GROUPS OF AUTOMATA

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Automata are a MODEL OF COMPUTATION:

INPUT ↓

Automaton ⇝ defines ⇝ [Initial Automaton] ⇝ defines ⇝ Function(INPUT) = = OUTPUT

↓ OUTPUT

Aim

1st part:

(1) INPUT and OUTPUT
(2) Automata and their visualization
(3) Initial Automaton and "actions"

Then we will analise the functions in detail.

Finally: interesting examples

INPUT and OUTPUT

- X = finite set of symbols

  Ex] $X = \{0, 1\}$

- X^* = set of words of $X = \{x_1 \cdot \ldots \cdot x_n \mid x_i \in X, n \in \mathbb{N}\} =$

  $= \{$ finite strings of $X\}$

- $|w| = |x_1 \cdot \ldots \cdot x_n| := n =$ lenght of w

- Monoid Structure

  $\emptyset :=$ empty word = identity respect to $\circ$

  $(x_1 \cdot \ldots \cdot x_n) \circ (y_1 \cdot \ldots \cdot y_m) = x_1 \cdot \ldots \cdot x_n y_1 \cdot \ldots \cdot y_m$

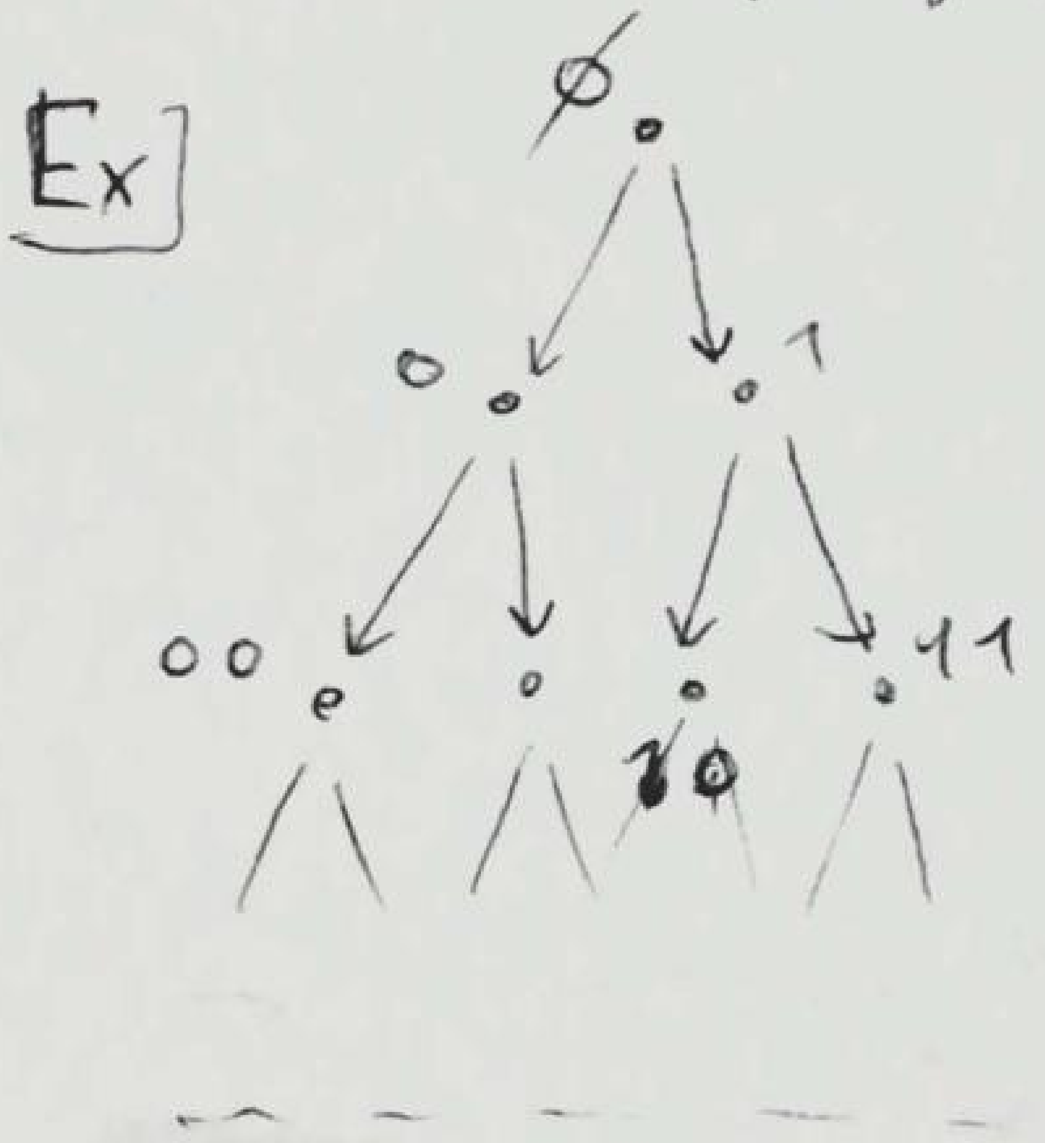
  Ex]: $\underbrace{001} \circ \underbrace{101} = \underbrace{001}\underbrace{101}$

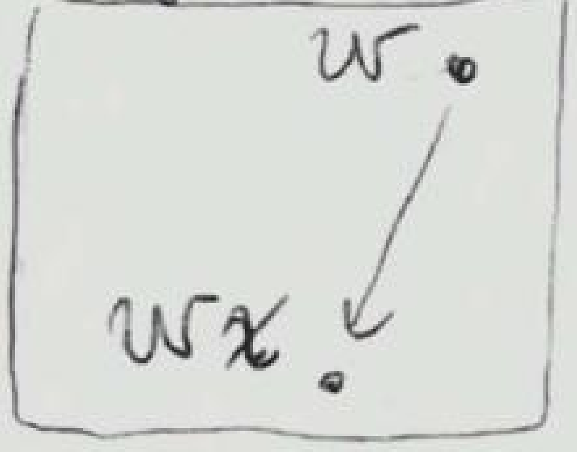
- Tree Structure

  $\emptyset$ = root

  v is son of w whenever $v = \underbrace{w\,|\,x}$, $x \in X$

Ex]





$n = 2$

$n = 3$

Observation:

$X^n = \{$words of lenght $n\}$

$= n$-th level of X

AUTOMATA

DEF A <u>SYNCHRONUS</u> <u>INVERTIBLE</u> <u>AUTOMATON</u> $\mathcal{A}$ is a tuple $\mathcal{A} = (X, Q, \lambda, \pi)$ where

(1) X is a finite set, the <u>INPUT</u> and <u>OUTPUT ALPHABET</u>

(2) Q is a set, the <u>SET OF STATES</u>

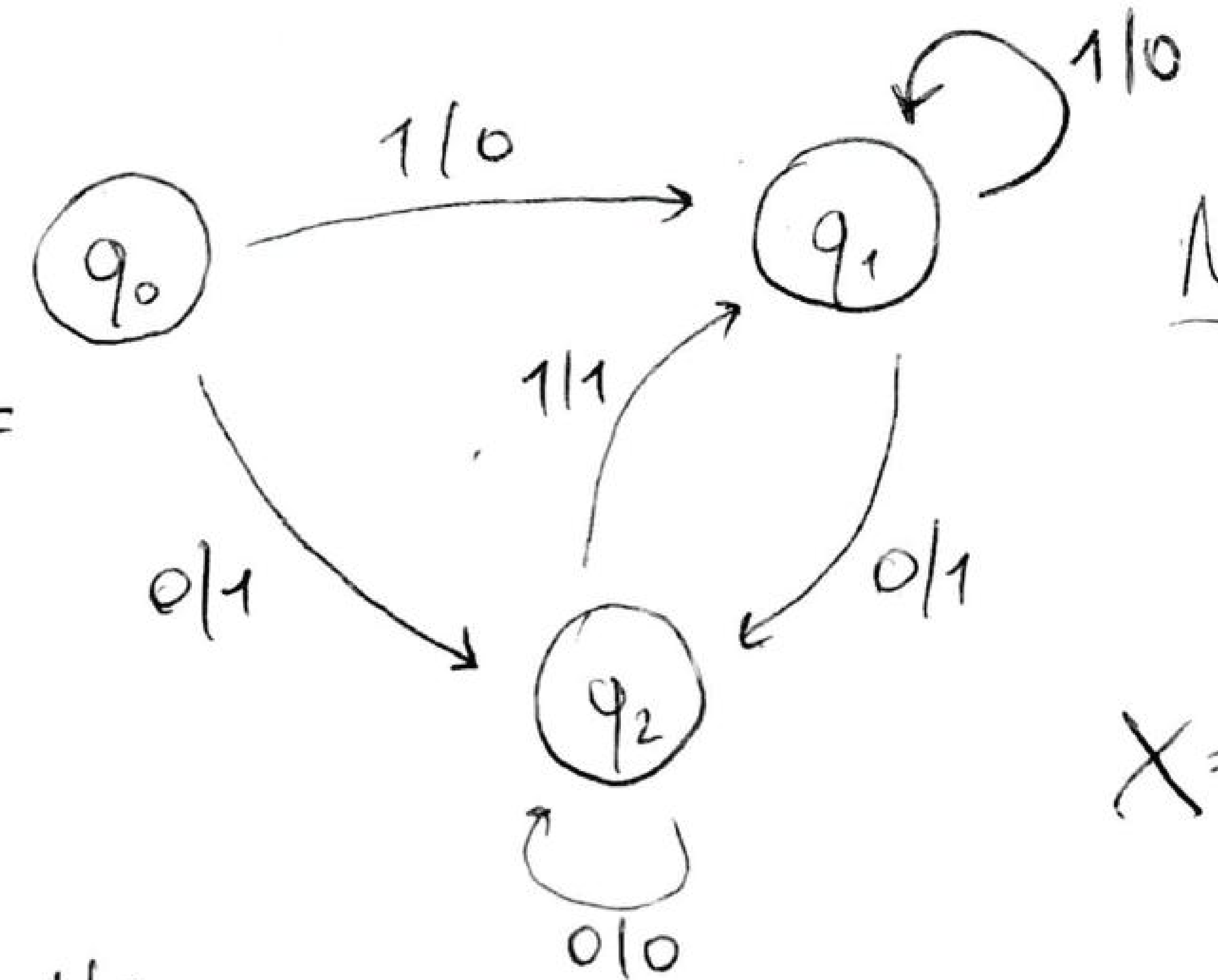
(3) $\pi : Q \times X \longrightarrow Q$ is the <u>TRANSITION</u> <u>FUNCTION</u>

