GROUPS OF AUTOMATA

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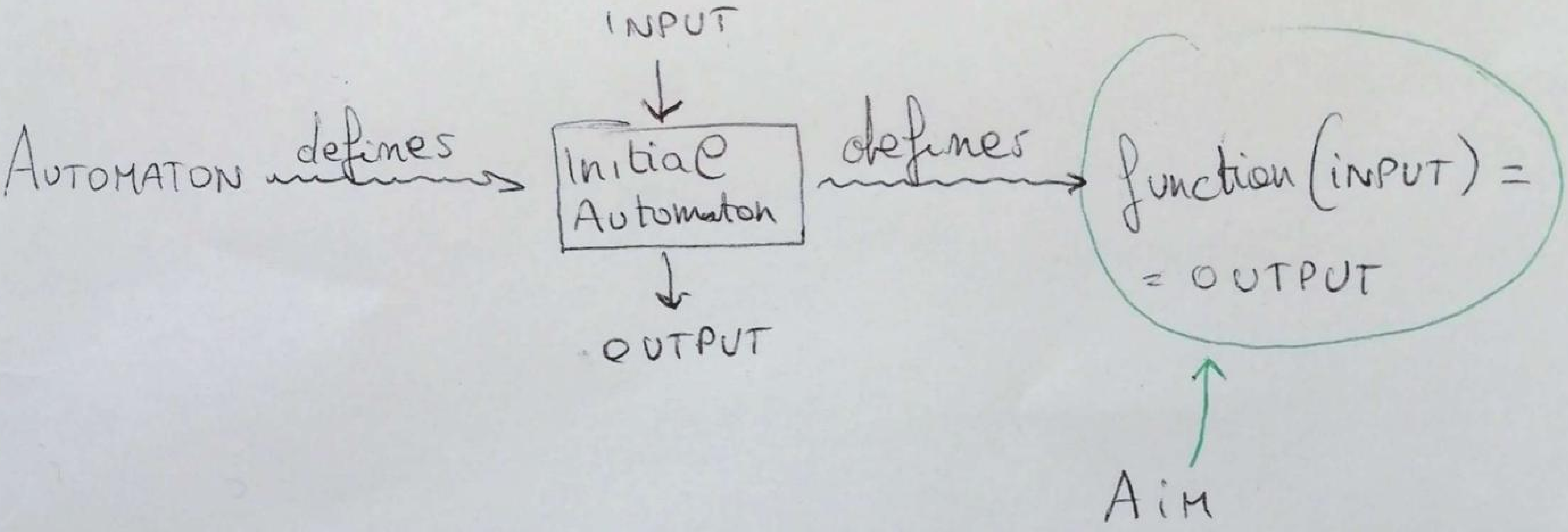
AUTOMATA ARE A MODEL OF COMPUTATION.



<u>ALPHABETS</u> (INPUT AND OUTPUT):

X = finite set of symbols   <u>Ex</u>: $X = \{0, 1\}$

X^* = set of words of $X = \{x_1 \cdot \ldots \cdot x_n \mid x_i \in X, n \in \mathbb{N}\}$

$|w| = |x_1 \cdot \ldots \cdot x_n| =$ lenght of the word $w = n$

$$\left.\begin{array}{l} (x_1 \cdot \ldots \cdot x_n) \circ (y_1 \cdot \ldots \cdot y_m) = x_1 \cdot \ldots \cdot x_n y_1 \cdot \ldots \cdot y_m \\ \emptyset := \text{empty word} \end{array}\right\} X^* \text{ MONOID}$$

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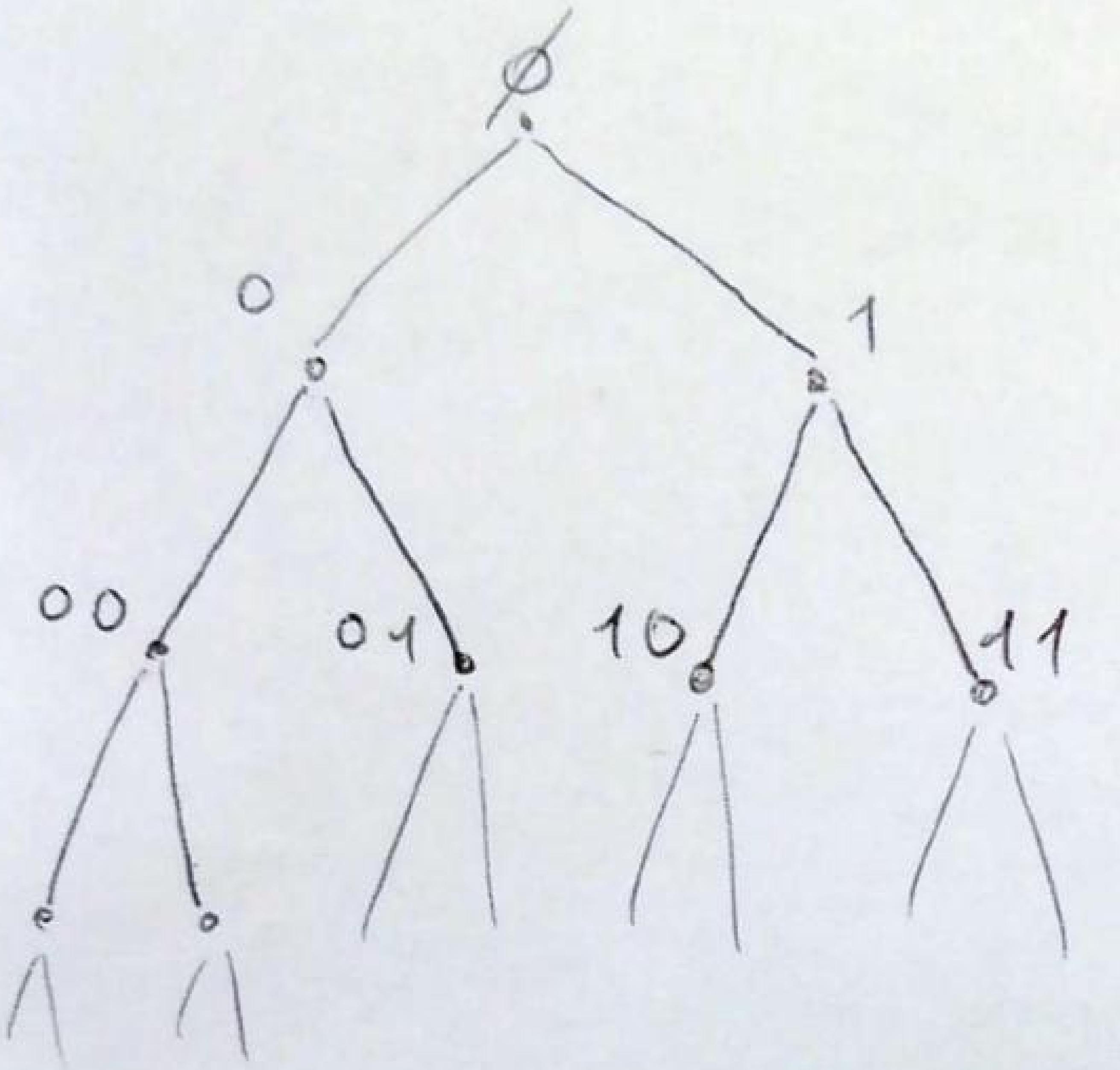
ALTERNATIVE WAY TO SEE X^*:

