Graph Conceptions of Properties

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KING'S College LONDON

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III. To overcome some shortcomings of such proposals, I will develop some examples of property-theoretic analogues of non-wellfounded set theory

IV. I will then ask whether “graph” conceptions of properties can support those formal frameworks

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Gödel (via Myhill):

> *There never were any set-theoretic paradoxes, but the property-theoretic paradoxes are still unresolved.*

Just like sets are constituted by their membership structure, properties are *partially* constituted by their instantiation structure.

- *Sets as extensional properties*. Associated with any set S there is a property of belonging to S. To generate such a rich realm of properties, some of the ZFC axioms are our (current) best bet (Jubien).

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- Iterative constructions. Bealer: 'for any credible motivation that can be given for [...] set theory, an analogous motivation, which is at least as satisfactory, can be given for the axioms in a corresponding logic for the predication relation'.
- *Unification of logical paradoxes*. Drawing a parallel between the iterative conception and resolutions of semantic paradoxes based on implicit quantifier restrictions (groundedness, contextualism). Predication/instantiation may not be an exception.

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Define a class

$$\begin{aligned} \mathrm{E} := \{e \mid\ & e \text{ is a collection of (extensional) pairs } (x, y), \\ & \text{every } u \in x \text{ is s.t. } (u, v) \in e \text{ for some } v \in y, \\ & \text{every } v \in y \text{ is s.t. } (u, v) \in e \text{ for some } u \in x\} \end{aligned}$$

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Fact (Friedman, and others). ZFC is relatively interpretable in $\mathrm{ZFC}^{\neq}$ (i.e. ZFC axiomatized with collection minus extensionality).

Another – arguably more – principled way to recover the axioms of ZFC is to assume a formulation of $\mathrm{ZFC}^{\neq}$ with *abstraction terms*, called $\mathrm{ZFC}^{\neq}_{\lambda}$, e.g.:

$$x \in \lambda x.\,(x \in u \wedge A) \leftrightarrow x \in u \wedge A,$$

on the background of a *logic for abstraction* featuring classical predicate logic, the existence of a denumerable plurality of abstracta:

$$\Box A :\leftrightarrow \lambda x.A = \top, \quad \text{governed by S5;}$$

$$\lambda\vec{x}.A = \lambda\vec{x}.B \leftrightarrow \Box(A \leftrightarrow B) \text{ – up to } \alpha\text{-conversion.}$$

(The purely logical part of $\mathrm{ZFC}^{\neq}_{\lambda}$ is proved complete – by Bealer – with respect to an algebraic semantics).

The iterative process of set-formation can be then paralleled for a suitable class of properties – *determined properties* – in the sense of $\mathrm{ZFC}^{\neq}_{\lambda}$:

$$\begin{aligned}\mathrm{Det}(x) :\leftrightarrow\ & \exists y\, x = \lambda v.\, v \in y, \\ & \Box\forall u(u \in x \to \Box u \in x), \\ & \forall u(u \text{ in the instantiation structure of } x \ \to \mathrm{Det}(u))\end{aligned}$$

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Fact. ZFC can be interpreted in $\mathrm{ZFC}^{\neq}_{\lambda}$ (even with replacement in place of collection).

For the proof: $\mathrm{Det}(x)$ is the domain formula. Crucially, for x, y determinate, instantiation is necessary instantiation. Therefore, if $\Box(u \in x \leftrightarrow u \in y)$, then by the modal axioms also $\lambda v.v \in x = \lambda v.v \in y$, i.e. $x = y$.

PRs – in fact, relations, properties, propositions – as *two-dimensional entities* (φ, S): φ is the *algebraic structure* of the PRs, akin to the syntactic structure of linguistic entities; S is its *instantiation class* (aboutness, range of significance) of φ: the collection of entities on which φ can instantiate.