(4) $\lambda : Q \times X \longrightarrow X$ is a function such that $\lambda(q, \cdot) : X \longrightarrow X$ is <u>bijective</u> ($\Rightarrow$ permutation), and it's called <u>OUTPUT FUNCTION</u>

[From now on AUTOMATON = SYNC. INV. AUTOMATON]

<u>Example 1</u>:

$\Gamma_1 :=$



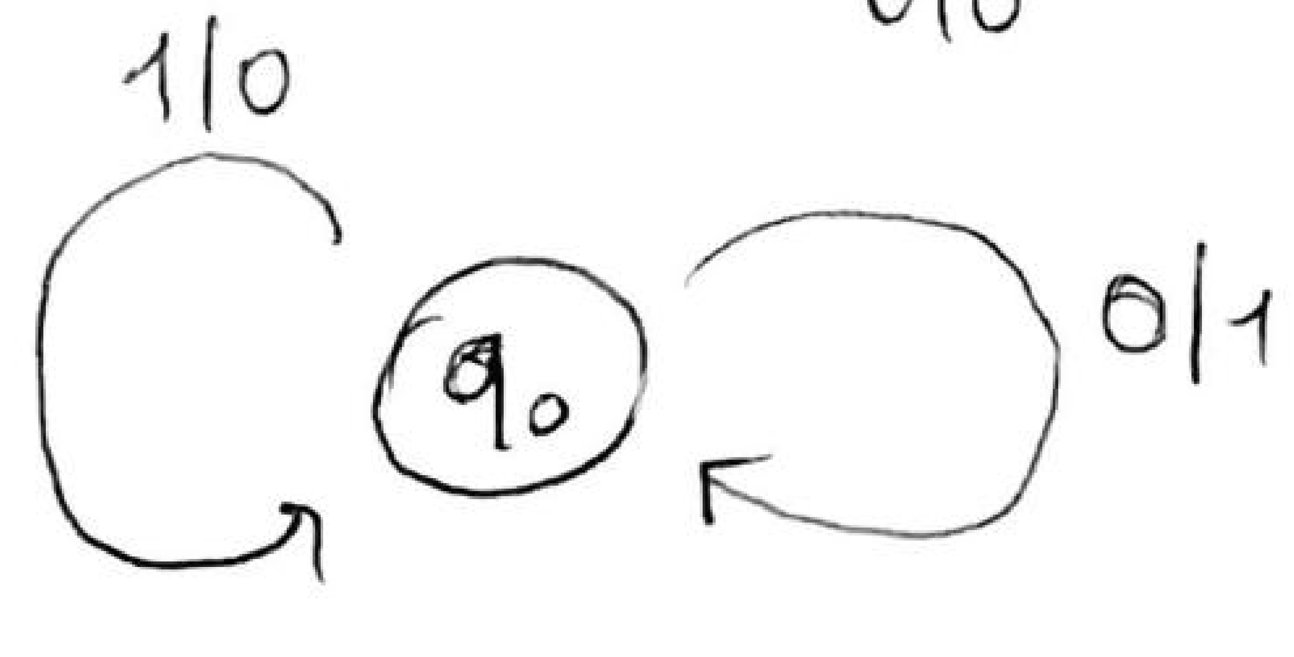
<u>Notation</u>:

INPUT LETTER | OUTPUT LETTER

$X = \{0, 1\}$

<u>Ex. 2</u>

$\Gamma_2 :=$



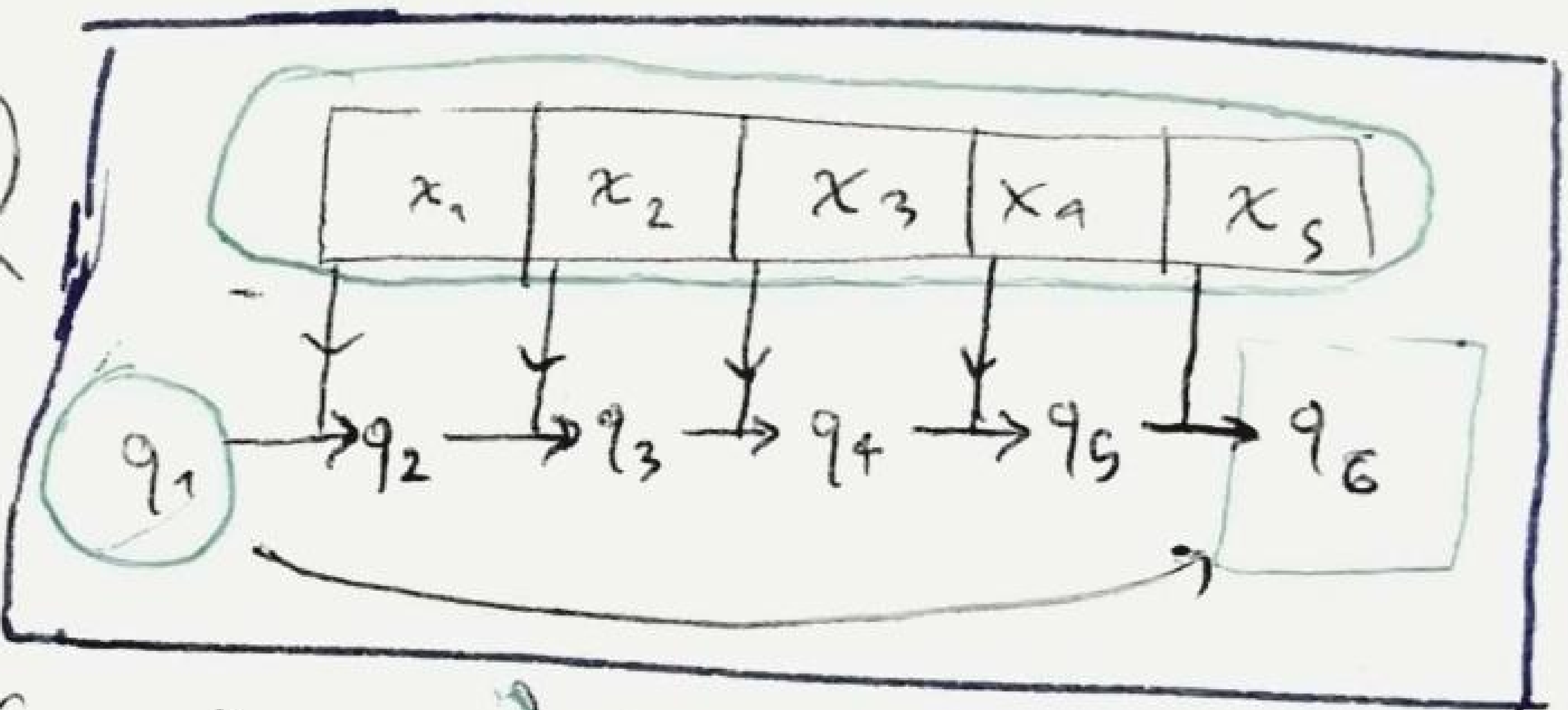
Extension of π and λ

$\pi(q, x) = q$ $\qquad$ $\pi : Q \times X \longrightarrow Q$

$\lambda(q, x) = y$ $\qquad$ $\lambda : Q \times X \longrightarrow X$

Recursive Extension:

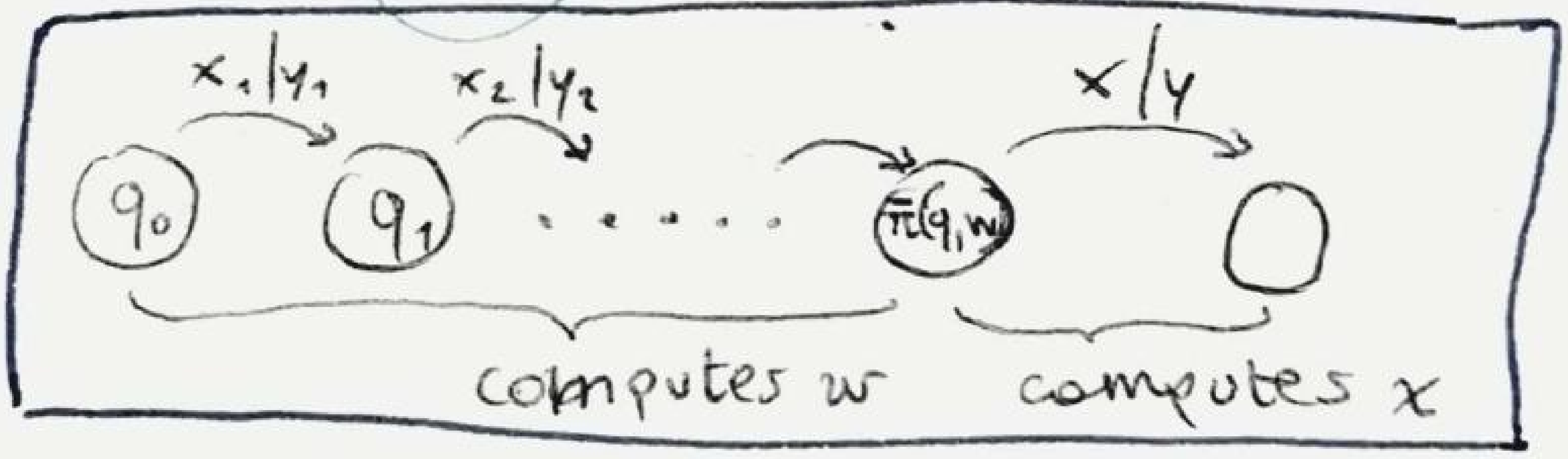
$\overline{\pi} : Q \times X^* \longrightarrow Q$



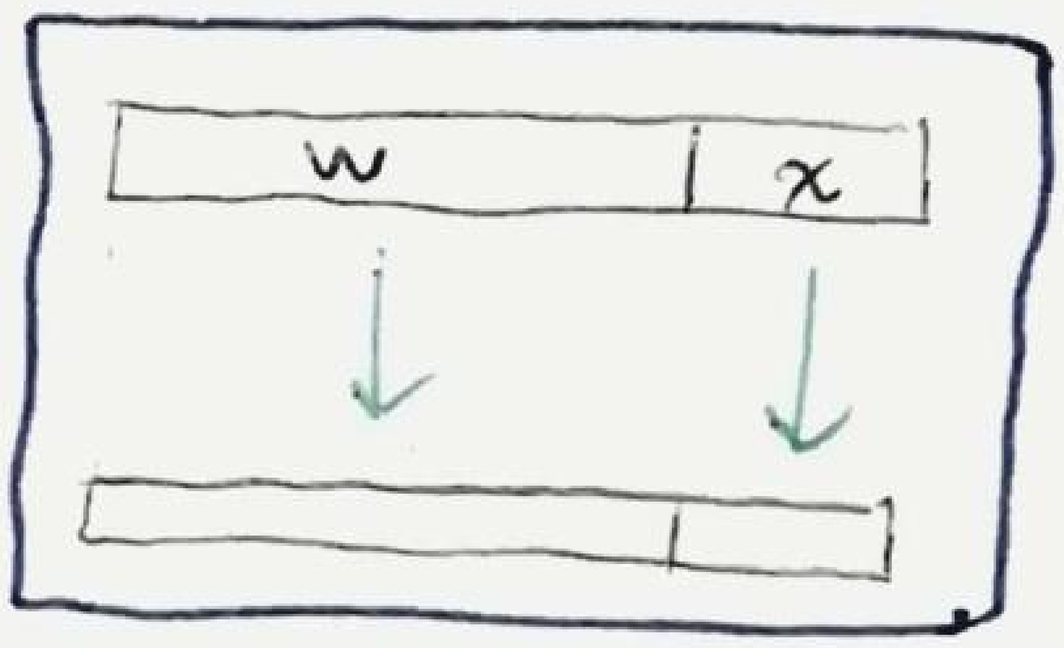
$$\begin{cases} \overline{\pi}(q, \emptyset) := q \\ \overline{\pi}(q, \underset{X^*}{\underset{\cap}{w}} \cdot \underset{X}{\underset{\cap}{x}}) := \pi(\overline{\pi}(q, w), x) \end{cases}$$

Ex] : page 2A

$\overline{\lambda} : Q \times X^* \longrightarrow X^*$



$$\begin{cases} \overline{\lambda}(q, \emptyset) := \emptyset \\ \overline{\lambda}(q, \underset{X^*}{\underset{\cap}{w}} \cdot \underset{X}{\underset{\cap}{x}}) := \underbrace{\overline{\lambda}(q, w)}_{\in X^*} \circ \underbrace{\lambda(\overline{\pi}(q, w), x)}_{\in X} \end{cases}$$



Notation: $\overline{\lambda}_q(w) := \overline{\lambda}(q, w)$

Ex] : page 2A

Def Given $\mathcal{A}$ automaton, $\mathcal{A}_{q_0}$, with fixed INITIAL STATE q_0, is called INITIAL AUTOMATON

Note (1) $\mathcal{A}_{q_0}$ defines $\bar{\lambda}_{q_0} : X^* \longrightarrow X^*$, called the ACTION of $\mathcal{A}_{q_0}$

(2) λ_q is bijective (on X) $\Rightarrow$ $\bar{\lambda}_q$ is bijective (on X^*)

Ex: page 2A

Povzetek:

Automaton $\mathcal{A}$ $\rightsquigarrow$ Initial Automaton $\mathcal{A}_{q_0}$ $\rightsquigarrow$ action $\bar{\lambda}_{q_0}$ $(X^* \longrightarrow X^*)$

Note $\forall \mathcal{A}$ we can define $|Q|$ different $\mathcal{A}_q$

Ex (1) Page 2A: (Γ_1) $\mathcal{A}$ $\rightsquigarrow \begin{cases} (\Gamma_1)_{q_0} \rightsquigarrow \bar{\lambda}_{q_0} \\ (\Gamma_1)_{q_1} \rightsquigarrow \bar{\lambda}_{q_1} \\ (\Gamma_2)_{q_2} \rightsquigarrow \bar{\lambda}_{q_2} \end{cases}$

(2) $(\Gamma_2) \rightsquigarrow (\Gamma_2)_q \rightsquigarrow \bar{\lambda}_q$

DEF] $f : X^* \longrightarrow X^*$ is <u>synchronous</u> <u>automatic</u> if is definable as the action of some initial automaton $\mathcal{A}_{q_0}$, i.d. (=to je) $f = \bar{\lambda}_{q_0}$.

