CHAPTER 1

Week 1
Sets, Classes, and The Cumulative Hierarchy

1. Sets

Sets are collections of objects. The collection of all students registered in the Advanced Logic module at King's College London in 2020 is a set. The collection of all prime numbers less than 1,000,000 is a set. This definition of set relies on a clear notion of 'object'. This, in turn, requires an uncontroversial metaphysics of objects, which we don't have. Luckily, for the purpose of developing mathematics – and therefore scientifically applicable mathematics – and portions of formal philosophy, we don't need that much.

There are two main ways to refer to sets. The first is by listing their elements – by enclosing them in curly brackets {, } –, as in

$$\{\text{Caravaggio}\}, \qquad \{\text{Michelangelo Merisi}\},$$
$$\{1, 2, 3\}, \qquad \{\sqrt{2}, \pi\}.$$

The second is by description, by means of the so called *abstraction terms*:

$$\{x \mid x \text{ is the painter of } \textit{The Calling of Saint Matthew}\}$$
$$\{x \mid x \text{ is a positive integer smaller than } 4\}$$

Notice, however, that *not all descriptions give rise to sets!* We will treat this point more carefully in the next section.

The identity conditions for sets – cf. Quine's 'there's no entity without identity' [**vOQ69**, p. 23] – are encapsulated in the well-known principle of extensionality:

PRINCIPLE OF EXTENSIONALITY: Two sets are identical if and only if they have the same elements. In symbols:

$$x = y \leftrightarrow \forall u(u \in x \leftrightarrow u \in y)$$

Sets are then completely characterized by their elements, and not by their *mode of presentation* – i.e. the way we refer to them. For instance:

$$\{\text{Caravaggio}\} = \{x \mid x \text{ is the painter of } \textit{The Calling of Saint Matthew}\},$$
$$\{x \mid x \text{ is a positive integer smaller than } 4\} = \{1, 2, 3\}.$$

Similarly, one has that (as far as we know):

$$\{x \mid x \text{ is an animal with a heart}\} = \{x \mid x \text{ is an animal with a kidney}\}.$$

Therefore, even though arguably the property of having a heart is not the same as the property of having a kidney, the *extension* of these two properties, i.e. the set of objects that have such properties, is the same. The philosophical question whether a scientific (broadly construed) worldview needs properties or concepts together with sets is still substantially open.[1]

As anticipated above, however, the entire universe of mathematical objects can be constructed without resorting to objects located in space and time such as Caravaggio. One of the fundamental building blocks of such construction is the following *set-existence principle*:

EMPTY SET PRINCIPLE: there is a set containing no elements:

$$\exists y \forall u (u \notin y)$$

EXERCISE 1.1. Show, using the EXTENSIONALITY PRINCIPLE, that there is a unique empty set.

Since there is only one empty set, we can safely denote it with the 'proper name' $\emptyset$. We collect a few simple facts concerning the empty set. As usual, we let

$$x \subseteq y \;:\leftrightarrow\; \forall u(u \in x \longrightarrow u \in y)$$
$$x \subset y \;:\leftrightarrow\; x \subseteq y \wedge x \neq y$$

FACT 1.

(i) *For any collection A, $\emptyset \subseteq A$*

(ii) $\emptyset \subseteq \emptyset$, $\emptyset \notin \emptyset$, $\{\emptyset\} \in \{\{\emptyset\}\}$, $\{\emptyset\} \nsubseteq \{\{\emptyset\}\}$.

PROOF. Exercise. *qed*

DEFINITION 1 (POWER SET). *The power set $\mathcal{P}(x)$ of a set x is the set $\{y \mid y \subseteq x\}$.*

[1] For an overview, see [**BM03**].

EXERCISE 1.2. Show that $\mathcal{P}(x)$, if it exists, is unique.

From the empty set, we can basically construct the entire universe of sets (and thus of mathematics). We only need to close the empty set under two basic operations. One is implicit in the definition of power set, and tells us that one *can always* collect subsets of a given set.

