Asset Purchases and Default-Inflation Risks in Noisy Financial Markets

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Abstract

This paper studies how central banks' asset purchases (APs) of nominal public debt affects perceived default and inflation risks when investors learn from market prices. We design a stylised model of monetary-fiscal interactions where a government issues nominal debt and repays it by collecting distortionary taxes, and a central bank issues money and buys bonds at market prices. Bond pricing takes place in financial markets and is disciplined by private arbitrage against money and a safe real asset. For a given market valuation, a stochastic and unobservable fundamental determines the government decision to default, which affects nominal bond returns. The return on money – and thus inflation – depends instead on the composition of the central bank balance sheet. In particular, when the central bank invests in bonds, a default generates inflation, which reduces the real value of debt service even further.

In our framework, APs are either irrelevant or reduce social welfare in the absence of beliefs heterogeneity. With heterogeneous beliefs instead we show that a positive amount of APs is optimal and increases welfare. This happens through two main channels. First, by crowding out the asset demand of more pessimistic agents, APs have a positive price effect which lowers interest rates, debt service and tax distortions in all states. This leverage on market optimism amplifies the beneficial effect of unexpected inflation in default states, taming the symmetric harmful effect of unexpected deflation in repayment states. Second, APs have a negative effect because they increase the precision of market information in default states. This makes surprise default less likely to happen, reducing the scope for beneficial unexpected inflation. We show that the net welfare effect of the two channels is positive when APs are small, and becomes negative as APs increase and the precision effect eventually overturns the benefits of leverage on optimism.

Keywords: Sovereign Default, Learning from Prices, Fiscal Dominance.

1 Introduction

The most striking worldwide change in the conduct of monetary policy of the last decade is certainly the permanent implementation of large scale asset purchases (APs henceforth), also known as quantitative easing. Most central banks introduced this non-conventional policy as a way to overcome the limits of the zero lower bound on overnight rates, achieving a downward pressure on long-term yields. APs soon took a macroprudential role too, offering a buffer to the sovereign bond market, hedging sudden shocks and the resulting large price fluctuations. In the Eurozone, sovereign bond purchases have been advocated as a powerful policy to fight financial fragmentation in sovereign debt markets and hedge against speculative attacks. Despite the growing practical importance of quantitative easing, economists are still debating its effectiveness. As bluntly put by Ben Bernanke in a famous speech: "the problem with quantitative easing is that it works in practice, but it does not work in theory."[1]

This paper provides a new rationale for how quantitative easing may work in theory by focusing on the effect of APs in nominal debt markets with heterogeneous beliefs on default risk. We nest a model of financial markets with noisy dispersed information à la Hellwig et al. (2006) in a stylised model of fiscal-monetary interactions, disciplined by private arbitrage. We first show that without belief heterogeneity there is no social benefit in implementing APs. With heterogeneous beliefs instead, a positive amount of APs may be optimal. The main benefit of such policy is to leverage market optimism by crowding out more pessimistic agents. The cost comes from making market information more precise, which limits the scope for non-distortionary taxation through unexpected inflation in cases of default. Intuitively, APs inflate prices in financial markets by selecting a more optimistic marginal agents, that is, the pivotal agent pricing the bonds is one with a more optimistic signal on the government repayment prospects. This reduces government debt service and tax distortions, but it also has information consequences. In fact, observing a low price while knowing that the central bank is already buying implies more accurate information that default is about to occur and inflation may emerge. As agents better anticipate default-inflation risks, bond returns adjust so that realized fiscal needs remain unaffected. The interaction between these two forces determines the optimal size for APs.

Basic setting. Our starting point is a stylised two-period model of fiscal-monetary interactions, disciplined by private investors' arbitrage on financial markets, where the government can default and the central bank issues money to buy assets. We refer to the first period as the short run,

[1]See Bernanke (January 16th, 2014).

and to the second period as the long run. A government issues nominal, defaultable bonds in the short run to finance its stochastic spending needs, and eventually repays such debt in the long run by raising taxes, whose real value depends on bond interest rates as well as inflation. As is standard in the sovereign default literature (e.g. Eaton and Gersovitz (1981), see Aguiar and Amador (2013) for a review), the government is benevolent and trades off tax distortions (increasing and convex) with the deadweight loss generated by a default. The central bank issues money in the short run whose long-run real value depends on that of its liabilities. The proceeds of money issuance can be invested in a safe real storage technology or in government bonds.[2] For tractability we focus on uncontingent AP policies, but our conclusions generalise to all situations where the central bank does not have more information than bond market participants. The private sector holds an endowment that is saved in the short run to be consumed in the long run. Investors save by investing in the asset yielding the highest expected return among bonds, money and the storage technology. The interest rate required by the financial market, the probability of default and the realised inflation rates are mutually endogenous objects. Our aim is to study under which conditions central bank APs may be welfare improving, i.e. help to decrease the expected tax distortion and cost of default in equilibrium.

We divide our analysis in two parts. We initially look at the set of equilibria when investors have homogeneous beliefs, and then shift to the environment with belief heterogeneity. The homogeneous beliefs analysis is particularly informative about the basic economic forces activated by assets purchases, and allows to cleanly identify the role played later on by information frictions.

Money-bond return correlation. First of all, APs introduce correlation between bond and money returns. The long-run real value of central bank liabilities depends on the real value of its investments. As the central bank invests in bonds, an event of default implies a balance sheet loss that depresses the long-run real value of money and generates inflation; on the contrary, when government debt is repaid, the central bank makes a profit that generates long-run deflationary pressures. This simple result sheds light on the empirical observation that APs do not necessarily lead to inflationary pressure; on the contrary, deflation is an outcome that is perfectly consistent with the central bank investing in assets that increase the value of its liabilities. It is worth highlighting the fact that we assume no transfers between the government and the central bank: the only flows between these entities are bond purchases and repayments. Our results would however still go through under the assumption that the central bank rebates its profits to the

[2] Although we do not explicitly model the possibility that the central bank can buy private assets, nothing prevents us from interpreting the government as the consolidated public sector.

government in case of repayment, while there remain no transfers from the government to the central bank in case of default.

Pricing under perfect information. Our first result is that APs are irrelevant under perfect foresight. This happens because, when there is no uncertainty, interest rate changes are purely nominal and do not affect real allocations since financial markets always price assets according to realized fundamentals. This results in the same level of tax distortions, no matter the degree of central bank intervention in financial markets. Effectively the case of perfect information echoes the neutrality result of Wallace (1981), which in our simple model holds even in presence of distortionary taxation.

Common uncertainty about default. When instead investors share the same uncertainty, APs reduce welfare. The result is a consequence of the correlation between money and bond returns induced by APs, against which financial markets cannot fully hedge as they price under uncertainty. Default-inflation correlation reduces real debt service obligations in default but increases them in repayment. This means that APs are unambiguously good in default states that are not perfectly anticipated by the private sector, as inflation is not entirely priced in the equilibrium bond price. On the other hand, APs are harmful in repayment states as they generate unanticipated deflation. Overall, APs increase the variance of real debt repayments, while their expected value remains constant because bonds are priced through a no-arbitrage condition with safe real storage that must hold for each investor. Given that tax distortions are increasing and convex, APs have an unambiguous welfare cost as they increase the variance of tax distortions' without affecting their mean.

Heterogeneous beliefs and APs. We then extend the model to allow for heterogeneous beliefs. Each agent receives a private signal about an exogenous, random fundamental that drives the deadweight cost of default and in turn triggers the government default decision. A second source of uncertainty is given by government bond supply, which is assumed to be random. Investors also learn from the market-clearing bond price, although they cannot tell apart the individual contribution of each of the two shocks to the price signal they observe. That is, bond prices may for example be high either because of low supply, or due to high demand (driven by high fundamentals). As is standard, the supply shock is needed to prevent the equilibrium price to be fully revealing.

We adopt a framework closely related to Hellwig et al. (2006), Albagli et al. (2021) and the sovereign debt application of Bassetto and Galli (2019), where the assumption of investors' risk neutrality and position limits allows to consider nonlinear asset payoffs (such as that of defaultable debt). This class of models features an extensive margin mechanism, where the equilibrium price depends on the beliefs of the marginal agent, that is, the agent who is indifferent between buying government debt or investing in the alternative assets.[3]

Trading-off market optimism and information. When conducting APs, the central bank buys at the market price, crowding out the bond demand of less optimistic investors. We show that such intervention generates two effects. First, it increases the market probability of repayment by selecting a more optimistic investor as the marginal agent that is pricing the bond. This results in a lower bond interest rate, which reduces government debt service and in turn the tax distortions, at any state. At the same time, APs create inflation volatility: deflation (inflation) increases (reduces) real debt service in repayment (default). Under general conditions, the positive effect of APs on the equilibrium price outweighs the negative effect on inflation volatility, so that the net welfare effect of this first channel is positive. The second effect of APs is that they increase the precision of the information revealed by market prices. We show that this takes the form of a truncation in the distribution of investors' posterior beliefs on the fundamental. This result, for which we find general conditions, extends the analysis of noisy information aggregation in financial market by allowing for truncated belief distributions, being a theoretical contribution per se. This beliefs truncation is asymmetric, allowing investors to better detect default states. As a consequence, debt prices more accurately reflect default-inflation realizations, reducing the scope for APs to lower realized fiscal needs through unanticipated inflation. It follows that this second effect of APs has a negative impact on welfare. In sum, by leveraging optimism APs shift the balance of default-inflation risks on the positive side, at the cost of narrowing the range of states for which these risks have an allocational impact. As a result, the optimal size of APs is positive but bounded.

2 Fiscal-Monetary Interactions with Private Arbitrage

In this section we introduce a two-period model of fiscal-monetary interactions with private arbitrage. Although extremely stylised, the model reproduces the basic results emphasized in

[3]This contrasts with the key mechanism in the vast CARA-Normal noisy REE literature, where equilibrium prices depend on the risk premium priced by risk-averse investors that solve a risk-return trade-off problem. See for example Iovino and Sergeyev (2018).

the literature on fiscal-monetary interactions, that is, a trade-off between inflation stability and financial fragility of public debt. On top of that, we introduce private arbitrage between saving asset, providing investors with an alternative to saving in public liabilities.

2.1 A Simple Model

The economy consists of a continuum of investors, a fiscal and a monetary authority. There are two periods $t \in \{1, 2\}$ intended to capture the short and the long term run of the economy. There is an homogeneous consumption good that is available as an endowment to investors in the first period, but can only be taxed in the second. In the first period, the fiscal authority (a government) issues debt in order to finance stochastic spending, the central bank issues money to finance storage or bond purchases; and investors save their endowment via a combination of bonds, money or storage. In the second period, the government repays debt by collecting distortionary taxes from investors, the central bank repays money balances to investors, and investors consume and suffer the eventual deadweight cost of default. Let us describe now the actions of every agent in the economy.

Government. In the first period, the government has stochastic spending needs S that follow a Uniform$[0, 1]$ distribution. We assume that the government needs to borrow to satisfy its spending needs. It then issues a nominal bond that gives R units of money in the second period in exchange of one unit of money in the first period. The nominal interest rate R is decided in the first period as the rate of return that lets supply and demand match in the bond market. The first period budget constraint of the government is then given by:

$$S = \frac{B}{P_1}, \tag{1}$$

where B denotes the total nominal offer of nominal bonds by the government given that the price level prevailing in the first period expressed in terms of money units is P_1.

Given a price level P_2 in the second period, the government pays back its debt obligations. It can generate resources by taxing or partially defaulting on its debt. The second period budget constraint of the government is

$$\frac{B}{P_2} R (1 - \delta h) = \widehat{\tau}$$

where $\widehat{\tau}$ is total taxes raised lump-sum by the government on investors and $\delta \in \{0, 1\}$ is an indicator function taking a value of 1 in case of default and 0 in case of repayment, so a haircut

$h \in (0,1)$ is applied to debt service upon default. It is useful to express the second period budget constraint of the government in units of first period real debt:

$$\frac{R}{\Pi(\delta)}(1-\delta h) = \tau, \tag{2}$$

where $\Pi(\delta) \equiv P_2/P_1$ is the inflation rate between the two periods, which we anticipate will depend on the default choice δ (among other things), and $\tau = \widehat{\tau}/(B/P_1)$ denotes the tax/debt ratio.[4] Henceforth, we will refer to the tax rate τ interchangeably as real debt service, real bond returns or government fiscal needs.