X^* as a tree (infinite):
(1) $\emptyset$ is the root
(2) w is son of v whenever $w = vx$ for some x in X

Ex: if $X = \{0,1\}$, X^* is



We observe:

$X^n = \{\text{words of lenght } n\}$
$= n$-th floor of X^*

DEF A SYNCHRONOUS INVERTIBLE AUTOMATON $\mathcal{A}$ is a tuple ($\simeq$ list) $\mathcal{A} = \langle X, Q, \pi, \lambda \rangle$ where:

(1) X is a finite set, the INPUT and OUTPUT ALPHABET

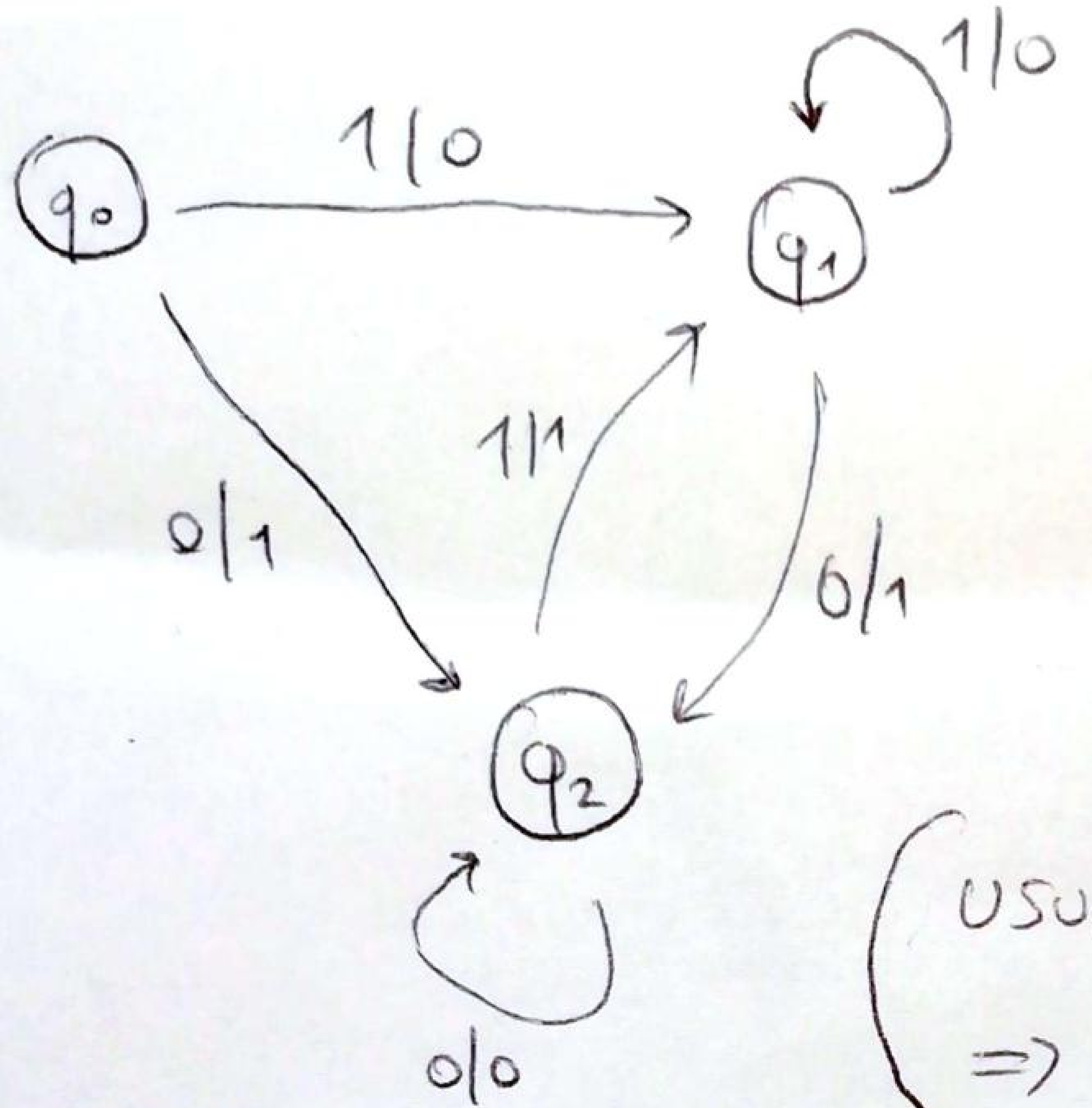
(2) Q is the SET OF STATES

(3) $\pi : Q \times X \longrightarrow Q$ is the TRANSITION FUNCTION

(4) $\lambda : Q \times X \longrightarrow X$ is a function, such that $\lambda(q; \cdot) : X \longrightarrow X$ is bijective, and it's called the OUTPUT FUNCTION