POWER SET PRINCIPLE: For any set x, $\mathcal{P}(x)$ exists and is a set.

The second operation is is a generalization of the well-known notion of *union* of sets. We know that

$$x \cup y := \{u \mid u \in x \vee u \in y\}$$

This operation (iterated finitely many times) enables us to define *finite unions*:

$$x_1 \cup \ldots \cup x_n \cup x_{n+1} := (x_1 \cup \ldots \cup x_n) \cup x_{n+1}.$$

But what about infinite ones? Suppose $x := \{x_i \mid i \in \mathbb{N}\}$. How do we collect all x_i together? We need a further operation

$$\bigcup x = \{u \mid (\exists y \in x)(u \in y)\}$$

Again implicit in this definition is the possibility of forming unions of all elements of any given set.[2]

PRINCIPLE OF UNION: For any set x, $\bigcup x$ exists and is a set.

Using the PRINCIPLE OF UNION, we can readily obtain the infinite union of all elements of x_i.

EXERCISE.

(i) Which collection is $\bigcup\{\{1\}, \{1, 2\}, \{1\}\}$?
(ii) Which collection is $\bigcup\{\{0\}, \{1\}, \{2\}, \{3\}, \ldots\}$?

The empty set, power sets, and unions give us basically all we need to outline the so called *cumulative hierarchy of sets*, the universe of all sets. I said 'outline', because a precise definition will only be available once the full axiomatic development of set theory will be given. However, at this

[2] A quick remark on notation. We write

$$(\exists u \in y)\, \varphi \qquad \text{for} \qquad \exists u(u \in y \wedge \varphi),$$

and

$$(\forall u \in y)\, \varphi \qquad \text{for} \qquad \forall u(u \in y \longrightarrow \varphi).$$

stage we can start by giving the first steps of this process, and leave the details for later.

We want to define the universe V of all sets in stages. So we put

$$\mathrm{V}_0 := \varnothing$$
$$\mathrm{V}_{n+1} := \mathcal{P}(\mathrm{V}_n)$$

Once we have completed the finite stages, we need to find a way to index later stages. This is the job of *Cantor's theory of ordinal numbers*, that we can only briefly introduce now. Roughly speaking, ordinals extend natural numbers $\mathbb{N}$ by introducing *infinite* numbers. The first infinite numbers is denoted with ω (read 'omega'). So we put:

$$\mathrm{V}_\omega := \bigcup\{\mathrm{V}_i \mid i \in \mathbb{N}\} = \{u \mid (\exists i \in \mathbb{N})(u \in \mathrm{V}_i)\}$$

The definition can then be iterated. So $\mathrm{V}_\omega + 1 := \mathcal{P}(\mathrm{V}_\omega)\ldots$ The resulting picture of V is displayed in Figure 1.

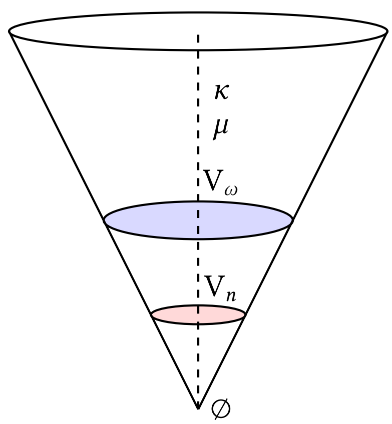

Figure 1. Cumulative hierarchy

Exercise. Show that $\mathrm{V}_3 = \mathrm{V}_2 \cup \mathcal{P}(\mathrm{V}_2)$. (Notice that this is true for every stage).

Exercise. Define the *rank* of a set x, written $\rho(x)$, to be the least α such that $x \subseteq \mathrm{V}_\alpha$. What is $\rho(\{\varnothing\})$? What is $\rho(\{\{\varnothing\},\{\{\{\varnothing\}\}\}\})$?

