ar - c]$$

Compute the maximum principle and the adjoint equation

$$
\begin{aligned}
H_a &= u'(c) - \mu = 0 \\
\dot{\mu} - \rho\mu &= -\mu r.
\end{aligned}
$$

Differentiate $H_a = 0$ one more time

$$\dot{c}\, u''(c) - \dot{\mu} = 0.$$

Combine this with the adjoint equation and the first expression for H_a

$$\dot{c}\, u''(c) = (\rho - r)\mu = (\rho - r)u'(c).$$

Rearranging

$$\frac{\dot{c}}{c} = \frac{u'(c)}{-c\, u''(c)}(r - \rho).$$

The fraction on the RHS is the inverse of relative risk aversion, so with a CRRA utility function it is equal to $1/\gamma$, which yields the Euler equation in continuous time

$$\frac{\dot{c}}{c} = \frac{r - \rho}{\gamma}.$$

Following the steps from above, the transversality condition is

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)a(t) = \lim_{t\to\infty} e^{-\rho t}u'(c(t))a(t) = 0.$$

The Euler equation is an homogeneous differential equation, which we can solve for c. These are the steps

$$
\begin{aligned}
\frac{\mathrm{d}c(t)}{\mathrm{d}t}\frac{1}{c(t)} &= \frac{r - \rho}{\gamma} \\
\int \frac{\mathrm{d}c(t)}{c(t)} &= \int \frac{r - \rho}{\gamma}\mathrm{d}t \\
c(t) &= e^{\left(\frac{r-\rho}{\gamma}\right)t}A \\
c(0) &= A = c_0 \\
c(t) &= e^{\left(\frac{r-\rho}{\gamma}\right)t}c_0.
\end{aligned}
$$

We can then plug this into the state dynamics

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = a(t)r - e^{\left(\frac{r-\rho}{\gamma}\right)t}c_0.$$

This is a non-homogeneous differential equation. The solution to the homogeneous part is

$$\begin{aligned}\int \frac{\mathrm{d}a(t)}{a(t)} &= \int r\mathrm{d}t \\ a(t) &= e^{rt}B.\end{aligned}$$

Guess $a(t) = e^{rt}v(t)$ as a particular solution. Then $\dot{a}(t) = re^{rt}v(t) + e^{rt}\dot{v}(t) = a(t)r + e^{rt}\dot{v}(t)$. Compare this with the state dynamics

$$\begin{aligned} a(t)r + e^{rt}\dot{v}(t) &= a(t)r - c(t) \\ e^{rt}\dot{v}(t) &= -e^{\left(\frac{r-\rho}{\gamma}\right)t}c_0 \\ \int \mathrm{d}v(t) &= -c_0 \int e^{-rt+\left(\frac{r-\rho}{\gamma}\right)t}\mathrm{d}t \\ v(t) &= -c_0\frac{\gamma}{r(1-\gamma)-\rho}e^{\left(\frac{r(1-\gamma)-\rho}{\gamma}\right)t} + D \end{aligned}$$

and we can set D to zero as we only need *one* particular solution. Finally, our general solution is the sum of the solution to the homogeneous equation and the particular solution

$$\begin{aligned} a(t) &= e^{rt}v(t) + e^{rt}B \\ &= -c_0\frac{\gamma}{r(1-\gamma)-\rho}e^{\left(\frac{r-\rho}{\gamma}\right)t} + e^{rt}B \\ a(0) &= -c_0\frac{\gamma}{r(1-\gamma)-\rho} + B \\ B &= a_0 + c_0\frac{\gamma}{r(1-\gamma)-\rho}. \end{aligned}$$

So finally

$$a(t) = -c_0\frac{\gamma}{r(1-\gamma)-\rho}e^{\left(\frac{r-\rho}{\gamma}\right)t} + e^{rt}\left(a_0 + c_0\frac{\gamma}{r(1-\gamma)-\rho}\right)$$

which we can plug into the TVC to solve for c_0 as a function of a_0. It is easier to first do so in some special cases.

Time preference equal to interest rate ($\rho = r$) and log utility ($\gamma = 1$). First, let $\rho = r$ (the rate of time preferences is equal to the interest rate on savings) and $\gamma = 1$ (log utility). Then we get

$$
\begin{aligned}
c(t) &= c_0 \\
a(t) &= \frac{c_0}{r} + e^{rt}\left(a_0 - \frac{c_0}{r}\right).
\end{aligned}
$$

To find the optimal value of c_0 let us plug our solution in the TVC

$$
\lim_{t\to\infty} e^{-rt}u'(c(t))a(t) = \lim_{t\to\infty} e^{-rt}c_0^{-1}\left[\frac{c_0}{r} + e^{rt}\left(a_0 - \frac{c_0}{r}\right)\right] = \left(a_0 - \frac{c_0}{r}\right) = 0
$$

which holds if an only if $c_0 = a_0 r$.

Consider the alternatives: if $c_0 > a_0 r$, then the agent would be consuming "too much" and would have a terminal value of savings that is negative, thus violating the no Ponzi game condition. If instead $c_0 < a_0 r$, then the agent would be consuming "too little": the no Ponzi game condition would not be violated, but the transversality condition above would not hold. This makes the *necessity* part of the TVC clear: there are many paths $\tilde{c}(t)$ with $\tilde{c}_0 < a_0 r$ that satisfy the constraints and the Euler equation, but are not optimal because there exists a plan $c(t)$ with $c_0 = a_0 r$ that yields a higher utility.

Log utility $\gamma = 1$. Then

$$
\begin{aligned}
c(t) &= e^{\left(\frac{r-\rho}{\gamma}\right)t}c_0 \\
a(t) &= \frac{c_0}{\rho}e^{(r-\rho)t} + e^{rt}\left(a_0 - \frac{c_0}{\rho}\right).
\end{aligned}
$$

Again, plugging this into the TVC we get

$$
\begin{aligned}
&\lim_{t\to\infty} e^{-\rho t}c(t)^{-1}a(t) = \\
&= \lim_{t\to\infty} e^{-\rho t}e^{(\rho-r)t}c_0^{-1}\left[\frac{c_0}{\rho}e^{(r-\rho)t} + e^{rt}\left(a_0 - \frac{c_0}{\rho}\right)\right] = \\
&= \lim_{t\to\infty} e^{-\rho t}\frac{1}{\rho} + a_0 - \frac{c_0}{\rho} = a_0 - \frac{c_0}{\rho} = 0
\end{aligned}
$$

which implies $c_0 = a_0\rho$.

General case. When no simplifying assumptions on parameters are made, we can still plug our general solution into the TVC. After a bit of algebra we obtain

$$c_0^{1-\gamma}\frac{\gamma}{\rho-r(1-\gamma)}\lim_{t\to\infty}e^{-\left(\frac{\rho-(1-\gamma)r}{\gamma}\right)t}+c_0^{\gamma}\left(a_0-c_0\frac{\gamma}{\rho-(1-\gamma)r}\right)=0.$$

Getting rid of the second term requires $c_0=a_0\frac{\rho-(1-\gamma)r}{\gamma}$. For this to make sense, we will then need that parameters are such that $\frac{\rho-(1-\gamma)r}{\gamma}>0$, so that $c(t)>0$. This condition also implies that the first term converges to zero as $t\to\infty$, which is also necessary for the TVC to be satisfied.