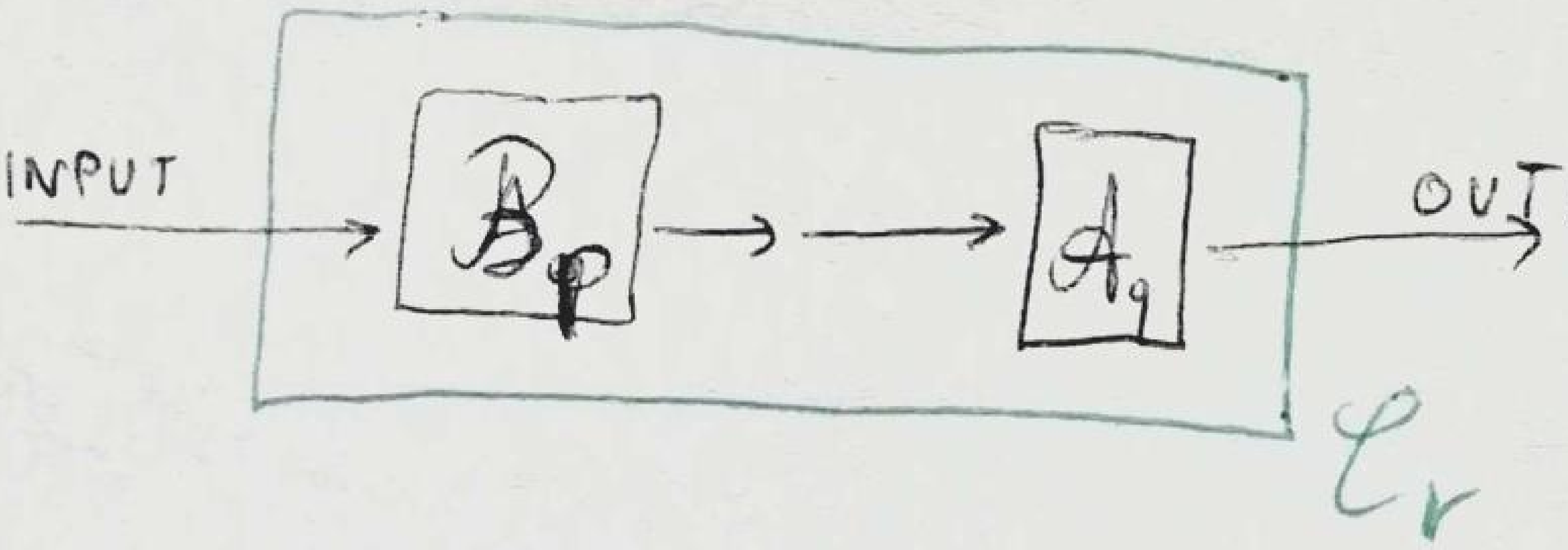
DEF] $S := \{ f : X^* \to X^* \,/\, f \text{ is synchronous automatic} \}$

Note] $f \in S \Rightarrow f = \bar{\lambda}_{q_0} \Rightarrow f$ is <u>bijective</u>

<u>We want to study S</u>

<u>Composition Lemma</u>] Given $\mathcal{A}_q$ and $\mathcal{B}_p$, initial automata on X, $\exists\ \mathcal{C}_r$, initial automaton, such that:

$$\begin{pmatrix} \text{Action of} \\ \mathcal{C}_r \end{pmatrix} = \begin{pmatrix} \text{Action} \\ \text{of} \\ \mathcal{A}_q \end{pmatrix} \circ \begin{pmatrix} \text{Action} \\ \text{of} \\ \mathcal{B}_p \end{pmatrix}$$



$\Rightarrow S$ is closed under composition!

Similarly we have:

$[f \in S \Rightarrow f^{-1} \in S]$

$\Rightarrow (id : X^* \longrightarrow X^*) \in S$

$\Rightarrow (S, \circ)$ is a group

CHARACTERIZATION OF THE ACTIONS OF AUTOMATA

Povzetek:

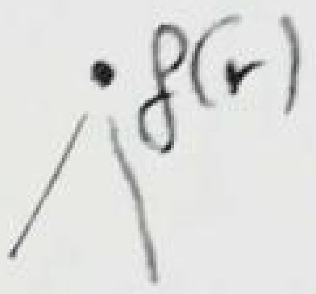
Automaton $\mathcal{A}$ $\rightsquigarrow$ Initial automaton $\mathcal{A}_{q_0}$ $\rightsquigarrow$ $\bar{l}_{q_0} : X^* \to X^*$

$$S := \{ f : X^* \to X^* \mid f \text{ is the action of some } \mathcal{A}_{q_0} \}$$

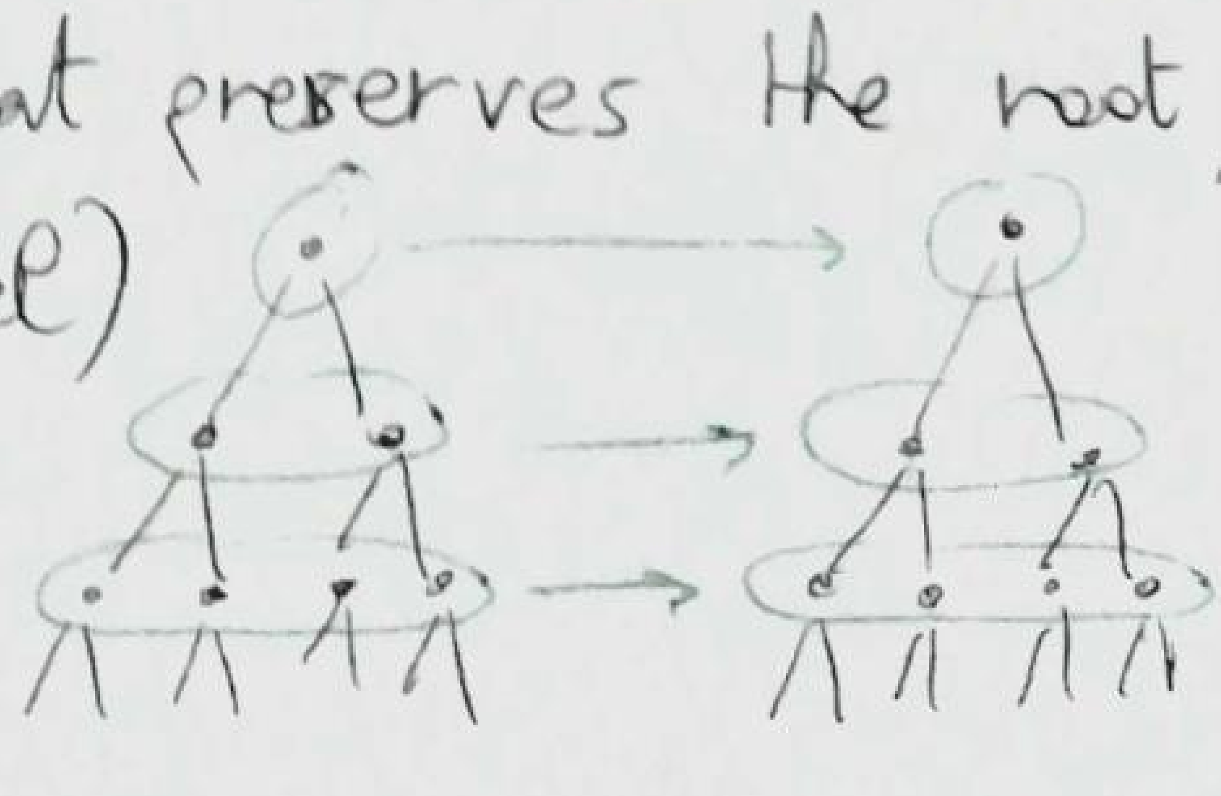
Remark] • Given $G = (V, E)$ graph, $f : V \to V$ is said to be a graph-homomorphism if preserve the adjacencies $[(v, w) \in E \implies (f(v), f(w)) \in E$

$$\begin{array}{ccc} v & & f(v) \\ \downarrow & \xrightarrow{\quad f \quad} & \downarrow \\ w & & f(w) \end{array}$$

• If G is a tree, f is said to be a graph-homorphism that preserves the root if (1) is a graph-homomorphism and if (2) $f(r) = r$, where r is the root of G

