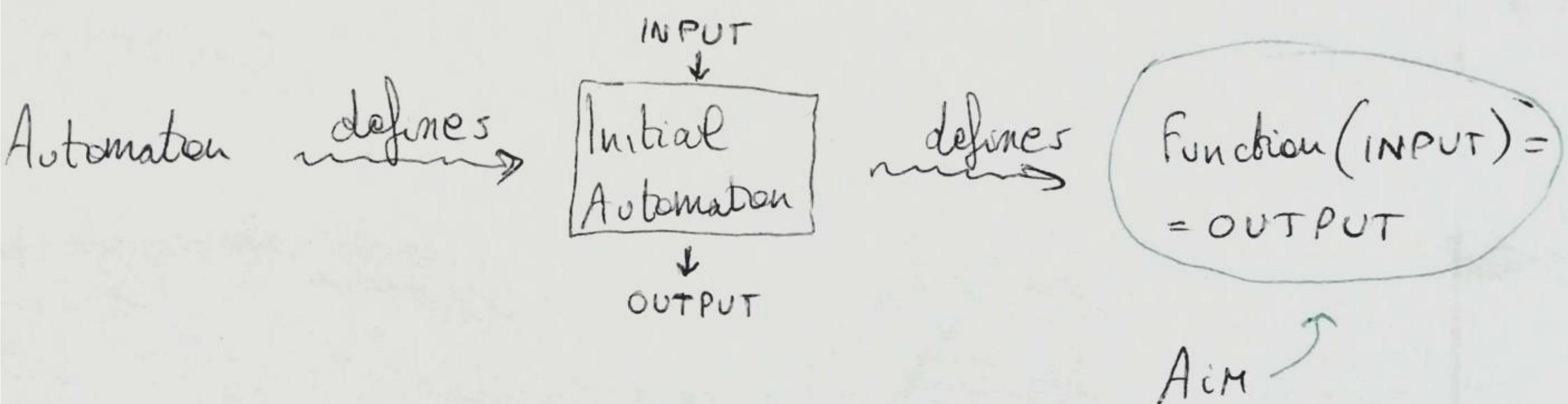


① GROUPS OF AUTOMATA

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Automata are a Model of Computation:



1st part:

- (1) INPUT and OUTPUT
- (2) Automata and their visualization
- (3) Initial Automation and "actions"

Then we will analyse the functions in detail.

Finally: interesting examples

INPUT and OUTPUT

- X = finite set of symbols

Ex] ~~X~~ $X = \{0, 1\}$

- X^* = set of words of $X = \{x_1 \cdots x_n \mid x_i \in X, n \in \mathbb{N}\} = \{\text{finite strings of } X\}$

- $|w| = |x_1 \cdots x_n| := n$ = length of w

- Monoid STRUCTURE

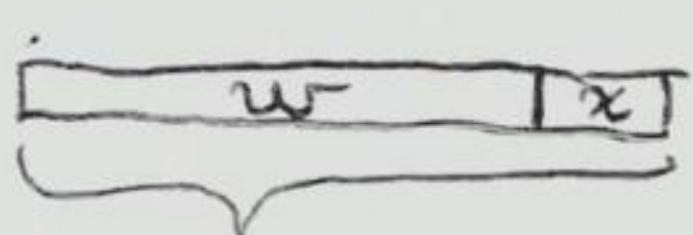
\emptyset := empty word = identity respect to \circ .

$$(x_1 \cdots x_n) \circ (y_1 \cdots y_m) = x_1 \cdots x_n y_1 \cdots y_m$$

Ex]: 001 \circ 101 = 001101

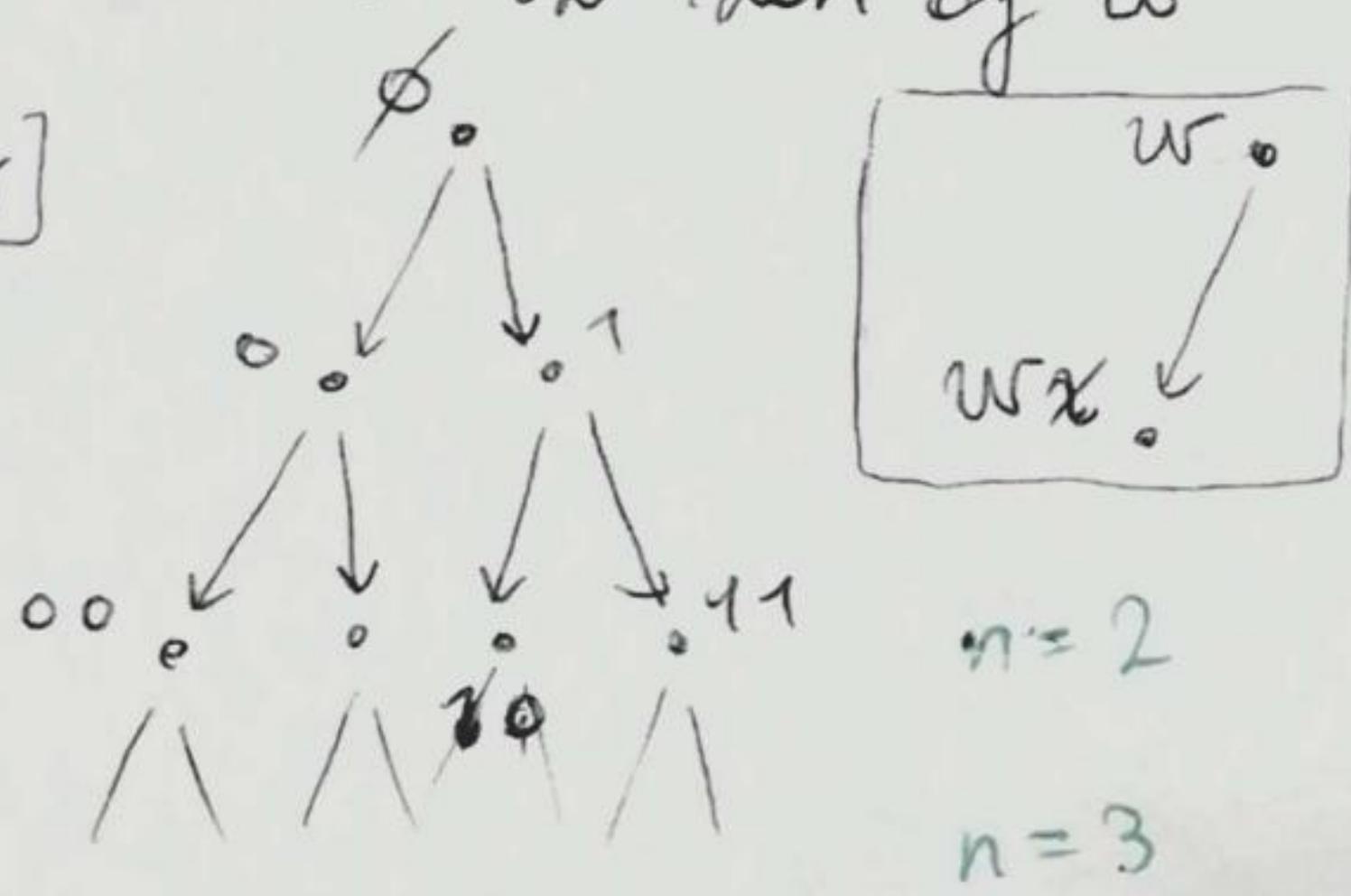
- TREE STRUCTURE

\emptyset = root



v is son of w whenever $v = w x$

Ex]



Observation:

$X^n = \{\text{words of length } n\}$
 $= n\text{-th level of } X$

AUTOMATA

2A

DEF A.

