# Substructural approaches to paradox and the logic of semantic groundedness

Carlo Nicolai

Slides available at https://carlonicolai.github.io

# OUTLINE

This is work in progress:

- Brief overview of some substructural approaches to paradox and their motivation
- Semantic groundedness and infinite derivations
- Restriction to reflexivity and the logic of semantic groundedness

# Part 1: paradox

We are interested in naïve or unrestricted rules for truth and (later) consequence. Similar remarks can be made for (non-extensional) class-membership, or property instantiation:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \mathrm{Tr}\ulcorner A \urcorner \Rightarrow \Delta}\ (\mathrm{Tr\,L}) \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \mathrm{Tr}\ulcorner A \urcorner, \Delta}\ (\mathrm{Tr\,R})$$

There is an important omission in the literature that I will admittedly follow. Given our language $L$ and *any* model $M$ for $L$, it is always assumed a set $N \subset |M|$ of distinguished names $\ulcorner \cdot \urcorner$ for $L$-sentences and a (bijective) denotation function $\mathsf{d}\colon N \to \mathsf{Sent}_L$ such that

$$\lambda := \mathsf{d}(\ulcorner \lambda \urcorner) = \neg \mathrm{Tr}\ulcorner \lambda \urcorner$$

It's completely unclear to me how this semantic approach could be integrated into a proof system by maintaining the intended properties.

# LIAR

Let $\lambda$ be $\neg\mathrm{Tr}\ulcorner\lambda\urcorner$, $\neg\lambda$ be $\mathrm{Tr}\ulcorner\lambda\urcorner$,

$$
\dfrac{\dfrac{\dfrac{\lambda \Rightarrow \lambda}{\lambda \Rightarrow \mathrm{Tr}\ulcorner\lambda\urcorner}\,(\mathrm{Tr\,R})}{\Rightarrow \neg\lambda}\,(\neg\mathrm{R,CL}) \qquad \dfrac{\dfrac{\lambda \Rightarrow \lambda}{\mathrm{Tr}\ulcorner\lambda\urcorner \Rightarrow \lambda}\,(\mathrm{Tr\,L})}{\neg\lambda \Rightarrow}\,(\neg\mathrm{L,CR})}{\Rightarrow}\,(\mathrm{Cut})
$$

# CURRY

Let $\varkappa$ be $\mathrm{Tr}\ulcorner\varkappa\urcorner \to \bot$,

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\mathrm{Tr}\ulcorner\varkappa\urcorner \Rightarrow \mathrm{Tr}\ulcorner\varkappa\urcorner \qquad \bot \Rightarrow \bot}{\mathrm{Tr}\ulcorner\varkappa\urcorner, \mathrm{Tr}\ulcorner\varkappa\urcorner \to \bot \Rightarrow \bot}\,(\to\mathrm{L})
}{\mathrm{Tr}\ulcorner\varkappa\urcorner \Rightarrow \bot}\,(\mathrm{CL})
}{\Rightarrow \mathrm{Tr}\ulcorner\varkappa\urcorner \to \bot}\,(\to\mathrm{R})
}{\Rightarrow \mathrm{Tr}\ulcorner\varkappa\urcorner}
\qquad
\dfrac{
\dfrac{\mathrm{Tr}\ulcorner\varkappa\urcorner \Rightarrow \mathrm{Tr}\ulcorner\varkappa\urcorner \qquad \bot \Rightarrow \bot}{\mathrm{Tr}\ulcorner\varkappa\urcorner, \mathrm{Tr}\ulcorner\varkappa\urcorner \to \bot \Rightarrow \bot}\,(\to\mathrm{L})
}{\mathrm{Tr}\ulcorner\varkappa\urcorner \Rightarrow \bot}\,(\mathrm{CL})
}{\Rightarrow \bot}\,(\mathrm{Cut})
$$