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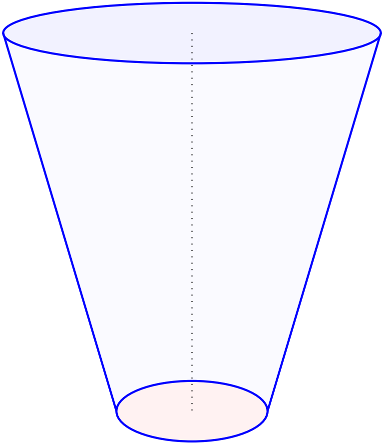
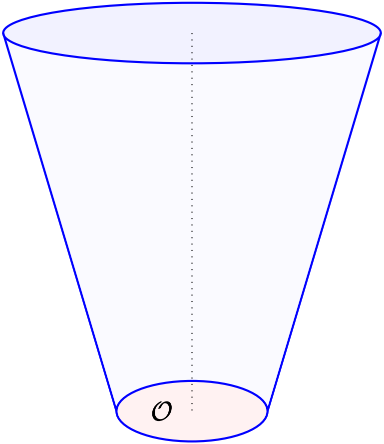
Two properties are identical iff

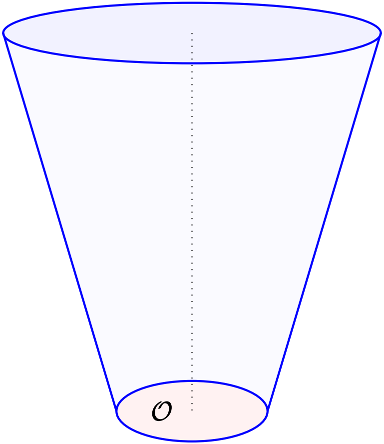
1. they have the same instantiation class (extensional)
2. they are structurally identical (non-extensional)





The language $\mathcal{L}$

Primitives $\mathcal{O}$:
$\mathrm{Var} = \{\underset{\cdot}{v}_i \mid i \in \omega\}$, $\underset{\cdot}{=}$, $\underset{\cdot}{\wedge}$, $\underset{\cdot}{\neg}$, $\underset{\cdot}{\forall}$, $\underset{\cdot}{P}$,...



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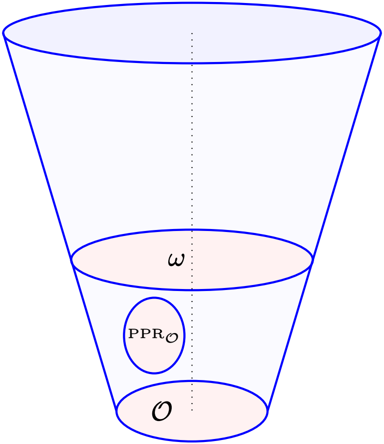

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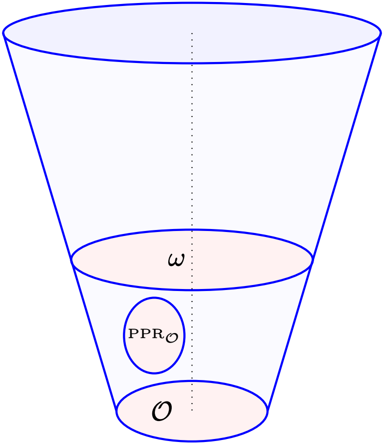
Transfinite induction definition of $\mathrm{PR}_{\mathcal{O}}$; for the algebraic structure, we only require the subformula relation (we can think of operations on codes).



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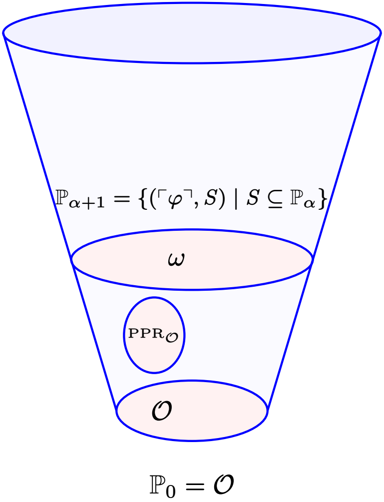

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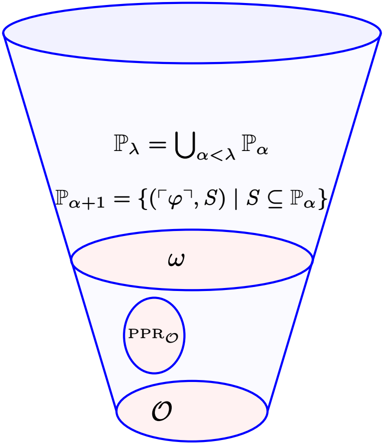