SYNCHRONOUS INVERTIBLE AUTOMATION is

a tuple $\mathcal{A} = (X, Q, \lambda, \pi)$ where

(1) X is a finite set, the INPUT and OUTPUT ALPHABET

(2) Q is a set, the SET OF STATES

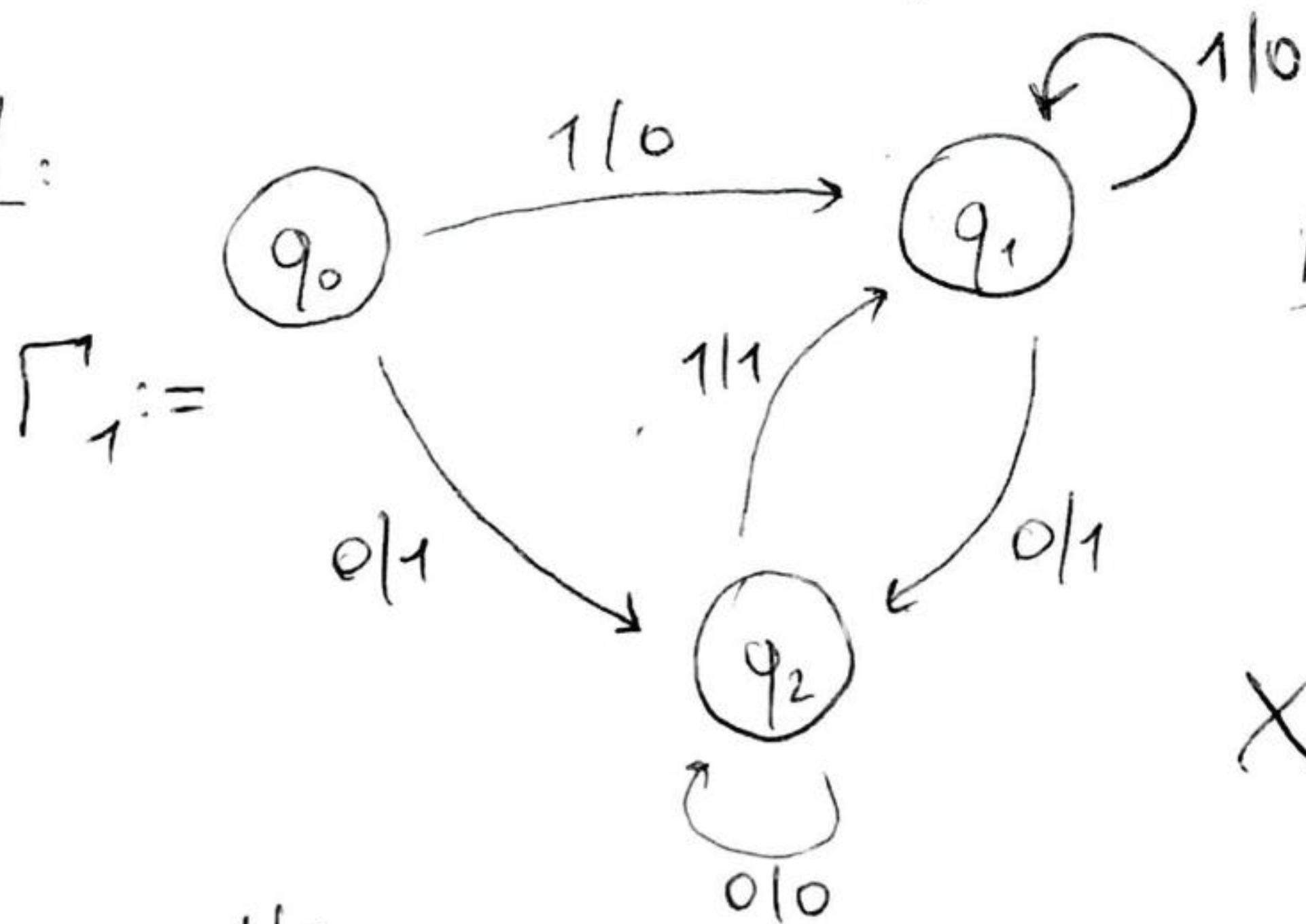
(3) $\pi : Q \times X \rightarrow Q$ is the TRANSITION FUNCTION

(4) $\lambda : Q \times X \rightarrow X$ is a function such that

$\lambda(q; \cdot) : X \rightarrow X$ is bijective (\Rightarrow permutation),
and it's called OUTPUT FUNCTION

[From now on AUTOMATON = Sync. Inv. AUTOMATON]

Example 1:

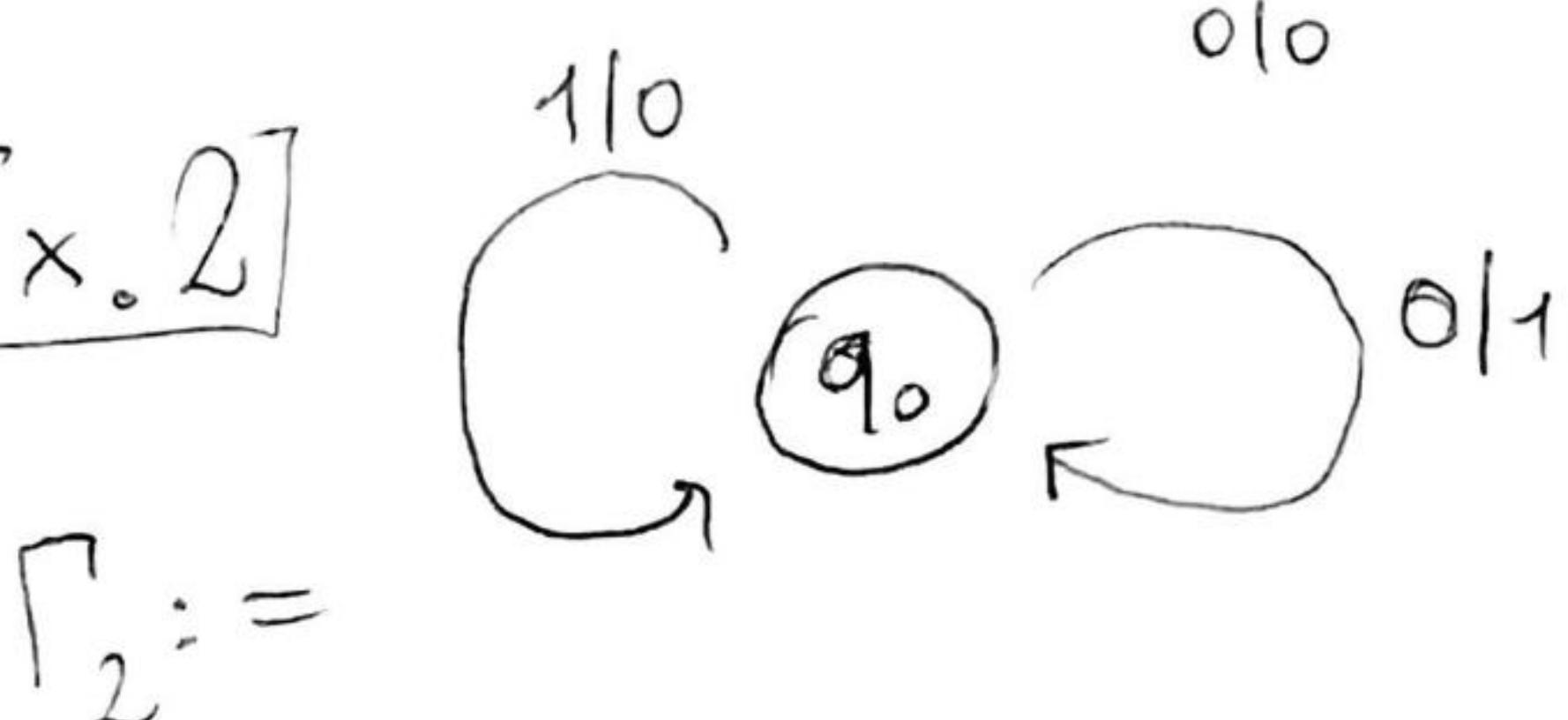


Notation:

INPUT LETTER		OUTPUT LETTER
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$$X = \{0, 1\}$$

Ex. 2



Extension of π and λ

(2B)