# INTERNAL CURRY

Let $\nu$ be $\mathsf{C}(\ulcorner \nu \urcorner, \ulcorner \bot \urcorner)$,

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\nu \Rightarrow \nu \qquad \bot \Rightarrow \bot}{\nu, \mathsf{C}(\ulcorner \nu \urcorner, \ulcorner \bot \urcorner) \Rightarrow \bot}\,(\mathrm{CL})
}{\nu \Rightarrow \bot}\,(\mathrm{CL})
}{\Rightarrow \mathsf{C}(\ulcorner \nu \urcorner, \ulcorner \bot \urcorner)}\,(\mathrm{CR})
}{\Rightarrow \nu}
\qquad
\dfrac{
\dfrac{\nu \Rightarrow \nu \qquad \bot \Rightarrow \bot}{\nu, \mathsf{C}(\ulcorner \nu \urcorner, \ulcorner \bot \urcorner) \Rightarrow \bot}\,(\mathrm{CL})
}{\nu \Rightarrow \bot}\,(\mathrm{CL})
}{\Rightarrow \bot}\,(\mathrm{Cut})
$$

# Part 2: substructural approaches

# NO GLUTS

We will not consider non-transitive approaches, in which the structural rule of cut is restricted.

The reason is that it is compatible with paradoxical sentences such as the Liar sentence $\lambda$ and its negation $\neg\lambda$ to be both provable. Of course allowing cut would trivialize the theory.

Nontransitive theorists such as Dave Ripley would argue that proofs are not about what we can assert, but about what we cannot strictly deny. I don't understand well this idea and I will leave this aside.

We let $\Gamma, \Delta$ be multisets of formulas (sentences in the infinitary systems). We limit ourselves to multiplicative connectives: the addition of additive quantifiers is straightforward.

# CONTRACTION-FREE NAÏVE TRUTH

$$\Gamma, P(t) \Rightarrow P(t), \Delta \quad [0]$$

$$\frac{\Gamma_0 \Rightarrow \Delta_0, A \quad [\alpha] \qquad A, \Gamma_1 \Rightarrow \Delta_1 \quad [\beta]}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \quad [\alpha + \beta]}\ (\text{CUT})$$

$$\Gamma \Rightarrow \top, \Delta \quad [0]$$

$$\Gamma, \bot \Rightarrow \Delta \quad [0]$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad [\alpha]}{\Gamma, \mathrm{Tr}\ulcorner A \urcorner \Rightarrow \Delta \quad [\alpha + 1]}\ (\mathrm{Tr}\,\mathrm{L})$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad [\alpha]}{\Gamma \Rightarrow \mathrm{Tr}\ulcorner A \urcorner, \Delta \quad [\alpha + 1]}\ (\mathrm{Tr}\,\mathrm{R})$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad [\alpha]}{\Gamma, \neg\varphi \Rightarrow \Delta \quad [\alpha]}\ (\neg\,\mathrm{L})$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad [\alpha]}{\Gamma \Rightarrow \neg\varphi, \Delta \quad [\alpha]}\ (\neg\,\mathrm{R})$$

$$\frac{\Gamma, A, B \Rightarrow \Delta \quad [\alpha]}{\Gamma, A \wedge B \Rightarrow \Delta \quad [\alpha]}\ (\wedge\mathrm{L})$$

$$\frac{\Gamma_0 \Rightarrow \Delta_0, A \quad [\alpha] \qquad \Gamma_1 \Rightarrow \Delta_1, B \quad [\beta]}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1, A \wedge B \quad [\alpha + \beta]}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma \Rightarrow \Delta, A, B \quad [\alpha]}{\Gamma \Rightarrow \Delta, A \vee B \quad [\alpha]}\ (\vee\mathrm{R})$$

$$\frac{\Gamma_0, A \Rightarrow \Delta_0 \quad [\alpha] \qquad \Gamma_1, B \Rightarrow \Delta_1 \quad [\beta]}{\Gamma_0, \Gamma_1, A \vee B \Rightarrow \Delta_0, \Delta_1 \quad [\alpha + \beta]}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma, A \Rightarrow \Delta, B \quad [\alpha]}{\Gamma \Rightarrow \Delta, A \rightarrow B \quad [\alpha]}\ (\rightarrow\mathrm{R})$$

$$\frac{\Gamma_0 \Rightarrow A, \Delta_0 \quad [\alpha] \qquad \Gamma_1, B \Rightarrow \Delta_1 \quad [\beta]}{\Gamma_0, \Gamma_1, A \rightarrow B \Rightarrow \Delta_0, \Delta_1 \quad [\alpha + \beta]}\ (\rightarrow\mathrm{L})$$

By adding only multiplicative quantifiers we obtain a variant of the theory of non-contractive truth defended by Zardini (2011).