[from now on AUTOMATON = SYNCHRONOUS INV. AUTOMATON]

Ex



$X = \{0, 1\}$

<u>Notation</u> :

input letter | output letter

(usually $|Q| < \infty$
$\Rightarrow$ drawable graph)

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we can extend π and λ :

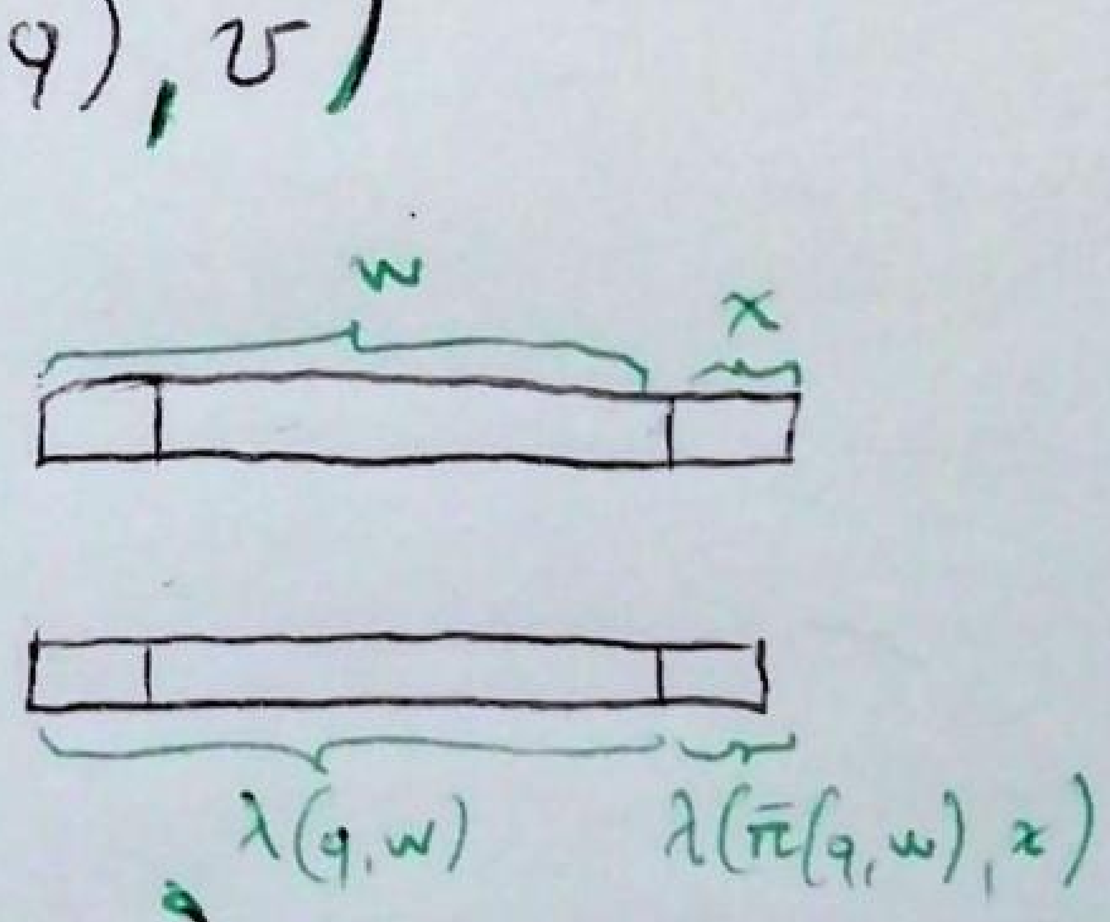
- $\bar{\pi} : Q \times X^* \longrightarrow X^*$

$$\begin{cases} \bar{\pi}(q, \phi) = q \\ \bar{\pi}(q, wx) = \pi(\bar{\pi}(q, w), x) \end{cases}$$

or equivalently

$$\bar{\pi}(q, xv) = \bar{\pi}(\pi(x, q), v)$$

- $\bar{\lambda} : Q \times X^* \longrightarrow X^*$



$$\begin{cases} \bar{\lambda}(q, \phi) = \phi \\ \bar{\lambda}(q, wx) = \bar{\lambda}(q, w)\, \lambda(\bar{\pi}(q, w), x) \end{cases}$$

~~or~~ equivalently

$$\bar{\lambda}(q, xv) = \bar{\lambda}(q, x) \cdot \bar{\lambda}(\pi(q, x), v)$$

DEF Given $\mathcal{A}$ automaton, $\mathcal{A}_{q_0}$ with a <u>fixed</u> <u>INITIAL STATE</u> $q_0 \in Q$, is called <u>INITIAL AUTOMATON</u>

NOTE (1) $\mathcal{A}_{q_0}$ defines $\overline{\lambda}_{q_0} : X^* \longrightarrow X^*$, its <u>ACTION</u>. $\left[\overline{\lambda}_{q_0}(w) = \overline{\lambda}(q_0, w)\right]$

