Groups of Automata

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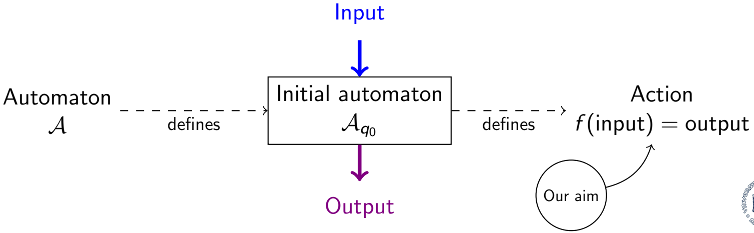
1. Introduction

The word automaton: from the greek "acting of one's own will".
Automata are important in:

- Information theory
- Theory of dynamical systems
- Algebra
- Others

My aim: study some of the groups constructed through a special class of them, the invertible deterministic Mealy automata, here called simply automata.



2. The automaton

Definition

An automaton is a 4-tuple $\mathcal{A} = \langle \mathsf{X}, \mathcal{Q}, \pi, \lambda \rangle$ where:

- $\mathsf{X} = \{x_1, \ldots, x_k\}$ is a finite set called the alphabet,
- $\mathcal{Q}$ is a set called the set of internal states of the automaton,
- $\pi : \mathsf{X} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is a function called the transition function,
- $\lambda : \mathsf{X} \times \mathcal{Q} \longrightarrow \mathsf{X}$ is a function such that $\lambda_q = \lambda(\cdot, q) : \mathsf{X} \longrightarrow \mathsf{X}$ is bijective, and is called the output function.

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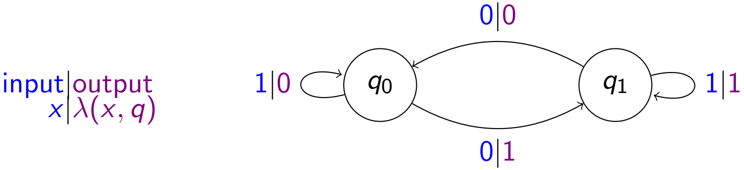

Figure: Moore diagram of a 2-state automaton over $\mathsf{X} = \{0, 1\}$

3. The initial automaton

Definition

$X^* = \{x_1 x_2 \dots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the dictionary.

Word composition: $x_1 \dots x_n . z_1 \dots z_n := x_1 \dots x_n z_1 \dots z_n$

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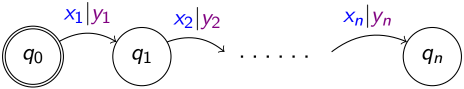
Definition

An initial automaton $\mathcal{A}_{q_0}$ is an automaton $\mathcal{A}$ with a fixed state q_0.
The action of $\mathcal{A}_{q_0}$ is the function $\bar{\lambda}_{q_0} : X^* \longrightarrow X^*$ with
$\bar{\lambda}_{q_0}(x_1x_2\dots x_n) = y_1y_2\dots y_n$.



4. The word tree X^*

Definition

Given $w, v \in X^*$, w is a child of v if and only if $w = v.x = vx$ for some letter $x \in X$.

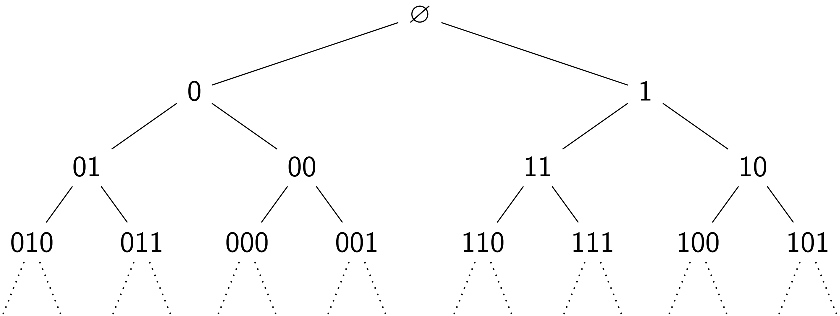


Figure: An example of the word tree X^* on $X = \{0, 1\}$.

