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Automatne Grupe

Delo diplomskega seminarja

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Groups of Automata

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Contents

Groups of Automata

Abstract

In this Bachelor Thesis we ~~review~~ present some interesting examples and results ~~of~~ on groups generated by Automata. First we treat the mathematical idea of input and output and we give the definition of Finite Synchronous Deterministic Mealy Automaton. We exploit it to generate groups and we link all these concepts with some abstract structures, such as as the tree homomorphism, or the wreath product. ~~Finally we~~ We ~~treat~~ include some interesting examples, ~~that~~ is the Infinite Dihedral Group and the Lamplighter Group. Finally, we present the classification of all groups generated by 2-state automata over a 2-letter alphabet.

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Keywords: Automata, Finite Automata, Word Spaces, Moore Diagram, Wreath Product.

Automatne Grupe

Razširjeni povzetek

Kasual

Ključne besede: Kasual ADD!

Gruppi di Automi

SINTESI ESTESA

Kasual

Parole chiave: Kasual

Part 1. Introduction

The word *automaton* comes from greek (plural *automat**A*** or *automatons*), and means "acting on one's self-will". Roughly speaking an Automaton is a very specific *model of computation*. We can heuristically say that a model of computation is a machine which, taking an input, spews out an output(1).

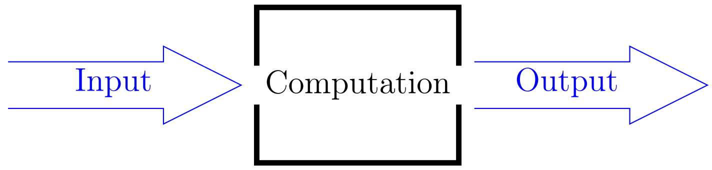

FIGURE 1. Model of Computation

So to first understand the topic we must define and understand the structure of input and output.

1. Words Spaces and Alphabet Trees

Data given and received will be somehow written and read, through some kind of symbols. Mathematically:

Definition 1. An **Alphabet X** is a finite set of elements called **letters**.

Definition 2. The set $\mathbf{X}^* := \{x_1 \dots x_n | n \in \mathbb{N}, x_i \in \mathbf{X}\}$ is called the **Set of Finite Words** or **Finite Dictionary**, and its elements are called **words**. The element with no letter, written as $\varnothing$, is called the *empty word*,

Definition 3. ~~Given words~~ Let $\mathbf{w} = x_1 \dots x_n$ and $\mathbf{u} = y_1 \dots y_m$ be words. The **lenght** of $\mathbf{w}$, written as $|\mathbf{w}|$, is n. The lenght of the empty word is 0. The **concatenation** of $\mathbf{w}$ and $\mathbf{u}$, written as $\mathbf{w} \circ \mathbf{u} = \mathbf{wu}$, is the word $x_1 \dots x_n y_1 \dots y_m$.

Example 1. *Let* $\mathbf{X}$ ~~*be*~~ $= \{0, 1\}$. Then $0100 \circ 111 = 0100111$ and $11 \circ 0101 = 110101$. *Let* $\mathbf{X}$ *be* ? $\{0, , 2\}$. Then $02 \circ 20 = 0220$. ? $\circ\, 2 = 2$.

Proposition 1. $(\mathbf{X}^*, \circ)$ *is a monoid, called the* ***Free Monoid*** on X.

Proof. The operation $\circ$ is associative with $\varnothing$ ~~as~~ being an identity element. $\square$

Better: The reader may now wonder:

A question might now come to the mind of the reader: can a word with an infinite lenght exist?

Definition 4. The **Set of Infinite Words** ~~of~~ or the **Infinite Dictionary** is the set $\mathbf{X}^\omega := \{x_1 \dots x_i \dots | x_i \in \mathbf{X}\} = \mathbf{X}^{\mathbb{N}}$.

~~*Observation*~~ Remark 1. ~~The concatenation can be extend.~~ ~~In fact taken~~ Note that if $\mathbf{u} = x_1 \dots x_n \in \mathbf{X}^*$ and $\mathbf{v} = y_1 \dots y_i \dots \in \mathbf{X}^\omega$, we can define $\mathbf{u} \circ \mathbf{v} := x_1 \dots x_n y_1 \dots y_i \dots \in \mathbf{X}^\omega$.

Definition 5. A word $\mathbf{w} = x_1 \dots x_n$ is the **beginning** or **prefix** of a word $\mathbf{u} \in \mathbf{X}^*$(or $\mathbf{u} \in \mathbf{X}^\omega$) if $\mathbf{u} = \mathbf{wv} = x_1 \dots x_n \mathbf{v}$ for some ~~$\mathbf{u}$~~ $\mathbf{v} \in \mathbf{X}^*$(or $\mathbf{v} \in \mathbf{X}^\omega$). In this case we set $\mathbf{v} = \mathbf{u} - \mathbf{w}$

Given $A \in \mathbf{X}^* \cup \mathbf{X}^\omega$, we denote $\mathcal{P}(A)$ the **longest common prefix of** A, that is uniquely defined.

A word $\mathbf{v} \in \mathbf{X}^\omega$ is called **almost periodic** if $\mathbf{v} = \mathbf{uwww} \dots$ for some ~~$\mathbf{u}$~~ $\mathbf{u}, \mathbf{w} \in \mathbf{X}^*$.

General comment: avoid starting a sentence from formula, start it from a word!

if you write "notice", provide a proof.

1.1. Topology on the Infinite Dictionaries. We can provide ~~this space~~ the set $\mathbf{X}^\omega$ with a distance, a metric; and consequently a topology, in order ~~then~~ to exploit the properties of some continous functions for our egoist means. ?

For every *decreasing* sequence of *positive* numbers $\widetilde{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$ such that $lim_{n\to\infty}\lambda_n =$? 0 we can find a metric $d_{\widetilde{\lambda}}$ on X^ω such that

$$d_{\widetilde{\lambda}}(\mathbf{w_1}, \mathbf{w_2}) = \lambda_n \tag{1}$$

~~defined on $\mathbf{X}^\omega$~~, where $n = |\mathcal{P}(\{\mathbf{w_1}, \mathbf{w_2}\})|$ is the longest common prefix is the length of the longest common prefix of the words $\mathbf{w_1}$ and $\mathbf{w_2}$. This tell us that every set $\mathbf{wX}^\omega := \{\mathbf{wu}|\mathbf{u} \in \mathbf{X}^\omega\}$ can be seen as a ball of radius $\lambda_{|\mathbf{w}|}$ with the center ~~on~~ in an arbitrary point $\mathbf{u} \in \mathbf{wX}^\omega$. In particular ~~notice that~~ it can be shown that the set $\mathbf{wX}^\omega$ is *clopen*(open and closed).