$$\pi(q, \star) = q$$

$$\pi: Q \times \mathbb{X} \longrightarrow \mathbb{Q}$$

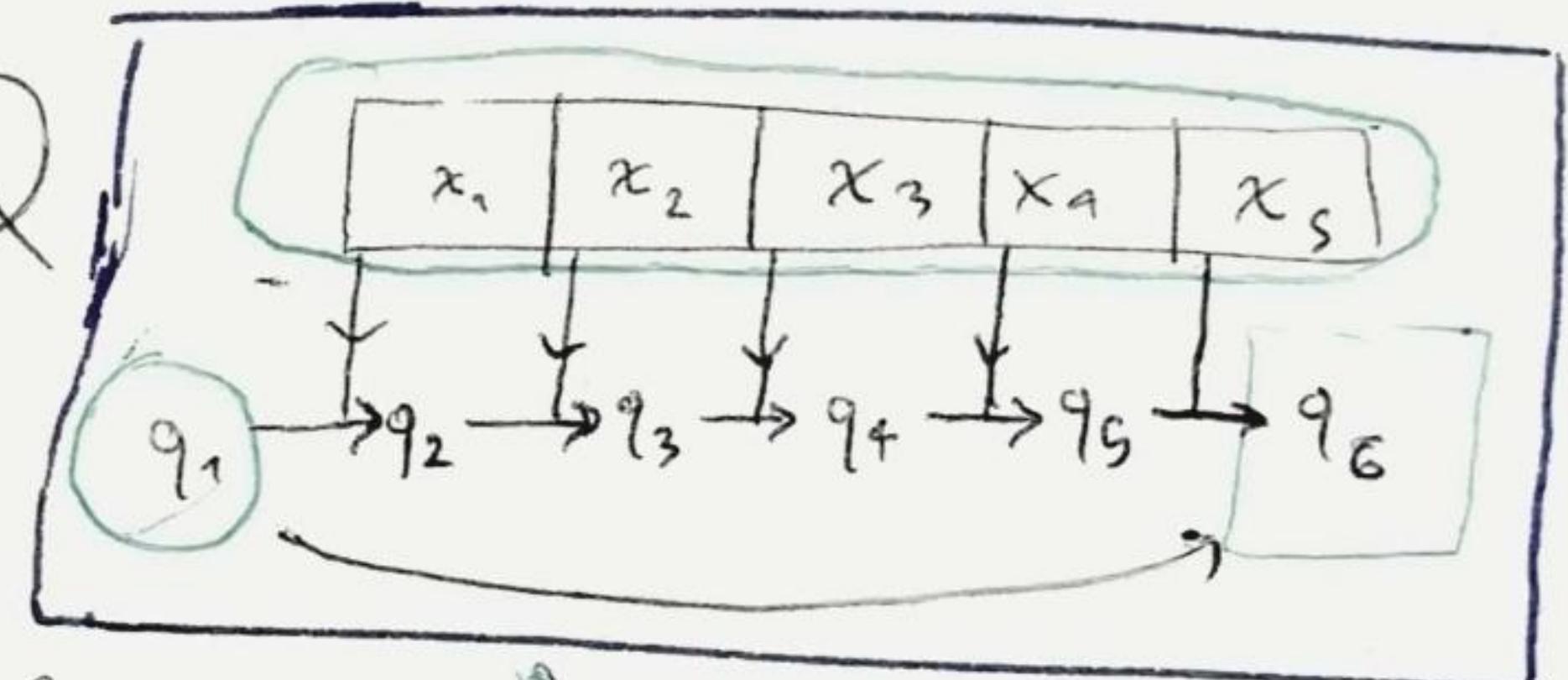
$$\lambda(q, x) = y$$

$$\lambda: Q \times \mathbb{X} \longrightarrow \mathbb{X}$$

Recursive Extension:

$$\bar{\pi}: Q \times (\mathbb{X}^*)^* \longrightarrow Q$$

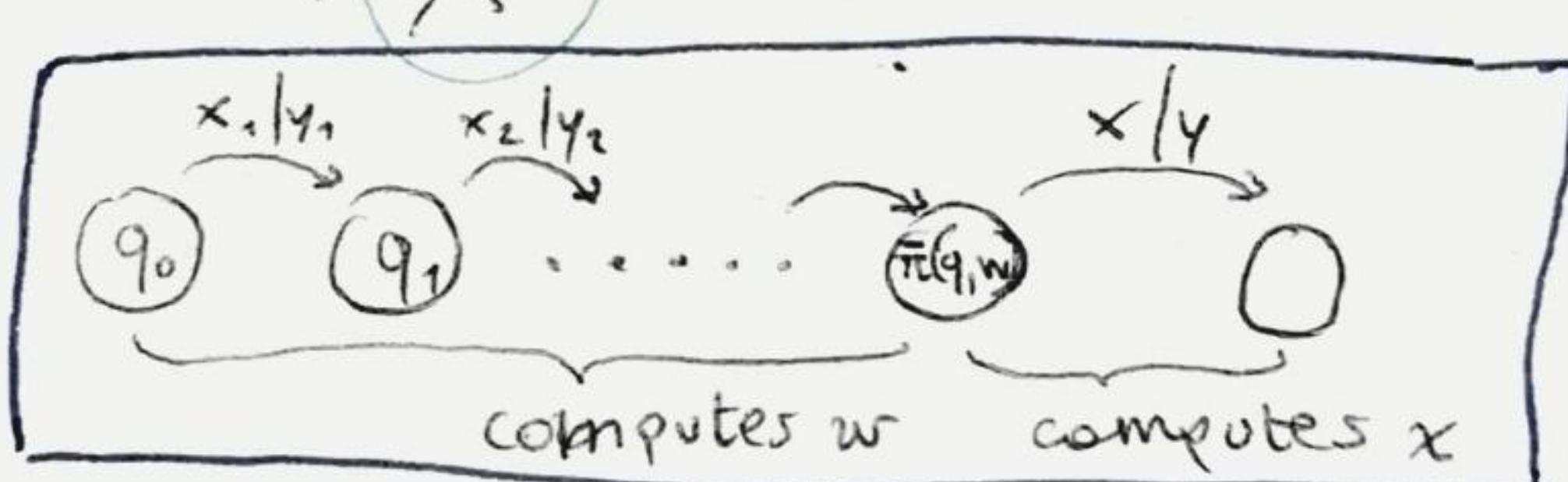
$$\left\{ \begin{array}{l} \bar{\pi}(q, \emptyset) := q \\ \bar{\pi}(q, w \cdot x) := \bar{\pi}(\bar{\pi}(q, w), x) \end{array} \right.$$



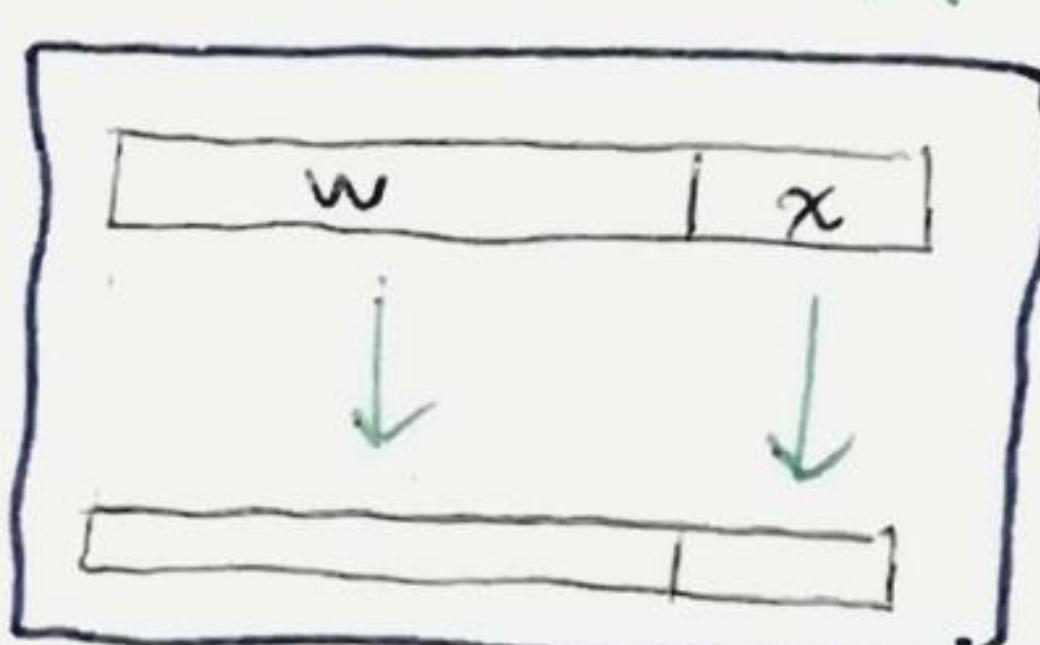
Ex]: page 2A

$$\bar{\lambda}: Q \times (\mathbb{X}^*)^* \longrightarrow (\mathbb{X}^*)^*$$

$$\left\{ \begin{array}{l} \bar{\lambda}(q, \emptyset) := \emptyset \\ \bar{\lambda}(q, w \cdot x) := \bar{\lambda}(q, w) \cdot \bar{\lambda}(\bar{\pi}(q, w), x) \end{array} \right.$$



$$\left\{ \begin{array}{l} \bar{\lambda}(q, w \cdot x) := \underbrace{\bar{\lambda}(q, w)}_{\in X^*} \cdot \underbrace{\bar{\lambda}(\bar{\pi}(q, w), x)}_{\in X^*} \end{array} \right.$$



Notation: $\bar{\lambda}_q(w) := \bar{\lambda}(q, w)$

Ex]: page 2A

DEF] Given A automaton, A_{q_0} , with fixed
INITIAL STATE q_0 , is called
INITIAL AUTOMATON ③

NOTE] (1) A_{q_0} defines $\bar{\lambda}_{q_0}: X^* \rightarrow X^*$, called
the ACTION of A_{q_0} .
(2) λ_q is bijective $\Rightarrow \bar{\lambda}_q$ is bijective
(on X) (on X^*)

Pozztek: Ex: page 2A

Automation \rightsquigarrow Initial Automaton A_{q_0} \rightsquigarrow action $\bar{\lambda}_{q_0}$
 $\rightsquigarrow (X^* \rightarrow X^*)$

