Groups of Automata

Carlo Lanzi Luciani

Mentors:
izr. prof. Ganna Kudryavtseva
prof. Alessandro Logar

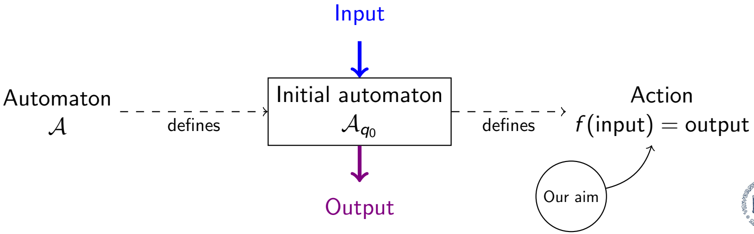
March 17, 2021

1. Introduction

The word automaton: from the greek "acting of one's own will".
Automata are important in:

- Information theory
- Theory of dynamical systems
- Algebra
- Others

My aim: study some of the groups constructed through a special class of them, the invertible finite deterministic Mealy automata, here called simply automata.



2. The automaton

Definition

An automaton is a 4-tuple $\mathcal{A} = \langle \mathsf{X}, \mathcal{Q}, \pi, \lambda \rangle$ where:

- $\mathsf{X} = \{x_1, \dots, x_k\}$ is a finite set called the alphabet,
- $\mathcal{Q} = \{q_1, \dots, q_n\}$ is a finite set called the set of internal states of the automaton,
- $\pi : \mathsf{X} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is a function called the transition function,
- $\lambda : \mathsf{X} \times \mathcal{Q} \longrightarrow \mathsf{X}$ is a function such that $\lambda_q = \lambda(\cdot, q) : \mathsf{X} \longrightarrow \mathsf{X}$ is bijective, and is called the output function.

2. The automaton

Definition

An automaton is a 4-tuple $\mathcal{A} = \langle \mathsf{X}, \mathcal{Q}, \pi, \lambda \rangle$ where:

- $\mathsf{X} = \{x_1, \ldots, x_k\}$ is a finite set called the alphabet,
- $\mathcal{Q} = \{q_1, \ldots, q_n\}$ is a finite set called the set of internal states of the automaton,
- $\pi : \mathsf{X} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is a function called the transition function,
- $\lambda : \mathsf{X} \times \mathcal{Q} \longrightarrow \mathsf{X}$ is a function such that $\lambda_q = \lambda(\cdot, q) : \mathsf{X} \longrightarrow \mathsf{X}$ is bijective, and is called the output function.

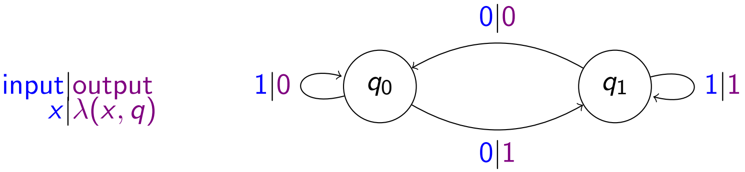


Figure: Moore diagram of a 2-state automaton over $\mathsf{X} = \{0, 1\}$

Definition

$X^* = \{x_1 x_2 \dots x_n : x_i \in X, n \in \mathbb{N} \cup \{0\}\}$ the dictionary.

Word composition: $x_1 \dots x_n . z_1 \dots z_n := x_1 \dots x_n z_1 \dots z_n$

$$q_0 \xrightarrow{x_1|y_1} q_1 \xrightarrow{x_2|y_2} \cdots\cdots \xrightarrow{x_n|y_n} q_n$$

Definition

An initial automaton $\mathcal{A}_{q_0}$ is an automaton $\mathcal{A}$ with a fixed state q_0.

The action of $\mathcal{A}_{q_0}$ is the function $\bar{\lambda}_{q_0} : X^* \longrightarrow X^*$ with

$\bar{\lambda}_{q_0}(x_1 x_2 \dots x_n) = y_1 y_2 \dots y_n$.

4.a Actions as tree-automorphisms

Definition

Given $w, v \in X^*$, w is a child of v if and only if $w = v.x = vx$ for some letter $x \in X$.

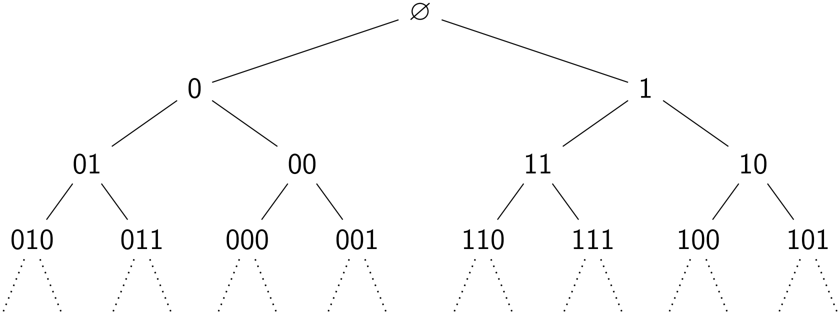


Figure: An example of the word tree X^* on $X = \{0, 1\}$.