9 Continuous Time Dynamic Programming

9.1 Finite horizon

Let's go back to a finite horizon problem

$$\begin{aligned}V(0,x_0)=\max_{x(t),a(t)}&\int_0^T r(t,x(t),a(t))\mathrm{d}t\\ &\dot{x}(t)=f(t,x(t),a(t))\\ &x(0)=x_0\text{ given}\\ &g(x(T))\geq 0.\end{aligned}$$

Consider the problem at a subset of the whole time interval $[\tau,T]\subset[0,T]$

$$\begin{aligned}V(\tau,x(\tau))=\max_{x(t),a(t)}&\int_\tau^T r(t,x(t),a(t))\mathrm{d}t\\ &\dot{x}(t)=f(t,x(t),a(t))\\ &x(\tau)\text{ given}\\ &g(x(T))\geq 0.\end{aligned}$$

Now, consider an even smaller subset of time which ends before T

$$
\begin{aligned}
V(\tau, x(\tau)) = \max_{x(t),a(t)} & \int_{\tau}^{\tau+h} r(t, x(t), a(t))\mathrm{d}t + V(\tau+h, x(\tau+h)) \\
& \dot{x}(t) = f(t, x(t), a(t)) \\
& x(\tau) \text{ given.}
\end{aligned}
$$

Note that the terminal condition disappears because it's only there at the end of time, i.e. when $\tau + h = T$.

We could in principle keep going and divide each V into smaller and smaller time intervals. This is an illustration of Bellman's principle of optimality: the continuation of an optimal plan is itself optimal, so we can break up a problem in smaller pieces and proceed from the end backwards, and we will always be looking for an optimal plan in the time interval at hand since the future already unfolds according to an optimal plan.

Take the last expression, bring the LHS to the right and divide through by h

$$
0 = \max_{x(t),a(t)} \left\{ \int_{\tau}^{\tau+h} \frac{r(t, x(t), a(t))}{h}\mathrm{d}t + \frac{V(\tau+h, x(\tau+h)) - V(\tau, x(\tau))}{x(\tau+h) - x(\tau)} \frac{x(\tau+h) - x(\tau)}{h} \right\}.
$$

Take the limit for $h \to 0$ and we get

$$
0 = \max_{a(\tau)} \left\{ r(\tau, x(\tau), a(\tau)) + V_x(\tau, x(\tau)) f(\tau, x(\tau), a(\tau)) + V_t(\tau, x(\tau)) \right\}. \tag{20}
$$

This is the Hamilton-Jacobi-Bellman (HJB) equation, the continuous time equivalent of the Bellman equation. It is a partial differential equation (PDE), with final condition $g(x(T)) \geq 0$. If the function V, which we still do not know, is differentiable with respect to x and t, then it must satisfy the HJB and the terminal condition. Note that, since we shrank the time interval to an infinitesimal step, we are not maximising with respect to the state variable anymore, because we are implicitly assuming that it cannot be changed in an instant, while the control variable can.

9.2 Infinite horizon

Let's go back to an infinite horizon problem starting from some period τ

$$
\begin{aligned}
V(\tau, x(\tau)) = \max_{x(t),a(t)} \int_{\tau}^{\infty} & e^{-\rho t} r(x(t), a(t)) \mathrm{d}t \\
\dot{x}(t) &= f(x(t), a(t)) \\
x(\tau) &\text{ given} \\
\lim_{t\to\infty} b(t)x(t) &\geq 0.
\end{aligned}
$$

This is the *present* value function, because we are discounting everything to $t = 0$. We can consider a rescaled version, the *current* value function

$$
V(x(\tau)) := e^{\rho\tau} V(\tau, x(\tau)) = \max_{x(t),a(t)} \int_{\tau}^{\infty} e^{-\rho(t-\tau)} r(x(t), a(t)) \mathrm{d}t
$$

which is not anymore a function of time, as the distance between $t = \tau$ and $t = 0$ becomes irrelevant. So we have defined a current, time-invariant value function which depends only on the initial value of the state variable, and which is such that

$$
\begin{aligned}
&V(x) = V(x, 0) \\
&e^{-\rho t} V(x) = V(x, t) \\
&V_t(t, x) = -\rho e^{-\rho t} V(x).
\end{aligned}
$$

We can derive the HJB using the same reasoning as before. With the present value function

$$
V(x(\tau), \tau) = \max_{x(t),a(t)} \int_{\tau}^{\tau+h} e^{-\rho t} r(x(t), a(t)) \mathrm{d}t + V(x(\tau + h), \tau + h)
$$

with the current value function

$$
e^{-\rho\tau} V(x(\tau)) = e^{-\rho\tau} \max_{x(t),a(t)} \int_{\tau}^{\tau+h} e^{-\rho(t-\tau)} r(x(t), a(t)) \mathrm{d}t + e^{-\rho(\tau+h)} V(x(\tau + h))
$$

simplifying and rearranging

$$
0 = \max_{x(t),a(t)} \int_{\tau}^{\tau+h} e^{-\rho(t-\tau)} r(x(t), a(t)) \mathrm{d}t + e^{-\rho h} V(x(\tau + h)) - V(x(\tau))
$$

dividing by h and taking the limit for $h \to 0$

$$0 = \max_{x(t),a(t)} \left\{ \int_{\tau}^{\tau+h} e^{-\rho(t-\tau)} \frac{r(x(t),a(t))}{h} \mathrm{d}t + \frac{V(x(\tau+h)) - V(x(\tau))}{x(\tau+h) - x(\tau)} \frac{x(\tau+h) - x(\tau)}{h} \right\}.$$

We get the HJB (omitting time dependence now)[21]

$$\rho V(x) = \max_{a} \left\{ r(x,a) + V'(x) f(x,a) \right\}. \tag{21}$$

This is a ODE! The LHS gives us the "flow" value function; term $r(x,a)$ gives us the "current" or instantaneous payoff, and term $V'(x)f(x,a)$ gives us the "capital gain", i.e. the marginal value of a change in the state variable.

To see the connection with the Hamiltonian, in the same way in which we have linked the sequential and recursive approaches in discrete time, take the FOC with respect to a in (21)

$$r_a(x,a) + V'(x) f_a(x,a) = 0$$

Define $\mu(t) = V'(x(t))$. First, note that once you replace $V'(x)$ with μ, the last equation is the maximum principle. Then, differentiate with respect to t

$$\dot{\mu}(t) = V''(x(t))\dot{x}(t).$$

Now differentiate the HJB, evaluated at the optimum so we get rid of the max operator, with respect to x (not t this time)

$$\rho V'(x) = r_x(x,a) + V''(x) f(x,a) + V'(x) f_x(x,a).$$

Plug the second- and third-last equations into the last and rearranging

$$\dot{\mu} - \rho\mu = -r_x - \mu f_x$$

[21]Note that the total differentiation of $e^{-\rho t}V(x(t))$ gives us two terms, by the chain rule

$$\lim_{h\to 0} \frac{e^{-\rho(t+h)}V(x(t+h)) - e^{-\rho t}V(x(t))}{h} = \frac{\mathrm{d}e^{-\rho t}V(x(t))}{\mathrm{d}t} = -\rho e^{-\rho t}V(x(t)) + e^{-\rho t}V'(x(t))\dot{x}(t).$$

which is exactly the adjoint equation. So we have established that there is a connection between the Hamiltonian and the HJB, and that the derivative of the value function is equal to the co-state variable, or the shadow value of the state variable.