~~*Observation*~~ Remark 2. It will be often useful to set $\widetilde{\lambda} = (\frac{1}{n})_{n\in\mathbb{N}}$.

1.2. Tree Structure of the Dictionaries. It's sometimes very useful to ~~imagine~~ represent $\mathbf{X}^*$ in the form of a tree graph: the ~~vertexes~~ vertices will be the elements of $\mathbf{X}^*$ with $\varnothing$ as the root. We then say that $\mathbf{v}$ is son a child of $\mathbf{u}$ *if and only if* $\mathbf{u} = \mathbf{v}x$ for some $x \in \mathbf{X}$. An example is 3. ?

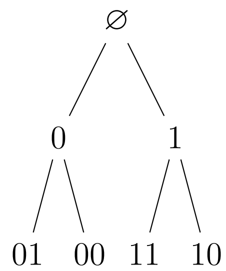

FIGURE 2. An example of a tree graph in the case $X = \{0, 1\}$

Norm 1. *From now on we will always use the alphabet* $\mathbf{X} = \{0, 1\}$*; this will also help the reader to better visualise the concept.* The case where $|X| = n \geq 2$ is similar.

The set X^n, the set of the words of lenght n, is called the *the n-th floor of* X^*(see **Fig.1**)?.

Notice that $\mathbf{X}^\omega$ is the very bottom of the graph of 3, therefore that's also called the *Boundary* ~~*tree*~~ of the tree.

Finally we define the notion of endomorphisms ~~on~~ of a tree, and some of their properties.

Definition 6. Given A, B trees, $f : A \longrightarrow B$ is a **tree-homomorphism** if it preserves the root and the adjacency of the ~~tree~~ vertices, i. e.:

- If $a \in A$ is the root, $f(a)$ is the root
- If (u, v) is an edge of A, also $(f(u), f(v))$ is an edge of B(that is, f *is a graph-homomorphism*).

If $A = B$, f is called *tree-endomorphism.* If $A = B$ and f is bijective we call it a *tree-automorphism.*

~~It's easy verifiable~~ It is easy to verify that all the tree-homomorphisms ~~of a tree are a~~ form a *semigroup* under the composition of functions, and all the tree-automorphisms ~~are a~~ form its subsemigroup ~~that~~ which is also a *group*.

Norm 2. *Very often we will simply* ~~*say*~~ write *"-morphism" instead of "tree -morphism". The meaning will be clear* ~~*by*~~ from *the context.*

2. Automata and Initial Automata

Now we will treat the formal definition of the very specific type of Automaton which we need, the *Deterministic Finite(Finite State) Synchronous Automaton*, or *Finite Mealy Automaton*, or *Finite Trasducer*. We will always call it simply *Automaton*, but the reader should know that this is a *very specific* case.

Definition 7. An **Automaton** is a ~~set~~ 4-tuple $\mathcal{A} =< \mathbf{X}, \mathcal{Q}, \pi, \lambda >$ where:

- $\mathbf{X}$ is an alphabet, usually referred ~~as~~ to the **Input** and/or **Output Alphabet**,
- $\mathcal{Q}$ is a set called the **Set of Internal States of the Automaton**,
- $\pi : \mathbf{X} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is a function called the **Transition Function**,
- $\lambda : \mathbf{X} \times \mathcal{Q} \longrightarrow \mathbf{X}$ is a function called the **Output Function**.

We say that $\mathcal{A}$ is a $|\mathcal{Q}|$-*state*-automaton *on* $\mathbf{X}$.

This technical description explains us how an automaton performs the action of transforming an input in an output(??)? We can imagine that for every *input letter* x we plug in the machine, and for every *state* q from which we decide to start, the machine moves to a state $p = \pi(x, q) \in \mathcal{Q}$ and ~~spits out~~ returns an *output letter* $y = \lambda(x, q) \in \mathbf{X}$. Don't worry, there is also a way to visualise ~~it~~ this.

Definition 8. Given an Automaton $\mathcal{A} =< \mathbf{X}, \mathcal{Q}, \pi, \lambda >$ we define ~~his~~ its **Moore Diagram** as the oriented graph $G = (\mathcal{Q}, \mathcal{E})$ where 2 states ~~(q_1, q_2)~~ q_1 and q_2 are connected whenever $\exists x \in \mathbf{X}$ *s.t.* $\pi(x, q_1) = q_2$ and the label reffered to each edge is $x|\lambda(x, q_1)$(=input|output). ?

I thank Edward F. Moore, for he gave us an easy-manageable representation of these objects. We can see here an example: BETTER: here is an example;

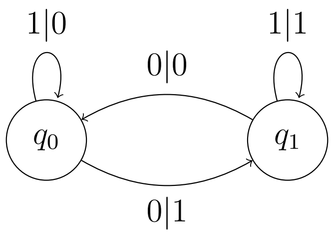

Figure 3. Example of a Moore Diagram of a 2-state-Automaton over the alphabet $\mathbf{X} = \{0, 1\}$

We observe that for every automaton ~~his~~ its Moore Diagram has ~~this characteristic:~~ the following property:

what is this?

(2) $\quad \forall q \in \mathcal{Q}$ and $\forall x \in X_I \quad \exists! e \in \mathcal{E}$ such that the left side of its label reads "x"

That is, for every vertex in the graph and for every letter of the input alphabet, exists a unique? an arrow, (from that vertex) through which we travel with that input. We can see many examples in 3 ? Moore Diagrams?

~~Observation~~ Remark 3. These Diagram are so important not just for the visual insight, but also because whenever a graph respects the condition (2), it *uniquely* defines an automaton. So given $\mathcal{M} := \{G \mid G \text{ is a Moore Diagram}\}$, there is a unique correspondence between $\mathcal{M}$ and the set of all a Automaton. This means that instead of defining this types of machine through some tedious list of outputs and transitions for each possible input $x \in \mathbf{X}$ and $q \in \mathcal{Q}$, we can simply draw them. Then to compute the outcome of (x, q), we just follow the arrow to $\pi(x, q)$ and watch look at the left side $\lambda(x, q)$ of the label.

? Figure 3?