By adding additive rules for additive connectives we obtain a notational variant of Grišin's set theory studied by Grišin 1982 and Cantini 2003.

**FACT**

Cut-elimination holds for contraction-free naïve truth.

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Lexicographic induction on the sum of the truth-levels, rank of the cut formula, length of the proof:

$$\dfrac{\dfrac{\begin{matrix} D_0 \\ \Gamma_0 \Rightarrow \Delta_0, A \;\; [\alpha] \end{matrix}}{\Gamma_0 \Rightarrow \Delta_0, \mathrm{Tr}\ulcorner A \urcorner \;\; [\alpha+1]} \qquad \dfrac{\begin{matrix} D_1 \\ A, \Gamma_1 \Rightarrow \Delta_1 \;\; [\beta] \end{matrix}}{\Gamma_1, \mathrm{Tr}\ulcorner A \urcorner \Rightarrow \Delta_1 \;\; [\beta+1]}}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \;\; [\alpha+\beta+2]}$$

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…becomes

$$\dfrac{\begin{array}{c}D_0\\ \Gamma_0 \Rightarrow \Delta_0, A \;\; [\alpha]\end{array} \qquad \begin{array}{c}D_1\\ A, \Gamma_1, A \Rightarrow \Delta_1 \;\; [\beta]\end{array}}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1 \;\; [\alpha+\beta]}$$

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# A PROBLEM?

What if we add rules for an additive connective, say:

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \sqcap B \Rightarrow \Delta}\ (\sqcap L) \qquad \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \sqcap B, \Delta}\ (\sqcap R)$$

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When we want to standardly reduce…

$$\frac{\dfrac{\Gamma \Rightarrow A, \Delta, C\ \ [\alpha_0] \qquad \Gamma \Rightarrow B, \Delta, C\ \ [\alpha_1]}{\Gamma \Rightarrow A \sqcap B, \Delta, C\ \ [\alpha(:= \alpha_0 + \alpha_1)]}\ (\sqcap\mathrm{R}) \qquad C, \Pi \Rightarrow \Sigma\ \ [\beta]}{\Gamma, \Pi \Rightarrow \Sigma, \Delta, A \sqcap B\ \ [\alpha + \beta]}\ (\mathrm{CUT})$$

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…to:

$$\dfrac{\dfrac{\Gamma \Rightarrow A, \Delta, C\,[\alpha_0] \qquad C, \Pi \Rightarrow \Sigma\,[\beta]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, A\,[\alpha_0 + \beta]} \qquad \dfrac{\Gamma \Rightarrow B, \Delta, C\,[\alpha_1] \qquad C, \Pi \Rightarrow \Sigma\,[\beta]}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, B\,[\alpha_1 + \beta]}}{\Gamma, \Pi \Rightarrow \Sigma, \Delta, A \sqcap B\,[\alpha + \beta \cdot 2]}\,(\sqcap R)$$

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$$\vdots$$

$$\Gamma \Rightarrow \mathrm{Tr}_n\ulcorner\top\urcorner, \Delta \ [n] \qquad\qquad \Pi \Rightarrow \mathrm{Tr}_m\ulcorner\top\urcorner, \Sigma \ [n-1]$$

$$\Gamma, \Pi \Rightarrow \Delta, \Sigma, \mathrm{Tr}_{n-1}\ulcorner\top\urcorner, \mathrm{Tr}_n\ulcorner\top\urcorner \ [2n-1]$$

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$$\vert$$

$$\Gamma, \Pi \Rightarrow \Delta, \Sigma, \mathrm{Tr}_n\ulcorner\top\urcorner [2n]$$

Contraction-free approaches have several drawbacks:

- The usual complaint, of which I'm guilty today, of not providing a theory that *proves* the Liar *exists*, is not only a form of laziness but a real *impossibility*. Da Re and Rosenblatt (2017) show that purely multiplicative vocabulary is incompatible with the diagonal lemma or a basic syntax such as Robinson's Q.

