

Groups of Automata

Carlo Lanzi Luciani

Mentors:

izr. prof. Ganna Kudryavtseva

prof. Alessandro Logar

June 7, 2021



1. Introduction

The word **automaton**: from the greek "acting of one's own will".
Automata are important in:

- ▶ Information theory
- ▶ Theory of dynamical systems
- ▶ Algebra
- ▶ Others

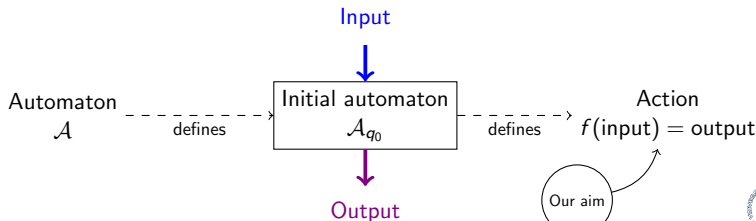


1. Introduction

The word **automaton**: from the greek "acting of one's own will".
Automata are important in:

- ▶ Information theory
- ▶ Theory of dynamical systems
- ▶ Algebra
- ▶ Others

My aim: study some of the groups constructed through a special class of them, the invertible deterministic Mealy automata, here called automata.



2. The automaton

Definition

An **automaton** is a 4-tuple $\mathcal{A} = \langle X, Q, \pi, \lambda \rangle$ where:

- ▶ $X = \{x_1, \dots, x_k\}$ is a finite set called the **alphabet**,
- ▶ Q is a set called the **set of internal states of the automaton**,
- ▶ $\pi : X \times Q \longrightarrow Q$ is called the **transition function**,
- ▶ $\lambda : X \times Q \longrightarrow X$ is such that $\lambda_q = \lambda(\cdot, q) : X \longrightarrow X$ is bijective, and is called the **output function**.



2. The automaton

Definition

An **automaton** is a 4-tuple $\mathcal{A} = \langle X, Q, \pi, \lambda \rangle$ where:

- ▶ $X = \{x_1, \dots, x_k\}$ is a finite set called the **alphabet**,
- ▶ Q is a set called the **set of internal states of the automaton**,
- ▶ $\pi : X \times Q \longrightarrow Q$ is called the **transition function**,
- ▶ $\lambda : X \times Q \longrightarrow X$ is such that $\lambda_q = \lambda(\cdot, q) : X \longrightarrow X$ is bijective, and is called the **output function**.

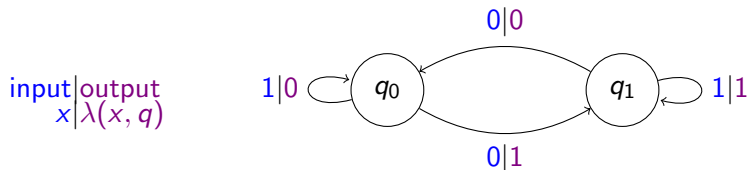


Figure: Moore diagram of a 2-state automaton over $X = \{0, 1\}$



3. The initial automaton

Definition

$X^* = \{x_1x_2 \dots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the **dictionary**.

Word composition: $x_1 \dots x_n \cdot z_1 \dots z_n := x_1 \dots x_n z_1 \dots z_n$



3. The initial automaton

Definition

$X^* = \{x_1 x_2 \dots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the **dictionary**.

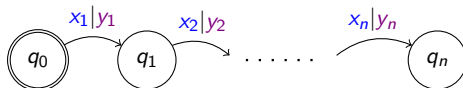
Word composition: $x_1 \dots x_n \cdot z_1 \dots z_n := x_1 \dots x_n z_1 \dots z_n$

Definition

An **initial automaton** \mathcal{A}_{q_0} is an automaton \mathcal{A} with a fixed state q_0 .

The **action of** \mathcal{A}_{q_0} is the function $\bar{\lambda}_{q_0} : X^* \rightarrow X^*$ with

$\bar{\lambda}_{q_0}(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n$.



4. The word tree X^*

Definition

$\underline{w} = w_1 \dots w_n$ is a child of $v_1 \dots v_m = \underline{v}$
 \iff
 $w_1 \dots w_n = v_1 \dots v_m x$ for some letter $x \in X$.

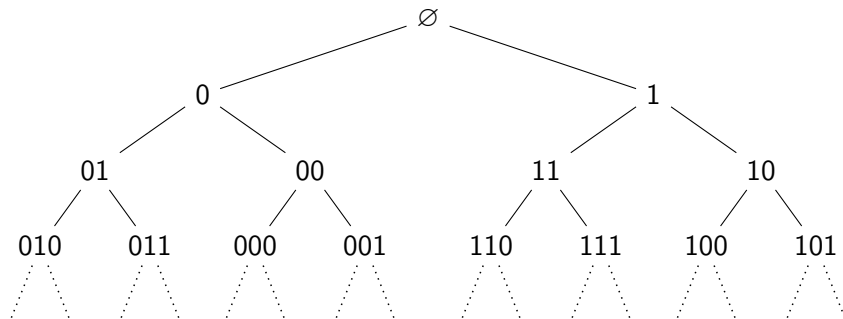


Figure: An example of the word tree X^* on $X = \{0, 1\}$.

