Conservativity of Truth via Free-Cut Elimination

Carlo Nicolai
(joint with Luca Castaldo)

KING'S College LONDON

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(slides at carlonicolai.github.io)

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- A theory of truth is obtained by axiomatizing a **unary predicate** Tr for truth (to form $\mathcal{L}_{\mathrm{Tr}}$), e.g. by axioms:

$$\forall \varphi, \psi (\mathrm{Tr}\,(\varphi \wedge \psi) \leftrightarrow \mathrm{Tr}\,\varphi \wedge \mathrm{Tr}\,\psi),$$

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- Theories of truth have been employed in different theoretical contexts:
  - Foundations and epistemology of mathematics (Feferman's predicativism, recursive saturation, Franzen, Beklemishev, Cieśliński, Horsten).
  - Truth as a (quasi-)logical notion (deflationism and the philosophy of truth, semantic and logical paradoxes).

- Conservativity over **the base theory** B:
  - Philosophical interest: truth **supervenes** on non-semantic facts. It's a 'logico-linguistic' device:
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    - Cieśliński: epistemic lightness of truth.
- Conservativity of truth **over comprehension** (and vice versa):
  - Philosophical interest: ontological reductions, theories of implicit commitment.
  - Mathematical interest: predicativism, Π^0_1-ordinal analysis.

Main Property

Let $T[B]$ be a theory of truth over B. If $T[B] \vdash \varphi$, for $\varphi \in \mathcal{L}_B$, and **there's no cut** on $\mathrm{Tr}\,\psi$ in this proof, then $B \vdash \varphi$.

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Potential advantages of cut-elimination proofs (over semantic alternatives):

- **Uniform**, finitistic proofs
- No resort, even instrumental, to semantic notions of truth
- Simple (if all is well...)

Theories

- Logical Rules (**G1**, context-sharing)

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$$\underset{A \text{ atomic}}{A \Rightarrow A} \qquad \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Logical inferences are among the **non-weak inferences**, whereas weakening, contraction, and exchange rules are called **weak inferences**.

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$$\Rightarrow t = t$$

$$s = t, A[s/v] \Rightarrow A[t/v]$$
$A \in \mathcal{L}_{\mathbb{N}}$ atomic

$$\mathrm{S}(x) = \overline{0} \Rightarrow$$

$$\mathrm{S}(x) = \mathrm{S}(y) \Rightarrow x = y$$

$$\frac{A(u), \Gamma \Rightarrow \Delta, A(\mathrm{S}(u))}{A(\overline{0}), \Gamma \Rightarrow \Delta, A(x)}$$
u eigenvariable,
$A \in \Delta_0(\mathcal{L}_{\mathbb{N}})$

$$\Rightarrow s = t$$
defining equations
for some elem. fun.

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$$\frac{\Gamma \Rightarrow \Delta, \neg \mathrm{Tr}\,\varphi}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,(\neg\varphi)} \qquad \frac{\neg \mathrm{Tr}\,\varphi, \Gamma \Rightarrow \Delta}{\mathrm{Tr}\,(\neg\varphi), \Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,\varphi \quad \Gamma \Rightarrow \Delta, \mathrm{Tr}\,\psi}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,(\varphi \wedge \psi)} \qquad \frac{\mathrm{Tr}\,\varphi, \mathrm{Tr}\,\psi, \Gamma \Rightarrow \Delta}{\mathrm{Tr}\,(\varphi \wedge \psi), \Gamma \Rightarrow \Delta}$$

$$\frac{\mathrm{Tr}\,\varphi(t), \Gamma \Rightarrow \Delta}{\mathrm{Tr}\,(\forall x \varphi), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,\varphi(u)}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,(\forall x \varphi)}$$

Truth-inferences are among the **non-weak inferences**.

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$$\frac{\Gamma \Rightarrow \Delta, \neg \mathrm{Tr}\,\varphi}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,(\neg\varphi)} \quad := \quad \frac{\Gamma \Rightarrow \Delta, \neg \mathrm{Tr}\,t \quad \Gamma \Rightarrow \Delta, \mathrm{Sent}(s) \quad \Gamma \Rightarrow \Delta, s = \neg t}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,s}$$

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- Arithmetical/Syntactic rules ($\mathcal{A}$-rules: $\mathrm{I}\Delta_0(\mathrm{exp})$)
- Rules for the truth predicate ($\varphi, \psi \in \mathcal{L}_{\mathbb{N}}$, cf. Halbach and Leigh):
- Substitution of identicals under Tr (externally):

$$\frac{\mathrm{Tr}\, t, \Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, s = t}{\mathrm{Tr}\, s, \Gamma \Rightarrow \Delta}$$

Also a **non-weak inference**.