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- The usual complaint, of which I'm guilty today, of not providing a theory that *proves* the Liar *exists*, is not only a form of laziness but a real *impossibility*. Da Re and Rosenblatt (2017) show that purely multiplicative vocabulary is incompatible with the diagonal lemma or a basic syntax such as Robinson's Q.
- The second important drawback is the lack of any *intuitive picture* of truth behind it. Crucially there is no known plausible semantics for naïve principles, only heavy metaphysics.

…however, there is an alternative.

# A CONCEPTION OF TRUTH (AND FALSITY)

# A conception of truth (and falsity)

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- A conjunction is true iff both conjuncts are true, false if at least one conjunct is false;
- A disjunction is true iff at least one disjunct is true, false iff both disjuncts are false;
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- A disjunction is true iff at least one disjunct is true, false iff both disjuncts are false;
- A universally quantified sentence is true iff all its instances are true, false iff at least one instance is false;
- A truth ascription $\mathrm{Tr}\ulcorner A \urcorner$ is true iff $A$ is true, false iff $A$ is false;
- A falsity ascription $\mathrm{F}\ulcorner A \urcorner$ is true iff $A$ is false, and false iff $A$ is true.

Translated into $L_2$, this conception translates into an operator $\Phi\colon P^2(\omega) \to P^2(\omega)$ such that

$$\Phi(X) := \langle \Phi(X)^+, \Phi(X)^- \rangle$$

$$\begin{aligned}
n \in \Phi(X)^+ :\Leftrightarrow\ & n = \ulcorner P(t) \urcorner \text{ and } \mathbb{N} \vDash P(t), \text{ or} \\
& n = \ulcorner \mathrm{Tr} \ulcorner \varphi \urcorner \urcorner \text{ and } \ulcorner \varphi \urcorner \in X^+, \text{ or} \\
& n = \ulcorner \mathrm{F} \ulcorner \varphi \urcorner \urcorner \text{ and } \ulcorner \varphi \urcorner \in X^-, \text{ or} \\
& n = \ulcorner \neg \varphi \urcorner \ \text{ and } \ \ulcorner \varphi \urcorner \in X^-, \text{ or} \\
& n = \ulcorner \varphi \wedge \psi \urcorner \ \text{ and } \ulcorner \varphi \urcorner \in X^+ \text{ and } \ulcorner \psi \urcorner \in X^+, \text{ or} \\
& n = \ulcorner \varphi \vee \psi \urcorner \ \text{ and } \ulcorner \varphi \urcorner \in X^+ \text{ or } \ulcorner \psi \urcorner \in X^+, \text{ or} \\
& n = \ulcorner \forall v \varphi \urcorner \ \text{ and } \ulcorner \varphi(\overline{m}) \urcorner \in X^+ \text{ for all } m \in \omega.
\end{aligned}$$

Translated into $L_2$, this conception translates into an operator $\Phi\colon P^2(\omega) \to P^2(\omega)$ such that

$$\Phi(X) := \langle \Phi(X)^+, \Phi(X)^- \rangle$$

$$\begin{aligned}
n \in \Phi(X)^- :\Leftrightarrow\ & n \text{ is not coding a sentence, or} \\
& n = \ulcorner P(t) \urcorner \text{ and } \mathbb{N} \not\models P(t), \text{ or} \\
& n = \ulcorner \mathrm{Tr} \ulcorner \varphi \urcorner \urcorner \text{ and } \ulcorner \varphi \urcorner \in X^-, \text{ or} \\
& n = \ulcorner \mathrm{F} \ulcorner \varphi \urcorner \urcorner \text{ and } \ulcorner \varphi \urcorner \in X^+, \text{ or} \\
& n = \ulcorner \neg\varphi \urcorner \text{ and } \ulcorner \varphi \urcorner \in X^+, \text{ or} \\
& n = \ulcorner \varphi \wedge \psi \urcorner \text{ and } \ulcorner \varphi \urcorner \in X^- \text{ or } \ulcorner \psi \urcorner \in X^-, \text{ or} \\
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& n = \ulcorner \forall v \varphi \urcorner \text{ and } \ulcorner \varphi(\overline{m}) \urcorner \in X^- \text{ for some } m \in \omega.
\end{aligned}$$