2. Classes

In introducing expressions of the form $\{x \mid \Phi(x)\}$, we have put no limits to the possibility of forming sets given a suitable specifying condition Φ. Such limits, however, exist.

FACT 2 (RUSSELL'S PARADOX). *The collection* $\mathrm{R} = \{x \mid x \notin x\}$ *is not a set.*

PROOF. Suppose R is a set. Since we assume classical logic, either $\mathrm{R} \in \mathrm{R}$ or $\mathrm{R} \notin \mathrm{R}$. If the former, then R is one of the sets that satisfies the condition $x \notin x$; therefore $\mathrm{R} \notin \mathrm{R}$, and so both $\mathrm{R} \in \mathrm{R}$ and $\mathrm{R} \notin \mathrm{R}$. If the latter, again R is one of the sets that satisfies the condition $x \notin x$. Therefore, $\mathrm{R} \in \mathrm{R}$ and $\mathrm{R} \notin \mathrm{R}$. In either case we obtain a contradiction. So R cannot be a set. *qed*

The totality of the collections we refer to are called *classes*. So all sets are classes, but not viceversa: classes that are not sets are called *proper classes*. It then follows from FACT 2 that R is a proper class.

But if not all conditions define a set, which ones do? The following principle gives us an answer.

> SUBSET PRINCIPLE: let $\Phi(\cdot)$ be a determinate property and x a set. Then $\{u \in x \mid \Phi(u)\}$ is a set.

But for the answer to be satisfactory, one also has to know what a determinate property is. To keep things simple,[3] the standard reply that set theory offers is that a property $\Phi(x)$ is determinate when it can be expressed in a specific formal language, the language $\mathcal{L}_\in$ of set theory. The SUBSET PRINCIPLE is then based on a *syntactic* restriction: the only properties that can be employed in defining sets are the ones that belong to a specific class of syntactic objects.

QUESTION 1. Is this notion of determinacy satisfactory?

COROLLARY 1. V *is not a set.*

PROOF. Suppose it is. Then by the SUBSET PRINCIPLE, the class $\{u \in \mathrm{V} \mid u \notin u\}$ is a set. But this contradicts FACT 2. *qed*

EXERCISE. Fill in the details of the proof of Corollary 1.

EXERCISE. Consider a set x and the set $y := \{u \in x \mid u \notin u\}$. Show that $y \notin x$.

[3]Notice that, depending on the answer, one might end up with a formulation of the subset principle that is difficult to even write down. For instance,

CHAPTER 2

Week 2
The Axioms of Set theory

The set theory that we study in the first part of the module – as we have seen, inspired by the cumulative hierarchy –, is called ZFC, standing for 'Zermelo-Fraenkel set theory with the axiom of Choice'. ZFC is a first-order theory. That is, it is obtained by extending pure predicate (aka first-order) logic with identity with a collection of first-order sentences, its *axioms*.

The language $\mathcal{L}_\in$ of ZFC is a language containing the usual *logical symbols*[1]

$$\neg, \vee, \wedge, \longrightarrow, \leftrightarrow, \forall, \exists, =$$

and *only one* nonlogical symbols, the membership (binary) relation $\in$. Therefore, formulas of $\mathcal{L}_\in$ are built from atomic formulas $x \in y$ and $x = y$ by closing them under the logical operations. When writing $\varphi(x_1, \dots, x_n)$, we will assume that all free variables in φ are among $x_1, \dots, x_n$.