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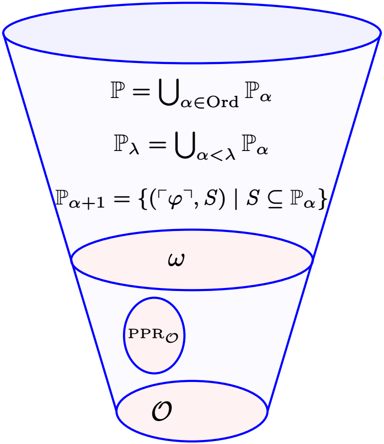

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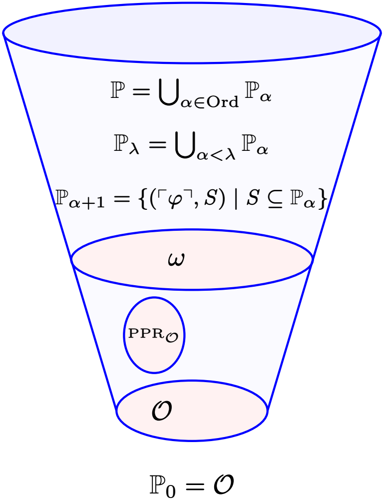

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Examples:

$\big((\underset{\cdot}{=}, \underset{\cdot}{v}_i, \underset{\cdot}{v}_i), \varnothing\big) \in \mathbb{P}_1$,

$\big((\underset{\cdot}{=}, \underset{\cdot}{v}_i, \underset{\cdot}{v}_i), \{\underset{\cdot}{\neg}\}\big) \in \mathbb{P}_1$

$\big((\underset{\cdot}{=}, \underset{\cdot}{v}_i, \underset{\cdot}{v}_i), S \subsetneq \mathrm{PPR}_{\mathcal{O}} \times \mathbb{P}_\omega\big) \in \mathbb{P}_\omega$

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- There is a property of being a property referred to in these slides.

properties and antifoundation

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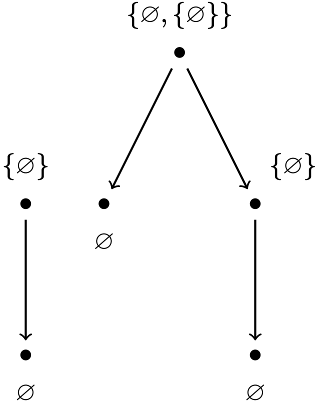

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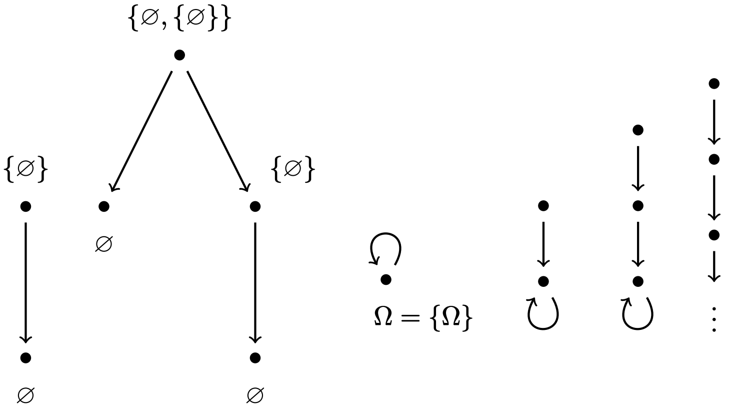

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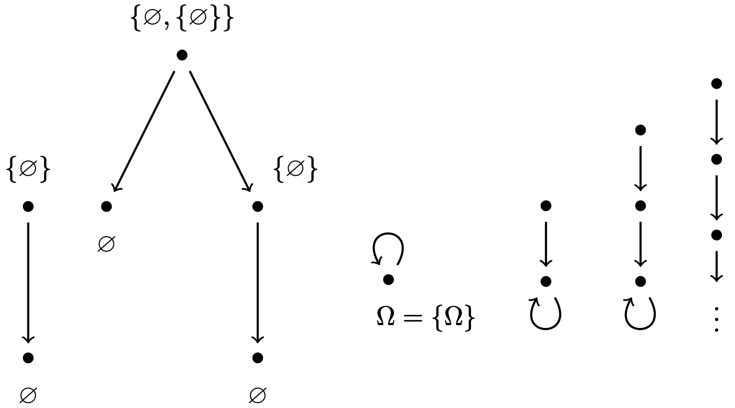