(2) $\overline{\lambda}_{q_0}$ is bijective on X^* $\left[\Leftarrow \lambda(q, \cdot) \text{ is bijective on } X\right]$

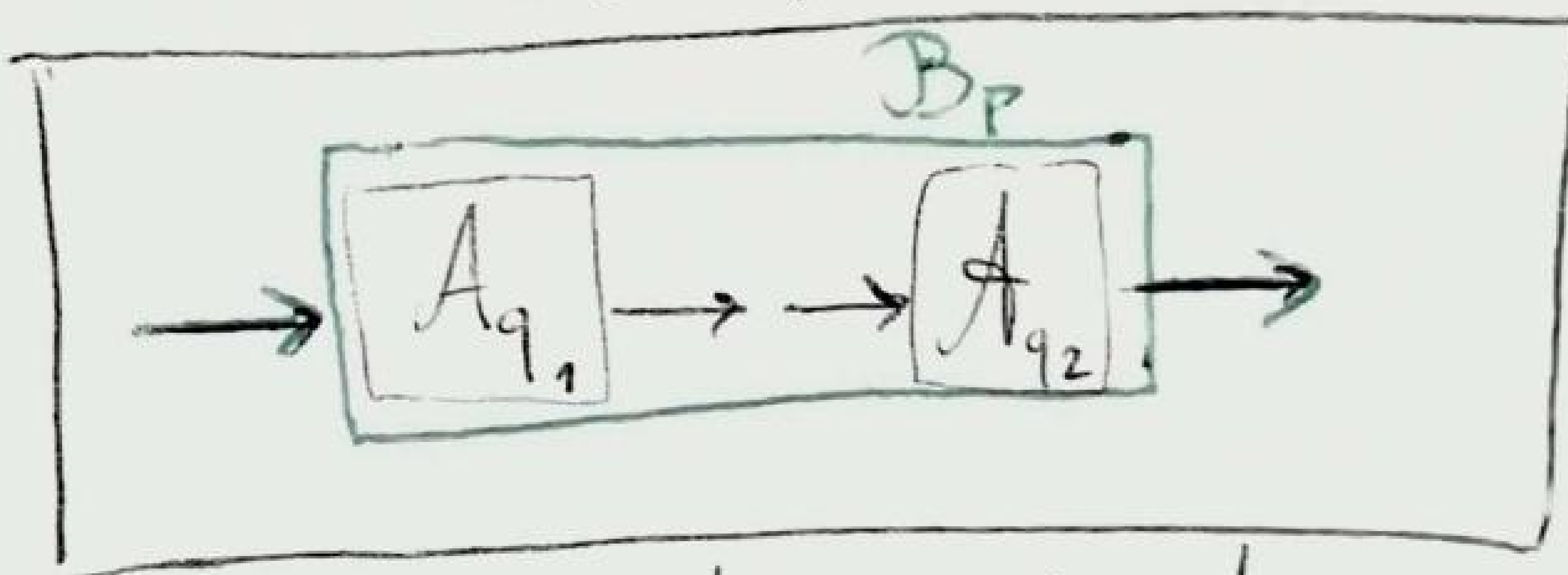
AUTOMATON ⇝ INITIAL AUTOMATON ⇝ ACTION OF $\mathcal{A}_{q_0}$

$\mathcal{A}$ $\qquad$ $\mathcal{A}_{q_0}$ $\qquad$ $\overline{\lambda}_{q_0} : X^* \longrightarrow X^*$

Example: pag 3

COMPOSITION LEMMA Given $\mathcal{A}_{q_1}$, $\mathcal{A}_{q_2}$ initial automata, $\exists\, \mathcal{B}_p$ initial automaton s.t.

$$\overline{\lambda}^{\mathcal{B}}_{p} = \overline{\lambda}^{\mathcal{A}_2}_{q_2} \circ \overline{\lambda}^{\mathcal{A}_1}_{q_1}$$



$\mathcal{B}_p$ is called <u>COMPOSITION</u> of $\mathcal{A}_{q_1}$ and $\mathcal{A}_{q_2}$

DEF $f : X^* \to X^*$ is <u>SYNCHRONOUS</u> <u>AUTOMATIC</u> if $\exists\, \mathcal{A}_q$ s.t. $f = \overline{\lambda}^{\mathcal{A}_q}_{q}$, so f is defined by an initial automaton (<u>Remark</u>: Automaton always invertible)

Note $\{f : X^* \longrightarrow X^* \mid f \text{ is SYNC. AUTOMATIC}\}$ is a group for the COMPOSITION LEMMA.

$\left[\text{if } f \text{ is synch. autom.} \Rightarrow f^{-1} \text{ is synch. autom}\right]$

