

Groups of Automata

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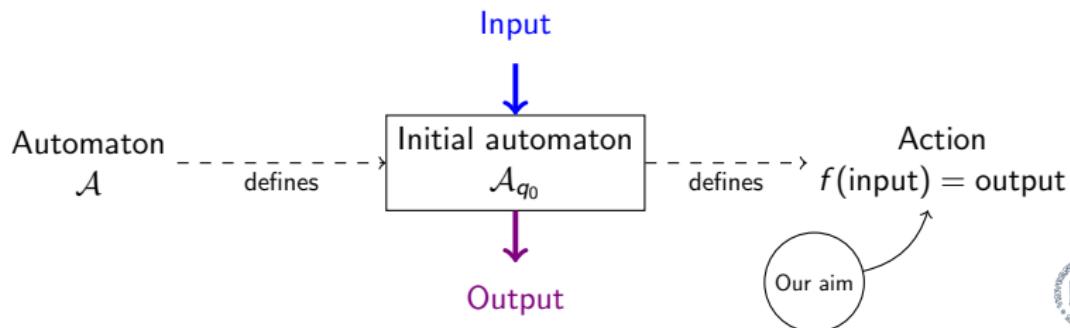
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1. Introduction

The word **automaton**: from the greek "acting of one's own will".
Automata are important in:

- ▶ Information theory
- ▶ Theory of dynamical systems
- ▶ Algebra
- ▶ Others

My aim: study some of the groups constructed through a special class of them, the invertible deterministic Mealy automata, here called simply automata.



2. The automaton

Definition

An **automaton** is a 4-tuple $\mathcal{A} = \langle X, Q, \pi, \lambda \rangle$ where:

- ▶ $X = \{x_1, \dots, x_k\}$ is a finite set called the **alphabet**,
- ▶ Q is a set called the **set of internal states of the automaton**,
- ▶ $\pi : X \times Q \longrightarrow Q$ is a function called the **transition function**,
- ▶ $\lambda : X \times Q \longrightarrow X$ is a function such that $\lambda_q = \lambda(\cdot, q) : X \longrightarrow X$ is bijective, and is called the **output function**.

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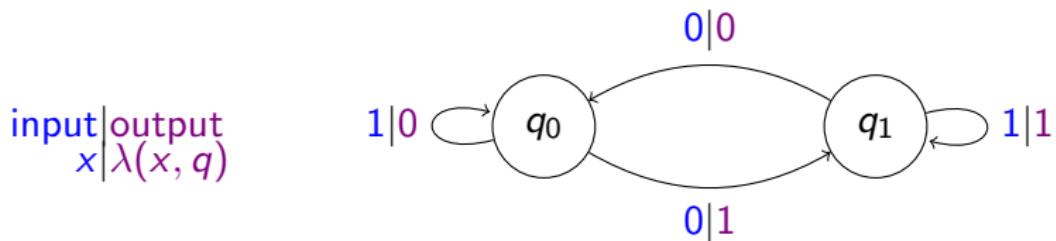


Figure: Moore diagram of an automaton with $Q = \{q_0, q_1\}$ over $X = \{0, 1\}$

3. The initial automaton

Definition

$X^* = \{x_1 x_2 \dots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the **dictionary**.

Word composition: $x_1 \dots x_n.z_1 \dots z_n := x_1 \dots x_n z_1 \dots z_n$

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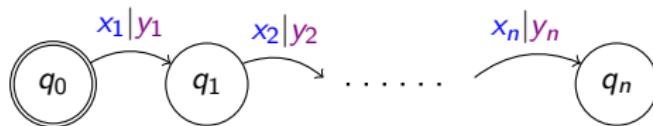
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Definition

An **initial automaton** \mathcal{A}_{q_0} is an automaton \mathcal{A} with a fixed state q_0 .

The **action of** \mathcal{A}_{q_0} is the function $\bar{\lambda}_{q_0} : X^* \rightarrow X^*$ with

$$\bar{\lambda}_{q_0}(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n.$$



4. The word tree X^*

Definition

Given $w, v \in X^*$, w is a child of v if and only if $w = v.x = vx$ for some letter $x \in X$.