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$$p \equiv (p_1, \ldots, p_n) \wedge q \equiv (q_1, \ldots, q_n) \rightarrow \big(p = q \leftrightarrow (p_i = q_i \wedge \forall z(\mathrm{I}(p, z) \leftrightarrow \mathrm{I}(q, z)))\big)$$

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H&L again: the theory CPR

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   - There is a property that is about infinitely many objects.
   - Separation, power set, union, pairs, replacement, choice, hold for any type of atomic property. E.g :
   $$\forall x \exists p \forall u \big((\mathrm{I}(p, u) \leftrightarrow \mathrm{I}(x, u) \wedge \varphi(u)) \wedge p \equiv (\underset{\cdot}{=}, \underset{\cdot}{v}_i, \underset{\cdot}{v}_i)\big)$$

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expressing AFA in a modified H&L

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Sets give us *graphs* (or better, $\underset{\cdot}{=}$-graphs):

1. A $\underset{\cdot}{=}$-*tagging* function is a mapping $\tau\colon G \longrightarrow \mathrm{PR}_{\mathcal{O}} \times \{(\underset{\cdot}{=}\underset{\cdot}{u}\underset{\cdot}{v}), \varnothing\}$ that assigns members of $\mathrm{PR}_{\mathcal{O}}$ and the empty instantiation class to childless nodes.
2. Let τ be given. A $\underset{\cdot}{=}$-*decoration* of $\mathcal{G}$ is a function d such that, for every node g:

$$d(g) = \begin{cases} \tau(g), & \text{for } g \text{ childless} \\ ((\underset{\cdot}{=}, \underset{\cdot}{u}, \underset{\cdot}{v}), \{d(g_0) \mid (g, g_0) \text{ is an edge}\}), & \text{else} \end{cases}$$

expressing AFA in a modified H&L

- $\mathtt{Set}(x) :\leftrightarrow x$ is a property of surface form $(\underset{\cdot}{=}, \underset{\cdot}{v}_i, \underset{\cdot}{v}_i)$
- $x \in y :\leftrightarrow \mathtt{Set}(y) \wedge \mathrm{I}(y, x)$

Sets give us *graphs* (or better, $\underset{\cdot}{=}$-graphs):

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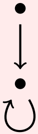
(AFA$\underset{\cdot}{=}$) Every tagged graph has a unique $\underset{\cdot}{=}$-*decoration* $((\underset{\cdot}{=}, \underset{\cdot}{u}, \underset{\cdot}{v}), X)$, to which it corresponds a unique PR of form $(\underset{\cdot}{=}, \underset{\cdot}{u}, \underset{\cdot}{v})$ and instantiation class X.

Example.



A $\underset{\cdot}{=}$-decoration the graphs above would assign to it the PR:
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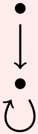

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Proposition. Assuming AFA$\underset{.}{=}$, there are PRs that belong to their instantiation class. However, no PR's instantiation class coincides with the entire universe!

However, we only have the axioms for $\underset{.}{=}$-decorations. As we (L&W) did for the other axioms, we need to relativize decorations to the *surface structure* Φ *of atomic PRs*:

Every tagged graph has a unique *Φ-decoration* (Φ, X), to which it corresponds a unique PR of form Φ and (unique) instantiation class X.

Let CPR* be the theory given so far. The following is only bookkeeping:

Proposition. CPR* relatively interprets ZFA (therefore, it's at least as strong as ZFA) .

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For the other direction, we would like to show in ZFA that, for each surface form $\ulcorner \varphi \urcorner$ and *class of indeterminates* X, the *system of equations*

$$x = (\ulcorner \varphi \urcorner, s_x)$$

(with s_x a set possibly containing indeterminates in X) has a unique solution in $\mathbb{V}_{\text{afa}}$ – i.e., a suitable substitution of x's with sets. But this is Aczel's solution lemma, which is equivalent to AFA.

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Claim. CPR* is interpretable in ZFA.

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In fact, we can even ask a simpler question to start with (closer to the analogue of Bealer's theory for properties):

Is there a conception of properties supporting the theory $\mathrm{Z}^-_{\neq} + \mathrm{AFA}_1$ (i.e. ZF formulated with collection, minus foundation, and extensionality, plus the existence part of AFA)?