CHARACTERIZATION OF SYNCHR. AUTOMATIC FUNCTIONS

Lemma f is synchronous automatic if and only if f is a tree-homomorphism on X^*

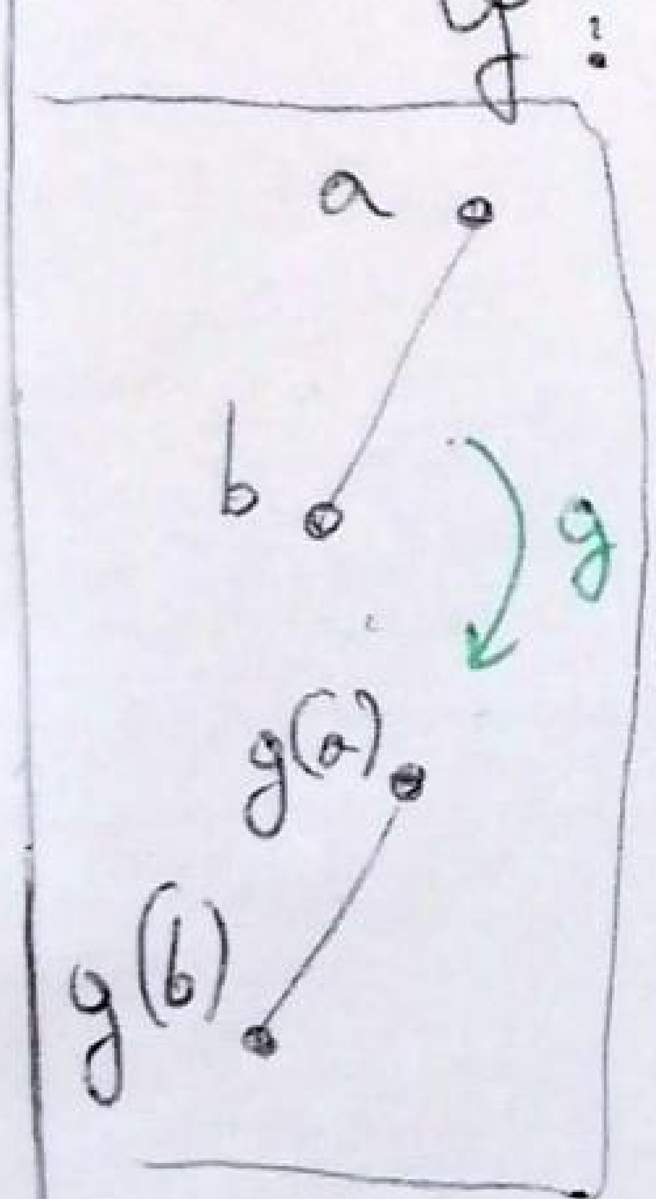
WHAT is a tree-homom?

DEF Given T tree, $g: T \to T$ is a tree-homom. if:

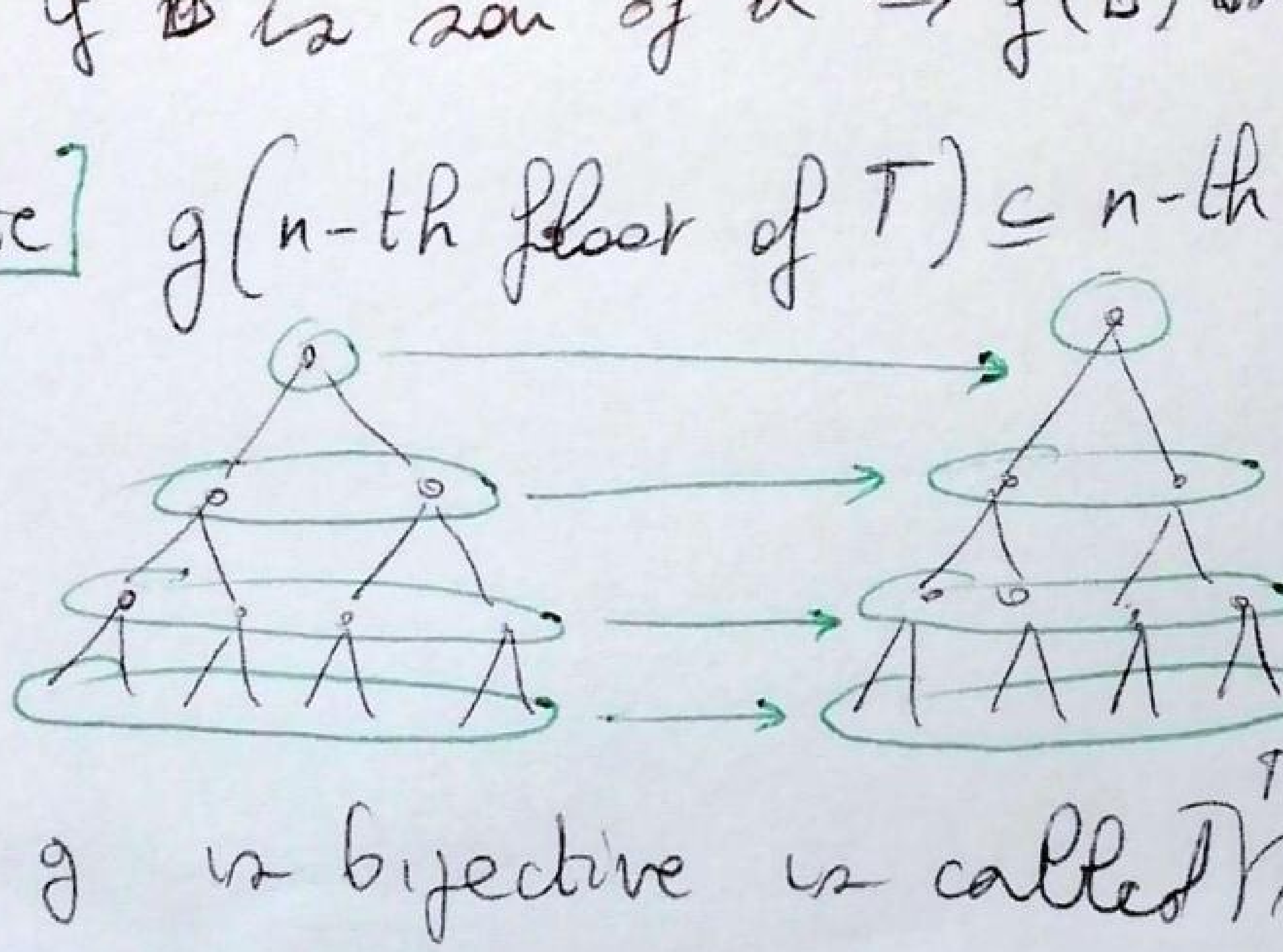
(1) preserves the root r: $g(r) = r$

(2) preserves descendant-relationship:

if ~~b~~ is son of $a \Rightarrow f(b)$ is son of $f(a)$



Note $g(n\text{-th floor of } T) \subseteq n\text{-th floor of } T$



Note • if g is bijective is called Tree-AUTOMORPHISM

• $\{$tree-automorphisms of $T\} = Aut(T)$ is a group under composition of functions

