Proof Theory in Philosophy

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Slides available at https://carlonicolai.github.io

Basics
- Paradox(es)
- Consistency via cut-elimination
- Objects of Truth

Systems of Truth
- Deflation and Conservation
- Classical v Nonclassical Kripkean truth
- Logical Pluralism

Extensions
- Reflection
- Modal Logic
- Modal Predicates

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What I'm <u>not</u> considering

I don't consider **proof-theoretic semantics**.

I only briefly touch upon **reductive proof-theory** in the philosophy of mathematics.

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Abstraction and Truth

'There never were any set-theoretic paradoxes, but the property theoretic paradoxes are still unresolved' (Gödel to Myhill)

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Naïve abstraction

$$\forall x\big(x \in \{v \mid \varphi(v)\} \leftrightarrow \varphi(x)\big)$$

Naïve Truth

$$\mathtt{Tr}\ulcorner A \urcorner \leftrightarrow A$$

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Naïve Truth

$$\mathtt{Tr}\ulcorner A \urcorner \leftrightarrow A$$

Here I assume that for any φ in the language there is a term $\{v \mid \varphi(v)\}$ with $\mathtt{FV}(\{v \mid \varphi(v)\}) = \mathtt{FV}(\varphi) \setminus \{v\}$. If φ is a sentence, I write $\ulcorner A \urcorner$ for 'the proposition expressed by A'.

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Liar

$\Gamma, \Delta, \Theta, \Lambda, \ldots$ are multisets of formulas.

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \mathtt{Tr}\ulcorner A \urcorner} \qquad \frac{A, \Gamma \Rightarrow D}{\mathtt{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\lambda \Leftrightarrow \neg\mathtt{Tr}\ulcorner \lambda \urcorner \qquad \neg\lambda \Leftrightarrow \mathtt{Tr}\ulcorner \lambda \urcorner$$

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$$\kappa \Leftrightarrow \mathtt{Tr}\ulcorner \kappa \urcorner \to \bot$$

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Internal Curry

'Consequence' predicate

$$\frac{\Gamma, A \Rightarrow B \qquad \Gamma, C \Rightarrow D}{\Gamma, A, \mathtt{C}(\ulcorner B \urcorner, \ulcorner C \urcorner) \Rightarrow D} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow \mathtt{C}(\ulcorner A \urcorner, \ulcorner B \urcorner)}$$

$$\nu \Leftrightarrow \mathtt{C}(\ulcorner \nu \urcorner, \ulcorner \bot \urcorner)$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v \Rightarrow v \qquad \bot \Rightarrow \bot}{v, \mathtt{C}(\ulcorner v \urcorner, \ulcorner \bot \urcorner) \Rightarrow \bot}}{v \Rightarrow \bot}}{\Rightarrow \mathtt{C}(\ulcorner v \urcorner, \ulcorner \bot \urcorner)}}{\Rightarrow v} \qquad \dfrac{\dfrac{v \Rightarrow v \qquad \bot \Rightarrow \bot}{v, \mathtt{C}(\ulcorner v \urcorner, \ulcorner \bot \urcorner) \Rightarrow \bot}}{v \Rightarrow \bot}}{\Rightarrow \bot}$$

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Cut-elimination for truth and abstraction

The main extension of the standard inductive strategy consists in the reduction of cuts of the following form:

$\mathtt{Tr}$ -rules principal in the last inferences

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \Delta, A}}{\Gamma \Rightarrow \Delta, \mathtt{Tr}\ulcorner A \urcorner} \qquad \dfrac{\dfrac{\mathcal{D}_1}{A, \Gamma \Rightarrow \Delta}}{\mathtt{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

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Tr -rules principal in the last inferences

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \Delta, A\end{matrix}}{\Gamma \Rightarrow \Delta, \mathtt{Tr}\ulcorner A \urcorner} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ A, \Gamma \Rightarrow \Delta\end{matrix}}{\mathtt{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

…which we would like to reduce to:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ A, \Gamma \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta}$$

This creates a problem because $\mathtt{Tr}\ulcorner A \urcorner$ is atomic whereas A is of arbitrary (logical) complexity.

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$\mathtt{Tr}$ -measures

- I will consider two ways of keeping track of applications of the truth rules in derivations: the first applies to nodes in the derivation tree, the second applies to single formulas within derivations.

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$\mathtt{Tr}$ -measures

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- In the first case:

$$\dfrac{\dfrac{\gamma_0 \Rightarrow \top \quad \alpha}{\gamma_0 \Rightarrow \mathtt{Tr}\ulcorner\top\urcorner \quad \alpha+1} \qquad \gamma_1 \Rightarrow \mathtt{Tr}\ulcorner\top\urcorner \quad \beta}{\gamma_0, \gamma_1 \Rightarrow \mathtt{Tr}\ulcorner\top\urcorner \wedge \mathtt{Tr}\ulcorner\top\urcorner \quad \mathtt{max}(\alpha, \beta)}$$

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- In the second case:

$$\dfrac{\dfrac{\gamma_0 \Rightarrow {}^{0}\top}{\gamma_0 \Rightarrow {}^{1}\mathtt{Tr}\ulcorner\top\urcorner} \qquad \gamma_1 \Rightarrow {}^{0}\mathtt{Tr}\ulcorner\top\urcorner}{\gamma_0, \gamma_1 \Rightarrow {}^{\mathtt{max}(1,n)}\mathtt{Tr}\ulcorner\top\urcorner \wedge \mathtt{Tr}\ulcorner\top\urcorner}$$

