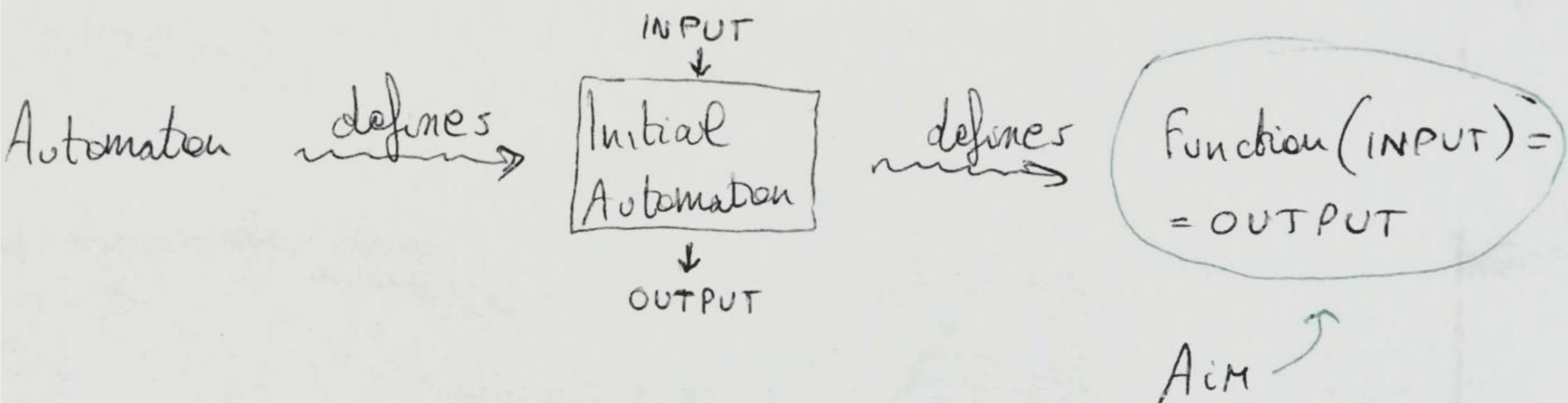


# GROUPS OF AUTOMATA

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Automata are a MODEL OF COMPUTATION:



1st part:

- (1) INPUT and OUTPUT
- (2) Automata and their visualization
- (3) Initial Automata and "actions"

Then we will analyse the functions in detail.

Finally: interesting examples



# INPUT and OUTPUT

①

- $X$  = finite set of symbols

Ex]  ~~$X$~~  =  $\{0, 1\}$

- $X^*$  = set of words of  $X = \{x_1 \cdot \dots \cdot x_n \mid x_i \in X, n \in \mathbb{N}\} = \{\text{finite strings of } X\}$

- $|w| = |x_1 \cdot \dots \cdot x_n| := n = \text{length of } w$

## Monoid Structure

$\phi := \text{empty word} = \text{identity respect to } \cdot$

$$(x_1 \cdot \dots \cdot x_n) \cdot (y_1 \cdot \dots \cdot y_m) = x_1 \cdot \dots \cdot x_n y_1 \cdot \dots \cdot y_m$$

Ex] 001  $\cdot$  101 = 001101

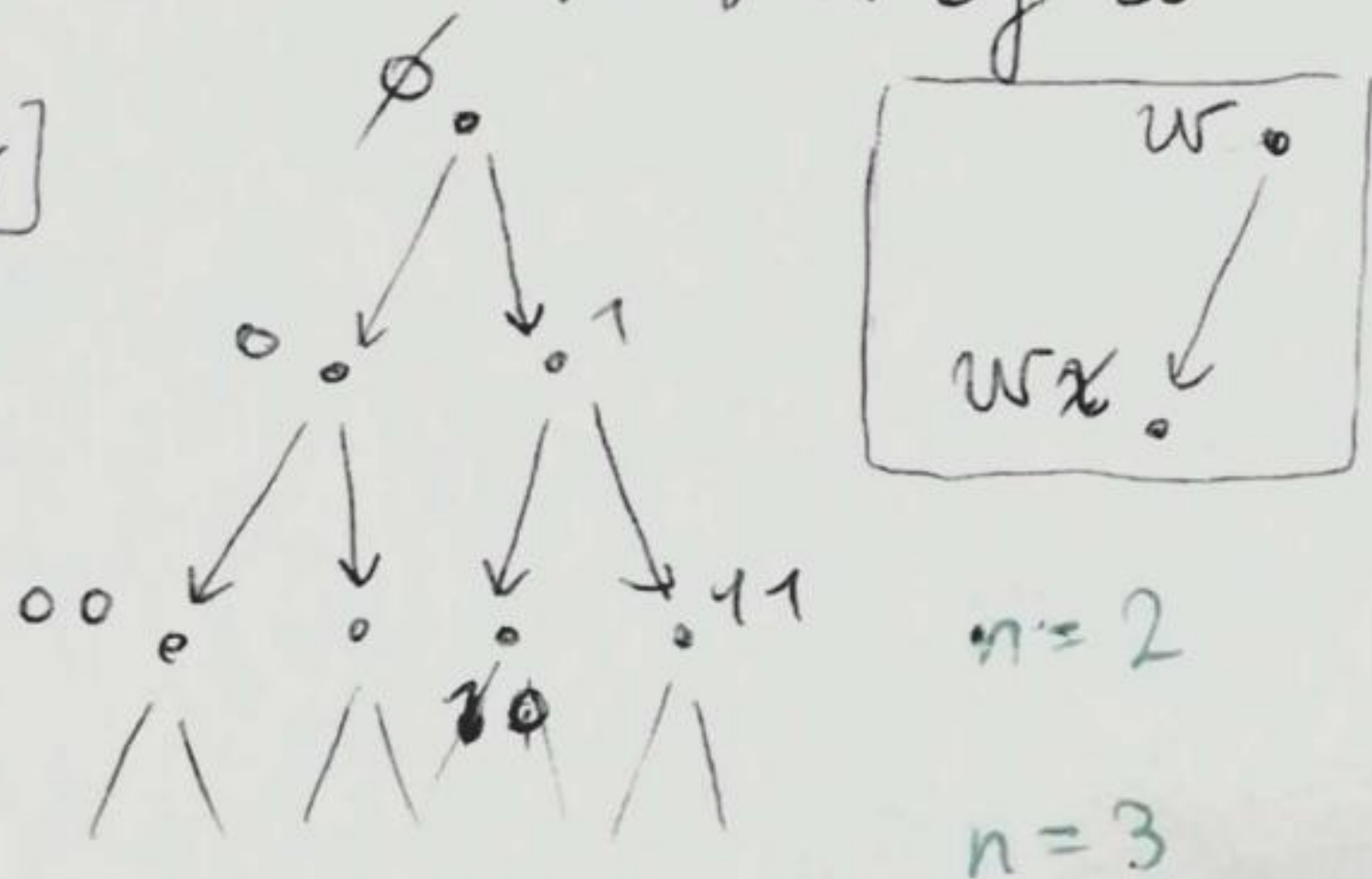
## TREE STRUCTURE

$\phi = \text{root}$

$v$  is son of  $w$  whenever  $v = w x$

$$\underbrace{w}_{\text{parent}} \underbrace{x}_{\text{child}}$$

Ex]



Observation:

$$X^n = \{\text{words of length } n\}$$

$\downarrow$

$$= n\text{-th level of } X$$



# AUTOMATA

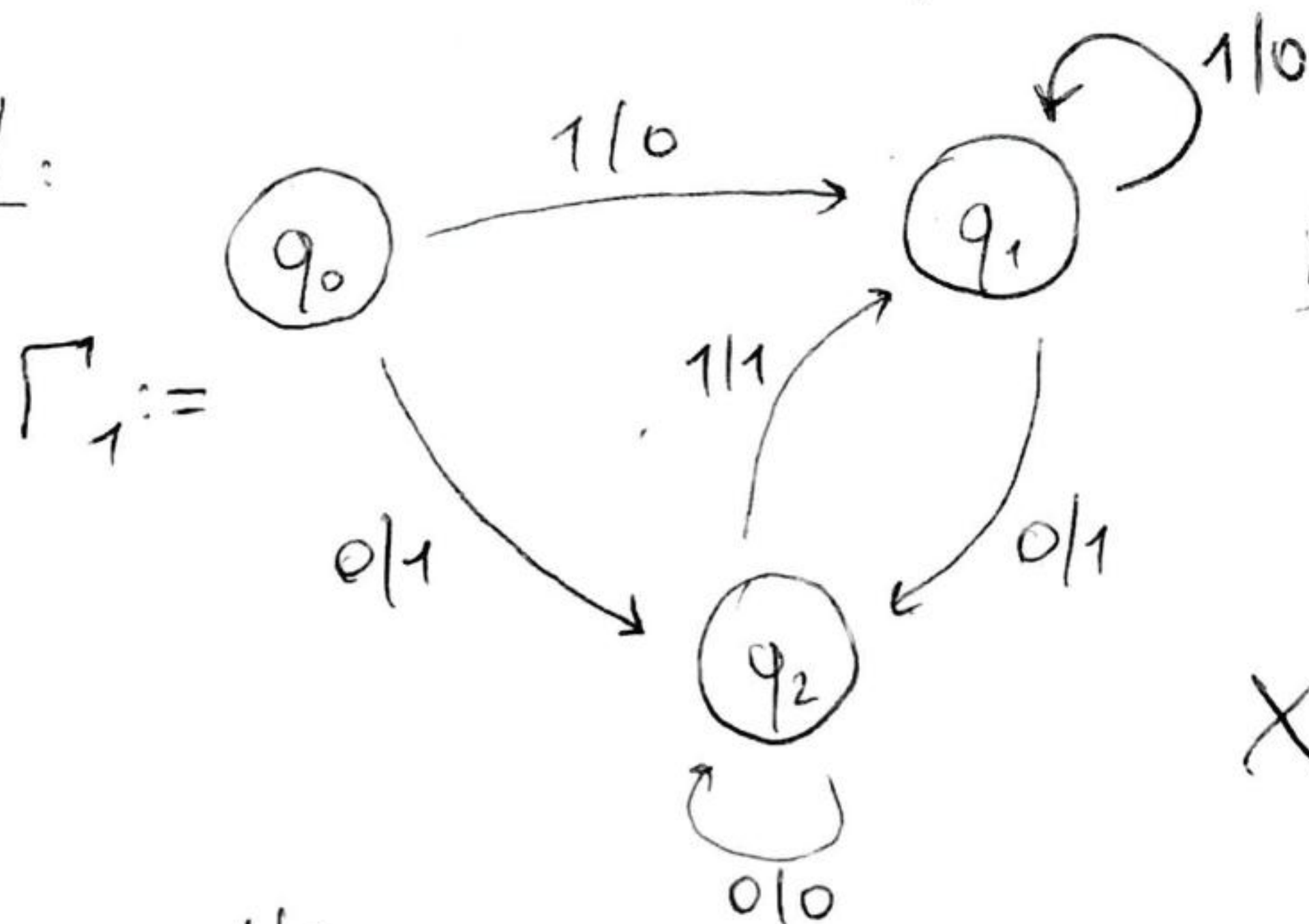
2a

**DEF** A SYNCHRONOUS INVERTIBLE AUTOMATON  $\mathcal{A}$  is a tuple  $\mathcal{A} = (X, Q, \lambda, \pi)$  where

- (1)  $X$  is a finite set, the INPUT and OUTPUT ALPHABET
- (2)  $Q$  is a set, the SET OF STATES
- (3)  $\pi : Q \times X \rightarrow Q$  is the TRANSITION FUNCTION
- (4)  $\lambda : Q \times X \rightarrow X$  is a function such that  $\lambda(q; \cdot) : X \rightarrow X$  is bijective ( $\Rightarrow$  permutation), and it's called OUTPUT FUNCTION

[From now on AUTOMATON = SYNC. INV. AUTOMATON]

Example 1:



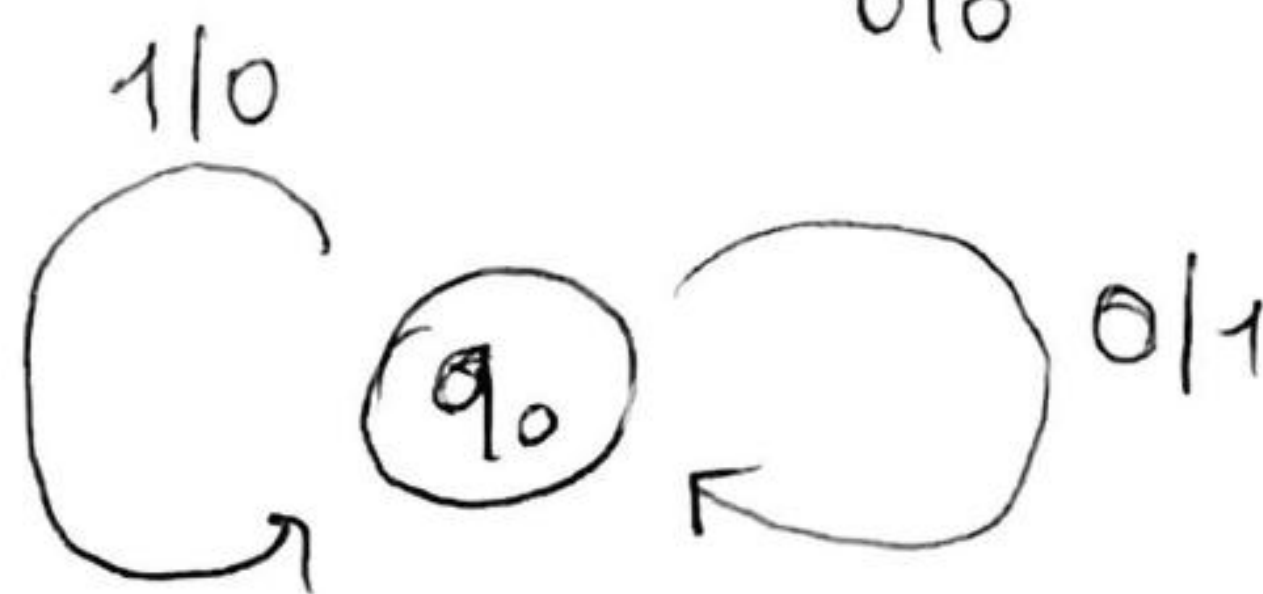
Notation:

INPUT LETTER / OUTPUT LETTER

$X = \{0, 1\}$

Ex. 2

$\Gamma_2 :=$





# Extension of $\pi$ and $\lambda$

(2B)