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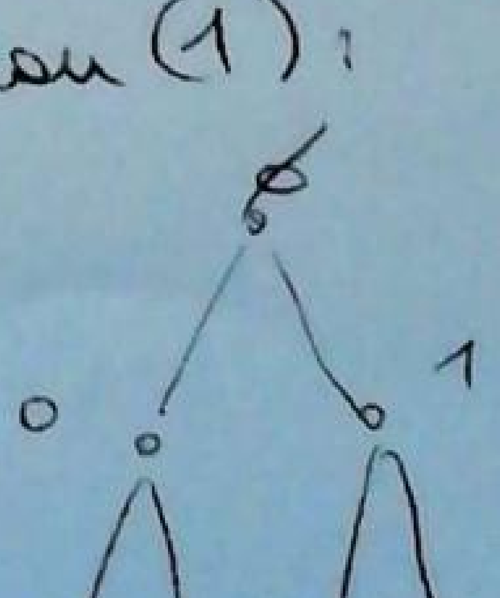
Proof of Lemma: (the tree is X^*)

"$\Rightarrow$" f is synch. automatic, so $\exists\ \mathcal{A}_q$ s.t. $f = \bar{\lambda}_q$, action of $\mathcal{A}_q$. Let's verify condition (1):

$f(\emptyset) = \bar{\lambda}_q(\emptyset) = \emptyset$ (root of X^*)

(page 3 formulas) ✓



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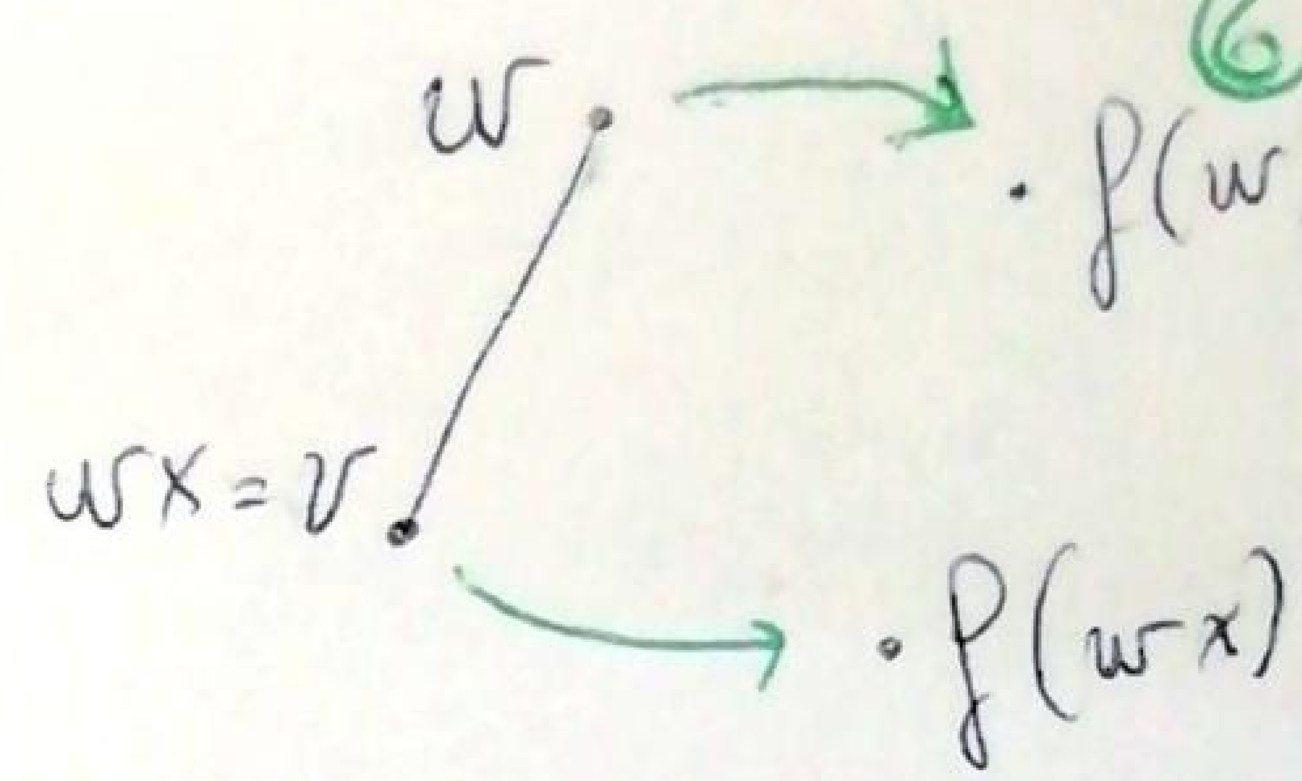
We want to verify condition (2).



