

Proof Theory in Philosophy

Carlo Nicolai



Slides available at <https://carlonicolai.github.io>

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Outline

Basics

- Paradox(es)
- Consistency via cut-elimination
- Objects of Truth

Systems of Truth

- Deflation and Conservation
- Classical v Nonclassical Kripkean truth
- Logical Pluralism

Extensions

- Reflection
- Modal Logic
- Modal Predicates

Basics

- Paradox(es)
- Consistency via cut-elimination
- Objects of Truth

Systems of Truth

- Deflation and Conservation
- Classical v Nonclassical Kripkean truth
- Logical Pluralism

Extensions

- Reflection
- Modal Logic
- Modal Predicates

References

What I'm not considering

I don't consider **proof-theoretic semantics**.

I only briefly touch upon **reductive proof-theory** in the philosophy of mathematics.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via

cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Abstraction and Truth

‘There never were any set-theoretic paradoxes, but the property theoretic paradoxes are still unresolved’ (Gödel to Myhill)

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Abstraction and Truth

'There never were any set-theoretic paradoxes, but the property theoretic paradoxes are still unresolved' (Gödel to Myhill)

Naïve abstraction

$$\forall x(x \in \{v \mid \varphi(v)\} \leftrightarrow \varphi(x))$$

Naïve Truth

$$\text{Tr} \ulcorner A \urcorner \leftrightarrow A$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Abstraction and Truth

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

'There never were any set-theoretic paradoxes, but the property theoretic paradoxes are still unresolved' (Gödel to Myhill)

Naïve abstraction

$$\forall x(x \in \{v \mid \varphi(v)\} \leftrightarrow \varphi(x))$$

Naïve Truth

$$\text{Tr} \ulcorner A \urcorner \leftrightarrow A$$

Here I assume that for any φ in the language there is a term $\{v \mid \varphi(v)\}$ with $\text{FV}(\{v \mid \varphi(v)\}) = \text{FV}(\varphi) \setminus \{v\}$. If φ is a sentence, I write $\ulcorner A \urcorner$ for 'the proposition expressed by A '.

Liar

$\Gamma, \Delta, \Theta, \Lambda, \dots$ are multisets of formulas.

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\lambda \Leftrightarrow \neg \text{Tr} \ulcorner \lambda \urcorner$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\neg \lambda \Leftrightarrow \text{Tr} \ulcorner \lambda \urcorner$$

Basics

Paradox(es)

Consistency via

cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Liar

$\Gamma, \Delta, \Theta, \Lambda, \dots$ are multisets of formulas.

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\lambda \Leftrightarrow \neg \text{Tr} \ulcorner \lambda \urcorner$$

$$\neg \lambda \Leftrightarrow \text{Tr} \ulcorner \lambda \urcorner$$

$$\frac{\lambda \Rightarrow \lambda}{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow \lambda}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Liar

$\Gamma, \Delta, \Theta, \Lambda, \dots$ are multisets of formulas.

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\lambda \Leftrightarrow \neg \text{Tr} \ulcorner \lambda \urcorner$$

$$\neg \lambda \Leftrightarrow \text{Tr} \ulcorner \lambda \urcorner$$

$$\frac{\lambda \Rightarrow \lambda}{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow \lambda}$$
$$\frac{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow \lambda}{\text{Tr} \ulcorner \lambda \urcorner, \neg \lambda \Rightarrow}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Liar

$\Gamma, \Delta, \Theta, \Lambda, \dots$ are multisets of formulas.

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\lambda \Leftrightarrow \neg \text{Tr} \ulcorner \lambda \urcorner$$

$$\neg \lambda \Leftrightarrow \text{Tr} \ulcorner \lambda \urcorner$$

$$\frac{\frac{\lambda \Rightarrow \lambda}{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow \lambda}}{\text{Tr} \ulcorner \lambda \urcorner, \neg \lambda \Rightarrow \text{Tr} \ulcorner \lambda \urcorner}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Liar

$\Gamma, \Delta, \Theta, \Lambda, \dots$ are multisets of formulas.

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\lambda \Leftrightarrow \neg \text{Tr} \ulcorner \lambda \urcorner$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\neg \lambda \Leftrightarrow \text{Tr} \ulcorner \lambda \urcorner$$

$$\frac{\frac{\frac{\lambda \Rightarrow \lambda}{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow \lambda}}{\text{Tr} \ulcorner \lambda \urcorner, \neg \lambda \Rightarrow}}{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow}}{\Rightarrow \neg \text{Tr} \ulcorner \lambda \urcorner}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Liar

$\Gamma, \Delta, \Theta, \Lambda, \dots$ are multisets of formulas.

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\lambda \Leftrightarrow \neg \text{Tr} \ulcorner \lambda \urcorner$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\neg \lambda \Leftrightarrow \text{Tr} \ulcorner \lambda \urcorner$$

$$\begin{array}{c}
\frac{\lambda \Rightarrow \lambda}{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow \lambda} \\
\frac{\text{Tr} \ulcorner \lambda \urcorner, \neg \lambda \Rightarrow}{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow} \\
\frac{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow}{\Rightarrow \neg \text{Tr} \ulcorner \lambda \urcorner} \\
\frac{\Rightarrow \neg \text{Tr} \ulcorner \lambda \urcorner}{\Rightarrow \lambda}
\end{array}
\Rightarrow
\begin{array}{c}
\frac{\lambda \Rightarrow \lambda}{\lambda \Rightarrow \text{Tr} \ulcorner \lambda \urcorner} \\
\frac{\lambda, \neg \text{Tr} \ulcorner \lambda \urcorner \Rightarrow}{\lambda, \lambda \Rightarrow} \\
\frac{\lambda, \lambda \Rightarrow}{\lambda \Rightarrow}
\end{array}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Liar

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\lambda \Leftrightarrow \neg \text{Tr} \ulcorner \lambda \urcorner$$

$$\neg \lambda \Leftrightarrow \text{Tr} \ulcorner \lambda \urcorner$$

$$\frac{\frac{\frac{\lambda \Rightarrow \lambda}{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow \lambda}}{\text{Tr} \ulcorner \lambda \urcorner, \neg \lambda \Rightarrow}}{\frac{\text{Tr} \ulcorner \lambda \urcorner \Rightarrow}{\Rightarrow \neg \text{Tr} \ulcorner \lambda \urcorner}}}{\Rightarrow \lambda} \Rightarrow \frac{\frac{\frac{\lambda \Rightarrow \lambda}{\lambda \Rightarrow \text{Tr} \ulcorner \lambda \urcorner}}{\lambda, \neg \text{Tr} \ulcorner \lambda \urcorner \Rightarrow}}{\frac{\lambda, \lambda \Rightarrow}{\lambda \Rightarrow}}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

Basics

Paradox(es)

Consistency via

cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

$$\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner}$$

Basics

Paradox(es)

Consistency via

cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

$$\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner}$$

Basics

Paradox(es)

Consistency via

cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

$$\frac{\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner} \quad \perp \Rightarrow \perp}{\kappa, \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \perp}$$

Basics

Paradox(es)

Consistency via
cut-elimination
Objects of Truth

Systems of Truth

Deflation and Conservation
Classical v Nonclassical
Kripkean truth
Logical Pluralism

Extensions

Reflection
Modal Logic
Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

$$\frac{\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner} \quad \perp \Rightarrow \perp}{\kappa, \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \perp} \quad \kappa, \kappa \Rightarrow \perp$$

Basics

Paradox(es)

Consistency via
cut-elimination
Objects of Truth

Systems of Truth

Deflation and Conservation
Classical v Nonclassical
Kripkean truth
Logical Pluralism

Extensions

Reflection
Modal Logic
Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

$$\frac{\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner} \quad \perp \Rightarrow \perp}{\kappa, \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \perp} \\ \frac{\kappa, \kappa \Rightarrow \perp}{\kappa \Rightarrow \perp}$$

Basics

Paradox(es)

Consistency via
cut-elimination
Objects of Truth

Systems of Truth

Deflation and Conservation
Classical v Nonclassical
Kripkean truth
Logical Pluralism

Extensions

Reflection
Modal Logic
Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

$$\frac{\frac{\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner} \quad \perp \Rightarrow \perp}{\kappa, \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \perp}}{\frac{\kappa, \kappa \Rightarrow \perp}{\kappa \Rightarrow \perp}} \Rightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

$$\frac{\frac{\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner} \quad \perp \Rightarrow \perp}{\kappa, \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \perp}}{\frac{\kappa, \kappa \Rightarrow \perp}{\kappa \Rightarrow \perp}} \Rightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \kappa$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

$$\frac{\frac{\frac{\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner} \quad \perp \Rightarrow \perp}{\kappa, \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \perp}}{\kappa, \kappa \Rightarrow \perp} \quad \frac{\kappa \Rightarrow \perp}{\Rightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp} \quad \frac{\Rightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp}{\Rightarrow \kappa} \quad \Rightarrow \perp$$

$$\frac{\frac{\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner} \quad \perp \Rightarrow \perp}{\kappa, \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \perp}}{\kappa \Rightarrow \perp}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth rules

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \ulcorner A \urcorner}$$

$$\frac{A, \Gamma \Rightarrow D}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow D}$$

$$\kappa \Leftrightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp$$

$$\frac{\frac{\frac{\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner} \quad \perp \Rightarrow \perp}{\kappa, \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \perp}}{\frac{\kappa, \kappa \Rightarrow \perp}{\kappa \Rightarrow \perp}} \Rightarrow \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp}{\Rightarrow \kappa} \quad \frac{\frac{\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr} \ulcorner \kappa \urcorner} \quad \perp \Rightarrow \perp}{\kappa, \text{Tr} \ulcorner \kappa \urcorner \rightarrow \perp \Rightarrow \perp}}{\kappa \Rightarrow \perp} \Rightarrow \perp$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Internal Curry