$$\pi(q, x) = q$$

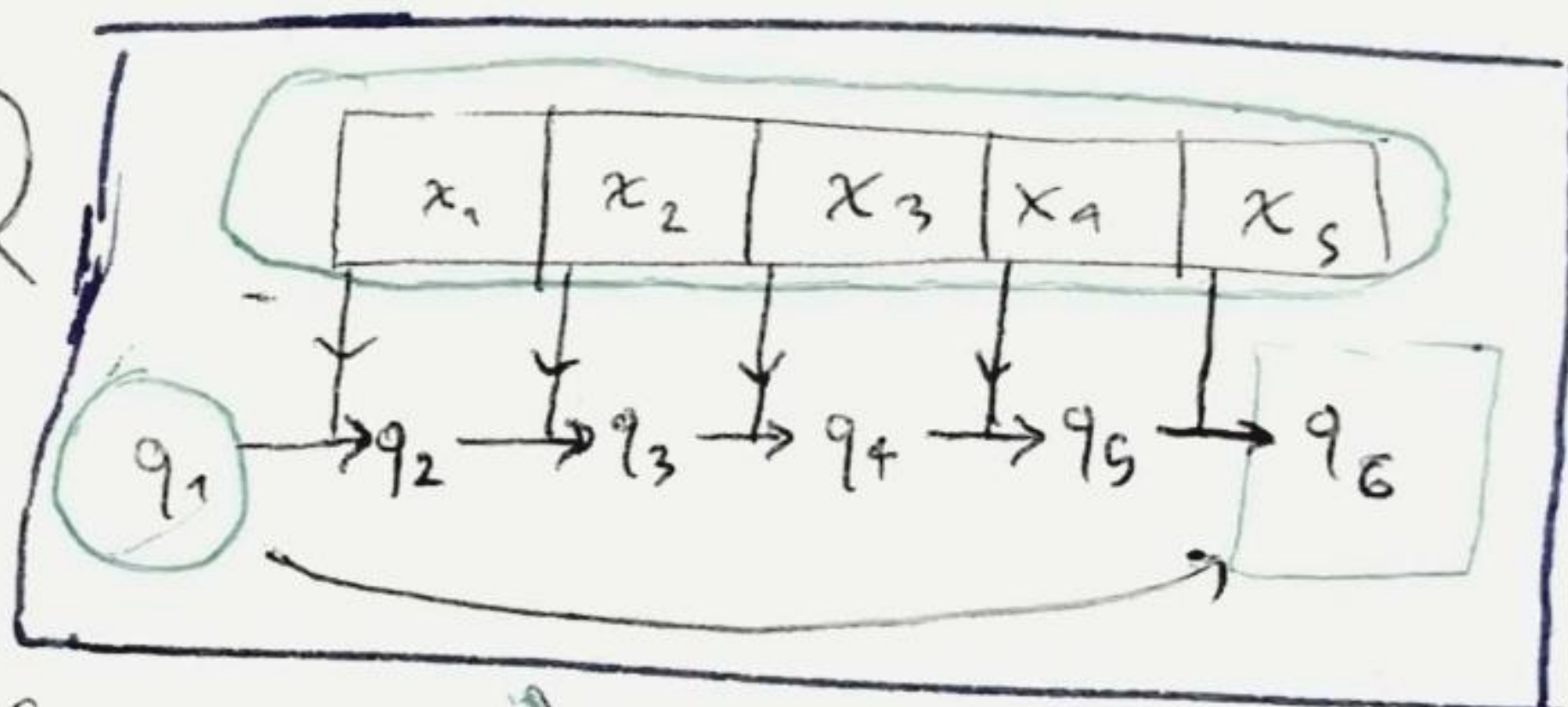
$$\pi: Q \times X \rightarrow Q$$

$$\lambda(q, x) = y$$

$$\lambda: Q \times X \rightarrow X$$

## Recursive Extension:

$$\bar{\pi}: Q \times X^* \rightarrow Q$$



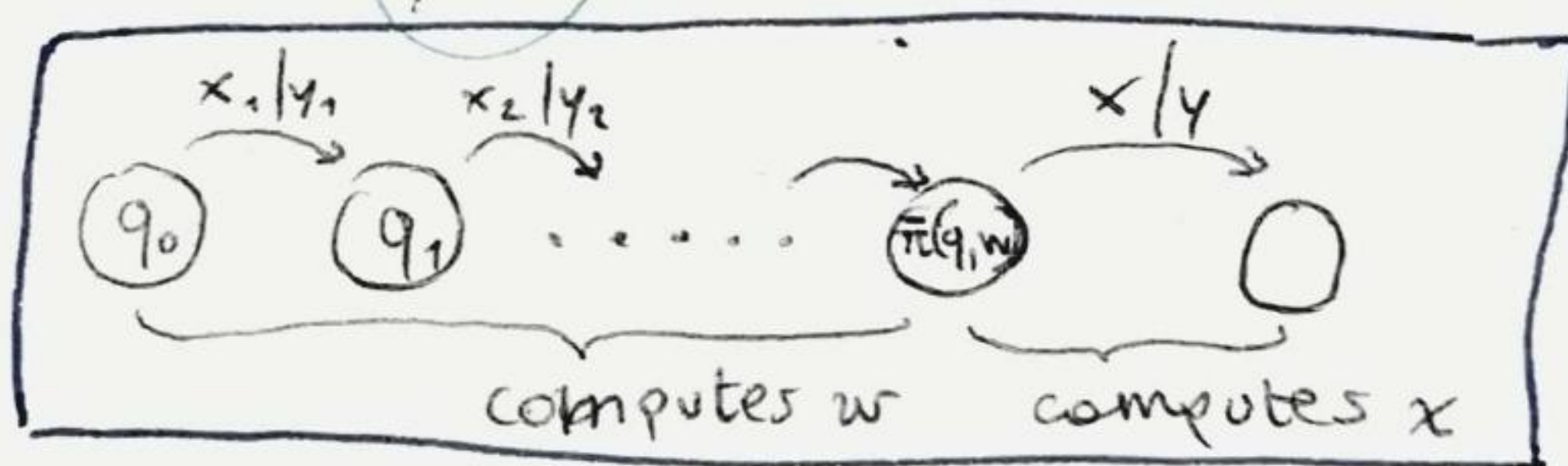
$$\bar{\pi}(q, \phi) := q$$

$$\bar{\pi}(q, w \cdot x) := \bar{\pi}(\bar{\pi}(q, w), x)$$

$\underbrace{w}_{\in X^*} \cdot \underbrace{x}_{\in X}$

Ex]: page 2A

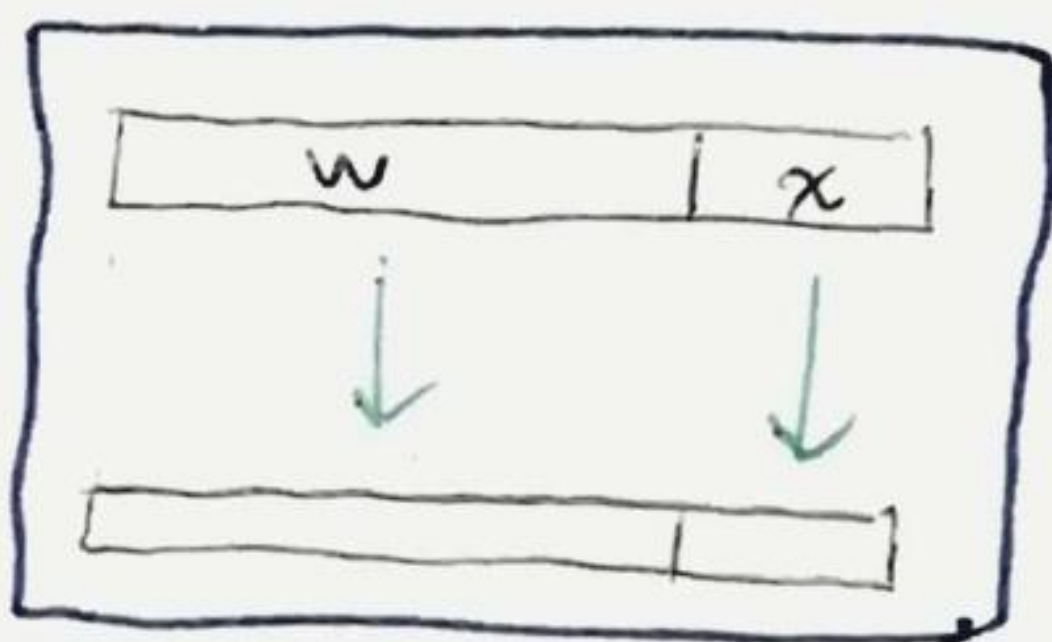
$$\bar{\lambda}: Q \times X^* \rightarrow X^*$$



$$\bar{\lambda}(q, \phi) := \phi$$

$$\bar{\lambda}(q, w \cdot x) := \bar{\lambda}(q, w) \cdot \bar{\lambda}(\bar{\pi}(q, w), x)$$

$\underbrace{\bar{\lambda}(q, w)}_{\in X^*} \cdot \underbrace{\bar{\lambda}(\bar{\pi}(q, w), x)}_{\in X}$



Notation:  $\bar{\lambda}_q(w) := \bar{\lambda}(q, w)$

Ex]: page 2A



DEF Given  $\mathcal{A}$  automaton,  $\mathcal{A}_{q_0}$ , with fixed  
INITIAL STATE  $q_0$ , is called  
INITIAL AUTOMATON

NOTE (1)  $\mathcal{A}_{q_0}$  defines  $\bar{\lambda}_{q_0}: X^* \rightarrow X^*$ , called  
 the ACTION of  $\mathcal{A}_{q_0}$

(2)  $\lambda_q$  is bijective  $\Rightarrow \bar{\lambda}_q$  is bijective  
 (on  $X$ ) (on  $X^*$ )

Ex: page 2A

Povzetek:

Automation  $\mathcal{A}$   $\rightsquigarrow$  Initial Automaton  $\mathcal{A}_{q_0}$   $\rightsquigarrow$  action  $\bar{\lambda}_{q_0}$   
 $(X^* \rightarrow X^*)$

Note  $\forall \mathcal{A}$  we can define  $|Q|$  different  $\mathcal{A}_q$

Ex (1) Page 2A:  $(\Gamma_1) \rightsquigarrow \begin{cases} (\Gamma_1)_{q_0} \rightsquigarrow \bar{\lambda}_{q_0} \\ (\Gamma_1)_{q_1} \rightsquigarrow \bar{\lambda}_{q_1} \\ (\Gamma_2)_{q_2} \rightsquigarrow \bar{\lambda}_{q_2} \end{cases}$

(2)  $(\Gamma_2) \rightsquigarrow (\Gamma_2)_q \rightsquigarrow \bar{\lambda}_q$



**DEF**  $f: X^* \rightarrow X^*$  is synchronous automatic if is definable as the action of some initial automaton  $A_{q_0}$ , i.e.  $(\exists q_0) f = \bar{\lambda}_{q_0}$ .