The unique choice of the government, which occurs in the second period, is whether to default or not on the debt inherited from the first period. The government is fully benevolent and always acts under perfect information. Formally, it chooses $\delta \in \{0,1\}$ to minimize a loss function which depends on the total distortions in the economy:

$$\mathcal{L} \equiv (1-\delta)\,\zeta\left(\frac{R}{\Pi(0)}\right) + \delta\left[\zeta\left(\frac{R}{\Pi(1)}(1-h)\right) + \eta(\theta)\right], \tag{3}$$

where $\zeta(\tau)$ is a welfare distortion depending on the real tax over debt ratio τ, and $\eta : \mathbb{R} \to \mathbb{R}_+$ is a deadweight default cost function that depends on the random variable θ. We will refer to θ as the "fundamental" of the economy, and assume it is distributed according to some cumulative distribution function F_θ which we will specify later. We assume that the distortion function $\zeta(\tau)$ is increasing and convex, i.e. $\zeta'(\cdot) > 0$ and $\zeta''(\cdot) > 0$, and that the default cost function is strictly increasing. It follows that the government defaults if and only if the fundamental θ is smaller than an endogenous threshold $\widehat{\theta}(R)$.

Although we design our model to allow for an endogenous default choice à la Calvo (1988), in this version of the paper we assume default is exogenous and simply depends on whether the random variable θ is below an exogenous threshold $\widehat{\theta}$. That is, the default variable follows $\delta = \mathbb{1}(\theta < \widehat{\theta})$ where $\mathbb{1}$ is the indicator function. This assumption greatly improves the tractability and transparency of the model. Since this implies that the default distortion is exogenous, in the rest of the exposition we will abstract from the term $\eta(\theta)$ when considering the welfare loss function $\mathcal{L}$.

[4] This specification is useful to express real government needs in terms of a tax rate on total spending or on real debt. An alternative would be to assume that spending is a constant fraction γ of random first-period output S, and tax revenues are expressed as a fraction $\tilde{\tau}$ of output. Simplifying yields $\gamma R/\Pi(1-\delta h) = \tilde{\tau}$, which is identical to equation (2) if we assume (without loss of generality) that $\gamma = 1$.

Investors. There is a unit mass of investors indexed by $i \in [0,1]$. In the first period an agent i faces the following budget constraint:

$$\frac{B^i}{P_1} + \frac{M^i}{P_1} + s^i \leq e_1 \tag{4}$$

where B^i denotes bonds holdings, M^i are money holdings, s^i is the quantity of consumption good stored in a storage technology, and e_1 is the agent's endowment in units of consumption good. Investors can thus save by either buying nominal assets – bonds and money – or investing in a real storage technology.

The second period budget constraint of agent i reads as:

$$c^i = R(1-\delta h)\frac{B^i}{P_2} + \frac{M^i}{P_2} + \rho s^i - \widehat{\tau} - \mathcal{L} \tag{5}$$

where c^i denotes individual consumption, which amounts to the real value of the bonds nominal gross returns $R(1-\delta h)B^i$, money M^i/P_2 and storage ρ, net of a lump sum tax $\widehat{\tau}$ and a deadweight loss term $\mathcal{L}$ that depends on the government's actions as explained above.

Investors compete on the market for nominal saving assets. Their choice boils down to the composition of their real saving portfolio in the first period. In particular, household i will invest in the asset to maximize the expected total return of her portfolio

$$\mathcal{R}_i \equiv \frac{1}{e_1}\mathbb{E}\left[\frac{R(1-\delta h)}{\Pi(\delta)}\frac{B^i}{P_1} + \frac{1}{\Pi(\delta)}\frac{M^i}{P_1}\,\Big|\,\Omega_i\right] + \frac{1}{e_1}s^i\rho, \tag{6}$$

where $\mathbb{E}[\cdot\,|\Omega_i]$ denotes the expectation operator conditional to the information set of agent i, to be specified later. While the return on storage is deterministic, that on money and bonds generally depends on the realization of the default event and on the size of the central bank intervention.

Monetary authority (CB). The monetary authority, or central bank, issues money in the first period and can invest money revenues into government bonds, i.e. by implementing APs, or in the storage technology. The first period budget constraint of the monetary authority reads as

$$\frac{M}{P_1} = \frac{B^{cb}}{P_1} + s^{cb} \tag{7}$$

with $s^{cb} \geq 0$, where M denotes nominal money supply, which without any loss of generality we take as fixed, and s^{cb} denote the central bank investment in the storage technology. The budget constraint of the central bank (7) determines the price level in period one.[5] Similarly to investors and differently from the government, the central bank does not consume in the first period and has access to a storage technology; on the other hand, like the government and differently from investors, it can issue liabilities. Let us denote with

$$\alpha \equiv \frac{B^{cb}}{M}$$

the asset purchase rule (AP rule), that is, the share of the central bank balance sheet invested in government debt rather than storage. As we will explain later, we assume that $\alpha \in [0, 1-h)$. In the second period, the budget constraint of the monetary authority is given by:

$$\rho s^{cb} + R(1-\delta h)\frac{B^{cb}}{P_2} = \frac{M}{P_2} \tag{8}$$

that is, the authority uses the revenues from bonds holdings and storage to repay money.

Timing, market clearing and equilibrium. We summarize here the actions and timing of agents in the economy. At the beginning of the first period all shocks $\{S, \phi\}$ realize and investors receive some information about it. Then, based on available information, and for a given AP rule α, each agent sets the interest rate R_i above which she is willing to buy government bonds. The markets for bonds, money and goods clear in the first period according to

$$B = \int B^i di + B^{cb} \quad \text{and} \quad M = \int M^i di \quad \text{and} \quad e_1 = S + s^i + s^{cb}, \tag{9}$$

which yields a market clearing bond return R and market clearing price P_1, respectively. In the second period the government eventually applies the default haircut h and sets taxes to repay its debt, the central bank repays its liabilities and investors consume. Goods market clearing in the second period requires:

$$\mathcal{C} \equiv \int c^i di = \rho(s^i + s^{cb}) - \mathcal{L}, \tag{10}$$

which determines the market clearing price level P_2 and in turn the realized gross inflation rate Π.

[5]That implies that the nominal quantity of bonds adjusts to satisfy real government spending needs.

Definition 1. *(Market equilibrium for a given AP rule) For given shock realizations* $\{S, \phi\}$*, AP rule* $\alpha \in [0, 1-h)$ *and investors' information sets* $\{\Omega_i\}_{i \in [0,1]}$*, a market equilibrium is defined as an allocation* $(S, \{c^i\}_{(0,1)})$*, a vector of prices* (R, P_1, P_2) *and choices* $\{M^i, B^i\}_{i \in [0,1]}$ *such that:*

i) the choice (M^i, B^i) *maximizes the expected return of household* i*'s portfolio (6), for each* i*,*

ii) all markets clear.

We consider equilibria for a given AP rule. Implicit in our choice is the idea that central banks follow a rule established ex ante, which does not provide any additional information to investors. We will comment in due course to which extent our framework can be extended to allow for more complex AP rules that are possibly contingent to market prices. The essential ingredient that we are going to retain along all our study is that APs do not have any signaling role, i.e they do not provide any additional information to investors.

Welfare. We define welfare as the present value of consumption by the government and the investors. Formally welfare is given by

$$S + \rho^{-1}\mathcal{C} = e_1 - \rho^{-1}\mathcal{L}, \tag{11}$$

which we get by putting together the two resource constraints, jointly with the $t = 1$ government budget constraint. Thus, for any given realization of the shocks $\{S, \phi\}$, welfare is equivalently measured by tax distortions. It is important to notice that, given welfare distortions $\mathcal{L}$ only depend on taxes, we are implicitly assuming that inflation does not have any deadweight cost per se, thus only affecting distortions through its effect on the real value of taxes. It follows that, in our framework, unanticipated inflation increases welfare by reducing real debt service and taxation, and the contrary holds for deflation.

The expression for welfare in equation (11) helps to identify some features of our model worth discussing. The problem in our economy is the intertemporal allocation of the endowment of consumption goods. In the first period it is the government who consumes, and since it cannot raise taxes at that time, it has to postpone taxation to the second period by issuing debt. Since taxes are lump sum, investors do not internalise that requiring an interest rate on government debt in $t = 1$ implies future taxes in $t = 2$. This generates an externality in that, for agent i to participate in the market for public liabilities, the following condition must hold for at least some pair $(B^i, M^i) \in \mathrm{R}^2_+$ at given prices (R, P_1, P_2)

$$\mathcal{R}_i \geq \rho. \tag{12}$$

This is a consequence of the decentralization of the consumption good saving problem.

It is simple to note that the unconstrained first best allocation in our economy entails a pure transfer of resources to the government directly in the first period, without generating any debt obligation and need to tax in the second. This is what an unconstrained central planner would choose to do, and would be implemented by setting an interest rate of zero on government debt. Such allocation cannot however be realised by the market because of the participation constraint (12). This constraint imposes that the return on public debt in expectation should be at least equal to the one on storage. Agents do not internalize taxes (and the related distortions), which are imposed on all investors irrespective of their individual saving choices. Indeed in our setting welfare is higher, the lower average fiscal needs are, i.e. the closer the expected real bond interest rate is to the first best level of zero.

Inflation and bond-money returns correlation. The central bank budget constraint (8) can be rearranged to express the return on money as

$$\frac{1}{\Pi(\delta)} = (1-\alpha)\rho + \alpha \frac{R(1-\delta h)}{\Pi(\delta)}. \tag{13}$$

That is, the real return on money is an average (with weight α) between the return on storage and the real return on bonds. Without APs (i.e. with $\alpha = 0$), inflation is constant and equal to the inverse of the real return of storage: money and storage yield identical real returns equal to ρ. The interpretation is that the central bank effectively anchors inflation to the return of storage by avoiding any interference in the bond market and only puts a real risk-free asset in its balance sheet.

With APs instead (i.e. with $\alpha > 0$), the real return of money and bonds correlate. This happens because an additional unit of APs, ceteris paribus, means a unit less of storage. By operating APs, the CB ties its balance sheet to the government default decision, so it makes excess profits over storage in repayment (low inflation), and losses in default (high inflation). The terminal real value of money is essentially the terminal real transfer to money holders, which will be higher when the CB realises profits. So a government repayment means low inflation, a default means high inflation, and this relationship is stronger, the larger the size of

APs.[6] This fiscal-monetary link, which is known by investors, implies that no matter the degree of uncertainty we consider, default beliefs necessarily correlate with inflation beliefs as long as $\alpha > 0$.

Rearranging the central bank budget constraint at $t = 2$, we can solve for inflation as a function of the bond interest rate R, the default indicator δ and the AP rule α:

$$\Pi(R, \alpha, \delta) = \frac{1 - \alpha R(1 - \delta h)}{\rho(1 - \alpha)}. \tag{14}$$

Since the price level must be non-negative, there exists an upper bound for APs given by $\alpha < 1/R(1 - \delta h)$. Because the AP policy α is chosen ex ante, we must use the lowest possible value of such upper bound, which means the choice set for APs is given by $\alpha \in [0, 1 - h)$.

3 Homogeneous Information

In this section, we study the effects of APs by assuming that investors have homogeneous information on the likelihood of a default, in other words $\Omega_i = \overline{\Omega}$ for any $i \in [0, 1]$. We explore the potential for APs to improve welfare and conclude that with homogeneous beliefs APs are in the best of cases irrelevant.

We will consider first the case where investors have perfect foresight on the default event: in this case APs are neutral, i.e. they do not affect allocations. We then consider the case where investors have common uncertainty on the default event. In this case, APs are detrimental as they increase the volatility of fiscal needs across default and repayment states, without affecting their expected level. Keeping investors' uncertainty fixed, APs would instead unambiguously be beneficial if they could occur only contingently to a default event.

3.1 Perfect Foresight on Default Decisions

The first case we investigate is that investors have perfect foresight on the government default shock, i.e. $\overline{\Omega} = \{\theta\}$. We obtain in this case a neutrality benchmark establishing the irrelevance of APs, as stated by the following.