'Consequence' predicate

$$\frac{\Gamma, A \Rightarrow B \quad \Gamma, C \Rightarrow D}{\Gamma, A, \mathcal{C}(\ulcorner B \urcorner, \ulcorner C \urcorner) \Rightarrow D}$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow \mathcal{C}(\ulcorner A \urcorner, \ulcorner B \urcorner)}$$

$$v \Leftrightarrow \mathcal{C}(\ulcorner v \urcorner, \ulcorner \perp \urcorner)$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Internal Curry

'Consequence' predicate

$$\frac{\Gamma, A \Rightarrow B \quad \Gamma, C \Rightarrow D}{\Gamma, A, \mathcal{C}(\ulcorner B \urcorner, \ulcorner C \urcorner) \Rightarrow D}$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow \mathcal{C}(\ulcorner A \urcorner, \ulcorner B \urcorner)}$$

$$v \Leftrightarrow \mathcal{C}(\ulcorner v \urcorner, \ulcorner \perp \urcorner)$$

$$\frac{v \Rightarrow v \quad \perp \Rightarrow \perp}{v, \mathcal{C}(\ulcorner v \urcorner, \ulcorner \perp \urcorner) \Rightarrow \perp}$$

$$\frac{v \Rightarrow \perp}{\Rightarrow \mathcal{C}(\ulcorner v \urcorner, \ulcorner \perp \urcorner)}$$

$$\Rightarrow v$$

$$\Rightarrow \perp$$

$$\frac{v \Rightarrow v \quad \perp \Rightarrow \perp}{v, \mathcal{C}(\ulcorner v \urcorner, \ulcorner \perp \urcorner) \Rightarrow \perp}$$

$$v \Rightarrow \perp$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Internal Curry

'Consequence' predicate

$$\frac{\Gamma, A \Rightarrow B \quad \Gamma, C \Rightarrow D}{\Gamma, A, C('B', 'C') \Rightarrow D}$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow C('A', 'B')}$$

$$v \Leftrightarrow C('v', '1')$$

$$\frac{v \Rightarrow v \quad 1 \Rightarrow 1}{v, C('v', '1') \Rightarrow 1}$$

$$\frac{v \Rightarrow 1}{\Rightarrow C('v', '1')}$$

$$\Rightarrow v$$

$$\Rightarrow 1$$

$$\frac{v \Rightarrow v \quad 1 \Rightarrow 1}{v, C('v', '1') \Rightarrow 1}$$

$$v \Rightarrow 1$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

**Consistency via
cut-elimination**

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Cut-elimination for truth and abstraction

The main extension of the standard inductive strategy consists in the reduction of cuts of the following form:

Tr -rules principal in the last inferences

$$\frac{\frac{\mathcal{D}_0}{\Gamma \Rightarrow \Delta, A} \quad \frac{\mathcal{D}_1}{A, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta, \text{Tr } \ulcorner A \urcorner} \quad \frac{\text{Tr } \ulcorner A \urcorner, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Cut-elimination for truth and abstraction

The main extension of the standard inductive strategy consists in the reduction of cuts of the following form:

Tr -rules principal in the last inferences

$$\frac{\frac{\mathcal{D}_0}{\Gamma \Rightarrow \Delta, A} \quad \frac{\mathcal{D}_1}{A, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta, \text{Tr } \ulcorner A \urcorner} \quad \frac{\text{Tr } \ulcorner A \urcorner, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

... which we would like to reduce to:

$$\frac{\mathcal{D}_0 \quad \mathcal{D}_1}{\Gamma \Rightarrow \Delta, A} \quad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

This creates a problem because $\text{Tr } \ulcorner A \urcorner$ is atomic whereas A is of arbitrary (logical) complexity.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Tr -measures

- ▶ I will consider two ways of keeping track of applications of the truth rules in derivations: the first applies to **nodes in the derivation tree**, the second applies to **single formulas within derivations**.

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Tr -measures

- ▶ I will consider two ways of keeping track of applications of the truth rules in derivations: the first applies to **nodes in the derivation tree**, the second applies to **single formulas within derivations**.
- ▶ In the first case:

$$\frac{\gamma_0 \Rightarrow \top \quad \alpha}{\gamma_0 \Rightarrow \text{Tr} \ulcorner \top \urcorner \quad \alpha + 1} \quad \gamma_1 \Rightarrow \text{Tr} \ulcorner \top \urcorner \quad \beta$$

$$\gamma_0, \gamma_1 \Rightarrow \text{Tr} \ulcorner \top \urcorner \wedge \text{Tr} \ulcorner \top \urcorner \quad \max(\alpha, \beta)$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Tr -measures

- ▶ I will consider two ways of keeping track of applications of the truth rules in derivations: the first applies to **nodes in the derivation tree**, the second applies to **single formulas within derivations**.
- ▶ In the first case:

$$\frac{\gamma_o \Rightarrow \top \quad \alpha}{\gamma_o \Rightarrow \text{Tr}^{\top} \quad \alpha + 1} \quad \gamma_1 \Rightarrow \text{Tr}^{\top} \quad \beta$$

$$\gamma_o, \gamma_1 \Rightarrow \text{Tr}^{\top} \wedge \text{Tr}^{\top} \quad \max(\alpha, \beta)$$

- ▶ In the second case:

$$\frac{\gamma_o \Rightarrow {}^o\top}{\gamma_o \Rightarrow {}^1\text{Tr}^{\top} \quad \gamma_1 \Rightarrow {}^o\text{Tr}^{\top}}$$

$$\gamma_o, \gamma_1 \Rightarrow \max(1, n) \text{Tr}^{\top} \wedge \text{Tr}^{\top}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Contraction-Free

Systems of truth and 'set theories' can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

Truth à la Grišin GT

$$\Gamma, \text{Tr } s \Rightarrow \text{Tr } s, \Delta [0]$$

$$\Gamma \Rightarrow \top, \Delta [0] \quad \Gamma, \perp \Rightarrow \Delta [0]$$

$$\frac{A, \Gamma \Rightarrow \Delta [\alpha]}{\text{Tr } \ulcorner A \urcorner, \Gamma \Rightarrow \Delta [\alpha + 1]}$$

$$\frac{\Gamma \Rightarrow \Delta, A [\alpha]}{\Gamma \Rightarrow \Delta, \text{Tr } \ulcorner A \urcorner [\alpha + 1]}$$

$$\frac{\Gamma \Rightarrow \Delta, A_i [\alpha]}{\Gamma \Rightarrow \Delta, A_o \sqcap A_i [\alpha]}$$

$$\frac{\Gamma \Rightarrow \Delta, A [\alpha] \quad \Gamma \Rightarrow \Delta, B [\beta]}{\Gamma \Rightarrow \Delta, A \sqcap B [\max(\alpha, \beta)]}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta [\alpha]}{A \star B, \Gamma \Rightarrow \Delta [\alpha]}$$

$$\frac{\Gamma \Rightarrow \Delta, A [\alpha] \quad \Theta \Rightarrow \Lambda, B [\beta]}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, A \star B [\alpha + \beta]}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Contraction-free

Systems of truth and 'set theories' can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

Lemma

Given cut-free derivations $\mathcal{D}_0 \vdash_{\text{GT}} \Gamma \Rightarrow \Delta, A$ and $\mathcal{D}_1 \vdash_{\text{GT}} A, \Theta \Rightarrow \Lambda$, there is a $\mathcal{D} \vdash_{\text{GT}} \Gamma, \Theta \Rightarrow \Delta, \Lambda$ with the Tr -rank ρ of \mathcal{D} is $\leq \rho(\mathcal{D}_0) + \rho(\mathcal{D}_1)$.

Proof Idea.

The induction is on $(\rho(\mathcal{D}_0) + \rho(\mathcal{D}_1), |A|, |\mathcal{D}_0| + |\mathcal{D}_1|)$. □

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Contraction-free

Systems of truth and ‘set theories’ can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

Two problems of the contraction-free approach:

- ▶ Viewed as a set theory, GS is inconsistent with extensionality, e.g defined as:

$$s \subseteq t \star t \subseteq s, t \in r \Rightarrow s \in r$$

This is often called **Grišin’s paradox**.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Contraction-free

Systems of truth and ‘set theories’ can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

Two problems of the contraction-free approach:

- ▶ Viewed as a set theory, GS is inconsistent with extensionality, e.g defined as:

$$s \subseteq t \star t \subseteq s, t \in r \Rightarrow s \in r$$

This is often called **Grišin’s paradox**.

- ▶ Viewed as a property theory or a truth theory, there is no known, **plausible semantics**.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Contraction-free

Systems of truth and ‘set theories’ can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

Two problems of the contraction-free approach:

- ▶ Viewed as a set theory, GS is inconsistent with extensionality, e.g defined as:

$$s \subseteq t \star t \subseteq s, t \in r \Rightarrow s \in r$$

This is often called **Grišin’s paradox**.

- ▶ Viewed as a property theory or a truth theory, there is no known, **plausible semantics**.

However, it needs to be added that it also features a ‘decent’ conditional (compared, e.g. to Field (2008)).

