LECTURE 1
PARADOX AND CONCEPTS

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CARLO NICOLAI
KING'S COLLEGE LONDON

1. Structure

In these lectures I want, essentially, to survey some of the most common formal approaches to *concept application* or *property predication*. It is an opinionated introduction, as I will forcefully argue for some of such options along the way.

There are three fundamental domains of application – although deeply intertwined – of such theories:

- *Logic and the foundations of mathematics*: the quest for a simple, but powerful theory of concept application in which one can retrieve a significant part of mathematics;
- *Semantics and Paradox*: a satisfactory resolution of the paradoxes of predication has an obvious impact on the propositional Liar and a formulation of a semantics for natural language;
- *Metaphysics*: a comprehensive *type-free* theory of this sort would amount to a much needed alternative to the revived interest in higher-order theorising in metaphysics.

2. Paradox

Following a powerful metaphor, reality is like a language: the language one would need to write the 'book of the world' (Sider, 2011). Its fundamental properties and relations are 'joint-carving': the structure of such *concepts* matches perfectly with the structure of reality. Traditionally, the problem of providing appropriate structure to concepts has been a fundamental problem in the philosophy of mathematics (Linnebo, 2018), and in semantics (Bealer and Mönnich, 2003).

The metaphor of the language of reality is dangerous though. Let $\mathcal{L}$ be a first order language. For any unary $\varphi(x)$ in $\mathcal{L}$, we assume a closed term $\{x \mid \varphi\}$ – read: 'the concept associated with the predicate $\varphi(v)$'.[1]

Proposition 1 (Russell). *The principle*

$$\text{(NA)} \qquad \forall x(x \in \{v \mid \varphi\} \leftrightarrow \varphi(x))$$

is classically inconsistent.

Proof. Apply (NA) to $\{x \mid x \notin x\}$. □

REMARK 1. Notice that neither extensionality nor any quantificational rule – if one formulates (NA) in quantifier-free predicate logic – is required. Alternatively, one may drop abstracts and simply resort to

$$\text{(NC)} \qquad \exists y \forall x(x \in y \leftrightarrow \varphi(x))$$

[1] Of course the paradox generalises to relations.

but the quantifier rules become essential in the derivation of the paradox.

The main question that I want to investigate in the lectures is: what's the best formal theory of conceptual structure?

3. Typing

Russell's own reaction to the paradox was to stratify the language. In addition to first-level variables $x, y, z, \ldots$, one has higher-order variables X^n for each type n, so that:

- if $s \in t$ is well-formed, then s is of type n and t is of type $n+1$;
- for a unary $\varphi(X^n)$, $\{X^n \mid \varphi\}$ is of type $n+1$.

Russell's simple theory of types (or higher-order logic) (HOL) has then axioms

$$\forall X^n(X^n \in \{X^n \mid \varphi\} \leftrightarrow \varphi(X^n)) \tag{TCA}$$

STT is consistent: given a set M_0 of individuals, a model $\mathcal{M} \vDash$ HOL is obtained, for instance, by letting $M_1 := \mathcal{P}(M_0), M_2 := \mathcal{P}(M_2)\ldots$ as domains for the different, typed variables, and interpreting properties and relations accordingly.

The typing strategy displays, however, severe drawbacks to be considered as a comprehensive theory:

- No space for cross-type concepts
- No self-application
- No quantification over hierarchies

To this camp belong the variety of type-theories employed in the standard possible worlds analysis of propositions, properties, relations.

4. Restricting the logic

What if the structure of concept application is inherently *nonclassical*?

Let $\mathcal{L}$ contain the logical symbols $\neg, \wedge, \vee, \exists, \forall, \equiv$ – $\rightarrow$ and $\leftrightarrow$ are defined as usual. We assume some explicit machinery to generate abstraction terms (Feferman, 1984).[2] I call $\mathcal{L}$ the *ground* language, $\mathcal{L}_\in := \mathcal{L} \cup \{\in\}$, $\mathcal{L}_\in^\equiv := \mathcal{L}_\in + \equiv$.