$v, w \in X^*$, v son of $w \Rightarrow$

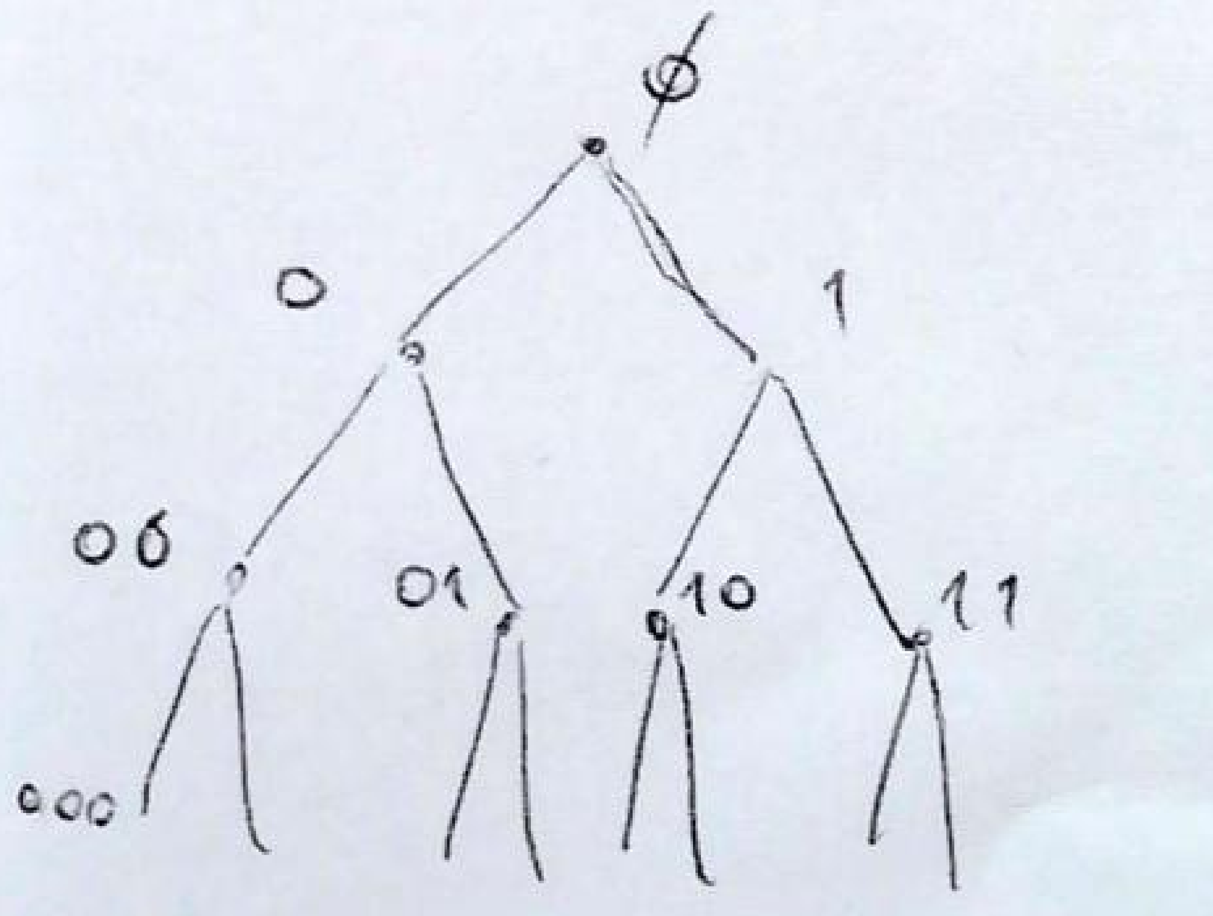
$\Rightarrow v = wx$ for some $x \in X$

$f(v) = f(wx) = \bar{\lambda}_q(wx) \underset{\substack{\uparrow \\ \text{pag 3} \\ \text{Form}}}{=} \bar{\lambda}_q(w) \cdot \bar{\lambda}_{\pi(q,w)}(x) =$

$= \bar{\lambda}_q(w) \cdot \underbrace{\lambda_{\pi(q,v)}(x)}_{\text{length} = 1} = f(w)\, y$ for some $y \in X$

$\Rightarrow f(v)$ is son of $f(w) \Rightarrow f$ is tree-homom.

"$\Leftarrow$" Let f be tree homom.. We want to build

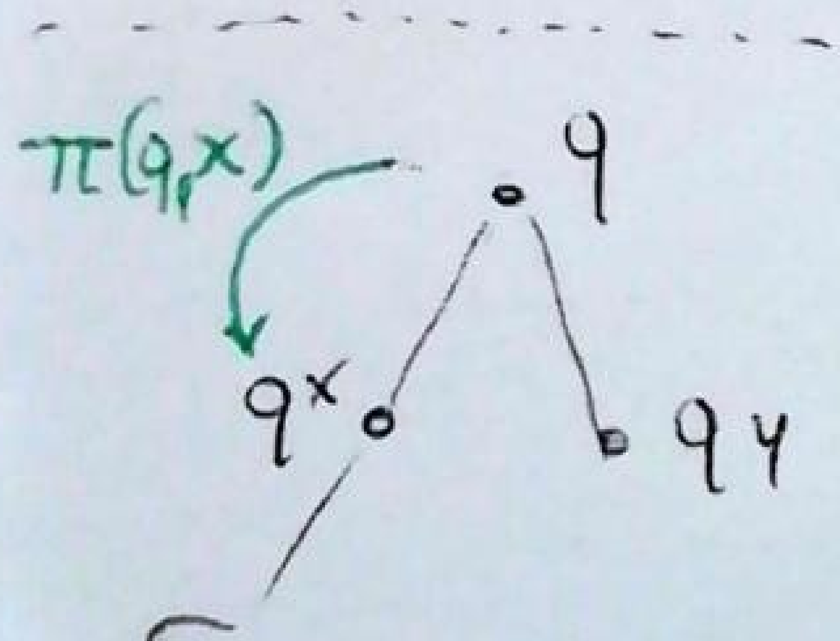


$\mathcal{A}$ s.t. $f = \bar{\lambda}_q$ for some q.