Contraction-Free

Systems of truth and 'set theories' can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

Truth à la Grišin GT

$$\Gamma, \mathtt{Tr}\, s \Rightarrow \mathtt{Tr}\, s, \Delta\ [0] \qquad \Gamma \Rightarrow \top, \Delta\ [0] \qquad \Gamma, \bot \Rightarrow \Delta\ [0]$$

$$\frac{A, \Gamma \Rightarrow \Delta\ [\alpha]}{\mathtt{Tr}\, \ulcorner A \urcorner, \Gamma \Rightarrow \Delta\ [\alpha + 1]} \qquad \frac{\Gamma \Rightarrow \Delta, A\ [\alpha]}{\Gamma \Rightarrow \Delta, \mathtt{Tr}\, \ulcorner A \urcorner\ [\alpha + 1]}$$

$$\frac{\Gamma \Rightarrow \Delta, A_i\ [\alpha]}{\Gamma \Rightarrow \Delta, A_0 \sqcap A_1\ [\alpha]} \qquad \frac{\Gamma \Rightarrow \Delta, A\ [\alpha] \qquad \Gamma \Rightarrow \Delta, B\ [\beta]}{\Gamma \Rightarrow \Delta, B\ [\mathtt{max}(\alpha, \beta)]}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta\ [\alpha]}{A \star B, \Gamma \Rightarrow \Delta\ [\alpha]} \qquad \frac{\Gamma \Rightarrow \Delta, A\ [\alpha] \qquad \Gamma \Rightarrow \Delta, B\ [\beta]}{\Gamma \Rightarrow \Delta, A \star B\ [\alpha + \beta]}$$

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Lemma

Given cut-free derivations $\mathcal{D}_0 \vdash_{GT} \Gamma \Rightarrow \Delta, A$ *and* $\mathcal{D}_1 \vdash_{GT} A, \Theta \Rightarrow \Lambda$, *there is a* $\mathcal{D} \vdash_{GT} \Gamma, \Theta \Rightarrow \Delta, \Lambda$ *with the* $\mathcal{Tr}$*-rank* ρ *of* $\mathcal{D}$ *is* $\leq \rho(\mathcal{D}_0) + \rho(\mathcal{D}_1)$.

Proof Idea.

The induction is on $(\rho(\mathcal{D}_0) + \rho(\mathcal{D}_1), |A|, |\mathcal{D}_0| + |\mathcal{D}_1|)$. $\square$

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Contraction-free

Systems of truth and 'set theories' can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

Two problems of the contraction-free approach:

- Viewed as a set theory, GS is inconsistent with extensionality, e.g defined as:

$$s \subseteq t \star t \subseteq s, t \in r \Rightarrow s \in r$$

  This is often called **Grišin's paradox**.

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However, it needs to be added that it also features a 'decent' conditional (compared, e.g. to Field (2008)).

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Fixed-Point Semantics

Given our language $\mathcal{L}_{\mathtt{Tr}} := \mathcal{L}_{\cup}\{\mathtt{Tr}\}$, we start with a (classical) model $\mathcal{M}$ of $\mathcal{L}$ such that $\ulcorner\varphi\urcorner^{\mathcal{M}} = \varphi$, and set, for $X \subset |\mathcal{M}|$:

$$
\begin{aligned}
a \in \Phi(X) \Leftrightarrow\ & a = \ulcorner\top\urcorner, \text{ or} \\
& a = \ulcorner\mathtt{Tr}\ulcorner\varphi\urcorner\urcorner \text{ and } \ulcorner\varphi\urcorner \in X, \text{ or} \\
& a = \ulcorner\neg\mathtt{Tr}\ulcorner\varphi\urcorner\urcorner \text{ and } \ulcorner\neg\varphi\urcorner \in X, \text{ or} \\
& a = \ulcorner\varphi \wedge \psi\urcorner \text{ and } \ulcorner\varphi\urcorner \in X \text{ and } \ulcorner\psi\urcorner \in X, \text{ or} \\
& a = \ulcorner\neg(\varphi \wedge \psi)\urcorner \text{ and } \ulcorner\neg\varphi\urcorner \in X \text{ or } \ulcorner\neg\psi\urcorner \in X, \text{ or} \ldots
\end{aligned}
$$

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\end{aligned}
$$

Let then $\Phi^0(X) = X, \Phi^{\alpha+1}(X) = \Phi(\Phi^\alpha(X)), \Phi^\lambda(X) = \bigcup_{\beta<\lambda}\Phi^\beta(X)$.