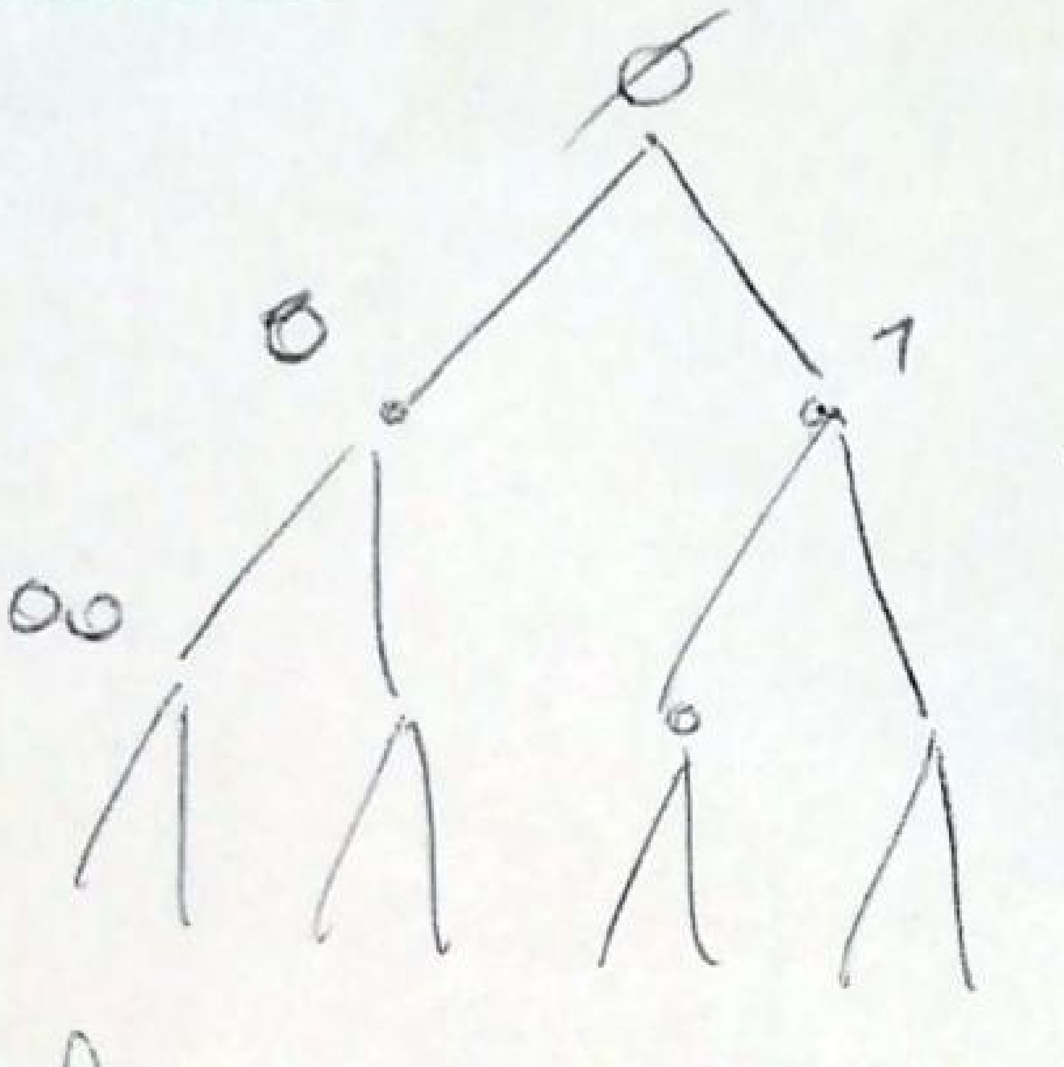


Note] If f is a graph-hom. that preserves the root, $f(\text{n-th level of } G) \subseteq (\text{n-th level})$

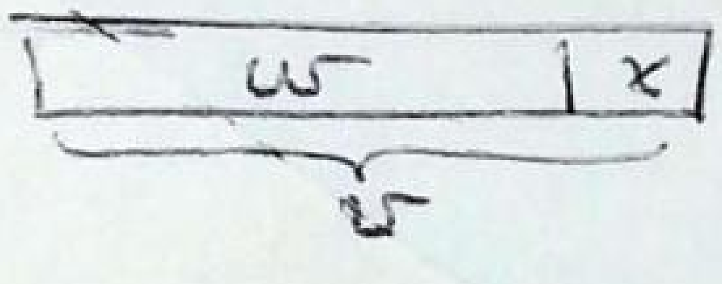


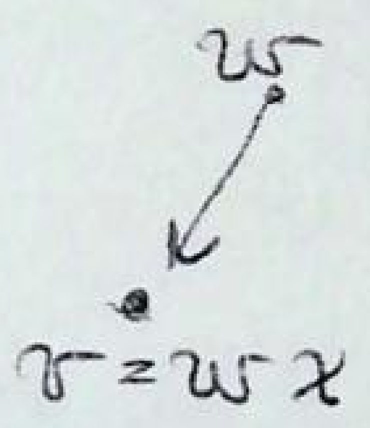
Proposition f is <u>synch. automatic</u> <u>if and only if</u> f is a graph-homom that preserves the root on X^*

Observation Remembering that X^* is a tree, what does it mean to be a graph-homom that preserve the root on X^*?

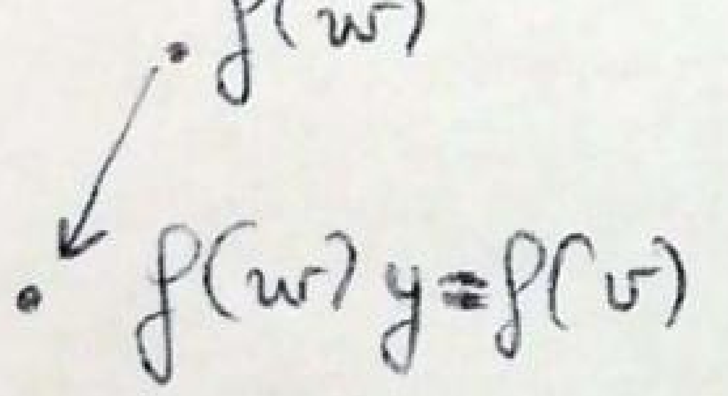


- Given a f with this property condition $f(r)=r$ means
  $\underline{f(\emptyset)=\emptyset}$
- The condition of being a map homomorphism? In X^*, v is son of w if and only if $v = wx$





So if we want $f(v)$ to be son of $f(w)$ means $f(v) = f(w)\,y$ for some $y \in X$.



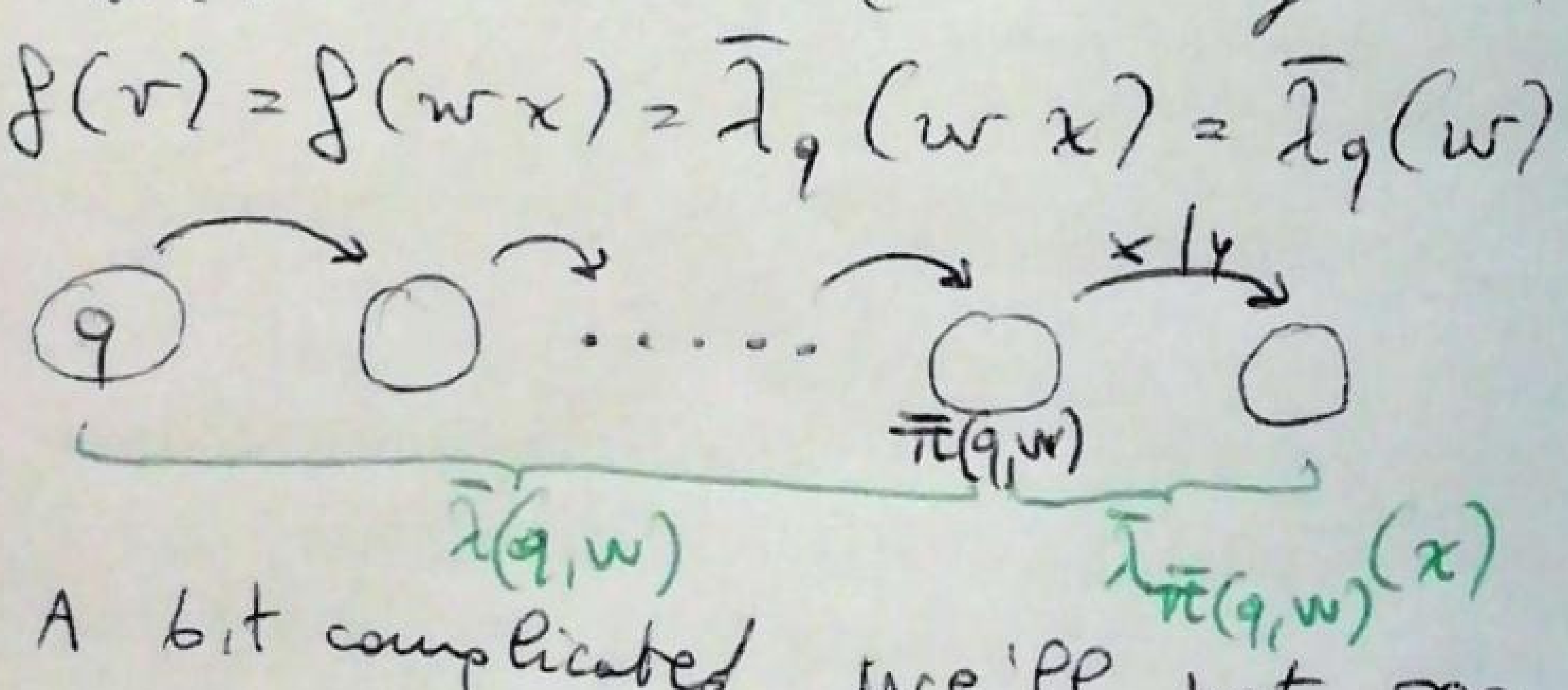
Dokaz $(\Longrightarrow)$

Given f synch. autom, we have $f = \overline{\lambda}_q$ for some a chan of some $\mathcal{A}_q$.

