Week 1
Sets, Classes, and The Cumulative Hierarchy

1. Sets

Sets are collections of objects. The collection of all students registered in the Advanced Logic module at King's College London in 2020 is a set. The collection of all prime numbers less than 1,000,000 is a set. This definition of set relies on a clear notion of 'object'. This, in turn, requires an uncontroversial metaphysics of objects, which we don't have. Luckily, for the purpose of developing mathematics – and therefore scientifically applicable mathematics – and portions of formal philosophy, we don't need that much.

There are two main ways to refer to sets. The first is by listing their elements – by enclosing them in curly brackets $\{,\}$ –, as in

$$\{\text{Caravaggio}\}, \qquad \{\text{Michelangelo Merisi}\},$$
$$\{1,2,3\}, \qquad \{\sqrt{2},\pi\}.$$

The second is by description, by means of the so called *abstraction terms*:

$$\{x \mid x \text{ is the painter of } \textit{The Calling of Saint Matthew}\}$$
$$\{x \mid x \text{ is a positive integer smaller than } 4\}$$

Notice, however, that *not all descriptions give rise to sets!* We will treat this point more carefully in the next section.

The identity conditions for sets – cf. Quine's 'there's no entity without identity' [**vOQ69**, p. 23] – are encapsulated in the well-known principle of extensionality:

PRINCIPLE OF EXTENSIONALITY: Two sets are identical if and only if they have the same elements. In symbols:

$$x = y \leftrightarrow \forall u(u \in x \leftrightarrow u \in y)$$

Sets are then completely characterized by their elements, and not by their *mode of presentation* – i.e. the way we refer to them. For instance:

$$\{\text{Caravaggio}\} = \{x \mid x \text{ is the painter of } \textit{The Calling of Saint Matthew}\},$$
$$\{x \mid x \text{ is a positive integer smaller than } 4\} = \{1, 2, 3\}.$$

Similarly, one has that (as far as we know):

$$\{x \mid x \text{ is an animal with a heart}\} = \{x \mid x \text{ is an animal with a kidney}\}.$$

Therefore, even though arguably the property of having a heart is not the same as the property of having a kidney, the *extension* of these two properties, i.e. the set of objects that have such properties, is the same. The philosophical question whether a scientific (broadly construed) worldview needs properties or concepts together with sets is still substantially open.[1]

As anticipated above, however, the entire universe of mathematical objects can be constructed without resorting to objects located in space and time such as Caravaggio. One of the fundamental building blocks of such construction is the following *set-existence principle*:

EMPTY SET PRINCIPLE: there is a set containing no elements:

$$\exists y \forall u (u \notin y)$$

EXERCISE 1. Show, using the EXTENSIONALITY PRINCIPLE, that there is a unique empty set.

Since there is only one empty set, we can safely denote it with the 'proper name' $\varnothing$. We collect a few simple facts concerning the empty set. As usual, we let

$$x \subseteq y \;:\leftrightarrow\; \forall u (u \in x \longrightarrow u \in y)$$
$$x \subset y \;:\leftrightarrow\; x \subseteq y \land x \neq y$$

FACT 1.

(i) *For any collection A, $\varnothing \subseteq A$*

(ii) $\varnothing \subseteq \varnothing$, $\varnothing \notin \varnothing$, $\{\varnothing\} \in \{\{\varnothing\}\}$, $\{\varnothing\} \nsubseteq \{\{\varnothing\}\}$.

PROOF. Exercise. *qed*

DEFINITION 1 (POWER SET). *The power set $\mathcal{P}(x)$ of a set x is the set $\{y \mid y \subseteq x\}$.*

[1] For an overview, see [**BM03**].

EXERCISE 2. Show that $\mathcal{P}(x)$, if it exists, is unique.

From the empty set, we can basically construct the entire universe of sets (and thus of mathematics). We only need to close the empty set under two basic operations. One is implicit in the definition of power set, and tells us that one *can always* collect subsets of a given set.

POWER SET PRINCIPLE: For any set x, $\mathcal{P}(x)$ exists and is a set.

The second operation is is a generalization of the well-known notion of *union* of sets. We know that

$$x \cup y := \{u \mid u \in x \vee u \in y\}$$

This operation (iterated finitely many times) enables us to define *finite unions*:

$$x_1 \cup \ldots \cup x_n \cup x_{n+1} := (x_1 \cup \ldots \cup x_n) \cup x_{n+1}.$$

But what about infinite ones? Suppose $x := \{x_i \mid i \in \mathbb{N}\}$. How do we collect all x_i together? We need a further operation

$$\bigcup x = \{u \mid (\exists y \in x)(u \in y)\}$$

Again implicit in this definition is the possibility of forming unions of all elements of any given set.

PRINCIPLE OF UNION: For any set x, $\bigcup x$ exists and is a set.

Using the PRINCIPLE OF UNION, we can readily obtain the infinite union of all elements of x_i.

EXERCISE.

(i) Which collection is $\bigcup\{\{1\},\{1,2\},\{1\}\}$?
(ii) Which collection is $\bigcup\{\{0\},\{1\},\{2\},\{3\},\ldots\}$?