Lemma (Kripke (1975),Martin and Woodruff (1975))

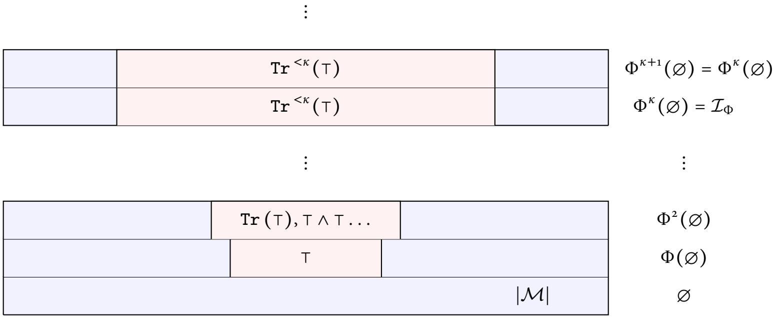
If $S \subseteq |\mathcal{M}|$ is a fixed-point of Φ, then for all $\varphi \in \mathcal{L}_{Tr}$:

$$\varphi \in S \textit{ iff } Tr\ulcorner\varphi\urcorner \in S$$

Fixed-Point Semantics

$\vdots$	
$\mathtt{Tr}^{<\kappa}(\top)$	$\Phi^{\kappa+1}(\varnothing) = \Phi^{\kappa}(\varnothing)$
$\mathtt{Tr}^{<\kappa}(\top)$	$\Phi^{\kappa}(\varnothing) = \mathcal{I}_{\Phi}$
$\vdots$	$\vdots$
$\mathtt{Tr}(\top), \top \wedge \top \dots$	$\Phi^{2}(\varnothing)$
$\top$	$\Phi(\varnothing)$
$\lvert\mathcal{M}\rvert$	$\varnothing$

Fixed-Point Semantics



The structure $(\mathcal{M}, \mathcal{I}_\Phi)$ gives rise to a three-valued model for $\mathcal{L}_{\mathtt{Tr}}$ with $\mathtt{Tr}$ a 'partial' predicate. Define

$$\mathcal{M} \vDash \Gamma \Rightarrow \Delta :\Leftrightarrow (\forall \gamma \in \Gamma)\ |\gamma|^{\mathcal{M}}_{\mathcal{I}_\Phi} \neq 0 \to (\exists \delta \in \Delta)\ |\gamma|^{\mathcal{M}}_{\mathcal{I}_\Phi} = 1$$

Restricting initial sequents

Already known in other contexts Kreuger (1994); Jäger and Stärk (1998); Schroeder-Heister (2016). This is contained in Nicolai (2018). Structural rules are absorbed.

Definition (LPT)

$$\Gamma, \bot \Rightarrow \Delta \qquad \Gamma \Rightarrow \top, \Delta$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \mathtt{Tr}\ulcorner A \urcorner} \qquad \frac{A, \Gamma \Rightarrow \Delta}{\mathtt{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow \Delta}$$

$$(\neg\text{L})\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\neg\text{R})\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$\vdots \qquad \vdots$

- Now $(\mathcal{M}, S) \models$ LPT for S a fixed point of Φ.

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$$\vdots \qquad \vdots$$

- Now $(\mathcal{M}, S) \vDash \text{LPT}$ for S a fixed point of Φ.
- The model $(\mathcal{M}, \mathcal{I}_\Phi)$ satisfies a fully operational, paracomplete version system of naïve truth based on Strong-Kleene logic (modulo definition of consequence).

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Back to cut elimination

When contraction is around, the notion of $\mathtt{Tr}$ -rank is not enough:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_{00}\\ \Gamma \Rightarrow \Delta, \mathtt{Tr}\ulcorner\psi\urcorner, \mathtt{Tr}\ulcorner\psi\urcorner\ [\alpha]\end{array}}{\Gamma \Rightarrow \Delta, \mathtt{Tr}\ulcorner\psi\urcorner\ [\alpha]} \qquad \begin{array}{c}\mathcal{D}_1\\ \mathtt{Tr}\ulcorner\psi\urcorner, \Theta \Rightarrow \Lambda\ [\beta]\end{array}}{\Gamma, \Theta \Rightarrow \Delta, \Lambda\ [\alpha + \beta]}$$

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Now the idea here would be that we transform the derivation in

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_{00}^{*}\\ \Gamma \Rightarrow \Delta, \psi, \psi\ [\alpha]\end{array} \qquad \begin{array}{c}\mathcal{D}_{1}^{*}\\ \psi, \Theta \Rightarrow \Lambda\ [\beta]\end{array}}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \psi\ [\alpha + \beta]} \qquad \begin{array}{c}\mathcal{D}_{1}^{*}\\ \psi, \Theta \Rightarrow \Lambda\ [\beta]\end{array}}{\Gamma, \Theta, \Theta \Rightarrow \Delta, \Lambda, \Lambda\ [2\alpha + \beta]}}{\Gamma, \Theta \Rightarrow \Delta, \Lambda\ [2\alpha + \beta]}$$

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$\mathtt{Tr}$ -complexity $\kappa(\cdot)$ of formulas

The ordinal $\mathtt{Tr}$ -complexity $\kappa_{\mathcal{D}}(\cdot)$ of a formula φ of $\mathcal{L}_{\mathtt{Tr}}$ in a derivation $\mathcal{D}$ is defined inductively as follows:

- formulas of $\mathcal{L}$ **have $\mathtt{Tr}$ -complexity** 0 in any $\mathcal{D}$;

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- If $\mathcal{D}$ ends with

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \mathtt{Tr}\ulcorner A \urcorner}$$

  then the complexity of formulas in Γ, Δ is unchanged and $\kappa_{\mathcal{D}}(\mathtt{Tr}\ulcorner A \urcorner) = \kappa_{\mathcal{D}}(A) + 1$ (similarly for ($\mathtt{Tr}$ -L)).