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$\mathrm{FSN}[B]^-$

Obtained from $\mathrm{CT}[B]^-$ by relaxing the restrictions on sentences: $\varphi, \psi \in \mathcal{L}_{\mathrm{Tr}}$ in, e.g.:

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Obtained from $\mathrm{FSN}[B]^-$ by removing the negation rules for truth, and adding **positive** and negative rules for connectives and (self-applicable) truth:

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Fact (Kotlarski, Krajewski, Lachlan 1981, Leigh 2015, Enayat & Visser 2015)

$\mathrm{CT}[\mathrm{PA}]^-$ is conservative over PA.

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Our focus will be on **methods** more than specific results. This promises to open the way to a number of new results.

Cut-Elimination with Tr

- The standard inductive strategy consists in a multiple induction on the complexity of the cut-formula and the length of the derivation. In the case of the truth rules one needs to reduce the following...

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner A \urcorner} \qquad \dfrac{A, \Gamma \Rightarrow \Delta}{\mathrm{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

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This creates a problem because $\mathrm{Tr}\ulcorner A \urcorner$ is atomic whereas A is of arbitrary (logical) complexity.

One needs to keep track of the number of truth-rules applied in proofs, and **induct (mainly) over such measure**:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A^{\alpha}}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner A \urcorner^{\alpha+1}} \qquad \dfrac{A^{\beta}, \Gamma \Rightarrow \Delta}{\mathrm{Tr}\ulcorner A \urcorner^{\beta+1}, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \dfrac{\Gamma \Rightarrow \Delta, A^{\alpha} \qquad A^{\beta}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

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This simple idea is surprisingly difficult to implement:

- It's important to have good notions of occurrence and ancestor of formulae in proofs.
- Absent those, one may have problems with implicit or explicit contraction (Halbach 1999)...
- It's possible fix the problem **locally** by employing deep and complex tools (e.g. Leigh 2015 via Kotlarski, Krajewski, Lachlan 1981 on $\mathrm{CT}[B]^{-}$).

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Resurrect Halbach's 1999 strategy and extend it to self-referential systems:

- Unlike Halbach, we adopt **finite sequences**, and a well-defined notions of ancestor
- Unlike Halbach and Leigh (and Gentzen), we adopt a **global reduction procedure**, modelled after Buss' way of proving the *free cut-elimination* theorem (no mix rule required).
- The methodology promises to be general: applicable to other logics as well.

Proofs

- The notion of **ancestor** is standard. It's the transitive, reflexive closure of the immediate ancestor relation.

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  E.g. in
  $$\frac{A,\ \gamma_0, \gamma_1 \Rightarrow \delta_0, \delta_1}{\gamma_0, \gamma_1 \Rightarrow \delta_0, \delta_1\ , \neg A}$$
  A is an **immediate ancestor** of $\neg A$, and **immediate ancestors** of occurrences of formulae in contexts in the lower sequent are their corresponding occurrence in the upper one.

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- A **direct ancestor** of ψ is an ancestor φ of ψ such that φ and ψ are the same formula:

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$$\frac{A,\ \gamma_0, \gamma_1 \Rightarrow \delta_0, \delta_1}{\gamma_0, \gamma_1 \Rightarrow \delta_0, \delta_1\ , \neg A}$$

  γ_0 is an **immediate ancestor** of γ_0, and in fact an **immediate direct ancestor** of γ_0.

Our complexity measure: $\mathcal{A}$-depth of a formula in a proof

Definition

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- If φ is a principal formula of a non-weak, non-$\mathcal{A}$ inference $\mathcal{I}$:

$$\mathcal{A}\text{-dp}(\varphi) = 1 + \max\{\mathcal{A}\text{-dp}(\psi) \mid \psi \text{ is an active formula of } \mathcal{I}\}.$$

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- If φ is in the lower sequent of an inference $\mathcal{I}$, and if either (a) $\mathcal{I}$ is a non-weak inference and φ is a side formula or (b) $\mathcal{I}$ is a weak inference, then

$$\mathcal{A}\text{-dp}(\varphi) = \max\{\mathcal{A}\text{-dp}(\varphi') \mid \varphi' \text{ is an im. di. anc. of } \varphi\}.$$

We let: $\max(\varnothing) = -\infty$ and $n + (-\infty) = (-\infty) + n = -\infty$. **Formulas obtained by weakening have $\mathcal{A}$-depth $-\infty$.**

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- If $\varphi \in \mathcal{L}_{\mathbb{N}}$, atomic, then $\mathcal{A}$-depth ≤ 1;
- If φ contains Tr, $\mathcal{A}\text{-dp}(\varphi) \neq 0$.

