Intensional paradoxes and the maxim of maximal recapture

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> *'There never were any set-theoretic paradoxes, but the property theoretic paradoxes are still unresolved'*
> *(Gödel via Myhill 1984)*

By intensional paradoxes one might refer to:

- semantic
- property-theoretic
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In the talk I'll essentially deal with the first two. The gist of the talk will mainly be methodological: paradoxical sentences are so far away from empirical testing or conceptual practices that not much consensus can be gained by reflecting on our intuitions about them.

Part I: Some theories of self-applicable predication

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- Nonclassical theories:
  - theories based on the logic K3 of 'gaps' (Field, Reinhardt);
  - theories based on the logic LP of 'gluts' (Priest, Beall);
- Classical theories of predication (Kripke, Feferman, Turner).

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They will capture the same such solution in a sense that will be made precise.

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Definition ($\mathrm{Basic}^{\mathsf{K}}$)

$\mathrm{Basic}^{\mathsf{K}}$ contains Q – conveniently formulated with $<$ – and axioms for sequences, projections, lenghts. Crucially:

$$\exists s(\mathrm{lh}(s) = 0)$$
$$\forall s(\mathrm{lh}(s) = n \to \exists s'(\mathrm{lh}(s') = n + 1 \wedge (\forall m < n)(s_m = s'_m)))$$

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Remarks:

- The superscript $(\cdot)^{\mathsf{K}}$ stands for 'classical logic'
- Basic$^{\mathsf{K}}$ is interpretable in Q, although not in a way that preserves identity since Q does not have a **general** notion of sequence;
- So far we have **no induction**, either on strings or on numbers.

Let's consider now a language $\mathcal{L}$ containing $\mathcal{L}_{\mathsf{Basic}}$ and a predicate $\mathsf{P}(x, y)$ for **predication**. From a canonical Gödel numbering of $\mathcal{L}$-expression we define

$$\lambda x\, A(x, \vec{y}) := \langle \ulcorner A \urcorner, \vec{y} \rangle$$

where $\ulcorner A \urcorner$ is a (term representing the) code of A. We mostly deal with unary wwfs $A(v)$.

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As it is well-known, $\mathsf{Basic}^{\mathsf{K}}$ formulated in $\mathcal{L}$ is **inconsistent** with full predication rules:

$$\frac{\Gamma \Rightarrow A(x), \Delta}{\Gamma \Rightarrow \mathsf{P}(\ulcorner A(v) \urcorner, x), \Delta} \qquad \frac{\Gamma, A(x) \Rightarrow \Delta}{\Gamma, \mathsf{P}(\ulcorner A(v) \urcorner, x) \Rightarrow \Delta}$$

This holds equally well for intuitionistic versions of the theory of sequences and predication rules.

Restricting the logic

Definition (Basic De Morgan Logic)

BDM is the fully structural logic obtained by considering and positive and negative rules for monotone connectives and quantifiers, e.g.:

$$\frac{\Gamma \Rightarrow \Delta, \neg A, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \qquad \frac{\Gamma, \neg A \Rightarrow \Delta \qquad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$$

and only double negation rules:

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A} \qquad \frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta}$$

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Definition (LP, K3)

LP and K3 are now obtained by adding to BDM, respectively:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$$

Definition (Basic^{K3}, Basic^{LP})

Basic^{K3} and Basic^{LP} are obtained by extending K3 and LP in $\mathcal{L}_{\text{Basic}}$, respectively, with

- Identity sequents (we can treat them classically given our applications)

$$\Rightarrow t = t \qquad\qquad s = t, A(s) \Rightarrow A(t)$$

- Sequents $\Rightarrow A$ for A an axiom of Q or of sequences.