Example 2. *In 3, given input* $\mathbf{w} = 0$ *and state* $q = q_0$*, then* $\pi(\mathbf{w}, q) = q_1$ *and* $\lambda(\mathbf{w}, q) = 1$*. ~~Or~~ If if* $\mathbf{w} = 1$ *and* $q = q_1$*, then* $\pi(\mathbf{w}, q) = q_1$ *and* $\lambda(\mathbf{w}, q) = 1$*.*

Proposition 2. *We can naturally extend the Domain of* π *and* λ *from single letters in* $\mathbf{X}$ *to words in* $\mathbf{X}^*$*:* recursively

- $\bar{\pi} : \mathbf{X}^* \times \mathcal{Q} :\longrightarrow \mathcal{Q}$

$$\bar{\pi}(\varnothing, q) = q \,,$$

$$\bar{\pi}(x\mathbf{w}, q) = \bar{\pi}(\mathbf{w}, \bar{\pi}(x, q)) \,.$$

- $\bar{\lambda} : \mathbf{X}^* \times \mathcal{Q} :\longrightarrow \mathbf{X}^*$

$$\bar{\lambda}(\varnothing, q) = \varnothing \,,$$

$$\bar{\lambda}(x\mathbf{w}, q) = \bar{\lambda}(x, q)\bar{\lambda}(\mathbf{w}, \bar{\pi}(x, q)) \,.$$

Remark ~~*Observation*~~ 4. The two definitions are equivalent to: Give exact reference

$$\bar{\pi}(\mathbf{w}x, q) = \bar{\pi}(x, \bar{\pi}(\mathbf{w}, q)) \,,$$

$$\bar{\lambda}(\mathbf{w}x, q) = \bar{\lambda}(\mathbf{w}, q)\bar{\lambda}(x, \bar{\pi}(\mathbf{w}, q)) \,.$$

Example 3. *We can compute* $\bar{\pi}$ *and* $\bar{\lambda}$ *following the arrows as before, and then maiking the composition of the single right side of the labels. In 3, given input* $\mathbf{w} = 0000$ *and state* $q = q_0$*, then* $\bar{\pi}(q, \mathbf{w}) = q_0$ *and* $\bar{\lambda}(q, \mathbf{w}) = 1010$*. Or if* $\mathbf{w} = 110$ *and* $q = q_1$*, then* $\bar{\pi}(q, \mathbf{w}) = q_0$ *and* $\bar{\lambda}(q, \mathbf{w}) = 110$*.* Figure 3

Now we ~~start~~ to point out a problem. For ~~each~~ an Automaton we don't have really an output till we don't decide where we start from. In other words in **??**, together with the input, we need to give a state(and different states, with the same input might get us different output). So what happens if we fix $q \in \mathcal{Q}$, and we imagine to start always there? Reword this, e.g. To make an automaton a word transducer, we need to specify an initial state.

Definition 9. If an Automaton $\mathcal{A}$ has a fixed state q_0 we call it an **Initial Automaton with Initial State** q_0 and we write it as $\mathcal{A}_{q_0}$. Each $\mathcal{A}_{q_0}$ naturally defines $\bar{\lambda}_{q_0} : \mathbf{X}^* \longrightarrow \mathbf{X}^*$, with $\bar{\lambda}_{q_0}(\mathbf{w}) := \bar{\lambda}(\mathbf{w}, q_0)$, called the **Action of the Automaton** $\mathcal{A}_{q_0}$. Two *Initial Automata* are said to be *equivalent* if they define the same actions.

Proposition 3. *The Action* $\bar{\lambda}_{q_0}$ *of an Initial Automaton preserves the lenght of words, i.e.* $|\bar{\lambda}_{q_0}(\mathbf{w})| = |\mathbf{w}|$*.*

Proof. ~~For induction on $n = |\mathbf{w}|$.~~ The statement can be easily proved by induction on $n = |w|$ □

Remark ~~*Observation*~~ 5. Given $\mathcal{A}_{q_0}$, we can define an *Infinite Action* $\bar{\lambda}_{q_0} : \mathbf{X}^\omega \longrightarrow \mathbf{X}^\omega$ by ~~from the same~~ similar recursive formulas, and we can consequently declare that two Initial Automata are ω*-equivalent* if they determine the same Infinite Action. Two Automata are equivalent *iff* are ω-equivalent. This comes naturally from the fact that the defining formula is the same. ?

Example 4. *In 2 there are two equivalent Initial Automata.* ?

Norm 3. *An Initial Automaton is usually drawn evidentiating the Initial State with a double circle around its vertex(2).* depicting ?

?

If an Automaton is a skeleton, an Initial Automaton is an alive working machine, is our model of computation, that for each input ~~gets~~ produces an output travelling through the states. I underline this concept: an Automaton doesn't define any function, till we don't fix a state q_0. Take care of distinguish the two in every definition, porposition and example that we will give. ? it is important to bear this in mind in all definitions,....

Part 2. Automata, Trees and Algebra

Here we will show how Automata can ~~build~~ define algebraic structures and how synchronous functions can be characterised.

Definition 2.1. Given automata $\mathcal{A}_1 =< X, \mathcal{Q}_1, \pi_1, \lambda_1 >$ and $\mathcal{A}_2 =< X, \mathcal{Q}_2, \pi_2, \lambda_2 >$ we define their *composition* $\mathcal{B} := \mathcal{A}_1 * \mathcal{A}_2 =< X, \mathcal{Q}_1 \times \mathcal{Q}_2, \pi, \lambda >$ with π and λ defined as follows:

- $\pi(x, (s_1, s_2)) = (\pi_1(x, s_1), \pi_2(\lambda_1(x, s_1), s_2))$,
- $\lambda(x, (s_1, s_2)) = \lambda_2(\lambda_1(x, s_1), s_2)$,

with $x \in X$ and $(s_1, s_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$.

This is something similar to put two machine ~~in serie~~ one next to the other. Informally this ~~means~~ means that we feed the second machine with the output of the first one(2), so we ~~use~~ join ~~the~~ both of them ~~as~~ into a single bigger and more complicated transducer. ? Notice that the two Automata *are not* Initial Automata.

~~*Observation*~~ Remark 6. With this operation the operation of composition the set of all Automata on an alphabet **X** becomes a *semigroup*.

? Reformulate this!