The empty set, power sets, and unions give us basically all we need to outline the so called *cumulative hierarchy of sets*, the universe of all sets. I said 'outline', because a precise definition will only be available once the full axiomatic development of set theory will be given. However, at this stage we can start by giving the first steps of this process, and leave the details for later.

We want to define the universe V of all sets in stages. So we put

$$\mathrm{V}_0 := \emptyset$$
$$\mathrm{V}_{n+1} := \mathcal{P}(\mathrm{V}_n)$$

Once we have completed the finite stages, we need to find a way to index later stages. This is the job of *Cantor's theory of ordinal numbers*, that we can only briefly introduce now. Roughly speaking, ordinals extend natural numbers $\mathbb{N}$ by introducing *infinite* numbers. The first infinite numbers is denoted with ω (read 'omega'). So we put:

$$\mathrm{V}_\omega := \bigcup\{\mathrm{V}_i \mid i \in \mathbb{N}\} = \{u \mid (\exists i \in \mathbb{N})(u \in \mathrm{V}_i)\}$$

The definition can then be iterated. So $\mathrm{V}_\omega + 1 := \mathcal{P}(\mathrm{V}_\omega)\ldots$. The resulting picture of V is displayed in Figure 1.

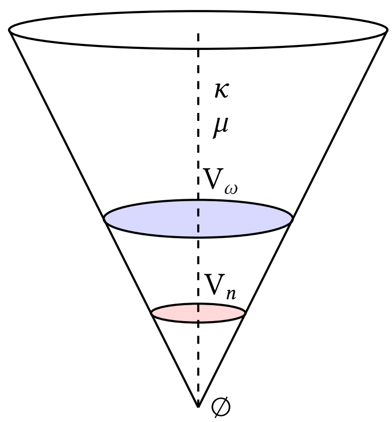

FIGURE 1. Cumulative hierarchy

EXERCISE. Show that $\mathrm{V}_3 = \mathrm{V}_2 \cup \mathcal{P}(\mathrm{V}_2)$. (Notice that this is true for every stage).

EXERCISE. Define the *rank* of a set x, written $\rho(x)$, to be the least α such that $x \subseteq \mathrm{V}_\alpha$. What is $\rho(\{\varnothing\})$? What is $\rho(\{\{\varnothing\}, \{\{\{\varnothing\}\}\}\})$?

2. Classes

In introducing expressions of the form $\{x \mid \Phi(x)\}$, we have put no limits to the possibility of forming sets given a suitable specifying condition Φ. Such limits, however, exist.

FACT 2 (RUSSELL'S PARADOX). *The collection* $\mathrm{R} = \{x \mid x \notin x\}$ *is not a set.*

PROOF. Suppose R is a set. Since we assume classical logic, either $\mathrm{R} \in \mathrm{R}$ or $\mathrm{R} \notin \mathrm{R}$. If the former, then R is one of the sets that satisfies the condition $x \notin x$; therefore $\mathrm{R} \notin \mathrm{R}$, and so both $\mathrm{R} \in \mathrm{R}$ and $\mathrm{R} \notin \mathrm{R}$. If the latter, again R is one of the sets that satisfies the condition $x \notin x$. Therefore, $\mathrm{R} \in \mathrm{R}$ and $\mathrm{R} \notin \mathrm{R}$. In either case we obtain a contradiction. So R cannot be a set. *qed*

Collections that are not sets are called *classes*. So all sets are classes, but not viceversa: classes that are not sets are called *proper classes*. It then follows from FACT 2 that R is a proper class.

But if not all conditions define a set, which ones do? The following principle gives us an answer.

> SUBSET PRINCIPLE: let $\Phi(\cdot)$ be a determinate property and x a set. Then $\{u \in x \mid \Phi(u)\}$ is a set.

But for the answer to be satisfactory, one also has to know what a determinate property is. To keep things simple,[2] the standard reply that set theory offers is that a property $\Phi(x)$ is determinate when it can be expressed in a specific formal language, the language $\mathcal{L}_\in$ of set theory. The SUBSET PRINCIPLE is then based on a *syntactic* restriction: the only properties that can be employed in defining sets are the ones that belong to a specific class of syntactic objects.

QUESTION 1. Is this notion of determinacy satisfactory?

COROLLARY 1. V *is not a set.*

PROOF. Suppose it is. Then by the SUBSET PRINCIPLE, the class $\{u \in \mathrm{V} \mid u \notin u\}$ is a set. But this contradicts FACT 2. *qed*

EXERCISE. Fill in the details of the proof of Corollary 1.

EXERCISE. Consider a set x and the set $y := \{u \in x \mid u \notin u\}$. Show that $y \notin x$.

[2]Notice that, depending on the answer, one might end up with a formulation of the subset principle that is difficult to even write down. For instance,

Bibliography

[BM03] George Bealer and Uwe Monnich, *Property theories*, Handbook of Philosophical Logic, Volume 10 (Dov Gabbay and Frans Guenthner, eds.), Kluwer Academic Publishers, 2003, pp. 143–248.

[vOQ69] Willard van Orman Quine, *Ontological relativity and other essays*, Columbia University Press, 1969.