9.2.1 Consumption-savings model

Let's go back to our example and see that we can derive the same Euler equation from the HJB. The HJB is

$$\rho V'(a) = \max_c \{u(c) + V'(a)[ar - c]\}.$$

Take the FOC

$$u'(c) = V'(a).$$

Differentiate the FOC with respect to time

$$u''(c)\dot{c} = V''(a)\dot{a} = V''(a)[ar - c].$$

Differentiate the HJB with respect to a

$$\rho V'(a) = V''(a)[ar - c] + V'(a)r.$$

Get rid of the value function derivatives

$$\rho u'(c) = u''(c)\dot{c} + u'(c)r$$

which finally yields

$$\frac{\dot{c}}{c} = \frac{u'(c)}{-c\, u''(c)}(r - \rho).$$

9.2.2 Neoclassical growth model

Variational approach. The time-0 problem is

$$
\begin{aligned}
V(k_0) = \max_{k(t),c(t)} \int_0^\infty & e^{-\rho t} u(c(t))\mathrm{d}t \\
& \dot{k}(t) = f(k(t)) - c(t) - \delta k(t) \\
& k(0) = k_0 \text{ given} \\
& k(t) \geq 0 \\
& \lim_{t\to\infty} k(t)b(t) \geq 0.
\end{aligned}
$$

The current value Hamiltonian

$$H(k, c, \lambda) = u(c) + \mu[f(k) - \delta k - c].$$

Maximum principle

$$u'(c) = \mu.$$

Differentiate with respect to time

$$u''(c)\dot{c} = \dot{\mu}.$$

Adjoint equation

$$\dot{\mu} - \rho\mu = -H_k = -\mu[f'(k) - \delta].$$

Put everything together

$$u''(c)\dot{c} - \rho u'(c) = -u'(c)[f'(k) - \delta]$$

which yields

$$\frac{\dot{c}}{c} = \frac{u'(c)}{-c\, u''(c)}(f'(k) - \delta - \rho).$$

Dynamic programming approach. The HJB is

$$\rho V(k) = \max_c \{u(c) + V'(k)[f(k) - c - \delta k]\}.$$

Take FOC

$$u'(c) = V'(k)$$

Differentiate the FOC with respect to time

$$u''(c)\dot{c} = V''(k)\dot{k} = V''(k)[f(k) - \delta k - c].$$

Differentiate the HJB with respect to k

$$\rho V'(k) = V''(k)[f(k) - \delta k - c] + V'(k)[f'(k) - \delta].$$

Get rid of the value function derivatives

$$\rho u'(c) = u''(c)\dot{c} + u'(c)[f'(k) - \delta]$$

which finally yields

$$\frac{\dot{c}}{c} = \frac{u'(c)}{-c\, u''(c)}(f'(k) - \delta - \rho).$$

Notice that this is the same as the Euler equation for the consumption-savings model once you replace the marginal rate of return on savings (r) with that on capital ($f'(k) - \delta$).

Characterising the solution. With CRRA, the Euler equation becomes

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}(f'(k) - \delta - \rho).$$

Together with the dynamics for the state

$$\dot{k} = f(k) - \delta k - c$$

we have a system of two ODEs in two unknowns (k, c). We can do a phase diagram, since it's easy to characterise when $\dot{c}$ and $\dot{k}$ are positive, negative or zero.

From the HJB, however, we can also get a *single* DE (rather than a system) which we can then solve directly, either by hand or with numerical methods. Consider the FOC for consumption

$$u'(c) = V'(k)$$

which implies that

$$c = (u')^{-1}(V'(k))$$

with log utility, this becomes

$$c = 1/V'(k).$$

Plug this inside the HJB

$$\rho V(k) = -\log V'(k) + V'(k)[f(k) - \delta k] - 1. \tag{22}$$

This is a nonlinear (see the logs) differential equation for $V(k)$. With numerical methods, this can be solved very quickly, in a way that is similar in spirit (but more efficient) to the value function iteration method we used in discrete time.

Remark. We will not go through it here, but there exist theorems that prove *(i)* existence and uniqueness of a solution to the HJB equation, even where there are non differentiabilities, under some conditions; and *(ii)* under what conditions the solutions of the dynamic programming approach (HJB) and the variational approach (Hamiltonian) coincide.

9.3 Numerical solutions

Bellman equations. As said before, equations like (21) are nonlinear differential equations that can be efficiently solved with a computer. Convergence is typically achieved in well-behaved problems, although there is no theorem equivalent to the CMT here that guarantees it. To solve for v in (21), the procedure is as follows:

- construct a grid for capital $\boldsymbol{k} = [k_1, ..., k_n]$, where Δk defines the distance between points
- construct an initial guess for v, i.e. a vector $\boldsymbol{v} = [v_1, ..., v_n]$
- compute $v'(k)$ (again, a vector $\boldsymbol{v'} = [v'_1, ..., v'_n]$) by taking finite differences of v. There are different options here: one can take backward differences

  $$v'_{i,B} = \frac{v_i - v_{i-1}}{\Delta k},$$

  forward differences

  $$v'_{i,F} = \frac{v_{i+1} - v_i}{\Delta k},$$

  or central differences

  $$v'_{i,C} = \frac{v_{i+1} - v_{i-1}}{2\ \Delta k}.$$

  We'll use different methods depending on that that gives the lowest absolute drift in the state variable ("upwind scheme"). If v is concave then we should get that $v'_{i,F} < v'_{i,B}$.

- use the FOC to get consumption c_i from v'_i

$$c_{i,j} = (u')^{-1}(v'_{i,j})$$

for $j \in \{F, C, B\}$. Since utility is also concave, we'll have that $c_{i,F} > c_{i,B}$.

- compute the drift in capital using the state dynamics equation

$$\dot{k}_{i,j} = f(k_i) - \delta k_i - c_{i,j}$$

again for $j \in \{F, C, B\}$. We'll have that $\dot{k}_{i,F} < \dot{k}_{i,B}$.

- to decide which method to use to compute v'_i and c_i
  - if $\dot{k}_{i,F} > 0$, use forward differencing
  - if $\dot{k}_{i,B} < 0$, use backward differencing
  - if $\dot{k}_{i,F} < 0 < \dot{k}_{i,B}$, assume we are in the steady state for capital, set $c_i = f(k_i) - \delta k_i$ and $v'_i = u'(f(k_i) - \delta k_i)$.
- compute the new value for v_i using the HJB and the recently obtained values for c_i and v'_i

$$\rho v_i^{n+1} = u(c_i^n) + (v_i^n)' \dot{k}_i^n.$$

This method is the most intuitive, but such a "full" update of the value function creates convergence problems. What works is to update the value function slowly, i.e. to set

$$v_i^{n+1} = v_i^n + \lambda[u(c_i^n) + (v_i^n)' \dot{k}_i^n - \rho v_i^n]$$

choosing a small value for the step size λ.

Plain ODEs. Simpler versions of the procedure above can be used to solve for simpler, first order ODEs of the form

$$\dot{x}(t) = f(x(t), t)$$
$$x(0) = x_0.$$

The simplest method is the Forward Euler scheme. Our ODE is to be solved for $t \in (0, T]$, and so we will look for the solution $x(t)$ at discrete time points $t_i = i\ \Delta$ for $i = 1, \ldots, n$, where clearly $t_{i+1} - t_i = \Delta$. For simplicity, let $x_i := x(t_i)$. We will approximate $\dot{x}(t)$ with a one-sided forward difference

$$\dot{x}(t_i) \approx \frac{x(t_{i+1}) - x_{t_i}}{\Delta} = \frac{x_{i+1} - x_i}{\Delta}.$$

The idea is that if you know x_i then you can compute x_{i+1} from

$$\frac{x_{i+1} - x_i}{\Delta} = f(x_i, t_i) \quad \Rightarrow \quad x_{i+1} = f(x_i, t_i)\Delta + x_i.$$

We have x_0, so we start with

$$x_1 = f(x_0, t_0)\Delta + x_0$$

and then we keep going iteratively.

10 Stochastic Dynamic Programming in Continuous Time

10.1 Review of Poisson processes

We now consider the simplest form of randomness in CT, which is Poisson processes. Consider a random variable z_t, with a discrete support given by $\{z_1, \ldots, z_n\}$, that has a transition intensity λ_{ij}. That means that a jump of z_t from value z_i to z_j is an event with a Poisson intensity λ_{ij}.