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Fixed-Point Semantics

Given our language $\mathcal{L}_{\text{Tr}} := \mathcal{L}_{\cup} \{ \text{Tr} \}$, we start with a (classical) model \mathcal{M} of \mathcal{L} such that $\ulcorner \varphi \urcorner^{\mathcal{M}} = \varphi$, and set, for $X \subset |\mathcal{M}|$:

$$a \in \Phi(X) \Leftrightarrow a = \ulcorner \top \urcorner, \text{ or}$$

$$a = \ulcorner \text{Tr} \ulcorner \varphi \urcorner \urcorner \text{ and } \ulcorner \varphi \urcorner \in X, \text{ or}$$

$$a = \ulcorner \neg \text{Tr} \ulcorner \varphi \urcorner \urcorner \text{ and } \ulcorner \neg \varphi \urcorner \in X, \text{ or}$$

$$a = \ulcorner \varphi \wedge \psi \urcorner \text{ and } \ulcorner \varphi \urcorner \in X \text{ and } \ulcorner \psi \urcorner \in X, \text{ or}$$

$$a = \ulcorner \neg(\varphi \wedge \psi) \urcorner \text{ and } \ulcorner \neg \varphi \urcorner \in X \text{ or } \ulcorner \neg \psi \urcorner \in X, \text{ or } \dots$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Fixed-Point Semantics

Given our language $\mathcal{L}_{\text{Tr}} := \mathcal{L}_{\cup} \{ \text{Tr} \}$, we start with a (classical) model \mathcal{M} of \mathcal{L} such that $\ulcorner \varphi \urcorner^{\mathcal{M}} = \varphi$, and set, for $X \subset |\mathcal{M}|$:

$$\begin{aligned} a \in \Phi(X) \Leftrightarrow & a = \ulcorner \top \urcorner, \text{ or} \\ & a = \ulcorner \text{Tr} \ulcorner \varphi \urcorner \urcorner \text{ and } \ulcorner \varphi \urcorner \in X, \text{ or} \\ & a = \ulcorner \neg \text{Tr} \ulcorner \varphi \urcorner \urcorner \text{ and } \ulcorner \neg \varphi \urcorner \in X, \text{ or} \\ & a = \ulcorner \varphi \wedge \psi \urcorner \text{ and } \ulcorner \varphi \urcorner \in X \text{ and } \ulcorner \psi \urcorner \in X, \text{ or} \\ & a = \ulcorner \neg(\varphi \wedge \psi) \urcorner \text{ and } \ulcorner \neg \varphi \urcorner \in X \text{ or } \ulcorner \neg \psi \urcorner \in X, \text{ or } \dots \end{aligned}$$

Let then $\Phi^0(X) = X$, $\Phi^{\alpha+1}(X) = \Phi(\Phi^\alpha(X))$, $\Phi^\lambda(X) = \bigcup_{\beta < \lambda} \Phi^\beta(X)$.

Lemma (Kripke (1975), Martin and Woodruff (1975))

If $S \subseteq |\mathcal{M}|$ is a fixed-point of Φ , then for all $\varphi \in \mathcal{L}_{\text{Tr}}$:

$$\varphi \in S \text{ iff } \text{Tr} \ulcorner \varphi \urcorner \in S$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Fixed-Point Semantics

			⋮
		$\text{Tr}^{<\kappa}(\top)$	
		$\text{Tr}^{<\kappa}(\top)$	
			⋮
	$\text{Tr}(\top), \top \wedge \top \dots$		
	\top		
		$ \mathcal{M} $	

$$\Phi^{\kappa+1}(\emptyset) = \Phi^{\kappa}(\emptyset)$$

$$\Phi^{\kappa}(\emptyset) = \mathcal{I}_{\Phi}$$

⋮

$$\Phi^2(\emptyset)$$

$$\Phi(\emptyset)$$

\emptyset

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Fixed-Point Semantics

⋮

	$\text{Tr}^{<\kappa}(\top)$	
	$\text{Tr}^{<\kappa}(\top)$	

$$\Phi^{\kappa+1}(\emptyset) = \Phi^{\kappa}(\emptyset)$$

$$\Phi^{\kappa}(\emptyset) = \mathcal{I}_{\Phi}$$

⋮

	$\text{Tr}(\top), \top \wedge \top \dots$	
	\top	
$ \mathcal{M} $		

⋮

$$\Phi^2(\emptyset)$$

$$\Phi(\emptyset)$$

$$\emptyset$$

The structure $(\mathcal{M}, \mathcal{I}_{\Phi})$ gives rise to a three-valued model for \mathcal{L}_{Tr} with Tr a ‘partial’ predicate. Define

$$\mathcal{M} \models \Gamma \Rightarrow \Delta :\Leftrightarrow (\forall \gamma \in \Gamma) |\gamma|_{\mathcal{I}_{\Phi}}^{\mathcal{M}} \neq 0 \rightarrow (\exists \delta \in \Delta) |\gamma|_{\mathcal{I}_{\Phi}}^{\mathcal{M}} = 1$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Restricting initial sequents

Already known in other contexts Kreuger (1994); Jäger and Stärk (1998); Schroeder-Heister (2016). This is contained in Nicolai (2018a). Structural rules are absorbed.

Definition (LPT)

$$\begin{array}{ll} \Gamma, \perp \Rightarrow \Delta & \Gamma \Rightarrow \top, \Delta \\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \text{Tr } \ulcorner A \urcorner} & \frac{A, \Gamma \Rightarrow \Delta}{\text{Tr } \ulcorner A \urcorner, \Gamma \Rightarrow \Delta} \\ (\neg\text{L}) \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} & (\neg\text{R}) \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \\ \vdots & \vdots \end{array}$$

- Now $(\mathcal{M}, S) \models \text{LPT}$ for S a fixed point of Φ .

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Restricting initial sequents

Already known in other contexts Kreuger (1994); Jäger and Stärk (1998); Schroeder-Heister (2016). This is contained in Nicolai (2018a). Structural rules are absorbed.

Definition (LPT)

$$\begin{array}{ll} \Gamma, \perp \Rightarrow \Delta & \Gamma \Rightarrow \top, \Delta \\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \text{Tr} \ulcorner A \urcorner} & \frac{A, \Gamma \Rightarrow \Delta}{\text{Tr} \ulcorner A \urcorner, \Gamma \Rightarrow \Delta} \\ (\neg\text{L}) \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} & (\neg\text{R}) \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \\ \vdots & \vdots \end{array}$$

- ▶ Now $(\mathcal{M}, S) \models \text{LPT}$ for S a fixed point of Φ .
- ▶ The model $(\mathcal{M}, \mathcal{I}_\Phi)$ satisfies a **fully operational, paracomplete** version system of naïve truth based on Strong-Kleene logic (modulo definition of consequence).

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Back to cut elimination

When contraction is around, the notion of **Tr -rank** is not enough:

$$\frac{\frac{\mathcal{D}_{00}}{\Gamma \Rightarrow \Delta, \text{Tr} \ulcorner \psi \urcorner, \text{Tr} \ulcorner \psi \urcorner \ [\alpha]}}{\Gamma \Rightarrow \Delta, \text{Tr} \ulcorner \psi \urcorner \ [\alpha]} \quad \frac{\mathcal{D}_1}{\text{Tr} \ulcorner \psi \urcorner, \Theta \Rightarrow \Lambda \ [\beta]} \\ \hline \Gamma, \Theta \Rightarrow \Delta, \Lambda \ [\alpha + \beta]$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Back to cut elimination

When contraction is around, the notion of **Tr -rank** is not enough:

$$\frac{\frac{\mathcal{D}_{\text{oo}}}{\Gamma \Rightarrow \Delta, \text{Tr} \ulcorner \psi \urcorner, \text{Tr} \ulcorner \psi \urcorner \ [\alpha]} \quad \frac{\mathcal{D}_1}{\text{Tr} \ulcorner \psi \urcorner, \Theta \Rightarrow \Lambda \ [\beta]}}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \ [\alpha + \beta]}$$

Now the idea here would be that we transform the derivation in

$$\frac{\frac{\frac{\mathcal{D}_{\text{oo}}^*}{\Gamma \Rightarrow \Delta, \psi, \psi \ [\alpha]} \quad \frac{\mathcal{D}_1^*}{\psi, \Theta \Rightarrow \Lambda \ [\beta]}}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \psi \ [\alpha + \beta]} \quad \frac{\mathcal{D}_1^*}{\psi, \Theta \Rightarrow \Lambda \ [\beta]}}{\Gamma, \Theta, \Theta \Rightarrow \Delta, \Lambda, \Lambda \ [2\alpha + \beta]} \quad \frac{\Gamma, \Theta, \Theta \Rightarrow \Delta, \Lambda, \Lambda \ [2\alpha + \beta]}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \ [2\alpha + \beta]}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Tr-complexity $\kappa(\cdot)$ of formulas

The ordinal Tr-complexity $\kappa_{\mathcal{D}}(\cdot)$ of a formula φ of \mathcal{L}_{Tr} in a derivation \mathcal{D} is defined inductively as follows:

- ▶ formulas of \mathcal{L} **have Tr-complexity** 0 in any \mathcal{D} ;

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Tr-complexity $\kappa(\cdot)$ of formulas

The ordinal Tr-complexity $\kappa_{\mathcal{D}}(\cdot)$ of a formula φ of \mathcal{L}_{Tr} in a derivation \mathcal{D} is defined inductively as follows:

- ▶ formulas of \mathcal{L} **have Tr-complexity** \mathbf{o} in any \mathcal{D} ;
- ▶ If \mathcal{D} is just $\Gamma, \varphi \Rightarrow \varphi, \Delta$ with $\varphi \in \mathcal{L}$, then $\kappa_{\mathcal{D}}(\psi) = \kappa_{\mathcal{D}}(\varphi) = \mathbf{o}$ for all $\psi \in \Gamma, \Delta$. Similarly for (\top) , (\perp) .

