LECTURE 2
A THEORY OF CIRCULAR PROPOSITIONAL FUNCTIONS

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1. Introduction

In the last lecture, two desiderata for a theory of 'concepts' arose naturally:

- the theory should be *type-free*
- the theory should be *classical*
- the theory should contain a *significant amount of mathematics*

In addition, it seems desirable that:

- a unified approach to the paradoxes of set/class and predication
- classes of models won't do: we need axioms to fulfil our universalist picture

2. Leitgeb and Welch's theory

Leitgeb and Welch (2011) introduced a theory of propositional functions (PF) – which is then extended to a theory of satisfaction and truth. The details of their account will be incorporated in my theory below, but here's a snapshot.

The fundamental objects of PF are constructed from a class $\mathcal{C}$ of basic concepts (e.g. the logical concepts of negation, conjuction...) in the following way:

$$\mathrm{P}_0 = \mathcal{C} \text{ i.e. the set of all concepts}$$

$$\mathrm{P}_{\alpha+1} = \{(u, A) \mid \mathtt{Fml}_{\mathcal{C}}(u) \wedge A \subseteq \mathrm{P}_\alpha\}$$

$$\mathrm{P}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{P}_\alpha$$

$$\mathbb{PF} = \bigcup_{\alpha\in\mathtt{ON}} \mathrm{P}_\alpha$$

With this definition at hand, one can define:

$\mathtt{PrF} := \mathbb{PF}$	the class of propositional functions
$\mathtt{Var} := \{\overline{v_i} \mid i \in \omega\}$	the class of conceptual variables
$\mathtt{App}_3 := \{(u, x, y, z) \mid u = (x, y, z)\}$	ternary concept application...
$\mathtt{About} := \{(x, y) \mid x \in \mathbb{PF} \wedge y \in \mathbb{PF} \wedge y \in x\}$	aboutness relation as inverse $\in$

REMARK 1.

(i) The last clause is crucial for the basic underlying picture. The fundamental theoretical relation for propositional function is the *aboutness* relation: it is essentially the inverse membership relation of ZFC.

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(ii) An immediate consequence of the definition of $\mathbb{PF}$ is that propositional functions are *well-founded*. Namely, they satisfy the schema:

(WF)
$$\forall x(\varphi(x) \to \mathtt{PrF}(x)) \to$$
$$(\exists y\, \varphi(y) \to \exists y_0(\varphi(y_0) \wedge \forall y(\varphi(y) \to \neg \mathtt{A}(y_0, y))))$$

This, in turn, immediately entails that *there are no circular propositional functions*: for instance, there is no propositional function p such that $\mathtt{A}(p,p)$. However, we can generate propositional functions of the form $p \approx \mathtt{App}_3(\overline{P}, \overline{p})$. In what sense such an x would not be about itself?

More generally, PF seems too restrictive: as we mentioned in the previous lecture, there are clear examples of theoretically relevant concepts (distinctness, belief, necessity, truth) that require self-constituency.

3. A Theory of Circular Propositional Functions: CPF

3.1. **Basis.** Our language contains the usual logical symbols, plus constants for constructing atomic and complex propositional functions.

Definition 1 (The language $\mathcal{L}_{\mathtt{PF}}$). The signature of $\mathcal{L}_{\mathtt{PF}}$ is specified by:

(i) Predicates: $\mathtt{PrF}, \mathtt{Var}, P$ (unary), $\mathtt{A}, =$ (binary), $\mathtt{App}_3$ (ternary), $\mathtt{App}_4$ (quaternary)

(ii) Constants: $\overline{\neg}$, $\overline{\wedge}$, $\overline{\vee}$, $\overline{\to}$, $\overline{\leftrightarrow}$, $\overline{\forall}$, $\overline{\exists}$, $\overline{=}$, $\overline{\mathtt{App}_3}$, $\overline{\mathtt{App}_4}$, $\overline{\mathtt{A}}$, $\overline{\mathtt{PrF}}$, $\overline{\mathtt{Var}}$, $\overline{P}$, $\overline{v_1}$, $\overline{v_2}$,...

I give now a summary of the axioms by groups:

group 1: basic concepts. Constants are distinct basic concepts, and they are not about anything.

group 2: identity and structure of PFs.

- PFs are identical if they are structurally so:

$$\mathtt{App}_3(w,u,v) \wedge \mathtt{App}_3(z,x,y) \to$$
$$\big(w = z \leftrightarrow ((u = x \wedge v = y) \wedge (\mathtt{PrF}(w) \leftrightarrow \mathtt{PrF}(z)) \wedge$$
$$(\mathtt{PrF}(w) \wedge \mathtt{PrF}(z) \to \forall o(\mathtt{A}(w,o) \leftrightarrow \mathtt{A}(z,o))))\big)$$

- structure of atomic PFs and of complex ones, e.g.:

$p \approx \mathtt{App}_3(\overline{P}, \overline{v_1})$ is an atomic PF,

$q \approx \mathtt{App}_4(\overline{\wedge}, q_0, q_1)$ is a complex PF, and it exists (and uniquely so) if q_0, q_1 exist...