Principles for Predication/Satisfaction for properties/unary formulas

(At+)	$\mathrm{P}(\ulcorner R(v) \urcorner, x) \Leftrightarrow R(x)$	
(At−)	$\mathrm{P}(\ulcorner \neg R(v) \urcorner, x) \Leftrightarrow \neg R(x)$	for R in $\mathcal{L}_{\mathrm{Basic}}$

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(P1) $\mathrm{P}(\ulcorner \mathrm{P}(u, v) \urcorner, (x, y)) \Leftrightarrow \mathrm{P}(x, y)$

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($\neg$1)	$\mathrm{P}(\neg\varphi, x) \Leftrightarrow \neg \mathrm{P}(\varphi, x)$	
($\neg$2)	$\mathrm{P}(\neg\neg\varphi, x) \Leftrightarrow \mathrm{P}(\varphi, x)$	
($\wedge$1)	$\mathrm{P}(\varphi \wedge \psi, x) \Leftrightarrow \mathrm{P}(\varphi, x) \wedge \mathrm{P}(\psi, x)$	
($\wedge$2)	$\mathrm{P}(\neg(\varphi \wedge \psi), x) \Leftrightarrow \mathrm{P}(\neg\varphi, x) \vee \mathrm{P}(\neg\psi, x)$	
	$\vdots$	

Paracomplete and paraconsistent satisfaction

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Definition ($\mathrm{K3P}_0$, LPP_0)

Extend respectively $\mathrm{Basic}^{\mathrm{K3}}$ and $\mathrm{Basic}^{\mathrm{LP}}$ with (At), (P1), ($\neg$1), ($\wedge$1) (and quantifier sequents).

Classical, Self-referential satisfaction

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Definition (KFP_0)

Extend $\mathsf{Basic}^{\mathsf{K}}$ with (At+)-(∧2) (and quantifier sequents).

Models

Fixed Point Model

A set $S \subseteq \mathbb{N}$ is a fixed point iff

$$(\varphi, n) \in S \quad \text{iff} \quad \mathrm{P}(\varphi, n) \in S$$

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Consistent Fixed Points

For S a consistent fixed point, i.e. for no φ, n: $(\varphi, n) \in S$ and $(\neg\varphi, n) \in S$,

$$(\mathbb{N}, S) \vDash \mathrm{KFP}_0 + \mathrm{P}(\neg\varphi, x) \Rightarrow \neg\mathrm{P}(\varphi, x) \quad \text{iff} \quad (\mathbb{N}, S) \vDash_{\mathrm{K3}} \mathrm{K3P}_0$$

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Complete Fixed point

For S a complete fixed point, i.e. for every φ, n: $(\varphi, n) \in S$ or $(\neg\varphi, n) \in S$,

$$(\mathbb{N}, S) \models \mathrm{KFP}_0 + \neg\mathrm{P}(\varphi, x) \Rightarrow \mathrm{P}(\neg\varphi, x) \quad \text{iff} \quad (\mathbb{N}, S) \models_{\mathrm{LP}} \mathrm{LPP}_0$$

Part II: Recapture(s)?

No restrictions

Fact

The unrestricted predication rules

$$\frac{\Gamma \Rightarrow A(x), \Delta}{\Gamma \Rightarrow \mathrm{P}(\ulcorner A(v) \urcorner, x), \Delta} \qquad \frac{\Gamma, A(x) \Rightarrow \Delta}{\Gamma, \mathrm{P}(\ulcorner A(v) \urcorner, x) \Rightarrow \Delta}$$

are admissible in $\mathrm{K3P}_0$ and LPP_0 (with $A(v) \in \mathcal{L}$).

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The classical theory $\mathrm{KFP_0}$ and its extensions with consistency / completeness principles don't have this property. The Russell property $\mathrm{R(x)} \leftrightarrow \neg \mathrm{P}(x, x)$ is such that:

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- KFP_0+COMP proves that the Russell's property **is** predicable of itself but the predicate $\mathrm{R}(x)$ does not apply to $\ulcorner \mathrm{R} \urcorner$.