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Tr-complexity $\kappa(\cdot)$ of formulas

The ordinal Tr-complexity $\kappa_{\mathcal{D}}(\cdot)$ of a formula φ of \mathcal{L}_{Tr} in a derivation \mathcal{D} is defined inductively as follows:

- ▶ formulas of \mathcal{L} **have Tr-complexity 0** in any \mathcal{D} ;
- ▶ If \mathcal{D} is just $\Gamma, \varphi \Rightarrow \varphi, \Delta$ with $\varphi \in \mathcal{L}$, then $\kappa_{\mathcal{D}}(\psi) = \kappa_{\mathcal{D}}(\varphi) = 0$ for all $\psi \in \Gamma, \Delta$. Similarly for (\top) , (\perp) .
- ▶ If \mathcal{D} ends with

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \text{Tr } \ulcorner A \urcorner}$$

then the complexity of formulas in Γ, Δ is unchanged and $\kappa_{\mathcal{D}}(\text{Tr } \ulcorner A \urcorner) = \kappa_{\mathcal{D}}(A) + 1$ (similarly for $(\text{Tr } -\text{L})$).

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Tr-complexity $\kappa(\cdot)$ of formulas

The ordinal Tr-complexity $\kappa_{\mathcal{D}}(\cdot)$ of a formula φ of \mathcal{L}_{Tr} in a derivation \mathcal{D} is defined inductively as follows:

- ▶ formulas of \mathcal{L} **have Tr-complexity 0** in any \mathcal{D} ;
- ▶ If \mathcal{D} is just $\Gamma, \varphi \Rightarrow \varphi, \Delta$ with $\varphi \in \mathcal{L}$, then $\kappa_{\mathcal{D}}(\psi) = \kappa_{\mathcal{D}}(\varphi) = 0$ for all $\psi \in \Gamma, \Delta$. Similarly for (\top) , (\perp) .
- ▶ If \mathcal{D} ends with

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \text{Tr} \ulcorner A \urcorner}$$

then the complexity of formulas in Γ, Δ is unchanged and

$$\kappa_{\mathcal{D}}(\text{Tr} \ulcorner A \urcorner) = \kappa_{\mathcal{D}}(A) + 1 \text{ (similarly for } (\text{Tr} \text{-L}) \text{)}.$$

- ▶ If \mathcal{D} ends with

$$\frac{\gamma_1^1, \dots, \gamma_n^1 \Rightarrow \delta_1^1, \dots, \delta_m^1, \varphi \quad \gamma_1^2, \dots, \gamma_n^2 \Rightarrow \delta_1^2, \dots, \delta_m^2, \psi}{\gamma_1^3, \dots, \gamma_n^3 \Rightarrow \delta_1^3, \dots, \delta_m^3, \varphi \wedge \psi}$$

then

$$\kappa_{\mathcal{D}}(\varphi \wedge \psi) = \max(\kappa_{\mathcal{D}}(\varphi), \kappa_{\mathcal{D}}(\psi))$$

$$\kappa_{\mathcal{D}}(\gamma_i^3) = \max(\kappa_{\mathcal{D}}(\gamma_i^1), \kappa_{\mathcal{D}}(\gamma_i^2)) \quad 1 \leq i \leq n$$

$$\kappa_{\mathcal{D}}(\delta_j^3) = \max(\kappa_{\mathcal{D}}(\delta_j^1), \kappa_{\mathcal{D}}(\delta_j^2)) \quad 1 \leq j \leq m$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Full cut elimination

Crucially, rules of LPT are **κ -invertible**, e.g.:

If $\mathcal{D} \vdash_{\text{LPT}} \Gamma^1, \text{Tr } \ulcorner A \urcorner \Rightarrow \Delta^1$, then there is $\mathcal{D}' \vdash_{\text{LPT}} A, \Gamma \Rightarrow \Delta$ with $\kappa_{\mathcal{D}'}(\Gamma) \leq \kappa_{\mathcal{D}}(\Gamma^1)$, $\kappa_{\mathcal{D}'}(\Delta) \leq \kappa_{\mathcal{D}}(\Delta^1)$, and

$$\kappa_{\mathcal{D}'}(A) \leq \kappa_{\mathcal{D}}(\text{Tr } \ulcorner A \urcorner), \text{ if } \kappa_{\mathcal{D}}(\text{Tr } \ulcorner A \urcorner) = 0;$$

$$\kappa_{\mathcal{D}'}(A) < \kappa_{\mathcal{D}}(\text{Tr } \ulcorner A \urcorner), \text{ if } \kappa_{\mathcal{D}}(\text{Tr } \ulcorner A \urcorner) \neq 0.$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Full cut elimination

Crucially, rules of LPT are **κ -invertible**, e.g.:

If $\mathcal{D} \vdash_{\text{LPT}} \Gamma^1, \text{Tr} \ulcorner A \urcorner \Rightarrow \Delta^1$, then there is $\mathcal{D}' \vdash_{\text{LPT}} A, \Gamma \Rightarrow \Delta$ with $\kappa_{\mathcal{D}'}(\Gamma) \leq \kappa_{\mathcal{D}}(\Gamma^1)$, $\kappa_{\mathcal{D}'}(\Delta) \leq \kappa_{\mathcal{D}}(\Delta^1)$, and

$$\kappa_{\mathcal{D}'}(A) \leq \kappa_{\mathcal{D}}(\text{Tr} \ulcorner A \urcorner), \text{ if } \kappa_{\mathcal{D}}(\text{Tr} \ulcorner A \urcorner) = 0;$$

$$\kappa_{\mathcal{D}'}(A) < \kappa_{\mathcal{D}}(\text{Tr} \ulcorner A \urcorner), \text{ if } \kappa_{\mathcal{D}}(\text{Tr} \ulcorner A \urcorner) \neq 0.$$

Lemma

Contraction is κ -admissible and length-admissible, e.g. : If $\mathcal{D} \vdash_{\text{LPT}}^n \Gamma^1, \varphi^1, \varphi^2 \Rightarrow \Delta^1$, then there is a $\mathcal{D}' \vdash_{\text{LPT}}^n \Gamma, \varphi \Rightarrow \Delta$ with

$$\kappa_{\mathcal{D}'}(\Gamma^1) \leq \kappa_{\mathcal{D}'}(\Gamma); \quad \kappa_{\mathcal{D}'}(\Delta^1) \leq \kappa_{\mathcal{D}'}(\Delta)$$

$$\kappa_{\mathcal{D}'}(\varphi) \leq \max(\kappa_{\mathcal{D}'}(\varphi^1), \kappa_{\mathcal{D}'}(\varphi^2)).$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Full cut elimination

Crucially, rules of LPT are **κ -invertible**, e.g.:

If $\mathcal{D} \vdash_{\text{LPT}} \Gamma^1, \text{Tr} \ulcorner A \urcorner \Rightarrow \Delta^1$, then there is $\mathcal{D}' \vdash_{\text{LPT}} A, \Gamma \Rightarrow \Delta$ with $\kappa_{\mathcal{D}'}(\Gamma) \leq \kappa_{\mathcal{D}}(\Gamma^1)$, $\kappa_{\mathcal{D}'}(\Delta) \leq \kappa_{\mathcal{D}}(\Delta^1)$, and

$$\kappa_{\mathcal{D}'}(A) \leq \kappa_{\mathcal{D}}(\text{Tr} \ulcorner A \urcorner), \text{ if } \kappa_{\mathcal{D}}(\text{Tr} \ulcorner A \urcorner) = 0;$$

$$\kappa_{\mathcal{D}'}(A) < \kappa_{\mathcal{D}}(\text{Tr} \ulcorner A \urcorner), \text{ if } \kappa_{\mathcal{D}}(\text{Tr} \ulcorner A \urcorner) \neq 0.$$

Lemma

Contraction is κ -admissible and length-admissible, e.g. : If $\mathcal{D} \vdash_{\text{LPT}}^n \Gamma^1, \varphi^1, \varphi^2 \Rightarrow \Delta^1$, then there is a $\mathcal{D}' \vdash_{\text{LPT}}^n \Gamma, \varphi \Rightarrow \Delta$ with

$$\kappa_{\mathcal{D}'}(\Gamma^1) \leq \kappa_{\mathcal{D}'}(\Gamma); \quad \kappa_{\mathcal{D}'}(\Delta^1) \leq \kappa_{\mathcal{D}'}(\Delta)$$

$$\kappa_{\mathcal{D}'}(\varphi) \leq \max(\kappa_{\mathcal{D}'}(\varphi^1), \kappa_{\mathcal{D}'}(\varphi^2)).$$

Proposition

If \mathcal{D}_0 is a cut-free proof of $\Gamma^1 \Rightarrow \Delta^1$, φ^1 in LPT, and \mathcal{D}_1 is a cut-free LPT-proof of $\varphi^2, \Gamma^2 \Rightarrow \Delta^2$, then there is a cut-free proof \mathcal{D} of $\Gamma^3 \Rightarrow \Delta^3$ with $\kappa_{\mathcal{D}}(\Gamma^3) \leq \max(\kappa_{\mathcal{D}_0}(\Gamma^1), \kappa_{\mathcal{D}_1}(\Gamma^2))$ and $\kappa_{\mathcal{D}}(\Delta^3) \leq \max(\kappa_{\mathcal{D}_0}(\Delta^1), \kappa_{\mathcal{D}_1}(\Delta^2))$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- ▶ The ideal of **semantic closure** is at odds with resources that outstrip the ones available in one's semantic theory. Cut-elimination procedures are usually formalizable in weak arithmetical systems.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- ▶ The ideal of **semantic closure** is at odds with resources that outstrip the ones available in one's semantic theory. Cut-elimination procedures are usually formalizable in weak arithmetical systems.
- ▶ When nonlogical initial sequents are around, full cut elimination is not in general available: however by eliminating cuts on formulas of the form $\text{Tr } \ulcorner A \urcorner$, one can obtain **conservativity proofs**.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- ▶ The ideal of **semantic closure** is at odds with resources that outstrip the ones available in one's semantic theory. Cut-elimination procedures are usually formalizable in weak arithmetical systems.
- ▶ When nonlogical initial sequents are around, full cut elimination is not in general available: however by eliminating cuts on formulas of the form $\text{Tr } \ulcorner A \urcorner$, one can obtain **conservativity proofs**.
- ▶ Another advantage of the approach with restricted initial sequents is that – unlike the contraction-free approaches – there are natural **infinitary systems** that arise from the logic and that give succinct presentations of Π_1^1 -sets.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth Bearers

- ▶ There are good reasons to require an ontology of bearers of truth prior to discussing principles of truth. We want to prove **in the object language** things like:

$$\forall \varphi, \psi \exists \chi (\chi = (\varphi \wedge \psi) \wedge \varphi \neq \chi)$$

This is usually guaranteed by assuming a theory of finite objects (as we shall see in a moment).