Proposition 1. *(Irrelevance benchmark) Suppose investors have perfect foresight, then* $R = 1$ *if* $\delta = 0$ *and* $R = 1/(1 - h)$ *if* $\delta = 1$*, for any AP rule* $\alpha \in [0, 1 - h)$*. In this case APs are allocation*

[6] Throughout the paper, we refer to inflation (deflation) as states where $\Pi > (<) \frac{1}{\rho}$.

irrelevant, i.e. bond and money real returns equate ρ irrespective of α.

For a given δ, consider the following cases: when $R(1-\delta h) < 1$, storage is strictly preferred to nominal assets, in which case markets do not clear; when instead $R(1-\delta h) > 1$, bonds are strictly preferred to money and storage, in which case competition among investors would lead to a lower R. Only when $R(1-\delta h) = 1$ investors are indifferent between storage, bonds and money, not only ex ante but also ex post as they perfectly observe default realizations. The real return on bonds and money are thus equal to ρ and are independent of α.

This first result is reminiscent of Wallace (1981)'s irrelevance result. However it is important to remark that Wallace's result is obtained in the context of an economy with no distortions. In this sense our result is stronger, in that irrelevance obtains also with respect to the level of distortions engendered by taxes, which under perfect information are $\mathcal{L} = \zeta(1)$ irrespective of α.

3.2 Exogenous Uncertainty on Default Decisions

We now consider the case where all investors attach the same probability $p \in [0,1]$ to the event that the government repays debt in full. Taking default probabilities as exogenous is useful to clarify the basic mechanism underlying our results. We consider the case of heterogeneous subjective default probabilities in Section 4.

Arbitrage under uncertainty. In this setting, competition in the financial market ensures no arbitrage between bonds and storage in expectation,

$$p\,\frac{R}{\Pi(R,\alpha,0)} \;+\; (1-p)\,\frac{R}{\Pi(R,\alpha,1)}(1-h) \;=\rho, \tag{15}$$

that is, the expected return on bonds is the same as the return on storage. Note that, because of (13), indifference between bonds and storage also implies indifference between money and storage, given by

$$p\,\frac{1}{\Pi(R,\alpha,0)} \;+\; (1-p)\,\frac{1}{\Pi(R,\alpha,1)} \;=\rho, \tag{16}$$

that is, the expected return on money is the same as the return on storage. In the following proposition we show that the arbitrage between bonds, money and storage is determined by a threshold condition on the repayment probability. In fact, by rearranging the no-arbitrage condition (15) we can define

$$p^* = \frac{1-R(1-h)}{h} \times \frac{\frac{1}{R}-\alpha}{1-\alpha} \tag{17}$$

being the value of the repayment probability such that the no-arbitrage condition holds for given α and R. The portfolio strategy of investors i is:

1. $\frac{B^i}{P_1} = e_1$, $\frac{M^i}{P_1} = s^i = 0$ if and only if $p > p^*$;
2. $\frac{B^i}{P_1} + \frac{M^i}{P_1} + s^i = e_1$ if and only if $p = p^*$;
3. $\frac{B^i}{P_1} = \frac{M^i}{P_1} = 0$, $s^i = e_1$ if and only if $p < p^*$.

Moreover, the absolute difference between $1/\rho$ and inflation is increasing in α, and so is its ex ante variance. In other words, there is a threshold probability of repayment such that for a higher probability, bond returns dominate money returns, that in turn dominate the return on storage; for a lower probability instead the opposite holds. Figure 1 illustrates graphically this result for a given α and R.

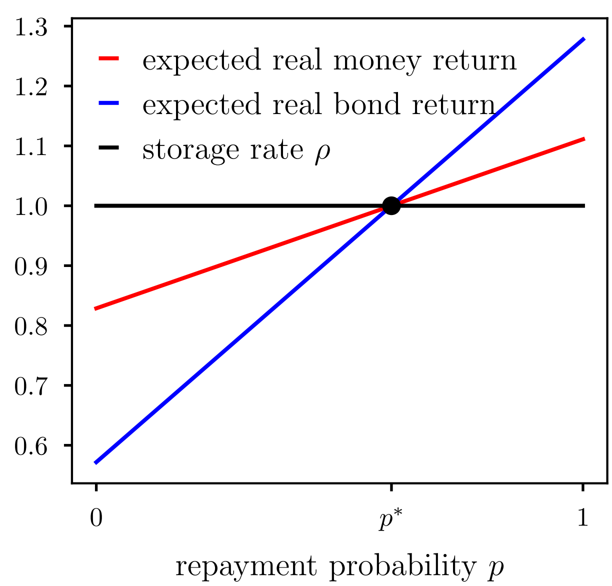

Figure 1: Illustration of the expected return on bonds (blue), money (red) and storage (dashed black) as a function of the repayment probability p, for given values of R and α. When $p = p^*$, the no-arbitrage condition holds and investors are indifferent across the three assets.

APs with common uncertainty. Let us now look at the the effect of APs in case investors are homogeneously uncertain about the default events. It is useful to remark right away that, with common uncertainty, APs would have a beneficial effect if they could be carried out only conditional on default states, i.e. if the central bank could discriminate. To see this, we can plug the gross inflation rate in equation (2) to obtain an explicit expression for the real return on bonds, which coincides with the tax rate that creates a deadweight distortion:

$$\tau(R, \alpha, \delta) = \frac{R(1-\delta h)}{\Pi(R, \alpha, \delta)} = \rho \frac{1-\alpha}{\frac{1}{R(1-\delta h)} - \alpha}. \tag{18}$$

Given equations (15) and (16), one can easily see that that when $p \in (0,1)$ it must be that $R > 1$ and $R(1-h) < 1$. It follows that $\tau(R, \alpha, 0)$ and $\tau(R, \alpha, 1)$ are respectively increasing and decreasing in α, for all equilibrium R. We embody this observation in the following lemma.

Lemma 1. *APs are beneficial in default states and harmful in repayments states as long as bonds are priced under uncertainty.*

Of course, the net effect on ex ante welfare depends on whether the welfare benefits outweigh the costs. We formalise this in the following proposition.

Proposition 2. *(Common uncertainty) Suppose repayment $\delta = 0$ happens with an exogenous probability $p \in (0,1)$, and this is common knowledge among agents, then welfare is decreasing in α, and the social optimum is given by $R = \frac{1}{1-h(1-p)}$ with $\alpha = 0$.*

Proof. See Appendix A.3. ∎

By embarking in APs the central bank decreases fiscal needs and in turn distortions in default states, since the realization of inflation is not completely anticipated by investors in their pricing. For the same reason, however, APs also increase distortions in repayment as unanticipated deflation occurs. Given that distortions are convex, the distortion increase in repayment always dominates the decrease upon default. This is why APs are never optimal.

4 Heterogeneous Information

We know study the case where investors' information about the government default event is heterogeneous. Recall that we are assuming that the government defaults when the random variable θ is below the cutoff $\widehat{\theta}$, and repays in full otherwise. We will specify the probability distribution of θ later, as the first part of our equilibrium characterisation does not depend on specific functional forms. We continue to consider the case where the central bank does not know θ and chooses an uncontingent policy α.

Investors observe a private signal on θ, as well as the market clearing price, which is an additional signal, public and endogenous, on the fundamental. Stochastic bond supply S plays the role of the additional source of noise that is needed to prevent the price from fully revealing θ.

Investors' Information and Bidding Strategies. Each agent i receives a noisy private signal $x_i = \theta + \sigma_x \xi_i$, where $\xi_i \sim N(0,1)$ and is iid over i's. Investors submit bids for government bonds and money that are contingent on the interest rate R and the central AP policy α. It follows that the information set of each investor i is now given by $\Omega_i = \{x_i, R, \alpha\}$. The investor's expected payoff of saving through government debt, money or storage depends on her subjective debt repayment probability, $\text{Prob}(\theta > \widehat{\theta} \mid x_i, R, \alpha)$, which we denote with $p(x_i, R, \alpha)$.

Let us now define the variable $\psi(R, \alpha, \delta) = \frac{1-\alpha}{\frac{1}{R(1-\delta h)} - \alpha}$, which represents the government real debt service adjusted by $1/\rho$ (or equivalently real bond returns, and the tax rate that determines tax distortions) as a function of the nominal interest rate R, the AP policy α and the default policy δ. Although in our setting τ and ψ are identical up to a constant, they need not be so in general, so we prefer to use different labels to maintain a conceptual different between tax rates and real bond returns. In fact, real bond returns are the key variable in the analysis of the Bayesian trading model that follows, which may be of separate interest from the macroeconomic model that microfounds our welfare results.

We can now write the subjective expectation of real bond returns for agent i as

$$\mathbb{E}[\psi \mid x_i, R, \alpha] \equiv p(x_i, R, \alpha) \frac{1-\alpha}{\frac{1}{R} - \alpha} + \Big(1 - p(x_i, R, \alpha)\Big) \frac{1-\alpha}{\frac{1}{R(1-h)} - \alpha}. \tag{19}$$

Note that this is the analog of equation (15) where we replaced p with $p(x_i, R, \alpha)$. Importantly, we make the assumption that investors' bond positions are limited to be inside the $[0,1]$ interval. Agent i's portfolio strategy is the following

- if $\mathbb{E}[\psi \mid x_i, R, \alpha] > 1$, the agent holds bonds and money: $b^i = 1, m^i = e_1 - 1$;
- if $\mathbb{E}[\psi \mid x_i, R, \alpha] = 1$, the agent is indifferent: $b^i + m^i + s^i = e_1$;
- if $\mathbb{E}[\psi \mid x_i, R, \alpha] < 1$, the agent stores all its endowment: $s^i = e_1$.

Conditional on the price and on the AP policy, investors' posterior beliefs over θ are strictly increasing in private signal x_i in the sense of first-order stochastic dominance. This implies that investors follow monotone strategies of the following type. Let $\widehat{x}(R,\alpha)$ be the private signal threshold at which

$$\mathbb{E}[\psi \mid x_i = \widehat{x}, R, \alpha] = p(\widehat{x}, R, \alpha)\frac{1-\alpha}{\frac{1}{R}-\alpha} + \Big(1 - p(\widehat{x}, R, \alpha)\Big)\frac{1-\alpha}{\frac{1}{R(1-h)}-\alpha} = 1 \tag{20}$$

and the expected payoff of bonds, money and storage is the same.[7] All investors with a signal $x_i \geq \widehat{x}(R,\alpha)$ will buy bonds up to the unit position limit, and invest the rest of their endowment in money. All those with a signal $x_i < \widehat{x}(R,\alpha)$ will invest all their endowment in storage. It follows that bonds will be priced off the marginal agent's no arbitrage condition (20).

The assumption that investors face bond position limits can be interpreted as a reduced form way to model an economy where investors do not have deep pockets and face liquidity or collateral constraints. For tractability reasons we assume that all investors face the same limits, but our key results would carry over to a more general setting where position bounds are heterogeneous, as long as the mapping between beliefs and bond asset positions remains invertible. It is worth noting that the assumption has the important implication that a non negligible mass of agents demand to hold money in equilibrium. With heterogeneous information, this would not be the case if there were no bond position bounds. To see this, note that all investors with a high enough private signal would want to save all their endowment in bonds, while all other investors want to store all of their endowment. Only the agent with signal $x_i = \widehat{x}(R,\alpha)$ is indifferent between the three assets. With a continuous distribution for $x_i|\theta$, this implies that a zero mass of investors would want to hold money.

Bonds and Money Market Clearing. Investors' portfolio decisions follow cutoff strategies, and we assume that a law of large numbers across investors applies as in Judd (1985). For a given value of the fundamental, the mass of investors buying bonds is given by

$$\text{Prob}(x_i \geq \widehat{x}(R,\alpha) \mid \theta) = \Phi\left(\frac{\theta - \widehat{x}(R,\alpha)}{\sigma_x}\right)$$

where Φ denotes the standard normal cumulative distribution function. If agent i buys bonds, she also holds money in the amount $\frac{M^i}{P_1} = e_1 - 1$. Those who do not buy bonds, do not demand

[7]Note that the rate of return on storage ρ cancels out as it shows up in the formula for $\frac{1}{\Pi}$ which we derived earlier.

any money either. Hence the money market clearing condition is given by

$$\Phi\left(\frac{\theta-\widehat{x}(R,\alpha)}{\sigma_x}\right)(e_1-1)=m, \tag{21}$$

where $m \equiv \frac{M}{P_1}$ denotes the real money supply in the first period.