5. Actions as tree-automorphisms

Proposition

A function $f : \mathrm{X}^* \longrightarrow \mathrm{X}^*$ is the action of some initial automaton if and only if is a tree-automorphism on the word tree X^*, i.e.:

- $f(\varnothing) = \varnothing$.
- if $\mathrm{w} \in \mathrm{X}^*$ is a child of v then $f(\mathrm{w})$ is a child of $f(\mathrm{v})$.
- f is bijective.

$$
\begin{tikzcd}
& \mathrm{v} \arrow[rr, dotted, bend left, "f"] \arrow[dl, dash] & & f(\mathrm{v}) \arrow[rr, dotted, bend left, "g"] \arrow[dl, dash] & & (g \circ f)(\mathrm{v}) \arrow[dl, dash] \\
\mathrm{w} = \mathrm{v}x \arrow[rr, dotted, bend left, "f"] & & f(\mathrm{v})y \arrow[rr, dotted, bend left, "g"] & & (g \circ f)(\mathrm{v})z &
\end{tikzcd}
$$

Proposition

If f, g are tree-automorphisms on X^*, then $g \circ f$ and f^{-1} are tree-automorphisms on X^*.

Proposition

The functions $f : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ defined by initial automata (i.e. tree-automorphisms), called synchronous automatic permutations, form a group denoted by $\mathcal{AUT}_{tree}(\mathsf{X}^*)$.

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Definition

Let $\mathcal{A} = \langle \mathsf{X}, \mathcal{Q}, \pi, \lambda \rangle$ be an automaton. The group defined by $\mathcal{A}$ is the group generated by the set $\{\bar{\lambda}_q : q \in \mathcal{Q}\}$.

$$\begin{array}{c}\mathcal{A} \text{ with} \\ \mathcal{Q} = \{q_1, \dots, q_n\}\end{array} \xrightarrow[\text{defines}]{} \begin{array}{c}\bar{\lambda}_{q_1} : \mathsf{X}^* \longrightarrow \mathsf{X}^* \\ \bar{\lambda}_{q_2} : \mathsf{X}^* \longrightarrow \mathsf{X}^* \\ \vdots \\ \bar{\lambda}_{q_n} : \mathsf{X}^* \longrightarrow \mathsf{X}^*\end{array} \xrightarrow[\text{generate}]{} \langle \{\bar{\lambda}_q : q \in \mathcal{Q}\} \rangle$$

7. Semidirect product and faithful actions

Definition

Let $(B, *_B), (N, *_N)$ be groups and $\varphi : B \longrightarrow \mathcal{AUT}(N)$ an homomorphism, where $\mathcal{AUT}(N)$ denotes the group of (group-)automorphisms on N. Then the semidirect product $B \ltimes_\varphi N$ is the group $(B \times N, *_\varphi)$, with composition rule:

$$\left(b_2, n_2\right) *_\varphi \left(b_1, n_1\right) = \left(\ b_2 *_B b_1,\ \varphi(b_1)(n_2) *_N n_1\right)$$

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Definition
Let B be a group and $\mathcal{S}(Y)$ be the simmetric group on Y. Then a faithful action φ of B on Y is a monomorphism $\varphi : B \longrightarrow \mathcal{S}(Y)$ and we sign $yb := \varphi(b)(y)$.

8. Wreath product

Direct sum

Let A be a group and Y a set. The direct sum of A on Y is the set

$$A^{(Y)} := \{(a_\omega)_{\omega \in Y} : a_\omega \in A \text{ and } a_\omega \neq 1_A \text{ only for a finite number of } \omega\}.$$

equipped with the component-wise operation of A. If Y is finite we have $A^{(Y)} = A^Y = A \times A \cdots \times A$.

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Wreath Product

Let B be a group which acts faithfully on Y and let A be a group. Then the wreath product $B \wr A$ is the semidirect product $B \ltimes_\Phi A^{(Y)} = (B \times A^{(Y)}, *)$ where the composition rule is:

$$\begin{aligned}(g, (u_y)_{y\in Y}) * (h, (v_y)_{y\in Y}) &:= (gh, \Phi(h)((u_y)_{y\in Y})(v_y)_{y\in Y}) := \\ &= (gh, (u_{yh}v_y)_{y\in Y}).\end{aligned}$$

Proposition

Let $X = \{x_1, \ldots, x_k\}$. Then the function $\psi : \mathcal{AUT}_{tree}(X^*) \longrightarrow \mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*) = \big(\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *\big)$ defined by

$$\psi(\bar{\lambda}_{q_0}) = \lambda_{q_0}(\bar{\lambda}_{\pi(x_1,q_0)}, \ldots, \bar{\lambda}_{\pi(x_n,q_0)})$$

is an isomorphism of groups.