The fundamental idea of this approach is to maintain a form of (NA)

$$\forall x(x \in \{v \mid \varphi\} \equiv \varphi(x)), \quad \varphi \in \mathcal{L}_\in \tag{NA$_\equiv$}$$

(with, crucially, $\equiv \neq \leftrightarrow$), by revising one's understanding of logical connectives. Here we consider the well-known *Strong Kleene* clauses – but other familiar monotone evaluation schemata work as well. One starts with a classical model $\mathcal{M}_0$ of $\mathcal{L}$ – one that satisfies our structural principles for abstraction.

Definition 1. Given $\mathcal{M} = (\mathcal{M}_0, E)$ – where $E \subseteq \mathrm{Form}^1_{\mathcal{L}_\in}$ is the interpretation of $\in$ –, a partial valuation is a function $|\cdot|_{\mathrm{sk}} \colon \mathrm{Sent}_{\mathcal{L}_\in} \longrightarrow \{1, 1/2, 0\}$ defined recursively as:

$$|\varphi|_{\mathrm{sk}} = \begin{cases} 1, & \text{if } \mathcal{M}_0 \vDash \varphi \\ 0, & \text{otherwise} \end{cases} \quad \varphi \text{ atomic of } \mathcal{L}$$

$$|s \in t|_{\mathrm{sk}} = 1, \text{ iff } E(s^{\mathcal{M}_0}, t^{\mathcal{M}_0}) \qquad |\neg\varphi|_{\mathrm{sk}} = 1 - |\varphi|_{\mathrm{sk}}$$

$$|\varphi \wedge \psi|_{\mathrm{sk}} = \min(|\varphi|_{\mathrm{sk}}, |\psi|_{\mathrm{sk}}) \qquad |\varphi \vee \psi|_{\mathrm{sk}} = \max(|\varphi|_{\mathrm{sk}}, |\psi|_{\mathrm{sk}})$$

[2]For instance, we can assume a basic theory of sequences and define:

$$\{x \mid \varphi(x, \vec{y})\} := (\ulcorner\varphi\urcorner, \vec{y})$$

where $\ulcorner\cdot\urcorner$ is some given coding procedure.

$$|\forall x\varphi|_{\mathrm{sk}} = \min\{\varphi(a) \mid a \in M\} \qquad |\exists x\varphi|_{\mathrm{sk}} = \max\{\varphi(a) \mid a \in M\}$$

One then defines $\mathcal{M} \vDash_{\mathrm{sk}} \varphi :\Leftrightarrow |\varphi|_{\mathrm{sk}} = 1$.

REMARK 2. The Strong Kleene truth functions are *monotonic*: if $|\vec{\varphi}|_{\mathrm{sk}} \leq |\vec{\psi}|_{\mathrm{sk}}$, then $\mathrm{F}(|\vec{\varphi}|_{\mathrm{sk}}) \leq \mathrm{F}(|\vec{\psi}|_{\mathrm{sk}})$.

An appropriate extension E for the predication relation is obtained by a standard inductive procedure. A model $\mathcal{M} = (\mathcal{M}_0, E)$ is *total* if $|s \in t|_{\mathrm{sk}} = 1$ or $|s \in t|_{\mathrm{sk}} = 0$ for any s, t. It is *partial* otherwise.

LEMMA 1. *Given any ground model $\mathcal{M}_0$ we can find a partial model $\mathcal{M} = (\mathcal{M}_0, E)$ of $\mathcal{L}_\in$ satisfying, for all a in M, and formulae $\varphi : \mathcal{L}_\in$:*

$$|a \in \{x \mid \varphi\}|_{\mathrm{sk}} = |\varphi(a)|_{\mathrm{sk}} \tag{1}$$

Proof. For $\mathcal{M} = (\mathcal{M}_0, E)$, define an operator $\Gamma(\mathcal{M}) = (\mathcal{M}, \Gamma(E))$ with, for all a in M:

$$(a, \{v \mid \varphi\}) \in \Gamma(E) \ \text{ if } \ (\mathcal{M}, E) \vDash_{\mathrm{sk}} \varphi(a)$$

