The least of all evils

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(slides at carlonicolai.github.io)

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theories

- $\mathcal{L}_{\mathrm{C}}$ results from expanding a first-order language $\mathcal{L}$ with the **binary** predicate C.

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- $\Gamma, \Delta, \Theta, \Lambda, \ldots$ are multisets of formulae of $\mathcal{L}_{\mathrm{C}}$.
- $\ulcorner \cdot \urcorner$ *somehow* yields a name for each sentence of the language.
- on occasion I will work with $\mathcal{L}_{\mathbb{N}}$, which yields a proper way of dealing with formal syntax

$$(\text{ref}^-)\quad \Gamma, A \Rightarrow A, \Delta \quad \text{with } A \in \mathrm{AtFml}_{\mathcal{L}}$$

$$(\text{cut})\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\neg\text{l})\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}$$

$$(\neg\text{r})\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

$$(\wedge\text{l})\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

$$(\wedge\text{r})\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$(\forall\text{l})\ \frac{\Gamma, \forall x A, A(s) \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta}$$

$$(\forall\text{r})\ \frac{\Gamma \Rightarrow A(y), \Delta}{\Gamma \Rightarrow \Delta, \forall x A}\ y \notin \mathrm{FV}(\Gamma, \Delta, \forall x A)$$

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Observation

This logic is the substructural 'dual' of **FDE**: ${}^{\Gamma\Rightarrow\Delta}/{}_{\Gamma'\Rightarrow\Delta'}$ is admissible iff $\bigwedge \Gamma \to \bigvee \Delta \Rightarrow \bigwedge \Gamma' \to \bigvee \Delta'$ is FDE-valid.

$$(\mathrm{Cl})\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, \mathrm{C}(\ulcorner A \urcorner, \ulcorner B \urcorner) \Rightarrow \Delta} \qquad (\mathrm{Cr})\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow \mathrm{C}(\ulcorner A \urcorner, \ulcorner B \urcorner), \Delta}$$

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... from which one can easily obtain (assuming $\Gamma \Rightarrow \Delta, \top$):

$$(\mathrm{Tr}\ \mathrm{l})\ \frac{A, \Gamma \Rightarrow \Delta}{\mathrm{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow \Delta} \qquad (\mathrm{Tr}\ \mathrm{r})\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner A \urcorner}$$

$\Gamma \Rightarrow \Delta, \varphi$ for φ an axiom of PA in $\mathcal{L}_{\mathbb{N}}$

$$(=\mathrm{r})\ \frac{\Gamma \Rightarrow s^\circ = t^\circ, \Delta}{\Gamma \Rightarrow \mathrm{Tr}\,(s = t), \Delta} \qquad (=\mathrm{l})\ \frac{\Gamma, s^\circ = t^\circ \Rightarrow \Delta}{\Gamma, \mathrm{Tr}\,(s = t) \Rightarrow \Delta}$$

$$(\mathrm{Tr}\,1)\ \frac{\Gamma \Rightarrow \mathrm{Tr}\,(x), \Delta}{\Gamma \Rightarrow \mathrm{Tr}\,\ulcorner \mathrm{Tr}\,(\dot{x})\urcorner, \Delta} \qquad (\mathrm{Tr}\,2)\ \frac{\Gamma, \mathrm{Tr}\,(x) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}\,\ulcorner \mathrm{Tr}\,(\dot{x})\urcorner \Rightarrow \Delta}$$

$$(\mathrm{Tr}\,\neg 1)\ \frac{\Gamma \Rightarrow \neg\mathrm{Tr}\,(\sigma), \Delta}{\Gamma \Rightarrow \mathrm{Tr}\,(\neg\sigma), \Delta} \qquad (\mathrm{Tr}\,\neg 2)\ \frac{\Gamma, \neg\mathrm{Tr}\,(\sigma) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}\,(\neg\sigma), \Delta}$$

$$(\mathrm{Tr}\,\wedge 1)\ \frac{\Gamma \Rightarrow \mathrm{Tr}\,(\sigma), \mathrm{Tr}\,(\tau), \Delta}{\Gamma \Rightarrow \mathrm{Tr}\,(\sigma \wedge \tau) \Rightarrow \Delta} \qquad (\mathrm{Tr}\,\wedge 2)\ \frac{\Gamma, \mathrm{Tr}\,(\sigma), \mathrm{Tr}\,(\tau) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}\,(\sigma \wedge \tau) \Rightarrow \Delta}$$

$$(\mathrm{Tr}\,\forall 1)\ \frac{\Gamma \Rightarrow \forall t\,\mathrm{Tr}\,(\sigma(t/v)), \Delta}{\Gamma \Rightarrow \mathrm{Tr}\,(\forall v\sigma), \Delta} \qquad (\mathrm{Tr}\,\forall 2)\ \frac{\Gamma, \forall t\,\mathrm{Tr}\,(\sigma(t/v)) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}\,(\forall v\sigma) \Rightarrow \Delta}$$