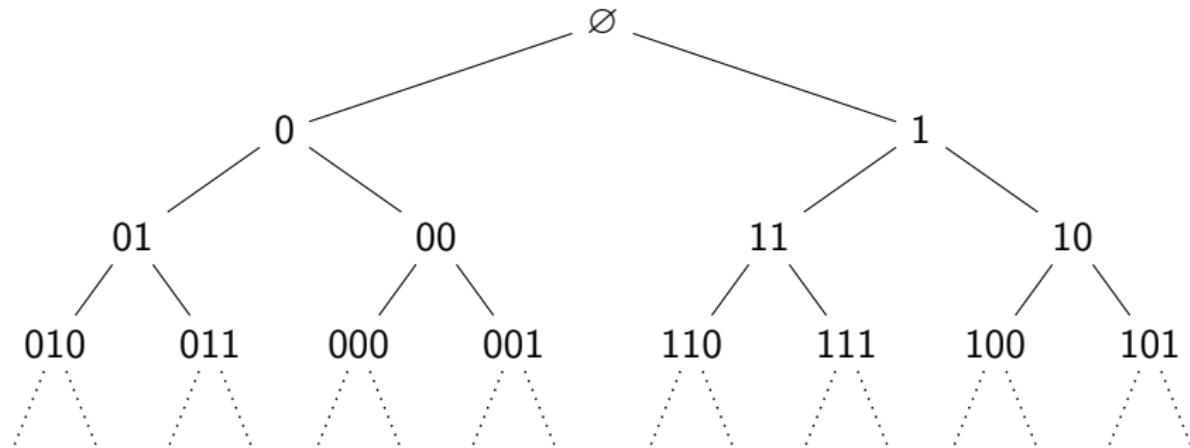


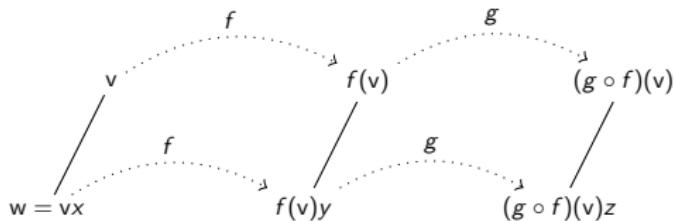
Figure: An example of the word tree X^* on $X = \{0, 1\}$.

5. Actions as tree-automorphisms

Proposition

A function $f : X^* \rightarrow X^*$ is the action of some initial automaton if and only if it is a **tree-automorphism** on the word tree X^* , i.e.:

- ▶ $f(\emptyset) = \emptyset$.
- ▶ if $w \in X^*$ is a child of v then $f(w)$ is a child of $f(v)$.
- ▶ f is bijective.



Proposition

If f, g are tree-automorphisms on X^* , then $g \circ f$ and f^{-1} are tree-automorphisms on X^* .

6. Groups defined by automata

Proposition

The functions $f : X^* \rightarrow X^*$ defined by initial automata (i.e. tree-automorphisms), called **synchronous automatic permutations**, form a group denoted by $\mathcal{AUT}_{\text{tree}}(X^*)$.

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Definition

Let $\mathcal{A} = \langle X, Q, \pi, \lambda \rangle$ be an automaton. The **group defined by \mathcal{A}** is the group generated by the set $\{\bar{\lambda}_q : q \in Q\}$.

$$\begin{array}{ccc} \mathcal{A} \text{ with } & & \bar{\lambda}_{q_1} : X^* \rightarrow X^* \\ Q = \{q_1, \dots, q_n\} & \xrightarrow{\text{--- defines ---}} & \bar{\lambda}_{q_2} : X^* \rightarrow X^* \\ & & \vdots \\ & & \xrightarrow{\text{--- generate ---}} \langle \{\bar{\lambda}_q : q \in Q\} \rangle \\ & & \bar{\lambda}_{q_n} : X^* \rightarrow X^* \end{array}$$

7. Semidirect product and faithful actions

Definition

Let $(B, *_B), (N, *_N)$ be groups and $\varphi : B \longrightarrow \text{AUT}(N)$ an homomorphism, where $\text{AUT}(N)$ denotes the group of (group-)automorphisms on N . Then **the semidirect product** $B \ltimes_{\varphi} N$ is the group $(B \times N, *_\varphi)$, with composition rule:

$$(b_2, n_2) *_\varphi (b_1, n_1) = (b_2 *_B b_1, \varphi(b_1)(n_2) *_N n_1)$$

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Definition

Let B be a group and $S(Y)$ be the simmetric group on Y . Then a **faithful action** φ of B on Y is a monomorphism $\varphi : B \longrightarrow S(Y)$ and we sign $yb := \varphi(b)(y)$.

8. Wreath product

Direct sum

Let A be a group and Y a set. The **direct sum of A on Y** is the set

$$A^{(Y)} := \{(a_\omega)_{\omega \in Y} : a_\omega \in A \text{ and } a_\omega \neq 1_A \text{ only for a finite number of } \omega\}.$$

equipped with the component-wise operation of A . If Y is finite we have $A^{(Y)} = A^Y = A \times A \times \dots \times A$.