Note] If we can define $|Q|$ different λ_q

Ex] (1) Page 2A: (Γ_1) \rightsquigarrow $\begin{cases} (\Gamma_1)_{q_0} & \rightsquigarrow \bar{\lambda}_{q_0} \\ (\Gamma_1)_{q_1} & \rightsquigarrow \bar{\lambda}_{q_1} \\ (\Gamma_1)_{q_2} & \rightsquigarrow \bar{\lambda}_{q_2} \end{cases}$

(2) (Γ_2) $\rightsquigarrow (\Gamma_2)_q$ $\rightsquigarrow \bar{\lambda}_q$

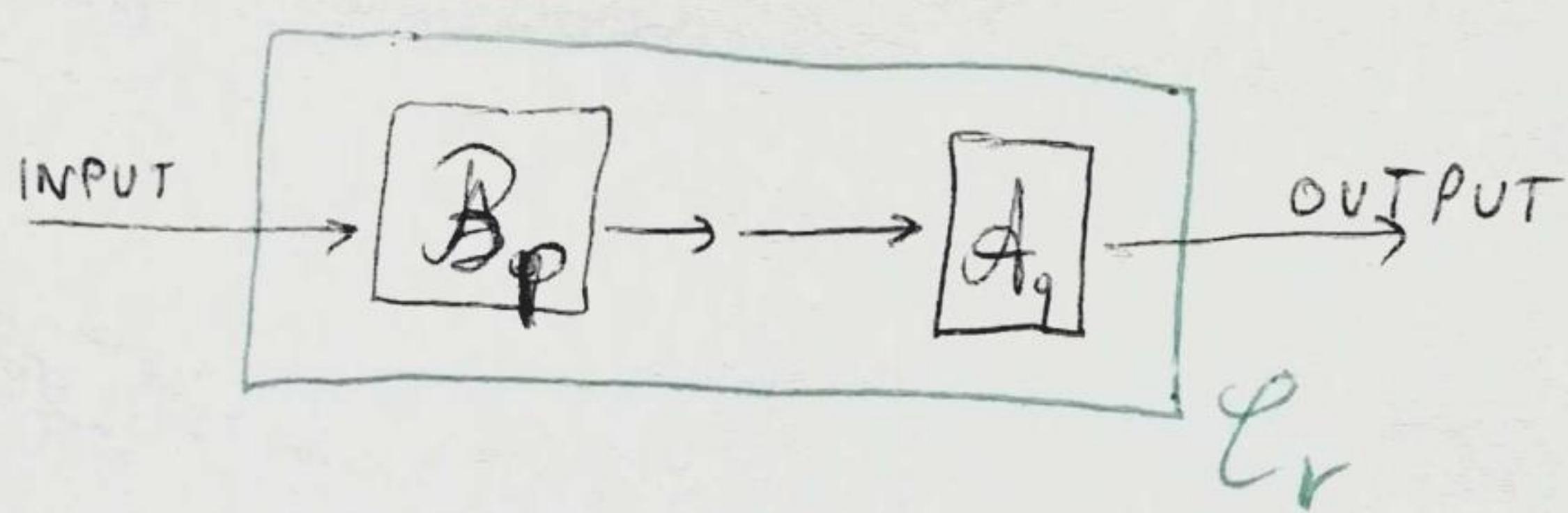
DEF] $f: X^* \rightarrow X^*$ is synchronous automatic
 if is definable as the action of some
 initial automaton A_{q_0} , i.d. (\Leftarrow je) $f = \bar{\lambda}_{q_0}$.

DEF] $S := \{f: X^* \rightarrow X^* \mid f \text{ is synchronous automatic}\}$

Note] $f \in S \Rightarrow f = \bar{\lambda}_{q_0} \Rightarrow f \text{ is bijective}$
We want to study S

Composition lemma] Given A_q and B_p , initial automata
 on X , $\exists C_r$, initial automaton, such that:

$$\left(\text{Action of } C_r \right) = \left(\text{Action of } A_q \right) \circ \left(\text{Action of } B_p \right)$$



$\Rightarrow S$ is closed under composition!

Similarly we have:

~~$[f \in S \Rightarrow f^{-1} \in S]$~~

~~$\Rightarrow (\text{id}: X^* \rightarrow X^*) \in S$~~

$\Rightarrow (S; \circ)$ is a group

CHARACTERIZATION OF THE ACTIONS

OF AUTOMATA

5A

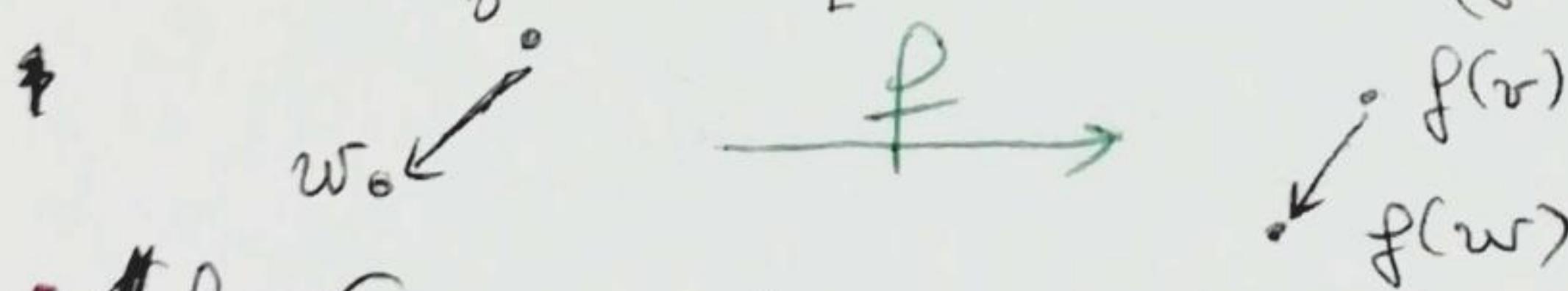
Povzetek:

Automaton $\xrightarrow{\Delta}$ initial automaton $\xrightarrow{\delta_{q_0}} \tilde{I}_{q_0}: X^* \rightarrow X^*$

$S := \{f: X^* \rightarrow X^* \mid f \text{ is the action of some } \tilde{A}_{q_0}\}$

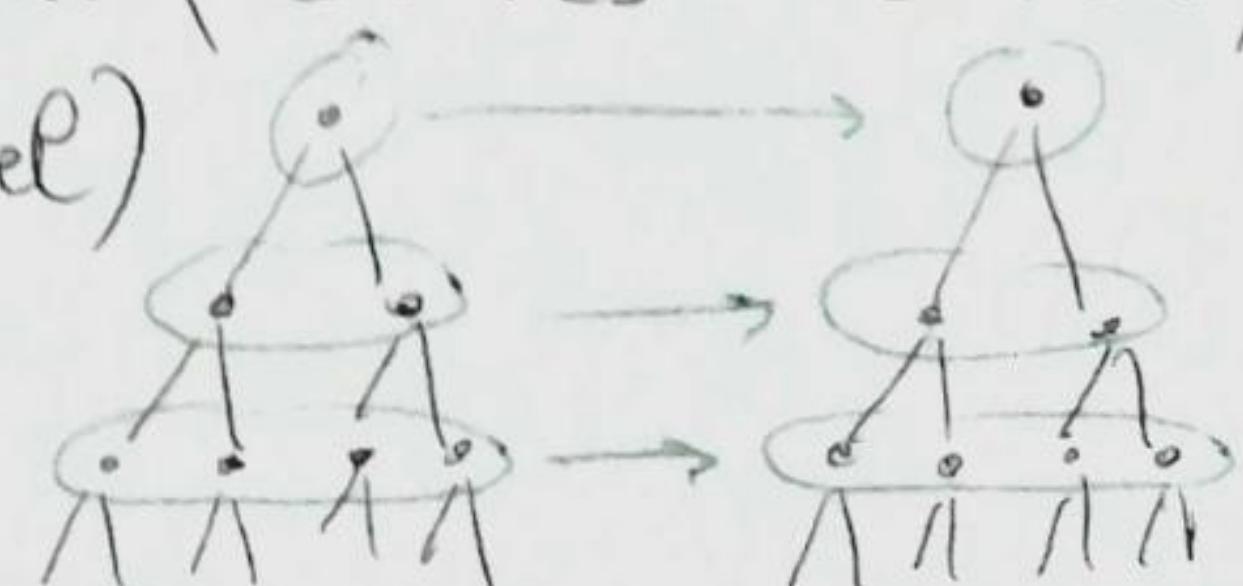
Remark] Given $G = (V, E)$ graph, $f: V \rightarrow V$ is said to be a graph-homomorphism if preserve

the adjacencies $\forall v \in V \quad [(v, w) \in E \Rightarrow (f(v), f(w)) \in E]$



