Groups of Automata

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June 7, 2021

1. Introduction

The word automaton: from the greek "acting of one's own will".
Automata are important in:

- Information theory
- Theory of dynamical systems
- Algebra
- Others

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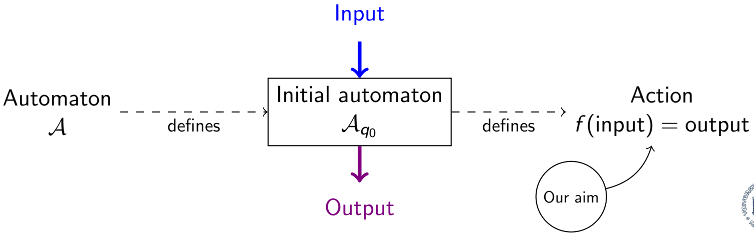
My aim: study some of the groups constructed through a special class of them, the invertible deterministic Mealy automata, here called automata.



2. The automaton

Definition

An automaton is a 4-tuple $\mathcal{A} = \langle \mathsf{X}, \mathcal{Q}, \pi, \lambda \rangle$ where:

- $\mathsf{X} = \{x_1, \ldots, x_k\}$ is a finite set called the alphabet,
- $\mathcal{Q}$ is a set called the set of internal states of the automaton,
- $\pi : \mathsf{X} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is called the transition function,
- $\lambda : \mathsf{X} \times \mathcal{Q} \longrightarrow \mathsf{X}$ is such that $\lambda_q = \lambda(\cdot, q) : \mathsf{X} \longrightarrow \mathsf{X}$ is bijective, and is called the output function.

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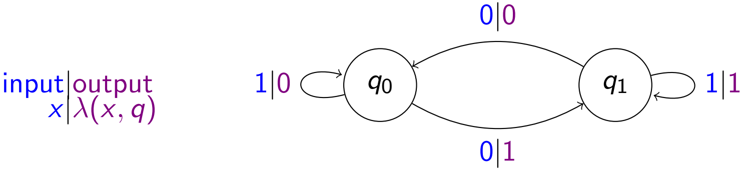

Figure: Moore diagram of a 2-state automaton over $\mathrm{X} = \{0, 1\}$

3. The initial automaton

Definition

$X^* = \{x_1x_2 \dots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the dictionary.

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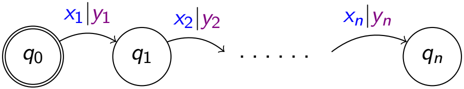
Definition

An initial automaton $\mathcal{A}_{q_0}$ is an automaton $\mathcal{A}$ with a fixed state q_0.
The action of $\mathcal{A}_{q_0}$ is the function $\bar{\lambda}_{q_0} : X^* \longrightarrow X^*$ with
$\bar{\lambda}_{q_0}(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n$.



4. The word tree X^*

Definition

$$\underline{w} = w_1 \dots w_n \text{ is a child of } v_1 \dots v_m = \underline{v}$$

$$\iff$$

$$w_1 \dots w_n = v_1 \dots v_m x \text{ for some letter } x \in X.$$

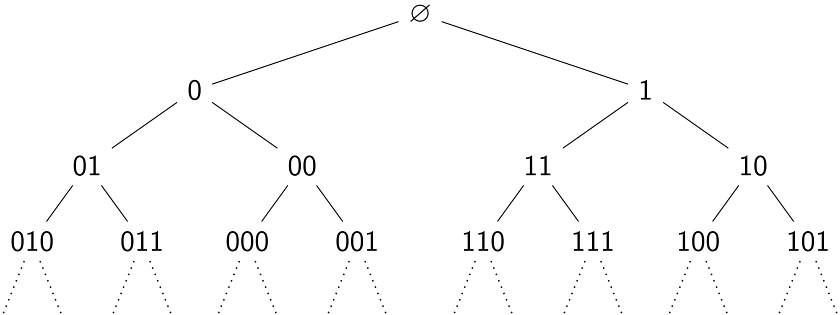
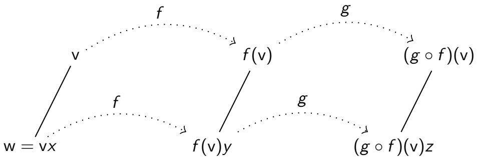


Figure: An example of the word tree X^* on $X = \{0, 1\}$.

Proposition

A function $f : X^* \longrightarrow X^*$ is the action of some initial automaton if and only if is a tree-automorphism on the word tree X^*, i.e.:

- $f(\varnothing) = \varnothing$.
- if $\underline{w} \in X^*$ is a child of $\underline{v}$ then $f(\underline{w})$ is a child of $f(\underline{v})$.
- f is bijective.



6. Groups defined by automata

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The functions $f : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ defined by initial automata (i.e. tree-automorphisms) form a group denoted by $\mathcal{AUT}_{tree}(\mathsf{X}^*)$.