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Wreath Product

Let B be a group which acts faithfully on Y and let A be a group.

Then the wreath product $B \wr A$ is the semidirect product

$B \ltimes_\Phi A^{(Y)} = (B \times A^{(Y)}, *)$ where the composition rule is:

$$\begin{aligned} (\mathbf{g}, (u_y)_{y \in Y}) * (\mathbf{h}, (v_y)_{y \in Y}) &:= (\mathbf{gh}, \Phi(\mathbf{h})((u_y)_{y \in Y}) (v_y)_{y \in Y}) := \\ &= (\mathbf{gh}, (u_y \mathbf{h} v_y)_{y \in Y}). \end{aligned}$$

9. Application to automata

Proposition

Let $X = \{x_1, \dots, x_k\}$. Then the function $\psi : \mathcal{AUT}_{tree}(X^*) \longrightarrow \mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*) = (\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *)$ defined by

$$\psi(\bar{\lambda}_{q_0}) = \lambda_{q_0}(\bar{\lambda}_{\pi(x_1, q_0)}, \dots, \bar{\lambda}_{\pi(x_n, q_0)})$$

is an isomorphism of groups.

10. System of formulas

Proposition

Let \mathcal{A} be an automaton with $\mathcal{Q} = \{q_1, \dots, q_n\}$ over $X = \{x_1, \dots, x_k\}$. Then \mathcal{A} is described by n recurrent formulas

$$f_{q_1} = \beta_{q_1}(h_{x_1, q_1}, \dots, h_{x_k, q_1}),$$

$$f_{q_2} = \beta_{q_2}(h_{x_1, q_2}, \dots, h_{x_k, q_2}),$$

⋮

$$f_{q_n} = \beta_{q_n}(h_{x_1, q_n}, \dots, h_{x_k, q_n}),$$

where each h_{x_i, q_j} is equal to some f_{q_l} and each $\beta_{q_j} \in \mathcal{S}(X)$. Conversely, each such set of n recursive formulas defines an automaton \mathcal{A} such that $\bar{\lambda}_{q_j} = f_{q_j}$ for every $q_j \in \mathcal{Q}$.

$$\begin{array}{lll} \bar{\lambda}_{q_1} = \lambda_{q_1}(\bar{\lambda}_{\pi(x_1, q_1)}, \dots, \bar{\lambda}_{\pi(x_n, q_1)}) & f_{q_1} = \beta_{q_1}(h_{x_1, q_1}, \dots, h_{x_k, q_1}) \\ \bar{\lambda}_{q_2} = \lambda_{q_2}(\bar{\lambda}_{\pi(x_1, q_2)}, \dots, \bar{\lambda}_{\pi(x_n, q_2)}) & f_{q_2} = \beta_{q_2}(h_{x_1, q_2}, \dots, h_{x_k, q_2}) \\ \vdots & \vdots \\ \bar{\lambda}_{q_n} = \lambda_{q_n}(\bar{\lambda}_{\pi(x_1, q_n)}, \dots, \bar{\lambda}_{\pi(x_n, q_n)}) & f_{q_n} = \beta_{q_n}(h_{x_1, q_n}, \dots, h_{x_k, q_n}) \end{array}$$



11. The classification theorem

Theorem

Let \mathcal{A} be a 2-state automaton over the alphabet $X = \{0, 1\}$ and G the group defined by this automaton. Then G is isomorphic to one of the following groups:

- ▶ the trivial group $\{1_G\}$,
- ▶ \mathbb{Z}_2 ,
- ▶ $\mathbb{Z}_2 \times \mathbb{Z}_2$,
- ▶ \mathbb{Z} ,
- ▶ the infinite dihedral group $\mathbb{Z}_2 \rtimes_{\phi} \mathbb{Z}$ (where $\phi(h)(z) := z(-1)^h$),
- ▶ the lamplighter group $\mathbb{Z} \wr \mathbb{Z}_2 = \mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}$ (where \mathbb{Z} acts on itself by $mz := z + m$).

12. Sketch of proof

Define the cases

Let $\mathcal{Q} = \{r, s\}$ and $a = \bar{\lambda}_r, b = \bar{\lambda}_s$, then

$$\mathcal{A} \xleftarrow{\text{---}} a = \tau^{i_1}(x_{11}, x_{12}) \\ b = \tau^{i_2}(x_{21}, x_{22})$$

where $x_{ij} \in \{a, b\}$ and $\tau^{i_1}, \tau^{i_2} \in \mathcal{S}(X) = \mathcal{S}(\{0, 1\})$. There are 64 possibilities. We proceed by analyzing a part of them.

Thank you for your attention!