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- If $\mathcal{D}$ ends with

$$\frac{\gamma_1^1, \ldots, \gamma_n^1 \Rightarrow \delta_1^1, \ldots, \delta_m^1, \varphi \qquad \gamma_1^2, \ldots, \gamma_n^2 \Rightarrow \delta_1^2, \ldots, \delta_m^2, \psi}{\gamma_1^3, \ldots, \gamma_n^3 \Rightarrow \delta_1^3, \ldots, \delta_m^3, \varphi \wedge \psi}$$

  then

$$\begin{aligned}
\kappa_{\mathcal{D}}(\varphi \wedge \psi) &= \mathtt{max}(\kappa_{\mathcal{D}}(\varphi), \kappa_{\mathcal{D}}(\psi)) \\
\kappa_{\mathcal{D}}(\gamma_i^3) &= \mathtt{max}(\kappa_{\mathcal{D}}(\gamma_i^1), \kappa_{\mathcal{D}}(\gamma_i^2)) \quad 1 \leq i \leq n \\
\kappa_{\mathcal{D}}(\delta_j^3) &= \mathtt{max}(\kappa_{\mathcal{D}}(\delta_j^1), \kappa_{\mathcal{D}}(\delta_j^2)) \quad 1 \leq j \leq m
\end{aligned}$$

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Full cut elimination

Crucially, rules of LPT are **κ-invertible**, e.g.:

If $\mathcal{D} \vdash_{\mathrm{LPT}} \Gamma^1, \mathtt{Tr}\ulcorner A \urcorner \Rightarrow \Delta^1$, then there is $\mathcal{D}' \vdash_{\mathrm{LPT}} A, \Gamma \Rightarrow \Delta$ with $\kappa_{\mathcal{D}'}(\Gamma) \leq \kappa_{\mathcal{D}}(\Gamma^1)$, $\kappa_{\mathcal{D}'}(\Delta) \leq \kappa_{\mathcal{D}}(\Delta^1)$, and

$$\kappa_{\mathcal{D}'}(A) \leq \kappa_{\mathcal{D}}(\mathtt{Tr}\ulcorner A \urcorner), \text{ if } \kappa_{\mathcal{D}}(\mathtt{Tr}\ulcorner A \urcorner) = 0;$$
$$\kappa_{\mathcal{D}'}(A) < \kappa_{\mathcal{D}}(\mathtt{Tr}\ulcorner A \urcorner), \text{ if } \kappa_{\mathcal{D}}(\mathtt{Tr}\ulcorner A \urcorner) \neq 0.$$

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Lemma

Contraction is κ-admissible and length-admissible, e.g. : If $\mathcal{D} \vdash^n_{\mathrm{LPT}} \Gamma^1, \varphi^1, \varphi^2 \Rightarrow \Delta^1$, then there is a $\mathcal{D}' \vdash^n_{\mathrm{LPT}} \Gamma, \varphi \Rightarrow \Delta$ with

$$\kappa_{\mathcal{D}'}(\Gamma^1) \leq \kappa_{\mathcal{D}'}(\Gamma); \quad \kappa_{\mathcal{D}'}(\Delta^1) \leq \kappa_{\mathcal{D}'}(\Delta)$$
$$\kappa_{\mathcal{D}'}(\varphi) \leq max(\kappa_{\mathcal{D}'}(\varphi^1), \kappa_{\mathcal{D}'}(\varphi^2)).$$

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$$\kappa_{\mathcal{D}'}(\varphi) \leq \mathit{max}(\kappa_{\mathcal{D}'}(\varphi^1), \kappa_{\mathcal{D}'}(\varphi^2)).$$

Proposition

If $\mathcal{D}_0$ is a cut-free proof of $\Gamma^1 \Rightarrow \Delta^1, \varphi^1$ in LPT, *and $\mathcal{D}_1$ is a cut-free* LPT*-proof of $\varphi^2, \Gamma^2 \Rightarrow \Delta^2$, then there is a cut-free proof $\mathcal{D}$ of $\Gamma^3 \Rightarrow \Delta^3$ with $\kappa_{\mathcal{D}}(\Gamma^3) \leq \mathit{max}(\kappa_{\mathcal{D}_0}(\Gamma^1), \kappa_{\mathcal{D}_1}(\Gamma^2))$ and $\kappa_{\mathcal{D}}(\Delta^3) \leq \mathit{max}(\kappa_{\mathcal{D}_0}(\Delta^1), \kappa_{\mathcal{D}_1}(\Delta^2))$.*

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- The ideal of **semantic closure** is at odds with resources that outstrip the ones available in one's semantic theory. Cut-elimination procedures are usually formalizable in weak arithmetical systems.

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- When nonlogical initial sequents are around, full cut elimination is not in general available: however by eliminating cuts on formulas of the form $\mathtt{Tr}\ulcorner A \urcorner$, one can obtain **conservativity proofs**.
- Another advantage of the approach with restricted initial sequents is that – unlike the contraction-free approaches – there are natural **infinitary systems** that arise from the logic and that give succinct presentations of Π^1_1-sets.

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Truth Bearers

- There are good reasons to require an ontology of bearers of truth prior to discussing principles of truth. We want to prove **in the object language** things like:

  $$\forall \varphi, \psi \, \exists \chi \, (\chi = (\varphi \dot{\wedge} \psi) \wedge \varphi \neq \chi)$$

  This is usually guaranteed by assuming a theory of finite objects (as we shall see in a moment).