Recall that government bond supply is denoted by S, is random and is distributed according to a Uniform$[0,1]$. Central bank bond demand is given by b^{cb}, and the bond market clearing condition is

$$\Phi\left(\frac{\theta-\widehat{x}(R,\alpha)}{\sigma_x}\right)+b^{cb}=S.$$

Plugging in the definition $\alpha \equiv b^{cb}/m$ together with the money market clearing condition, we can rewrite the bond market clearing condition

$$\Phi\left(\frac{\theta-\widehat{x}(R,\alpha)}{\sigma_x}\right)[1+\alpha(e_1-1)]=S. \tag{22}$$

Solving for the cutoff private signal yields

$$\widehat{x}(R,\alpha)=\theta-\sigma_x\Phi^{-1}\left(\frac{S}{d(\alpha)}\right) \tag{23}$$

where we define $d(\alpha)\equiv 1+\alpha(e_1-1)$ to keep notation compact.

We know that the marginal agent's signal is a deterministic function of the equilibrium interest rate R and the AP policy α via no-arbitrage condition (20). Equation (23) thus simply states that the marginal agent's private signal $\widehat{x}(R,\alpha)$ must be equal, in equilibrium, to a nonlinear function of the AP policy variable α and the two exogenous shocks θ and S. Henceforth, we denote realisations of such function with $z = Z(\theta, S, \alpha)$, which we refer to as the "state variable" or "market signal" going forward.

Condition (23) shows that in equilibrium the price constitutes an endogenous public signal of the fundamental, with a peculiar noise term that depends on the AP policy α. Lemma 2 describes the properties of this noise term, and shows how APs affect its mean, variance and support.

Lemma 2 (Market Signal Noise)**.** *We define the market signal noise as the random variable*

$$\nu(S,\alpha)\equiv\sigma_x\Phi^{-1}\left(\frac{S}{d(\alpha)}\right).$$

The term $\nu(S,\alpha)$ follows the normal distribution $N(0,\sigma_x^2)$ with the truncated support $Supp_\nu = (-\infty, \overline{\nu}(\alpha)]$ where $\overline{\nu}(\alpha) \equiv \sigma_x \Phi^{-1}\left(\frac{1}{d(\alpha)}\right)$. Its first and second moments are $\mu_\nu(\alpha) = -\lambda(\overline{\nu}(\alpha))$ and $\sigma_\nu^2(\alpha) = 1 - \lambda(\overline{\nu}(\alpha))\left[\overline{\nu}(\alpha) + \lambda(\overline{\nu}(\alpha))\right]$, where $\lambda(x) \equiv \frac{\phi(x)}{\Phi(x)}$ denotes the inverse of the reverse hazard rate function, and ϕ is the standard normal probability density function. For the standard normal distribution it is well known that $-1 < \lambda'(x) < 0$. This implies the following useful results: $\frac{d}{d\alpha}\nu(\alpha) < \frac{\partial}{\partial\alpha}\nu(S,\alpha) < 0$, $\frac{d}{d\alpha}\mu_\nu(\alpha) < 0$, $\sigma_\nu^2(\alpha) < \mathbb{V}(S) = 1$, $\frac{d}{d\alpha}\sigma_\nu^2(\alpha) < 0$.

Proof. In the appendix (at this link). ■

It follows that APs affect the distribution of the information conveyed by the price. The cumulative distribution function of the market signal z conditional on θ is given by

$$F_{z|\theta}(y|\theta) = \begin{cases} 1 - \frac{\Phi\left(\frac{\theta-y}{\sigma_x}\right)}{\Phi\left(\frac{\bar{\nu}(\alpha)}{\sigma_x}\right)} & \text{for} \quad z \geq \theta - \bar{\nu}(\alpha) \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

We discuss in depth the equilibrium implications of the effect of APs on the market signal and its truncation in the following subsection.

Henceforth, we focus on equilibria where z and R convey the same information, conditional on α. In such instance, conditioning beliefs on the endogenous price is equivalent to conditioning them on the exogenous variable z.

4.1 Asset Purchases and Learning from Prices

To derive agents' posterior beliefs on the fundamental, we must consider the effect of APs on the market clearing condition and on the information conveyed by the equilibrium price. Specifically, here we show the two main effects of APs: first, they amplify the importance of optimistic investors by acting as a bond demand multiplier; second; they fully reveal the fundamentals in some circumstances.

To help intuition, we now make the assumption that the fundamental θ is a binary random variable that takes value θ^H with probability q and θ^L with probability $1-q$. We assume the government repays in the former case and defaults in the latter, i.e. $\theta^L < \widehat{\theta} < \theta^H$.

Figure 2 plots the state variable z, and in turn the marginal agent's signal $\widehat{x}$, as a function of supply S, the fundamental θ and the AP policy α. Solid and dashed curves respectively denote the repayment case ($\theta = \theta^H$) and the default case ($\theta = \theta^L$). We consider the case without APs

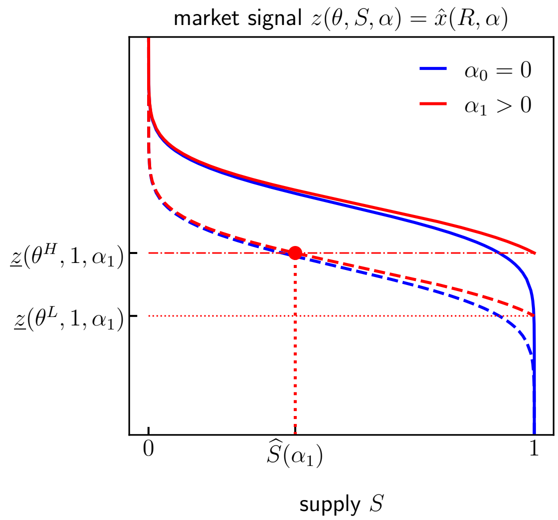

Figure 2: Illustration of the marginal agent's signal as a function of supply S, fundamental θ and AP policy α. The solid curves represent $z(\theta^H, S, \alpha)$, the dashed curves represent $z(\theta^L, S, \alpha)$. The two red horizontal lines show the lowest possible level of z compatible with θ^H (dash-dotted line) and θ^L (dotted line).

(blue curves, $\alpha_0 = 0$) and with a positive AP policy (red curves, $\alpha_1 > 0$). The graph illustrates the two main effects of central bank APs: first, they reduce the set of market signals compatible with some realisations of the fundamental; second, they increase the value of the market signal related to any (θ, S) pair. We now explain these mechanisms in detail.

Let us focus first on the case *without* APs: the support of the market signal z is the real line, and any value of z is compatible with both realisations of the fundamental. Mathematically, this happens because the support of the supply shock is the same as the support of the mass of buyers (which follows the private signal distribution), regardless of the realisation of θ. Investors update their beliefs using the information contained in the price, and in this case each agent i's posterior repayment probability is given by

$$\text{Prob}(\theta = \theta^H \mid x_i, z, \alpha = 0) = \frac{\frac{q}{\sigma_{post}}\phi\left(\frac{\theta^H - \frac{x_i+z}{2}}{\sigma_{post}}\right)}{\frac{q}{\sigma_{post}}\phi\left(\frac{\theta^H - \frac{x_i+z}{2}}{\sigma_{post}}\right) + \frac{1-q}{\sigma_{post}}\phi\left(\frac{\theta^L - \frac{x_i+z}{2}}{\sigma_{post}}\right)} \quad \forall z \in \mathbb{R}$$

where $\sigma_{post} \equiv \frac{\sigma_x}{\sqrt{2}}$.

Consider now the case *with* APs: first, the value of z associated to a (θ, S) pair increases; second, the support of z features a right truncation point, as not all values of z are compatible with any realisation of the fundamental. Let us denote with $\underline{z}(\theta, S, \alpha)$ the lowest possible value of z compatible with a given (θ, S, α) triplet. In the example of Figure 2, values of z below $\underline{z}(\theta^L, 1, \alpha_1)$ (the horizontal red dotted line) are not possible anymore, while values between $\underline{z}(\theta^L, 1, \alpha_1)$ and $\underline{z}(\theta^H, 1, \alpha_1)$ (the horizontal red dash-dotted line) are only compatible with $\theta = \theta^L$. Mathematically, this can be seen clearly in equation (22): APs act as a multiplier, effectively putting an upper bound of $1/d(\alpha) < 1$ on the net bond supply available to buyers, and in turn on their mass.

To help intuition, recall that by observing the price agents effectively learn the private signal received by the marginal agent. What they cannot infer is the ordinal position of the marginal agent; if they could, they would back out the fundamental. Without APs, the largest possible supply shock is $S = 1$, which requires all agents to buy, and in turn the marginal agent to be the "last", that with the worst signal. When the private signal is distributed on the whole real line, the worst signal is equal to minus infinity *regardless* of the fundamental. When instead the central bank participates in the bond market, the largest possible supply to agents becomes $1/d(\alpha) < 1$: APs effectively reduce net bond supply. This has a positive effect on beliefs: the larger the AP policy, the higher the position of the marginal agent needed for demand to meet net supply, the more optimistic the belief distribution being used to price the bonds. But an upper bound on net supply implies a lower bound on the position of the marginal agent, which will differ depending on θ. When agents observe a sufficiently bad market signal, they realise a default is coming because they can tell that the most pessimistic marginal agent compatible with repayment (i.e. that corresponding to the largest possible supply shock) could not possibly have received such a bad signal. In other words, since central bank intervention improves the price (everything else equal), when one still sees a bad price, it can only mean that a default is coming.

It follows that, when $\theta = \theta^L$, a sufficiently large supply shock (above $\widehat{S}(\alpha_1)$ in the example) implies a full revelation of the fundamental and the resolution of all uncertainty. Agent i's posterior repayment probabilities will be as in the case of no APs for z large enough, and zero otherwise:

$$\text{Prob}(\theta = \theta^H | x_i, z, \alpha) = \begin{cases} \dfrac{\frac{q}{\sigma_{post}} \phi\left(\frac{\theta^H - \frac{x_i + z}{2}}{\sigma_{post}}\right)}{\frac{q}{\sigma_{post}} \phi\left(\frac{\theta^H - \frac{x_i + z}{2}}{\sigma_{post}}\right) + \frac{1-q}{\sigma_{post}} \phi\left(\frac{\theta^L - \frac{x_i + z}{2}}{\sigma_{post}}\right)} & \text{for } z \geq \underline{z}(\theta^H, 1, \alpha) \\ 0 & \text{for } z \in [\underline{z}(\theta^L, 1, \alpha), \underline{z}(\theta^H, 1, \alpha)). \end{cases} \tag{25}$$

Note that the posterior probability of (25) is general, that is, it is valid for any values of $\alpha \in [0, 1-h)$, since $\lim_{\alpha \to 0} \underline{z}(\theta, 1, \alpha) = -\infty$ for all θ.

It is worth to make a few remarks related to our findings so far. First, all these observations remain valid in the more general case where θ has a continuous, unbounded support; we use the binary because it makes intuition more transparent. Second, APs have an asymmetric effect on information revelation. This is the case whenever APs reduce net bond supply available to investors in a multiplicative way. In other words, when investors observe a high market signal, they still cannot tell apart whether that is driven by good fundamentals or by a low supply shock, given the lower bound of the latter is zero regardless of α. That is, default states may be perfectly revealed for some (S, α) pairs, while repayment states never are. Third, the range of supply shocks that fully reveal θ^L is increasing in the AP policy α. Proposition 5 in the appendix shows that this result is general, as it holds for any private signal distribution that satisfies the monotone likelihood ratio property. Fourth, when APs do not fully reveal the fundamental, they have a positive effect on repayment probabilities. Note that, as is clear from (25), the AP policy does not change repayment probabilities *given* the signals x_i, z and $z \geq \underline{z}(\theta^H, 1, \alpha)$. For a given value of the exogenous shocks (θ, S), what APs do affect is the value of the market signal z and the identity of the marginal agent with private signal $x_i = z$. This is driven by the fact that the central bank crowds out pessimistic investors, investing more of the money holdings of the optimistic agents into bonds. As a result, the identity of the marginal agent pricing the bond moves up towards a more optimistic one.

We now explain how the double effect of APs on information revelation and default beliefs affects tax distortions and welfare.