~~*Observation*~~ Remark 7. (Action of $(\mathcal{A}_2)_{q_2}$)$\circ$(Action of $(\mathcal{A}_1)_{q_1}$)=Action of $(\mathcal{A}_1 * \mathcal{A}_2)_{(q_1,q_2)}$ =Action of $(\mathcal{B})_{(q_1,q_2)}$ for all $q_1 \in \mathcal{Q}_1$ and for all $q_2 \in \mathcal{Q}_2$. This means that the operation on the set of *Automata* extends to an operation on the set of *Initial Automata*! However there are certain characteristic for which this operation is not yet very maneuverable, ? for example the fact that there can be many different Automata which, for every pair of states (q_1, q_2) describe the same Action. ?

Reformulate this!

3. Synchronous Automatic functions

~~Here~~ In this section, given an Action of an Initial Automaton, we describe and study its properties.

Definition 10. A transformation on $\mathbf{X}^*$(i.e. a function $f : \mathbf{X}^* \longrightarrow \mathbf{X}^*$) is called **Finite Synchronous Automatic** if it is the (finite) action ~~to~~ of some Initial Automaton $\mathcal{A}_{q_0}$, i.e. if $f = \bar{\lambda}_{q_0}$.

Definition 11. A transformation on $\mathbf{X}^\omega$(i.e. a function $f : \mathbf{X}^\omega \longrightarrow \mathbf{X}^\omega$) is called **Infinite Synchronous Automatic** if it is the infinite action of some Initial Automaton.

Proposition 3.1. *The Finite Synchronous Automatic transformations form a semigroup $\mathcal{SF}(X)$(S stands for "Synchronous" and F for "Finite"). The Infinite Synchronous Automatic transformations form a semigroup isomorphic to $\mathcal{SF}(X)$.*

Proof. The first point comes from the fact that the Composition of *Initial* Automata is an *Initial* Automaton, therefore $\mathcal{SF}(X)$ is closed under composition of functions. The second point arise from 5. ? $\square$

Observation 8. If we are interested to describe the set of all possible actions, it is the same to study $\mathcal{SF}(X)$ or its isomorphic copy. Thus we will just generally speak about **Synchronous Automatic** Transformations and we will usually refer to the finite case. Different usage will be clear from the context.

we provide

Now ~~let's see~~ an important characterization. of synchronous automatic transformations.

A transformation $f: X^* \to X^*$

Proposition 4. ~~f~~ *is Synchronous Automatic iff f is a tree-homomorphism on* $\mathbf{X}^*$ *(3)*. ?

Proof. Just for the purpose of this proof, and just in the second part, we will use a more general definition of Automaton, allowing $\mathcal{Q}$ to be infinite.

Since

(⇐) f is synchronous automatic, ~~therefore~~ there is an action $\bar{\lambda}_{q_0}$ of some Initial Automaton such that $f = \bar{\lambda}_{q_0}$. We need to show that $\bar{\lambda}_{q_0}$ (1)preserves the root and (2)is a graph endomorphism. ~~For~~ by the definition $f(\varnothing) = \bar{\lambda}_{q_0}(\varnothing) = \varnothing$, thus (1) holds. Now (2): if $\mathbf{v}$ is ~~son~~ child of $\mathbf{w}$(i.e. $\mathbf{v} = \mathbf{w}x$ for some $x \in \mathbf{X}$) is it true that $f(\mathbf{v})$ is ~~son~~ a child of $f(\mathbf{w})$(i.e. $f(\mathbf{v}) = f(\mathbf{w})y$ for some $y \in \mathbf{X}$)?

we prove

$$(3) \quad f(\mathbf{v}) = f(\mathbf{w}x) = \bar{\lambda}_{q_0}(\mathbf{w}x) = \bar{\lambda}(\mathbf{w}x, q_0) = \\ = \bar{\lambda}(\mathbf{w}, q_0)\bar{\lambda}(x, \bar{\pi}(q_0, \mathbf{w})) = f(\mathbf{w})\bar{\lambda}(x, \bar{\pi}(q_0, \mathbf{w}))$$

But $|\bar{\lambda}(x, \bar{\pi}(q_0, \mathbf{w}))| = 1$ because every action is lenght-preserving, thus $y = \bar{\lambda}(x, \bar{\pi}(q_0, \mathbf{w})) \in \mathbf{X}$, so $f(\mathbf{v}) = f(\mathbf{w}x) = f(\mathbf{w})y$, so (2) holds as well.

(⇒) Let $f : \mathbf{X}^* \longrightarrow \mathbf{X}^*$ be a tree-endomorphism. We must find an Initial Automaton such that its action is exactly f. We define $\mathcal{A} =< X, \mathcal{Q}, \pi, \lambda >:=< X, \mathbf{X}^*, \pi, \lambda >(\mathcal{Q} = \mathbf{X}^*$ is *infinite*) with $\pi(\mathbf{q} \in \mathbf{X}^*, x \in \mathbf{X}^*) := qx$ and $\lambda(\mathbf{q} \in \mathbf{X}^*, x \in \mathbf{X}^*) := f(qx) - f(q)$.

First: is the output function λ well defined, i.e. is the subtraction $f(\mathbf{q}x) - f(\mathbf{q})$ well defined? Yes, because f is a tree-endomorphism, so $f(\mathbf{q}x)$ is son of $f(\mathbf{q})$. Second: is $\bar{\lambda}_{\varnothing}$, the action of $\mathcal{A}_{\varnothing}$, our function? For induction on $n = |\mathbf{w}|$, with $\mathbf{w} \in \mathbf{X}^*$. reformulate this! Formulate precisely what you prove by induction!

(Case $n = 0$**)** $\bar{\lambda}(\varnothing, \varnothing) = \varnothing = f(\varnothing)$.

(Case $n \to n+1$**)** Given $\mathbf{w} \in \mathbf{X}^* \setminus \{\varnothing\}$, it can be written as $\mathbf{v}x$, with $\mathbf{v} \in \mathbf{X}^*$ and $x \in \mathbf{X}$. Then $\bar{\lambda}(\varnothing, \mathbf{v}x) = \bar{\lambda}(\varnothing, \mathbf{v})\bar{\lambda}(\bar{\pi}(\varnothing, \mathbf{v}), x) = f(\mathbf{v})\bar{\lambda}(\mathbf{v}, x) = f(\mathbf{v})[f(\mathbf{v}x) - f(\mathbf{v})]$. □

tree-endomorphism?

Proposition 5. *If f is an endomorphism on* $\mathbf{X}^*$*, then $f(X^n) \subset X^n$.*

Proof. By induction on n. □ This can be easily proved by induction on n.

