Groups of Automata

Carlo Lanzi Luciani

Mentors:
izr. prof. Ganna Kudryavtseva
prof. Alessandro Logar

June 6, 2021
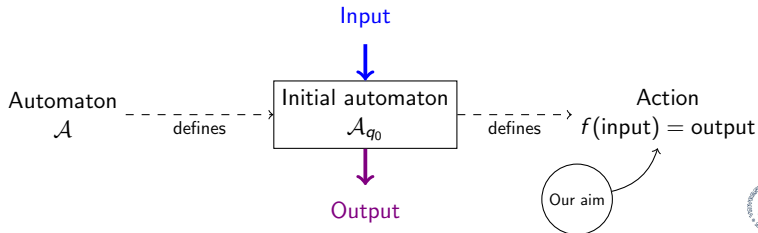
1. Introduction

The word automaton: from the greek "acting of one's own will".
Automata are important in:

- ▶ Information theory
- ▶ Theory of dynamical systems
- ▶ Algebra
- ▶ Others

My aim: study some of the groups constructed through a special class of them, the invertible deterministic Mealy automata, here called simply automata.

Input

Automaton
$\mathcal{A}$ — — — → Initial automaton   — — — →   Action
          defines $\mathcal{A}_{q_0}$ defines $f(\text{input}) = \text{output}$

Output

Our aim

2. The automaton

Definition

An automaton is a 4-tuple $\mathcal{A} = \langle X, \mathcal{Q}, \pi, \lambda \rangle$ where:

- $X = \{x_1, \ldots, x_k\}$ is a finite set called the alphabet,
- $\mathcal{Q}$ is a set called the set of internal states of the automaton,
- $\pi : X \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is a function called the transition function,
- $\lambda : X \times \mathcal{Q} \longrightarrow X$ is a function such that $\lambda_q = \lambda(\cdot, q) : X \longrightarrow X$ is bijective, and is called the output function.

2. The automaton

Definition

An automaton is a 4-tuple $\mathcal{A} = \langle X, \mathcal{Q}, \pi, \lambda \rangle$ where:

- $X = \{x_1, \ldots, x_k\}$ is a finite set called the alphabet,
- $\mathcal{Q}$ is a set called the set of internal states of the automaton,
- $\pi : X \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is a function called the transition function,
- $\lambda : X \times \mathcal{Q} \longrightarrow X$ is a function such that $\lambda_q = \lambda(\cdot, q) : X \longrightarrow X$ is bijective, and is called the output function.
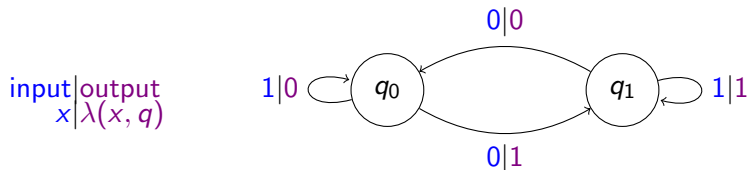
input|output
$x|\lambda(x, q)$

$$1|0 \quad q_0 \qquad 0|0 \qquad q_1 \quad 1|1$$

$$0|1$$

Figure: Moore diagram of a 2-state automaton over $X = \{0, 1\}$

3. The initial automaton

Definition

$X^* = \{x_1 x_2 \ldots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the dictionary.

Word composition: $x_1 \ldots x_n . z_1 \ldots z_n := x_1 \ldots x_n z_1 \ldots z_n$

3. The initial automaton

Definition
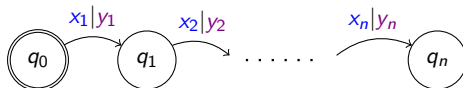$X^* = \{x_1 x_2 \dots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the dictionary.
Word composition: $x_1 \dots x_n . z_1 \dots z_n := x_1 \dots x_n z_1 \dots z_n$

Definition
An initial automaton $\mathcal{A}_{q_0}$ is an automaton $\mathcal{A}$ with a fixed state q_0.
The action of $\mathcal{A}_{q_0}$ is the function $\bar{\lambda}_{q_0} : X^* \longrightarrow X^*$ with
$\bar{\lambda}_{q_0}(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n$.