It is possible to formulate compositional extensions of **LPC**, e.g.:

$$\frac{\Gamma, \mathrm{C}(\sigma, \delta) \Rightarrow \Delta \qquad \Gamma, \mathrm{C}(\tau, \delta) \Rightarrow \Delta}{\Gamma, \mathrm{C}(\sigma \wedge \tau, \delta) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \mathrm{C}(\sigma, \delta), \mathrm{C}(\tau, \delta), \Delta}{\Gamma \Rightarrow \mathrm{C}(\sigma \wedge \tau, \delta), \Delta}$$

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semantics

Nicolai&Rossi 2018

LPC – and its compositional extension – has a natural model $(\mathbb{N}, X_\Phi)$ where X_Φ is a fixed point of a monotone operator Φ such that:

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- with $(\mathbb{N}, X) \vDash^* \Gamma \Rightarrow \Delta$ iff all $\gamma \in \Gamma$ are true/both/neither only if there is a $\delta \in \Delta$ that is true/both, we have:

$$\#(\Gamma, A \Rightarrow B, \Delta) \in X \;\; \textbf{iff} \;\; \#(\Gamma \Rightarrow \mathrm{C}(\ulcorner A \urcorner, \ulcorner B \urcorner), \Delta) \in X.$$

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Observation

$(\mathbb{N}, X_{\Phi\restriction\mathrm{Tr}}) \vDash \mathrm{KF}\restriction$ iff $(\mathbb{N}, X_{\Phi\restriction\mathrm{Tr}}) \vDash \mathrm{PKF}\restriction$ iff $(\mathbb{N}, X_{\Phi\restriction\mathrm{Tr}}) \vDash^* \mathrm{RKF}\restriction$

LPC$^\infty$ is obtained from LPC in $\mathcal{L}_{\mathbb{N}}$ by omitting free variables and:

- adding axioms $\Gamma \Rightarrow r = s, \Delta$ and $\Gamma, r = s \Rightarrow \Delta$; where r, s are closed terms of $\mathcal{L}_{\mathrm{C}}^{\mathbb{N}}$ and, respectively, $r^{\mathbb{N}} = s^{\mathbb{N}}$ and $r^{\mathbb{N}} \neq s^{\mathbb{N}}$.
- Replacing ($\forall$r) with:

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Proposition

For $\mathcal{I}_{\Phi}$ the (Π^1_1-complete) minimal fixed point of Φ, one has (by employing cut-elimination):

$$A \Rightarrow B \in \mathcal{I}_{\Phi} \quad \textbf{iff} \quad \vdash^{\alpha}_{\mathsf{LPC}^{\infty}} \Rightarrow \mathrm{C}(\ulcorner A \urcorner, \ulcorner B \urcorner) \text{ for some } \alpha < \omega_1^{\mathsf{ck}}$$

proof theory

- Cut-elimination for theories with non-logical rules such as (Cl) and (Cr) is a piece of cake for non-contractive systems (Cantini 2003, Grišin 1982): derivation trees don't 'grow' in the reduction, so one can simply count the applications of Tr- or C-rules.

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- When contraction is admissible, one has to find finer grained measures, and the task becomes notoriously hard. To my knowledge, there is no **simple** reduction strategy for fully structural, transparent theories of truth (and consequence), either for full or partial cut elimination. But...

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- ...there is one for LPC (and extensions thereof), and RKF$\restriction$.