Remark

~~*Observation*~~ 9. The last proposition is a graph persepective on the lenght-preserving condition.

and next

Definition 12. Given a *tree-endomorphism* $g : \mathbf{X}^* \longrightarrow \mathbf{X}^*$, we can define its **restriction in v** as the function $g_{|\mathbf{v}} : \mathbf{X}^* \longrightarrow \mathbf{X}^*$ defined by the equality:

$$g(\mathbf{v}\mathbf{w}) = g(\mathbf{v})g_{|\mathbf{v}}(\mathbf{w})$$

I think $f(X^n) = X^n$?

what does this mean?

Observation 10. Since g is a tree-endomorphism, and since $\mathbf{v}\mathbf{X}^* \cong g(\mathbf{v})\mathbf{X}^* \cong \mathbf{X}^*$, we have that $g : \mathbf{v}\mathbf{X}^* \longrightarrow g(\mathbf{v})\mathbf{X}^*$ is identifiable as $g|_{\mathbf{v}} : \mathbf{X}^* \longrightarrow \mathbf{X}^*$, and the latter one is consequently a tree-endomorphism.

In the language of Automata what does this becomes? ← Reformulate this!

E.g. we now give a description of the restriction g_v in terms of automata.

Proposition 6. *If $\bar{\lambda}_{q_0} : \mathbf{X}^* \longrightarrow \mathbf{X}^*$ is the action of $\mathcal{A}_{q_0}$, then, for every $\mathbf{v} \in \mathbf{X}^*$, the action of $\mathcal{A}_{\bar{\pi}(\mathbf{v},q_0)}$ is given by $(\bar{\lambda}_{q_0})|_{\mathbf{v}} = \bar{\lambda}_{\bar{\pi}(\mathbf{v},q_0)}$, so the restriction of $\bar{\lambda}_{q_0}$ in $\mathbf{v}$.* ← verb is missing.

Proof. Given $\mathbf{v}, \mathbf{w} \in \mathbf{X}^*$ we can easily prove by induction on $n = |\mathbf{w}|$ that $g(\mathbf{v}\mathbf{w}) := \bar{\lambda}_{q_0}(\mathbf{v}\mathbf{w}) = \bar{\lambda}_{q_0}(\mathbf{v})\bar{\lambda}_{\bar{\pi}(\mathbf{v},q_0)}(\mathbf{w}) = g(\mathbf{v})\bar{\lambda}_{\bar{\pi}(\mathbf{v},q_0)}(\mathbf{w})$, consequently $g|_{\mathbf{v}} = \bar{\lambda}_{\bar{\pi}(\mathbf{v},q_0)}$. □

4. Groups generated by an Automaton

BETTER: automata

please try to write more formally!

Our groups will live inside $\mathcal{SF}(X)$, so we will need to have *invertible* elements, i.e. *invertible* transformations f in $\mathcal{SF}(X)$. This means imposing ~~requiring~~ a special condition on all the possible Initial Automata $\mathcal{A}_{q_0}$ describing f, i.e. $\bar{\lambda}_{q_0}$ is invertible. To formalise ~~and approach~~ this concept there will be a bit of technical work.

The groups we will consider are subgroups of $\mathcal{SF}(X)$.

Definition 13. Given an Initial Automaton $\mathcal{A}_{q_0}$, a state q is *accessible* if there exists a word $\mathbf{w} \in \mathbf{X}$ such that $\bar{\pi}(\mathbf{w}, q_0) = q$. We can also say that q *is accessible respect to* with q_0. (OR: from q_0)

Practically this means that in the Moore Diagram there's a path from q_0 to q for ~~each~~ the vertex $q \in \mathcal{Q}$.

Definition 14. An Initial Automaton $\mathcal{A}_{q_0}$ is called *accessible* if each $q \in \mathcal{Q}$ is accessible with respect to q_0. An Automaton is called *accessible* if each Initial Automaton by it definable is accessible.

Proposition 7. *Given an Automaton $\mathcal{A} =< X, \mathcal{Q}, \pi, \lambda >$ and a state $q_0 \in \mathcal{Q}$, $\bar{\lambda}_{q_0}$ is an invertible function if and only if for every accessible state $q \in \mathcal{Q}$ (respect to* with *q_0) $\lambda_q : \mathbf{X} \longrightarrow \mathbf{X}$ is invertible.*

Proof. ($\Rightarrow$) suppose that $\bar{\lambda}_{q_0}$ is an invertible function. Let us ~~Let's~~ take an accessible state $q \in \mathcal{Q}$ and a ~~the~~ word $\mathbf{w}$ such that $\bar{\pi}_{q_0}(\mathbf{w}) = q$. Let's observe $\lambda_q : \mathbf{X} \longrightarrow \mathbf{X}$. Is it injective? Given $x \neq y$, is it true that $\lambda_q(x) \neq \lambda_q(y)$? Supposing absurdly that $\lambda_q(x) = \lambda_q(y)$, we would have that

Consider the transformation

from the converse

we check that λ_q is injective.

$$\bar{\lambda}_{q_0}(\mathbf{w}x) = \bar{\lambda}_{q_0}(\mathbf{w})\underbrace{\bar{\lambda}_{\bar{\pi}(\mathbf{w},q_0)}(x)} = \bar{\lambda}_{q_0}(\mathbf{w})\underbrace{\lambda_q(x)} = \bar{\lambda}_{q_0}(\mathbf{w})\lambda_q(y) = \bar{\lambda}_{q_0}(\mathbf{w}y)$$

Consequently we would loose the injectivity of $\bar{\lambda}_{q_0}$, against the hypothesis of its invertibility. Analogously we can see that λ_q is surjective: let's take ~~the~~ a word $\mathbf{w} \in \mathbf{X}^*$ such that $\bar{\pi}(\mathbf{w}, q_0) = q$ and $y \in \mathbf{X}$. We search ~~a~~ an $x \in \mathbf{X}$ such that $\lambda_q(x) = y$. Since $\bar{\lambda}_{q_0}$ is invertible and synchronous automatic exists a unique preimage of the word $\bar{\lambda}_{q_0}(\mathbf{w})y$, and it's of the form $\mathbf{w}x$ for some x. Therefore: $\bar{\lambda}_{q_0}(\mathbf{w})y = \bar{\lambda}_{q_0}(\mathbf{w}x) = \bar{\lambda}_{q_0}(\mathbf{w})\lambda_q(x)$. → it is

the word $\bar{\lambda}_{q_0}(w)y$ has a unique preimage

($\Leftarrow$) $\bar{\pi}_{q_0}(\mathbf{w}) = \bar{\pi}_{q_0}(x_1 \ldots x_n)$ lands necessarily on an accessible state q for each $\mathbf{w} \in \mathbf{X}^*$. We know that $\lambda_p : \mathbf{X} \longrightarrow \mathbf{X}$ is invertible (i.e is a permutation of $\mathbf{X}$) for each p accessible, including ~~included~~ all the states on the path to q.