(1) $f(\emptyset) = \overline{\lambda}_q(\emptyset) = \emptyset$ ✓

(2) Given $v = wx$ (v son of w) we have:

$$f(v) = f(wx) = \overline{\lambda}_q(wx) = \overline{\lambda}_q(w) \cdot \underbrace{\lambda_{\pi(q,w)}(x)}_{} = f(w)\underbrace{y}_{\in X}$$



$(\Longleftarrow)$ A bit complicated, we'll not see it ☐

Corollary $S = \{$bijective graph-homom. that preserve $\emptyset$ on $X^*\}$

DEF Given $\mathcal{A}$ automaton we can define $|Q|$ initial automata $\mathcal{A}_q$, so $|Q|$ actions $\bar{\lambda}_q \in S$.

The <u>group generated by $\mathcal{A}$</u> is defined as:

$$G(\mathcal{A}) := \langle \{ \bar{\lambda}_q \mid q \in Q \text{ of } \mathcal{A} \} \rangle \leq S$$

$$\text{Automaton} \rightsquigarrow \begin{cases} \mathcal{A}_{q_1} \\ \vdots \\ \mathcal{A}_{q_n} \end{cases} \begin{matrix} \rightsquigarrow \\ \\ \rightsquigarrow \end{matrix} \begin{cases} \bar{\lambda}_{q_1} \\ \vdots \\ \bar{\lambda}_{q_n} \end{cases} \longrightarrow \langle \bar{\lambda}_{q_1}, \dots, \bar{\lambda}_{q_n} \rangle \leq S$$

Ex: page 2A, (4) Γ_1 defines $G(\Gamma_1) := \langle \bar{\lambda}_{q_0}, \bar{\lambda}_{q_1}, \bar{\lambda}_{q_2} \rangle$

Interesting Results and Examples

PROPOSITION Given $X = \{0, 1\}$, and $\mathcal{A}$, 2-state automaton on X, $G(\mathcal{A})$ must be isomorphic to one of this groups:

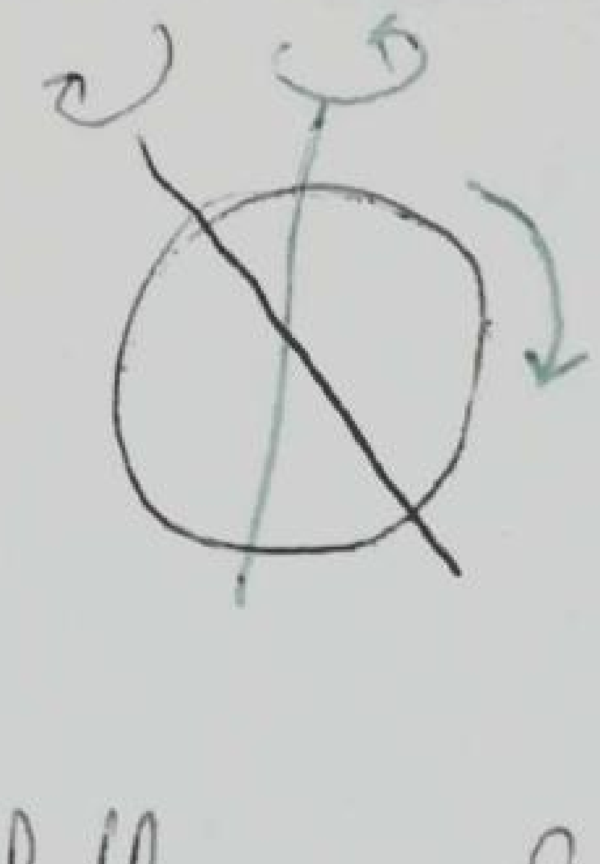
(1) $\{1_G\}$

(2) $\mathbb{Z}_2$

(3) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

(4) $\mathbb{Z}$

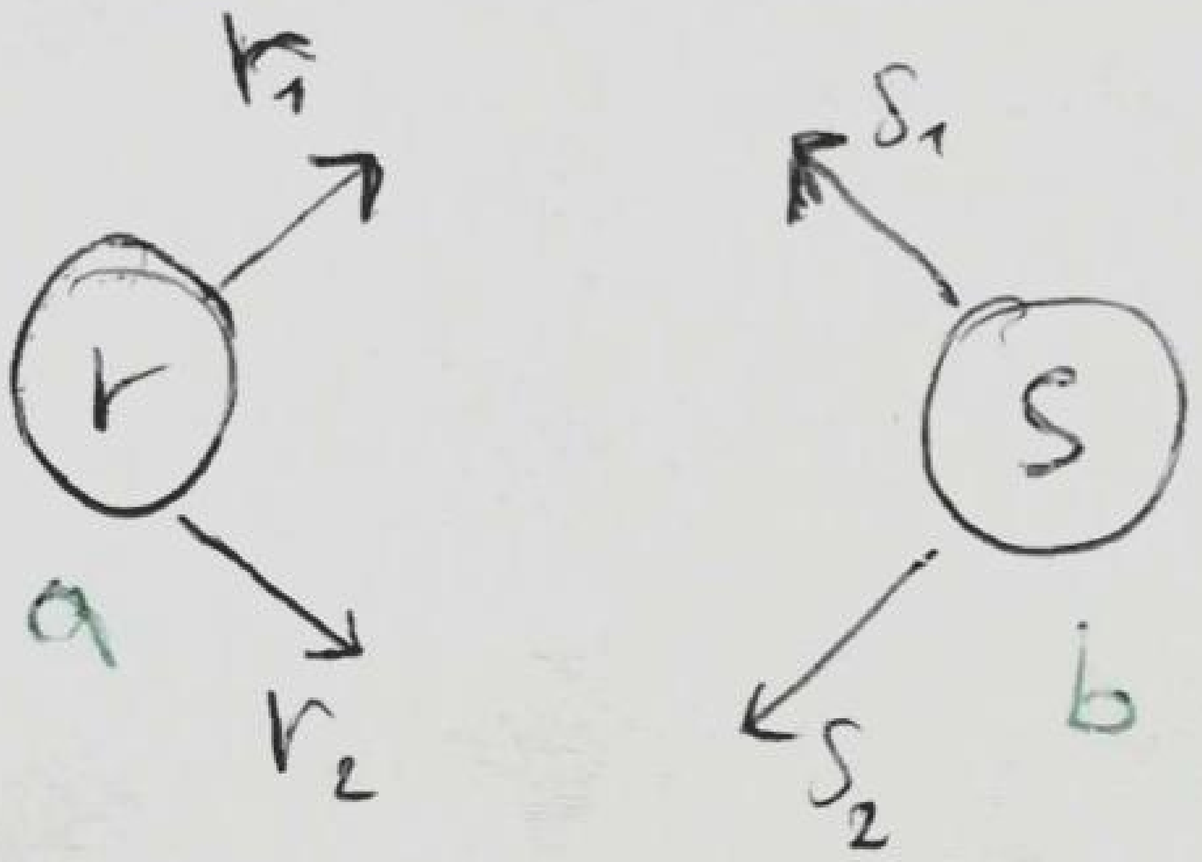
(5) D_∞ = Infinite dihedral group = {symm. of the circle}



(6) $\mathbb{Z} \wr \mathbb{Z}_2$ = Lamplighter group

Sketch of proof: Q, the set of states

$$Q=\{r,s\}\ ,\ X=\{0,1\}.\ \mathcal{A}=(X,Q,\pi,\lambda)$$



• $\lambda_s, \lambda_r : X \to X$ are permutation of a 2-element set $X \Rightarrow \lambda_s, \lambda_r \in S_2$, $S_2 = \{id = 1, \sigma\}$

$$\begin{cases}\sigma(0)=1\\ \sigma(1)=0\end{cases}$$

• each arrow in $\{r_1, r_2, s_1, s_2\}$ can point on an element of $Q=\{r,s\}$

So all the possible $\mathcal{A}$, with $Q=\{r,s\}$, $X=\{0,1\}$ are the ones in witch $\lambda_s, \lambda_r \in S_2$ (uniquely determines λ), and $r_1, r_2, s_1, s_2 \in \{r,s\}$ (uniquely determines π) $\Longrightarrow$

$$\Longrightarrow \text{they are } \underset{\lambda_s}{2}\cdot\underset{\lambda_r}{2}\cdot\underset{r_1}{2}\cdot\underset{r_2}{2}\cdot\underset{s_1}{2}\cdot\underset{s_2}{2} = 2^6.$$

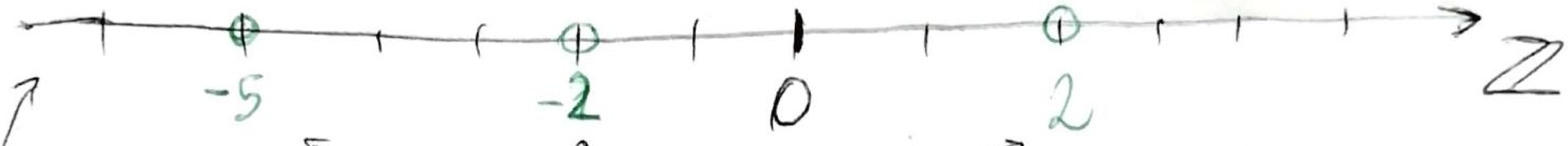
Analysing each case we get [$G(\mathcal{A})$ is isomorphic to one ~~case~~ of the latter cases].