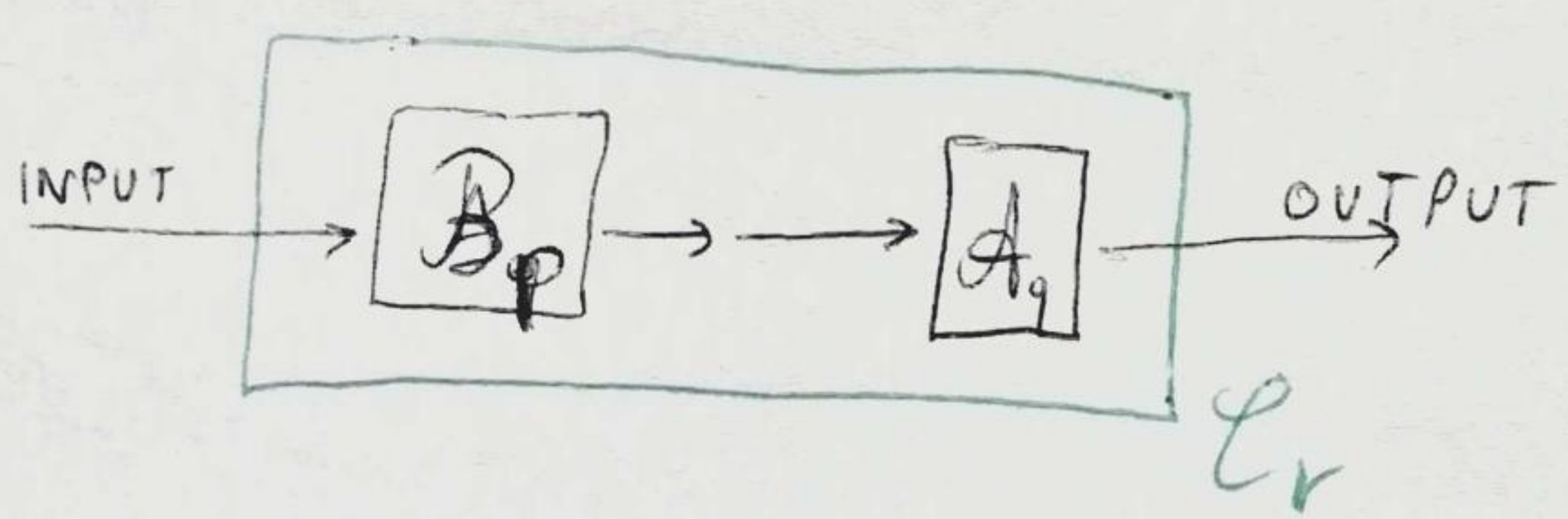
**DEF**  $S := \{f: X^* \rightarrow X^* \mid f \text{ is synchronous automatic}\}$

**Note**  $f \in S \Rightarrow f = \bar{\lambda}_{q_0} \Rightarrow f$  is bijective

We want to study  $S$

Composition Lemma Given  $A_q$  and  $B_p$ , initial automata on  $X$ ,  $\exists C_r$ , initial automaton, such that:

$$\left( \text{Action of } C_r \right) = \left( \text{Action of } A_q \right) \circ \left( \text{Action of } B_p \right)$$



$\Rightarrow S$  is closed under composition!

Similarly we have:

$$[f \in S \Rightarrow f^{-1} \in S]$$

$$\Rightarrow (\text{id}: X^* \rightarrow X^*) \in S$$

$\Rightarrow (S, \circ)$  is a group



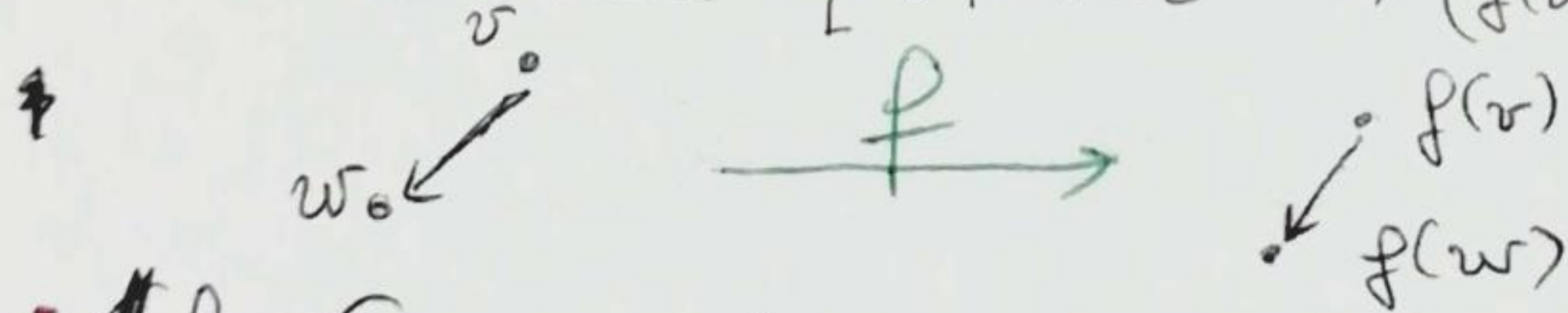
# CHARACTERIZATION OF THE ACTIONS OF AUTOMATA

Povzetek:

Automation  $\rightsquigarrow$  Initial automaton  $\rightsquigarrow \hat{I}_{q_0} : X^* \rightarrow X^*$

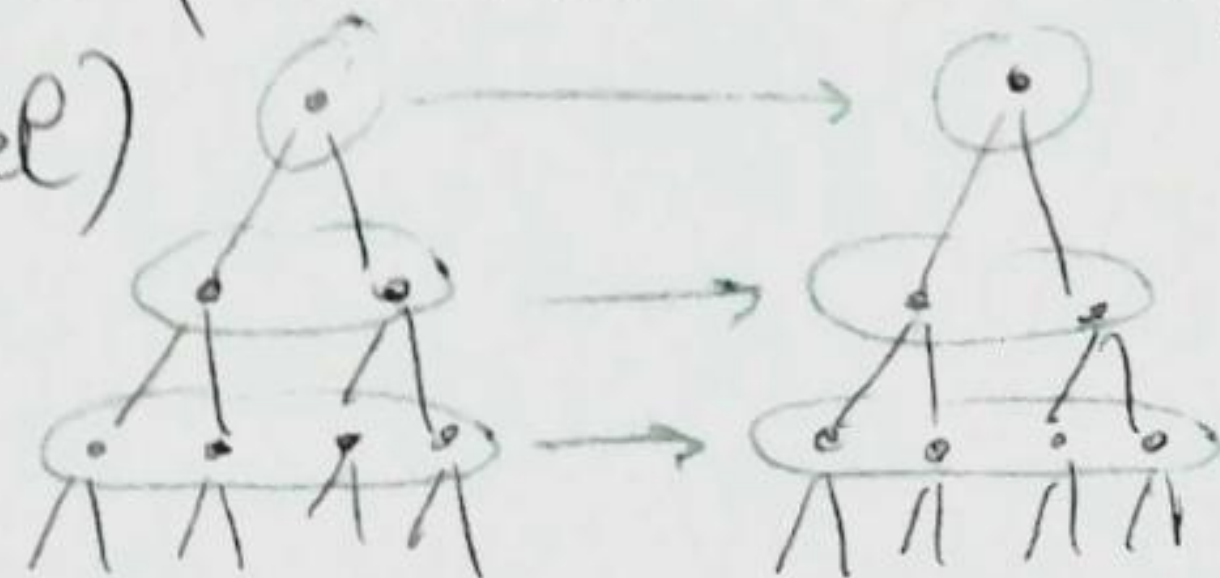
$$S := \{ f : X^* \rightarrow X^* \mid f \text{ is the action of some } A_{q_0} \}$$

Remark • Given  $G = (V, E)$  graph,  $f : V \rightarrow V$  is said to be a graph-homomorphism if preserve the adjacencies  $[(v, w) \in E \Rightarrow (f(v), f(w)) \in E]$



• If  $G$  is a tree,  $f$  is said to be a graph-homomorphism that preserves the root if (1) is a graph-homomorphism and if (2)  $f(r) = r$ , where  $r$  is the root of  $G$

Note If  $f$  is a graph-hom. that preserves the root,  $f(n\text{-th level of } G) \subseteq (n\text{-th level})$