$\mathcal{A}$-depth of a cut (our version)

The $\mathcal{A}$-depth of a cut inference is the sum of the $\mathcal{A}$-depths of the two occurrences of its cut formula.

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Anchored/Free Cut

A cut is **anchored** provided that one of the occurrences of its cut-formula has $\mathcal{A}$-depth zero. A cut is **free** if one of the following holds:

- One of the occurrences of the cut formula has $\mathcal{A}$-depth $-\infty$.
- Its cut formula is atomic, and one of the occurrences of the cut formula has $\mathcal{A}$-depth ≥ 1.
- It is not anchored.

Note that Tr**-cuts are free**.

A key ingredient of the reduction procedure is to show that non-weak inferences leading up to cut formulae can be replaced with applications of **weak inferences**, and specifically of Weakening. Such weakened formulae will have minimal $\mathcal{A}$-depth ($-\infty$).

Cuts are then performed on the premisses of the final cut, as in the local procedure, but to preserve the structure of the derivation **all relevant inferences in which the cut-formula originates** need to be processed.

Lemma

Let $\mathcal{D} \vdash^n_{\mathrm{CT}[B]^-} \Gamma \Rightarrow \Delta$. Let $\Gamma' \Rightarrow \Delta'$ be obtained from $\Gamma \Rightarrow \Delta$ by removing from it an arbitrary subset of formulae with $\mathcal{A}$-depth $-\infty$. Then we can find $\mathcal{D}' \vdash^n_{\mathrm{CT}[B]^-} \Gamma' \Rightarrow \Delta'$ with no cuts of $\mathcal{A}$-depth $-\infty$.

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Proof. By induction on the number of inferences in $\mathcal{D}$.

E.g. suppose $\mathcal{D}$ ends with

$$\frac{\Gamma \Rightarrow \Delta, s = t}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,(s = t)}$$

- If $\mathcal{A}$-dp$(\mathrm{Tr}\,(s = t)) = -\infty$, and $\mathrm{Tr}\,(s = t)$ **needs to be** deleted, then also $\mathcal{A}$-dp$(s = t) = -\infty$ and apply the IH;
- If $\mathcal{A}$-dp$(\mathrm{Tr}\,(s = t)) = -\infty$ and **does not need to be deleted**, we apply the IH first and then weakening.

Reduction Lemma for $\mathrm{CT}[B]^-$

Suppose $\mathcal{D}$ is an $\mathrm{CT}[B]^-$-derivation of $\Gamma \Rightarrow \Delta$ that ends with a Tr-Cut of $\mathcal{A}$-depth $d \geq 2$ and the subderivations $\mathcal{D}_0$ and $\mathcal{D}_1$ of $\mathcal{D}$ are Tr-cut free. Then there is a Tr-cut free $\mathrm{CT}[B]^-$-derivation $\mathcal{D}'$ of $\Gamma \Rightarrow \Delta$ (with superexponential increase in height).

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The proof is by induction on the number of **non-weak inferences and sub-induction on the $\mathcal{A}$-depth of the derivation.**

We want to eliminate cuts such as:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{00} \\ \Gamma \Rightarrow \Delta, \neg \mathrm{Tr}\,\psi\end{matrix} \quad \begin{matrix}\mathcal{D}_{01} \\ \Gamma \Rightarrow \Delta, \varphi = \neg\psi\end{matrix}}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,\varphi} \qquad \dfrac{\begin{matrix}\mathcal{D}_{10} \\ \neg \mathrm{Tr}\,\psi, \Gamma \Rightarrow \Delta\end{matrix} \quad \begin{matrix}\mathcal{D}_{11} \\ \Gamma \Rightarrow \Delta, \varphi = \neg\psi\end{matrix}}{\mathrm{Tr}\,\varphi, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

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$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{00} \\ \Gamma \Rightarrow \Delta, \neg \mathrm{Tr}\,\psi\end{matrix} \quad \begin{matrix}\mathcal{D}_{01} \\ \Gamma \Rightarrow \Delta, \varphi = \neg\psi\end{matrix}}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,\varphi} \qquad \dfrac{\begin{matrix}\mathcal{D}_{10} \\ \neg \mathrm{Tr}\,\psi, \Gamma \Rightarrow \Delta\end{matrix} \quad \begin{matrix}\mathcal{D}_{11} \\ \Gamma \Rightarrow \Delta, \varphi = \neg\psi\end{matrix}}{\mathrm{Tr}\,\varphi, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