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Definition

Let $\mathcal{A} = \langle \mathsf{X}, \mathcal{Q}, \pi, \lambda \rangle$ be an automaton. The group defined by $\mathcal{A}$ is the group generated by the set $\{\bar{\lambda}_q : q \in \mathcal{Q}\}$.

$$\begin{array}{c} \mathcal{A} \text{ with} \\ \mathcal{Q} = \{q_1, \dots, q_n\} \end{array} \xrightarrow[\text{defines}]{} \begin{array}{c} \bar{\lambda}_{q_1} : \mathsf{X}^* \longrightarrow \mathsf{X}^* \\ \bar{\lambda}_{q_2} : \mathsf{X}^* \longrightarrow \mathsf{X}^* \\ \vdots \\ \bar{\lambda}_{q_n} : \mathsf{X}^* \longrightarrow \mathsf{X}^* \end{array} \xrightarrow[\text{generate}]{} \langle \{\bar{\lambda}_q : q \in \mathcal{Q}\} \rangle$$

7. Wreath product and automata

Definition

Let $\mathcal{S}(\mathsf{X})$ be the simmetric group on $\mathsf{X} = \{x_1, \ldots, x_k\}$. Then the wreath product $\mathcal{S}(\mathsf{X}) \wr \mathcal{AUT}_{tree}(\mathsf{X}^*)$ is the group $\left(\mathcal{S}(\mathsf{X}) \times \mathcal{AUT}_{tree}(\mathsf{X}^*)^{\mathsf{X}}, *\right)$ where the multiplication rule is:

$$\gamma(c_{x_1}, \ \ldots \ , c_{x_k}) \ * \ \alpha(a_{x_1}, \ \ldots \ , a_{x_k}) :=$$
$$\gamma \circ \alpha \ \left(c_{\alpha(x_1)} \circ a_{x_1}, \ \ldots \ , c_{\alpha(x_k)} \circ a_{x_k}\right)$$

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Proposition

Let $\mathsf{X} = \{x_1, \ldots, x_k\}$. Then the function $\psi : \mathcal{AUT}_{tree}(\mathsf{X}^*) \longrightarrow \mathcal{S}(\mathsf{X}) \wr \mathcal{AUT}_{tree}(\mathsf{X}^*) = \left(\mathcal{S}(\mathsf{X}) \times \mathcal{AUT}_{tree}(\mathsf{X}^*)^{\mathsf{X}}, *\right)$ defined by

$$\psi(\bar{\lambda}_{q_0}) = \lambda_{q_0}(\bar{\lambda}_{\pi(x_1,q_0)}, \ldots, \bar{\lambda}_{\pi(x_n,q_0)})$$

is an isomorphism of groups.

Proposition

Let $\mathcal{A}$ be an automaton with $\mathcal{Q} = \{q_1, \ldots, q_n\}$ over $\mathrm{X} = \{x_1, \ldots, x_k\}$. Then $\mathcal{A}$ is described by n recurrent formulas

$$
\begin{aligned}
f_{q_1} &= \beta_{q_1}(h_{x_1,q_1}, \ldots, h_{x_k,q_1}), \\
f_{q_2} &= \beta_{q_2}(h_{x_1,q_2}, \ldots, h_{x_k,q_2}), \\
&\vdots \\
f_{q_n} &= \beta_{q_n}(h_{x_1,q_n}, \ldots, h_{x_k,q_n}),
\end{aligned}
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where each h_{x_i,q_j} is equal to some f_{q_l} and each $\beta_{q_j} \in \mathcal{S}(\mathrm{X})$.

8. System of formulas

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where each h_{x_i,q_j} is equal to some f_{q_l} and each $\beta_{q_j} \in \mathcal{S}(X)$. Conversely, each such set of n recursive formulas defines an automaton $\mathcal{A}$ such that $\bar{\lambda}_{q_j} = f_{q_j}$ for every $q_j \in \mathcal{Q}$.

$$
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9. The classification theorem

Theorem

Let $\mathcal{A}$ be a 2-state automaton over the alphabet $X = \{0, 1\}$ and G the group defined by this automaton. Then G is isomorphic to one of the following groups:

- the trivial group $\{1_G\}$,
- $\mathbb{Z}_2$,
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- $\mathbb{Z}$,
- the infinite dihedral group $\mathcal{D}_\infty = (\mathbb{Z}_2 \times \mathbb{Z}, *)$, where:

$$(h, z_1) * (k, z_2) = (h +_{\mathbb{Z}_2} k, (-1)^k z_1 + z_2),$$

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- the lamplighter group $\mathcal{L} = (\mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}, *)$, where:

$$(z_2, (h_i)_{i\in\mathbb{Z}}) * (z_1, (k_i)_{i\in\mathbb{Z}}) = (z_2 + z_1, (h_{i+z_1} +_{\mathbb{Z}_2} k_i)_{i\in\mathbb{Z}}).$$

10. Sketch of proof

Define the cases

Let $\mathcal{Q} = \{r, s\}$ and $a = \bar{\lambda}_r, b = \bar{\lambda}_s$, then

$$\mathcal{A} \dashleftarrow\dashrightarrow \begin{array}{l} a = \tau^{i_1}(x_{11}, x_{12}) \\ b = \tau^{i_2}(x_{21}, x_{22}) \end{array}$$

where $x_{ij} \in \{a, b\}$ and $\tau^{i_1}, \tau^{i_2} \in \mathcal{S}(\mathrm{X}) = \mathcal{S}(\{0, 1\})$. There are 64 possibilities. We proceed by analysing part of them.

Thank you for your attention!