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  $$\forall \varphi, \psi \, \exists \chi \, (\chi = (\varphi \dot{\wedge} \psi) \wedge \varphi \neq \chi)$$

  This is usually guaranteed by assuming a theory of finite objects (as we shall see in a moment).
- Notice that this is imposing non-trivial constraints. More 'philosophical' theories of truth are often formulated in terms of **propositions**, and not sentence types (Horwich, 1998; Soames, 1998; Jago, 2018). This rules out that propositions are coarse-grained, e.g. sets of possible worlds.

Arithmetic

- Peano arithmetic (PA) is the preferred base theory for systems of truth. It is usually formulated in $\mathcal{L}_{\mathbb{N}} = \{0, \mathtt{S}, +, \times\}$ and features equations for its primitives, e.g.

$$(x + 0) = x \qquad\qquad x + \mathtt{S}y = \mathtt{S}(x + y)$$

and the first-order induction schema

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(\mathtt{S}x)) \to \forall x\, \varphi(x) \qquad \text{for } \varphi \in \mathcal{L}_{\mathbb{N}}$$

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- Alternatively, one can employ a theory of strings and concatenation $\frown$ with two atoms a, b based on **Tarski's axiom**

$$a^\frown y = u^\frown v \leftrightarrow \exists w\big((x = u^\frown w \wedge v = w^\frown y) \vee (u = x^\frown w \wedge y = w^\frown v)\big)$$

  With first-order string induction, the two theories are mutually interpretable. An accessible source is Ganea (2009).

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Arithmetic

- Peano arithmetic (PA) is the preferred base theory for systems of truth. It is usually formulated in $\mathcal{L}_{\mathbb{N}} = \{0, \mathtt{S}, +, \times\}$ and features equations for its primitives, e.g.

$$(x + 0) = x \qquad\qquad x + \mathtt{S}y = \mathtt{S}(x + y)$$

and the first-order induction schema

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(\mathtt{S}x)) \to \forall x\, \varphi(x) \qquad \text{for } \varphi \in \mathcal{L}_{\mathbb{N}}$$

- Alternatively, one can employ a theory of strings and concatenation $\frown$ with two atoms a, b based on **Tarski's axiom**

$$a^\frown y = u^\frown v \leftrightarrow \exists w\big((x = u^\frown w \wedge v = w^\frown y) \vee (u = x^\frown w \wedge y = w^\frown v)\big)$$

With first-order string induction, the two theories are mutually interpretable. An accessible source is Ganea (2009).
- Finite set theories are also a convenient choice. For instance
  - Kaye and Wong (2006) show that PA and $\mathrm{KF} \setminus \{\mathrm{Inf}\}+$ 'every set has a transitive closure' are **bi-interpretable**;
  - similarly, a neat set theory based by Świerczkowski (2003) based on the **adjunction operation** $x \triangleleft y \mapsto x \cup \{y\}$ is bi-interpretable with PA.

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Doing with less

- Ultimately, what we require to establish the basic properties of the truth bearers are **a good notion of sequence**, and **a minimum of induction** to handle suitable forms of recursion.

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- For the former the notion of a **sequential** theory is enough – see Visser (2010) for a comprehensive overview. A theory is sequential if it interprets – with no relativization of quantifiers – the theory AS given by the empyt set axiom and adjunction – which is as strong as Robinson's Q.

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Doing with less

- Ultimately, what we require to establish the basic properties of the truth bearers are **a good notion of sequence**, and **a minimum of induction** to handle suitable forms of recursion.
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- As to induction, since all the relevant syntactic notions (terms, formulas, proofs) are p-time decidable, the theory S^1_2 by Buss (1986) suffices. However, many of the results that I will treat below are specific to PA (or equivalents), and it is object of current research to check which results are stable over $T \supseteq S^1_2$.

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Ordinals

- A good base theory will also provide a satisfactory representation of ordinals. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal Γ_0:

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  - α is principal if it cannot be expressed as $\zeta + \eta$ for $\zeta, \eta < \alpha$. Define:

$$\begin{aligned} C(0) &:= \text{`the class of principal ordinals'} \\ C(\alpha + 1) &:= \text{`the class of fixed points of the function enumerating } C(\alpha)\text{'} \\ C(\lambda) &:= \bigcap_{\zeta < \lambda} C(\zeta) \quad \text{for } \lambda \text{ a limit ordinal} \end{aligned}$$

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  - The *Veblen functions* φ_α are the enumerating functions of $C(\alpha)$. The class of *strongly critical* ordinals SC contains precisely the ordinals α that are themselves α-critical. Γ_ζ indicates the ζ-th strongly critical ordinal.

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  - The *Veblen functions* φ_α are the enumerating functions of $C(\alpha)$. The class of *strongly critical* ordinals SC contains precisely the ordinals α that are themselves α-critical. Γ_ζ indicates the ζ-th strongly critical ordinal.
  - Principal ordinals α that are not themselves strongly critical are such that $\alpha = \varphi_\zeta \eta$ for $\eta, \zeta < \alpha$. Therefore, by this fact and Cantor's normal form theorem, ordinals $< \Gamma_0$ can be uniquely determined as words of the alphabet $(0, +, \varphi\cdot\cdot)$.

