# Groups of Automata

Carlo Lanzi Luciani

Mentors:
izr. prof. Ganna Kudryavtseva
prof. Alessandro Logar

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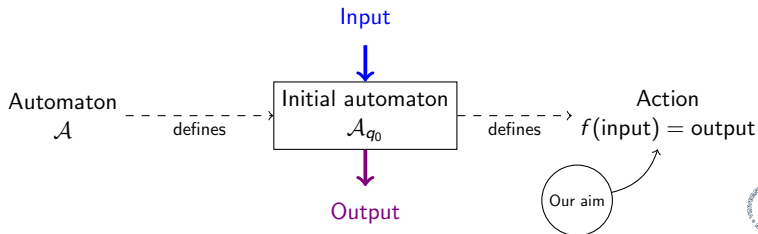
# 1. Introduction

The word automaton: from the greek "acting of one's own will". Automata are important in:

- ▶ Information theory
- ▶ Theory of dynamical systems
- ▶ Algebra
- ▶ Others

My aim: study some of the groups constructed through a special class of them, the invertible deterministic Mealy automata, here called simply automata.

# 2. The automaton

## Definition

An automaton is a 4-tuple $\mathcal{A} = \langle X, \mathcal{Q}, \pi, \lambda \rangle$ where:

- $X = \{x_1, \ldots, x_k\}$ is a finite set called the alphabet,
- $\mathcal{Q}$ is a set called the set of internal states of the automaton,
- $\pi : X \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is a function called the transition function,
- $\lambda : X \times \mathcal{Q} \longrightarrow X$ is a function such that $\lambda_q = \lambda(\cdot, q) : X \longrightarrow X$ is bijective, and is called the output function.

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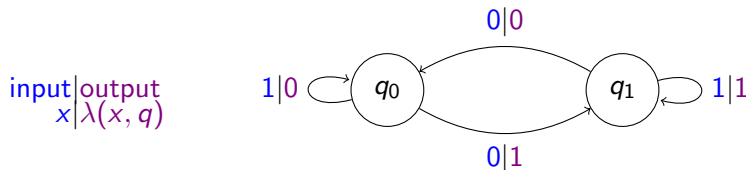
$$0|0$$

input|output
$x|\lambda(x, q)$

$1|0$    $q_0$      $q_1$    $1|1$

$$0|1$$

Figure: Moore diagram of a 2-state automaton over $X = \{0, 1\}$

Definition

$X^* = \{x_1 x_2 \ldots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the dictionary.

Word composition: $x_1 \ldots x_n.z_1 \ldots z_n := x_1 \ldots x_n z_1 \ldots z_n$

# 3. The initial automaton

### Definition

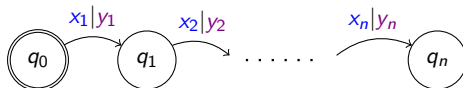$X^* = \{x_1 x_2 \ldots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the dictionary.

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### Definition

An initial automaton $\mathcal{A}_{q_0}$ is an automaton $\mathcal{A}$ with a fixed state $q_0$.

The action of $\mathcal{A}_{q_0}$ is the function $\bar{\lambda}_{q_0} : X^* \longrightarrow X^*$ with

$\bar{\lambda}_{q_0}(x_1 x_2 \ldots x_n) = y_1 y_2 \ldots y_n$.

# 4. The word tree X*

## Definition

Given $w, v \in X^*$, $w$ is a child of $v$ if and only if $w = v.x = vx$ for some letter $x \in X$.
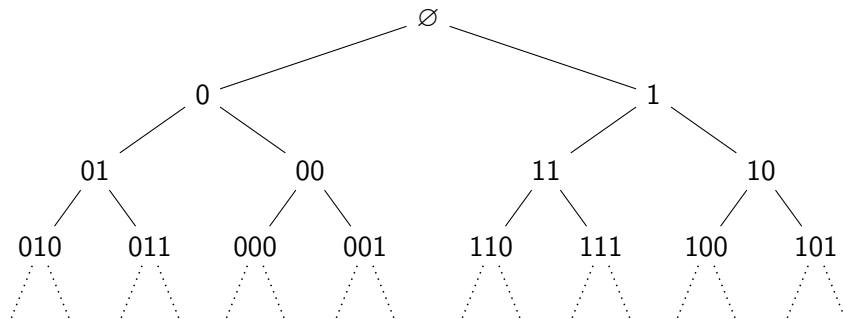