Γ is monotone: $E_0 \subseteq E_1$ entails $\Gamma(E_0) \subseteq \Gamma(E_1)$. Therefore, it has a fixed point (Knaster-Tarski Theorem). Starting with $\emptyset$, we obtain for instance the least fixed point $\mathcal{I}_E$ satisfying (1). $\square$

A formal system satisfied by such $(\mathcal{M}_0, \mathcal{I}_E)$ is readily obtained. Strong Kleene logic, extended with a basic (classical) structural theory of abstracts, can be consistently extended with the schema ($\mathrm{NA}_\equiv$), where

$$(\mathcal{M}_0, E) \vDash_{\mathrm{sk}^*} \varphi \equiv \psi \ \text{ iff } \ |\varphi|^E_{\mathrm{sk}} = |\psi|^E_{\mathrm{sk}}.$$

REMARK 3. The choice of Strong Kleene is a natural one, but it's not at all the only one. Paraconsistent, substructural approaches can be developed in a similar way. The shortcomings of the SK approach apply to these approaches as well.

Such shortcomings are fundamentally two:

- Strong-Kleene logic is clumsy: it does not have theorems such as $\varphi \to \varphi$, $\varphi \to \neg\neg\varphi$, $\varphi \to \varphi \vee \psi$. Actually, it does not have theorems at all.
- If one is after a fundamental theory, giving up classical logic means a substantial abductive disadvantage. This can be clearly measured (Halbach and Nicolai, 2018; Nicolai, 2018).
- Even if one wanted to endorse ($\mathrm{NA}_\equiv$), $\equiv$ cannot be used in the properties and relations one wants to instantiate. This is a critical asymmetry. The alternative is to endorse rules of inference:

  $$\frac{x \in \{v \mid \varphi\}}{\varphi(x)} \qquad\qquad \frac{\varphi(x)}{x \in \{v \mid \varphi\}}$$

  This, however, would insert new metatheoretic elements into the theory without addressing the weakness concern.

5. Type-free, classical predication

The nonclassical theory characterised by ($\mathrm{NA}_\equiv$) above is on a clear message:

MATCH: to every predication (of $\mathcal{L}_\in$) there corresponds a concept application.

We have seen, however, that there are good reasons to abandon nonclassical approaches that underlie MATCH. We now consider a classical approach due essentially to Peter Aczel and Sol Feferman that is still based on the semantic picture above.

To formulate it, we define a new language $\mathcal{L}_{\in}^{+-}$ which contains, together with $\in$ and all the primitives for the abstraction support, its *dual* $\bar{\in}$. *Positive* formulas can be build from $\mathcal{L}_{\in}$-formulas in the following way:

- bring any φ in prenex disjunctive normal form;
- replace, in the result, any occurrence of $\neg \in$ by $\bar{\in}$, and $\neg\bar{\in}$ by $\in$.

For any φ of $\mathcal{L}_{\in}$, we call φ^+ the result of this process, and let $\varphi^- := (\neg\varphi)^+$.

Definition 2 (PNA). The theory of positive abstraction is obtained by extending the abstraction base with

$$\forall x(x \in \{v \mid \varphi\} \leftrightarrow \varphi^+(x)) \tag{PA$^+$}$$

$$\forall x(x \,\bar{\in}\, \{v \mid \varphi\} \leftrightarrow \varphi^-(x)) \tag{PA$^-$}$$

for any formula of $\mathcal{L}_{\in}$.

In addition, one can stipulate that our predication relation is consistent:

$$\forall x, y \,\neg(x \in y \wedge x \,\bar{\in}\, y) \tag{CONS}$$

Lemma 2. PNA *is consistent. The fixed point construction of Lemma 1 can be adapted to satisfy* (PA$^+$), (PA$^-$), (CONS).