Grounded truth is obtained by closing $\langle\emptyset,\emptyset\rangle$ under $\Phi$:

### MINIMAL FIXED POINT

There is an $\alpha$ such that $\Phi^{\alpha}(\langle\emptyset,\emptyset\rangle)=\Phi^{\beta}(\langle\emptyset,\emptyset\rangle)$ for all $\beta\geq\alpha$. We call it $\mathrm{I}_{\Phi}$. Crucially

$$\varphi\in\mathrm{I}_{\Phi}\quad\text{iff}\quad\mathrm{Tr}\ulcorner\varphi\urcorner\in\mathrm{I}_{\Phi}$$

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### KRIPKE, BURGESS 1986

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$\mathrm{I}_{\Phi}$ is $\Pi^1_1$-complete and its closure ordinal is $\omega_1^{ck}$.

This suggests to look at infinitary proof systems for $\mathrm{I}_{\Phi}$.

## STRONG KLEENE SYSTEM $\mathsf{SK}^{\omega}$

| | |
|---|---|
| | $\vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow A, \Delta$ for $A$ a literal |
| | $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow A, \Delta$ for $A$ a literal and $\mathbb{N} \vDash A$ |
| | $\vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow \Delta$ for $A$ a literal and $\mathbb{N} \nvDash A$ |
| | $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow \mathrm{F}(t), \Delta$ if $t^{\mathbb{N}}$ is not a sentence |
| (Tr 1) | if $\vdash^{\alpha}_{\rho} (\Gamma) \Rightarrow A, (\Delta)$ then $\vdash^{\beta}_{\sigma} \Gamma \Rightarrow \mathrm{Tr}\ulcorner A \urcorner, \Delta$, with $\alpha < \beta, \rho < \sigma$ |
| (F1) | if $\vdash^{\alpha}_{\rho} (\Gamma) \Rightarrow (\Delta), \neg A$, then $\vdash^{\beta}_{\sigma} \Gamma \Rightarrow \mathrm{F}\ulcorner A \urcorner, \Delta$, with $\alpha < \beta, \rho < \sigma$ |
| (Tr 2) | if $\vdash^{\alpha}_{\rho} (\Gamma), A \Rightarrow (\Delta)$, then $\vdash^{\beta}_{\sigma} \Gamma, \mathrm{Tr}\ulcorner A \urcorner \Rightarrow \Delta$, with $\alpha < \beta, \rho < \sigma$ |
| (F2) | if $\vdash^{\alpha}_{\rho} (\Gamma), \neg A \Rightarrow (\Delta)$, then $\vdash^{\beta}_{\sigma} \Gamma, \mathrm{F}\ulcorner A \urcorner \Rightarrow \Delta$, with $\alpha < \beta, \rho < \sigma$ |
| ($\wedge$R) | if $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow A, \Delta$ and $\vdash^{\beta}_{\rho} \Gamma \Rightarrow B, \Delta$, then $\vdash^{\gamma}_{\rho} \Gamma, A \wedge B$ for $\alpha, \beta < \gamma$ |
| ($\neg\wedge$R) | if $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow \neg A_i, \Delta$, then $\vdash^{\gamma}_{\rho} \Gamma, \neg(A \wedge B)$ for $\alpha, \beta < \gamma, i = 0, 1$ |
| | $\vdots$ |
| ($\omega$) | if $(\forall n \in \omega)(\exists \alpha < \beta) \vdash^{\alpha}_{\rho} \Gamma, \varphi(\overline{n})$, then $\vdash^{\beta}_{\rho} \Gamma, \forall x \varphi$ |