10. System of formulas

Proposition

Let $\mathcal{A}$ be an automaton with $\mathcal{Q} = \{q_1, \dots, q_n\}$ over $X = \{x_1, \dots, x_k\}$. Then $\mathcal{A}$ is described by n recurrent formulas

$$\begin{aligned}
f_{q_1} &= \beta_{q_1}(h_{x_1,q_1}, \dots, h_{x_k,q_1}), \\
f_{q_2} &= \beta_{q_2}(h_{x_1,q_2}, \dots, h_{x_k,q_2}), \\
&\vdots \\
f_{q_n} &= \beta_{q_n}(h_{x_1,q_n}, \dots, h_{x_k,q_n}),
\end{aligned}$$

where each h_{x_i,q_j} is equal to some f_{q_l} and each $\beta_{q_j} \in \mathcal{S}(X)$. Conversely, each such set of n recursive formulas defines an automaton $\mathcal{A}$ such that $\bar{\lambda}_{q_j} = f_{q_j}$ for every $q_j \in \mathcal{Q}$.

$$\mathcal{A} \longleftrightarrow \begin{array}{c}
\bar{\lambda}_{q_1} = \lambda_{q_1}(\bar{\lambda}_{\pi(x_1,q_1)}, \dots, \bar{\lambda}_{\pi(x_n,q_1)}) \\
\bar{\lambda}_{q_2} = \lambda_{q_2}(\bar{\lambda}_{\pi(x_1,q_2)}, \dots, \bar{\lambda}_{\pi(x_n,q_2)}) \\
\vdots \\
\bar{\lambda}_{q_n} = \lambda_{q_n}(\bar{\lambda}_{\pi(x_1,q_n)}, \dots, \bar{\lambda}_{\pi(x_n,q_n)})
\end{array} \longleftrightarrow \begin{array}{c}
f_{q_1} = \beta_{q_1}(h_{x_1,q_1}, \dots, h_{x_k,q_1}) \\
f_{q_2} = \beta_{q_2}(h_{x_1,q_2}, \dots, h_{x_k,q_2}) \\
\vdots \\
f_{q_n} = \beta_{q_n}(h_{x_1,q_n}, \dots, h_{x_k,q_n})
\end{array}$$

11. The classification theorem

Theorem

Let $\mathcal{A}$ be a 2-state automaton over the alphabet $X = \{0, 1\}$ and G the group defined by this automaton. Then G is isomorphic to one of the following groups:

- the trivial group $\{1_G\}$,
- $\mathbb{Z}_2$,
- $\mathbb{Z}_2 \times \mathbb{Z}_2$,
- $\mathbb{Z}$,
- the infinite dihedral group $\mathbb{Z}_2 \ltimes_\phi \mathbb{Z}$ (where $\phi(h)(z) := z(-1)^h$),
- the lamplighter group $\mathbb{Z} \wr \mathbb{Z}_2 = \mathbb{Z} \ltimes \mathbb{Z}_2^{(\mathbb{Z})}$ (where $\mathbb{Z}$ acts on itself by $mz := z + m$).

12. Sketch of proof

Define the cases

Let $\mathcal{Q} = \{r, s\}$ and $a = \bar{\lambda}_r, b = \bar{\lambda}_s$, then

$$\mathcal{A} \dashleftarrow\!\dashrightarrow \begin{array}{l} a = \tau^{i_1}(x_{11}, x_{12}) \\ b = \tau^{i_2}(x_{21}, x_{22}) \end{array}$$

where $x_{ij} \in \{a, b\}$ and $\tau^{i_1}, \tau^{i_2} \in \mathcal{S}(\mathrm{X}) = \mathcal{S}(\{0, 1\})$. There are 64 possibilities. We proceed by analyzing a part of them.

Thank you for your attention!