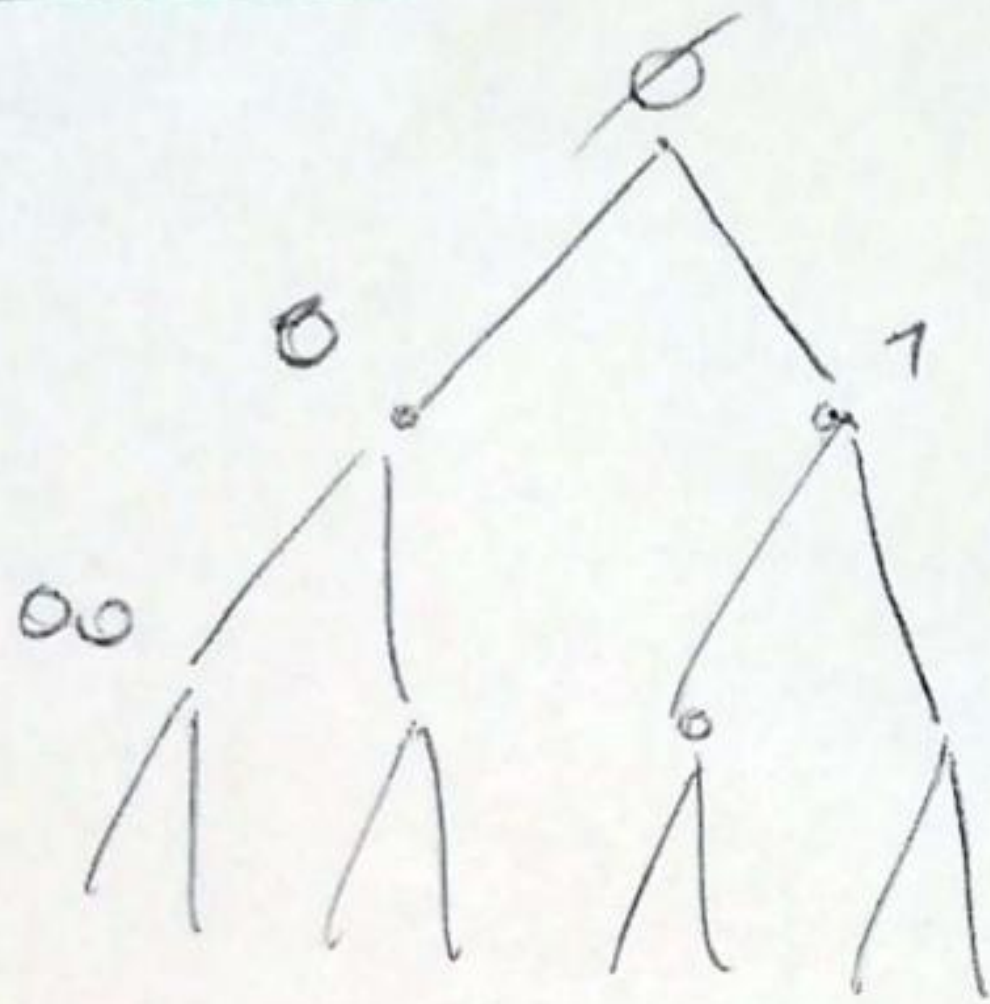




Proposition  $f$  is synch. automatic if and only if  $f$  is a graph-homomorphism that preserves the root on  $X^*$

(53)

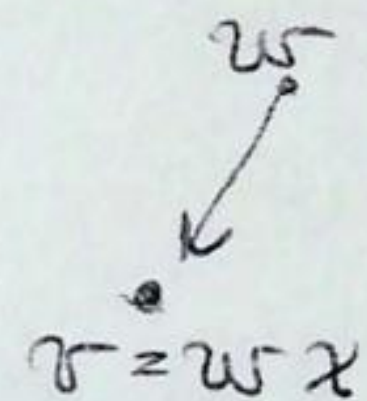
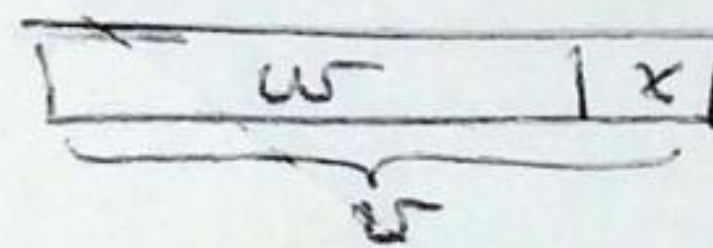
Observation Remembering that  $X^*$  is a tree, what does it mean to be a graph-homomorphism that preserve the root on  $X^*$ ?



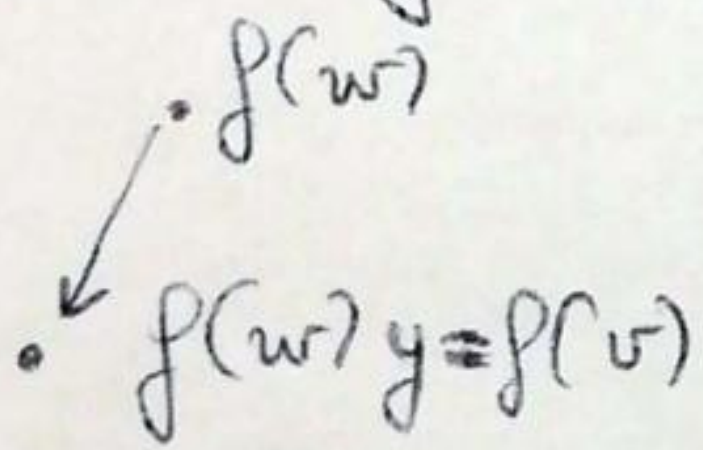
• Given a  $f$  with this property condition  $f(r) = r$  means

$$f(\emptyset) = \emptyset$$

• The condition of being a map homomorphism? In  $X^*$ ,  $v$  is son of  $w$  if and only if  $v = wx$



So if we want  $f(v)$  to be son of  $f(w)$  means  $f(v) = f(w)y$  for some  $y \in X$ .



Dokaz ( $\Rightarrow$ )

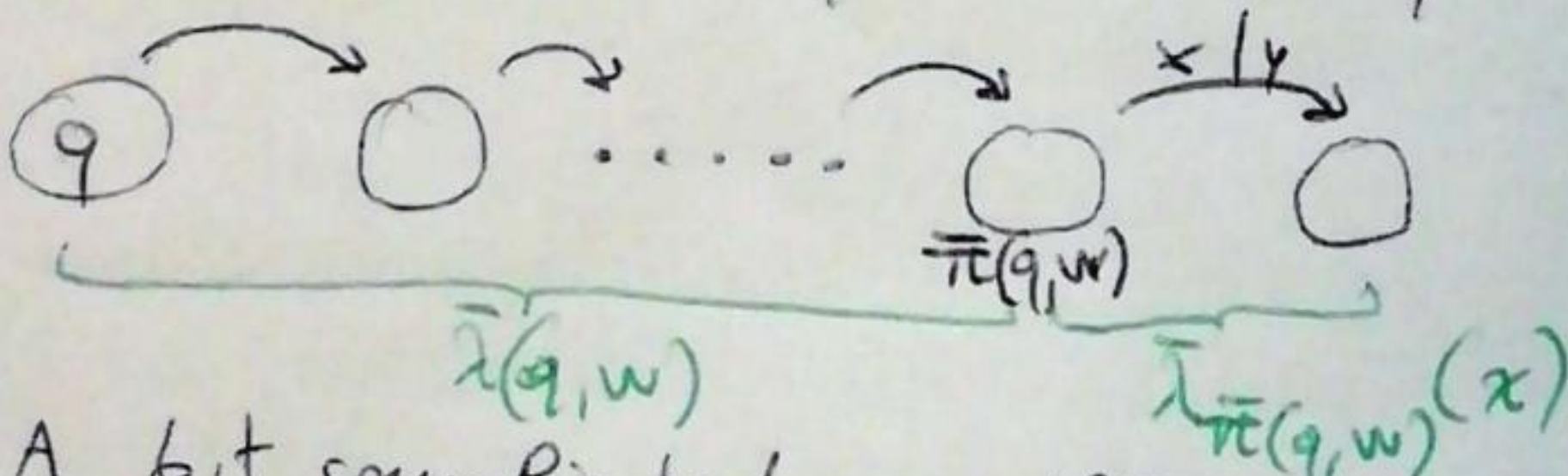
Given  $f$  synch. autom, we have  $f = \bar{\lambda}_q$