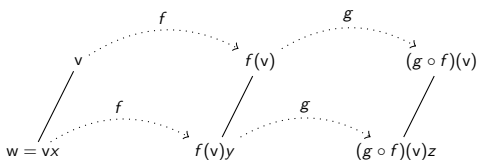


5. Actions as tree-automorphisms

Proposition

A function $f : X^* \rightarrow X^*$ is the action of some initial automaton if and only if it is a **tree-automorphism** on the word tree X^* , i.e.:

- ▶ $f(\emptyset) = \emptyset$.
- ▶ if $\underline{w} \in X^*$ is a child of \underline{v} then $f(\underline{w})$ is a child of $f(\underline{v})$.
- ▶ f is bijective.



6. Groups defined by automata

Proposition

The functions $f : X^* \longrightarrow X^*$ defined by initial automata (i.e. tree-automorphisms) form a group denoted by $\mathcal{AUT}_{tree}(X^*)$.



6. Groups defined by automata

Proposition

The functions $f : X^* \longrightarrow X^*$ defined by initial automata (i.e. tree-automorphisms) form a group denoted by $\mathcal{AUT}_{tree}(X^*)$.

Definition

Let $\mathcal{A} = \langle X, \mathcal{Q}, \pi, \lambda \rangle$ be an automaton. The **group defined by \mathcal{A}** is the group generated by the set $\{\bar{\lambda}_q : q \in \mathcal{Q}\}$.

$$\begin{array}{ccc} \mathcal{A} \text{ with } & & \bar{\lambda}_{q_1} : X^* \longrightarrow X^* \\ \mathcal{Q} = \{q_1, \dots, q_n\} & \xrightarrow{\text{defines}} & \bar{\lambda}_{q_2} : X^* \longrightarrow X^* \\ & & \vdots \\ & & \bar{\lambda}_{q_n} : X^* \longrightarrow X^* \end{array} \quad \xrightarrow{\text{generate}} \quad \langle \{\bar{\lambda}_q : q \in \mathcal{Q}\} \rangle$$



7. Wreath product and automata

Definition

Let $\mathcal{S}(X)$ be the symmetric group on $X = \{x_1, \dots, x_k\}$. Then the wreath product $\mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*)$ is the group $(\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *)$ where the multiplication rule is:

$$\begin{aligned} \gamma(c_{x_1}, \dots, c_{x_k}) * \alpha(a_{x_1}, \dots, a_{x_k}) := \\ \gamma \circ \alpha(c_{\alpha(x_1)} \circ a_{x_1}, \dots, c_{\alpha(x_k)} \circ a_{x_k}) \end{aligned}$$



7. Wreath product and automata

Definition

Let $\mathcal{S}(X)$ be the symmetric group on $X = \{x_1, \dots, x_k\}$. Then the wreath product $\mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*)$ is the group $(\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *)$ where the multiplication rule is:

$$\begin{aligned} \gamma(c_{x_1}, \dots, c_{x_k}) * \alpha(a_{x_1}, \dots, a_{x_k}) := \\ \gamma \circ \alpha(c_{\alpha(x_1)} \circ a_{x_1}, \dots, c_{\alpha(x_k)} \circ a_{x_k}) \end{aligned}$$

Proposition

Let $X = \{x_1, \dots, x_k\}$. Then the function $\psi : \mathcal{AUT}_{tree}(X^*) \longrightarrow \mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*) = (\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *)$ defined by

$$\psi(\bar{\lambda}_{q_0}) = \lambda_{q_0}(\bar{\lambda}_{\pi(x_1, q_0)}, \dots, \bar{\lambda}_{\pi(x_n, q_0)})$$

is an isomorphism of groups.



8. System of formulas

Proposition

Let \mathcal{A} be an automaton with $\mathcal{Q} = \{q_1, \dots, q_n\}$ over $X = \{x_1, \dots, x_k\}$. Then \mathcal{A} is described by n recurrent formulas

$$f_{q_1} = \beta_{q_1}(h_{x_1, q_1}, \dots, h_{x_k, q_1}),$$

$$f_{q_2} = \beta_{q_2}(h_{x_1, q_2}, \dots, h_{x_k, q_2}),$$

$$\vdots$$

$$f_{q_n} = \beta_{q_n}(h_{x_1, q_n}, \dots, h_{x_k, q_n}),$$

where each h_{x_i, q_j} is equal to some f_{q_l} and each $\beta_{q_j} \in \mathcal{S}(X)$.