Let's refresh the concepts of intensity and arrival probability. A *point process* is an increasing sequences of random (time) points $0 < t_1 < t_2 < \ldots < t_n < \ldots$, each of which is a random variable t_i indicating the time at which the i-th occurrence of an event has happened. The random *counting function* N_t indicates the number of events happened up to t. A point process n_t, or its counting function N_t, is a Poisson process if $\{N_t, t \geq 0\}$ is a process with stationary independent increments. Independence means that the increments $N_{t_i} - N_{t_{i-1}}$ for all i are independent; stationarity means that $N_t - N_s$ depends only on $t - s$. If N_t is the counting function of a Poisson process, then

$$\mathrm{P}(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!} \quad \text{for } k \in \mathbb{N}$$

that is, the number of events taking places in a time interval follows a Poisson distribution with rate parameter $\lambda(t - s)$. The parameter λ is the intensity of the Poisson process, and it gives the average number of events in a *unit* of time ($t - s = 1$):

$$\mathbb{E}[N_{t+1} - N_t] = \lambda.$$

Now, let an event be a "jump" from z_i to z_j as mentioned earlier. The probability of observing such a jump during a period of time of length h is given by

$$\mathrm{P}(N_{t+h} - N_t = 1) = e^{-\lambda h}\lambda h \approx \lambda h$$

where the approximation (first order Taylor expansion) holds when h is small. The probability of observing zero jumps in such a period is $e^{-\lambda h}$, which is approximately equal to $1 - \lambda h$ up to a first order. The probability of observing two jumps in such a period is $e^{-\lambda h}\frac{(\lambda h)^2}{2}$, which is approximately zero up to a first order.

10.2 Stochastic HJB

Consider now, to make things more transparent, the usual example of the consumption savings model. Here the agent receives stochastic labour earnings w, where $w \in \{w_l, w_h\}$ and the jumps from one value to the other are Poisson with intensities $\lambda_{lh} = \lambda_{hl} = \lambda$. Let's derive the stochastic HJB in this case with the usual procedure.

The problem between t and $t + h$, conditional on $w(t) = w_i$ is

$$e^{-\rho t}V(a(t), w_i) = e^{-\rho t} \max_{a(s),c(s)} \int_t^{t+h} e^{-\rho(s-t)}u(c(s))\mathrm{d}s + e^{-\rho(t+h)}\left[(1 - \lambda h)V(a(t+h), w_i) + \lambda h V(a(t+h), w_j)\right]$$

where w_j is the other realisation of w. Rearrange the expression:

$$\begin{aligned} 0 =& e^{-\rho t} \max_{a(s),c(s)} \int_t^{t+h} e^{-\rho(s-t)}u(c(s))\mathrm{d}s \; + \; e^{-\rho(t+h)}V(a(t+h), w_i) - e^{-\rho t}V(a(t), w_i) \\ &+ e^{-\rho(t+h)}\lambda h[V(a(t+h), w_j) - V(a(t+h), w_i)] \end{aligned}$$

Dividing through by h everywhere, taking the limit for $h \to 0$, and omitting time subscripts, we get

$$\rho V(a, w_i) = \max_c u(c) + V_a(a, w_i)[ar - c + w_i] + \lambda[V(a, w_j) - V(a, w_i)].$$

Clearly, there exists another HJB for w_j, where the i and j wage subscripts are inverted. This HJB is identical to the deterministic one, except for the fact that now we have partial derivatives with respect to the deterministic state, and that there is the additional last term that represents wage risk.

10.3 Stochastic Euler equation

How do we derive the Euler equation in this setting? First, let's take the FOC for consumption

$$u'(c) = V_a(a, w_i).$$

Second let's differentiate the HJB with respect to assets

$$\rho V_a(a, w_i) = V_{aa}(a, w_i)[ar - c + w_i] + V_a(a, w_i)r + \lambda[V_a(a, w_j) - V_a(a, w_i)]. \tag{23}$$

Now we have to change our approach because we can't differentiate the FOC with respect to time and keep going, because all variables of interest (consumption and savings) now follow stochastic paths. The road to follow is to try to get an analog of the discrete time Euler equation with uncertainty, which relates the expected growth rate of marginal utility to the interest rate and the discount factor

$$\frac{\mathbb{E}[u'(c_{t+1})]}{u'(c_t)} = \frac{1}{\beta R}.$$

To do so, let's define the *infinitesimal generator* $\mathcal{A}$ of some arbitrary function of the states $f(a, w)$ the following *expected* time derivative:

$$\mathcal{A}f(a(t), w(t)) = \lim_{h \to 0} \frac{\mathbb{E}_t[f(a(t+h), w(t+h))] - f(a(t), w(t))}{h}$$

In words, we have to be careful in using time derivatives now, and we need expected time derivatives rather than deterministic ones such as the $\dot{c}$ we have used earlier. In our current example, we know that we can approximate that expectation as

$$\mathbb{E}[f(a(t+h), w(t+h))|w(t) = w_i] \approx (1 - \lambda h)f(a(t+h), w_i) + \lambda h f(a(t+h), w_j).$$

Consider a first-order Taylor expansion

$$f(a(t+h), w) \approx f(a(t), w) + f_a(a(t), w)\dot{a}h$$

and let's use it in our expectation

$$\begin{aligned}\mathbb{E}[f(a(t+h),w(t+h))|w(t)=w_i] \approx &(1-\lambda h)[f(a(t),w_i)+f_a(a(t),w_i)\dot{a}h]\\ &+\lambda h[f(a(t),w_j)+f_a(a(t),w_j)\dot{a}h]\\ \approx &f(a(t),w_i)+\lambda h[f(a(t),w_j)-f(a(t),w_i)]+f_a(a(t),w_i)\ \dot{a}\ h\end{aligned}$$

where the last approximation comes from the fact that the terms with h^2 are of second order and so we can ignore them because they will be very small.

So finally we have our operator definition reduced to

$$\mathcal{A}f(a,w_i)=\lambda[f(a,w_j)-f(a,w_i)]+\dot{a}f_a(a,w_i).$$

Now let us apply it to $f(a,w)=V_a(a,w)$:

$$\mathcal{A}V_a(a,w_i)=\lambda[V_a(a,w_j)-V_a(a,w_i)]+\dot{a}V_{aa}(a,w_i)$$

and let us plug it in (23)

$$(\rho-r)V_a(a,w_i)=\mathcal{A}V_a(a,w_i)$$

which gives us

$$\frac{\mathcal{A}V_a(a,w_i)}{V_a(a,w_i)}=\rho-r$$

or

$$\frac{\mathcal{A}u'(c)}{u'(c)}=\rho-r.$$

That is, the expected percentage change in marginal utility is a function of the interest rate and the rate of time preference, exactly as we saw in the discrete time case.

10.4 Kolmogorov forward equation

With income shocks, we have in hand a heterogeneous agent model. Some agents will earn little, some a lot, so we'll have a non-trivial asset market clearing condition, and a non-trivial distribution of wealth across agents. With the Kolmogorov forward (KF) equation, we can characterise such distribution and derive its law of motion.