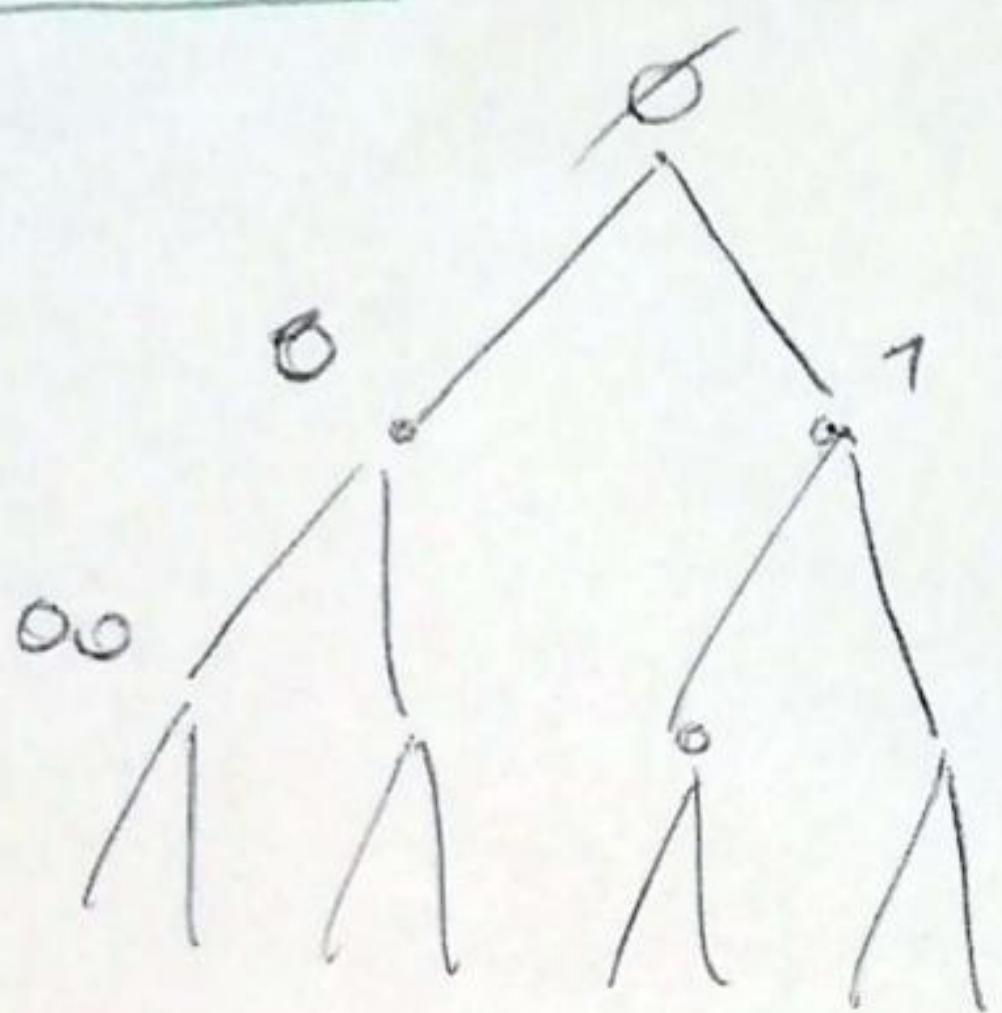
or • If G is a tree, f is said to be a graph-homomorphism that preserves the root if (1) it is a graph-homomorphism and if (2) $f(r) = r$, where r is the root of G

Note] If f is a graph-hom. that preserves the root, $f(n\text{-th level of } G) \subseteq (n\text{-th level})$

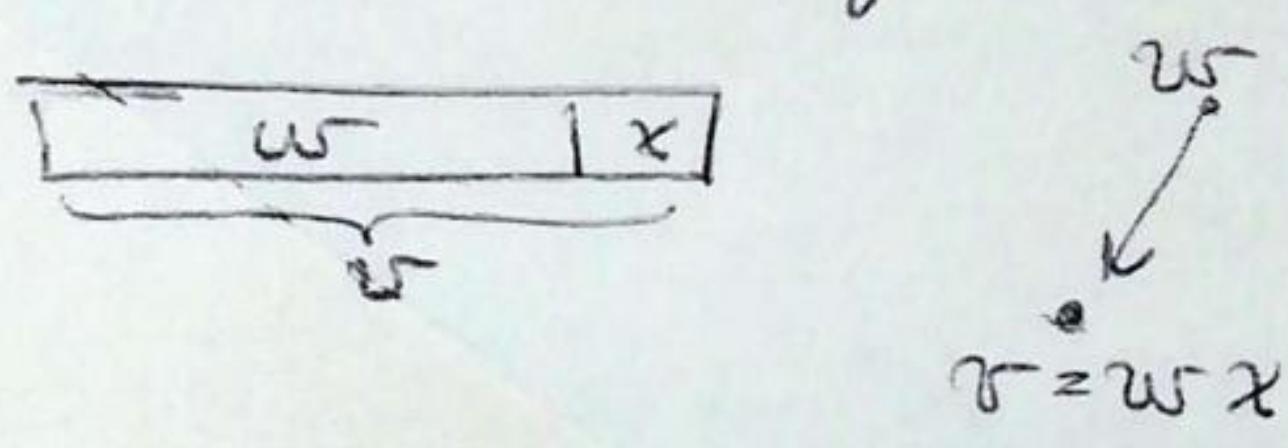


Proposition] f is synch. automatic if and only if (53)
 f is a graph-homomorphism that preserves the root
on X^*

Observation] Remembering that X^* is a tree, what
does it mean to be a graph-homomorphism
that preserve the root on X^* ?



- Given a f with this property condition $f(r) = r$ means $f(\phi) = \phi$
- The condition of being a map homomorphism? In X^* , v is son of w if and only if $v = wx$



So if we want $f(v)$ to be son of $f(w)$ means
 $f(v) = f(w)y$ for some $y \in X$.

$\begin{array}{l} f(w) \\ \swarrow \\ f(w)y = f(v) \end{array}$

Dokaz (\Rightarrow)

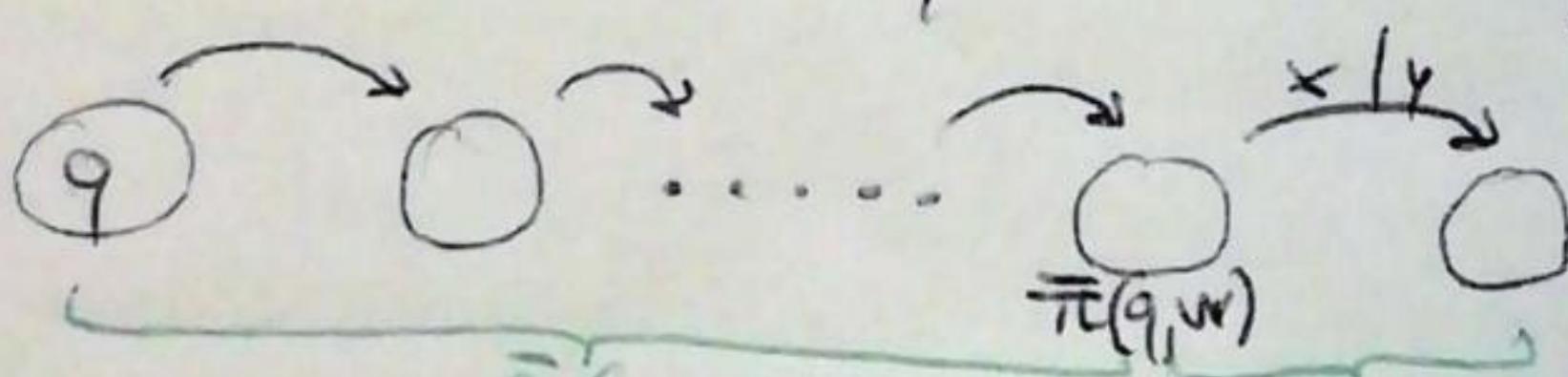
Given f synch. autom., we have $f = \bar{\lambda}_q$

for some action of some $\bar{\lambda}_q$.