space

Start sentences from a word.

by induction on n

Now ~~for induction on n~~ we will prove that $\bar{\lambda}_{q_0}$ is invertible on $\mathbf{X}^n$, ~~thus~~ Consequently it will be invertible on $\bigcup_{n\in\mathbb{N}} \mathbf{X}^n = \mathbf{X}^*$.

we have $\bar{\lambda}_{q_0}$

$(n=1)$ On $\mathbf{X}$ $\bar{\lambda}_{q_0} = \lambda_{q_0}$ therefore is invertible for hypothesis.

$(n \to n+1)$ Let's suppose $\bar{\lambda}_{q_0}$ is invertible on $\mathbf{X}^n$. If $\mathbf{v} \in \mathbf{X}^{n+1}$ then $\mathbf{v} = \mathbf{w}x \in \mathbf{X}^n \times \mathbf{X}$ with that $|\mathbf{w}| = n$. Thus $\bar{\lambda}_{q_0}(\mathbf{v}) = \bar{\lambda}_{q_0}(\mathbf{w}x) = \bar{\lambda}_{q_0}(\mathbf{w})\bar{\lambda}_{\bar{\pi}(\mathbf{w},q_0)}(x) = \bar{\lambda}_{q_0}(\mathbf{w})\lambda_p(x)$ for some p. We observe now that if we change $\mathbf{w}$ or x, we will obtain a different image through $\bar{\lambda}_{q_0}$ on $\mathbf{X}^n$ and through λ_{q_0} on $\mathbf{X}$(injectivity). And if we search for the preimage of a word $\bar{\mathbf{w}}\bar{x} \in \mathbf{X}^{n+1}$, we know that there exists a preimage $\mathbf{w}$ of $\bar{\mathbf{w}}$ through $\bar{\lambda}_{q_0}$ and a preimage x of $\bar{x}$ through $\lambda_{\bar{\pi}(\mathbf{w},q_0)}$. If we glue them together we ~~have:~~ obtain ? with respect to ?

let us

space

$$\bar{\lambda}_{q_0}(\mathbf{w}x) = \bar{\lambda}_{q_0}(\mathbf{w})\lambda_{\bar{\pi}(\mathbf{w},q_0)}(x) = \bar{\mathbf{w}}\bar{x}$$

So the surjectivity is proven.

□

Definition 15. An *Initial Automaton* $\mathcal{A}_{q_0}$ is **invertible** if its action is invertible. An *Automaton* $\mathcal{A}$ is **Invertible** if $\mathcal{A}_{q_0}$ is invertible for each $q \in \mathcal{Q}$.

? **Norm 4.** *Henceforth* ~~*with*~~ by an *Automaton and* an *Initial Automaton we mean* an *Accessible Invertible Automaton and Accessible Invertible Initial Automaton*, respectively

? **Norm 5.** *We observe that now* an *λ is a permutation of* ? *X. Exploiting this we will sometime use a different label-notation for the Moore Diagram. Given an edge $q \longrightarrow p$, on its label there will be written not $x|\lambda(x,q)$ but x, and on the vertex q not q but $\lambda(\cdot, q)$.* ?

Suppose that $\lambda : X \to X$ is invertible. Then it is a permutation on X

This is not clear!

Example 5. ADD!

Observation 11. Since now we consider only ~~our Automata are always~~ invertible automata, their actions, in term of functions on trees, are **automorphism**s, so bijective endomorphisms. This means that they *permute* the children of each node, and the respective lower trees.

Now we can finally introduce the wannabe-protagonist ? of the topic.

Definition 16. Given an Automaton $\mathcal{A} =< \mathbf{X}, \mathcal{Q}, \pi, \lambda >$, we can define $|\mathcal{Q}|$ Initial Automata, which define $|\mathcal{Q}|$ actions $\bar{\lambda}_q$ on $\mathbf{X}^*$ inside $\mathcal{SF}(X)$. ~~We call~~ By the **Group Generated by** $\mathcal{A}$, the subgroup $\mathcal{GA}(\mathbf{X})$ of $\mathcal{SF}(X)$ generated by all the actions $\bar{\lambda}_q$ with $q \in \mathcal{Q}$, ~~or in symbols:~~ for each $q \in \mathcal{Q}$ we mean

$$\mathcal{GA}(\mathbf{X}) :=< \{\bar{\lambda}_q : \mathbf{X}^* \longrightarrow \mathbf{X}^* | q \in \mathcal{Q}\} >. \quad \leqslant ~~\mathcal{SF}(X)~~$$

(or introduce notation $\leq$).

Part 3. Some ~~advanced~~ Group Theory

We need some advanced notions of Group Theory to understand and represent (in mathematics to understand and to represent are very often the same concept, and we could open a very interesting philosophical and historical debate on this declaration) some examples we are going to see. to consider.

you mean present?

5. Actions, Semidirect Products, and Wreath Products

These quite complicated structures, regardless of the skeptisism of the reader, arise very often naturally in Algebra. Particularly in environments involving some kind of recursion or selfsimilarity, as Automata do. The order of construction will be: ?

This is particularly the case

What is norm? I never encountered this! BETTER: Definition or NOTATION.

FIGURE 4. A pattern

?

Norm 6. *Given a set X, $\mathcal{S}(X)$ denotes the* ***Symmetrical Group of*** *X, so the group of all permutations $\sigma : X \longrightarrow X$.*

5.1. Step 0: Actions and Embeddings in Permutation Groups.

Definition 17. Let us take a group G and a set X. Then a function $\tau : G\times X \longrightarrow X$ is called an **Action of G on X** if:

- $\tau(1, x) = x$ for every $x \in X$
- $\tau(g, \tau(h, x)) = \tau(g *_G h, x)$ for every $x \in X$ and $g, h \in G$

what is this?

$\tau(gh, x) = \tau(g, \tau(h,x))$ DID you mean this?

In this case we write $gx := \tau(g, x)$.