To derive the KF equation, we once again start from a discrete time approximation. Let

$$s_i(t) = a_t r - c_t - +w_i$$

denote the savings function at time t of an individual with wage w_i and wealth a (which we omit to keep notation light). It follows that

$$a_{t+h} = a_t + hs_i(t). \tag{24}$$

Our goal is to derive the density of the wealth distribution $g_i(a,t)$, i.e. the mass of people with wealth a and wage w_i at moment t. Let us however first start with the CDF of such distribution

$$G_i(x,t) = \mathrm{P}(a_t < x, w_t = w_i).$$

Ignore for a second the possibility that wage earnings jump. Using the discrete dynamics in (24) we can write

$$G_i(x,t+h) = \mathrm{P}(a_{t+h} < x, w_{t+h} = w_i) = \mathrm{P}(a_t < x - hs_i(t), w_{t+h} = w_i) = G_i(x - hs_i(t), t).$$

Reintroducing the stochastic earning process

$$G_i(x,t+h) = G_i(x - hs_i(t), t)(1 - \lambda h) + G_j(x - hs_j(t), t)\lambda h$$

subtract $G_i(x,t)$ from both sides and divide through by h

$$\frac{G_i(x,t+h) - G_i(x,t)}{h} = \frac{G_i(x - hs_i(t), t) - G_i(x,t)}{h} + \lambda[G_j(x - hs_j(t), t) - G_i(x - hs_i(t), t)]$$

take the limit for $h \to 0$

$$\frac{\partial}{\partial t} G_i(x,t) = -s_i(t)\frac{\partial}{\partial a} G_i(x,t) + \lambda[G_j(x,t) - G_i(x,t)].$$

This equation already has a clear intuition: the time change in the share of people with wage w_i and below a certain wealth level x (LHS) is given by inflow/outflows due to continuous changes in wealth (first term on the RHS) and due to discrete jumps in wage earnings (second term on the RHS). If we differentiate with respect to a, we can get the KF equation with the densities

$$\frac{\partial}{\partial t} g_i(x,t) = -\frac{\partial}{\partial a}[s_i(t) g_i(x,t)] + \lambda[g_j(x,t) - g_i(x,t)].$$

Finally, the *stationary* distribution $g_i(a)$ is that which, by construction, does not change over time. We can write a KF equation for that too

$$0 = -\frac{\partial}{\partial a}[s_i(a) g_i(a)] + \lambda[g_j(a) - g_i(a)].$$

This is a partial differential equation! If you can solve it, then you can derive the stationary wealth distribution without having to simulate a model over time.

11 Real Business Cycle Theory

The behaviour of macroeconomic variables (GDP, consumption, hours worked, ...) is typically decomposed in two components, a trend and a cycle. The study of the trend is growth theory, which focuses on what determines the long-run behaviour of the economy. We do not cover it here. The study of the cycle is RBC theory, which focuses on what determines short-run fluctuations around the trend, and how policy can affect those.

Before the so-called RBC revolution, the main doctrine was Keynesiansm, which postulated that investors were driven by non-rational “animal spirits”, and studied the aggregate effects of these confidence swings on the economy, and especially the government role in correcting these spirits, by stimulating or cooling off demand. On top of that, and this is the famous critique by Lucas (1976), Keynesian models studied policy by taking the patterns of agents’ behaviour as given (e.g. their savings or consumption function).

The main point of the RBC revolution (Lucas (1977), Kydland and Prescott (1982) and Long Jr and Plosser (1983)) was that, on the contrary, the behaviour of optimising agents does change as a function of economic policy. These papers proposed macroeconomic models that had three key components: *(i)* they were micro-founded, modelling behaviour at the individual level; *(ii)* they assumed a single shock, techonology, that would be the sole driver of short-run fluctuations; and *(iii)* they were all “real”, in the sense that they did not consider the role and behaviour of monetary variables. The important other feature of RBC theory is efficiency: perfect competition and frictionless markets are assumed, so the welfare theorems do apply.

Of course RBC theory has competitors in claiming the ability to explain short-run fluctuations. The most relevant alternatives are New-Keynesian models, which are monetary models where price rigidity acts as the main mechanism of shock propagation; and sunspot theories, where the existence of micro-founded multiple equilibria implies that cycles are oscillations between them.

Much of modern macroeconomics uses the basic RBC model as a frictionless benchmark, on which one can add all sort of frictions (financial, information, search, rationality) to improve the model's ability to explain recent (or less so) facts.

11.1 Some stylised facts

Kydland and Prescott (1982) paved the way by filtering the data, identifying cycles, and then documenting a number of stylised facts, akin to the Kaldor facts you have studies with respect to growth.

To de-trend variables, they used the Hodrick-Prescott (HP) filter. The filter works in the following way

$$\min_{\{\bar{y}_t\}_{t=1}^T} \sum_{t=1}^{T} (y_t - \bar{y}_t)^2$$

subject to

$$\sum_{t=2}^{T-1} \left[(\bar{y}_{t+1} - \bar{y}_t) - (\bar{y}_t - \bar{y}_{t-1})\right]^2 \leq K.$$

Setting $K = 0$ gives the following Lagrangian

$$\mathcal{L} = \sum_{t=1}^{T} \left\{ (y_t - \bar{y}_t)^2 - \mu \left[(\bar{y}_{t+1} - \bar{y}_t) - (\bar{y}_t - \bar{y}_{t-1})\right]^2 + (y_T - \bar{y}_T)^2 + (y_1 - \bar{y}_1)^2 \right\}.$$

Choosing μ here amounts to choosing the importance of the trend smoothness (the constraint) relative to trend fit (the objective). The typical values used in the literature are 1600 for quarterly data and 400 for annual data. Once one has derived a sequence for $\bar{y}_t$, the cycle can be obtained from $y_t - \bar{y}_t$.

Let $\tilde{\sigma}_x = \frac{\sigma_x}{\mu_x}$ denote the coefficient of variation, so we can express standard deviations in percentage term of means. Kydland and Prescott's facts are the following.

1. $\tilde{\sigma}_c < \tilde{\sigma}_y$ (consumption smoothing)

2. $\tilde{\sigma}_d > \tilde{\sigma}_y$ (durable consumption is volatile)
3. $\tilde{\sigma}_i \approx 3\tilde{\sigma}_y$ (investment is very volatile)
4. $\tilde{\sigma}_{tb} > \tilde{\sigma}_y$ (trade balance, i.e. exports minus imports, is volatile)
5. $\tilde{\sigma}_n \approx \tilde{\sigma}_y$ (total hours worked are similar to output)
6. $\tilde{\sigma}_e \approx \tilde{\sigma}_y$ (employment is similar to output)
7. $\tilde{\sigma}_k << \tilde{\sigma}_y$ (the capital stock is slow-moving)
8. $\tilde{\sigma}_w < \tilde{\sigma}_{y/n}$ (output per hour is more volatile than real hourly wages).

11.2 The basic RBC model

We now introduce the framework we will use. We will follow a "cookbook": first, we specify the model we use, including its functional forms (utility, production, etc...); second, we calibrate the model, i.e. we pick the parameters from external estimates; third, we solve the model, which is typically done using numerical methods; fourth, we simulate the model and analyse its outcomes.

We will use a stochastic version of the neoclassical growth model, with the quantitative goal of "calibrating it" to try and match the facts we mentioned above.

$$\max_{\{c_t,n_t,l_t,i_t,k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

subject to

$$\begin{aligned}
&c_t + i_t = z_t F(k_t, n_t) \\
&F(k_t, n_t) = k_t^{\alpha} n_t^{1-\alpha} \\
&i_t = k_{t+1} - (1-\delta)k_t \\
&l_t + n_t = 1 \\
&\log(z_{t+1}) = \rho \log(z_t) + \epsilon_{t+1}.
\end{aligned}$$

As said, the goal here is to specifiy functional forms, solve the model, pick some values for the parameters, and examine the model's quantitative performance. To pick parameters, we will use here external sources, such as micro data or long-run trend data. Note that this is different from moment matching, where parameters are picked in order to minimise the distance between the moments generated by a simulation of the model and those measured in the data.

