

# Groups of Automata

Carlo Lanzi Luciani

Mentors:  
izr. prof. Ganna Kudryavtseva  
prof. Alessandro Logar

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# 1. Introduction

The word **automaton**: from the greek "acting of one's own will".  
Automata are important in:

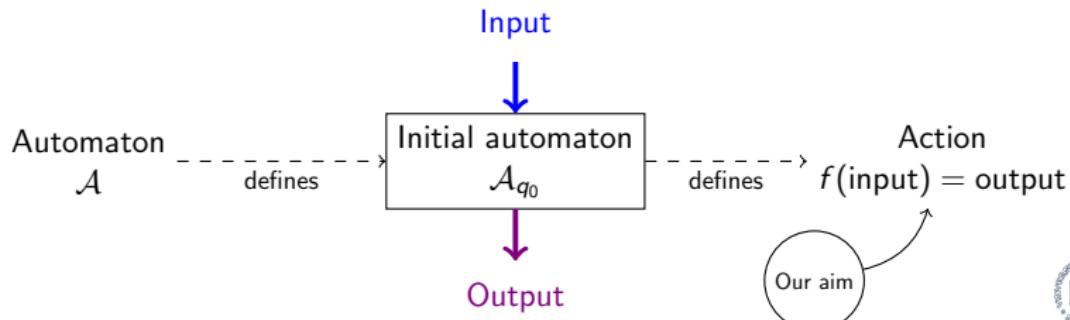
- ▶ Information theory
- ▶ Theory of dynamical systems
- ▶ Algebra
- ▶ Others

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- ▶ Theory of dynamical systems
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- ▶ Others

My aim: study some of the groups constructed through a special class of them, the invertible deterministic Mealy automata, here called automata.



## 2. The automaton

### Definition

An **automaton** is a 4-tuple  $\mathcal{A} = \langle X, Q, \pi, \lambda \rangle$  where:

- ▶  $X = \{x_1, \dots, x_k\}$  is a finite set called the **alphabet**,
- ▶  $Q$  is a set called the **set of internal states of the automaton**,
- ▶  $\pi : X \times Q \rightarrow Q$  is called the **transition function**,
- ▶  $\lambda : X \times Q \rightarrow X$  is such that  $\lambda_q = \lambda(\cdot, q) : X \rightarrow X$  is bijective, and is called the **output function**.

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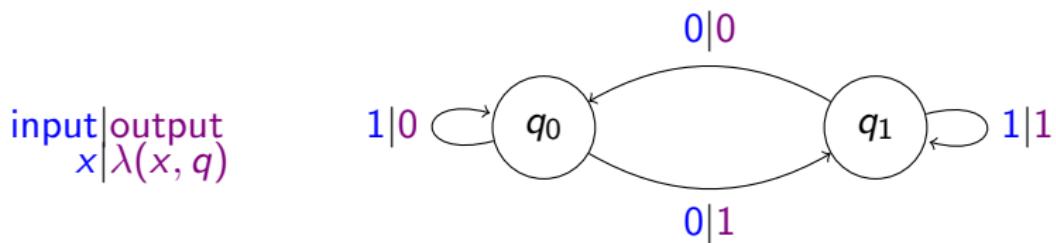


Figure: Moore diagram of a 2-state automaton over  $X = \{0, 1\}$

### 3. The initial automaton

#### Definition

$X^* = \{x_1 x_2 \dots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$  the **dictionary**.

Word composition:  $x_1 \dots x_n.z_1 \dots z_n := x_1 \dots x_n z_1 \dots z_n$

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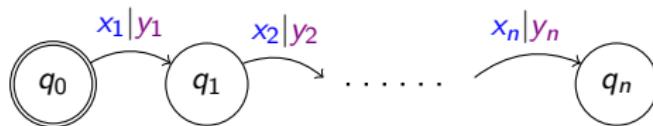
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#### Definition

An **initial automaton**  $\mathcal{A}_{q_0}$  is an automaton  $\mathcal{A}$  with a fixed state  $q_0$ .

The **action of**  $\mathcal{A}_{q_0}$  is the function  $\bar{\lambda}_{q_0} : X^* \rightarrow X^*$  with

$$\bar{\lambda}_{q_0}(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n.$$



#### 4. The word tree $X^*$

##### Definition

$\underline{w} = w_1 \dots w_n$  is a child of  $\underline{v} = v_1 \dots v_m$

$\iff$

$w_1 \dots w_n = v_1 \dots v_m x$  for some letter  $x \in X$ .