[Trick : $Q := X^*$ infinite]

$\mathcal{A} = \langle X, Q, \pi, \lambda \rangle := \langle X, X^*, \pi, \lambda \rangle$

with π and λ so defined:



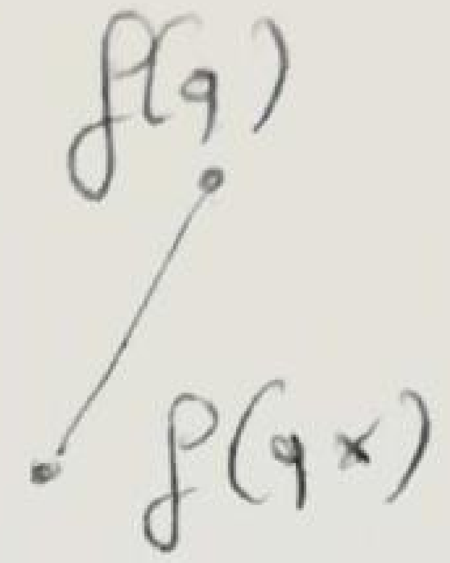
$$\begin{cases} \pi(q,x) = qx \\ \lambda(q,x) = f(qx) - f(q) \end{cases}$$

[$\underline{\text{Subtraction on } X^*}$: if $w = uv$ (*) $\Rightarrow w - u := v$]

¿ Does (*) condition hold for λ? i.e.

¿ is $f(q)$ beginning of $f(qx)$?

f is tree-homom. $\Rightarrow f(qx)$ is son of $f(q)$



$\Rightarrow \quad f(qx) = f(q)z \quad$ for some $z \in X$

$\Rightarrow \quad f(qx) - f(q) = z. \quad [\Rightarrow \lambda$ is well defined$]$

Claim: $f = \bar{\lambda}_\emptyset$. $[\bar{\lambda} \neq \lambda]$. For induction on n = lenght of w

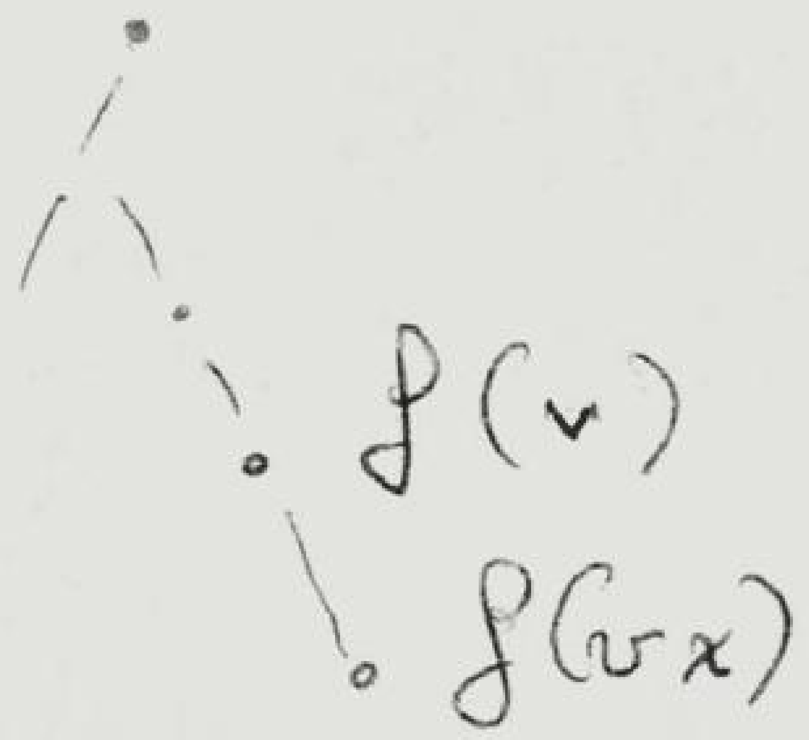
$\underline{n=0}$: $\bar{\lambda}(\emptyset, \emptyset) = \emptyset = f(\emptyset)$ ✓

$\underline{n \rightsquigarrow n+1}$: if $w \in X^* \setminus \{\emptyset\}$, w can be written as $\underset{X^*}{v}\,\underset{X}{x}$.

$$\bar{\lambda}(\emptyset, vx) \underset{\text{pag}(3)}{=} \bar{\lambda}(\emptyset, v) \cdot \bar{\lambda}(\pi(\emptyset, v), x) = f(v) \cdot \bar{\lambda}(\overset{v}{\cancel{\quad}}, x) =$$

$$= f(v) \cdot [f(vx) - f(v)]$$

$f(vx)$ ✓



Povzetek:

Automata $\mathcal{A}$ $\rightsquigarrow$ init. Automata $\mathcal{A}_q$ $\longmapsto$ Actions $f = \bar{\lambda}_q$ $\longleftrightarrow$ tree Autom. on X^*

DEF] Given $\mathcal{A}$ automaton, we define the GROUP GENERATED BY $\mathcal{A}$, as the group whose generators are the actions of all the possible Initial Automata definable on $\mathcal{A}$

i.e. $G\mathcal{A}(\lambda) := \langle\{ \bar{\lambda}_q : X^* \to X^* \mid q \in Q \}\rangle$

Ex:] Automaton on page 3 defines a group with 3 generators

Proposition: Let $\mathcal{A}$ be a 2-state automaton on $X=\{0,1\}$. Then $G_{\mathcal{A}}(X)$ must be isomorphic to one of these groups:

(1) $\{1_G\}$

(2) $\mathbb{Z}_2$

(3) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

(4) $\mathbb{Z}$

(5) $\mathbb{D}_\infty = \{\text{symmetries of the circle}\}$

(6) $\mathbb{Z} \wr \mathbb{Z}_2 = L_2 =$ Lamplighter group