- structural induction over PFs: if some condition holds of atomic PFs, and is preserved under the construction of complex ones, it holds of all PFs.

group 3: existence of PFs.

- there is a PF that is about basic concepts.
- there is a PF that is about infinitely many objects.
- separation, power set, union, pairs, replacement, choice, hold for any type of atomic PF. For instance, in the case of separation, we have:

$$\forall x \exists y \forall u\big((\mathtt{A}(y,u) \leftrightarrow \mathtt{A}(x,u) \wedge \varphi(u)) \wedge y \approx \mathtt{App}_3(\overline{P}, \overline{v_i})\big)$$

The final axiom is the defining one of the theory CPF. We want to express the anti-foundation axiom for propositional functions.[1]

(AFA) Every (directed, tagged) graph corresponds to a unique set

[1]For more details on (AFA), see the original Forti and Honsell (1983), Aczel (1988), Devlin (1993), Barwise and Moss (1996), Moss (2018).

But how to formulate the axiom in our language?

3.2. **Bootstrapping.** First, we want to capture the notion of a *(directed, tagged) graph.*

To define the notion of *set* (possibly, with PFs as urelemente), by taking a proxy PF, say $\mathtt{App}_3(\overline{P}, \overline{v_1})$, and focus on its pure aboutness structure.

Definition 2 (Set).

(i) $\mathtt{Set}(x) :\leftrightarrow x \approx \mathtt{App}_3(\overline{P}, \overline{v_1})$

(ii) $x \in y :\leftrightarrow \mathtt{Set}(y) \wedge \mathtt{A}(y, x)$

Notice that, by the axioms of group 3, all the usual ZF-existence axioms hold for the entities satisfied by $\mathtt{Set}(x)$. The notion of a *graph* can then be standardly defined: a *directed graph* is an object of the form $\mathcal{G} = (G, R)$, with G a set of nodes – some canonical, set-theoretic representative will do –, and R an binary relation on G.

Definition 3 (Tagging, Decoration).

(i) A *tagging* function is a mapping $\tau \colon G \longrightarrow \mathcal{C} \cup \{\varnothing\}$ that assigns members of $\mathcal{C} \cup \{\varnothing\}$ to childless nodes.

(ii) Let τ be given. A *decoration* of $\mathcal{G}$ is a function d such that, for $g \in G$:

$$d(g) = \begin{cases} \tau(g), & \text{for } g \text{ childless} \\ \{d(g_0) \mid (g, g_0) \in R\}, & \text{else} \end{cases}$$

In (ii), of course, the notion of set employed is the one relative to our predicate $\mathtt{Set}(x)$ above.

remark 2. The notions of graph, decoration are all relative to the notion of set defined. Therefore, instead of $\mathtt{Set}$, graph, decoration, one should really talk about $\mathtt{Set}_\Phi$, Φ-graph, Φ-decoration, for Φ some conceptual structure form.

One may be tempted to say that our job is now done. But this is only partially so. we can certainly formulate a first version of our anti-foundation axiom:

(afa1) Every tagged $\overline{P}$-graph has a unique decoration.

The label $\overline{P}$-graph is intended to emphasize that the aboutness structure that we read off from the graph is based on a notion of set based on PFs of the form $\mathtt{App}_3(\overline{P}, \overline{v_1})$.

Lemma 1. afa1 *entails that there is a PF* $p \approx \mathtt{App}_3(\overline{P}, \overline{v})$ *such that* $\mathtt{A}(p, p)$*.*

Lemma 1, however, does not tell us anything about possible circular functions of a different conceptual structure. For instance, we cannot conclude from our axioms that there is a propositional function $p \approx \mathtt{App}_3(\overline{\mathtt{PrF}}, \overline{v_1})$ such that $\mathtt{A}(p, p)$.

This shortcoming, however, can be easily overcome (it seems). We can generalise (afa1) to all atomic propositional functions:

(afa_Φ)

For any atomic propositional function form Φ: every Φ-graph has a unique Φ-decoration.

Corollary 1. $\text{afa}_{\mathtt{PrF}}$ *entails that there is a PF* $q \approx \mathtt{App}_3(\overline{\mathtt{PrF}}, \overline{v_1})$ *such that* $\mathtt{A}(q, q)$*.*

This obviously generalises to the other atomic ones and to complex PFs that share the same conceptual form. One problem that is still open is to find a good notion of decoration for complex PFs that combine different conceptual structures.

3.3. **Strength.** So the theory CPF is classical, it is type-free in a non-trivial sense. In addition, what I said in the previous section entails that the mathematical content of the theory is substantial. In fact:

PROPOSITION 1.

(i) *CPF is mutually interpretable with ZFA*

(ii) *Theorefore, CPF interprets ZFC (in a somewhat natural way).*

Proof. Ad (i): by considering our definition of $\mathtt{Set}_{\overline{P}}$, one can check that all axioms of ZFA are satisfied. In the other direction, one lets $\mathcal{C}$ be a urelements, and interprets PFs as pairs $(\varphi, \mathcal{G})$ where $\mathcal{G}$ gives the appropriate aboutness structure to the PF.

Ad (ii): One interprets ZFC-sets as well-founded $\overline{P}$-graphs. □

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