Suppose all **direct ancestors** of $\mathrm{Tr}\,\varphi$ in the proof originate via a **negation truth-rule.**

We want to eliminate cuts such as:

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Consider any such inference creating direct ancestors of $\mathrm{Tr}\,\varphi$:

$$\dfrac{\neg \mathrm{Tr}\,\psi, \Theta \Rightarrow \Lambda \qquad \Theta \Rightarrow \Lambda, \neg\psi = \varphi}{\mathrm{Tr}\,\varphi, \Theta \Rightarrow \Lambda}$$

We want to eliminate cuts such as:

$$\dfrac{\dfrac{\overset{\mathcal{D}_{00}}{\Gamma \Rightarrow \Delta, \neg \mathrm{Tr}\,\psi} \quad \overset{\mathcal{D}_{01}}{\Gamma \Rightarrow \Delta, \varphi = \neg\psi}}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,\varphi} \qquad \dfrac{\overset{\mathcal{D}_{10}}{\neg \mathrm{Tr}\,\psi, \Gamma \Rightarrow \Delta} \quad \overset{\mathcal{D}_{11}}{\Gamma \Rightarrow \Delta, \varphi = \neg\psi}}{\mathrm{Tr}\,\varphi, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

The idea is to replace each such inference with:

$$\mathrm{WL}\ \dfrac{\neg \mathrm{Tr}\,\psi, \Theta \Rightarrow \Lambda}{\neg \mathrm{Tr}\,\psi, \Theta, \mathrm{Tr}\,\varphi \Rightarrow \Lambda}$$

so that $\mathcal{A}$-dp$(\mathrm{Tr}\,\varphi) = -\infty$.

We want to eliminate cuts such as:

$$\dfrac{\dfrac{\overset{\mathcal{D}_{00}}{\Gamma \Rightarrow \Delta, \neg \mathrm{Tr}\,\psi} \quad \overset{\mathcal{D}_{01}}{\Gamma \Rightarrow \Delta, \varphi = \neg\psi}}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,\varphi} \qquad \dfrac{\overset{\mathcal{D}_{10}}{\neg \mathrm{Tr}\,\psi, \Gamma \Rightarrow \Delta} \quad \overset{\mathcal{D}_{11}}{\Gamma \Rightarrow \Delta, \varphi = \neg\psi}}{\mathrm{Tr}\,\varphi, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

In the best case, by doing a symmetric move on the right hand derivation, and **propagating down** the new occurrences of $\neg\mathrm{Tr}\,\psi$, one can perform the cut:

$$\dfrac{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,\varphi, \neg\mathrm{Tr}\,\psi \qquad \neg\mathrm{Tr}\,\psi, \mathrm{Tr}\,\varphi, \Gamma \Rightarrow \Delta}{\mathrm{Tr}\,\varphi, \Gamma \Rightarrow \Delta, \mathrm{Tr}\,\varphi}$$

since the $\mathcal{A}$-depth of the cut is $< d$.

We want to eliminate cuts such as:

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Finally, by the lemma on formulae with $\mathcal{A}$-depth $-\infty$, one could obtain the required proof of $\Gamma \Rightarrow \Delta$ by eliminating the occurrences of $\mathrm{Tr}\,\varphi$.

However... cuts can also look like this:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_{00}\\ \Gamma \Rightarrow \Delta, \neg \mathrm{Tr}\,\psi\end{array} \quad \begin{array}{c}\mathcal{D}_{01}\\ \Gamma \Rightarrow \Delta, \varphi = \neg\psi\end{array}}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\,\varphi} \qquad \dfrac{\begin{array}{c}\mathcal{D}_{10}\\ \neg \mathrm{Tr}\,\chi, \Gamma \Rightarrow \Delta\end{array} \quad \begin{array}{c}\mathcal{D}_{11}\\ \Gamma \Rightarrow \Delta, \varphi = \neg\chi\end{array}}{\mathrm{Tr}\,\varphi, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