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth Bearers

- ▶ There are good reasons to require an ontology of bearers of truth prior to discussing principles of truth. We want to prove **in the object language** things like:

$$\forall \varphi, \psi \exists \chi (\chi = (\varphi \wedge \psi) \wedge \varphi \neq \chi)$$

This is usually guaranteed by assuming a theory of finite objects (as we shall see in a moment).

- ▶ Notice that this is imposing non-trivial constraints. More ‘philosophical’ theories of truth are often formulated in terms of **propositions**, and not sentence types (Horwich, 1998; Soames, 1998; Jago, 2018). This rules out that propositions are coarse-grained, e.g. sets of possible worlds.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Arithmetic

- ▶ Peano arithmetic (PA) is the preferred base theory for systems of truth. It is usually formulated in $\mathcal{L}_{\mathbb{N}} = \{0, S, +, \times\}$ and features equations for its primitives, e.g.

$$(x + 0) = x \qquad x + Sy = S(x + y)$$

and the first-order induction schema

$$\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x \varphi(x) \qquad \text{for } \varphi \in \mathcal{L}_{\mathbb{N}}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Arithmetic

- ▶ Peano arithmetic (PA) is the preferred base theory for systems of truth. It is usually formulated in $\mathcal{L}_{\mathbb{N}} = \{0, S, +, \times\}$ and features equations for its primitives, e.g.

$$(x + 0) = x \qquad x + Sy = S(x + y)$$

and the first-order induction schema

$$\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x \varphi(x) \qquad \text{for } \varphi \in \mathcal{L}_{\mathbb{N}}$$

- ▶ Alternatively, one can employ a theory of strings and concatenation $\hat{\cdot}$ with two atoms a, b based on **Tarski's axiom**
$$a \hat{\cdot} y = u \hat{\cdot} v \leftrightarrow \exists w((x = u \hat{\cdot} w \wedge v = w \hat{\cdot} y) \vee (u = x \hat{\cdot} w \wedge y = w \hat{\cdot} v))$$
With first-order string induction, the two theories are **mutually interpretable**. An accessible source is Ganea (2009).

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Arithmetic

- ▶ Peano arithmetic (PA) is the preferred base theory for systems of truth. It is usually formulated in $\mathcal{L}_{\mathbb{N}} = \{0, S, +, \times\}$ and features equations for its primitives, e.g.

$$(x + 0) = x \qquad x + S y = S(x + y)$$

and the first-order induction schema

$$\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x \varphi(x) \qquad \text{for } \varphi \in \mathcal{L}_{\mathbb{N}}$$

- ▶ Alternatively, one can employ a theory of strings and concatenation $\hat{\cdot}$ with two atoms a, b based on **Tarski's axiom**
$$a \hat{\cdot} y = u \hat{\cdot} v \leftrightarrow \exists w((x = u \hat{\cdot} w \wedge v = w \hat{\cdot} y) \vee (u = x \hat{\cdot} w \wedge y = w \hat{\cdot} v))$$
With first-order string induction, the two theories are **mutually interpretable**. An accessible source is Ganea (2009).
- ▶ Finite set theories are also a convenient choice. For instance
 - ▶ Kaye and Wong (2006) show that PA and $KF \setminus \{\text{Inf}\} +$ 'every set has a transitive closure' are **bi-interpretable**;
 - ▶ similarly, a neat set theory based by Świerczkowski (2003) based on the **adjunction operation** $x \triangleleft y \mapsto x \cup \{y\}$ is bi-interpretable with PA.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Doing with less

- ▶ Ultimately, what we require to establish the basic properties of the truth bearers are **a good notion of sequence**, and **a minimum of induction** to handle suitable forms of recursion.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Doing with less

- ▶ Ultimately, what we require to establish the basic properties of the truth bearers are **a good notion of sequence**, and **a minimum of induction** to handle suitable forms of recursion.
- ▶ For the former the notion of a **sequential** theory is enough – see Visser (2010) for a comprehensive overview. A theory is sequential if it interprets – with no relativization of quantifiers – the theory **AS** given by the **empty set** axiom and **adjunction** – which is as strong as Robinson's **Q**.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Doing with less

- ▶ Ultimately, what we require to establish the basic properties of the truth bearers are **a good notion of sequence**, and **a minimum of induction** to handle suitable forms of recursion.
- ▶ For the former the notion of a **sequential** theory is enough – see Visser (2010) for a comprehensive overview. A theory is sequential if it interprets – with no relativization of quantifiers – the theory **AS** given by the **empty set** axiom and **adjunction** – which is as strong as Robinson's **Q**.
- ▶ As to induction, since all the relevant syntactic notions (terms, formulas, proofs) are **p-time decidable**, the theory **S₂¹** by Buss (1986) suffices. However, many of the results that I will treat below are specific to **PA** (or equivalents), and it is object of current research to check which results are stable over $T \supseteq S_2^1$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Ordinals

- ▶ A good base theory will also provide a satisfactory **representation of ordinals**. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal Γ_0 :

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Ordinals

- ▶ A good base theory will also provide a satisfactory **representation of ordinals**. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal Γ_0 :
 - ▶ α is **principal** if it cannot be expressed as $\zeta + \eta$ for $\zeta, \eta < \alpha$. Define:

$C(o) :=$ ‘the class of principal ordinals’

$C(\alpha + 1) :=$ ‘the class of fixed points of the function enumerating $C(\alpha)$ ’

$C(\lambda) := \bigcap_{\zeta < \lambda} C(\zeta)$ for λ a limit ordinal

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Ordinals

- ▶ A good base theory will also provide a satisfactory **representation of ordinals**. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal Γ_0 :

- ▶ α is **principal** if it cannot be expressed as $\zeta + \eta$ for $\zeta, \eta < \alpha$. Define:

$C(o) :=$ ‘the class of principal ordinals’

$C(\alpha + 1) :=$ ‘the class of fixed points of the function enumerating $C(\alpha)$ ’

$C(\lambda) := \bigcap_{\zeta < \lambda} C(\zeta)$ for λ a limit ordinal

- ▶ The *Veblen functions* φ_α are the enumerating functions of $C(\alpha)$. The class of *strongly critical* ordinals SC contains precisely the ordinals α that are themselves α -critical. Γ_ζ indicates the ζ -th strongly critical ordinal.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Ordinals

- ▶ A good base theory will also provide a satisfactory **representation of ordinals**. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal Γ_0 :

- ▶ α is **principal** if it cannot be expressed as $\zeta + \eta$ for $\zeta, \eta < \alpha$. Define:

$C(o) :=$ ‘the class of principal ordinals’

$C(\alpha + 1) :=$ ‘the class of fixed points of the function enumerating $C(\alpha)$ ’

$C(\lambda) := \bigcap_{\zeta < \lambda} C(\zeta)$ for λ a limit ordinal

- ▶ The *Veblen functions* φ_α are the enumerating functions of $C(\alpha)$. The class of *strongly critical* ordinals SC contains precisely the ordinals α that are themselves α -critical. Γ_ζ indicates the ζ -th strongly critical ordinal.
- ▶ Principal ordinals α that are not themselves strongly critical are such that $\alpha = \varphi_\zeta \eta$ for $\eta, \zeta < \alpha$. Therefore, by this fact and Cantor’s normal form theorem, ordinals $< \Gamma_0$ can be uniquely determined as words of the alphabet $(o, +, \varphi \cdot)$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Ordinals

- ▶ A good base theory will also provide a satisfactory **representation of ordinals**. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal Γ_0

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Ordinals

- ▶ A good base theory will also provide a satisfactory **representation of ordinals**. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal Γ_0
- ▶ A notation system for Γ_0 is of the form $(OT, PT, |\cdot|, <)$, with
 - ▶ OT the set of natural number 'codes' for ordinals $< \Gamma_0$
 - ▶ $OT \subseteq OT$ the set of codes of principal ordinals
 - ▶ $|\cdot|: OT \rightarrow ON$
 - ▶ $n < m \Leftrightarrow n \in OT \wedge m \in OT \wedge |n| < |m|$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Ordinals

- ▶ A good base theory will also provide a satisfactory **representation of ordinals**. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal Γ_0
- ▶ A notation system for Γ_0 is of the form $(OT, PT, |\cdot|, <)$, with
 - ▶ OT the set of natural number ‘codes’ for ordinals $< \Gamma_0$
 - ▶ $OT \subseteq \mathbb{N}$ the set of codes of principal ordinals
 - ▶ $|\cdot|: OT \rightarrow \mathbb{N}$
 - ▶ $n < m \iff n \in OT \wedge m \in OT \wedge |n| < |m|$
- ▶ Using standard coding techniques one can show that $OT, PT, <$ are **primitive recursive**. Actually, Beckmann et al. (2003) show that they can be showed to be p-time and represented in S_2^1 – notice that I **do not** mean that Γ_0 can be well-founded in S_2^1 !