(1) $f(\phi) = \bar{\lambda}_q(\phi) = \phi \quad \checkmark$

(2) Given $v = wx$ (v son of w) we have:

$$f(v) = f(wx) = \bar{\lambda}_q(wx) = \bar{\lambda}_q(w) \cdot \bar{\lambda}_{\bar{\pi}(q,w)}(x) = f(w)y$$



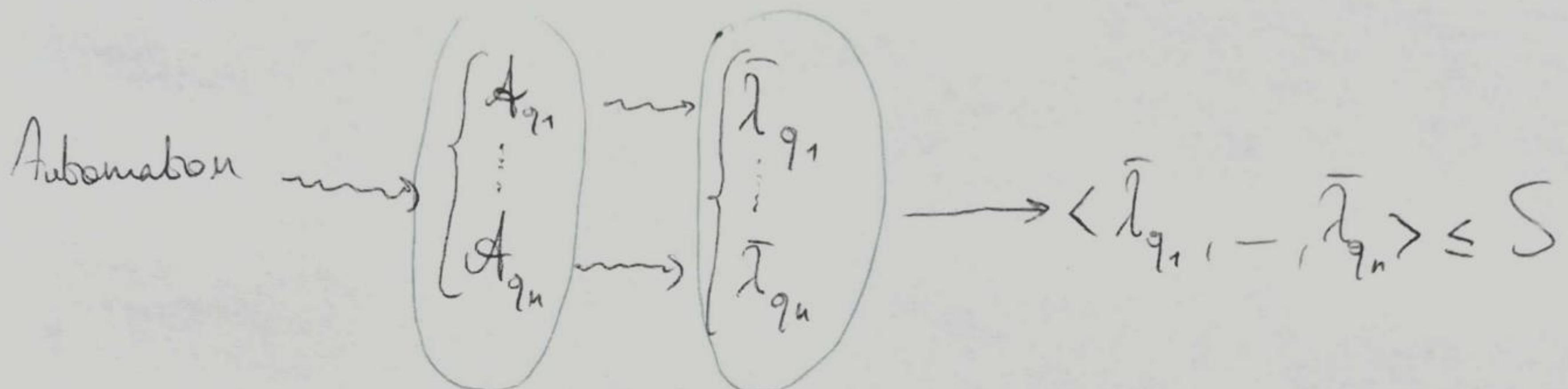
(\Leftarrow) A bit complicated, we'll not see it \square

Corollary] $S = \{\text{bijective graph-homom. that preserve } \phi \text{ on } X^*\}$

DEF Given a automaton we can define $|Q|$ mutual automata \bar{A}_q , so $|Q|$ actions $\bar{\lambda}_q \in S$.

The group generated by \bar{A} is defined as:

$$G(\bar{A}) := \langle \{ \bar{\lambda}_q \mid q \in Q \text{ of } A \} \rangle \leq S$$

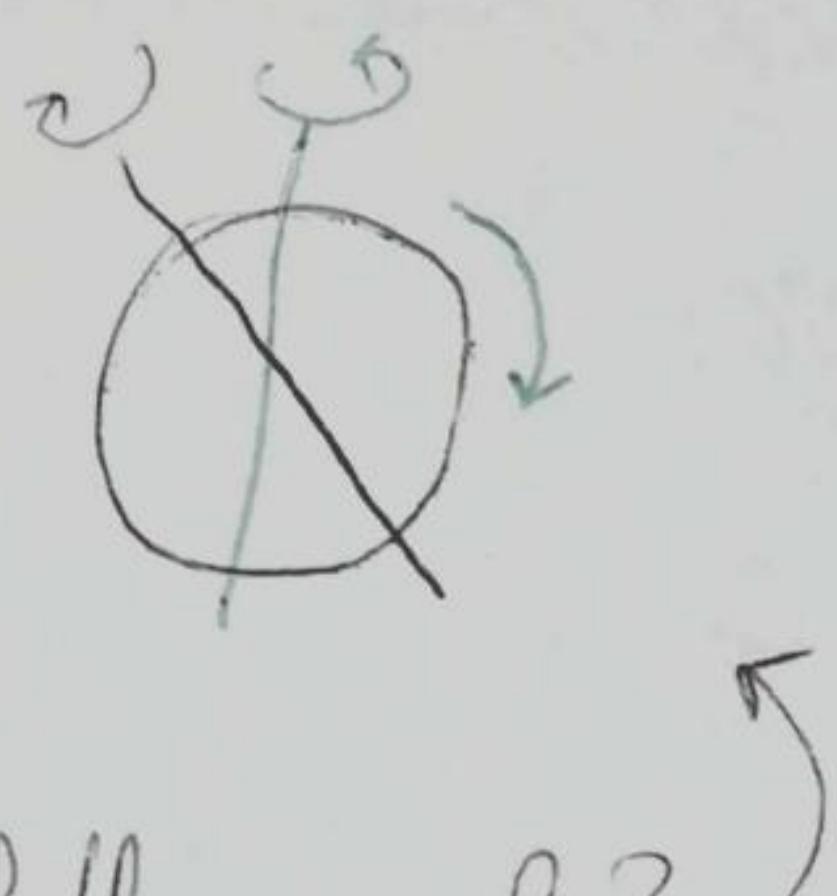


Ex: page 2A, (1) Γ_1 defines $G(\Gamma_1) := \langle \bar{\lambda}_{q_0}, \bar{\lambda}_{q_1}, \bar{\lambda}_{q_2} \rangle$

Interesting Results and Examples

Proposition] Given $X = \{0, 1\}$, and \bar{A} , 2-state automaton on X , $G(\bar{A})$ must be isomorphic to one of these groups:

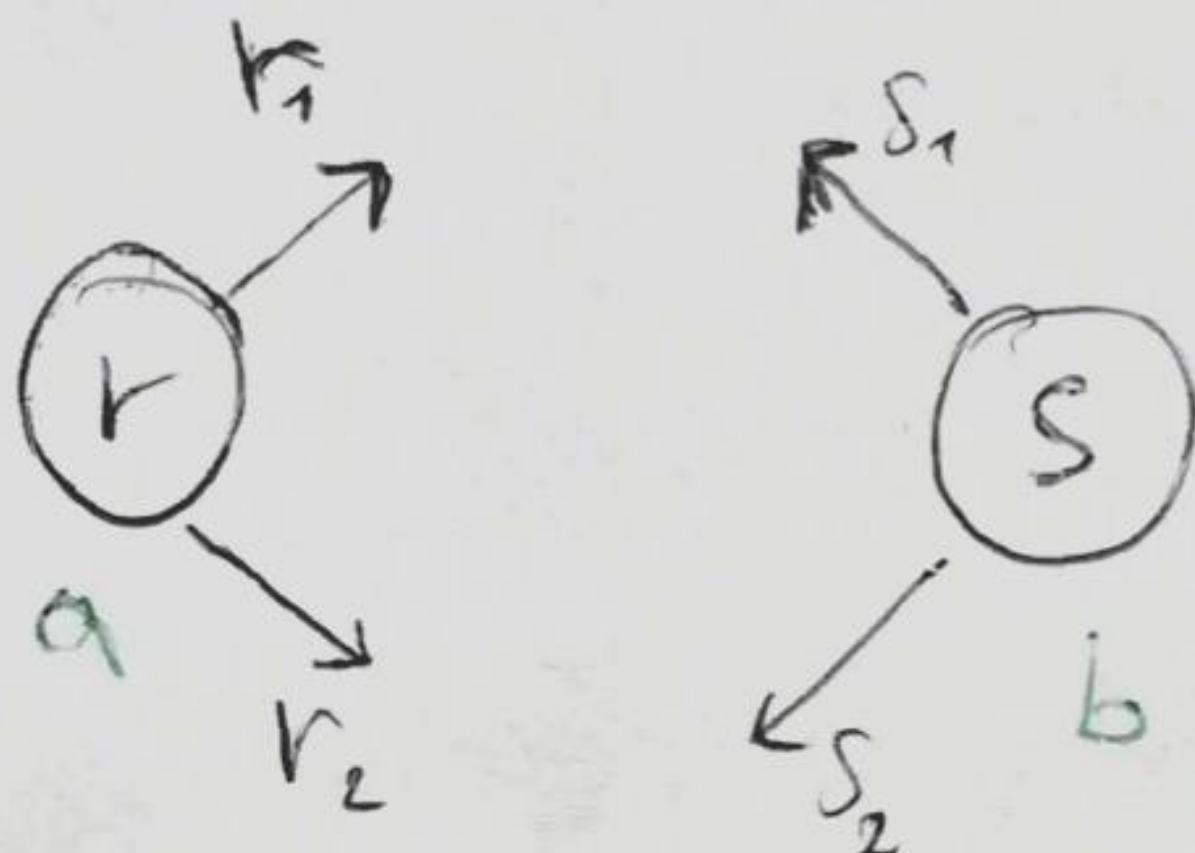
- (1) $\{1_G\}$
- (2) \mathbb{Z}_2
- (3) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$
- (4) \mathbb{Z}
- (5) $D_\infty = \text{Infinite dihedral group} = \{\text{symm. of the circle}\}$
- (6) $\mathbb{Z} \wr \mathbb{Z}_2 = \text{lampighter group}$



(7)

Sketch of proof: Q , the set of states

$$Q = \{r, s\}, X = \{0, 1\}. A = (X, Q, \pi, \alpha)$$



- $\lambda_s, \lambda_r : X \rightarrow X$ are permutation of a 2-element set $X \Rightarrow \lambda_s, \lambda_r \in S_2$,

$$S_2 = \{\text{id} = 1, \sigma\}$$

~~$\lambda_s = \lambda_r$~~ $\begin{cases} \sigma(0) = 1 \\ \sigma(1) = 0 \end{cases}$