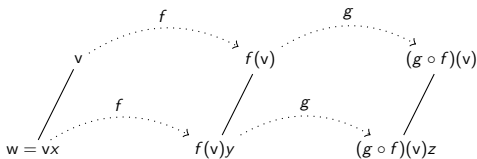Figure: An example of the word tree $X^*$ on $X = \{0, 1\}$.

# 5. Actions as tree-automorphisms

## Proposition

A function $f : X^* \longrightarrow X^*$ is the action of some initial automaton if and only if is a tree-automorphism on the word tree $X^*$, i.e.:

- $f(\varnothing) = \varnothing$.
- if $w \in X^*$ is a child of $v$ then $f(w)$ is a child of $f(v)$.
- f is bijective.



## Proposition

If $f, g$ are tree-automorphisms on $X^*$, then $g \circ f$ and $f^{-1}$ are tree-automorphisms on $X^*$.

# 6. Groups defined by automata

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The functions $f : X^* \longrightarrow X^*$ defined by initial automata (i.e. tree-automorphisms), called synchronous automatic transformations, form a group denoted by $\mathcal{AUT}_{tree}(X^*)$.

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## Definition

Let $\mathcal{A} = \langle X, \mathcal{Q}, \pi, \lambda \rangle$ be an automaton. The group defined by $\mathcal{A}$ is the group generated by the set $\{\bar{\lambda}_q : q \in \mathcal{Q}\}$.

$$\mathcal{A} \text{ with} \atop \mathcal{Q} = \{q_1, \ldots, q_n\} \quad \xdashrightarrow{\text{defines}} \quad \begin{matrix} \bar{\lambda}_{q_1} : X^* \longrightarrow X^* \\ \bar{\lambda}_{q_2} : X^* \longrightarrow X^* \\ \vdots \\ \bar{\lambda}_{q_n} : X^* \longrightarrow X^* \end{matrix} \quad \xdashrightarrow{\text{generate}} \quad \langle \{\bar{\lambda}_q : q \in \mathcal{Q}\} \rangle$$

# 7. Application to automata

### Definition

Let $\mathcal{S}(X)$ be the simmetric group on $X = \{x_1, \ldots, x_k\}$. Then the wreath product $\mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*)$ is the group $\left(\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *\right)$ where the multiplication rule is:

$$\gamma(c_{x_1}, \ldots, c_{x_k}) * \alpha(a_{x_1}, \ldots, a_{x_k}) :=$$
$$\gamma \circ \alpha \left(c_{\alpha(x_1)} \circ a_{x_1}, \ldots, c_{\alpha(x_k)} \circ a_{x_k}\right)$$

### Proposition

Let $X = \{x_1, \ldots, x_k\}$. Then the function $\psi : \mathcal{AUT}_{tree}(X^*) \longrightarrow \mathcal{S}(X) \wr \mathcal{AUT}_{tree}(X^*) = \left(\mathcal{S}(X) \times \mathcal{AUT}_{tree}(X^*)^X, *\right)$ defined by

$$\psi(\bar{\lambda}_{q_0}) = \lambda_{q_0}\left(\bar{\lambda}_{\pi(x_1, q_0)}, \ldots, \bar{\lambda}_{\pi(x_n, q_0)}\right)$$

is an isomorphism of groups.