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- If $\mathcal{D}$ ends with

$$\frac{\gamma_1^1, \ldots, \gamma_n^1 \Rightarrow \delta_1^1, \ldots, \delta_m^1, A \qquad B, \gamma_1^2, \ldots, \gamma_n^2 \Rightarrow \delta_1^2, \ldots, \delta_m^2}{\gamma_1^3, \ldots, \gamma_n^3, \mathrm{C}(\ulcorner A \urcorner, \ulcorner B \urcorner) \Rightarrow \delta_1^3, \ldots, \delta_m^3}$$

  then

$$\kappa(\mathrm{C}(\ulcorner A \urcorner, \ulcorner B \urcorner)) = \max(\kappa(A), \kappa(B)) + 1$$
$$\kappa(\gamma_i^3) = \max(\kappa(\gamma_i^1), \kappa(\gamma_i^2)) \;\; 1 \leq i \leq n$$
$$\kappa(\delta_j^3) = \max(\kappa(\delta_j^1), \kappa(\delta_j^2)) \;\; 1 \leq j \leq m$$

Lemma (Inversion)

All rules of LPC are κ-invertible: the C-complexity of the complex formulae is ***no greater*** *than the one of their constituents, and* ***stricly smaller*** *for $\kappa > 0$.*

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For instance: If $\mathcal{D} \vdash_{\mathsf{LPC}} \Gamma^1, \mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \Delta^1$, then there are $\mathcal{D}' \vdash_{\mathsf{LPC}} \Gamma^{10} \Rightarrow \Delta^{10}, \varphi$ and $\mathcal{D}'' \vdash_{\mathsf{LPC}} \psi, \Gamma^{11} \Rightarrow \Delta^{11}$ with $\kappa(\Gamma^{10}), \kappa(\Gamma^{11}) \leq \kappa(\Gamma^1)$, with $\kappa(\Delta^{10}), \kappa(\Delta^{11}) \leq \kappa(\Delta^1)$, and

$$\kappa(\varphi), \kappa(\psi) \leq \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)), \text{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) = 0, \text{ or}$$
$$\kappa(\varphi), \kappa(\psi) < \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)), \text{ if } \kappa(\mathrm{C}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)) > 0.$$

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Lemma

All ***truth rules*** *of* $\mathsf{RKF}\restriction$ *are κ-invertible – again, with a strict inequality if the active formula has Tr-complexity > 0.*

Proposition ($I\Delta_0 + \mathrm{superexp}$)

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Proof strategy. For φ the cut formula, the proof is by main induction on $\kappa(\varphi)$, and side inductions on the complexity of φ and on the level of the cut (i.e. $d_0 + d_1$). This yields the usual superexponential bound.

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Corollary

LPC$^\infty$ admits full cut elimination.

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Let $\mathcal{D}_0$, $\mathcal{D}_1$ be cut-free derivations, in $\mathsf{RKF}\!\upharpoonright$, *of* $\Gamma^1 \Rightarrow \Delta^1, \mathrm{Tr}\,(t)$ *and* $\mathrm{Tr}\,(t), \Gamma^2 \Rightarrow \Delta^2$ *respectively. Then we can find a derivation $\mathcal{D}$ of* $\Gamma \Rightarrow \Delta$ *with* $\kappa(\Gamma^k) \leq \kappa(\Gamma)$ *and* $\kappa(\Delta^k) \leq \tau(\Delta)$ *and* $k \in \{1, 2\}$.

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Lemma (Tr-cut elimination)

Let $\mathcal{D}_0$, $\mathcal{D}_1$ be cut-free derivations, in RKF$\upharpoonright$, *of* $\Gamma^1 \Rightarrow \Delta^1, \mathrm{Tr}\,(t)$ *and* $\mathrm{Tr}\,(t), \Gamma^2 \Rightarrow \Delta^2$ *respectively. Then we can find a derivation $\mathcal{D}$ of* $\Gamma \Rightarrow \Delta$ *with* $\kappa(\Gamma^k) \leq \kappa(\Gamma)$ *and* $\kappa(\Delta^k) \leq \tau(\Delta)$ *and* $k \in \{1, 2\}$.

Corollary ($I\Delta_0$(superexp))

RKF$\upharpoonright$ is **conservative** over PA: if $\Rightarrow \varphi$ is derivable in RKF$\upharpoonright$ and $\varphi \in \mathcal{L}_{\mathbb{N}}$, then there is a derivation of $\Rightarrow \varphi$ without Tr-cuts, i.e. a PA-derivation of $\Rightarrow \varphi$. A fortiori, RKF$\upharpoonright$ is consistent if PA is.

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- The version of $\mathsf{CT}\upharpoonright$ with restricted initial sequents – a subtheory of $\mathsf{RKF}\upharpoonright$ – admits, by contrast, a smooth cut-elimination procedure.
- What I presented for truth can be extended to a *compositional theory of consequence* in the style of $\mathsf{RKF}\upharpoonright$.

- I've presented a cluster of nonclassical systems for (transparent) truth and consequence.
- Such (substructural) theories stand out among their fellow villains:
  - Unlike other substructural options, they have a nice semantics
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