Although we only have one type of objects in ZFC, we will avail ourselves with a way of referring to *classes*, especially *proper classes* – in particular, we will use capital letters for classes. In fact, this is what we have already done in the previous section, when for instance referring to

$$\mathrm{V} = \{x \mid x = x\},$$

or

$$\mathrm{R} = \{x \mid x \notin x\}.$$

Recall that the main difference between classes and sets is the way they are formed. The former satisfy the so-called *comprehension principle*:

$$\exists X \forall u(u \in X \leftrightarrow \varphi(u, v_1, \dots, v_n)),$$

where $v_1, \dots, v_n$ are called *parameters* (essentially, class talk is shorthand for formulas that *define them*, so that R is shorthand for $x \notin x$, and V is shorthand for $x = x$). The latter (sets) satisfy restricted principles such as

[1]Notice that identity is treated as a logical symbol.

the SUBSET PRINCIPLE that we have considered before, and that will correspond to a specific axiom schema that we will introduce in a moment.

1. Extensionality

The first axiom is a familiar one. It is often thought to be *constitutive* of the *concept* of set. Sets are fully characterized by their elements:

(EXTENSIONALITY) $\forall x \forall y (x = y \leftrightarrow \forall u (u \in x \leftrightarrow u \in y))$

'sets are identical iff they have the same elements'.

EXERCISE 2.1. In fact, only the direction

$$\forall x \forall y (\forall u (u \in x \leftrightarrow u \in y) \longrightarrow x = y)$$

of extensionality would suffice for our development of set theory. Why?

2. Pairing

The first set-existence axiom that we consider enables us to form sets containing two given elements.

(PAIRING) $\forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u = x \vee u = y))$

'Given x and y, there is the set $\{x, y\}$.'

LEMMA 1.

(i) *Given sets x, y, the set $\{x, y\}$ obtained by pairing is unique.*

(ii) *Given a set x, the* singleton $\{x\}$ *exists and is unique.*

PROOF. Exercise. For (ii), you need to use both PAIRING and EXTENSIONALITY. *qed*

It is an immediate consequence of EXTENSIONALITY that, for any x, y, $\{x, y\} = \{y, x\}$. In fact, the PAIRING AXIOM is often referred to as the axiom of *unordered* pairing. *Ordered pairs* – written (x, y) – are characterized by the principle

$$(x, y) = (u, v) \leftrightarrow (x = u \wedge y = v) \tag{1}$$

Ordered pairs (x, y) can be defined in ZFC as the set $\{\{x\}, \{x, y\}\}$ – this is the so-called Kuratowski definition of ordered pair. Of course, given any sets x, y, PAIRING enable us to construct $\{\{x\}, \{x, y\}\}$, and EXTENSIONALITY ensures its uniqueness.

EXERCISE 2.2. Show that Kuratowski's definition of ordered pair satisfies (1).

What we have said generalizes to triples, quadruples, quintuples, …It suffices to put

$$\begin{aligned}(x, y, z) &:= ((x, y), z) := \{\{(x, y)\}, \{(x, y), z\}\}\\ (x, y, z, u) &:= ((x, y, z), u) := \{\{(x, y, z)\}, \{(x, y, z), u\}\}\\ &\vdots \qquad\qquad \vdots\end{aligned}$$

Exercise 2.3. Show that

$$(x_1, \dots, x_n) = (y_1, \dots, y_n) \leftrightarrow x_1 = y_1 \wedge \dots \wedge x_n = y_n.$$

3. Union

The principle of union considered in the previous chapter is also an axiom of ZFC:

(UNION)

$\forall x \exists y \forall u (u \in y \leftrightarrow (\exists z \in x)\, u \in z)$

'Give a set x, the set $\bigcup x = \{u \mid (\exists z \in x)\, u \in z\} = \bigcup\{z \mid z \in x\}$ exists'.

In the previous chapter we have introduced the union of x, $\bigcup x$, as a generalization of the union of two sets. In ZFC, the order is reversed. We can define:

$$\begin{aligned}x \cup y &:= \bigcup\{x, y\}\\ x \cup y \cup z &:= (x \cup y) \cup z\\ &\vdots\end{aligned}$$

Exercise 2.4. Show that: $\bigcup\{x\} = x$; $\bigcup(x \cup y) = \bigcup x \cup \bigcup y$.