8. System of formulas

Proposition

Let \mathcal{A} be an automaton with $\mathcal{Q} = \{q_1, \dots, q_n\}$ over $X = \{x_1, \dots, x_k\}$. Then \mathcal{A} is described by n recurrent formulas

$$\begin{aligned} f_{q_1} &= \beta_{q_1}(h_{x_1, q_1}, \dots, h_{x_k, q_1}), \\ f_{q_2} &= \beta_{q_2}(h_{x_1, q_2}, \dots, h_{x_k, q_2}), \\ &\vdots \\ f_{q_n} &= \beta_{q_n}(h_{x_1, q_n}, \dots, h_{x_k, q_n}), \end{aligned}$$

where each h_{x_i, q_j} is equal to some f_{q_l} and each $\beta_{q_j} \in \mathcal{S}(X)$. Conversely, each such set of n recursive formulas defines an automaton \mathcal{A} such that $\bar{\lambda}_{q_j} = f_{q_j}$ for every $q_j \in \mathcal{Q}$.

$$\begin{array}{ccc} \bar{\lambda}_{q_1} = \lambda_{q_1}(\bar{\lambda}_{\pi(x_1, q_1)}, \dots, \bar{\lambda}_{\pi(x_n, q_1)}) & & f_{q_1} = \beta_{q_1}(h_{x_1, q_1}, \dots, h_{x_k, q_1}) \\ \bar{\lambda}_{q_2} = \lambda_{q_2}(\bar{\lambda}_{\pi(x_1, q_2)}, \dots, \bar{\lambda}_{\pi(x_n, q_2)}) & & f_{q_2} = \beta_{q_2}(h_{x_1, q_2}, \dots, h_{x_k, q_2}) \\ & \vdots & \vdots \\ \bar{\lambda}_{q_n} = \lambda_{q_n}(\bar{\lambda}_{\pi(x_1, q_n)}, \dots, \bar{\lambda}_{\pi(x_n, q_n)}) & & f_{q_n} = \beta_{q_n}(h_{x_1, q_n}, \dots, h_{x_k, q_n}) \end{array}$$



9. The classification theorem

Theorem

Let \mathcal{A} be a 2-state automaton over the alphabet $X = \{0, 1\}$ and G the group defined by this automaton. Then G is isomorphic to one of the following groups:

- ▶ the trivial group $\{1_G\}$,
- ▶ \mathbb{Z}_2 ,
- ▶ $\mathbb{Z}_2 \times \mathbb{Z}_2$,



9. The classification theorem

Theorem

Let \mathcal{A} be a 2-state automaton over the alphabet $X = \{0, 1\}$ and G the group defined by this automaton. Then G is isomorphic to one of the following groups:

- ▶ the trivial group $\{1_G\}$,
- ▶ \mathbb{Z}_2 ,
- ▶ $\mathbb{Z}_2 \times \mathbb{Z}_2$,
- ▶ \mathbb{Z} ,
- ▶ the infinite dihedral group $\mathcal{D}_\infty = (\mathbb{Z}_2 \times \mathbb{Z}, *)$, where:

$$(\textcolor{red}{h}, z_1) * (\textcolor{red}{k}, z_2) = (\textcolor{red}{h} +_{\mathbb{Z}_2} \textcolor{red}{k}, (-1)^{\textcolor{red}{k}} z_1 + z_2),$$

- ▶ the lamplighter group $\mathcal{L} = (\mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}, *)$, where:

$$(\textcolor{red}{z}_2, (h_i)_{i \in \mathbb{Z}}) * (\textcolor{red}{z}_1, (k_i)_{i \in \mathbb{Z}}) = (\textcolor{red}{z}_2 + \textcolor{red}{z}_1, (h_{i+\textcolor{red}{z}_1} +_{\mathbb{Z}_2} k_i)_{i \in \mathbb{Z}})$$



10. Sketch of proof

Define the cases

Let $Q = \{r, s\}$ and $a = \bar{\lambda}_r, b = \bar{\lambda}_s$, then

$$\mathcal{A} \leftarrow \text{-----} \rightarrow \begin{array}{l} a = \tau^{i_1}(x_{11}, x_{12}) \\ b = \tau^{i_2}(x_{21}, x_{22}) \end{array}$$

where $x_{ij} \in \{a, b\}$ and $\tau^{i_1}, \tau^{i_2} \in \mathcal{S}(X) = \mathcal{S}(\{0, 1\})$. There are 64 possibilities. We proceed by analysing part of them.



Thank you for your attention!