4.2 Equilibrium Characterisation and Welfare

Equilibrium Interest Rate. As explained earlier, in equilibrium the bonds will be priced by the marginal agent. Having defined all equilibrium objects, we can rewrite the equilibrium condition that defines the interest rate $R(z, \alpha)$ as a function of the state variable z and the AP policy α:

$$p(z,\alpha)\frac{1-\alpha}{\frac{1}{R}-\alpha} + \Big(1 - p(z,\alpha)\Big)\frac{1-\alpha}{\frac{1}{R(1-h)}-\alpha} = 1 \tag{26}$$

where $p(z, \alpha) = \text{Prob}(\theta = \theta^H \mid x_i = z, z, \alpha)$. Note that this "market" repayment probability, i.e. the posterior beliefs on $\theta | x_i = z, z$ of the marginal agent, are analogous to the posterior beliefs on θ given $x_i = z$ only, but have half the variance ($\sigma^2_{post} = \sigma^2_x/2$). This is due to the fact that the former uses two sources of information, the private signal and the price, while the latter uses only one. It is useful to make this comparison because the beliefs on $\theta|z$ can be interpreted as the posterior belief distribution of an agent who only gets the private signal $x_i = z$, or those of an external observer that only observes the price but receives no private signal.[8]

Our ultimate goal is to understand the effect of APs on tax distortions. To do so, it is helpful to start by illustrating the effect of APs on the equilibrium nominal interest rate R.

Proposition 3. *An increase in central bank APs has an ambiguous effect on the equilibrium interest rate:*

1. *it reduces R for all (θ, S) where uncertainty is not resolved;*
2. *it weakly increases R for all (θ, S) where uncertainty is resolved.*

Mathematically, consider an initial level of APs given by $\alpha_0 \in [0, 1-h)$. Let $\mathcal{Z}^n(\alpha_0) = \{(\theta, S) \ : \ z(\theta, S, \alpha_0) \in [\underline{z}(\theta^L, 1, \alpha_0), \underline{z}(\theta^H, 1, \alpha_0))\}$ denote the region of the shocks state space where the market signal is fully revealing, there is no uncertainty about default and $R = 1/(1-h)$. Similarly, let $\mathcal{Z}^u(\alpha_0) = \{(\theta, S) \ : \ z(\theta, S, \alpha) \in [\underline{z}(\theta^H, 1, \alpha_0), +\infty)\}$ denote the region of the state space where uncertainty remains, and $R \in (1, 1/(1-h))$.[9]

A larger AP program $\alpha_1 > \alpha_0$ has the following effects

- *it expands the set of states where uncertainty is resolved, and shrinks its complement:*

$$\mathcal{Z}^n(\alpha_0) \subset \mathcal{Z}^n(\alpha_1) \quad \text{and} \quad \mathcal{Z}^u(\alpha_0) \supset \mathcal{Z}^u(\alpha_1).$$

- *it increases the equilibrium interest rate R for $\mathcal{Z}^n(\alpha_1) \setminus \mathcal{Z}^n(\alpha_0)$;*
- *it reduces the equilibrium interest rate R for $\mathcal{Z}^u(\alpha_1)$;*
- *lastly, it leaves the equilibrium interest rate R unchanged at $R = 1/(1-h)$ for $\mathcal{Z}^n(\alpha_0)$.*

Proof. In the appendix (at this link). ■

While the equilibrium interest rate is a variable of interest, what is ultimately relevant for tax distortions and welfare are the mean and variance (when distortions are quadratic) of the real interest rates and government debt repayments, which we analyse next.

[8]The difference between the "market" and "external" belief distributions and its implications for asset payoffs are carefully analysed in Albagli et al. (2021).

[9]Recall that if $\alpha_0 = 0$ then $\mathcal{Z}^n(\alpha_0) = \emptyset$ and $\mathcal{Z}^u(\alpha_0) = \mathbb{R}^2$.

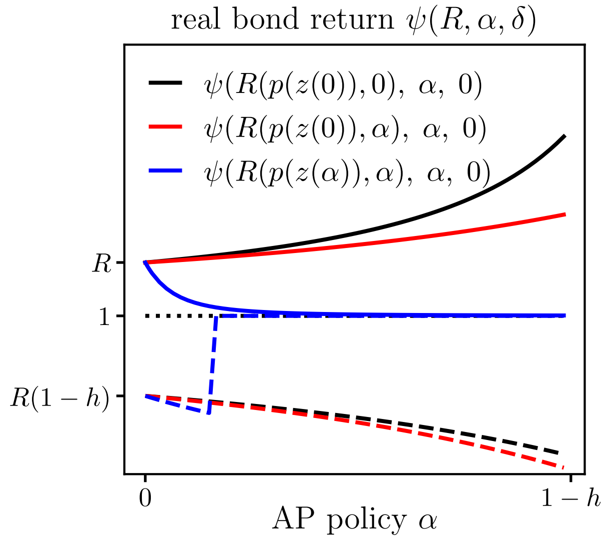

Figure 3: Real tax rate $\tau(R, \alpha, \delta)$ as a function of the AP policy α.

Figure 3 shows an example of the effect of APs on real bond returns and real debt service ψ, which we defined earlier, and in turn on the tax rate τ. The solid and dashed curves respectively denote real debt service in repayment and default.[10] To construct the figure, we pick two pairs (θ^H, S^H) and (θ^L, S^L) that result in the same market signal in the absence of APs (that is, $z(\theta^H, S^H, 0) = z(\theta^L, S^L, 0)$). At $\alpha = 0$ net inflation is equal to $1/\rho$ regardless of default, so real debt service is simply given by the nominal interest rate. All curves plot real debt service as a function of the AP policy α. The dotted flat line illustrates the value of real debt service under perfect foresight, when $\psi(R, \alpha, \delta) = R(1 - \delta h) = 1$. The black curves are computed keeping the equilibrium interest rate constant at its $\alpha = 0$ level. This isolates the direct effect of APs on real debt service via inflation and deflation, abstracting from any equilibrium effect coming from R. The red curves are computed keeping repayment probabilities constant. This adds to the black curves the additional equilibrium effect of APs on R via the effect of inflation and deflation on real bond returns and the no arbitrage condition. Finally, the blue curves show the actual real debt service, including the third effect of α on ψ via its equilibrium impact on R through the marginal trader's subjective repayment probabilities. Note that, in case of default, there is the further effect of APs on repayment probabilities that makes the blue dashed curve jump up to one, as APs make the market signal fully reveal the default fundamental.

As the figure shows clearly, the effect of APs on the marginal agent identity is crucial for the

[10]The figure legend only displays curves upon repayment for simplicity.

dynamics of real bond returns and taxes. When α is not too large, APs reduce the tax burden in *both* default and repayment states, which unambiguously increases welfare by reducing tax distortions. Note that this is not true for the case where we keep p fixed, where APs only increase the volatility of tax distortions and reduce welfare. This is the result we state in Proposition 2 and is highlighted by the red curves in the figure. When instead α is large enough, APs fully reveal the default state and take away the beneficial effect of unanticipated default and inflation on taxes. The following lemma formalises this argument.

Lemma 3. *Under certain parametric assumptions, the effect of APs on real debt service has the same sign of that on the nominal interest rate R described in Proposition 3.*

Proof(to be completed). So far, we can prove that this is true for any parametrisation for $\alpha = 0$ and $\alpha \to 1 - h$. ■

Ex-ante welfare. We have provided some intuition on the effects of APs on real debt service using a specific value of the market signal as an example. We now move to evaluate the welfare effect of APs from an ex ante perspective, that is, integrating over all possible combinations of the fundamental and supply shocks.

Figure 4 illustrates some of the relevant equilibrium variables as a function of the supply shock S (on the horizontal axis), the fundamental shock θ (solid and dashed curves again correspond to θ^H and θ^L respectively) and the AP policy α (blue curves correspond to $\alpha_0 = 0$, orange curves correspond to $\alpha_1 > 0$, and green curves correspond to $\alpha_2 > \alpha_1$). The top left panel plots the market signal $z(\theta, S, \alpha)$ and is the analogous of Figure 2. The horizontal dashed lines again denote the full revelation cutoffs $\underline{z}(\theta^H, 1, \alpha)$. The bottom left panel plots the probability of repayment of the marginal agent. In repayment states, APs unambiguously increase p, making the price a more accurate reflection of the fundamental. In default APs increase p and thus make the price less accurate for low supply shocks, and fully reveal the default state otherwise. The top right panels plots the equilibrium nominal interest rate. As explained earlier, the behaviour of repayment probabilities are a crucial driver of R, so it not surprising that p and R move in a very similar way. Lastly, the bottom right panel illustrates tax distortions, which for this example are assumed to have the functional form $\zeta(\tau) = \tau^2$. When $\theta = \theta^H$, APs are beneficial because they reduce real debt service, as we pointed out earlier with Figure 3. When $\theta = \theta^L$ instead we can see clearly how the information effect of APs reduces welfare. When agents cannot infer the state with certainty, $R < 1/(1-h)$ and default generates inflation. Since inflation is not costly per se and reduces real debt service, this reduces tax distortions down to a level which is below

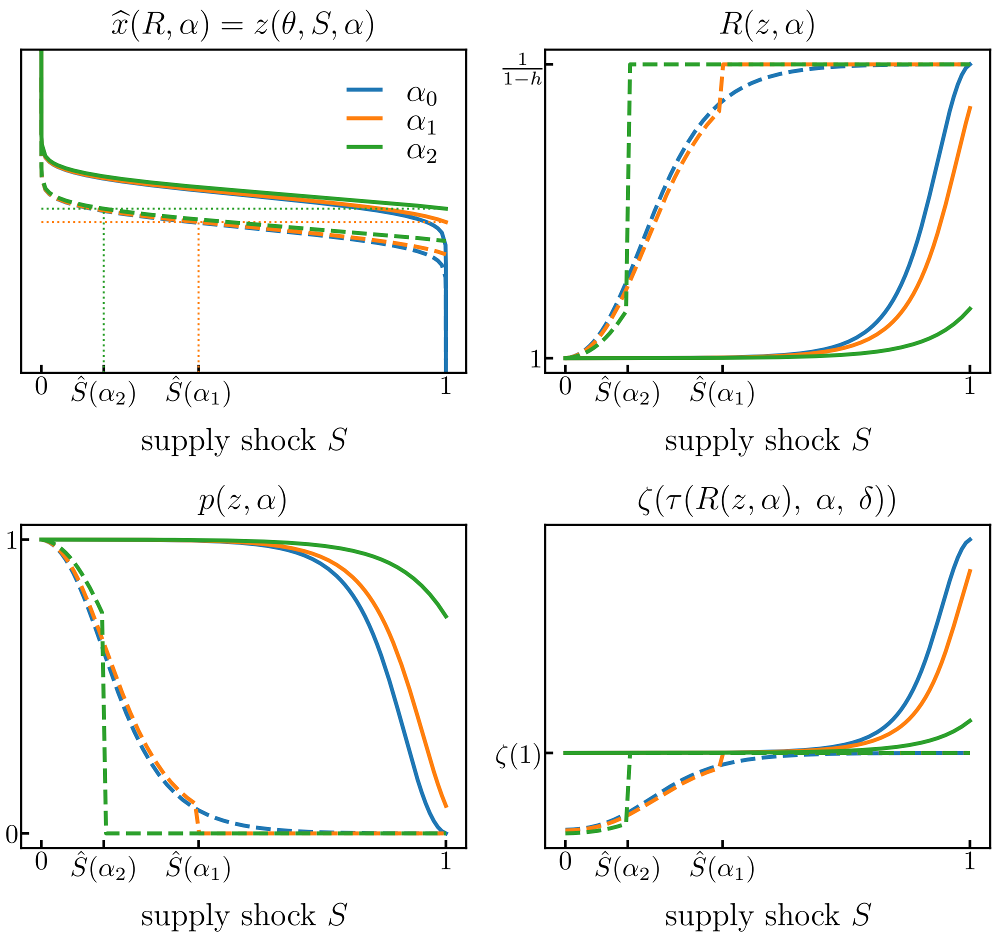

Figure 4: Equilibrium variables as a function of supply shock S (on the horizontal axis), fundamental shock θ (solid and dashed curves correspond to θ^H and θ^L respectively) and the AP policy α (blue, orange and green curves correspond to $\alpha_0 = 0$, $\alpha_1 > 0$ and $\alpha_2 > \alpha_1$ respectively). To lighten up notation we omit the arguments of z in the titles of all but the top left panel. The example assumes a tax distortion function $\zeta(\tau) = \tau^2$.

the perfect foresight value of $\zeta(1)$. When instead uncertainty is resolved, debt is priced correctly, inflation is anchored and tax distortions take their perfect foresight value.