Exercise 2.5. Show that $\mathrm{P} := \{(x, y) \mid x, y \in \mathrm{V}\}$, the class of all ordered pairs, is a proper class. (Hint: we need to use the axiom of union twice).

4. Separation (Subset)

The next set-existence axiom is the precise counterpart of the SUBSET PRINCIPLE that we have considered in the previous section: for any formula $\varphi(u, v)$ of $\mathcal{L}_\in$:

(SEPARATION) $\forall x \exists y \forall u (u \in y \leftrightarrow u \in x \wedge \varphi(u, v))$

'given a set x, the subset of x satisfying φ is a set'.

The first thing to notice about SEPARATION is that, unlike PAIRING, it is not a single sentence, but is is an *axiom schema*, that is a recipe to generate infinitely many sentences. In fact, the expression $\varphi(u)$ in SEPARATION is a placeholder for an arbitrary formula of $\mathcal{L}_{\in}$.[2]

An immediate consequence of SEPARATION is the existence of the *intersection* and of the set-theoretic difference of two sets. Given sets x and y, the former can be defined as

$$x \cap y := \{u \in x \mid u \in y\},$$

whereas the latter as

$$x \setminus y := \{u \in x \mid u \notin y\}.$$

EXERCISE 2.6. Show that, given sets x, y, $x \cap y$ and $x \setminus y$ are unique.

EXERCISE 2.7. The *symmetric difference* of x and y, written $x \triangle y$, is defined as

$$(x \setminus y) \cup (y \setminus x).$$

Is the equation $\{\emptyset, \{\emptyset\}\} \triangle \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{\emptyset\}$ true or false?

The operation of intersection gives us a way to understand why the axiom schema of SEPARATION is often called 'Subset Axiom'. Consider in fact a class:

$$X = \{u \mid \varphi(u, v_1, \dots, v_n)\}.$$

Then SEPARATION tells us that

$$\exists y \forall u (u \in y \leftrightarrow u \in x \wedge \varphi(u, v_1, \dots, v_n)),$$

that is that the set

$$y = \{u \mid u \in x \wedge u \in X\} = X \cap x$$

exists.

Two sets x and y are called *disjoint* if $x \cap y = \emptyset$. We define, for x a nonempty set:

$$\bigcap x := \{u \mid (\forall y \in x)\, u \in y\}.$$

FACT 3.

(i) *For any x, y: $x \cap y = \bigcap\{x, y\}$.*

(ii) *For any class A of sets, $\bigcap A$ is a set.*

PROOF. Exercise. (Hint for (ii): it's important that A is a class of *sets*, and that there is an $a \in A$). *qed*

[2]This situation should not be unfamiliar: the rules of natural deduction are in fact rule schemata.

The notation just introduced enables us to extend our conceptual inventory for pairs and sequences. Give a pair $x = (u, v)$, we let

$$(x)_0 = u, \qquad (x)_1 = v$$

These operations – often called *projection functions* – can be properly defined in our setting by letting

$$(2) \qquad (x)_0 = \bigcup\bigcap x, \qquad (x)_1 = \begin{cases} \bigcup(\bigcup x \setminus \bigcap x), & \text{if } \bigcup x \neq \bigcap x; \\ \bigcup\bigcup x, & \text{otherwise.} \end{cases}$$

EXERCISE 2.8. Show that, for $x = (u, v)$, our definition of the projection functions (2) does indeed yield the correct results.

A final important aspect of our formulation of SEPARATION is that it only contains one parameter – the variable v in our formulation. However, we can show that, given PAIRING and projection, a more general formulation of SEPARATION is available, featuring arbitrarily many (finite) parameters:

(SEPARATION*) $$\forall x \exists y \forall u (u \in y \leftrightarrow u \in x \wedge \chi(u, v_1, \dots, v_n))$$

for any formula $\chi(u, v_1, \dots, v_n)$ of $\mathcal{L}_\in$.