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Schemata

For ordinals $\alpha < \Gamma_o$, we denote with a the corresponding numeral in the representation of OT and we do not distinguish between ordinal functions such as the Veblen functions and their representations.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Schemata

For ordinals $\alpha < \Gamma_0$, we denote with a the corresponding numeral in the representation of OT and we do not distinguish between ordinal functions such as the Veblen functions and their representations. The system (OT, PT, $<$) enables us to formulate the following principles of transfinite induction:

$$\begin{aligned}(\text{TI}_{\mathcal{L}_{\text{Tr}}}^{\varepsilon_0}) \quad & \frac{\forall a < b \, \phi(a), \Gamma \Rightarrow \Delta, \phi(b)}{\Gamma \Rightarrow \Delta, \forall a < \varepsilon_0 \, \phi(a)} \\[1em](\text{TI}_{\mathcal{L}_{\text{Tr}}}^{<\omega^\omega}) \quad & \frac{\forall a < b \, \phi(a), \Gamma \Rightarrow \Delta, \phi(b)}{\Gamma \Rightarrow \Delta, \forall a < c \, \phi(a)} \quad \text{for all } \gamma (= |c|) < \omega^\omega \\[1em](\text{TI}_{\mathcal{L}_{\text{Tr}}}^{<\varepsilon_0}) \quad & \frac{\forall a < b \, \phi(a), \Gamma \Rightarrow \Delta, \phi(b)}{\Gamma \Rightarrow \Delta, \forall a < c \, \phi(a)} \quad \text{for all } \gamma < \varepsilon_0\end{aligned}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth as primitive

- ▶ Truth-theoretic deflationism holds that truth is **not a genuine property** and that its function is mainly that of a **generalizing device** (Quine, 1970; Field, 1994; Horwich, 1998).

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth as primitive

- ▶ Truth-theoretic deflationism holds that truth is **not a genuine property** and that its function is mainly that of a **generalizing device** (Quine, 1970; Field, 1994; Horwich, 1998).
- ▶ Unlike other notions that have been taken to be primitive for lack of consensus over a definition – e.g. knowledge, see Williamson (2000) – **Tarski's theorem** uncontroversially establishes this (Halbach, 2014, Ch. 1).

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Truth as primitive

- ▶ Truth-theoretic deflationism holds that truth is **not a genuine property** and that its function is mainly that of a **generalizing device** (Quine, 1970; Field, 1994; Horwich, 1998).
- ▶ Unlike other notions that have been taken to be primitive for lack of consensus over a definition – e.g. knowledge, see Williamson (2000) – **Tarski's theorem** uncontroversially establishes this (Halbach, 2014, Ch. 1).
- ▶ Truth is a fundamental **semantic** concept. A theory of meaning for natural language expressions is not much more than a (Tarskian) **theory of truth** for it (Davidson, 1984).

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Tarskian Truth

The theory of truth in \mathcal{L}_{Tr} that Davidson had in mind extends PA with the following:

Definition (CT)

$$\forall s, t (\text{Tr}(s = t) \leftrightarrow s^\circ = t^\circ)$$

$$\forall \varphi \in \mathcal{L} (\text{Tr}(\neg \varphi) \leftrightarrow \neg \text{Tr} \varphi)$$

$$\forall \varphi, \psi \in \mathcal{L} (\text{Tr}(\varphi \wedge \psi) \leftrightarrow \text{Tr} \varphi \wedge \text{Tr} \psi)$$

$$\forall v, \forall \varphi(v) \in \mathcal{L} (\text{Tr}(\forall v \varphi) \leftrightarrow \forall x \text{Tr} \varphi(\dot{x}))$$

$$\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x) \quad \text{with } \varphi(v) \in \mathcal{L}_{\text{Tr}}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Tarskian Truth

The theory of truth in \mathcal{L}_{Tr} that Davidson had in mind extends PA with the following:

Definition (CT)

$$\forall s, t (\text{Tr}(s = t) \leftrightarrow s^\circ = t^\circ)$$

$$\forall \varphi \in \mathcal{L} (\text{Tr}(\neg \varphi) \leftrightarrow \neg \text{Tr} \varphi)$$

$$\forall \varphi, \psi \in \mathcal{L} (\text{Tr}(\varphi \wedge \psi) \leftrightarrow \text{Tr} \varphi \wedge \text{Tr} \psi)$$

$$\forall v, \forall \varphi(v) \in \mathcal{L} (\text{Tr}(\forall v \varphi) \leftrightarrow \forall x \text{Tr} \varphi(\dot{x}))$$

$$\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x) \quad \text{with } \varphi(v) \in \mathcal{L}_{\text{Tr}}$$

Important variations are obtained by tweaking the **induction schema**:

- ▶ **CT** \upharpoonright (a.k.a. **CT⁻**) is obtained by restricting induction to \mathcal{L}
- ▶ **CT_{int}** is obtained by adopting the **internal induction schema**

$$\forall \varphi(v) (\text{Tr} \varphi(0/v) \wedge \forall y (\text{Tr} \varphi(\dot{y}/v) \rightarrow \text{Tr} \varphi(\dot{S}y/v)) \rightarrow \forall x \text{Tr} \varphi(\dot{x}/v))$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Conservativeness

Thesis (Shapiro, 1998; Ketland, 1999)

CT proves $\text{Con}(\text{PA})$, therefore deflationism is untenable.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Conservativeness

Thesis (Shapiro, 1998; Ketland, 1999)

CT proves $\text{Con}(\text{PA})$, therefore deflationism is untenable.

This contrasts with:

Proposition (Kotlarski et al. (1981); Visser and Enayat (2015); Leigh (2015))

CT_{int} *is a conservative extension of* PA.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Conservativeness

Thesis (Shapiro, 1998; Ketland, 1999)

CT proves $\text{Con}(\text{PA})$, therefore deflationism is untenable.

This contrasts with:

Proposition (Kotlarski et al. (1981); Visser and Enayat (2015); Leigh (2015))

CT_{int} *is a conservative extension of* PA.

The discussion took a strong technical turn, brilliantly summarized in Cieliski (2017) – with many original contributions. It's worth mentioning:

Proposition (Enayat and Pakhomov (2018))

$\text{CT} \upharpoonright$ *plus 'disjunctive correctness', i.e.*

$$\forall s (\text{Tr}(\bigvee_{i < s} \varphi_i) \leftrightarrow \exists i < s \text{Tr} \varphi_i)$$

is the same theory as $\text{CT}[\text{I}\Delta_0]$, *and therefore proves* $\text{Con}(\text{PA})$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Conservativeness

- ▶ Despite the technical interest, the debate seems to be built on shaky foundations. Virtually **no deflationist** has thoroughly defended the claim that truth **has to be** conservative over the base theory.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Conservativeness

- ▶ Despite the technical interest, the debate seems to be built on shaky foundations. Virtually **no deflationist** has thoroughly defended the claim that truth **has to be** conservative over the base theory.
- ▶ By contrast, it has repeatedly been argued that **truth has to be nonconservative**, but in a way that is **distinctively metalinguistic**, i.e. it does not interfere with the subject matter of the base theory over which truth is built.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Conservativeness

- ▶ Despite the technical interest, the debate seems to be built on shaky foundations. Virtually **no deflationist** has thoroughly defended the claim that truth **has to be** conservative over the base theory.
- ▶ By contrast, it has repeatedly been argued that **truth has to be nonconservative**, but in a way that is **distinctively metalinguistic**, i.e. it does not interfere with the subject matter of the base theory over which truth is built.
- ▶ This has led to the programme of **'disentangling'** syntactic quantifiers from quantifiers over natural numbers:

Proposition (Nicolai (2015, 2016))

If one formulates CT \upharpoonright as a two-sorted theory, with 'syntactic' quantifiers and 'number-theoretic' quantifiers, and truth applying only over syntactic objects, then:

- ▶ *The theory of truth becomes **trivially** conservative over PA;*
- ▶ *This version of CT \upharpoonright plus 'all axioms of PA are true' is **mutually interpretable** with PA + Con(PA).*

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

**Classical v Nonclassical
Kripkean truth**

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Feferman's project

To give a nice presentation of the **reflective closure** of **PA** – and possibly of further ‘natural’ theories: i.e. the (truth-)theory that makes **explicit all that is implicit in the acceptance of PA.**

Basics

Paradox(es)

Consistency via

cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Feferman's project

To give a nice presentation of the **reflective closure** of PA – and possibly of further ‘natural’ theories: i.e. the (truth-)theory that makes explicit all that is implicit in the acceptance of PA.