The ex-ante optimal AP policy thus trades off the benefits of APs under repayment, with the net impact of APs upon default (which is generally welfare reducing, although it may not be so).

Mathematically the expected welfare loss is given by

$$\mathbb{E}[\mathcal{L}(\theta, S, \alpha)] = q \int_0^1 \zeta\Big(\tau(R(\theta^H, s, \alpha), \alpha, 0), \Big) dS$$
$$(1-q)\left\{[1-\widehat{S}(\alpha)]\zeta(1) + \int_0^{S(\alpha)} \zeta\Big(\psi^d(R(\theta^L, s, \alpha), \alpha, 1), \Big) dS\right\}.$$

The first and second rows correspond to the expected tax distortions conditional on θ^H and θ^L respectively. The second row consists of two components because we are separating the region where default is perfectly anticipated and real debt service is equal to one (the first term) from the region where uncertainty still remains (the second term). We now enunciate our main result.

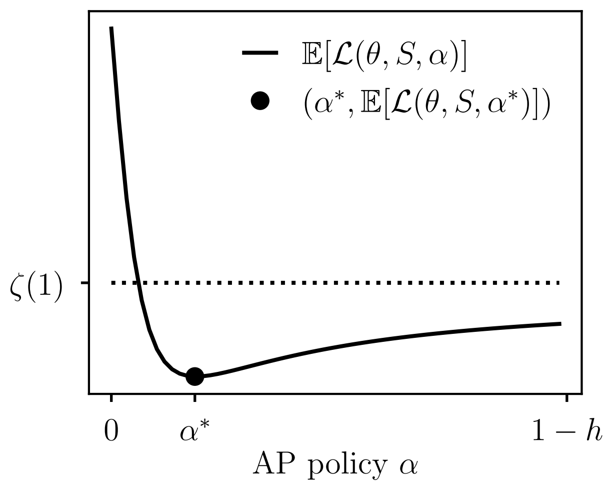

Figure 5: Ex-ante expected tax distortions as a function of the asset purchase policy α.

Proposition 4. *The AP policy that is optimal ex ante is non-zero and bounded. Mathematically, let α^* be the ex-ante optimal AP policy. Then*

$$\alpha^* \in (0, 1-h).$$

Proof (sketch, to be completed). We aim to prove that the optimal AP policy is non-zero and bounded. We show that the derivative of the expected welfare loss with respect to α is negative at $\alpha = 0$, and eventually becomes positive as α grows. First, we perform a full proof. Second, we employ a first-order Taylor expansion around the curvature parameter of the distortion function $\zeta(\cdot)$, to highlight the role of distortion convexity in the result. ■

Figure 5 illustrates the proposition with a numerical example. APs can indeed improve upon the perfect foresight benchmark displayed by the horizontal dotted line. In particular, the marginal effect of APs on welfare evaluated at $\alpha = 0$ is strong, which allows us to say that *some* central bank asset purchases are always optimal.

Appendix A Proofs and Derivations

A.1 Derivation of Posterior Beliefs

We know

$$z = \theta - \nu.$$

The error term $\nu \equiv \sigma_x \Phi^{-1}\left(\frac{S}{d(\alpha)}\right)$ follows a truncated normal with upper bound $\bar{\nu}(\alpha) \equiv \sigma_x \Phi^{-1}\left(\frac{1}{d(\alpha)}\right)$. Its *actual* pdf (adjusted by the truncation term) is

$$f_\nu(y; \bar{\nu}) = \frac{1}{\sigma_x \sqrt{2\pi}} \frac{\exp\left\{-\frac{1}{2}\left(\frac{y}{\sigma_x}\right)^2\right\}}{\Phi\left(\frac{\bar{\nu}}{\sigma_x}\right)} \quad \text{for } y < \bar{\nu}, \text{ and } 0 \text{ otherwise.} \tag{27}$$

which is consistent with equation (36). The price signal also has a truncated normal conditional density:

$$f_{z|\theta}(y \mid \theta) = f_\nu(\theta - y) \quad \text{for } y > \theta - \bar{\nu}, \text{ and } 0 \text{ otherwise.}$$

The "prior" we now wish to update is

$$\tilde{f}(\theta) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp\left\{-\frac{1}{2}\left(\frac{\theta - \tilde{\mu}}{\tilde{\sigma}}\right)^2\right\} \quad \text{for } \theta \in \mathbb{R}.$$

which encompasses two cases:

- we want to update $\theta \mid x$, in which case $\begin{cases} \tilde{\mu} = \frac{\tau_\theta \mu_\theta + \tau_x x}{\tau_\theta + \tau_x} \\ \tilde{\sigma}^2 = \frac{\sigma_x^2 + \sigma_\theta^2}{\sigma_x^2 \sigma_\theta^2}. \end{cases}$ This is the relevant case for investors with private information.
- we want to update the prior of θ, in which case $\begin{cases} \tilde{\mu} = \mu_\theta \\ \tilde{\sigma}^2 = \sigma_\theta^2. \end{cases}$ This is the relevant case when deriving the marginal distribution of z when we compute ex-ante welfare.

We need to use Bayes' rule:

$$f(\theta \mid z(, x)) = \frac{f(z \mid \theta) f(\theta(\mid x))}{f(z)} \quad \text{for } \theta < z + \bar{\nu}, \text{ and } 0 \text{ otherwise.}$$

open up the densities

$$f\left(\theta|z(,x)\right)=f\left(z\right)^{-1}\times\frac{1}{\Phi(\bar{\nu}/\sigma_x)}\frac{1}{\sigma_x}\frac{e^{-\frac{1}{2}\left(\frac{\theta-z}{\sigma_x}\right)^2}}{\sqrt{2\pi}}\times\frac{1}{\tilde{\sigma}}\frac{e^{-\frac{1}{2}\left(\frac{\theta-\tilde{\mu}}{\tilde{\sigma}}\right)^2}}{\sqrt{2\pi}}=$$

second step

$$=f\left(z\right)^{-1}\frac{1}{2\pi}\frac{1}{\sigma_x\tilde{\sigma}}e^{-\frac{1}{2}\left(\left(\frac{\theta-z}{\tilde{\sigma}}\right)^2+\left(\frac{\theta-\tilde{\mu}}{\sigma_x}\right)^2\right)}\frac{1}{\Phi(\bar{\nu}/\sigma_x)}=$$

third step

$$=f\left(z\right)^{-1}\frac{1}{2\pi}\frac{1}{\sigma_x\tilde{\sigma}}\exp\left\{-\frac{1}{2}\left(\frac{(\theta^2+z^2-2\theta z)\tilde{\sigma}^2+(\theta^2+\tilde{\mu}^2-2\theta\tilde{\mu})\sigma_x^2}{\sigma_x^2\tilde{\sigma}^2}\right)\right\}\frac{1}{\Phi(\bar{\nu}/\sigma_x)}.$$

The stuff inside the exponential

$$\theta^2\frac{\sigma_x^2+\tilde{\sigma}^2}{\sigma_x^2\tilde{\sigma}^2}-2\theta\frac{(z\tilde{\sigma}^2+\tilde{\mu}\sigma_x^2)}{\sigma_x^2\tilde{\sigma}^2}+\frac{z^2\tilde{\sigma}^2+\tilde{\mu}^2\sigma_x^2}{\sigma_x^2\tilde{\sigma}^2}$$

manipulate it

$$\frac{\sigma_x^2+\tilde{\sigma}^2}{\sigma_x^2\tilde{\sigma}^2}\left[\theta^2-2\theta\frac{(z\tilde{\sigma}^2+\tilde{\mu}\sigma_x^2)}{\sigma_x^2+\tilde{\sigma}^2}+\left(\frac{z\tilde{\sigma}^2+\tilde{\mu}\sigma_x^2}{\sigma_x^2+\tilde{\sigma}^2}\right)^2-\left(\frac{z\tilde{\sigma}^2+\tilde{\mu}\sigma_x^2}{\sigma_x^2+\tilde{\sigma}^2}\right)^2+\frac{z^2\tilde{\sigma}^2+\tilde{\mu}^2\sigma_x^2}{\sigma_x^2+\tilde{\sigma}^2}\right]$$

collect terms

$$\frac{\sigma_x^2+\tilde{\sigma}^2}{\sigma_x^2\tilde{\sigma}^2}\left(\theta-\frac{(z\tilde{\sigma}^2+\tilde{\mu}\sigma_x^2)}{\sigma_x^2+\tilde{\sigma}^2}\right)^2+\frac{(z-\tilde{\mu})^2}{\sigma_x^2+\tilde{\sigma}^2}$$

put τ's in

$$(\tilde{\tau}+\tau_x)\left(\theta-\frac{z\tau_x+\tilde{\mu}\tilde{\tau})}{\tilde{\tau}+\tau_x}\right)^2+\frac{\tau_x\tilde{\tau}}{\tau_x+\tilde{\tau}}(z-\tilde{\mu})^2$$

Go back

$$=f\left(z\right)^{-1}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_x\tilde{\sigma}}\frac{1}{\Phi(\bar{\nu}/\sigma_x)}e^{-\frac{1}{2}\left(\frac{\theta-\frac{z\tau_x+\tilde{\mu}\tilde{\tau})}{\tilde{\tau}+\tau_x}}{1/\sqrt{\tilde{\tau}+\tau_x}}\right)^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{z-\tilde{\mu}}{\sqrt{\frac{\tau_x+\tilde{\tau}}{\tau_x\tilde{\tau}}}}\right)^2}$$

make it ready for integration

$$f(\theta\mid z)=\frac{f\left(z\right)^{-1}}{\Phi(\bar{\nu}/\sigma_x)}\times\frac{1}{\sqrt{2\pi}}\sqrt{\tau_x+\tilde{\tau}}e^{-\frac{1}{2}\left(\frac{\theta-\frac{z\tau_x+\tilde{\mu}\tilde{\tau})}{\tilde{\tau}+\tau_x}}{1/\sqrt{\tilde{\tau}+\tau_x}}\right)^2}\times\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\tau_x\tilde{\tau}}{\tilde{\tau}+\tau_x}}e^{-\frac{1}{2}\left(\frac{z-\tilde{\mu}}{\sqrt{\frac{\tau_x+\tilde{\tau}}{\tau_x\tilde{\tau}}}}\right)^2}$$

Then we integrate over θ, remembering that $Supp(\theta \mid z) = (-\infty, z + \bar{\nu})$:

$$\int_{-\infty}^{z+\bar{\nu}} f(\theta \mid z) d\theta = \frac{f\,(z)^{-1}}{\Phi(\bar{\nu}/\sigma_x)} \times \int_{-\infty}^{z+\bar{\nu}} \frac{1}{\sqrt{2\pi}} \sqrt{\tau_x + \tilde{\tau}} e^{-\frac{1}{2}\left(\frac{\theta - \frac{z\tau_x + \tilde{\mu}\tilde{\tau})}{\tilde{\tau}+\tau_x}}{1/\sqrt{\tilde{\tau}+\tau_x}}\right)^2} d\theta \times \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\tau_x \tilde{\tau}}{\tilde{\tau} + \tau_x}} e^{-\frac{1}{2}\left(\frac{z-\tilde{\mu}}{\sqrt{\frac{\tau_x+\tilde{\tau}}{\tau_x\tilde{\tau}}}}\right)^2}$$

which yields

$$1 = \frac{f\,(z)^{-1}}{\Phi(\bar{\nu}/\sigma_x)} \times \Phi\left(\frac{\tilde{\tau}}{\sqrt{\tau_x + \tilde{\tau}}}(z - \tilde{\mu}) + \bar{\nu}\sqrt{\tau_x + \tilde{\tau}}\right) \times \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\tau_x \tilde{\tau}}{\tilde{\tau} + \tau_x}} e^{-\frac{1}{2}\left(\frac{z-\tilde{\mu}}{\sqrt{\frac{\tau_x+\tilde{\tau}}{\tau_x\tilde{\tau}}}}\right)^2}$$

where the LHS was set to 1 because the posterior density of $\theta|z$ must integrate to 1 over its whole support. Note that the integral on the RHS reduces to a standard normal CDF thanks to the change of variable $y = [\theta - (1 - w_e)\mu_\theta - w_e z]/\sigma_e$.