# IRREFLEXIVE SYSTEM $\mathsf{TS}^{\omega}$

| | |
|---|---|
| | $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow A, \Delta$ for $A$ a literal and $\mathbb{N} \vDash A$ |
| | $\vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow \Delta$ for $A$ a literal and $\mathbb{N} \nvDash A$ |
| | $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow \mathrm{F}(t), \Delta$ if $t^{\mathbb{N}}$ is not a sentence |
| $(\neg\mathrm{L})$ | if $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow A, \Delta$, then $\vdash^{\beta}_{\rho} \Gamma, \neg A \Rightarrow \Delta$, with $\alpha < \beta$ |
| $(\neg\mathrm{R})$ | if $\vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow \Delta$, then $\vdash^{\beta}_{\rho} \Gamma \Rightarrow \neg A, \Delta$ with $\alpha < \beta$ |
| (Tr 1) | if $\vdash^{\alpha}_{\rho} (\Gamma) \Rightarrow A, (\Delta)$, then $\vdash^{\beta}_{\sigma} \Gamma \Rightarrow \mathrm{Tr}\ulcorner A \urcorner, \Delta$, with $\alpha < \beta, \rho < \sigma$ |
| (F1) | if $\vdash^{\alpha}_{\rho} (\Gamma) \Rightarrow (\Delta), \neg A$, then $\vdash^{\beta}_{\sigma} \Gamma \Rightarrow \mathrm{F}\ulcorner A \urcorner, \Delta$, with $\alpha < \beta, \rho < \sigma$ |
| (Tr 2) | if $\vdash^{\alpha}_{\rho} (\Gamma), A \Rightarrow (\Delta)$, then $\vdash^{\beta}_{\sigma} \Gamma, \mathrm{Tr}\ulcorner A \urcorner \Rightarrow \Delta$, with $\alpha < \beta, \rho < \sigma$ |
| (F2) | if $\vdash^{\alpha}_{\rho} (\Gamma), \neg A \Rightarrow, (\Delta)$ then $\vdash^{\beta}_{\sigma} \Gamma, \mathrm{F}\ulcorner A \urcorner \Rightarrow \Delta$, with $\alpha < \beta, \rho < \sigma$ |
| $(\wedge\mathrm{R})$ | if $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow A, \Delta$ and $\vdash^{\beta}_{\rho} \Gamma \Rightarrow B, \Delta$, then $\vdash^{\gamma}_{\rho} \Gamma, A \wedge B$ for $\alpha, \beta < \gamma$ |
| | $\vdots$ |
| $(\omega)$ | if $(\forall n \in \omega)(\exists \alpha < \beta) \vdash^{\alpha}_{\rho} \Gamma, \varphi(\overline{n})$, then $\vdash^{\beta}_{\rho} \Gamma, \forall x \varphi$ |

By a straightforward induction on the ordinal stages of the construction of $\mathrm{I}_\Phi$ and on the length of the derivations in $\mathsf{SK}^\omega$ and $\mathsf{TS}^\omega$:

**PROPOSITION**

- $\varphi \in \mathrm{I}_\Phi^+$ iff $\mathsf{SK}^\omega \vdash \Rightarrow \varphi$ iff $\mathsf{TS}^\omega \vdash \Rightarrow \varphi$;
- $\varphi \in \mathrm{I}_\Phi^-$ iff $\mathsf{SK}^\omega \vdash \Rightarrow \neg\varphi$ iff $\mathsf{TS}^\omega \vdash \Rightarrow \neg\varphi$;

By a straightforward induction on the ordinal stages of the construction of $\mathrm{I}_\Phi$ and on the length of the derivations in $\mathsf{SK}^\omega$ and $\mathsf{TS}^\omega$:

**PROPOSITION**

- $\varphi \in \mathrm{I}_\Phi^+$ iff $\mathsf{SK}^\omega \vdash \Rightarrow \varphi$ iff $\mathsf{TS}^\omega \vdash \Rightarrow \varphi$;
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More precisely, the induction aims at matching the ordinal norm

$$|A| := \mathsf{min}\{\alpha < \omega_1^{ck} \mid A \in \mathrm{I}_\Phi^+\}$$

and heights $\vdash^\alpha_\rho \Rightarrow A$ of derivations in the systems, clearly with $\alpha \geq \rho$. Similarly for $\mathrm{I}_\Phi^-$.