The first thing we need to measure is the most important: the only shock in this economy, total factor productivity (TFP). To measure it in the data, take the production function $y_t = z_t F(k_t, n_t)$, perform total differentiation

$$\mathrm{d}y_t = F(k_t, n_t)\mathrm{d}z_t + z_t F(k_t, n_t)\mathrm{d}k_t + z_t F(k_t, n_t)\mathrm{d}n_t.$$

Then divide each term by y_t and simplify some things

$$\frac{\mathrm{d}y_t}{y_t} = \frac{\mathrm{d}z_t}{z_t} + \frac{z_t F_k(k_t, n_t)}{y_t}\mathrm{d}k_t + \frac{z_t F_n(k_t, n_t)}{y_t}\mathrm{d}n_t.$$

With perfect competition, the firm's FOCs imply

$$\begin{aligned} w_t &= z_t F_n(k_t, n_t) \\ r_t^k &= z_t F_k(k_t, n_t) \end{aligned}$$

so we get

$$\frac{\mathrm{d}y_t}{y_t} = \frac{\mathrm{d}z_t}{z_t} + \frac{r_t^k k_t}{y_t}\frac{\mathrm{d}k_t}{k_t} + \frac{w_t n_t}{y_t}\frac{\mathrm{d}n_t}{n_t}.$$

We assume that $F(k, n) = k^\alpha n^{1-\alpha}$, i.e. it is constant returns to scale, so $kF_k + nF_n = F$ and the factors share in output is constant and given by $\frac{r_t^k k_t}{y_t} = \alpha$ and $\frac{w_t n_t}{y_t} = 1 - \alpha$. Hence

$$\frac{\mathrm{d}z_t}{z_t} = \frac{\mathrm{d}y_t}{y_t} - \alpha\frac{\mathrm{d}k_t}{k_t} - (1-\alpha)\frac{\mathrm{d}n_t}{n_t}$$

so we can get the time series for TFP percentage changes using the time series for output, capital and labour, as well as their factor shares. Once we have the time series for z_t, we can estimate the AR(1) process it follows.

Now on to functional forms. The functional form we pick for utility is

$$u(c, l) = \frac{\left(\frac{c^{1-\theta}}{1-\theta} l^\theta\right)^{1-\sigma} - 1}{1-\sigma}.$$

Here θ is the relative importance of leisure vs consumption, and σ is the degree of relative risk aversion (and its inverse is the intertemporal elasticity of substitution). The functional form for production is a standard CRS function, as specified above.

Even when studying cycles, one may want to include growth factors. One example is to have population growth (e.g. at some constant rate η) as well as labour-augmenting technical progress, which amounts to exponential growth in TFP of the form

$$z_t = z_0(1+\gamma)^{t(1-\alpha)}e^{\omega_t}$$
$$\omega_t = \rho\omega_{t-1} + \epsilon_t.$$

With these assumptions, to study cycles we'd have to de-trend the model, which is deterministic in this case. Here for simplicity we will abstract from this and consider a fully stationary economy where there is no trend growth but only fluctuations.

The equilibrium equations are the labour supply equation

$$\frac{\theta}{1-\theta}\frac{c_t}{l_t} = w_t$$

the Euler equation

$$u_c(t) = \beta\mathbb{E}_t[z_{t+1}r^k_{t+1} + 1 - \delta]u_c(t+1)$$

and the firm's FOCs we stated earlier.

Calibration. Now, let's see how we pick values for parameters.

The parameters of the TFP process are estimated on US data (for 1950-1990 in the original paper) in the way we discussed earlier. We have $\rho = 0.972$ and $\sigma^2_\epsilon = 0.0072$.

Taking the law of motion for capital in steady state

$$1 = 1 - \delta + \frac{i_{ss}}{k_{ss}}$$

so using the long run average for $\frac{i_{ss}}{k_{ss}} = 0.076$ we can pin down the value for δ.

We set $\alpha = 1/3$ to match the long run capital share in production, which is fairly stable over time.

To compute utility function parameters, we use the optimality conditions. The risk aversion coefficient is a tricky parameter to estimate, even more so when (like here) it both represents risk aversion and IES[22]. Estimates range from 0.5 to 5 approximately, with the asset pricing

[22] There are other types of utility functions, such as Epstein-Zin, that separate the two objects.

literature typically using even higher values. We'll assume log utility, for simplicity and because it is sort of in the middle of the range. To pick β, we use the Euler equation in steady state which yields

$$\frac{1}{\beta} = \alpha \frac{y_{ss}}{k_{ss}} + 1 - \delta.$$

Using a long run output/capital ratio of 0.3012 we get $\beta = 0.976$.

To pick θ, we use the labour supply equation in steady state after substituting out wages

$$\frac{\theta}{1-\theta} \frac{1-l_{ss}}{l_{ss}} = (1-\alpha)\frac{y_{ss}}{c_{ss}}.$$

We get the long run value for the consumption/output ratio from the data as $\frac{y_{ss}}{c_{ss}} = 1.173$, use the long run value for labour and leisure (respectively 1/3 and 2/3 of a day) and solve directly for $\theta = 0.61$. The value of θ is a controversial subject, because it is a parameter that is object to much study in the labour literature. For instance, the value we derived implies a Frisch wage elasticity of labour supply, that is the wage elasticity of n_t when we keep constant the marginal utility of wealth (hence the marginal utility of consumption, in this model). We'd get a value of 2 (can you show it?), which is very much at odds with micro studies that find a value close to zero for this elasticity.

We have now set all of the parameters. The next step is to solve the model, to get policy function for consumption, investment, labour and leisure. Then we can simulate the model: start from some value for k_0, draw a long series of TFP shocks, compute time series for all other variables, compute moments (means, standard deviations, correlations) and check how they compare to their data counterparts.

11.3 Perturbation methods

The solution methods that can be used to solve the model are either global or local. Global methods, such as guess and verify or value function iteration, yield exact (often non-linear) solutions. Local methods are typically called perturbation methods: we approximate the system around some point (usually the steady state), and then we solve the simpler, approximated problem.

We now look at the latter. The way we approximate the system is through log-linearisation around the steady state. We'll use the following notation: x_t denotes a variable, x denotes its steady-state, and $\hat{x}_t$ denotes its log-deviation from steady state, which is approximately equal to its percentage deviation from steady state since $\hat{x}_t := \log \frac{x_t}{x} \approx \frac{x_t - x}{x}$.[23]

To approximate a generic power expression, we'll use the rule $x_t^\alpha = x^\alpha(1 + \alpha \hat{x}_t)$. This comes from

$$x_t^\alpha = x^\alpha e^{\alpha \log x_t/x} = x^\alpha e^{\alpha \hat{x}_t}$$

and the following first-order Taylor expansion

$$e^{\alpha \hat{x}_t} \approx e^{\alpha 0} + \alpha e^{\alpha 0}(\hat{x}_t - 0) = 1 + \alpha \hat{x}_t.$$

Let us approximate the production function, as an example. Start from

$$y_t = z_t k_t^\alpha n_t^{1-\alpha}$$

replace variables with their approximations

$$y(1 + \hat{y}_t) = z(1 + \hat{z}_t) k^\alpha (1 + \alpha \hat{k}_t) n^{1-\alpha}(1 + (1-\alpha)\hat{n}_t)$$

simplify the steady state stuff, and then do all the cross products

$$1 + \hat{y}_t = 1 + \hat{z}_t + \alpha k_t + (1-\alpha)\hat{n}_t$$

where we got rid of all products of log deviations (e.g. $\hat{x}_t \hat{y}_t$) because they are approximately zero up to a first-order. We get the log-linearised version of the production function

$$\hat{y}_t = \hat{z}_t + \alpha \hat{k}_t + (1-\alpha)\hat{n}_t. \tag{25}$$

[23]The approximation here comes from a first-order Taylor expansion:

$$\log x_t \approx \log x + \frac{1}{x}(x_t - x)$$

which implies

$$\log \frac{x_t}{x} \approx \frac{x_t - x}{x}.$$

There is another, alternative approach. Take logs of the production function directly

$$\log y_t = \log z_t + \alpha \log k_t + (1-\alpha)\log n_t.$$

Replace with first-order approximations around the steady state, using $\hat{x}_t \approx \frac{x_t - x}{x}$

$$\log y + \hat{y}_t = \log z + \hat{z}_t + \alpha \log k + \alpha \hat{k}_t + (1-\alpha)\log n + (1-\alpha)\hat{n}_t$$

get rid of the steady state variables and get to the same conclusion as with the previous method.

