Groups of Automata

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1.Introduction

Automata are found to be important in:

- Information Theory
- Theory of Dynamical System
- Algebra
- Other areas

My Aim

Study some of the groups constructed through a special class of them, the Mealy Automata

2.Alphabet and the Free Monoid

Let X be finite set, called the *Alphabet*. Then we have:

- $X^* = \{x^1 x^2 \dots x^n : x^i \in X, n \in \mathbb{N}\}$ the *Free Monoid*
- Composition of words: $\mathbf{w} \circ \mathbf{v} := \mathbf{wv}$
- The Empty Word $\varnothing$
- Lenght of words: $\mathbf{w} = x_1 \dots x_n, |\mathbf{w}| := n$

3.An Example with Moore Diagrams

Moore Diagrams

We put $G = (\mathcal{Q}, E)$ with
$E := \{(q_i, q_j) | \exists \mathbf{w} \in X : \pi(\mathbf{w}, q_i) = q_j\}$

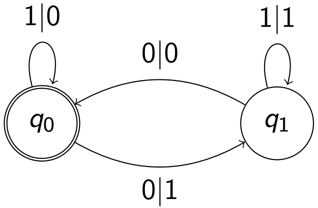

Figure: Example of a Moore Diagram of a 2-state Synchonous Automaton over the alphabet $X_I = X_O = X = \{0, 1\}$

4.Synchronous Automaton

Definition

A **Finite Synchronous Automaton** is a set $\mathcal{A} = < X_I, X_O, \mathcal{Q}, \pi, \lambda >$ where:

- X_I and X_O are *finite* sets called respectively the **Input and Output Alphabets**
- $\mathcal{Q}$ is a *finite* set called the **Set of Internal States of the Automaton**
- $\pi : X_I \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is a function called the **Transition Function**
- $\lambda : X_I \times \mathcal{Q} \longrightarrow X_O$ is a function called the **Output Function**

5.Extension of π and λ

Observation

We can naturally extend the Domain of π and λ:

- $\pi : X_I^* \times \mathcal{Q} :\longrightarrow \mathcal{Q}$

$$\pi(\varnothing, q) = q$$

$$\pi(\mathbf{w}x, q) = \pi(\mathbf{w}, \pi(x, q))$$

- $\lambda : X_I^* \times \mathcal{Q} :\longrightarrow X_O^*$

$$\lambda(\varnothing, q) = \varnothing$$

$$\lambda(\mathbf{w}x, q) = \lambda(\mathbf{w}, \pi(x, q))\lambda(x, q)$$

6.Action of an Automaton

Definition

If an Automaton $\mathcal{A}$ has a fixed state q_0 we call it an *Initial Automaton* and we write it as $\mathcal{A}_{q_0}$

Observation

Each $\mathcal{A}_{q_0}$ naturally defines $f : {X_I}^* \longrightarrow {X_O}^*$, with $f(\mathbf{w}) := \lambda(\mathbf{w}, q_0)$, called *Action of the Automaton* $\mathcal{A}_{q_0}$

7.Composition of Automata

Definition

Given $\mathcal{A}_1 =< X_I, X_{IO}, \mathcal{Q}_1, \pi_1, \lambda_1 >$ and $\mathcal{A}_2 =< X_{IO}, X_O, \mathcal{Q}_2, \pi_2, \lambda_2 >$ we define their *composition* $\mathcal{B} = \mathcal{A}_1 * \mathcal{A}_2 =< X_I, X_O, \mathcal{Q}_1 \times \mathcal{Q}_2, \pi, \lambda >$ with π and λ as follows:

- $\pi(x, (s_1, s_2)) = (\pi_1(x, s_1), \pi_2(\lambda_1(x, s_1), s_2))$
- $\lambda(x, (s_1, s_2)) = \lambda_2(\lambda_1(x, s_1), s_2)$

Observe: (Action of $\mathcal{A}_2$)$\circ$(Action of $\mathcal{A}_1$)=Action of $\mathcal{A}_1 * \mathcal{A}_2$

8.Some Algebraic Results

Consequences

From now on we assume that $X = X_I = X_O$:

- The functions $f : X^* \longrightarrow X^*$ defined by Mealy Automata(called **synchronous automatic**) form a *semigroup*
- Let $\mathcal{A}$ be a Synchronous Automaton with its action f. It's invertible **(exists $\mathcal{A}'$ with its action f' such that** $f \circ f' = id$**)** *if and only if* $\lambda(\cdot, q)$ is invertible (Memento: $\lambda(\cdot, q)$ is exactly the action of the Automaton)

$$x|y \longrightarrow y|x$$

9.Groups of Synchronous Automata

Definition

Let $\mathcal{A}$ be a Synchronous Invertible Automata. The *Group Defined by the Invertible Automaton* $\mathcal{A}$ is the group generated by the actions defined by $\{\mathcal{A}_q : q \in \mathcal{Q}\}$

Proposition

Let $\mathcal{A}$ be a 2-state Automaton over the alphabet $X = \{0, 1\}$ and G the group defined by this Automaton. Then G is isomorphic to one of the following group:

- *the trivial group* $\{1_G\}$
- $\mathbb{Z}_2$
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$
- $\mathbb{Z}$
- *the Infinite Dihedral group* $\mathbb{D}_\infty$
- *the Lamplighter group* $\mathbb{Z} \wr \mathbb{Z}_2$

10.Conclusion

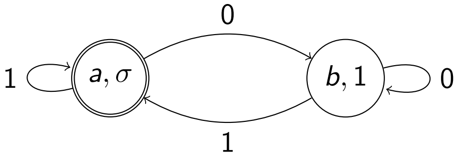

Figure: Automaton which defines the Lamplighter group

Actual Progress

At present, in the case of a *3-state Automaton* over a *2-letter Alphabet* are classified just the finite, abelian and free groups

Thank you for your attention!