In fact, we can let

$$\psi(u, v) := \exists v_1 \dots \exists v_n (v = (v_1, \dots, v_n) \wedge \chi(u, v_1, \dots, v_n)).$$

SEPARATION then entails that

$$\exists y \forall u (u \in y \leftrightarrow u \in x \wedge \psi(u, v_1, \dots, v_n)).$$

EXERCISE 2.9. Fill in the details in this argument showing that SEPARATION entails SEPARATION*.

5. Emptyset

In the previous chapter we have introduced the EMPTY SET PRINCIPLE. As an axiom, it is redundant, as it is a theorem of ZFC as presented here:

(EMPTYSET) $$\exists y \forall u (u \notin y).$$

The axiom of INFINITY below in fact will entail the existence of at least one set, that is it will entail that $\exists x\, x = x$.

LEMMA 2. *If* $\exists x\, x = x$, *then* $\exists y \forall u (u \notin y)$.

PROOF. We know by assumption that there is at least one set. Call one such set a. Define $\emptyset = \{u \in a \mid u \neq u\}$. I leave it as an exercise to complete the details (i.e.: which principles are employed in my reasoning?). *qed*

6. Power Set

Also the power set principle has a counterpart in the axioms of ZFC:

(POWER SET) $\forall x \exists y \forall u (u \in y \leftrightarrow u \subseteq x)$

'for any x, its power set $\mathcal{P}(x)$ exists and is a set.'

The *product* of two sets is defined as:

$$x \times y := \{(u, v) \mid u \in x \wedge v \in y\},$$

that is, $x \times y$ is the set of pairs whose first component belongs to x, and whose second component belongs to y. Notice that

$$x \times y \subseteq \mathcal{P}\mathcal{P}(x \cup y),$$

so that we can use SEPARATION to show that $x \times y$ is always a set:

$$x \times y = \{u \in \mathcal{P}\mathcal{P}(x \cup y) \mid \exists v \exists w (u = (v, w) \wedge v \in x \wedge w \in y\}. \tag{3}$$

EXERCISE 2.10. Explain why one needs to employ $\subseteq \mathcal{P}\mathcal{P}(x \cup y)$ in (3) to apply SEPARATION.

The definition of product can be generalized in a straighforward way:

$$x^n := x_1 \times \ldots \times x_n := (x_1 \times \ldots \times x_{n-1}) \times x_n.$$

DEFINITION 2. *An n-*ary relation *is a set of tuples $(u_1, \ldots, u_n)$, and as such is a subset of some $x_1 \times \ldots \times x_n$. An n-*ary relation *R^n on a set x is such that $R^n \subseteq x^n$.*

The *domain* of a binary relation R is the set

$$\mathrm{dom}(R) = \{u \mid (u, v) \in R\},$$

and its *range* the set

$$\mathrm{ran}(R) = \{v \mid (u, v) \in R\}.$$

EXERCISE 2.11. Show that $\mathrm{dom}(R) \subseteq \bigcup\bigcup R$ and that $\mathrm{ran}(R) \subseteq \bigcup\bigcup R$.

As a consequence, $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ are sets by SEPARATION and POWER SET (why?).

A binary relation $R \subseteq x^2$ is:

reflexive, if $(\forall u \in x)\, Ruu$,

irreflexive,	if $(\forall u \in x)\, \neg Ruu$,
symmetric,	if $(\forall u, v \in x)(Ruv \longrightarrow Rvu)$,
antisymmetric,	if $(\forall u, v \in x)((Ruv \wedge Rvu) \longrightarrow u = v)$,
connected,	if $(\forall u, v \in x)(u = v \vee Ruv \vee Rvu)$,
transitive,	if $(\forall u, v, w \in x)(Ruv \wedge Rvw \longrightarrow Ruw)$.