Let us now approximate all relevant equilibrium condition: the resource constraint

$$\frac{c}{y}\hat{c}_t + \frac{k}{y}\hat{k}_{t+1} = \hat{z}_t + \alpha\hat{k}_t + (1-\alpha)\hat{n}_t + (1-\delta)\frac{k}{y}\hat{k}_t \tag{26}$$

the Euler equation

$$\mathbb{E}_t\hat{c}_{t+1} - \hat{c}_t = \beta\alpha\frac{y}{k}\mathbb{E}_t\left[\hat{z}_{t+1} + (1-\alpha)\hat{n}_{t+1} + (\alpha-1)\hat{k}_{t+1}\right] \tag{27}$$

the labour supply equation

$$\hat{c}_t + \frac{n}{1-n}\hat{n}_t = \hat{z}_t + \alpha\hat{k}_t - \alpha\hat{n}_t \tag{28}$$

and the law of motion for TFP

$$\hat{z}_{t+1} = \rho\hat{z}_t + \epsilon_{t+1}. \tag{29}$$

Now that we have a different, approximated model in front of us, we can look for its solution, i.e., policy functions for consumption, future capital and hours worked. We will see two methods: the method of undetermined coefficients (Uhlig (2001)), and the Blanchard and Kahn (1980) method.

11.3.1 Method of undetermined coefficients

This method consists in guessing that the controls are linear functions of the state variables, with coefficients to be solved for. Guess

$$\hat{c}_t = \gamma_c\hat{k}_t + \mu_c\hat{z}_t$$
$$\hat{n}_t = \gamma_n\hat{k}_t + \mu_n\hat{z}_t$$
$$\hat{k}_{t+1} = \gamma_k\hat{k}_t + \mu_k\hat{z}_t.$$

In other words we're guessing a linear system $\hat{\boldsymbol{a}}_t = H\hat{\boldsymbol{x}}_t$ where $\hat{\boldsymbol{a}}_t$ is a 3×1 vector of controls, H is a 3×2 matrix of coefficients, and $\hat{\boldsymbol{x}}_t$ is a 3×1 vector of states.

Plug the guesses into the resource constraint

$$\frac{c}{y}[\gamma_x \hat{k}_t + \mu_c \hat{z}_t] + \frac{k}{y}[\gamma_k \hat{k}_t + \mu_k \hat{z}_t] = \hat{z}_t + \alpha \hat{k}_t + (1-\alpha)[\gamma_n \hat{k}_t + \mu_n \hat{z}_t] + (1-\delta)\frac{k}{y}\hat{k}_t$$

Collecting terms

$$\hat{k}_t \left[\frac{c}{y}\gamma_c + \frac{k}{y}\gamma_k - \alpha - (1-\delta)\frac{k}{y} - (1-\alpha)\gamma_n\right] = \hat{z}_t \left[-\frac{c}{y}\mu_c - \frac{k}{y}\mu_k + 1 + (1-\alpha)\mu_n\right].$$

Since this equation must hold at all points of the system, i.e. for any combination of values for $\hat{k}_t$ and $\hat{z}_t$, both square brackets must equal zero, which gives us two equations in 6 unknown parameters.

Repeat the same process for the Euler equation

$$\begin{aligned}
&\mathbb{E}\left[\gamma_c(\gamma_k \hat{k}_t + \mu_k \hat{z}_t) + \mu_c(\rho \hat{z}_t + \epsilon_{t+1})\right] - (\gamma_c \hat{k}_t + \mu_c \hat{z}_t) = \\
&= \beta\alpha\frac{k}{y}\mathbb{E}_t\left[(\rho \hat{z}_t + \epsilon_{t+1}) + (1-\alpha)\Big(\gamma_n(\gamma_k \hat{k}_t + \mu_k \hat{z}_t) + \mu_n(\rho \hat{z}_t + \epsilon_{t+1}) + (\alpha-1)(\gamma_k \hat{k}_t + \mu_k \hat{z}_t)\Big)\right].
\end{aligned}$$

Use the fact that $\mathbb{E}_t \epsilon_{t+1} = 0$, get rid of expectations and collect terms

$$\begin{aligned}
&\hat{k}_t \left\{\gamma_c\gamma_k - \gamma_c - \beta\alpha\frac{y}{k}\left[(1-\alpha)\gamma_n\gamma_k + (\alpha-1)\gamma_k\right]\right\} = \\
&= \hat{z}_t \left\{-\mu_c\rho + \mu_c + \beta\alpha\frac{y}{k}\left[\rho + (1-\alpha)\Big(\gamma_n\mu_k + \mu_n\rho\Big) + (\alpha-1)\mu_k\right]\right\}.
\end{aligned}$$

As before, since this equation must hold at all points of the system, both square brackets must equal zero, which gives us two more equations in 6 unknown parameters.

We repeat the same process for the labour supply equation

$$\gamma_c \hat{k}_t + \mu_c \hat{z}_t + \frac{n}{1-n}\Big(\gamma_n \hat{k}_t + \mu_n \hat{z}_t\Big) = \hat{z}_t + \alpha \hat{k}_t - \alpha(\gamma_n \hat{k}_t + \mu_n \hat{z}_t)$$

which becomes

$$\hat{k}_t \left\{\gamma_c + \frac{n}{1-n}\gamma_n - \alpha + \alpha\gamma_n\right\} = \hat{z}_t \left\{-\mu_c - \mu_n\frac{n}{1-n} + 1 - \alpha\mu_n\right\}$$

which again gives us two equations in four parameters (rather than six, there are no γ_k, μ_k here).

What we do now is to solve for the coefficients. Let's simplify this example by assuming that labour supply is inelastic, so we lose the labour supply equation and the parameters γ_n, μ_n. Take the terms for $\hat{k}_t$ in the resource constraint and Euler equation

$$\frac{c}{y}\gamma_c + \frac{k}{y}\gamma_k = \alpha + (1-\delta)\frac{k}{y}$$
$$\gamma_c(\gamma_k - 1) = \beta\alpha\frac{k}{y}(\alpha - 1)\gamma_k.$$

Put them together to get

$$\gamma_k^2 - \gamma_k\left(1 + \alpha\frac{y}{k} + 1 - \delta + \alpha\beta\frac{c}{y}\right) + \left(\alpha + (1-\delta)\frac{k}{y}\right) = 0$$

which is a second order equation in γ_k that admits two solutions. How do we know which solution is the right one for our policy function parameter? For typical parametrisations, we get that

$$0 < \gamma_{k,1} < 1 < \gamma_{k,2}.$$

Suppose you choose $\gamma_{k,2}$. You'd get that

$$\hat{k}_{t+1} = \gamma_{k,2}\hat{k}_t + \mu_k\hat{z}_t$$

iterating backwards

$$\hat{k}_{t+1} = \gamma_{k,2}^{t+1}\hat{k}_0 + \sum_{j=1}^{t}(\gamma_{k,2})^{j-1}\mu_k\hat{z}_{t-j}.$$

Since we chose the solution larger than 1, this implies that capital will explode to infinity (unless $\hat{k}_0 = 0$, which is unlikely). This cannot be a solution, and you can use the TVC to see this formally. Thus we'll pick $\gamma_{k,1}$, and then derive all other undetermined coefficients.