Recapture

The problem with $K3P_0$ and LPP_0 approaches is well-known: principles such as the the law of excluded middle – in the case of $K3P_0$ – or material modus ponens – for LPP_0:

> *'nothing like sustained ordinary reasoning can be carried out in either logic' (Feferman 1984)*

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In our framework, the paracomplete/paracosistent theorist's reassuring remarks to the mathematician can be translated as follows:

Lemma (Recapture)

For $A \in \mathcal{L}_{\mathrm{Basic}}$:

- $\mathrm{K3P}_0$ proves $\Rightarrow A \vee \neg A$;
- LPP_0 proves $A \wedge \neg A \Rightarrow$.

In particular, this means that if one restricts the attention to $\mathcal{L}_{\mathrm{Basic}}$, one obtains that $\mathrm{Basic}^{\mathrm{K}}$, $\mathrm{Basic}^{\mathrm{LP}}$, and $\mathrm{Basic}^{\mathrm{K3}}$ are identical theories and classical reasoning can be used unrestrictedly.

Schemata

Our understanding of the language of arithmetic is such that we anticipate that the Induction Axiom Schema will persist through all changes. [...] We are always capable of generating new Induction Axioms by expanding the language. (McGee, How we learn mathematical language, p. 58)

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What happens if we add induction schemata to our theories?

$$\frac{\Gamma, A(x) \Rightarrow A(x+1), \Delta}{\Gamma, A(0) \Rightarrow A(y), \Delta}$$

for x not free in $A(0), \Gamma, \Delta$ and y arbitrary. We call the resulting theories KFP, K3P, LPP.

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Notice that this is not a logical *divertissement*: there are foundational programs (Aczel, Feferman, Schütte, Turner) that wish to start with natural numbers and justify higher mathematics via self-referential predication/satisfaction axioms. Similar remarks can be made, with a bit of effort, for ZFC and classes over it.

Transfinite Induction

We assume a notation $(O, \prec)$ for the ordinals up to the Feferman - Schütte ordinal Γ_0. Transfinite induction up to any $\beta < \Gamma_0$ is the schema, for $\delta < \beta$ and $A(v) \in \mathcal{L}$,

$$\frac{\Gamma, \forall\gamma \prec \alpha\, A(\gamma) \Rightarrow A(\alpha), \Delta}{\Gamma \Rightarrow \forall\gamma \prec \delta\, A(\gamma), \Delta}\ \mathsf{TI}_{\mathcal{L}}(< \beta)$$

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Proposition

By the classical Gentzen argument, KFP proves $\mathsf{TI}_{\mathcal{L}}(<\varepsilon_0)$, whereas K3P and LPP can only prove $\mathsf{TI}_{\mathcal{L}}(<\omega^\omega)$. The former is proof-theoretically equivalent to $\mathsf{ACA}_{<\varepsilon_0}$, the second to $\mathsf{ACA}_{<\omega^\omega}$.

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And crucially:

Corollary

$\mathsf{Basic}^{\mathsf{LP}} + \mathsf{Ind}(\mathcal{L})$, and $\mathsf{Basic}^{\mathsf{K3}} + \mathsf{Ind}(\mathcal{L})$ can only prove $\mathsf{TI}_{\mathcal{L}}(<\omega^\omega)$, whereas $\mathsf{Basic}^{\mathsf{K}} + \mathsf{Ind}(\mathcal{L})$ proves $\mathsf{TI}_{\mathcal{L}}(<\varepsilon_0)$.

Non-semantic consequences /1

If one carefully looks at the proof-theory of these systems, it's clear that also in terms of mathematical consequences they are very different:

Proposition

- KFP proves $\mathsf{TI}_{\mathcal{L}_{\mathbf{Basic}}}(< \phi_{\varepsilon_{\mathbf{0}}}0)$ (same as $\mathsf{ACA}_{<\varepsilon_{\mathbf{0}}}$);
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But the defender of gaps or gluts is within their rights to ask whether the the loss of the classically accepted pattern of reasoning by transfinite induction is in fact a **mathematically significant** one.