Example 6. *$\mathcal{S}(X)$ acts on X, in fact $\sigma x := \sigma(x)$ $\quad \forall \sigma \in \mathcal{S}(X)$ and $\forall x \in X$*

make more precise! $\tau : S(X)\times X \to X$, $\tau(\sigma, x) = \sigma(x)$

To practically work with actions is much more comfortable to use a more abstract characterization:

Proposition 8. *Given a group G and a set X, $\tau : G \times X \longrightarrow X$ is an action iff $T : G \longrightarrow \mathcal{S}(X)$ defined by $(T(g))(x) := \tau(g, x) =: \phi_g(x)$ is a homomorphism of groups.*

START This from a word!

Proof. ($\Rightarrow$): $T(g)(x) := \tau(g, x)$, therefore, for the property of τ we have $T(g*h)(x) = \tau(g * h, x) = \tau(g, \tau(h, x)) = T(g)(\tau(h, x)) = T(g) \circ T(h)(x)$ for every $x \in X$
($\Leftarrow$): Analogous to the other sense. $\square$

5.2. Step 1: Faithful Actions and Permutation Groups.

Definition 18. A **Permutation Group** is a pair (B, Y) ~~such that~~ where B is a group that can be embedded in $\mathcal{S}(Y)$, the Symmetric Group ~~of~~ on Y.

~~*Observation*~~ Remark 12. Take care: The Symmetric Group $S(Y)$ is the set of **ALL** permutations, of Y while a Permutation Group (B,Y) is, mangling the definition, a subgroup of ~~the Symmetric Group.~~ $S(Y)$

a group a set

Definition 19. A **Faithful Action** *of G on X* is an action $f : G \times X \longrightarrow X$ such that, given $g \neq h$ in G, exists $x \in X$ ~~for which~~ such that $gx \neq hx$. We then say that *G acts faithfully on X.*

The group there

Proposition 9. *G acts faithfully on X iff (G, X) is a permutation group.*

a set

Proof. ($\Rightarrow$): Let's take $\Phi : G \longrightarrow \mathcal{S}(X)$ defined by $\Phi(g) = \phi_g$ where $\phi_g(x) := gx$. ~~We can easily verify its injectivity~~, thus the thesis. ($\Leftarrow$): G is embeddable in $\mathcal{S}(X)$, thus we identify each g with ~~its unique copy~~ ϕ, where $\phi_g(x) := gx$. Since ~~the copy~~ ϕ is unique, we have that G acts faithfully on X. $\square$

what about surjectivity?

Example 7 (Translations)**.** *Given (V, A) with A, affine space built on V, $\mathbb{K}$-vector space, then $Tr := \{t_v : A \longrightarrow A | t_v(p) = p + v\}$, the group of translations, is acting faithfully on A.*

a vector space over a field K?

Observation 13. Note that if X is a group $\mathcal{AUT}(X) \subsetneq \mathcal{S}(X)$. Therefore in general if (G, X) is a permutation group and X is a group, the injective endomorphism $\Phi : G \longrightarrow \mathcal{S}(X)$ *doesn't necessarily* induce an endomorphism $G \longrightarrow \mathcal{AUT}(X)$.

what is this? Please define before using this!

This definition should be before remark 13

5.3. Step 2: Semidirect Products

Definition 20. Given two groups H, N, with operations $*_H$ and $*_N$ and an homomorphism $\varphi : H \longrightarrow \mathcal{AUT}(N)$, where $\mathcal{AUT}(N)$ is the group $\{f : N \longrightarrow N | f \quad is \quad bijective\}$, we can define on the set $H \times N$ the ~~subsequent~~ following operation:

$$\star_\varphi : ((h_1, n_1), (h_2, n_2)) \longmapsto (\quad h_1 *_H h_2 \quad , \quad n_1 *_N (\varphi(h_1))(n_2) \quad)$$

We call $(H \times N, \star_\varphi)$ the **Semidirect Product of N and H relative to φ** and we write it down as $H \ltimes_\varphi N$.

Proof? At least say that proof is a routine calculation

Proposition 10. *$H \ltimes_\varphi N$ is a group, where the identity is $(1_H, 1_N)$ and, for each (h,n)* $^{-1}$ *~~the inverse is~~ $(h^{-1}, (\varphi(h))(n^{-1}))$.* for each $(h,n) \in H \ltimes_\varphi N$.

Example 8 (Dihedral Groups)**.** *Among the first groups studied there were the groups* one can consider *of symmetries, i.e. given a geometric ~~shape~~* object *A, the set of all geometrical transformation which ~~left~~* leave *A unchanged. Let A be a regular polygon ~~of~~* with *n sides, ~~and you will~~* Then we *obtain the group of symmetries $\mathcal{D}_n$, the so called* ***n-Dihedral Group****. ~~In it we can have~~* There are *two types of transformations, the rotation of $\frac{k\pi}{n}$ degrees around the centre of the polygon, ~~or~~* and *the reflection ~~around~~* in it with respect to *one of the n possible axes of simmetry. It turns out that a ~~possible way to represent this group is through~~* this group is isomorphic to *the Semidirect Product $\mathbb{Z}_2 \ltimes_\varphi \mathbb{Z}_n$, where $\varphi(0)(z) := id_{\mathbb{Z}_n}(z) = z$ and $\varphi(1)(z) := inv_{\mathbb{Z}_n}(z) := -z$. So $\mathbb{Z}_2 \times \mathbb{Z}_n \ni (0, n_1) * (h_2, n_2) = (h_2, n_1 + n_2)$ and $(1, n_1) * (h_2, n_2) = (h_2, n_1 - n_2)$. Practically, if we have $(h,k) \in \mathbb{Z}_2 \times \mathbb{Z}_n$, h tells us if we are reflecting the figure or not, and k tells us we are rotating of $\frac{k\pi}{n}$ radiants.*

? h encodes the reflection and k the rotation

5.4. Step 3: Restricted and Unrestricted Wreath Products

Definition 21. Given a group A and an arbitrary set of ind~~exes~~ices Y we define:

(Direct Product) $A^Y = \prod_{\omega \in Y} A := \{\bar{a} = (a_\omega)_{\omega \in Y} : a \in A\}$,

(Direct Sum) $A^{(Y)} = \bigoplus_{\omega \in Y} A := \{\tilde{a} = (a_\omega)_{\omega \in Y} : a \in A \wedge a \neq 1_A$ just for a finite number of indexes$\}$?

In case $|Y|$ is finite there's no difference between the two spaces structures.

~~*Observation*~~ Remark 14. If A is a group we can extend its operation on the two structures component-wise, to obtain so another 'bigger' group.