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- A notation system for Γ_0 is of the form $(\mathtt{OT}, \mathtt{PT}, |\cdot|, \prec)$, with
  - $\mathtt{OT}$ the set of natural number 'codes' for ordinals $< \Gamma_0$
  - $\mathtt{OT} \subseteq \mathtt{OT}$ the set of codes of principal ordinals
  - $|\cdot|: \mathtt{OT} \to \mathtt{ON}$
  - $n \prec m :\leftrightarrow n \in \mathtt{OT} \wedge m \in \mathtt{OT} \wedge |n| < |m|$

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  - $|\cdot| \colon \mathtt{OT} \to \mathtt{ON}$
  - $n \prec m :\leftrightarrow n \in \mathtt{OT} \wedge m \in \mathtt{OT} \wedge |n| < |m|$
- Using standard coding techniques one can show that $\mathtt{OT}, \mathtt{PT}, \prec$ are primitive recursive. Actually, Beckmann et al. (2003) show that they can be showed to be p-time and represented in S^1_2 – notice that I **do not** mean that Γ_0 can be well-founded in S^1_2!

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Schemata

For ordinals $\alpha < \Gamma_0$, we denote with a the corresponding numeral in the representation of $\mathtt{OT}$ and we do not distinguish between ordinal functions such as the Veblen functions and their representations.

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Schemata

For ordinals $\alpha < \Gamma_0$, we denote with a the corresponding numeral in the representation of $\mathtt{OT}$ and we do not distinguish between ordinal functions such as the Veblen functions and their representations. The system $(\mathtt{OT}, \mathtt{PT}, \prec)$ enables us to formulate the following principles of transfinite induction:

$$(\mathrm{TI}^{\varepsilon_0}_{\mathcal{L}_{\mathtt{Tr}}}) \qquad \frac{\forall a \prec b\, \phi(a), \Gamma \Rightarrow \Delta, \phi(b)}{\Gamma \Rightarrow \Delta, \forall a \prec \varepsilon_0\, \phi(a)}$$

$$(\mathrm{TI}^{<\omega^\omega}_{\mathcal{L}_{\mathtt{Tr}}}) \qquad \frac{\forall a \prec b\, \phi(a), \Gamma \Rightarrow \Delta, \phi(b)}{\Gamma \Rightarrow \Delta, \forall a \prec c\, \phi(a)} \qquad \text{for all } \gamma (= |c|) < \omega^\omega$$

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Truth as primitive

- Truth-theoretic deflationism holds that truth is **not a genuine property** and that its function is mainly that of a **generalizing device** (Quine, 1970; Field, 1994; Horwich, 1998).

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- Unlike other notions that have been taken to be primitive for lack of consensus over a definition – e.g. knowledge, see Williamson (2000) – Tarski's theorem uncontroversially establishes this (Halbach, 2014, Ch. 1).

Truth as primitive

- Truth-theoretic deflationism holds that truth is **not a genuine property** and that its function is mainly that of a **generalizing device** (Quine, 1970; Field, 1994; Horwich, 1998).
- Unlike other notions that have been taken to be primitive for lack of consensus over a definition – e.g. knowledge, see Williamson (2000) – Tarski's theorem uncontroversially establishes this (Halbach, 2014, Ch. 1).
- Truth is a fundamental **semantic** concept. A theory of meaning for natural language expressions is not much more than a (Tarskian) theory of truth for it (Davidson, 1984).

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Tarskian Truth

The theory of truth in $\mathcal{L}_{\mathtt{Tr}}$ that Davidson had in mind extends PA with the following:

Definition (CT)

$$\forall s, t(\mathtt{Tr}\,(s = t) \leftrightarrow s^{\circ} = t^{\circ})$$

$$\forall \varphi \in \mathcal{L}(\mathtt{Tr}\,(\neg\varphi) \leftrightarrow \neg\mathtt{Tr}\,\varphi)$$

$$\forall \varphi, \psi \in \mathcal{L}(\mathtt{Tr}\,(\varphi \wedge \psi) \leftrightarrow \mathtt{Tr}\,\varphi \wedge \mathtt{Tr}\,\psi)$$

$$\forall v, \forall \varphi(v) \in \mathcal{L}(\mathtt{Tr}\,(\forall v \varphi) \leftrightarrow \forall x\,\mathtt{Tr}\,\varphi(\dot{x}))$$

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x\,\varphi(x) \qquad \text{with } \varphi(v) \in \mathcal{L}_{\mathtt{Tr}}$$

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$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x\, \varphi(x) \qquad \text{with } \varphi(v) \in \mathcal{L}_{\mathtt{Tr}}$$

Important variations are obtained by tweaking the induction schema:

- CT $\restriction$ (a.k.a. CT^-) is obtained by restricting induction to $\mathcal{L}$
- $\mathrm{CT}_{\mathrm{int}}$ is obtained by adopting the internal induction schema

$$\forall \varphi(v)(\mathtt{Tr}\,\varphi(o/v) \wedge \forall y(\mathtt{Tr}\,\varphi(\dot{y}/v) \to \mathtt{Tr}\,\varphi(\dot{\mathrm{S}y}/v)) \to \forall x \mathtt{Tr}\,\varphi(\dot{x}/v))$$

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Conservativeness

Thesis (Shapiro, 1998; Ketland, 1999)

CT proves Con(PA), therefore deflationism is untenable.