For some action of some  $q$ .

$$(1) f(\emptyset) = \bar{\lambda}_q(\emptyset) = \emptyset \quad \checkmark$$

(2) Given  $v = wx$  ( $v$  son of  $w$ ) we have:

$$f(v) = f(wx) = \bar{\lambda}_q(wx) = \bar{\lambda}_q(w) \cdot \bar{\lambda}_{\pi(q,w)}(x) = f(w)y \quad y \in X$$



( $\Leftarrow$ ) A bit complicated, we'll not see it

Corollary  $S = \{\text{bijective graph-homom. that preserve } \emptyset \text{ on } X^*\}$

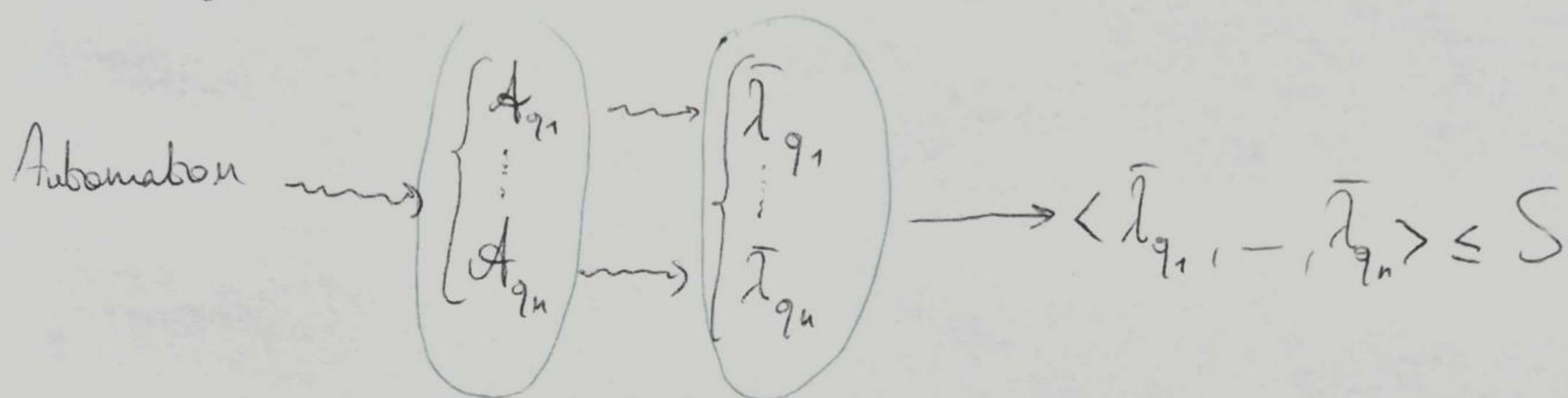


DEF Given  $\mathcal{A}$  automaton we can define  $|Q|$  initial automata  $\mathcal{A}_q$ , so  $|Q|$  actions  $\bar{\lambda}_q \in S$ .

(6)

The group generated by  $\mathcal{A}$  is defined as:

$$G(\mathcal{A}) := \langle \{\bar{\lambda}_q \mid q \in Q \text{ of } \mathcal{A}\} \rangle \leq S$$

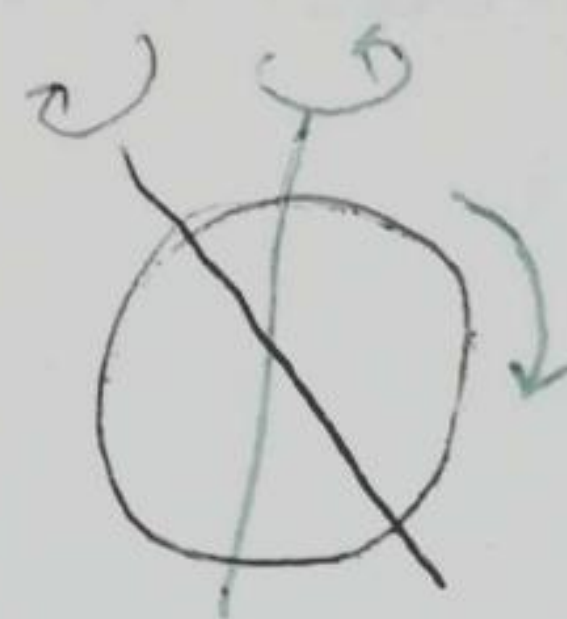


Ex: page 2A, (1)  $\Gamma_1$  defines  $G(\Gamma_1) := \langle \bar{\lambda}_{q_0}, \bar{\lambda}_{q_1}, \bar{\lambda}_{q_2} \rangle$

### Interesting Results and Examples

Proposition Given  $X = \{0, 1\}$ , and  $\mathcal{A}$ , 2-state automaton on  $X$ ,  $G(\mathcal{A})$  must be isomorphic to one of these groups:

- (1)  $\{1_G\}$
- (2)  $\mathbb{Z}_2$
- (3)  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$
- (4)  $\mathbb{Z}$
- (5)  $D_\infty = \text{Infinite dihedral group} = \{\text{symm. of the circle}\}$
- (6)  $\mathbb{Z} \rtimes \mathbb{Z}_2 = \text{lamplighter group}$

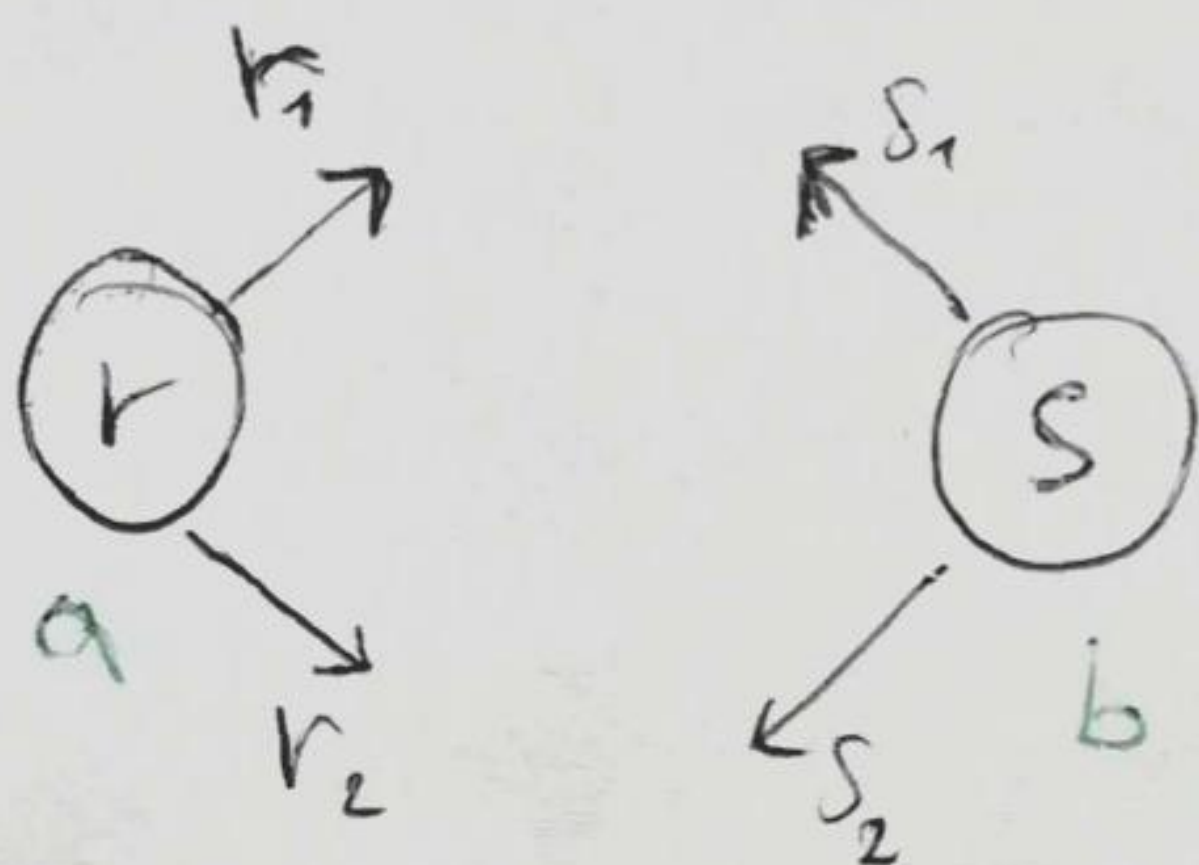




Sketch of proof:  $Q$ , the set of states

(7)

$$Q = \{r, s\}, \quad X = \{0, 1\}. \quad \mathcal{A} = (X, Q, \pi, \lambda)$$



•  $\lambda_s, \lambda_r: X \rightarrow X$  are permutations of a 2-element set  $X \Rightarrow \lambda_s, \lambda_r \in S_2$ ,  
 $S_2 = \{id = 1, \sigma\}$