If we don't allow for side-formulae, Tr-paths are always traceable and we have, by induction on $(\rho, \mathsf{compl}(A), \alpha, \beta)$,

## CUT ADMISSIBILITY

- If $\mathsf{SK}^\omega \vdash^\alpha_\rho \Gamma \Rightarrow \Delta, A$ and $\mathsf{SK}^\omega \vdash^\beta_\rho A, \Gamma \Rightarrow \Delta$, then $\mathsf{SK}^\omega \vdash_\rho \Gamma \Rightarrow \Delta$;
- If $\mathsf{TS}^\omega \vdash^\alpha_\rho \Gamma \Rightarrow \Delta, A$ and $\mathsf{TS}^\omega \vdash^\beta_\rho A, \Gamma \Rightarrow \Delta$, then $\mathsf{TS}^\omega \vdash_\rho \Gamma \Rightarrow \Delta$.

But are these two approaches equivalent when it comes to the reasoning allowed by the notion of semantic groundedness given by $I_\Phi$?

# Part 3: the logic of grounded concepts

The short answer is that the notion of sentential truth is not rich enough to make finer distinctions.

### THE ‘EXTERNAL’ LOGIC OF $\mathrm{I}_\Phi$

For any $\varphi, \psi \in L_{\mathrm{Tr},\mathrm{F}}$,

$$\mathsf{EL}_\Phi := \{(\varphi, \psi) \mid \text{ if } \varphi \in \mathrm{I}^+_\Phi, \text{ then } \psi \in \mathrm{I}^+_\Phi\}$$

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As before, the distinctions resurface if one introduces a predicate $\mathsf{C}(\cdot, \cdot)$ for consequence, requiring that, in the same spirit as the rules for $\mathrm{Tr}(\cdot)$, we satisfy:

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \mathsf{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta}\ (\mathrm{CR}) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \mathsf{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta}\ (\mathrm{CL})$$

It's clear that there is no hope of extending $\mathsf{SK}^{\omega}$ with the naïve rules of consequence.

We get straight the internal Curry back.

$$\vdash^{\alpha}_{\rho} \varkappa \Rightarrow \varkappa$$

$$\vdash^{\beta}_{\rho} \bot \Rightarrow \bot$$

$$\vdash^{\gamma}_{\sigma} \varkappa, \mathsf{C}(\ulcorner \varkappa \urcorner, \ulcorner \bot \urcorner) \Rightarrow \bot \qquad\qquad \gamma > \alpha, \beta, \sigma > \rho$$

$$\vdash^{\delta}_{\sigma} \varkappa \Rightarrow \bot$$

$$\vdash^{\epsilon}_{\tau} \Rightarrow \varkappa$$

## THE 'INTERNAL' LOGIC OF $\mathrm{I}_\Phi$

$$\mathsf{IL}_\Phi := \{(\varphi, \psi) \mid \mathsf{TS}^\omega \vdash \varphi \Rightarrow \psi\}$$

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The intuitive idea behind the ‘internal’ logic is that $(A, B) \in \mathsf{IL}_\Psi$ if

‘either $A$ is determinately false (grounded in a non-semantic falsity) or $B$ is determinately true (grounded in a non-semantic truth)’

## THE 'INTERNAL' LOGIC OF $\mathrm{I}_{\Phi}$

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It turns our that $\mathsf{IL}_{\Phi}$, in the form of the 'navigating device' $\Rightarrow$ for $\mathsf{TS}^{\omega}$ can now be consistently internalized. To see this, however, we generalize $\Phi$.

Starting with the language $L \cup \{\mathsf{C}\}$. Define $\Psi : P(\omega) \to P(\omega)$ as

$$
\begin{aligned}
n \in \Psi(X) :\Leftrightarrow\ & n = (\Gamma; \varphi, \Delta) \text{ with } \varphi \text{ an } L\text{-literal } \mathbb{N} \nvDash \varphi, \text{ or} \\
& n = (\Gamma, \varphi; \Delta) \text{ with } \varphi \text{ an } L\text{-literal and } \mathbb{N} \vDash \varphi \text{ or} \\
& n = (\Gamma, \varphi \wedge \psi; \Delta) \text{ and } (\Gamma, \varphi; \Delta) \in X \text{ and } (\Gamma, \psi; \Delta) \in X, \text{ or} \\
& n = (\Gamma; \varphi \wedge \psi, \Delta) \text{ and } (\Gamma, \varphi; \Delta) \in X \text{ or } (\Gamma, \psi; \Delta) \in X, \text{ or} \\
& \qquad \vdots \\
& n = (\Gamma, \mathsf{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner); \Delta) \text{ and } (\Gamma; \varphi, \psi; \Delta) \in X \text{ or} \\
& n = (\Gamma; \mathsf{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta) \text{ and } (\Gamma, \varphi; \Delta) \in X \text{ and} \\
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& (\Gamma; \psi, \Delta) \in X.
\end{aligned}
$$