Non-semantic consequences /2

INDEC is the assertion that:

> *every countable scattered indecomposable linear ordering is either indecomposable to the left, or indecomposable to the right.*

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Proposition (Eastaugh, N.)

RCA+INDEC is proof-theoretically equivalent to KFP. It follows that KFP can 'nicely' interpret RCA+INDEC, but neither K3P nor LPP can.

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- Under the presupposition that the schema of mathematical induction is to be understood as (weakly) open-ended, mathematical patterns of reasoning are severely crippled when one moves to from classical to non-classical theories of predication.
- Even if one is suspicious about the status of 'axioms' or logical principles such as iterations of comprehension, there are theorems, such as INDEC, that separate the two clusters of theories.

Part III: Bridges

Reflection

One way to fill this mathematical gap is to resort to Feferman's theory of implicit commitment. We can enlarge K3P or LPP with a reflection principle of the form, for $S = \mathsf{K3P}, \mathsf{LPP}$,

$$\text{(R)}\ \frac{\mathrm{Pr}^2_S(\ulcorner \Gamma\dot{x} \Rightarrow \Delta\dot{x}\urcorner, \ulcorner \Theta\dot{x} \Rightarrow \Lambda\dot{x}\urcorner) \qquad \Gamma(x) \Rightarrow \Delta(x)}{\Theta(x) \Rightarrow \Lambda(x)}$$

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Proposition (Fischer, N., Horsten)

- Finitely many iterations of reflection over S give us $\mathrm{TI}_{\mathcal{L}}(< \omega^{\omega^2})$;
- $< \varepsilon_0$-iterations of reflection give us the same transfinite induction as KFP and variants thereof.

Reflection

One way to fill this mathematical gap is to resort to Feferman's theory of implicit commitment. We can enlarge K3P or LPP with a reflection principle of the form, for $S = \mathrm{K3P}, \mathrm{LPP}$,

$$(\mathrm{R})\ \frac{\mathrm{Pr}^2_S(\ulcorner \Gamma \dot{x} \Rightarrow \Delta \dot{x} \urcorner, \ulcorner \Theta \dot{x} \Rightarrow \Lambda \dot{x} \urcorner) \qquad \Gamma(x) \Rightarrow \Delta(x)}{\Theta(x) \Rightarrow \Lambda(x)}$$

Proposition (Fischer, N., Horsten)

- Finitely many iterations of reflection over S give us $\mathrm{TI}_{\mathcal{L}}(< \omega^{\omega^2})$;
- $< \varepsilon_0$-iterations of reflection give us the same transfinite induction as KFP and variants thereof.

An obvious problem with this strategy is that it is available also to the classical logician, especially the one willing to accept gluts internally.

A conditional

We can extend Basic De Morgan Logic with a new conditional satisfying

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \qquad\qquad \frac{\Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}$$

Definition (HBP)

We extend $\mathsf{Basic}^{\mathsf{BDM}^{\to}}$ with the axiom

$$\Rightarrow \mathrm{P}(\ulcorner A \urcorner, x) \leftrightarrow A(x)$$

for A containing P but not $\to$ itself.

A conditional

We can extend Basic De Morgan Logic with a new conditional satisfying

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \qquad\qquad \frac{\Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta}$$

Definition (HBP)

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for A containing P but not $\rightarrow$ itself.

These rules are sound with respect to the class of fixed point models for satisfaction / predication extracted from Leitgeb's system HYPE (roughly a 'hyperintensional conditional').

By carefully reproducing Gentzen's argument one should be able to obtain:

Conjecture

HBP proves $\mathsf{TI}_{\mathcal{L}}(<\varepsilon_0)$ (indeed already in $\mathsf{Basic}^{\mathsf{BDM}^{\rightarrow}}$).

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If true, it is still not clear whether this claim may help the gap / glut theorist, unless it is established that there is no classical, natural counterpart to HBP and its extensions with compositional clauses.

In addition, we cannot have unrestricted principles anymore.

THANK YOU.