4.b Actions as tree-automorphisms

Proposition

A function $f : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ is the action of some initial automaton if and only if is a tree-automorphism on the word tree X^*, i.e.:

- $f(\varnothing) = \varnothing$.
- if $\mathsf{w} \in \mathsf{X}^*$ is a child of v then $f(\mathsf{w})$ is a child of $f(\mathsf{v})$.
- f is bijective.

$$
\begin{tikzcd}
\mathsf{v} \arrow[rr, dotted, bend left, "f"] \arrow[d, dash] & & f(\mathsf{v}) \arrow[rr, dotted, bend left, "g"] \arrow[d, dash] & & (g \circ f)(\mathsf{v}) \arrow[d, dash] \\
\mathsf{w} = \mathsf{v}x \arrow[rr, dotted, bend left, "f"] & & f(\mathsf{v})y \arrow[rr, dotted, bend left, "g"] & & (g \circ f)(\mathsf{v})z
\end{tikzcd}
$$

Proposition

If f, g are tree-automorphism on X^*, then $g \circ f$ and f^{-1} are tree-automorphisms on X^*.

5. Groups defined by automata

Proposition

The functions $f : \mathsf{X}^* \longrightarrow \mathsf{X}^*$ defined by initial automata (i.e. tree-automorphisms), called synchronous automatic transformations, form a group denoted by $\mathcal{AUT}_{tree}(\mathsf{X}^*)$.

Definition

Let $\mathcal{A} = \langle \mathsf{X}, \mathcal{Q}, \pi, \lambda \rangle$ be an automaton. The group defined by $\mathcal{A}$ is the group generated by the set $\{\bar{\lambda}_q : q \in \mathcal{Q}\}$.

$$\begin{array}{c} \mathcal{A} \text{ with} \\ \mathcal{Q} = \{q_1, \ldots, q_n\} \end{array} \xrightarrow[\text{defines}]{\quad} \begin{array}{c} \bar{\lambda}_{q_1} : \mathsf{X}^* \longrightarrow \mathsf{X}^* \\ \bar{\lambda}_{q_2} : \mathsf{X}^* \longrightarrow \mathsf{X}^* \\ \vdots \\ \bar{\lambda}_{q_n} : \mathsf{X}^* \longrightarrow \mathsf{X}^* \end{array} \xrightarrow[\text{generate}]{\quad} \langle \{\bar{\lambda}_q : q \in \mathcal{Q}\} \rangle$$

6. Wreath product

Definition

Let $\mathcal{S}(\mathsf{X})$ be the simmetric group on $\mathsf{X} = \{x_1, \ldots, x_k\}$. Then the wreath product $\mathcal{S}(\mathsf{X}) \wr \mathcal{AUT}_{tree}(\mathsf{X}^*)$ is the group $\left(\mathcal{S}(\mathsf{X}) \times \mathcal{AUT}_{tree}(\mathsf{X}^*)^{\mathsf{X}}, *\right)$ where the multiplication rule is:

$$\gamma(c_{x_1}, \ldots, c_{x_k}) * \alpha(a_{x_1}, \ldots, a_{x_k}) :=$$
$$\gamma \circ \alpha \left(c_{\alpha(x_1)} \circ a_{x_1}, \ldots, c_{\alpha(x_k)} \circ a_{x_k}\right)$$

Proposition

The group of the synchronous automatic transformations $\mathcal{AUT}_{tree}(\mathsf{X}^*)$ is isomorphic to $\mathcal{S}(\mathsf{X}) \wr \mathcal{AUT}_{tree}(\mathsf{X}^*)$.

7. Recursive formulas

Proposition

Let $\mathcal{A}$ be an automaton such that $\mathcal{Q} = \{q_1, \ldots, q_n\}$ and $\mathsf{X} = \{x_1, \ldots, x_k\}$. Then $\mathcal{A}$ is described by n recurrent formulas

$$\begin{aligned} f_{q_1} &= \beta_{q_1}(h_{x_1,q_1}, \ldots, h_{x_k,q_1}), \\ f_{q_2} &= \beta_{q_2}(h_{x_1,q_2}, \ldots, h_{x_k,q_2}), \\ &\vdots \\ f_{q_n} &= \beta_{q_n}(h_{x_1,q_n}, \ldots, h_{x_k,q_n}), \end{aligned}$$

where each h_{x_i,q_j} is equal to some f_{q_l} and each $\beta_{q_j} \in \mathcal{S}(\mathsf{X})$. Conversely, each such set of n recursive formulas defines an automaton $\mathcal{A}$ such that $\bar{\lambda}_{q_l} = f_{q_l}$ for every $q_l \in \mathcal{Q}$.

$$\mathcal{A} \longleftrightarrow \begin{array}{c} \bar{\lambda}_{q_1} = \lambda_{q_1}(\bar{\lambda}_{\pi(x_1,q_1)}, \ldots, \bar{\lambda}_{\pi(x_n,q_1)}) \\ \bar{\lambda}_{q_2} = \lambda_{q_2}(\bar{\lambda}_{\pi(x_1,q_2)}, \ldots, \bar{\lambda}_{\pi(x_n,q_2)}) \\ \vdots \\ \bar{\lambda}_{q_n} = \lambda_{q_n}(\bar{\lambda}_{\pi(x_1,q_n)}, \ldots, \bar{\lambda}_{\pi(x_n,q_n)}) \end{array}$$

8. The classification theorem

Theorem

Let $\mathcal{A}$ be a 2-state automaton over the alphabet $X = \{0, 1\}$ and G the group defined by this automaton. Then G is isomorphic to one of the following groups:

- the trivial group $\{1_G\}$,
- $\mathbb{Z}_2$,
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$,
- $\mathbb{Z}$,
- the infinite dihedral group $\mathcal{D}_\infty$,
- the lamplighter group $\mathcal{L}$.

Thank you for your attention!