Interesting group:

$$\mathbb{Z} \wr \mathbb{Z}_2 := \left(\mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})} \; ; \; \star \right)$$

- $\mathbb{Z}_2^{(\mathbb{Z})} := \{ (b_i)_{i \in \mathbb{Z}} \mid b_i \in \mathbb{Z}_2 = \{0,1\}, b_i = 1$ just for a finite set of indexes $I \}$

Practically:



Infinite dark road $\mathbb{Z}$

indexes of the ~~open~~ lamps turned on (indexes in I)

Ex] The previous represented element is $(\tilde{b}_i)_{i \in \mathbb{Z}}$, where $b_{-5}, b_{-2}, b_2 = 1$.
[We can sign $(\tilde{b}_i)_{i \in \mathbb{Z}}$ with $\{-5, -2, 2\} = I$]

- $\star$ is not a direct product!

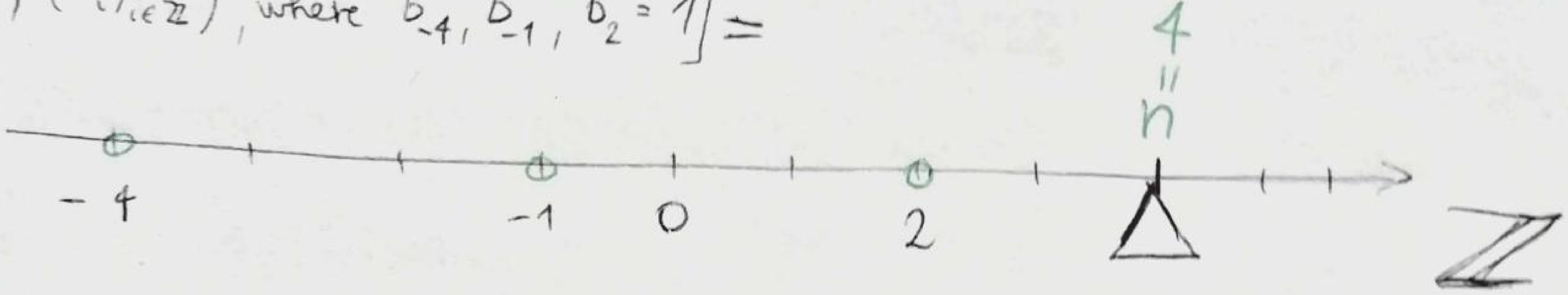
$$\left(n_1 \, , \, (b_i)_{i \in \mathbb{Z}} \right) \star \left(n_2 \, , \, (q_i)_{i \in \mathbb{Z}} \right) :=$$

$$= \left(n_1 + n_2 \, , \, \left(b_i + q_{i + n_1} \right)_{i \in \mathbb{Z}} \right)$$

Visualization:

- Of an element $(n, (b_i)_{i\in\mathbb{Z}}) \in \mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}$

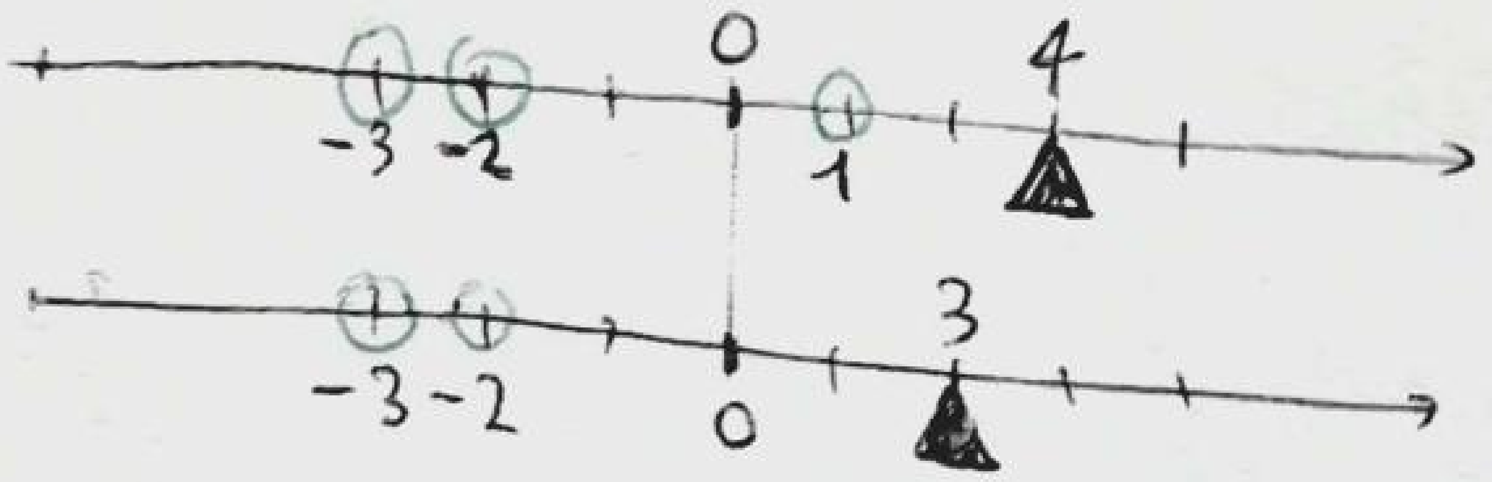
$\left[(4, (b_i)_{i\in\mathbb{Z}}), \text{ where } b_{-4}, b_{-1}, b_2 = 1\right] =$



n = position of the "lamplighter on the infinite road

- Of the product $\star$:

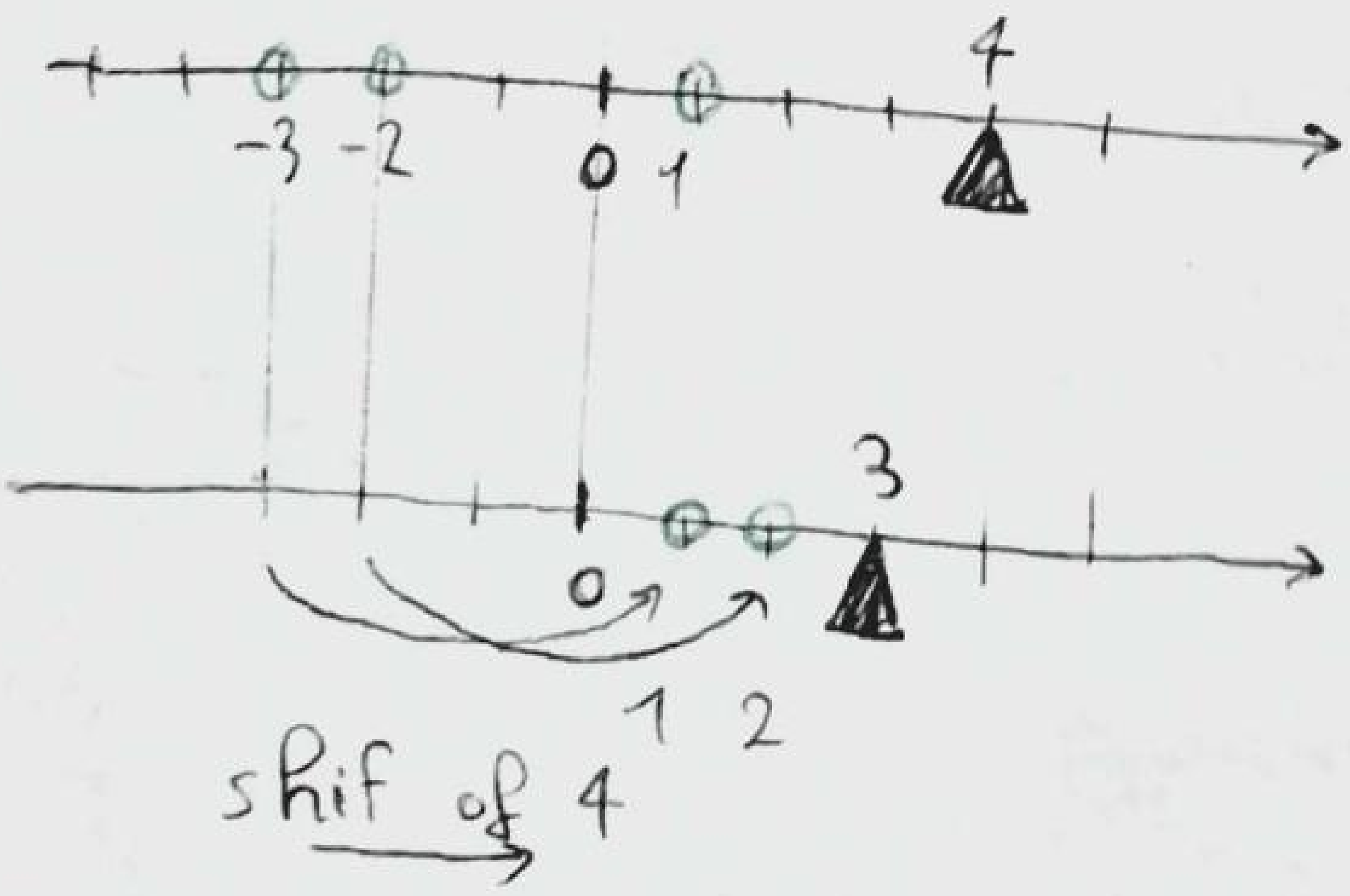
$$\left(\overset{n_1}{4} ; \overset{(b_i)_{i\in\mathbb{Z}}}{\{-3;-2;1\}}\right) \star \left(\overset{n_2}{3} ; \overset{(q_i)_{i\in\mathbb{Z}}}{\{-3;-2\}}\right) =$$