- each arrow in $\{r_1, r_2, s_1, s_2\}$ can point on an element of $Q = \{r, s\}$

So all the possible A , with $Q = \{r, s\}, X = \{0, 1\}$ are the ones in which $\lambda_s, \lambda_r \in S_2$ (uniquely determined), and $r_1, r_2, s_1, s_2 \in \{r, s\}$ (uniquely determines π) \Rightarrow

$$\Rightarrow \text{they are } 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6.$$

$$\lambda_s \quad \lambda_r \quad r_1 \quad r_2 \quad s_1 \quad s_2$$

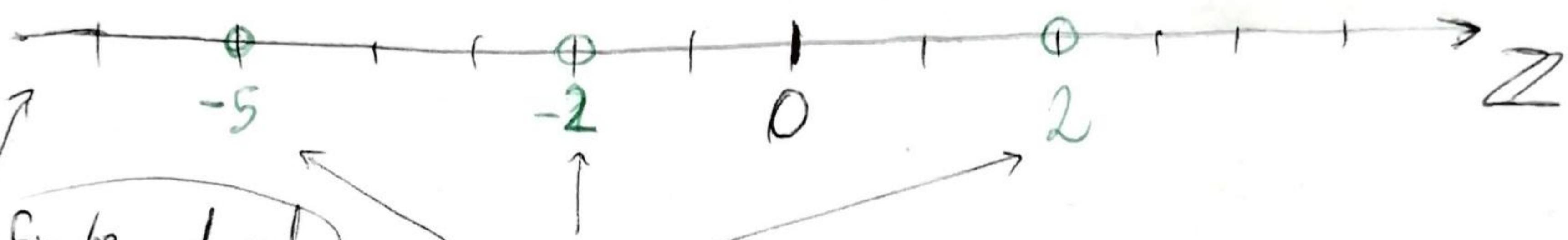
Analysing each case we get $[G(A)$ is isomorphic to ~~one~~ \square of the latter cases].

Interesting group:

$$\mathbb{Z} \wr \mathbb{Z}_2 := (\mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})} ; *)$$

$$\mathbb{Z}_2^{(\mathbb{Z})} := \{(b_i)_{i \in \mathbb{Z}} \mid b_i \in \underline{\mathbb{Z}_2 = \{0, 1\}}, b_i = 1 \text{ just for a } \underline{\text{finite set of indexes } I}\}$$

Practically:



Infinite dark road \mathbb{Z}

indexes of the ~~open~~ lamps turned on
(indexes in I)

Ex] The previous represented element is

$$[(\tilde{b}_i)_{i \in \mathbb{Z}} \text{, where } \tilde{b}_{-5}, \tilde{b}_{-2}, \tilde{b}_2 = 1]$$

[We can sign $(\tilde{b}_i)_{i \in \mathbb{Z}}$ with $\{-5, -2, 2\} = I$]

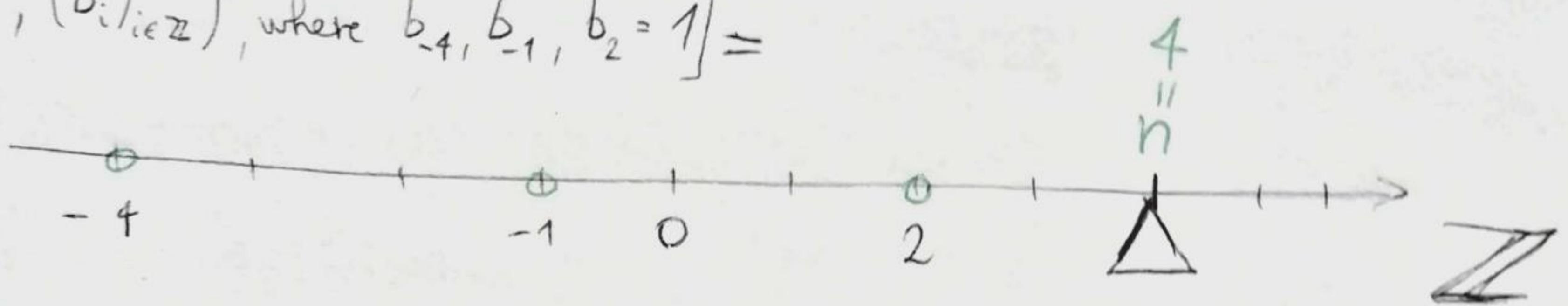
* is not a direct product!

$$(n_1, (b_i)_{i \in \mathbb{Z}}) * (n_2, (q_i)_{i \in \mathbb{Z}}) :=$$

$$= (n_1 + n_2, (b_i + q_i + h_1)_{i \in \mathbb{Z}})$$

Visualization:

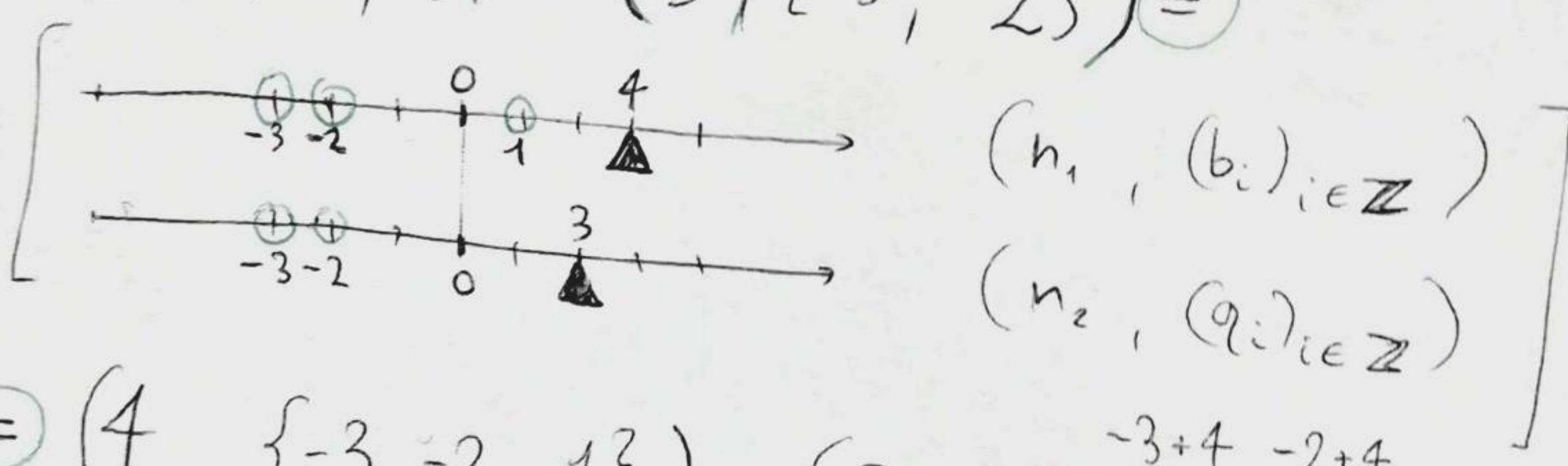
- Of an element $(n, (b_i)_{i \in \mathbb{Z}}) \in \mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}$
 $\left[(4, (b_i)_{i \in \mathbb{Z}}), \text{ where } b_{-4}, b_{-1}, b_2 = 1 \right] =$