4. The word tree X^*

Definition

Given $w, v \in X^*$, w is a child of v if and only if $w = v.x = vx$ for some letter $x \in X$.
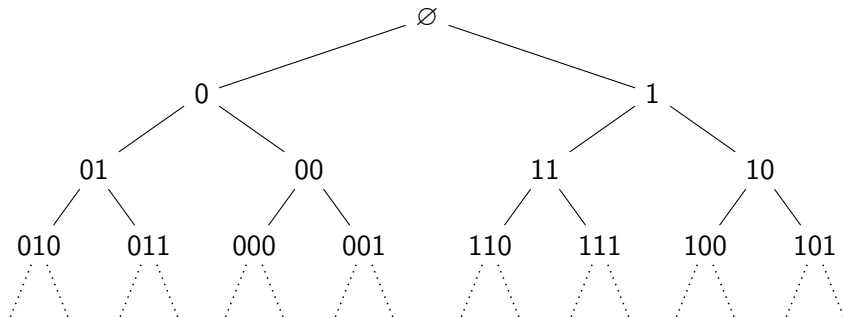

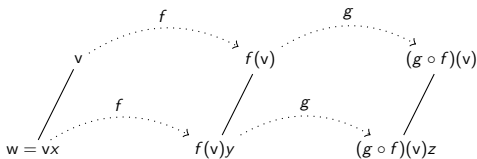Figure: An example of the word tree X^* on $X = \{0, 1\}$.

5. Actions as tree-automorphisms

Proposition
A function $f : X^* \longrightarrow X^*$ is the action of some initial automaton if and only if is a tree-automorphism on the word tree X^*, i.e.:

- $f(\varnothing) = \varnothing$.
- if $w \in X^*$ is a child of v then $f(w)$ is a child of $f(v)$.
- f is bijective.

Proposition
If f, g are tree-automorphisms on X^*, then $g \circ f$ and f^{-1} are tree-automorphisms on X^*.

6. Groups defined by automata

Proposition

The functions $f : X^* \longrightarrow X^*$ defined by initial automata (i.e. tree-automorphisms), called synchronous automatic permutations, form a group denoted by $\mathcal{AUT}_{tree}(X^*)$.

6. Groups defined by automata

Proposition

The functions $f : X^* \longrightarrow X^*$ defined by initial automata (i.e. tree-automorphisms), called synchronous automatic permutations, form a group denoted by $\mathcal{AUT}_{tree}(X^*)$.

Definition

Let $\mathcal{A} = \langle X, \mathcal{Q}, \pi, \lambda \rangle$ be an automaton. The group defined by $\mathcal{A}$ is the group generated by the set $\{\bar{\lambda}_q : q \in \mathcal{Q}\}$.

$$
\begin{array}{ccccc}
& & \bar{\lambda}_{q_1} : X^* \longrightarrow X^* & & \\
\mathcal{A} \text{ with} & & \bar{\lambda}_{q_2} : X^* \longrightarrow X^* & & \\
\mathcal{Q} = \{q_1, \ldots, q_n\} & \xrightarrow{\text{defines}} & \vdots & \xrightarrow{\text{generate}} & \langle \{\bar{\lambda}_q : q \in \mathcal{Q}\}\rangle \\
& & \bar{\lambda}_{q_n} : X^* \longrightarrow X^* & &
\end{array}
$$

7. Application to automata

Definition
Let $\mathcal{S}(X)$ be the simmetric group on $X = \{x_1, \ldots, x_k\}$. Then the wreath product $\mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*)$ is the group $\left(\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *\right)$ where the multiplication rule is:

$$\gamma(c_{x_1}, \ldots, c_{x_k}) * \alpha(a_{x_1}, \ldots, a_{x_k}) :=$$
$$\gamma \circ \alpha \left(c_{\alpha(x_1)} \circ a_{x_1}, \ldots, c_{\alpha(x_k)} \circ a_{x_k}\right)$$

Proposition
Let $X = \{x_1, \ldots, x_k\}$. Then the function $\psi : \mathcal{AUT}_{tree}(X^*) \longrightarrow \mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*) = \left(\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *\right)$ defined by

$$\psi(\bar{\lambda}_{q_0}) = \lambda_{q_0}(\bar{\lambda}_{\pi(x_1, q_0)}, \ldots, \bar{\lambda}_{\pi(x_n, q_0)})$$

is an isomorphism of groups.