We can now back out the marginal density of z, reintroducing dependence on α where present

$$f(z) = \frac{\Phi\left(\frac{\tilde{\tau}}{\sqrt{\tau_x+\tilde{\tau}}}(z - \tilde{\mu}) + \bar{\nu}(\alpha)\sqrt{\tau_x + \tilde{\tau}}\right)}{\Phi(\bar{\nu}(\alpha)/\sigma_x)} \times \frac{1}{\sqrt{2\pi}\sqrt{\sigma_x^2 + \tilde{\sigma}^2}} e^{-\frac{1}{2}\left(\frac{z-\tilde{\mu}}{\sqrt{\sigma_x^2+\tilde{\sigma}^2}}\right)^2} \tag{28}$$

This marginal density is important for the derivations of ex-ante welfare starting in equation (**??**). Also, recall that the CDF at the numerator regards the upper bound of the distribution of θ conditional on z; while the CDF at the denominator regards the upper bound of the unconditional distribution of $\nu(\epsilon)$. On a separate PDF we prove, using the fact that $\int \Phi(a + bx)\phi(x)dx = \Phi\left(\frac{a}{\sqrt{1+b^2}}\right)$, that integrating this marginal density with respect to z over $(-\infty, +\infty)$ yields 1.

If we think of the marginal of z from the external observer's point of view, who does not take private signals into account, then

$$f(z) = \int^{z+\bar{\nu}} f(z \mid \theta) f_{prior}(\theta) d\theta$$

and going through the algebra we get the same as (28) where $\tilde{\mu} = \mu_\theta$ and $\tilde{\sigma} = \sigma_\theta$, i.e. we only rely on the prior distribution of θ.

We can then conclude by writing down the posterior density of $\theta \mid z(, x)$ for all $\theta < z + \bar{\nu}$:

$$f\,(\theta|z) = \frac{\sqrt{\tau_x + \tilde{\tau}}}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\left(\frac{\theta - \frac{z\tau_x + \tilde{\mu}\tilde{\tau})}{\tilde{\tau}+\tau_x}}{1/\sqrt{\tilde{\tau}+\tau_x}}\right)^2}}{\Phi(z + \bar{\nu})} \quad \text{for } \theta < z + \bar{\nu}, \text{ and 0 otherwise.}$$

Clearly, the unconditional variance of z is larger than the conditional variance of $\theta \mid z$:

$$\sigma_x^2 + \tilde{\sigma} > \frac{1}{\tilde{\tau} + \tau_x}.$$

GG – derivation

the truncated normal with upper bound b from above as the form

$$f(v; \mu, \sigma, b) = \frac{1}{\sigma} \frac{e^{-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}} \left(\int_{-\infty}^{\frac{b-\mu}{\sigma}} \frac{e^{-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}} dv \right)^{-1}$$

Given that

$$z = \theta - v$$

where

$$f(\theta) = \frac{1}{\sigma_\theta} \frac{e^{-\frac{1}{2}\left(\frac{\theta}{\sigma_\theta}\right)^2}}{\sqrt{2\pi}},$$

then

$$f\left(\theta|z\right) = \frac{f_{trunc}\left(z|\theta\right) f\left(\theta\right)}{f_{unknown}\left(z\right)} = \frac{f_{trunc}\left(v\right) f\left(\theta\right)}{f_{unknown}\left(z\right)}$$

$$
\begin{aligned}
f\left(\theta|z\right) &= f\left(z\right)^{-1}\frac{1}{\sigma}\frac{e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^{2}}}{\sqrt{2\pi}}\frac{1}{\sigma_{\theta}}\frac{e^{-\frac{1}{2}\left(\frac{\theta}{\sigma_{\theta}}\right)^{2}}}{\sqrt{2\pi}}\left(\int_{-\infty}^{\frac{b}{\sigma}}\frac{e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^{2}}}{\sqrt{2\pi}}dv\right)^{-1} = \\
&= f\left(z\right)^{-1}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sigma_{\theta}}\frac{e^{-\frac{1}{2}\left(\left(\frac{\theta}{\sigma_{\theta}}\right)^{2}+\left(\frac{v}{\sigma}\right)^{2}\right)}}{\sqrt{2\pi}}\left(\int_{-\infty}^{\frac{b}{\sigma}}\frac{e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^{2}}}{\sqrt{2\pi}}dv\right)^{-1} = \\
&= f\left(z\right)^{-1}\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sigma_{\theta}}\frac{e^{-\frac{1}{2}\left(\frac{v^{2}\sigma_{\theta}^{2}+\theta^{2}\sigma^{2}}{\sigma^{2}\sigma_{\theta}^{2}}\right)}}{\sqrt{2\pi}}\left(\int_{-\infty}^{\frac{b}{\sigma}}\frac{e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^{2}}}{\sqrt{2\pi}}dv\right)^{-1} \\
&= f\left(z\right)^{-1}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sigma^{2}\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}}\frac{1}{\sigma\sigma_{\theta}}\frac{\exp\left\{-\frac{1}{2}\left(\frac{\left(\theta-\frac{\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}z\right)^{2}}{\frac{\sigma^{2}\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}}\right)\right\}}{\sqrt{2\pi}\sqrt{\frac{\sigma^{2}\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}}}e^{-\frac{1}{2}}\left(\frac{z^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}\right)\left(\int_{-\infty}^{\frac{b}{\sigma}}\frac{e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^{2}}}{\sqrt{2\pi}}dv\right)^{-1} \\
&= f\left(z\right)^{-1}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{1}{\sigma^{2}+\sigma_{\theta}^{2}}}\frac{\exp\left\{-\frac{1}{2}\left(\frac{\left(\theta-\frac{\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}z\right)^{2}}{\frac{\sigma^{2}\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}}\right)\right\}}{\sqrt{2\pi}\sqrt{\frac{\sigma^{2}\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}}}e^{-\frac{1}{2}}\left(\frac{z^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}\right)\left(\int_{-\infty}^{\frac{b}{\sigma}}\frac{e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^{2}}}{\sqrt{2\pi}}dv\right)^{-1}
\end{aligned}
$$

Then I integrate over θ

$$
\begin{aligned}
\int_{-\infty}^{\bar{\theta}}f\left(\theta|z\right)d\theta &= f\left(z\right)^{-1}\frac{1}{\sqrt{2\pi}}\sqrt{\frac{1}{\sigma^{2}+\sigma_{\theta}^{2}}}\left[\int_{-\infty}^{\bar{\theta}}\frac{\exp\left\{-\frac{1}{2}\left(\frac{\left(\theta-\frac{\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}z\right)^{2}}{\frac{\sigma^{2}\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}}\right)\right\}}{\sqrt{2\pi}\sqrt{\frac{\sigma^{2}\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}}}d\theta\right]e^{-\frac{1}{2}}\left(\frac{z^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}\right)\left(\int_{-\infty}^{\frac{b}{\sigma}}\frac{e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^{2}}}{\sqrt{2\pi}}dv\right. \\
f\left(z\right) &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\sigma^{2}+\sigma_{\theta}^{2}}}e^{-\frac{1}{2}}\left(\frac{z^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}\right)F_{\theta|z}\left(\bar{\theta}\right)\left(\int_{-\infty}^{\frac{b}{\sigma}}\frac{e^{-\frac{1}{2}\left(\frac{v}{\sigma}\right)^{2}}}{\sqrt{2\pi}}dv\right)^{-1}
\end{aligned}
$$

and proove that z is distributed according to the same truncated normal.

$$
f\left(\theta|z\right)=\frac{\exp\left\{-\frac{1}{2}\left(\frac{\left(\theta-\frac{\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}z\right)^{2}}{\frac{\sigma^{2}\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}}\right)\right\}}{\sqrt{2\pi}\sqrt{\frac{\sigma^{2}\sigma_{\theta}^{2}}{\sigma^{2}+\sigma_{\theta}^{2}}}}F_{\theta|z}\left(\bar{\theta}\right)^{-1}
$$

We have shown that

$$
\begin{aligned}
f(\theta|z) &= \frac{f_{trunc}(z|\theta)\, f(\theta)}{f_{unknown}(z)} = \frac{f_{trunc}(v)\, f(\theta)}{f_{unknown}(z)} = \frac{\frac{f(v)f(\theta)}{f(z)}}{F_{\theta|z}\left(\bar{\theta}\right)} \\
f_{unknown}(z) &= \frac{\frac{f(v)}{F(\bar{v})} f(\theta)}{\frac{f(v)f(\theta)}{f(z)}} F_{\theta|z}\left(\bar{\theta}\right) = \frac{f(z)}{F(\bar{v})} \int_{-\infty}^{\bar{\theta}} \frac{f(z|\theta)\, f(\theta)}{f(z)} d\theta \\
&= \frac{\int_{-\infty}^{\bar{\theta}} f(z|\theta)\, f(\theta)\, d\theta}{F(\bar{v})} = \frac{F\left(\bar{\theta}\right)}{F\left(\bar{\theta} - z\right)} f(z) = \frac{F(z + \bar{v})}{F(\bar{v})} f(z)
\end{aligned}
$$

since

$$
\frac{\int_{-\infty}^{\bar{\theta}} f(z|\theta)\, f(\theta)\, d\theta}{\int_{-\infty}^{\bar{\theta}} f(\theta)\, d\theta} = \int_{-\infty}^{\bar{\theta}} f(z|\theta) \frac{f(\theta)}{F\left(\bar{\theta}\right)} d\theta = f(z)
$$

Then we note that

$$
\begin{aligned}
\frac{\partial \left(\frac{F(z+\bar{v})}{F(\bar{v})} f(z)\right)}{\partial \alpha} &= \bar{v}_{\alpha} \left(\frac{\phi(z + \bar{v})\, F(\bar{v}) - \phi(\bar{v})\, F(z + \bar{v})}{F(\bar{v})^2}\right) f(z) \\
&= \underbrace{\bar{v}_{\alpha}}_{<0} \frac{F(z + \bar{v})}{F(\bar{v})} \left(\frac{\phi(z + \bar{v})}{F(z + \bar{v})} - \frac{\phi(\bar{v})}{F(\bar{v})}\right) f(z)
\end{aligned}
$$

A.2 Price Level Existence and Determinacy

We examine how price level existence and determinacy restricts the value of α.

Assumption 1. *The budget constraint of the central bank at $t = 2$ is given by equation* (**??**) *Consider two cases:*

1. *Positive storage and bounded α, i.e. $\alpha \in \left[0, \frac{1}{R(1-\delta h)}\right)$.*

   In this case, the RHS of equation (14) must be positive. We can divide through and we get a unique solution for the inverse of inflation:

$$
\frac{1}{\Pi_2} = \rho \frac{1 - \alpha}{1 - \alpha R(1 - \delta h)}.
$$

- ***if APs are default-contingent**, then in repayment we need $\alpha < 1/R$, while in default any value of $\alpha \in [0, 1)$ is admissible (since $R < (1-h)^{-1}$ and in turn $\frac{1}{R(1-h)} >$*

1). A value of $\alpha = 1$ is admissible in principle but would entail zero storage, so we consider it in the following case.

- ***if APs are non-contingent**, then we need $\alpha < 1/R$ which is the tighter constraint. Since R can in principle take any value inside the $[1, 1/(1-h)]$ interval for $\theta \in \mathbb{R}$, then we need*

$$\alpha < 1 - h.$$

2. *Zero storage, i.e. $\alpha = 1$*

   In this case, all money issuance is destined to APs, and the real value of storage is zero. There are two possibilities:

   - *if $R(1 - \delta h) = 1$, inflation is indeterminate as both sides of equation (14) are zero regardless of the value of Π_2. **In this case, α and Π_2 are not one-to-one as they usually are.***
   - *if instead $R(1 - \delta h) \neq 1$, then inflation is $\Pi_2 \to +\infty$, as that is the only way to set the RHS of equation (14) to zero.*

A.3 Propositions Proofs

Proof of Proposition 2. The effect of α cannot be considered independently of the no-arbitrage condition. Rewrite the no arbitrage condition in a way that is useful for the implicit function theorem

$$p \underbrace{\frac{1-\alpha}{\frac{1}{R} - \alpha}}_{\psi^r} + (1-p) \underbrace{\frac{1-\alpha}{\frac{1}{R(1-h)} - \alpha}}_{\psi^d} - 1 = 0 \tag{29}$$

where ψ^r, ψ^d are shorthand notation for real fiscal needs in repayment and default.