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This contrasts with:

Proposition (Kotlarski et al. (1981); Visser and Enayat (2015); Leigh (2015))

$\mathrm{CT}_{\mathrm{int}}$ *is a conservative extension of* PA.

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$\mathrm{CT}_{\mathrm{int}}$ *is a conservative extension of* PA.

The discussion took a strong technical turn, brilliantly summarized in Cieliski (2017) – with many original contributions. It's worth mentioning:

Proposition (Enayat and Pakhomov (2018))

CT $\restriction$ *plus 'disjunctive correctness', i.e.*

$$\forall s(\,Tr(\bigvee_{i<s} \varphi_i) \leftrightarrow \exists i < s\; Tr\,\varphi_i)$$

is the same theory as $\mathrm{CT}[\mathrm{I}\Delta_0]$, *and therefore proves* Con(PA).

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- Despite the technical interest, the debate seems to be built on shaky foundations. Virtually **no deflationist** has thoroughly defended the claim that truth **has to be** conservative over the base theory.

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- By contrast, it has repeatedly been argued that **truth has to be nonconservative**, but in a way that is **distinctively metalinguistic**, i.e. it does not interfere with the subject matter of the base theory over which truth is built.

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- By contrast, it has repeatedly been argued that **truth has to be nonconservative**, but in a way that is **distinctively metalinguistic**, i.e. it does not interfere with the subject matter of the base theory over which truth is built.
- This has led to the programme of **'disentangling'** syntactic quantifiers from quantifiers over natural numbers:

Proposition (Nicolai (2015, 2016))

If one formulates $\mathrm{CT}\restriction$ *as a two-sorted theory, with 'syntactic' quantifiers and 'number-theoretic' quantifiers, and truth applying only over syntactic objects, then:*

- *The theory of truth becomes* ***trivially*** *conservative over* PA*;*
- *This version of* $\mathrm{CT}\restriction$ *plus 'all axioms of* PA *are true' is* ***mutually interpretable*** *with* $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$.

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Feferman's project

To give a nice presentation of the **reflective closure** of PA – and possibly of further 'natural' theories: i.e. the (truth-)theory that makes explicit all that is implicit in the acceptance of PA.

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First attempt: $CT_{<a}$, $a \leq \Gamma_0$

$$\mathcal{L}_0 := \mathcal{L}_{\mathtt{Tr}} \qquad \mathcal{L}_{<\gamma} := \mathcal{L} \cup \{\mathtt{Tr}_b \mid b < \gamma\}$$

With $b < \gamma$:

$$\forall s, t(\mathtt{Tr}_b(s = t) \leftrightarrow s^\circ = t^\circ)$$
$$\forall \varphi \in \mathcal{L}_{<b}\,(\mathtt{Tr}_b(\neg\varphi) \leftrightarrow \mathtt{Tr}_b(\neg\varphi))$$
$$\forall \varphi, \psi \in \mathcal{L}_{<b}\,(\mathtt{Tr}_b(\varphi \wedge \psi) \leftrightarrow \mathtt{Tr}_b\varphi \wedge \mathtt{Tr}_b\psi)$$
$$\forall v, \forall \varphi(v) \in \mathcal{L}(\mathtt{Tr}_b(\forall v\varphi) \leftrightarrow \forall x\,\mathtt{Tr}_b\varphi(\dot{x}))$$
$$\forall \varphi \in \mathcal{L}_{a<b}(\mathtt{Tr}_b\mathtt{Tr}_a\varphi \leftrightarrow \mathtt{Tr}_b\varphi)$$
$$\forall d \prec b, \forall \varphi \in \mathcal{L}_{<d}(\mathtt{Tr}_b\mathtt{Tr}_d\varphi \leftrightarrow \mathtt{Tr}_b\varphi)$$

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Feferman's project

- The project of isolating $\mathrm{CT}_{<\varepsilon_0}$ or $\mathrm{CT}_{<\Gamma_0}$ as **natural stopping points**, that was congenial to Feferman's project, depended essentially on other results, such as Feferman's and Schütte's independent characterization of Γ_0, or the provable well-orderings of PA.

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- The next step was to find an independent characterization of such theories:

Definition (KF)

$$\forall s, t(\mathtt{Tr}\,(s = t) \leftrightarrow s^\circ = t^\circ)$$
$$\forall s, t(\mathtt{Tr}\,(s \neq t) \leftrightarrow s^\circ \neq t^\circ)$$
$$\forall t(\mathtt{Tr}\,\mathtt{Tr}\,t \leftrightarrow \mathtt{Tr}\,t^\circ)$$
$$\forall t(\mathtt{Tr}\,\mathtt{Tr}\,\neg t \leftrightarrow \mathtt{Tr}\,\neg t^\circ)$$
$$\forall \varphi \in \mathcal{L}_{\mathtt{Tr}}\,(\mathtt{Tr}\,(\neg\neg\varphi) \leftrightarrow \mathtt{Tr}\,\varphi)$$
$$\forall \varphi, \psi \in \mathcal{L}_{\mathtt{Tr}}\,(\mathtt{Tr}\,(\varphi \wedge \psi) \leftrightarrow \mathtt{Tr}\,\varphi \wedge \mathtt{Tr}\,\psi)$$
$$\forall \varphi, \psi \in \mathcal{L}_{\mathtt{Tr}}\,(\mathtt{Tr}\,\neg(\varphi \wedge \psi) \leftrightarrow \mathtt{Tr}\,\neg\varphi \vee \mathtt{Tr}\,\neg\psi)$$
$$\forall v, \forall \varphi(v) \in \mathcal{L}_{\mathtt{Tr}}\,(\mathtt{Tr}\,(\forall v\varphi) \leftrightarrow \forall x\,\mathtt{Tr}\,\varphi(\dot{x}))$$
$$\forall v, \forall \varphi(v) \in \mathcal{L}_{\mathtt{Tr}}\,(\mathtt{Tr}\,(\neg\forall v\varphi) \leftrightarrow \exists x\,\mathtt{Tr}\,\neg\varphi(\dot{x}))$$