Proof. The basic idea is to construct – as in the case of $\mathcal{I}_E$ above – a pair $(\mathcal{I}_E^+, \mathcal{I}_E^-)$ from atomic $\mathcal{L}$-formulas – therefore, positive formulas – by iterating clauses that preserve the positive status in iterating $\in$ and $\bar{\in}$. Then one simply 'closes off' $(\mathcal{M}_0, \mathcal{I}_E^+, \mathcal{I}_E^-)$ by assigning value 0 also to undefined pairs object-formulas. □

Back to the conceptual point, to the question: which predicates of our language correspond to concepts? PNA has a quick answer, the positive ones. However, this restriction does not have a clear motivation independent from the containment of the role of negation in the paradoxical phenomena.

Exercise 1. What happens to the concept $r = \{v \mid v \notin v\}$ in PNA?

Our fundamental conceptual structure, for all we know, should include logical and mathematical concepts. It is especially with this latter feature that we see in full the shortcomings of theories like PNA. First, one has:

Proposition 2. PNA *conservatively extends the abstraction base.*

Proof. The model construction above is essentially a model expansion argument: any $\mathcal{M}_0$ can be expanded to a suitable $(\mathcal{M}_0, \mathcal{I}_E^+, \mathcal{I}_E^-)$. □

One can nonetheless try to develop a theory of mathematical objects, in particular *classes*, in PNA:

$$\mathrm{Cl}(x) :\leftrightarrow \forall u(u \in x \vee u \,\bar{\in}\, x)$$

Since $\mathrm{Cl}(x)$ is positive, $x \in \{v \mid \mathrm{Cl}(v)\} \leftrightarrow \mathrm{Cl}(x)$ holds.

Some positive developments are possible:

Lemma 3.

(i) *There are empty and universal classes.*
(ii) *Classes are closed under intersection, union, complement, union and intersection of a family of sets.*

Proof. Ad (i): Let $V = \{v \mid v = v\}$ and $\varnothing = \{v \mid v \neq v\}$. It is worth noting that V determines some non-trivial instances of self-application: $V \in V$.

Ad (ii): we show the complement. Given a class c, let $-c := \{v \mid v \notin c\}$. One needs to show that, for any u, $u \in -c \vee u \mathrel{\bar{\in}} -c$. Suppose not. Since c is a class, this assumption is equivalent to saying that there is some u such that $u \in c \wedge u \mathrel{\bar{\in}} c$. □

EXERCISE 2. Does $\varnothing \mathrel{\bar{\in}} \varnothing$ hold?

However, these should count as success:

- $\mathrm{Cl} \notin \mathrm{Cl}$: the class of all classes is not a class.
- classes are not closed under power set $\mathcal{P}(a) = \{v \mid \mathrm{Cl}(v) \wedge v \subseteq a\}$ – since $x \in \mathcal{P}(V) \leftrightarrow x \in \mathrm{Cl}$.
- The natural number system cannot be developed in PNA.

REMARK 4. One could try to take the natural numbers for granted, and develop a theory which is akin in strength to the theory PUTB of Halbach (2014). However, this theory has only $\mathrm{ACA}_{<\varepsilon_0}$ as upper-bound.

References

Bealer, G. and Mönnich, U. (2003). Property theories. In Gabbay, D. and Guenthner, F., editors, *Handbook of Philosophical Logic, Volume 10*, pages 143–248. Kluwer Academic Publishers.

Feferman, S. (1984). Towards useful type-free theories i. *Journal of Symbolic Logic*, 49(1):75–111.

Halbach, V. (2014). *Axiomatic theories of truth. Revised edition.* Cambridge University Press.

Halbach, V. and Nicolai, C. (2018). On the costs of nonclassical logic. *Journal of Philosophical Logic*, 47(2):227–257.

Linnebo, O. (2018). *Thin Objects.* Oxford: Oxford University Press.

Nicolai, C. (2018). Provably true sentences across axiomatizations of kripke's theory of truth. *Studia Logica*, 106(1):101–130.

Sider, T. (2011). *Writing the Book of the World.* Oxford University Press.

carlonicolai6@gmail.com ; carlo.nicolai@kcl.ac.uk