Then the effects of α on R are given by

$$R_\alpha \equiv \frac{dR}{d\alpha} = -\frac{H_\alpha}{H_R} = -\frac{p\psi^r_\alpha + (1-p)\psi^d_\alpha}{p\psi^r_R + (1-p)\psi^d_R} < 0 \tag{30}$$

where ψ^i_j denotes the partial derivative of fiscal needs with respect to variable j under repayment $i = r$ or default $i = d$. We can prove that this derivative is negative. First, we know that $H_R > 0$ since both $\psi^r_R > 0$ and $\psi^d_R > 0$. Second, we show that $H_\alpha > 0$. To do so, note that $\psi^r > 1 > \psi^d$

and let us rearrange the no-arbitrage condition as

$$p = \frac{1-\psi^d}{\psi^r - \psi^d}.$$

The derivation is the following

$$\begin{aligned} H_\alpha = & p\psi^r_\alpha + (1-p)\psi^d_\alpha > 0 \\ & \psi^d_\alpha + \frac{1-\psi^d}{\psi^r - \psi^d}(\psi^r_\alpha - \psi^d_\alpha) > 0 \end{aligned}$$

we multiply for $\psi^r - \psi^d > 0$ to get an equivalent condition

$$\begin{aligned} \psi^r_\alpha(1-\psi^d) + \psi^d_\alpha(\psi^r - 1) > 0 \\ \frac{\psi^r_\alpha}{-\psi^d_\alpha}\frac{1-\psi^d}{\psi^r - 1} > 1 \end{aligned}$$

The effects of α_0, α_1 (more shorthand notation for when looking at the effect of α differentiating between the case $\delta = 0$ or $\delta = 1$) on R are given by

$$R_{(\alpha_0)} \equiv \frac{dR}{d\alpha_0} = -\frac{H_{\alpha_0}}{H_R} = -\frac{p\psi^r_{(\alpha_0)}}{p\psi^r_R + (1-p)\psi^d_R} < 0 \tag{31}$$

$$R_{(\alpha_1)} \equiv \frac{dR}{d\alpha_1} = -\frac{H_{\alpha_1}}{H_R} = -\frac{(1-p)\psi^d_{(\alpha_1)}}{p\psi^r_R + (1-p)\psi^d_R} > 0. \tag{32}$$

since:

$$\psi^r_R = \frac{1-\alpha_0}{\left(\frac{1}{R} - \alpha_0\right)^2} R^{-2} > 0; \qquad \psi^d_R = \frac{1-\alpha_1}{\left(\frac{1}{R(1-h)} - \alpha_1\right)^2}\frac{R^{-2}}{1-h} > 0 \tag{33}$$

$$\psi^r_{(\alpha_0)} = \frac{1-\frac{1}{R}}{\left(\frac{1}{R} - \alpha_0\right)^2} > 0 \qquad \psi^d_{(\alpha_1)} = \frac{1-\frac{1}{R(1-h)}}{\left(\frac{1}{R(1-h)} - \alpha_1\right)^2} < 0 \tag{34}$$

with $\psi^\alpha > \psi^r_R > \psi^d_R > 0 > \psi^\alpha_R$ for any $\alpha < 1/R$.

Applying the formulas of equation (33) we get

$$\begin{aligned}\frac{R(R-1)}{(1-\alpha R)^2} \times \frac{[1-\alpha R(1-h)]^2}{R(1-h)[1-R(1-h)]} \times \frac{1-R(1-h)}{1-\alpha R(1-h)} \times \frac{1-\alpha R}{R-1} &> 1 \\ \frac{1-\alpha R(1-h)}{(1-h)(1-\alpha R)} &> 1 \\ 1 &> 1-h.\end{aligned}$$

The expected distortion is given by

$$ED \equiv p\zeta\left(\psi^r\right) + (1-p)\zeta\left(\psi^d\right). \tag{35}$$

The marginal change with respect to α is given by

$$\begin{aligned}\frac{dED}{d\alpha} &= \zeta_r' p\left[\psi_\alpha^r + \psi_R^r R_\alpha\right] + \zeta_d'(1-p)\left[\psi_\alpha^d + \psi_R^d R_\alpha\right] \\ &= R_\alpha\left[\zeta_r' p\psi_R^r + \zeta_d'(1-p)\psi_R^d\right] + \zeta_r' p\psi_\alpha^r + \zeta_d'(1-p)\psi_\alpha^d \\ &> R_\alpha\zeta_r'\left[p\psi_R^r + (1-p)\psi_R^d\right] + \zeta_r' p\psi_\alpha^r + \zeta_d'(1-p)\psi_\alpha^d \\ &= -(1-p)\psi_\alpha^d(\zeta_r' - \zeta_d') > 0\end{aligned}$$

This shows that $\alpha = 0$ is optimal for any R. As a result $R(\alpha = 0) = 1/(1-h(1-p))$ is the unique interest rate satisfying the arbitrage condition.

■

Proof of Lemma 2. Note that we can write

$$\begin{aligned}F_\nu(y) &= P\left(\sigma_x\Phi^{-1}\left(\frac{S}{d(\alpha)}\right) < y\right) = P\Big(S < d(\alpha)\Phi(y/\sigma_x)\Big) = \\ &= \begin{cases}\frac{d(\alpha)\Phi(y/\sigma_x)-0}{d(\alpha)\Phi(y_{max}/\sigma_x)-0} = \frac{\Phi(y/\sigma_x)}{\Phi(\overline{\nu}(\alpha)/\sigma_x)} & \text{for } y \le y_{max} = \overline{\nu}(\alpha) \\ 1 & \text{for } y > \overline{\nu}(\alpha)\end{cases}\end{aligned}$$

where $\bar{\nu}(\alpha) \equiv \sigma_x\Phi^{-1}\left(\frac{1}{d(\alpha)}\right)$ and therefore $\frac{1}{\Phi(\overline{\nu}(\alpha)/\sigma_x)} = d(\alpha)$. The market signal noise density is then given by

$$f_\nu(y) = \frac{1}{\sigma_x}\frac{\phi(y/\sigma_x)}{\Phi(\overline{\nu}/\sigma_x)}. \tag{36}$$

We use this density in equation (36) to derive posterior beliefs. The distribution moments follow from standard properties of a truncated standard normal distribution. First, note that $\lambda(x)$ is a version of the standard normal inverse Mills ratio (specifically, $\lambda(-x) = \phi(x)/[1 - \Phi(x)]$ which is the inverse Mills ratio).

That $\lambda'(x) > -1$ can be proved using the fact that the variance of ν is $\sigma_\nu^2(\overline{\nu}) = 1 + \lambda(\overline{\nu}) > 0$ for any right tail truncation point $\overline{\nu}$. $\lambda'(x) < 0$ is a well-known result: it is trivial for $x > 0$, and can be proved using Markov's and Chebychev's inequalities for $x < 0$. The derivative of ν with respect to α is given by

$$\nu_\alpha(\epsilon, \alpha) = -\frac{\sigma_x S(e_1 - 1)}{\phi\Big(\nu(\epsilon, \alpha)/\sigma_x\Big)[1 + \alpha(e_1 - 1)]^2}. \tag{37}$$

The derivative of the truncation point $\overline{\nu}$ with respect to α is given by

$$\overline{\nu}'(\alpha) = -\frac{\sigma_x(e_1 - 1)}{\phi\Big(\overline{\nu}(\alpha)/\sigma_x\Big)[1 + \alpha(e_1 - 1)]^2}. \tag{38}$$

To show that the expected value of ν is decreasing in α, simply note that $\mu_\nu'(\alpha) = -\lambda'\Big(\overline{\nu}(\alpha)\Big)\,\overline{\nu}'(\alpha) < 0$. The result that $\sigma_\nu^2(\alpha) < 1$ follows from the fact that $\lambda'(x) \in (-1, 0)$. The result that $\sigma_\nu^2(\alpha)$ is decreasing in α is rather convoluted so we omit it for brevity. ■

Proposition 5. *Let $[\widehat{S}(\alpha), 1]$ denote the interval where S fully reveals θ^L. The width of such interval is increasing in the AP policy α, since $\frac{\partial \widehat{S}}{\partial \alpha} < 0$.*

Proof. Given some private signal cumulative distribution function F, the cutoff supply shock $\widehat{S}$ is defined as

$$\theta^L - \sigma_x F^{-1}\left(\frac{\widehat{s}}{d(\alpha)}\right) = \theta^H - \sigma_x F^{-1}\left(\frac{1}{d(\alpha)}\right).$$

Rearrange for $\widehat{s}$

$$\widehat{S} = d(\alpha) F\left(F^{-1}\left(\frac{1}{d(\alpha)}\right) - \frac{\theta^H - \theta^L}{\sigma_x}\right).$$

Derive $\widehat{S}$ with respect to α

$$\frac{\partial \widehat{S}}{\partial \alpha} = d'(\alpha) F\left(F^{-1}\left(\frac{1}{d(\alpha)}\right) - \frac{\theta^H - \theta^L}{\sigma_x}\right) - \frac{d'(\alpha)}{d(\alpha)^2}\frac{f\left(F^{-1}\left(\frac{1}{d(\alpha)}\right) - \frac{\theta^H - \theta^L}{\sigma_x}\right)}{f\left(\Phi^{-1}\left(\frac{1}{d(\alpha)}\right)\right)} < 0$$

which is equivalent to

$$\frac{f(x)}{F(x)} < \frac{f\left(x - \frac{\theta^H - \theta^L}{\sigma_x}\right)}{F\left(x - \frac{\theta^H - \theta^L}{\sigma_x}\right)}$$

where $x = F^{-1}\left(\frac{1}{d(\alpha)}\right)$. It is well known that if $f\left(x - \frac{\theta^H - \theta^L}{\sigma_x}\right)$ has the monotone likelihood ratio property with respect to $f(x)$, then f has a monotone decreasing reverse hazard rate function. ■

Proof of Proposition 3. To show that $\mathcal{Z}^n(\alpha_0) \subset \mathcal{Z}^n(\alpha_1)$, note that the set of fully revealing states is given by $\{\theta^L\} \times (\widehat{S}(\alpha), 1]$, where $\widehat{S}(\alpha)$ is decreasing in α as per Proposition 5. In these states, the probability of repayment goes from some $p(z, \alpha) \in (0, 1)$ to $p(z, \alpha) = 0$, and it is straightforward to show that $\frac{\partial R}{\partial p} < 0$.

To show that APs reduce R in $\mathcal{Z}^u(\alpha_1)$, we need to show that $\frac{dR}{d\alpha} < 0$ when $p(z, \alpha) \in (0, 1)$. The total derivative of R with respect to α is given by

$$\frac{dR}{d\alpha} = \frac{\partial R}{\partial \alpha} + \frac{\partial R}{\partial p}\frac{\partial p}{\partial z}\frac{\partial z}{\partial \alpha}.$$

We know that $\frac{\partial R}{\partial \alpha} < 0$ from the previous results in (30). We then use the implicit function theorem to show that $\frac{\partial z}{\partial \alpha} > 0$

$$\frac{\partial z}{\partial \alpha} = \sigma_x \frac{d'(\alpha)}{d(\alpha)^2} \frac{S}{\phi\left(\Phi\left(\frac{S}{d(\alpha)}\right)\right)} > 0,$$

and $\frac{\partial p}{\partial z} > 0$

$$\frac{\partial p(z, \alpha)}{\partial z} = \frac{\frac{q}{\sigma_{post}}\phi\left(\frac{\theta^H - z}{\sigma_{post}}\right)\frac{1-q}{\sigma_{post}}\phi\left(\frac{\theta^L - z}{\sigma_{post}}\right)\frac{\theta^H - \theta^L}{\sigma_{post}}}{\left[\frac{q}{\sigma_{post}}\phi\left(\frac{\theta^H - z}{\sigma_{post}}\right) + \frac{1-q}{\sigma_{post}}\phi\left(\frac{\theta^L - z}{\sigma_{post}}\right)\right]^2} = p(z, \alpha)\left(1 - p(z, \alpha)\right)\frac{\theta^H - \theta^L}{\sigma_{post}} > 0.$$

Finally, it is straightforward to prove that $\frac{\partial R}{\partial p} < 0$, which concludes the proof. ■

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