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Properties of KF

- Semantically KF fits nicely with Kripke's fixed-point semantics (Kripke, 1975),

$$(\mathbb{N}, S) \vDash \mathrm{KF} \ \text{ iff } \ S \text{ is a fixed point of Kripke's theory of truth}$$

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- The full $\mathtt{Tr}$ -schema is available for meaningful predicates satisfying $\mathtt{D}(x) :\leftrightarrow \mathtt{Tr}\,x \vee \mathtt{Tr}\,\neg x$, i.e. for all $A \in \mathcal{L}_{\mathtt{Tr}}$:

$$\mathrm{KF} \vdash \mathtt{D}(\ulcorner A \urcorner) \rightarrow (\mathtt{Tr}\,\ulcorner A \urcorner \leftrightarrow A)$$

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$$\mathrm{KF} \vdash \mathtt{D}(\ulcorner A \urcorner) \rightarrow (\mathtt{Tr}\, \ulcorner A \urcorner \leftrightarrow A)$$

Proposition (Feferman (1991); Cantini (1989))

KF *is proof-theoretically equivalent to* $(\Pi^0_1\text{–CA})_{<\varepsilon_0}$.

Proof Idea.

Lower bound: PA in $\mathcal{L}_{\mathtt{Tr}}$ proves $\mathrm{TI}^{<\varepsilon_0}_{\mathcal{L}_{\mathtt{Tr}}}$. Now KF proves:

$$\varphi \in \mathcal{L}_{<a} \rightarrow \mathtt{D}(\varphi) \Rightarrow \varphi \in \mathcal{L}_a \rightarrow \mathtt{D}(\varphi)$$

An application of $\mathrm{TI}^{<\varepsilon_0}_{\mathcal{L}_{\mathtt{Tr}}}$ yields an embedding of $\mathrm{CT}_{<\varepsilon_0}$, which suffices. □

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Properties of KF

- Semantically KF fits nicely with Kripke's fixed-point semantics (Kripke, 1975),

$$(\mathbb{N}, S) \models \mathrm{KF} \ \text{ iff } \ S \text{ is a fixed point of Kripke's theory of truth}$$

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$$(\mathbb{N}, S) \vDash \mathrm{KF} \text{ iff } S \text{ is a fixed point of Kripke's theory of truth}$$

- The full $\mathtt{Tr}$ -schema is available for meaningful predicates satisfying $\mathtt{D}(x) :\leftrightarrow \mathtt{Tr}\, x \vee \mathtt{Tr}\, \neg x$, i.e. for all $A \in \mathcal{L}_{\mathtt{Tr}}$:

$$\mathrm{KF} \vdash \mathtt{D}(\ulcorner A \urcorner) \to (\mathtt{Tr}\, \ulcorner A \urcorner \leftrightarrow A)$$

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Proposition (Feferman (1991); Cantini (1989))

KF *is proof-theoretically equivalent to* $(\Pi^0_1\text{–CA})_{<\varepsilon_0}$.

Proof Idea.

Upper bound: One formulates KF in a Tait (one-sided) infinitary calculus, and analyzes quai-normal derivations, i.e. derivations with only cuts on $\mathtt{Tr}\, t$ and $\neg\mathtt{Tr}\, t$ and proves in $\mathrm{CT}_{<\varepsilon_0}$ that

$$\text{if } \mathrm{KF}^\infty \vdash^\alpha \mathtt{Tr}\, \ulcorner A \urcorner, \text{ then } \mathtt{Tr}_{2^\alpha} \ulcorner A \urcorner$$

□

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Symmetries

- One important drawback of KF is that its internal theory $\{\varphi \in \mathcal{L}_{\mathtt{Tr}} \mid \mathrm{KF} \vdash \varphi\}$ is different from its theorems: for instance $\mathrm{KF} \vdash \lambda \vee \neg\lambda$ but $\mathrm{KF} \nvdash \mathtt{Tr}\ulcorner \lambda \vee \neg\lambda \urcorner$.

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- To overcome this:

Reinhardt's thesis

One should adopt an **instrumental** reading of KF. Its conceptual core is given by its **internal theory**.

Lemma (Halbach and Horsten (2006))

There are A's in $\mathcal{L}_{Tr}$ *such that* $\mathrm{KF} \vdash Tr\ulcorner A \urcorner$ *but the proof essentially employs B's such that* $\mathrm{KF} \nvdash Tr\ulcorner B \urcorner$.

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