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In this case, we need to **uniformize (down to one)** different 'sentences' in premises, such as χ and ψ. E.g. working on $\mathcal{D}_0$:

$$\cfrac{\cfrac{\Theta \Rightarrow \Lambda, \neg\mathrm{Tr}\,\psi \qquad \cfrac{\begin{array}{c}\text{Arithmetic (prop. of }\neg\text{),}\\ \text{Tr-Rep with minimal }\mathcal{A}\text{-depth}\\ \mathrm{Tr}\,\chi, \Theta \Rightarrow \Lambda, \mathrm{Tr}\,\psi\end{array}}{\neg\mathrm{Tr}\,\psi, \Theta \Rightarrow \Lambda, \neg\mathrm{Tr}\,\chi}}{\Theta \Rightarrow \Lambda, \neg\mathrm{Tr}\,\chi}}{\Theta \Rightarrow \Lambda, \neg\mathrm{Tr}\,\chi, \mathrm{Tr}\,\varphi}$$

And... cuts can also look like this:

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The idea is to replace relevant occurrences of $\mathrm{Tr}\,\varphi$ in $\mathcal{D}_0$ with $\mathrm{Tr}\,t$:

$$\dfrac{\dfrac{\mathcal{D}'_{00}}{\varphi = t, \Gamma \Rightarrow \Delta, \neg \mathrm{Tr}\,\psi} \qquad \dfrac{\mathcal{D}_{01} \qquad \mathcal{D}_{11}}{\varphi = t, \Gamma \Rightarrow \Delta, t = \neg\psi}}{\varphi = t, \Gamma \Rightarrow \Delta, \mathrm{Tr}\,t}$$

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We then employ $\mathcal{D}_{10}$, $\mathcal{D}_{11}$ to obtain a derivation of $\Gamma \Rightarrow \Delta$.

When reducing both **Types 1 and 2**, the global transformation requires to deal with cases in which a direct ancestor of the cut formula originates in inferences of the form:

$$\frac{\mathrm{Tr}\,t, \Pi \Rightarrow \Xi \qquad \Pi \Rightarrow \Xi, \varphi = t}{\mathrm{Tr}\,\varphi, \Pi \Rightarrow \Xi}$$

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An analogue of **Type 2**-reduction is employed. We first obtain:

$$\frac{\varphi = t, \Pi, \Gamma \Rightarrow \Delta, \Xi, \neg\mathrm{Tr}\, \psi \quad \varphi = t, \Pi, \Gamma \Rightarrow \Delta, \Xi, t = \neg\psi}{\varphi = t, \Pi, \Gamma \Rightarrow \Delta, \Xi, \mathrm{Tr}\, t}$$

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By Cut and Weakening, we obtain a derivation of

$$\mathrm{Tr}\, \varphi, \Pi, \Gamma \Rightarrow \Delta, \Xi$$

with $\mathcal{A}\text{-dp}(\mathrm{Tr}\, \varphi) = -\infty$.

By induction on the height of the proof, applying the *Reduction Lemma*:

Cut-Elimination

Suppose $\mathcal{D}$ is a $\mathrm{CT}[B]^-$-derivation whose max $\mathcal{A}$-depth of its Tr-cuts is $\leq d$, where $d \geq 2$. Then there is a $\mathrm{CT}[B]^-$-derivation $\mathcal{D}'$ which contains no Tr-cuts and has super-exponential increase over $\mathcal{D}$.

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The procedure above, quite surprisingly, adapts with little modification to self-applicable truth, in particular to the rules of $\mathrm{KF}[B]^-$ and (less surprisingly) $\mathrm{FSN}[B]^-$.

The Dark Side

If there's a mistake in our uniformization procedure, it's already in $\mathrm{CT}[B]^-$...

Conservativity

If $\mathrm{CT}[B]^- \vdash \varphi$ with $\varphi \in \mathcal{L}_{\mathbb{N}}$, then φ can be derived in $\mathrm{CT}[B]^-$ without Tr-cuts. Therefore $\mathrm{CT}[B]^-$ is a **conservative extension** B.

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Again, the procedure above is applicable to $\mathrm{FSN}[B]^-$ and $\mathrm{KF}[B]^-$, witnessing their conservativity over B.

Once Tr-cuts have been removed (but potentially not all **free-cuts**), we can apply the algorithm by Buss & Beckmann 2011 to obtain:

Free-Cut Elimination

All **free-cuts** can be eliminated in $\mathrm{CT}[B]^-$ ($\mathrm{FSN}[B]^-$ and $\mathrm{KF}[B]^-$).

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- Our procedure works for theories formulated in classical logic, and we cannot see any obstacle to generalize it to theories formulated in logics other than classical (e.g. PKF^{-}).
- The results also provide **relative interpretability results** (for reflexive base theories, see Fischer 2009).