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Incidentally, there's also a question about the strength of this theory. Friedman's proof employs foundation to show that bisimulations behave as expected.

- Is ZF consistent relative to $Z^-_{\neq} + AFA_1$?
- Is the addition of abstraction terms logically stronger?

If one takes properties to be entities constituted by their instantiation structure:

> Properties are what is depicted by arbitrary graphs.

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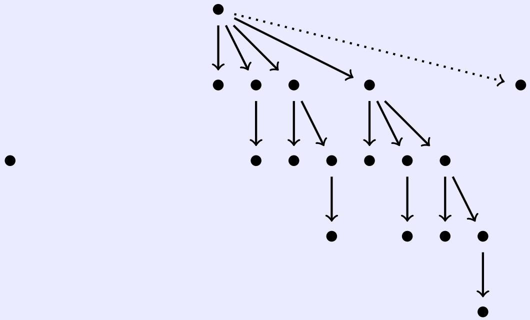
The existence of properties corresponding to the graphs:



By adapting work of Luca Incurvati, one can show that one can derive the axioms of $Z^-_{\neq}$ from the following assumptions:

The following theory of *properties* $(x, y, z, \ldots)$ and *trees* $(t_1, t_2, t_3, \ldots)$:

$$\forall x \exists t\, \mathrm{Dep}(x, t)$$

$$t_1 \leq t_2 \wedge t_2 \leq t_3 \rightarrow t_1 \leq t_3$$

$$\exists t_3 (t_1 \leq t_3 \wedge t_2 \leq t_3)$$

$$\mathrm{Dep}(x, t) \rightarrow (\forall y \in x)(\mathrm{Subtree}(y, t))$$

$$\mathrm{Dep}(x, t) \rightarrow (\forall y \subseteq x)(\mathrm{Subtree}(y, t))$$

$$\exists t \forall y (\varphi(y) \rightarrow \mathrm{Subtree}(y, t)) \rightarrow \exists x \forall y (y \in x \leftrightarrow \varphi(y))$$

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If one is considering the modified L&W theory, one should look at further considerations for *uniqueness*:

(AFA$_2$) There is at most one property decorating any graph.

- 'one should be able to move from a graph to a property unambiguously', 'graphs are our only guide' (potentially troublesome for the conception in general, but OK for the instantiation class)
- AFA$_2$ is just a generalization of extensionality

Analogously to nwf set theory, these non-wellfounded property theories address paradox is by banning entities whose instantiation structure is too “complex” – Cantor’s ‘totality of everything thinkable’ comes to mind.

Properties are *hereditarily simple* in their instantiation structure.

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How to understand ‘complex’, or ‘not simple’? One option is to formulate a version of the limitation of size principle:

A system is complex if it has the same size as V_{afa}.

It’s fairly clear that there’s no *universal property* (other axioms for non-wellfounded sets may be more attractive).

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If property theory is foundational, then it has to *eliminate* mathematical entities. But it cannot eliminate graphs and the associated set theory. Therefore, it cannot be foundational.

At least one can "eliminate" set-theoretic entities...

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Proposition. Principles 1-6 together with the 'mixed' axioms for trees and properties above derive the axioms of $\mathrm{Z}^{-}_{\neq}$.

The idea is that all relevant notions such as *subgraph, graph isomorphism, path* are now directly axiomatized.

A consistency proof for the theory is likely to require substantial resources.

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- This, however, may not be unlike other conceptions of properties, like the possible worlds analysis, in which the set-theoretic machinery of possible worlds guides property-theoretic theorizing.

- Ultimately, if properties are constituted – at least partially – by their instantiation structure, there seems to be nothing wrong in assuming a mathematical model of such a structure to establish conditions for their existence and constitution, unless one intends to eliminate mathematical objects altogether via properties.

I. I have been concerned with untyped theories of properties (formulated in first-order logic)

II. I considered some proposals to develop property theory in the same vein as standard set theory and the associated iterative conception

III. To overcome some shortcomings of such proposals, I developed some examples of property-theoretic analogues of non-wellfounded set theory

IV. I then asked whether "graph" conceptions of properties can support those formal frameworks