First attempt: $\text{CT}_{<a}, \alpha \leq \Gamma_0$

$$\mathcal{L}_o := \mathcal{L}_{\text{Tr}}$$

$$\mathcal{L}_{<c} := \mathcal{L} \cup \{\text{Tr}_b \mid b < c\}$$

With $b < c$:

$$\forall s, t (\text{Tr}_b(s = t) \leftrightarrow s^\circ = t^\circ)$$

$$\forall \varphi \in \mathcal{L}_{<b} (\text{Tr}_b(\neg\varphi) \leftrightarrow \neg\text{Tr}_b\varphi)$$

$$\forall \varphi, \psi \in \mathcal{L}_{<b} (\text{Tr}_b(\varphi \wedge \psi) \leftrightarrow \text{Tr}_b\varphi \wedge \text{Tr}_b\psi)$$

$$\forall v, \forall \varphi(v) \in \mathcal{L} (\text{Tr}_b(\forall v\varphi) \leftrightarrow \forall x \text{Tr}_b\varphi(\dot{x}))$$

$$\forall \varphi \in \mathcal{L}_{<a < b} (\text{Tr}_b\text{Tr}_a\varphi \leftrightarrow \text{Tr}_a\varphi)$$

$$\forall d < b, \forall \varphi \in \mathcal{L}_{<d} (\text{Tr}_b\text{Tr}_d\varphi \leftrightarrow \text{Tr}_d\varphi)$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Feferman's project

- ▶ The project of isolating $CT_{<\varepsilon_0}$ or $CT_{<\Gamma_0}$ as **natural stopping points**, that was congenial to Feferman's project, depended essentially on other results, such as Feferman's and Schütte's independent characterization of Γ_0 , or the provable well-orderings of **PA**.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Feferman's project

- ▶ The project of isolating $\text{CT}_{<\varepsilon_0}$ or $\text{CT}_{<\Gamma_0}$ as **natural stopping points**, that was congenial to Feferman's project, depended essentially on other results, such as Feferman's and Schütte's independent characterization of Γ_0 , or the provable well-orderings of PA.
- ▶ The next step was to find an independent characterization of such theories:

Definition (KF)

$$\forall s, t (\text{Tr}(s = t) \leftrightarrow s^\circ = t^\circ)$$

$$\forall s, t (\text{Tr}(s \neq t) \leftrightarrow s^\circ \neq t^\circ)$$

$$\forall t (\text{Tr Tr } t \leftrightarrow \text{Tr } t^\circ)$$

$$\forall t (\text{Tr Tr } \neg t \leftrightarrow \text{Tr } \neg t^\circ)$$

$$\forall \varphi \in \mathcal{L}_{\text{Tr}} (\text{Tr}(\neg\neg\varphi) \leftrightarrow \text{Tr } \varphi)$$

$$\forall \varphi, \psi \in \mathcal{L}_{\text{Tr}} (\text{Tr}(\varphi \wedge \psi) \leftrightarrow \text{Tr } \varphi \wedge \text{Tr } \psi)$$

$$\forall \varphi, \psi \in \mathcal{L}_{\text{Tr}} (\text{Tr} \neg(\varphi \wedge \psi) \leftrightarrow \text{Tr } \neg\varphi \vee \text{Tr } \neg\psi)$$

$$\forall v, \forall \varphi(v) \in \mathcal{L}_{\text{Tr}} (\text{Tr}(\forall v \varphi) \leftrightarrow \forall x \text{Tr } \varphi(\dot{x}))$$

$$\forall v, \forall \varphi(v) \in \mathcal{L}_{\text{Tr}} (\text{Tr}(\neg \forall v \varphi) \leftrightarrow \exists x \text{Tr } \neg \varphi(\dot{x}))$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Properties of KF

- Semantically KF fits nicely with Kripke's fixed-point semantics (Kripke, 1975),

$(\mathbb{N}, S) \models \text{KF}$ iff S is a fixed point of Kripke's theory of truth

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

**Classical v Nonclassical
Kripkean truth**

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Properties of KF

- Semantically KF fits nicely with Kripke's fixed-point semantics (Kripke, 1975),

$(\mathbb{N}, S) \models \text{KF}$ iff S is a fixed point of Kripke's theory of truth

- The full Tr -schema is available for **meaningful** predicates satisfying $D(x) :\leftrightarrow \text{Tr } x \vee \text{Tr } \neg x$, i.e. for all $A \in \mathcal{L}_{\text{Tr}}$:

$$\text{KF} \vdash D(\ulcorner A \urcorner) \rightarrow (\text{Tr } \ulcorner A \urcorner \leftrightarrow A)$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Properties of KF

- Semantically KF fits nicely with Kripke's fixed-point semantics (Kripke, 1975),

$(\mathbb{N}, S) \models \text{KF}$ iff S is a fixed point of Kripke's theory of truth

- The full Tr -schema is available for **meaningful** predicates satisfying $D(x) : \leftrightarrow \text{Tr } x \vee \text{Tr } \neg x$, i.e. for all $A \in \mathcal{L}_{\text{Tr}}$:

$$\text{KF} \vdash D(\ulcorner A \urcorner) \rightarrow (\text{Tr } \ulcorner A \urcorner \leftrightarrow A)$$

Proposition (Feferman (1991); Cantini (1989))

KF is proof-theoretically equivalent to $(\Pi_1^0\text{-CA})_{<\varepsilon_0}$.

Proof Idea.

Lower bound: PA in \mathcal{L}_{Tr} proves $\text{TI}_{\mathcal{L}_{\text{Tr}}}^{<\varepsilon_0}$. Now KF proves:

$$\varphi \in \mathcal{L}_{<a} \rightarrow D(\varphi) \Rightarrow \varphi \in \mathcal{L}_a \rightarrow D(\varphi)$$

An application of $\text{TI}_{\mathcal{L}_{\text{Tr}}}^{<\varepsilon_0}$ yields an embedding of $\text{CT}_{<\varepsilon_0}$, which suffices. □

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Properties of KF

- Semantically KF fits nicely with Kripke's fixed-point semantics (Kripke, 1975),

$(\mathbb{N}, S) \models \text{KF}$ iff S is a fixed point of Kripke's theory of truth

- The full Tr -schema is available for **meaningful** predicates satisfying $D(x) : \leftrightarrow \text{Tr } x \vee \text{Tr } \neg x$, i.e. for all $A \in \mathcal{L}_{\text{Tr}}$:

$$\text{KF} \vdash D(\ulcorner A \urcorner) \rightarrow (\text{Tr } \ulcorner A \urcorner \leftrightarrow A)$$

Proposition (Feferman (1991); Cantini (1989))

KF is proof-theoretically equivalent to $(\Pi_1^0\text{-CA})_{<\varepsilon_0}$.

Proof Idea.

Upper bound: One formulates KF in a Tait (one-sided) infinitary calculus, and analyzes **quasi-normal** derivations, i.e. derivations with only cuts on $\text{Tr } t$ and $\neg \text{Tr } t$ and proves in $\text{CT}_{<\varepsilon_0}$ that

$$\text{if } \text{KF}^\infty \vdash^\alpha \text{Tr } \ulcorner A \urcorner, \text{ then } \text{Tr } {}_2^\alpha \ulcorner A \urcorner$$



Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Symmetries

- ▶ One important drawback of KF is that its **internal theory** $\{\varphi \in \mathcal{L}_{\text{Tr}} \mid \text{KF} \vdash \varphi\}$ is different from its theorems: for instance $\text{KF} \vdash \lambda \vee \neg \lambda$ but $\text{KF} \not\vdash \text{Tr} \ulcorner \lambda \vee \neg \lambda \urcorner$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Symmetries

- ▶ One important drawback of KF is that its **internal theory** $\{\varphi \in \mathcal{L}_{\text{Tr}} \mid \text{KF} \vdash \varphi\}$ is different from its theorems: for instance $\text{KF} \vdash \lambda \vee \neg\lambda$ but $\text{KF} \not\models \text{Tr} \ulcorner \lambda \vee \neg\lambda \urcorner$.
- ▶ To overcome this:

Reinhardt's thesis

One should adopt an **instrumental** reading of KF. Its conceptual core is given by its **internal theory**.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Symmetries

- ▶ One important drawback of KF is that its **internal theory** $\{\varphi \in \mathcal{L}_{Tr} \mid KF \vdash \varphi\}$ is different from its theorems: for instance $KF \vdash \lambda \vee \neg\lambda$ but $KF \not\vdash Tr \ulcorner \lambda \vee \neg\lambda \urcorner$.
- ▶ To overcome this:

Reinhardt's thesis

One should adopt an **instrumental** reading of KF. Its conceptual core is given by its **internal theory**.

Lemma (Halbach and Horsten (2006))

There are A 's in \mathcal{L}_{Tr} such that $KF \vdash Tr \ulcorner A \urcorner$ but the proof essentially employs B 's such that $KF \not\vdash Tr \ulcorner B \urcorner$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Kripke-Feferman in four-valued logic

- From classical logic, **remove** both negation introduction rules are replace them with, e.g.:

$$\frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)} \qquad \frac{\neg\varphi, \Gamma \Rightarrow \Delta \qquad \neg\psi, \Gamma \Rightarrow \Delta}{\neg(\varphi \wedge \psi), \Gamma \Rightarrow \Delta}$$

One obtains a logic in the vicinity of **FDE**.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Kripke-Feferman in four-valued logic

- From classical logic, **remove** both negation introduction rules are replace them with, e.g.:

$$\frac{\Gamma \Rightarrow \Delta, \neg\phi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\phi \wedge \psi)} \quad \frac{\neg\phi, \Gamma \Rightarrow \Delta \quad \neg\psi, \Gamma \Rightarrow \Delta}{\neg(\phi \wedge \psi), \Gamma \Rightarrow \Delta}$$

One obtains a logic in the vicinity of **FDE**.

- As to truth:

PKF

$$s^\circ = t^\circ \Leftrightarrow \text{Tr}(s = t)$$

$$\neg \text{Tr } \phi \Leftrightarrow \text{Tr } \neg \phi$$

$$\text{Tr}(\phi \wedge \psi) \Leftrightarrow \text{Tr } \phi \wedge \text{Tr } \psi$$

$$\text{Tr Tr } \phi \Leftrightarrow \text{Tr } \phi$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Kripke-Feferman in four-valued logic

- From classical logic, **remove** both negation introduction rules are replace them with, e.g.:

$$\frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)} \quad \frac{\neg\varphi, \Gamma \Rightarrow \Delta \quad \neg\psi, \Gamma \Rightarrow \Delta}{\neg(\varphi \wedge \psi), \Gamma \Rightarrow \Delta}$$

One obtains a logic in the vicinity of **FDE**.