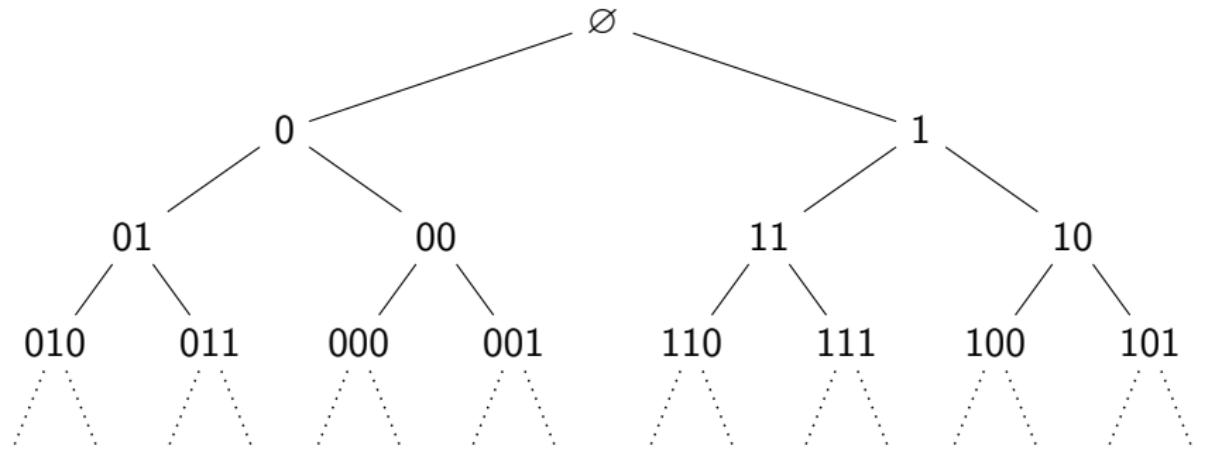


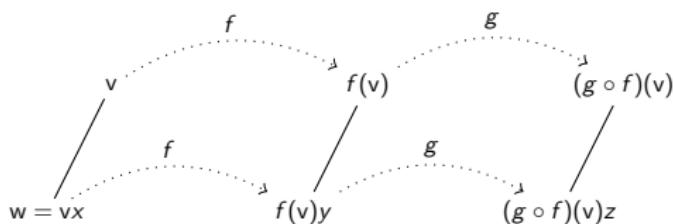
Figure: An example of the word tree  $X^*$  on  $X = \{0, 1\}$ .

## 5. Actions as tree-automorphisms

### Proposition

A function  $f : X^* \rightarrow X^*$  is the action of some initial automaton if and only if it is a **tree-automorphism** on the word tree  $X^*$ , i.e.:

- ▶  $f(\emptyset) = \emptyset$ .
- ▶ if  $\underline{w} \in X^*$  is a child of  $\underline{v}$  then  $f(\underline{w})$  is a child of  $f(\underline{v})$ .
- ▶  $f$  is bijective.



## 6. Groups defined by automata

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The functions  $f : X^* \rightarrow X^*$  defined by initial automata (i.e. tree-automorphisms) form a group denoted by  $\mathcal{AUT}_{tree}(X^*)$ .

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### Definition

Let  $\mathcal{A} = \langle X, Q, \pi, \lambda \rangle$  be an automaton. The **group defined by  $\mathcal{A}$**  is the group generated by the set  $\{\bar{\lambda}_q : q \in Q\}$ .

$$\begin{array}{ccc} \mathcal{A} \text{ with } & & \bar{\lambda}_{q_1} : X^* \rightarrow X^* \\ Q = \{q_1, \dots, q_n\} & \xrightarrow{\text{----- defines -----}} & \bar{\lambda}_{q_2} : X^* \rightarrow X^* \\ & & \vdots \\ & & \bar{\lambda}_{q_n} : X^* \rightarrow X^* \end{array} \xrightarrow{\text{----- generate -----}} \langle \{\bar{\lambda}_q : q \in Q\} \rangle$$

## 7. Wreath product and automata

### Definition

Let  $\mathcal{S}(X)$  be the symmetric group on  $X = \{x_1, \dots, x_k\}$ . Then the wreath product  $\mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*)$  is the group  $(\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *)$  where the multiplication rule is:

$$\begin{aligned}\gamma(c_{x_1}, \dots, c_{x_k}) * \alpha(a_{x_1}, \dots, a_{x_k}) := \\ \gamma \circ \alpha (c_{\alpha(x_1)} \circ a_{x_1}, \dots, c_{\alpha(x_k)} \circ a_{x_k})\end{aligned}$$

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### Proposition

Let  $X = \{x_1, \dots, x_k\}$ . Then the function  $\psi : \mathcal{AUT}_{tree}(X^*) \longrightarrow \mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*) = (\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *)$  defined by

$$\psi(\bar{\lambda}_{q_0}) = \lambda_{q_0}(\bar{\lambda}_{\pi(x_1, q_0)}, \dots, \bar{\lambda}_{\pi(x_n, q_0)})$$

is an isomorphism of groups.