Note that once we have derived all of the coefficients, we can express the policy functions not only in log-deviations (as we defined it) but also in levels, since

$$\hat{k}_{t+1} = \gamma_k\hat{k}_t + \mu_k\hat{z}_t$$

implies

$$k_{t+1} = k + \gamma_k(k_t - k) + \mu_k k(z_t - 1).$$

If we had the global solution too, we could then compare the exact and approximate solutions. We'll see that they are pretty close when k_t is close to the steady state, and they diverge significantly once we get further away from the steady state.

11.3.2 Blanchard-Kahn method

This is a method that is closely related the one we just saw. Write down the system of equilibrium conditions (Euler equation, resource constraint, labour supply equation) in matrix form

$$A \begin{bmatrix} \hat{\boldsymbol{x}}_{t+1} \\ \hat{\boldsymbol{a}}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{\boldsymbol{x}}_{t} \\ \hat{\boldsymbol{a}}_{t} \end{bmatrix} + D\hat{\boldsymbol{v}}_{t+1}$$

where $\hat{\boldsymbol{a}}_t$ is a vector of controls, $\hat{\boldsymbol{x}}_t$ is a vector of states, and $\hat{\boldsymbol{v}}_{t+1}$ is a vector of errors (which will be the TFP innovation for the exogenous state variable, and mean-zero prediction errors that imply future control variable are not known with certainty as of today).

For example, consider our example with inelastic labour. We'd have

$$\begin{bmatrix} k/y & 0 & 0 \\ 0 & 1 & 0 \\ \alpha\beta\frac{k}{y}(1-\alpha) & -\alpha\beta\frac{k}{y} & 1 \end{bmatrix} \begin{bmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha & 1 & -\frac{c}{y} \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{k}_{t} \\ \hat{z}_{t} \\ \hat{c}_{t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ d \end{bmatrix} \begin{bmatrix} \hat{v}^k_{t+1} \\ \hat{v}^z_{t+1} \\ \hat{v}^c_{t+1} \end{bmatrix}.$$

Invert the LHS matrix and rewrite in reduced form

$$\begin{bmatrix} \hat{\boldsymbol{x}}_{t+1} \\ \hat{\boldsymbol{a}}_{t+1} \end{bmatrix} = F \begin{bmatrix} \hat{\boldsymbol{x}}_{t} \\ \hat{\boldsymbol{a}}_{t} \end{bmatrix} + G\hat{\boldsymbol{v}}_{t+1}$$

where $F := A^{-1}B$ and $G := A^{-1}D$. Consider the Jordan decomposition of $F = HJH^{-1}$ where J is a diagonal matrix whose diagonal elements are the eigenvalues of F, which we denote with $\lambda_1, \cdots, \lambda_n$. We typically assume that the ordering of the system is such that the λs are sorted from the smallest to the largest in absolute value.

In general (away from our example), let h denote the number of eigenvalues that are larger than 1 in absolute value, and let m denote the number of control variables. The BK conditions are the following:

- if $h = m$, then the system has a unique, stable solution;
- if $h > m$, then there exist no solution;
- if $h < m$, then there exist infinite solutions.

We will understand what is the reason behind this result. After the Jordan decomposition, we have

$$\begin{bmatrix} \hat{\boldsymbol{x}}_{t+1} \\ \hat{\boldsymbol{a}}_{t+1} \end{bmatrix} = HJH^{-1} \begin{bmatrix} \hat{\boldsymbol{x}}_{t} \\ \hat{\boldsymbol{a}}_{t} \end{bmatrix} + G\hat{\boldsymbol{v}}_{t+1}.$$

Pre-multiply by H^{-1} and get

$$\begin{bmatrix} \widetilde{\boldsymbol{x}}_{t+1} \\ \widetilde{\boldsymbol{a}}_{t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{x}}_{t} \\ \widetilde{\boldsymbol{a}}_{t} \end{bmatrix} + \widetilde{G}\hat{\boldsymbol{v}}_{t+1}$$

where $\begin{bmatrix} \widetilde{\boldsymbol{x}}_{t} \\ \widetilde{\boldsymbol{a}}_{t} \end{bmatrix} = H^{-1} \begin{bmatrix} \hat{\boldsymbol{x}}_{t} \\ \hat{\boldsymbol{a}}_{t} \end{bmatrix}$ and $\widetilde{G} = H^{-1}G$. Let us take expectations on both sides

$$\mathbb{E}_t \begin{bmatrix} \widetilde{\boldsymbol{x}}_{t+1} \\ \widetilde{\boldsymbol{a}}_{t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{x}}_{t} \\ \widetilde{\boldsymbol{a}}_{t} \end{bmatrix}.$$

Since J is diagonal, we can look at the system block by block, where the first and second block are related to the state and control variables respectively. In the NGM example we only have one control variable, so let us focus on the second block which is now the last row

$$\mathbb{E}_t \widetilde{a}_{3,t+1} = \lambda_3 \widetilde{a}_{3,t}. \tag{30}$$

Rearrange for $\widetilde{a}_{3,t}$ and iterate forward to get

$$\widetilde{a}_{3,t} = \frac{1}{\lambda_3^s} E_t \widetilde{a}_{3,t+s}.$$

If the BK conditions are satisfied, $|\lambda_3| > 1$ and $|\lambda_1| < |\lambda_2| < 1$, so in the limit it must be that $\widetilde{a}_{3,t} = 0$ for all t. This is equivalent to saying that we need to "kill" the unstable (eigenvalue outside the unit circle) part of the system to rule out explosive paths which would not give us a stable system that converges to the steady state in the absence of shocks.

This yields

$$h_{31}\hat{k}_t + h_{32}\hat{z}_t + h_{33}\hat{c}_t = 0$$

where h_{ij} denotes the ij-th element of matrix H^{-1}. This becomes

$$\hat{c}_t = -\frac{h_{31}}{h_{33}}\hat{k}_t - \frac{h_{32}}{h_{33}}\hat{z}_t$$

which is clearly the policy function for consumption.

When the BK conditions are satisfied, we have m control variables and $h = m$ eigenvalues outside the unit circle. By ruling out such explosive paths, we'll have m conditions relating control variables to state variables. Once we have solved for the policy functions of the control variables, it is simple to back out the implied policy functions for the endogenous state variables.

Finally, in our example the policy function for consumption also implies that the prediction errors are such that $\widetilde{G}_{(3,:)}\hat{\boldsymbol{v}}_{t+1} = 0$, i.e. the prediction errors are only a function of the exogenous shocks. In general, we'll have m conditions pinning down the m prediction errors associated with the control variables.

When $m < h$, we have too many explosive eigenvalues (or too few control variables) and no solution exists that makes the system stable.

When $m > h$, there exist infinite stable solutions: the control variables are not uniquely pinned down as a function of the state variables, and the prediction errors associated to the control variables are also not pinned down uniquely. In that case, the path of the system depends on self-fulfilling beliefs. To see this, consider equation (30) and suppose $|\lambda_3| \in (0, 1)$. Then

$$\widetilde{a}_{3,t} = \frac{1}{\lambda_3}\mathbb{E}_t\widetilde{a}_{3,t+1} \neq 0$$

so the current value of the control variable has multiple solutions and depends on the states as well as the expectation of its future value. So agents beliefs determine equilibrium outcomes, and prediction errors are not independent of fundamental variables.

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