$(n_1, (b_i)_{i\in\mathbb{Z}})$

$(n_2, (q_i)_{i\in\mathbb{Z}})$

$$= \left(\underset{n_1}{4}, \underset{(b_i)_{i\in\mathbb{Z}}}{\{-3,-2,-1\}}\right) + \left(\underset{n_2}{3}, \underset{(q_i+n_1)_{i\in\mathbb{Z}}}{\{\overset{-3+4}{1}, \overset{-2+4}{2}\}}\right) =$$

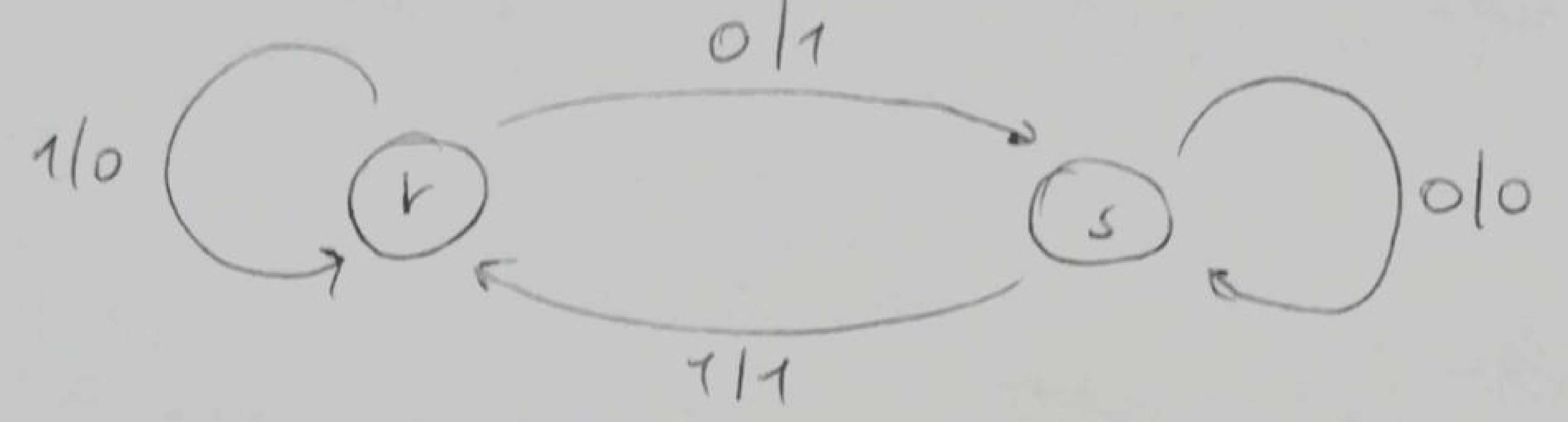


$(n_1, (b_i)_{i\in\mathbb{Z}})$

$(n_2 ; (q_i + n_1)_{i\in\mathbb{Z}})$

$$= (4+3 ; \{-3;-2;2\})$$

The automaton which defines $\mathbb{Z} \wr \mathbb{Z}_2$:



$=: \mathcal{A}$

a b

$$a := \overline{\lambda}_r \quad , \quad b := \overline{\lambda}_s$$

$$\mathbb{Z} \wr \mathbb{Z}_2 = \langle a , b \rangle = G(\mathcal{A})$$

Let's watch closer:

$$\begin{cases} a(0v) = 1b(v) \\ a(1v) = 0a(v) \end{cases} \qquad \begin{cases} b(0v) = 0b(v) \\ b(1v) = 1a(v) \end{cases}$$

$$\lambda_r : X \to X \qquad \lambda_s : X \to X$$

$$\lambda_r = \sigma \in S_2 \qquad \lambda_s = id \in S_2$$

$$\Rightarrow \quad \begin{array}{c} b^{-1} \\ : \\ X^* \\ \downarrow \\ X^* \end{array} = \begin{cases} b^{-1}(0v) = 0b^{-1}(v) \\ b^{-1}(1v) = 1a^{-1}(v) \end{cases}$$

$$c := b^{-1} \cdot a = a \circ b^{-1}$$

$$\begin{cases} \underline{c(0v)} = a \circ b^{-1}(0v) = a(0\, b^{-1}(v)) = 1\, \overbrace{b \circ b^{-1}}^{id}(v) = \underline{1v} \\ \underline{c(1v)} = a \circ b^{-1}(1v) = a(1\, b^{-1}(v)) = 0\, \underbrace{b \circ b^{-1}}_{id}(v) = \underline{0v} \end{cases}$$

- we see $\langle a, b \rangle = \langle \underbrace{b^{-1}a}_{\in\, a}, b \rangle = \underline{\langle c, b \rangle} = G(A)$

  $b^{-1}a = c$

- $\mathbb{Z}_2 = (X ; +_{\mathbb{Z}_2})$

$$0 + 0 = 0$$
$$1 + 1 = 0$$
$$1 + 0 = 1$$

$$c(x_1 x_2 x_3 \ldots) = (x_1 + 1) x_2 x_3 \ldots$$

we search an explicit formula for b:

$$b(x_1 x_2 x_3 \ldots) = y_1 y_2 y_3 \ldots \qquad y_n = ?$$

we claim (A) $b(x_1 x_2 \ldots x_n \overset{(n+1)}{0} x_{n+2} \ldots) =$
$= y_1 y_2 \ \ y_n y_{n+1} b(x_{n+2} \ldots)$

(B) $b(x_1 x_2 \ldots x_n \overset{(n+1)}{1} x_{n+2} \ldots) =$
$= y_1 y_2 \ldots y_n y_{n+1} a(x_{n+2} \ldots)$

Proof (A) Watch diagram of A. Whenever we encounter a "0", b acts on the next letter

(B) Analogous

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We claim $b(x_1 x_2 - x_n -) = \underset{y_1}{x_1} \underbrace{(x_2+x_1)}_{y_2} - \underbrace{(x_n+x_{n-1})}_{y_n} -$

Proof: For induction on n.

$n=1$ We set $x_0 = 0 \Rightarrow y_1 = x_1 = x_1 + 0 = x_1 + x_0$

$n \rightsquigarrow n+1$ 4 cases: $x_n x_{n+1} \in \{00, 01, 10, 11\}$

Case 00: $y_n = \cancel{x_n} + \cancel{x_{n+1}}$; $y_{n+1} = ?$

$$b(x_1 - x_{n-1} \overset{n}{0} \overset{(n+1)}{0} x_{n+2} -) \overset{(4)}{=} y_1 - y_n b(0 x_{n+2} -) =$$

$$= y_1 - y_n 0 y_{n+2} -$$

$$y_{n+1} = 0 = 0 + 0 = x_n + x_{n+1}$$

The other cases are analogous ✓

$$\Rightarrow c(x_1 x_2 x_3 -) = \underbrace{(x_1+1)}_{\sigma(x_1)} x_2 x_3 -$$

$$b(x_1 x_2 x_3 -) = x_1 (x_2 + x_1)(x_3 + x_2) - =$$

$$= x_1 \quad x_2 \quad x_3 \quad x_4 -$$

$$+$$

[shift to right] $\qquad x_1 \quad x_2 \quad x_3 -$

It can be prooved, with these formulas, that

$$\langle b, c \rangle = \mathbb{Z} \wr \mathbb{Z}_2$$