8. System of formulas

Proposition

Let $\mathcal{A}$ be an automaton with $\mathcal{Q} = \{q_1, \ldots, q_n\}$ over $X = \{x_1, \ldots, x_k\}$. Then $\mathcal{A}$ is described by n recurrent formulas

$$f_{q_1} = \beta_{q_1}(h_{x_1,q_1}, \ldots, h_{x_k,q_1}),$$
$$f_{q_2} = \beta_{q_2}(h_{x_1,q_2}, \ldots, h_{x_k,q_2}),$$
$$\vdots$$
$$f_{q_n} = \beta_{q_n}(h_{x_1,q_n}, \ldots, h_{x_k,q_n}),$$

where each h_{x_i,q_j} is equal to some f_{q_l} and each $\beta_{q_j} \in \mathcal{S}(X)$. Conversely, each such set of n recursive formulas defines an automaton $\mathcal{A}$ such that $\bar{\lambda}_{q_j} = f_{q_j}$ for every $q_j \in \mathcal{Q}$.

$\mathcal{A} \longleftarrow \text{-} \text{-} \text{-} \text{-} \longrightarrow$

$$\bar{\lambda}_{q_1} = \lambda_{q_1}(\bar{\lambda}_{\pi(x_1,q_1)}, \ldots, \bar{\lambda}_{\pi(x_n,q_1)})$$
$$\bar{\lambda}_{q_2} = \lambda_{q_2}(\bar{\lambda}_{\pi(x_1,q_2)}, \ldots, \bar{\lambda}_{\pi(x_n,q_2)})$$
$$\vdots$$
$$\bar{\lambda}_{q_n} = \lambda_{q_n}(\bar{\lambda}_{\pi(x_1,q_n)}, \ldots, \bar{\lambda}_{\pi(x_n,q_n)})$$

$\longleftarrow \text{-} \text{-} \text{-} \text{-} \longrightarrow$

$$f_{q_1} = \beta_{q_1}(h_{x_1,q_1}, \ldots, h_{x_k,q_1})$$
$$f_{q_2} = \beta_{q_2}(h_{x_1,q_2}, \ldots, h_{x_k,q_2})$$
$$\vdots$$
$$f_{q_n} = \beta_{q_n}(h_{x_1,q_n}, \ldots, h_{x_k,q_n})$$

9. The classification theorem

Theorem

Let $\mathcal{A}$ be a 2-state automaton over the alphabet $X = \{0, 1\}$ and G the group defined by this automaton. Then G is isomorphic to one of the following groups:

- the trivial group $\{1_G\}$,
- $\mathbb{Z}_2$,
- $\mathbb{Z}_2 \times \mathbb{Z}_2$,
- $\mathbb{Z}$,
- the infinite dihedral group $\mathcal{D}_\infty = (\mathbb{Z}_2 \times \mathbb{Z}, *)$, where:

$$(h, z_1) * (k, z_2) = (h +_{\mathbb{Z}_2} k, (-1)^k z_1 + z_2),$$

- the lamplighter group $\mathcal{L} = (\mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}, *)$, where:

$$(z_2, (h_i)_{i \in \mathbb{Z}}) * (z_1, (k_i)_{i \in \mathbb{Z}}) = (z_2 + z_1, (h_{i+z_1} +_{\mathbb{Z}_2} k_i)_{i \in \mathbb{Z}})$$

10. Sketch of proof

Define the cases

Let $\mathcal{Q} = \{r, s\}$ and $a = \bar{\lambda}_r$, $b = \bar{\lambda}_s$, then

$$\mathcal{A} \longleftarrow - - - - - - - - - - - - \longrightarrow \begin{array}{l} a = \tau^{i_1}(x_{11}, x_{12}) \\ b = \tau^{i_2}(x_{21}, x_{22}) \end{array}$$

where $x_{ij} \in \{a, b\}$ and $\tau^{i_1}, \tau^{i_2} \in \mathcal{S}(\mathsf{X}) = \mathcal{S}(\{0, 1\})$. There are 64 possibilities.

10. Sketch of proof

Define the cases

Let $\mathcal{Q} = \{r, s\}$ and $a = \bar{\lambda}_r, b = \bar{\lambda}_s$, then

$$\mathcal{A} \xleftarrow{} \begin{array}{l} a = \tau^{i_1}(x_{11}, x_{12}) \\ b = \tau^{i_2}(x_{21}, x_{22}) \end{array}$$

where $x_{ij} \in \{a, b\}$ and $\tau^{i_1}, \tau^{i_2} \in \mathcal{S}(\mathsf{X}) = \mathcal{S}(\{0, 1\})$. There are 64 possibilities.

Case analysis

Let τ be the transposition of $\mathcal{S}(\mathsf{X})$. Every case is analogous to one of the 24 cases where $a \in \{\tau(a, a), \tau(a, b), \tau(b, b)\}$. We proceed by analysing part of them.

Thank you for your attention!