~~id = 1~~  $\begin{cases} \sigma(0) = 1 \\ \sigma(1) = 0 \end{cases}$

• each arrow in  $\{r_1, r_2, s_1, s_2\}$  can ~~point~~ point on an element of  $Q = \{r, s\}$

So all the possible  $\mathcal{A}$ , with  $Q = \{r, s\}$ ,  $X = \{0, 1\}$  are the ones in which  $\lambda_s, \lambda_r \in S_2$  (uniquely determined  $\lambda$ ), and  $r_1, r_2, s_1, s_2 \in \{r, s\}$

(uniquely determines  $\pi$ )  $\Rightarrow$

$$\Rightarrow \text{they are } \underset{\lambda_s}{2} \cdot \underset{\lambda_r}{2} \cdot \underset{r_1}{2} \cdot \underset{r_2}{2} \cdot \underset{s_1}{2} \cdot \underset{s_2}{2} = 2^6.$$

Analysing each case we get  $[G(\mathcal{A}) \text{ is isomorphic to one case of the latter cases}]$ .



Interesting group :

$$\mathbb{Z} \wr \mathbb{Z}_2 := (\mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}, \star)$$

- $\mathbb{Z}_2^{(\mathbb{Z})} := \{ (b_i)_{i \in \mathbb{Z}} \mid b_i \in \mathbb{Z}_2 = \{0, 1\}, b_i = 1 \text{ just for a finite set of indexes } I \}$

Practically:



Infinite dark road  $\mathbb{Z}$

indexes of the ~~open~~ lamps turned on (indexes in  $I$ )

Ex] The previous represented element is

$((\tilde{b}_i)_{i \in \mathbb{Z}})$ , where  $b_{-5}, b_{-2}, b_2 = 1$ .

[We can sign  $(\tilde{b}_i)_{i \in \mathbb{Z}}$  with  $\{-5, -2, 2\} = I$ ]

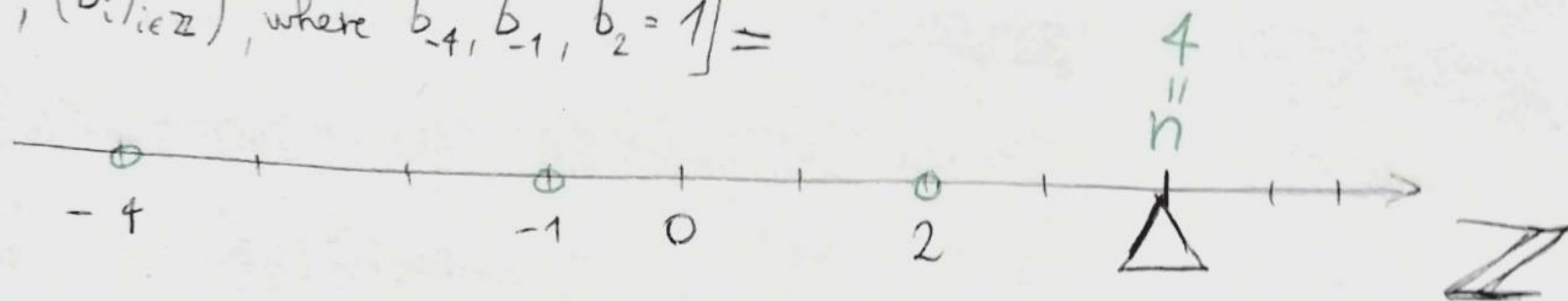
- $\star$  is not a direct product!

$$\begin{aligned} (n_1, (b_i)_{i \in \mathbb{Z}}) \star (n_2, (q_i)_{i \in \mathbb{Z}}) &:= \\ &= (\textcircled{n_1} + n_2, (b_i + q_{i + \textcircled{n_1}})_{i \in \mathbb{Z}}) \end{aligned}$$



# Visualization:

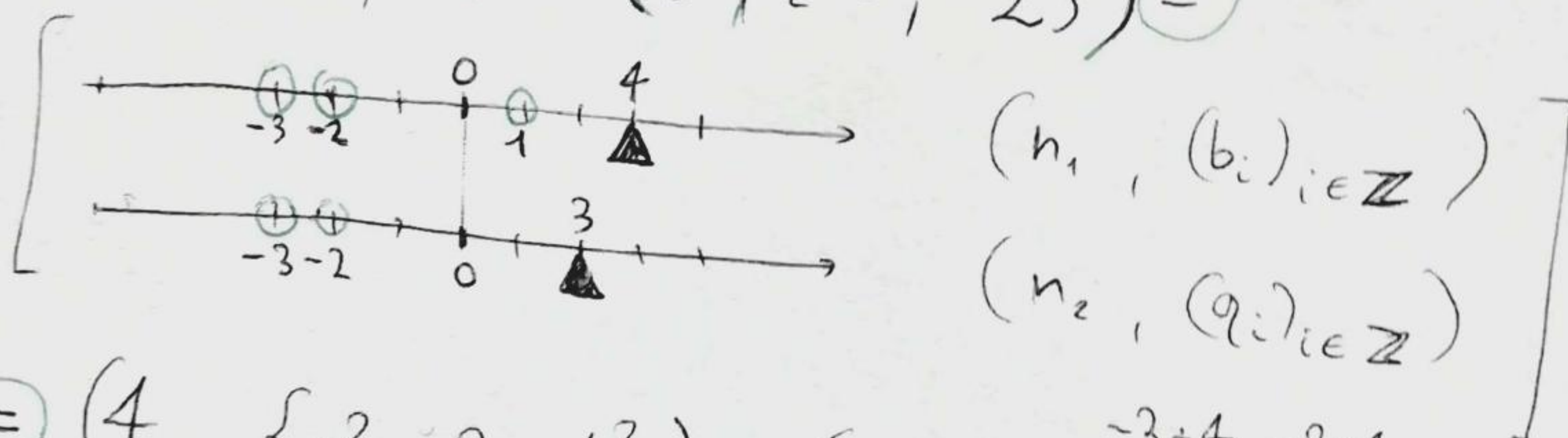
- OF an element  $(n, (b_i)_{i \in \mathbb{Z}}) \in \mathbb{Z} \times \mathbb{Z}_2^{(\mathbb{Z})}$   
 $[(4, (b_i)_{i \in \mathbb{Z}}), \text{ where } b_{-4}, b_{-1}, b_2 = 1] =$



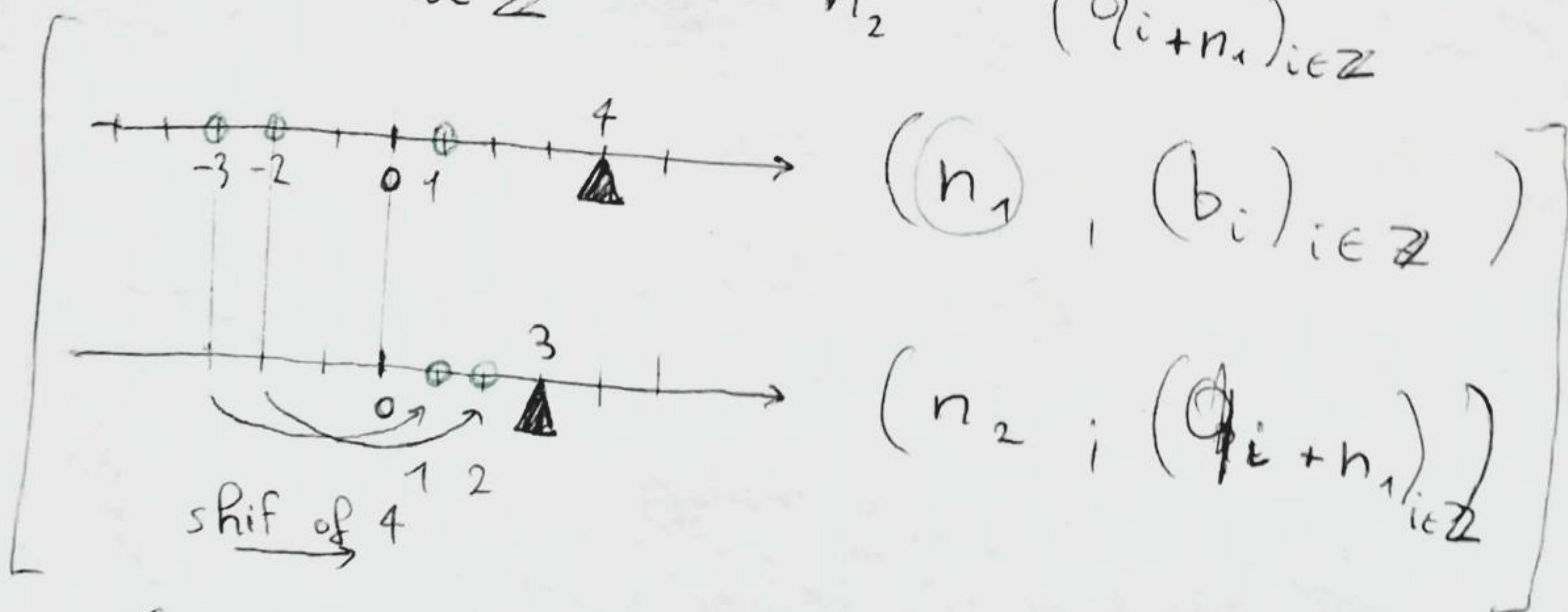
$n$  = position of the "camp light" on the infinite road