## 8. System of formulas

### Proposition

Let  $\mathcal{A}$  be an automaton with  $\mathcal{Q} = \{q_1, \dots, q_n\}$  over  $X = \{x_1, \dots, x_k\}$ . Then  $\mathcal{A}$  is described by  $n$  recurrent formulas

$$f_{q_1} = \beta_{q_1}(h_{x_1, q_1}, \dots, h_{x_k, q_1}),$$

$$f_{q_2} = \beta_{q_2}(h_{x_1, q_2}, \dots, h_{x_k, q_2}),$$

⋮

$$f_{q_n} = \beta_{q_n}(h_{x_1, q_n}, \dots, h_{x_k, q_n}),$$

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where each  $h_{x_i, q_j}$  is equal to some  $f_{q_l}$  and each  $\beta_{q_j} \in \mathcal{S}(X)$ . Conversely, each such set of  $n$  recursive formulas defines an automaton  $\mathcal{A}$  such that  $\bar{\lambda}_{q_j} = f_{q_j}$  for every  $q_j \in \mathcal{Q}$ .

$$\begin{array}{lll} \bar{\lambda}_{q_1} = \lambda_{q_1}(\bar{\lambda}_{\pi(x_1, q_1)}, \dots, \bar{\lambda}_{\pi(x_n, q_1)}) & f_{q_1} = \beta_{q_1}(h_{x_1, q_1}, \dots, h_{x_k, q_1}) \\ \bar{\lambda}_{q_2} = \lambda_{q_2}(\bar{\lambda}_{\pi(x_1, q_2)}, \dots, \bar{\lambda}_{\pi(x_n, q_2)}) & f_{q_2} = \beta_{q_2}(h_{x_1, q_2}, \dots, h_{x_k, q_2}) \\ \vdots & \vdots \\ \bar{\lambda}_{q_n} = \lambda_{q_n}(\bar{\lambda}_{\pi(x_1, q_n)}, \dots, \bar{\lambda}_{\pi(x_n, q_n)}) & f_{q_n} = \beta_{q_n}(h_{x_1, q_n}, \dots, h_{x_k, q_n}) \end{array}$$



## 9. The classification theorem

### Theorem

Let  $\mathcal{A}$  be a 2-state automaton over the alphabet  $X = \{0, 1\}$  and  $G$  the group defined by this automaton. Then  $G$  is isomorphic to one of the following groups:

- ▶ the trivial group  $\{1_G\}$ ,
- ▶  $\mathbb{Z}_2$ ,
- ▶  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ,

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- ▶ the trivial group  $\{1_G\}$ ,
- ▶  $\mathbb{Z}_2$ ,
- ▶  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ,
- ▶  $\mathbb{Z}$ ,
- ▶ the infinite dihedral group  $\mathcal{D}_\infty = (\mathbb{Z}_2 \times \mathbb{Z}, *)$ , where:

$$(\textcolor{red}{h}, z_1) * (\textcolor{red}{k}, z_2) = (\textcolor{red}{h +_{\mathbb{Z}_2} k}, (-1)^{\textcolor{red}{k}} z_1 + z_2),$$

- ▶ the lamplighter group  $\mathcal{L} = (\mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}, *)$ , where:

$$(\textcolor{red}{z_2}, (h_i)_{i \in \mathbb{Z}}) * (\textcolor{red}{z_1}, (k_i)_{i \in \mathbb{Z}}) = (\textcolor{red}{z_2 + z_1}, (h_{i+z_1} +_{\mathbb{Z}_2} k_i)_{i \in \mathbb{Z}}).$$



## 10. Sketch of proof

Define the cases

Let  $\mathcal{Q} = \{r, s\}$  and  $a = \bar{\lambda}_r, b = \bar{\lambda}_s$ , then

$$\mathcal{A} \xleftarrow{\text{---}} a = \tau^{i_1}(x_{11}, x_{12}) \\ b = \tau^{i_2}(x_{21}, x_{22})$$

where  $x_{ij} \in \{a, b\}$  and  $\tau^{i_1}, \tau^{i_2} \in \mathcal{S}(X) = \mathcal{S}(\{0, 1\})$ . There are 64 possibilities. We proceed by analysing part of them.

# Thank you for your attention!