Think if you need this!

Norm 7. *If the reader is already familiar with the concepts we will treat, note that we will use a notation inverted respect to western researchers, that means: Western Notation: $A \wr B \quad A \rtimes_\varphi B$ Grigorchuk Notation: $B \wr A \quad B \ltimes_\varphi A$. This choice is made because in the case of an infinite wreath or semidirect product* it *is useful to have a finite number of symbols on our left.*

well, I disagree with "Western"

Example 9.

? where A is a group ?

Now ~~let's~~ let us take (B, Y), A groups, where (B, Y) is a permutation group. If we construct A^Y we have a group on which B can act faithfully, so we have an injective homomorphism $\Phi : B \longrightarrow \mathcal{S}(A^Y)$, through the action of B on the indexes of A^Y. If we prove that $\Phi(B) \subset \mathcal{AUT}(A^Y)$ we have everything we need to construct $B \ltimes_\Phi A^Y = H \ltimes_\varphi N$. The same can a be done substituting A^Y with $A^{(Y)}$.
Let's formalise this:

TO AVOID THESE BLACK BOXES, YOU CAN PUT \sloppy TO THE PREAMPLE → overfull lines

Let (B,Y) be a transformation group and A a group.

Proposition 11. *~~Given (B,Y) and A groups,~~ ~~with (B,Y) permutation group,~~ We can extend the action of B ~~on~~ to A^Y, so (B,A^Y) ~~is~~ a permutation group. ~~Plus~~, B is a subgroup of $\mathcal{AUT}(A^Y)$ with $\mathcal{AUT}(A^Y)$ that becomes group respect to the component-wise operation on A. The same con be done substituting A^Y with $A^{(Y)}$.* ?

in addition

Proof. If by is the action of B on Y, we can define the action q of B on A^Y as follow:

$$q(b \in B, \overline{a} \in A^Y) = q_b(\overline{a}) = b\overline{a} = b(a_y)_{y\in Y} := (a_{by})_{y\in Y} = (a_y)_{b^{-1}y\in Y}$$

(1) We have to prove that q is a faithful, i.e. that $q_b \in \mathcal{S}(A^Y)$: if $\overline{a} = (a_y)_{y\in Y} \neq (x_y)_{y\in Y} = \overline{x} \Rightarrow \exists y' \in Y$ such that $a_{y'} \neq x_{y'}$. Let's now look $q_b(\overline{a})$ at the index by'. We find that at the by'-th component $a_{by'}$ is different from $x_{by'}$. So q_b is injective. The surjectivity is very simple: if we have $(a_y)_{y\in Y}$ the element $(a_{b^{-1}y})_{y\in Y}$ is its counter-image.

(2) We have now to prove that q_b is an automorphism: $q_b(\overline{a} \star \overline{x}) = q_b((a_y \star x_y)_{y\in Y}) = (a_{by} \star x_{by})_{y\in Y} = (a_{by})_{y\in Y} \star (x_{by})_{y\in Y} = q_b(\overline{a}) \star q_b(\overline{x})$

□

Definition 22. Given a permutation group (B,Y) and a group A we call **Wreath Product**, and we sign it $B \wr A$ one of these two structures:

Reword this

- The **Unrestricted Wreath Product**, ~~so~~ is? the semidirect product on $B \times A^Y$ where the required homomorphism is the application $\varphi : B \longrightarrow \mathcal{S}(A^Y)$ defined by $(\varphi(b))(\overline{a} = (a_y)_{y\in Y}) = (a_{by})_{y\in Y}$. ?
- The **Restricted Wreath Product**, so the semidirect product on $(B, A^{(Y)})$ where the required homomorphism? is the application $\varphi : B \longrightarrow \mathcal{S}(A^{(Y)})$ defined by $(\varphi(b))(\widetilde{a} = (a_y)_{y\in Y}) = (a_{by})_{y\in Y}$?

Therefore, having $(b_1, \overline{p}), (b_2, \overline{q})$ in $B \times A^Y$ or in $B \times A^{(Y)}$, their product is:

$$(b_1, \overline{p}) *_\phi (b_2, \overline{q}) = (b_1, (p_y)_{y\in Y}) *_\phi (b_2, (q_y)_{y\in Y}) :=$$
$$= (b_1 *_B b_2, \overline{p} * (\phi(b_1))(\overline{q})) = (b_1 *_B b_2, (p_y * q_{b_1y})_{y\in Y})$$

If we take ~~a~~ two group B, A, we can anyway consider their wreath product considering (B,B) a permutation group, where B acts faithfully on ~~him~~ itself by left multiplication.

Observation 15. From the context it will be understandable, or it will be stated, of which of the two structure are we talking about. Notice that if Y is finite there's no difference between the restricted and unrestricted wreath product.

After all this technical work we need to breathe with some example.

"in nature"
I am not sure :)

Example 10 (Generalised Symmetric Groups)**.**

Observation 16. Wreath products arise quite often in mathematics ~~and nature~~. Interesting examples involve groups used to understand and solve Sudoku or the Rubik's Cube. Otherwise John Rodes in [**?**] states many examples of the applications of the theory of Automata, which as we will see, are deeply bounded with the structure of the Wreath Product.

6. Free Groups

Often in Group Theory, we ask ourselves: is it possible to build a group G with a set S of characteristics? And then we have to search for a specimen of G "in the wild", exactly how Grigorchuk did for his group. But if the set S is simple enough, so our requirements are all of a certain type, Free Groups give us a way to plasm the searched object directly from S! The difference is thin.

Part 4. Classification of every 2-state Automaton over a 2-letter alphabet

7. Overtoure

7.1. Case of Z 2.

7.2. Klein Group.

7.3. case 3.

7.4. case 4.

7.5. Infinite Dihedral Group.

7.6. Lamplighter Group.

References

General comments:

1. First Do The following:
formulate The Classification Theorem
and before the formulation define
all the groups that are mentioned there:
pay special attention to the lamplighter group.

2. Then: go on to the proof of
the classification Theorem.
I suggest that you include only
a part of the proof (in particular,
that you skip the part with the
lamplighter group, and maybe
even more).

,

3. Then: Enter changes, as I

specified, to the first part.
If some definition is not used in the sequel, think that you maybe exclude it

---

4. Remember: your thesis should not be too long!!!

[if you want a longer thesis, this can be your Master's thesis later on]

For now, only a shorter version is needed!

Overall, the beginning is OK, go on!