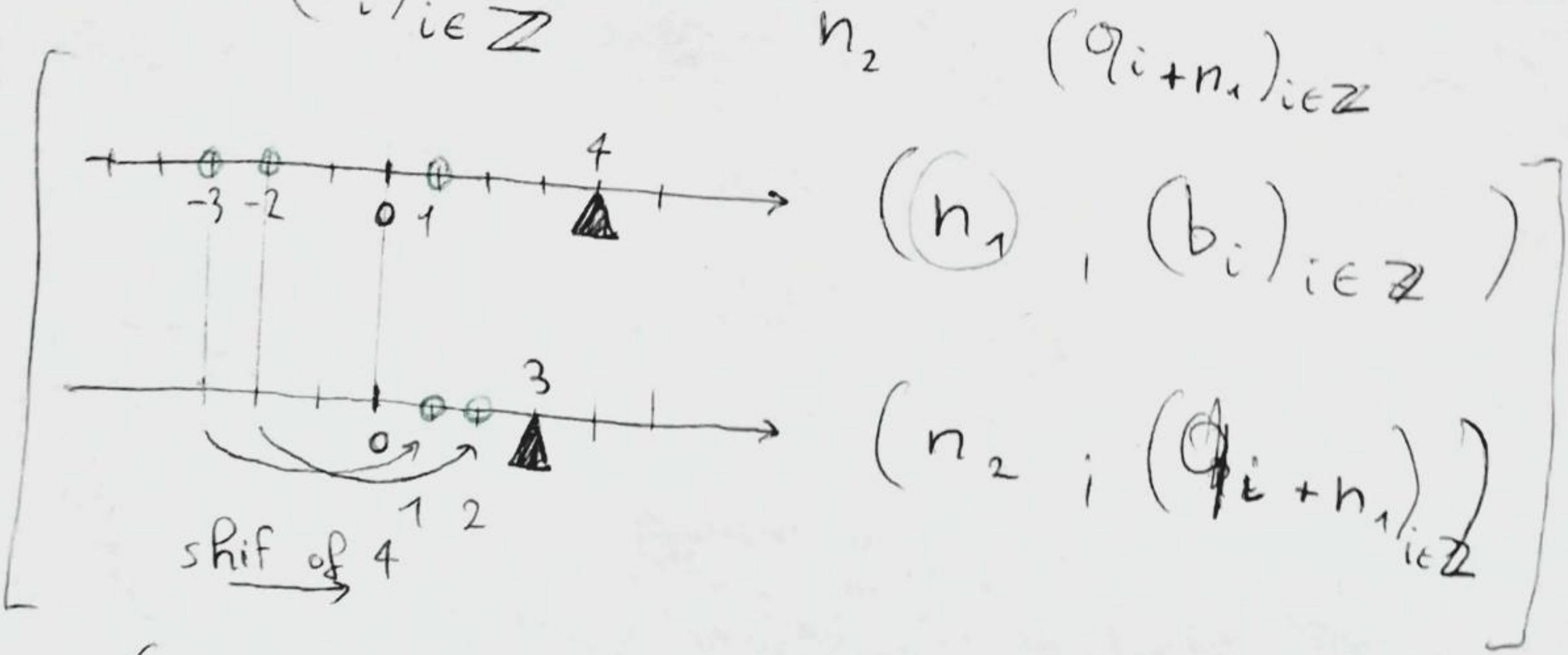
n = position of the "Lamplighter" on the infinite road

- Of the product $*$:

$$(4; \{ -3; -2; 1 \}) * (3; \{ -3; -2 \}) =$$

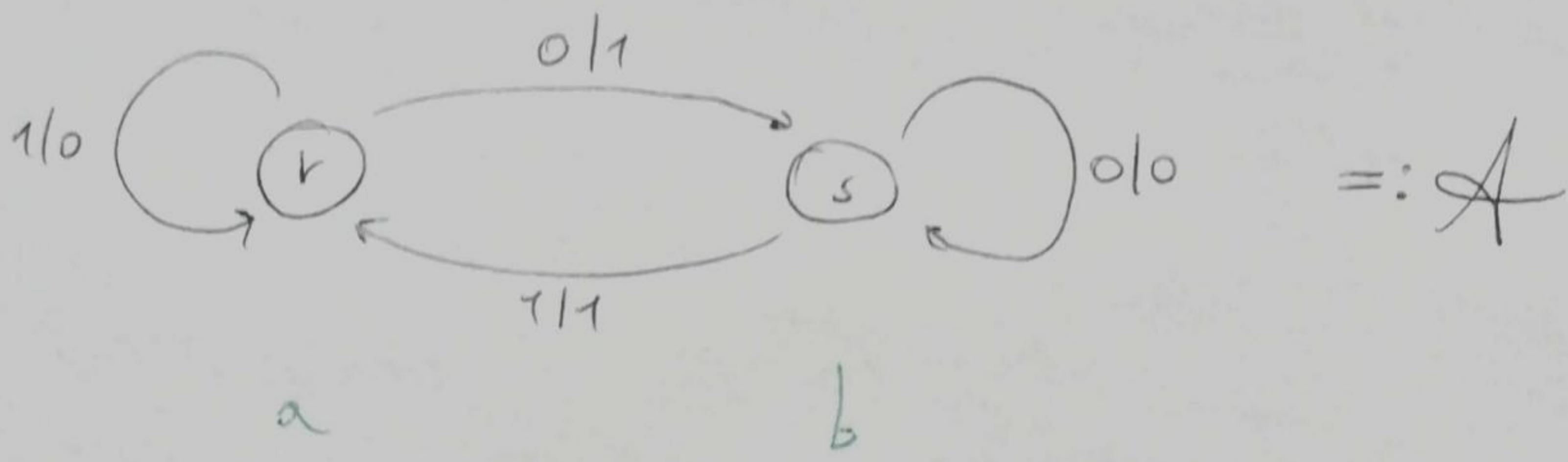


$$= (4; \{ -3, -2, -1 \}) + (3; \{ -3+4, -2+4 \}) [=]$$



$$= (4+3; \{ -3; -2; 2 \})$$

The automaton which defines \mathbb{Z}/\mathbb{Z}_2 :



$$a := \bar{\lambda}_r, \quad b := \bar{\lambda}_s$$

$$\mathbb{Z}/\mathbb{Z}_2 = \langle a, b \rangle = G(\mathcal{A})$$

let's watch closer:

$$\begin{cases} a(0v) = 1b(v) \\ a(1v) = 0b(v) \end{cases} \quad \begin{cases} b(0v) = 0b(v) \\ b(1v) = 1a(v) \end{cases}$$

$$\lambda_r: X \rightarrow X$$

$$\lambda_r = \sigma \in S_2$$

$$\lambda_s: X \rightarrow X$$

$$\lambda_s = id \in S_2$$

$$\Rightarrow b^{-1} = \begin{cases} b^{-1}(0v) = 0b^{-1}(v) \\ b^{-1}(1v) = 1a^{-1}(v) \end{cases}$$

\downarrow
 X^*

$$c := b^{-1} \cdot a = a \circ b^{-1}$$

$$\begin{cases} c(0v) = a \circ b^{-1}(0v) \circ a(0 \underbrace{b^{-1}(v)}_{\text{id}}) = 1 \underbrace{b \circ b^{-1}(v)}_{\text{id}} = 1v \\ c(1v) = a \circ b^{-1}(1v) = a(1 \underbrace{b^{-1}(v)}_{\text{id}}) = 0 \underbrace{b \circ b^{-1}(v)}_{\text{id}} = 0v \end{cases}$$

• we see $\langle a, b \rangle = \langle b^{-1}a, b \rangle = \langle c, b \rangle = G(A)$

$$b^{-1}a = \underset{\text{a}}{c}$$

$$\cdot \mathcal{Z}_2 = (x; +_{\mathcal{Z}_2}) \quad 0+0=0 \\ 1+1=0 \\ 1+0=1$$

$$c(x_1 x_2 x_3 \dots) = (x_1 + 1) x_2 x_3 \dots$$

we search an explicit formula for b :

$$b(x_1 x_2 x_3 \dots) = y_1 y_2 y_3 \dots \quad y_n = ?$$

$$\text{we claim } (A) b(x_1 x_2 \dots x_n \overset{(n+1)}{\bigcirc} x_{n+2} \dots) =$$

$$= y_1 y_2 \dots y_n y_{n+1} b(x_{n+2} \dots)$$

$$(B) b(x_1 x_2 \dots x_n \overset{(n+1)}{1} x_{n+2} \dots) =$$

$$= y_1 y_2 \dots y_n y_{n+1} a(x_{n+2} \dots)$$

Proof

(A) Watch diagram of A. Whenever we encounter a "0", b acts on the next letter

(B) Analogous

$$\text{We claim } b(x, x_2 - x_n -) = x_1 \underbrace{(x_2 + x_1)}_{y_1} - \underbrace{(x_n + x_{n-1})}_{\underbrace{y_2}_{y_n}} - \quad (12)$$

Proof: For induction on n .

$$n=1 \text{ We set } x_0 = 0 \Rightarrow y_1 = x_1 = x_1 + 0 = x_1 + x_0$$

$$n \rightsquigarrow n+1 \text{ 4 cases: } x_n x_{n+1} \in \{00, 01, 10, 11\}$$

Case 00:

$$y_n = \cancel{x_n} + \cancel{x_{n+1}} ; y_{n+1} = ? \\ b(x_1 - x_{n-1} \overset{n}{\underset{00}{\circ}} x_{n+2} -) \stackrel{(A)}{=} y_1 - y_n b(0 x_{n+2} -) = \\ = y_1 - y_n \circ y_{n+2} -$$

$$y_{n+1} = 0 = 0 + 0 = x_n + x_{n+1}$$

The other cases are analogous ✓

$$\Rightarrow c(x, x_2 x_3 -) = \underbrace{(x_1 + 1)}_{\sigma(x_1)} x_2 x_3 -$$

$$b(x, x_2 x_3 -) = x_1 (x_2 + x_1)(x_3 + x_2) - = \\ = x_1 x_2 x_3 x_4 - +$$

$$[\text{shift to right}] \quad x_1 x_2 x_3 - *$$

It can be proved, with these formulas, that

$$\langle b, c \rangle = \mathbb{Z} \setminus \mathbb{Z}_2$$