# 8. System of formulas

## Proposition

Let $\mathcal{A}$ be an automaton with $\mathcal{Q} = \{q_1, \ldots, q_n\}$ over $X = \{x_1, \ldots, x_k\}$. Then $\mathcal{A}$ is described by $n$ recurrent formulas

$$f_{q_1} = \beta_{q_1}(h_{x_1,q_1}, \ldots, h_{x_k,q_1}),$$
$$f_{q_2} = \beta_{q_2}(h_{x_1,q_2}, \ldots, h_{x_k,q_2}),$$
$$\vdots$$
$$f_{q_n} = \beta_{q_n}(h_{x_1,q_n}, \ldots, h_{x_k,q_n}),$$

where each $h_{x_i,q_j}$ is equal to some $f_{q_l}$ and each $\beta_{q_j} \in \mathcal{S}(X)$. Conversely, each such set of $n$ recursive formulas defines an automaton $\mathcal{A}$ such that $\bar{\lambda}_{q_j} = f_{q_j}$ for every $q_j \in \mathcal{Q}$.

$$\mathcal{A} \dashleftarrow\dashrightarrow \quad \begin{array}{l} \bar{\lambda}_{q_1} = \lambda_{q_1}(\bar{\lambda}_{\pi(x_1,q_1)}, \ldots, \bar{\lambda}_{\pi(x_n,q_1)}) \\ \bar{\lambda}_{q_2} = \lambda_{q_2}(\bar{\lambda}_{\pi(x_1,q_2)}, \ldots, \bar{\lambda}_{\pi(x_n,q_2)}) \\ \qquad\qquad \vdots \\ \bar{\lambda}_{q_n} = \lambda_{q_n}(\bar{\lambda}_{\pi(x_1,q_n)}, \ldots, \bar{\lambda}_{\pi(x_n,q_n)}) \end{array} \quad \dashleftarrow\dashrightarrow \quad \begin{array}{l} f_{q_1} = \beta_{q_1}(h_{x_1,q_1}, \ldots, h_{x_k,q_1}) \\ f_{q_2} = \beta_{q_2}(h_{x_1,q_2}, \ldots, h_{x_k,q_2}) \\ \qquad\qquad \vdots \\ f_{q_n} = \beta_{q_n}(h_{x_1,q_n}, \ldots, h_{x_k,q_n}) \end{array}$$

# 9. The classification theorem

### Theorem

Let $\mathcal{A}$ be a 2-state automaton over the alphabet $X = \{0, 1\}$ and $G$ the group defined by this automaton. Then $G$ is isomorphic to one of the following groups:

- the trivial group $\{1_G\}$,
- $\mathbb{Z}_2$,
- $\mathbb{Z}_2 \times \mathbb{Z}_2$,
- $\mathbb{Z}$,
- the infinite dihedral group $\mathcal{D}_\infty = (\mathbb{Z}_2 \times \mathbb{Z}, *)$, where:

$$(h, z_1) * (k, z_2) = (h +_{\mathbb{Z}_2} k, (-1)^k z_1 + z_2),$$

- the lamplighter group $\mathcal{L} = (\mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}, *)$, where:

$$(z_2, (h_i)_{i \in \mathbb{Z}}) * (z_1, (k_i)_{i \in \mathbb{Z}}) = (z_2 + z_1, (h_{i +_{z_1}} +_{\mathbb{Z}_2} k_i)_{i \in \mathbb{Z}})$$

# 10. Sketch of proof

Define the cases

Let $\mathcal{Q} = \{r, s\}$ and $a = \bar{\lambda}_r, b = \bar{\lambda}_s$, then

$$\mathcal{A} \longleftarrow\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\longrightarrow \begin{array}{l} a = \tau^{i_1}(x_{11}, x_{12}) \\ b = \tau^{i_2}(x_{21}, x_{22}) \end{array}$$

where $x_{ij} \in \{a, b\}$ and $\tau^{i_1}, \tau^{i_2} \in \mathcal{S}(X) = \mathcal{S}(\{0, 1\})$. There are 64 possibilities.

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Case analysis

Let $\tau$ be the transposition of $\mathcal{S}(X)$. Every case is analogous to one of the 24 cases where $a \in \{\tau(a, a), \tau(a, b), \tau(b, b)\}$. We proceed by analysing part of them.

Thank you for your attention!