As before, we call $\mathrm{I}_\Psi$ the minimal fixed point of $\Psi$.

## $\mathsf{TSC}^{\omega}$

It is the infinitary system extending $\mathsf{TS}^{\omega}$ and 'read off' from $\Psi$. Crucially we can have full rules for negation and unrestricted rules from $\mathsf{C}$.

- if $\vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow \Delta$, then $\vdash^{\beta}_{\rho} \Gamma \Rightarrow \neg A, \Delta$ with $\beta > \alpha$,
- if $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow A, \Delta$, then $\vdash^{\beta}_{\rho} \Gamma, \neg A \Rightarrow \Delta$ with $\beta > \alpha$,
- if $\vdash^{\alpha}_{\rho} \Gamma, A \Rightarrow B, \Delta$, then $\vdash^{\beta}_{\sigma} \Gamma \Rightarrow \mathsf{C}(\ulcorner A \urcorner, \ulcorner B \urcorner), \Delta$ with $\beta > \alpha, \sigma > \rho$
- if $\vdash^{\alpha}_{\rho} \Gamma \Rightarrow \varphi, \Delta$, and $\vdash^{\beta}_{\rho} \Gamma, \psi \Rightarrow \Delta$, then $\vdash^{\gamma}_{\sigma} \Gamma \Rightarrow \mathsf{C}(\ulcorner A \urcorner, \ulcorner B \urcorner)\, \gamma > \alpha, \beta, \sigma > \rho$

**THEOREM**

$(A,B) \in \mathrm{I}_\Psi$ iff $\mathsf{TSC}^\omega \vdash \Rightarrow \mathsf{C}(\ulcorner A \urcorner, \ulcorner B \urcorner)$ iff $\mathsf{TSC}^\omega \vdash A \Rightarrow B$.

THEOREM

$(A,B) \in \mathrm{I}_\Psi$ iff $\mathsf{TSC}^\omega \vdash \Rightarrow \mathsf{C}(\ulcorner A \urcorner, \ulcorner B \urcorner)$ iff $\mathsf{TSC}^\omega \vdash A \Rightarrow B$.

By letting

$$\begin{aligned} \mathrm{Tr}\ulcorner A \urcorner &:= \mathsf{C}(\ulcorner \top \urcorner, \ulcorner A \urcorner) \\ \mathrm{F}\ulcorner A \urcorner &:= \mathsf{C}(\ulcorner A \urcorner, \ulcorner \bot \urcorner) \end{aligned}$$

COROLLARY

For $A \in L_{\mathrm{Tr},\mathrm{F}} \cap L_C$:

$$\begin{aligned} &\mathsf{TS}^\omega \vdash \Rightarrow A \quad \text{iff} \quad \mathsf{TSC}^\omega \vdash \Rightarrow A \quad \text{iff} \quad A \in \mathrm{I}_\Phi^+ \\ &\mathsf{TS}^\omega \vdash \Rightarrow \neg A \quad \text{iff} \quad \mathsf{TSC}^\omega \vdash \Rightarrow \neg A \quad \text{iff} \quad A \in \mathrm{I}_\Phi^- \end{aligned}$$

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# SUMMING UP

- The standard picture of semantic groundedness is perhaps the most widespread solution to paradoxes.
- It is usually tied with a paracomplete logic which restricts negation (or implication).
- We have just seen that this is only part of the story, and if we can rightly say that its external logic is indeed paracomplete, its internal logic is essentially substructural and in particular irreflexive.
- However, if one aims at a unified solution to paradox, we might (but I won't go so far) even say that the logic of semantic groundedness is irreflexive.