- OF the product  $\star$ :

$$\left( 4, \{ -3, -2, 1 \} \right) \star \left( 3, \{ -3, -2 \} \right) =$$



$$= \left( 4, \{ -3, -2, -1 \} \right) + \left( 3, \{ 1, 2 \} \right) =$$

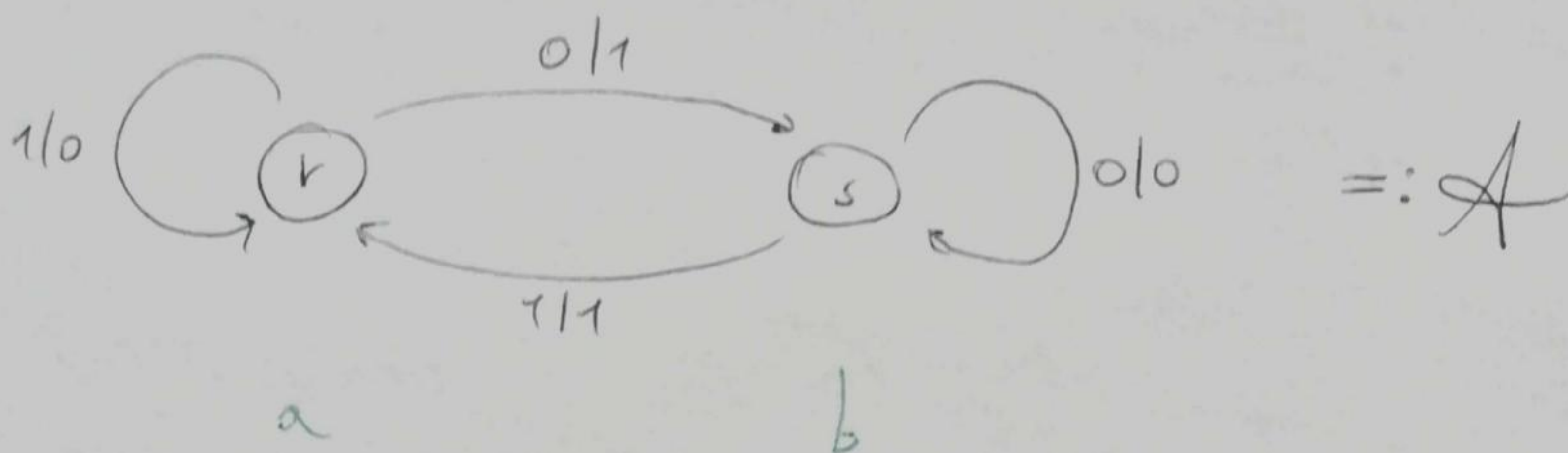


$$= (4+3, \{ -3, -2, 2 \} )$$



The automaton which defines  $\mathbb{Z} \int \mathbb{Z}_2$ :

(10)



$$a := \bar{\lambda}_r, \quad b := \bar{\lambda}_s$$

$$\mathbb{Z} \int \mathbb{Z}_2 = \langle a, b \rangle = G(\mathcal{A})$$

let's watch closer:

$$\begin{cases} a(0v) = 1b(v) \\ a(1v) = 0b(v) \end{cases}$$

$$\begin{cases} b(0v) = 0b(v) \\ b(1v) = 1a(v) \end{cases}$$

$$\lambda_r: X \rightarrow X$$

$$\lambda_r = \sigma \in S_2$$

$$\lambda_s: X \rightarrow X$$

$$\lambda_s = \text{id} \in S_2$$

$$\Rightarrow \begin{matrix} b^{-1} \\ \vdots \\ X^* \\ \downarrow \\ X^p \end{matrix} = \begin{cases} b^{-1}(0v) = 0b^{-1}(v) \\ b^{-1}(1v) = 1a^{-1}(v) \end{cases}$$



$$c := b^{-1} \cdot a = a \circ b^{-1}$$

$$\begin{cases} c(0v) = a \circ b^{-1}(0v) = a(0 \underbrace{b^{-1}(v)}^{\text{id}}) = 1 \underbrace{b \cdot b^{-1}(v)}^{\text{id}} = 1v \\ \underline{c(1v)} = a \circ b^{-1}(1v) = a(1 \underbrace{b^{-1}(v)}^{\text{id}}) = 0 \underbrace{b \cdot b^{-1}(v)}^{\text{id}} = 0v \end{cases}$$

• we see  $\langle a, b \rangle = \langle \underbrace{b^{-1}a}_{\underline{c}}, b \rangle = \langle \underline{c}, b \rangle = G(\underline{A})$

•  $\Sigma_2 = (X; +_{\Sigma_2})$

$$0+0=0$$

$$1+1=0$$

$$1+0=1$$

$$c(x_1 x_2 x_3 -) = (x_1 + 1) x_2 x_3 -$$

we search an explicit formula for  $b$ :

$$b(x_1 x_2 x_3 -) = y_1 y_2 y_3 - \quad y_n = ?$$

$$\begin{aligned} \text{we claim (A)} \quad b(x_1 x_2 - x_n \overset{(n+1)}{0} x_{n+2} -) &= \\ &= y_1 y_2 \dots y_n y_{n+1} b(x_{n+2} -) \end{aligned}$$

$$\begin{aligned} \text{(B)} \quad b(x_1 x_2 - x_n \overset{(n+1)}{1} x_{n+2} -) &= \\ &= y_1 y_2 - y_n y_{n+1} a(x_{n+2} -) \end{aligned}$$

Proof

(A) Watch diagram of  $A$ . Whenever we encounter a "0",  $b$  acts on the next letter

(B) Analogous



(12)

We claim  $b(x_1 x_2 \dots x_n \text{---}) = x_1 \underbrace{(x_2 + x_1)}_{y_2} \text{---} \underbrace{(x_n + x_{n-1})}_{y_n} \text{---}$

Proof: For induction on  $n$ .

$n=1$  We set  $x_0 = 0 \Rightarrow y_1 = x_1 = x_1 + 0 = x_1 + x_0$

$n \mapsto n+1$  4 cases:  $x_n, x_{n+1} \in \{00, 01, 10, 11\}$

Case 00:  $y_n = \cancel{x_n} + \cancel{x_{n+1}} ; y_{n+1} = ?$

$$b(x_1 \text{---} x_{n-1} \overset{n}{0} \overset{(n+1)}{0} x_{n+2} \text{---}) \stackrel{(4)}{=} y_1 \text{---} y_n b(0 x_{n+2} \text{---}) =$$

$$= y_1 \text{---} y_n 0 y_{n+2} \text{---}$$

$$y_{n+1} = 0 = 0 + 0 = x_n + x_{n+1}$$

The other cases are analogous ✓

$$\Rightarrow c(x_1 x_2 x_3 \text{---}) = \underbrace{(x_1 + 1)}_{\sigma(x_1)} x_2 x_3 \text{---}$$

$$b(x_1 x_2 x_3 \text{---}) = x_1 (x_2 + x_1) (x_3 + x_2) \text{---} =$$

$$= x_1 x_2 x_3 x_4 \text{---} +$$

[shift to right]

$$x_1 x_2 x_3 \text{---}$$

It can be proved, with these formulas, that

$$\langle b, c \rangle = \mathbb{Z} \int \mathbb{Z}_2$$