A binary relation is an *equivalence relation* if it is reflexive, symmetric, and transitive. For $\equiv$ an equivalence relation on x, and for any $u \in x$,

$$[u] = \{v \in x \mid u \equiv v\}$$

is called the *equivalence class* of u.

DEFINITION 3. *A binary relation f is* a function *if*

$$(x, y) \in f \wedge (x, z) \in f \longrightarrow y = z.$$

We introduce some notation for functions:

- f is a function *on* x if $x = \mathrm{dom}(f)$.
- f is an n-ary function on x if $\mathrm{dom}(f) = x^n$.
- f is a function *from* x to y (written: $f : x \longrightarrow y$) if $\mathrm{dom}(f) = x$ and $\mathrm{ran}(f) \subseteq y$.
- the set of *all functions* from x to y is written y^x.

EXERCISE 2.12. Considering that $y^x \subseteq \mathcal{P}(x \times y)$, show that y^x is a set.

- f is a function *onto* y, if $\mathrm{ran}(f) = y$.
- f is *one-to-one* if $f(x) = f(y) \longrightarrow x = y$.
- the *restriction* of f to x, written $f \restriction x$, is defined as:

$$f \restriction x = \{(u, v) \in f \mid u \in x\}.$$

- for f and g functions and $\mathrm{ran}(g) \subseteq \mathrm{dom}(f)$, the *composition* of f and g, written $f \circ g$, is the function with $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$ such that

$$\text{for all } x \in \mathrm{dom}(g),\, f \circ g(x) = f(g(x)).$$

- the *image* of a set x under f is

$$f''(x) = \{v \mid (\exists u \in x)\, v = f(u)\},$$

and the *inverse image* of x is

$$f_{-1}(x) = \{u : f(u) \in x\}$$

- for f a one-to-one function, the the *inverse* of f, written f^{-1}, is such that

$$f^{-1}(x) = y \text{ if and only if } f(y) = x.$$

A *family* F of sets – i.e. a collection of sets – is *disjoint* iff, for all $x, y \in F$, $x \cap y = \emptyset$. A *partition* of a set x is a disjoint family P of nonempty sets such that

$$x = \bigcup\{u \mid u \in P\}.$$

Exercise 2.13. For $\equiv$ an equivalence relation on x, define the *quotient of x by $\equiv$* as

$$x/\equiv := \{[y] \mid y \in x\}.$$

Show that $x/\equiv$ is a partition of x.

Finally, consider a partition P of x. It can be shown that it defines an equivalence relation, by letting:

$$u \equiv v \text{ if and only if } (\exists y \in P)(u \in y \wedge v \in y)$$

(exercise: verify this last claim, that is verify that it indeed satisfies the properties of an equivalence relation).

7. Other axioms

There are four additional axioms of ZFC that we will consider in detail in the following weeks, but that we will only mention for completeness.

The first is the main set existence axiom of ZFC – and the one that enables us to dispense with the empty set axiom: it states that there is an infinite set. Call a set x *inductive* if

$$\emptyset \in x \wedge (\forall u \in x)(u \cup \{u\} \in x).$$

Then:

(INFINITY) There is an inductive set.

The remaining two axioms are:

(REPLACEMENT)

$$\forall x \forall u \forall v(\varphi(x, u, w) \wedge \varphi(x, v, w) \longrightarrow u = v) \longrightarrow \forall x \exists y \forall u(u \in y \leftrightarrow (\exists v \in x)\varphi(v, u, w))$$

'If the class F defined by φ is a function and $\mathrm{dom}(F)$ is a set, then $\mathrm{ran}(F)$ is a set'.

(CHOICE) For every family of nonempty sets there is a choice function.

(FOUNDATION) Every non-empty set has an $\in$-minimal element.

DEFINITION 4. ZFC *is the first-order theory in* $\mathcal{L}_\in$ *whose axioms are* EXTENSIONALITY, PAIRING, UNION, SEPARATION, POWER-SET, INFINITY, REPLACEMENT, CHOICE, FOUNDATION.

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