EPISTEMIC STABILITY AND COMMITMENT IN MATHEMATICAL THEORIES

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1. Invitation

Our best scientific and philosophical theories commit us to the existence of mathematical objects and to theorizing about them (eg Quine (1995), Putnam (1975)).

But how does the mathematical landscape look like? Are there portions of it that display a privileged ontological and/or epistemological status?

Much more humbly, I will try to address the following question:

> Can someone coherently adopt a foundational stance on mathematical objects and methods of proof whose justification and commitments do not require resources exceeding what is allowed by such stance?

2. Inexhaustibility and Implicit Commitment

Mathematical knowledge is often thought to be *inexhaustible* (Franzén, 2004). This is mostly because, by accepting a collection of axioms Ax_S for a theory S, and by recognizing the correctness of one's logical rules, one is *committed* to the claim 'all theorems of S are true', and therefore to the consistency of S. However:

FACT 1 (Essentially Gödel). *For S a formal mathematical theory containing basic arithmetic:*

(i) *S cannot prove the claim 'S is consistent': $S \nvdash \mathtt{Con}(S)$.*

(ii) *Over S, 'all theorems of S are true' entails $\mathtt{Con}(S)$:*

$$S + \mathtt{RFN}(S) \vdash \mathtt{Con}(S)$$

with $\mathtt{RFN}(S) = \{\forall x(\mathtt{Prov}_S(\ulcorner \varphi(\overline{x}) \urcorner) \to \varphi(x)) \mid \varphi \in \mathcal{L}_S\}$.

This leads to the following thesis (Feferman, 1991; Dean, 2015):

> IMPLICIT COMMITMENT THESIS (ICT): Anyone who accepts the axioms of a mathematical theory S is thereby also committed to accepting various additional statements which are expressible in the language of S but which are formally independent of its axioms.

3. A Challenge to ICT: Epistemic Stability

Several positions in the foundations of mathematics aim to some sort of *stability*.

A *foundational stance* is an informal conception of a mathematical domain or mode of reasoning, corresponding to a principled position in the philosophy of mathematics that one might coherently adopt (Mount and Waxman, 2019).

A *foundational equivalence thesis* for a formal mathematical theory S requires that there is a foundational stance that is extensionally equivalent to S (cf Table 1).[1]

	STANCE	OBJECT	FORMAL SYSTEM
Nelson (1986)	Ultrafinitism	numbers, functions, proofs	$\mathtt{I}\Delta_0 + \omega_1$ or equivalent
Tait (1981)	Finitism	numbers, functions, proofs	Primitive Recursive Arithmetic ($\mathtt{PRA}$)
Isaacson (1987)	First-Orderism	finite mathematics	Peano Arithmetic ($\mathtt{PA}$)
Feferman (2005)	Predicativism	numbers, sets thereof	Arithmetic Transfinite Recursion ($\mathtt{ATR}_0$)
Dedekind	Second-Orderism	numbers, sets thereof	Second-Order Arithmetic ($\mathtt{Z}_2$)

Table 1. Foundational Equivalence Theses for $\mathbb{N}$ and sets thereof

A system S is *epistemically stable* if there is a coherent rationale for a theorist to *accept* a mathematical theory without being *committed* to claims that are not consequences of its axioms.

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[1] One should think about these informal equivalence thesis as specific instances of a more general class including the Church-Turing thesis.

ICT and EPISTEMIC STABILITY are clearly at odds: pick any theory S which corresponds to some foundational stance according to some foundational equivalence thesis. If a theorist accepts S, by ICT she is committed to the consistency of S which, once properly formalized, amounts to an extensions of its theorems by fact 1(i), thereby violating EPISTEMIC STABILITY. So, which one is true?

4. A theory of implicit commitment

Let T, T' be (effective presentations of) first-order theories in $\mathcal{L}_{\mathbb{N}}$ (the language of arithmetic). We define the functor $\mathcal{C}(\cdot)$ that, given a theory, returns its *minimal, implicit* epistemic commitments.

REFLECTION: If T proves that all instances of a formula $\varphi(v)$ of $\mathcal{L}_{\mathbb{N}}$ are axioms, then $\mathcal{C}(T)$ proves that all numbers satisfy φ. More precisely:

$$\text{if } T \vdash \forall n(\ulcorner \varphi(\overline{n}) \urcorner \in T), \text{ then } \mathcal{C}(T) \vdash \forall n\, \varphi(n).$$

INVARIANCE: If T and T' are equi-consistent (proof-theoretically equivalent) – and there is an algorithm that tells us so –, then $\mathcal{C}(T) = \mathcal{C}(T')$.

INVARIANCE is based on the simple idea that, if T and T' have the same consequences, modulo simple translations of their languages, then they have the same implicit commitments.

REFLECTION is more subtle. Given an external presentation (axiomatization) of T, if T itself can formalize (codify) the act of axiomatizing T (externally) for *each* axiom φ, then T proves (internally) that all objects satisfy each such φ.

INVARIANCE and REFLECTION appear much weaker than the outright acceptance of the soundness of T. However:

PROPOSITION 1. *Given any reasonable T containing basic number theory, $\mathcal{C}(T)$ proves that T is sound: $\mathcal{C}(T) \vdash \mathtt{RNF}(T)$.*

I take proposition 1 to support ICT against ES. It may be reasonable to require an argument to justify the step from T to 'T is sound'. However, what is required to satisfy the conditions in proposition 1 appears to be even more uncontroversial.

5. Some objections and replies

Objection. One has to distinguish between what are the methods of proof and objects that are allowed by one's foundational stance, and what is allowed to formulate such stance. The latter may be stronger.

Reply. This will lead to strange consequences, close to self-defeat. Suppose that you are a $\mathtt{PRA}$-finitist. Then you may not be able to: (i) formulate your own foundational standpoint, or worse (ii) once given an axiom of $\mathtt{PRA}$, recognize that it is so.

Objection. Some foundational stances (eg Feferman's Predicativism) are 'closed under reflection', in the sense that by adding $\mathtt{RFN}(T)$ to one of its preferred systems will not increase their strength.

Reply. A first reaction is that even if one shows that one particular stance is not affected by prop. 1, still many claims of epistemic stability are affected. However, that some theories are 'closed under reflection' is more a feature of presentation than a real phenomenon. Given invariance, $\mathtt{ATR}_0$ is affected by prop. 1.

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