- As to truth:

PKF

$$\begin{aligned}s^\circ = t^\circ &\Leftrightarrow \text{Tr}(s = t) \\ \neg\text{Tr } \varphi &\Leftrightarrow \text{Tr } \neg\varphi \\ \text{Tr}(\varphi \wedge \psi) &\Leftrightarrow \text{Tr } \varphi \wedge \text{Tr } \psi \\ \text{Tr Tr } \varphi &\Leftrightarrow \text{Tr } \varphi\end{aligned}$$

- The theories are closely related:

ω -categoricity

For S a fixed point of Kripke's theory of truth,

$$(\mathbb{N}, S) \models \text{KF} \text{ iff } (\mathbb{N}, S) \models_{\text{FDE}} \text{PKF}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Kripke-Feferman in four-valued logic

Unlike KF, $\text{PKF} \vdash \text{Tr} \ulcorner A \urcorner$ iff $\text{PKF} \vdash A$, for all $A \in \mathcal{L}_{\text{Tr}}$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Kripke-Feferman in four-valued logic

Unlike KF, $\text{PKF} \vdash \text{Tr} \ulcorner A \urcorner$ iff $\text{PKF} \vdash A$, for all $A \in \mathcal{L}_{\text{Tr}}$.

Proposition (Halbach and Horsten (2006))

PKF proves the same arithmetical sentences as $\text{CT}_{<\omega^\omega}$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Kripke-Feferman in four-valued logic

Unlike KF, $\text{PKF} \vdash \text{Tr} \ulcorner A \urcorner$ iff $\text{PKF} \vdash A$, for all $A \in \mathcal{L}_{\text{Tr}}$.

Proposition (Halbach and Horsten (2006))

PKF proves the same arithmetical sentences as $\text{CT}_{<\omega^\omega}$.

Proposition (Feferman (1991))

KF proves the same arithmetical sentences of $\text{CT}_{<\varepsilon_0}$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Kripke-Feferman in four-valued logic

Unlike KF, $\text{PKF} \vdash \text{Tr} \ulcorner A \urcorner$ iff $\text{PKF} \vdash A$, for all $A \in \mathcal{L}_{\text{Tr}}$.

Proposition (Halbach and Horsten (2006))

PKF proves the same arithmetical sentences as $\text{CT}_{<\omega^\omega}$.

Proposition (Feferman (1991))

KF proves the same arithmetical sentences of $\text{CT}_{<\varepsilon_0}$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Kripke-Feferman in four-valued logic

Unlike KF, $\text{PKF} \vdash \text{Tr} \ulcorner A \urcorner$ iff $\text{PKF} \vdash A$, for all $A \in \mathcal{L}_{\text{Tr}}$.

Proposition (Halbach and Horsten (2006))

PKF proves the same arithmetical sentences as $\text{CT}_{<\omega^\omega}$.

Proposition (Feferman (1991))

KF proves the same arithmetical sentences of $\text{CT}_{<\varepsilon_0}$.

Recalling Reinhardt's thesis:

Corollary

The internal theory of KF and PKF differ considerably.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Kripke-Feferman in four-valued logic

Unlike KF, $\text{PKF} \vdash \text{Tr} \ulcorner A \urcorner$ iff $\text{PKF} \vdash A$, for all $A \in \mathcal{L}_{\text{Tr}}$.

Proposition (Halbach and Horsten (2006))

PKF proves the same arithmetical sentences as $\text{CT}_{<\omega^\omega}$.

Proposition (Feferman (1991))

KF proves the same arithmetical sentences of $\text{CT}_{<\varepsilon_0}$.

Recalling Reinhardt's thesis:

Corollary

The internal theory of KF and PKF differ considerably.

Proposition (Nicolai (2018b))

- ▶ $\text{PKF} = \{A \in \mathcal{L}_{\text{Tr}} \mid \text{KF}_{\text{int}} \vdash \text{Tr} \ulcorner A \urcorner\}$
- ▶ $\{A \in \mathcal{L}_{\text{Tr}} \mid \text{KF} \vdash \text{Tr} \ulcorner A \urcorner\} = \text{PKF} + \text{TI}_{\mathcal{L}_{\text{Tr}}}^{<\varepsilon_0}$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

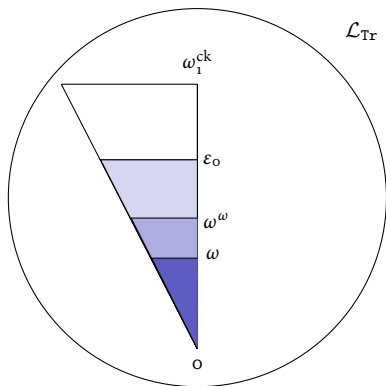
Extensions

Reflection

Modal Logic

Modal Predicates

References


 \mathcal{L}_{Tr}

	\mathcal{L}	\mathcal{L}_{Tr}
PKF, KF_{int}	$< \varphi_{\omega} o$	$\varphi_o \omega$ (or ω^{ω})
KF	$< \varphi_{\varepsilon_o} o$	$\varphi_1 o$ (or ε_o)

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

The costs of nonclassical logic

- ▶ This asymmetry between the amount of transfinite induction provable in PKF and KF has been taken – see Halbach (2014) and Halbach and Nicolai (2018) – as substantiating Feferman's claim that

'nothing like sustained ordinary reasoning can be carried out [in such logic]' (Feferman, 1984)

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

The costs of nonclassical logic

- ▶ This asymmetry between the amount of transfinite induction provable in PKF and KF has been taken – see Halbach (2014) and Halbach and Nicolai (2018) – as substantiating Feferman's claim that

'nothing like sustained ordinary reasoning can be carried out [in such logic]' (Feferman, 1984)

- ▶ This seems to be supported by:

Halbach and Nicolai (2018)

$\text{PKF} \upharpoonright \mathcal{L} = \{ \varphi \in \mathcal{L}_{\text{Tr}} \mid \text{KF} \upharpoonright \mathcal{L} \}.$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

The costs of nonclassical logic

- ▶ This asymmetry between the amount of transfinite induction provable in **PKF** and **KF** has been taken – see Halbach (2014) and Halbach and Nicolai (2018) – as substantiating Feferman's claim that

'nothing like sustained ordinary reasoning can be carried out [in such logic]' (Feferman, 1984)

- ▶ This seems to be supported by:

Halbach and Nicolai (2018)

$\text{PKF} \upharpoonright \mathcal{L} = \{ \varphi \in \mathcal{L}_{\text{Tr}} \mid \text{KF} \upharpoonright \mathcal{L} \}.$

- ▶ A possible rejoinder is that transfinite induction open to **semantic notions** cannot be taken as compromising **mathematical** reasoning.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

INDEC is the assertion that:

every countable scattered indecomposable linear ordering is either indecomposable to the left, or indecomposable to the right.

It was proved to be true by Pierre Jullien in 1969.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

INDEC is the assertion that:

every countable scattered indecomposable linear ordering is either indecomposable to the left, or indecomposable to the right.

It was proved to be true by Pierre Jullien in 1969.

Lemma (Montalbán, Friedman)

INDEC is implied by Σ_1^1 -CA, which is Π_2^1 -conservative over $\text{ACA}_{<\varepsilon_0}$.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

INDEC is the assertion that:

every countable scattered indecomposable linear ordering is either indecomposable to the left, or indecomposable to the right.

It was proved to be true by Pierre Jullien in 1969.

Lemma (Montalbán, Friedman)

INDEC is implied by Σ_1^1 -CA, which is Π_2^1 -conservative over $\text{ACA}_{<\varepsilon_0}$.

Proposition (Eastaugh, N.)

$\text{RCA} + \text{INDEC}$ is proof-theoretically equivalent to KFP . It follows that KFP can ‘nicely’ interpret $\text{RCA} + \text{INDEC}$, but KF cannot.

Basics

Paradox(es)

Consistency via

cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- ▶ We have seen that $CT_{<\varepsilon_0}$, $CT_{<\Gamma_0}$, KF, and in part PKF can be seen as attempt to characterize the **reflective closure** of PA.
- ▶ There is a more direct strategy involving **reflection principles**, i.e. claims of the form

RFN(S)

if something is provable in a theory S , then it's true, or

$$\forall x(\text{Prov}_S(\ulcorner A(x) \urcorner) \rightarrow A(x))$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- ▶ We have seen that $CT_{<\varepsilon_0}$, $CT_{<\Gamma_0}$, KF, and in part PKF can be seen as attempt to characterize the **reflective closure** of PA.
- ▶ There is a more direct strategy involving **reflection principles**, i.e. claims of the form

RFN(S)

if something is provable in a theory S , then it's true, or

$$\forall x(\text{Prov}_S(\ulcorner A(x) \urcorner) \rightarrow A(x))$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- ▶ We have seen that $CT_{<\varepsilon_0}$, $CT_{<\Gamma_0}$, KF, and in part PKF can be seen as attempt to characterize the **reflective closure** of PA.
- ▶ There is a more direct strategy involving **reflection principles**, i.e. claims of the form

RFN(S)

if something is provable in a theory S , then it's true, or

$$\forall x(\text{Prov}_S(\ulcorner A(x) \urcorner) \rightarrow A(x))$$

- ▶ It turns out that if one focuses on **classical** biconditionals

$$\text{PTB} \upharpoonright \mathcal{L} = \{ \text{Tr} \ulcorner A \urcorner \leftrightarrow A \mid A \text{ positive of } \mathcal{L}_{\text{Tr}} \}$$

one obtains:

Proposition (Horsten and Leigh (2017))

$$\text{RFN}^2(\text{PTB} \upharpoonright \mathcal{L}) \supseteq \text{KF}.$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- ▶ There seems to be a conceptual problem with this strategy:

If one formulates reflection – as it seems plausible – as:

$$\forall \varphi (\text{Prov}_{\text{PTB}}(\varphi) \rightarrow \text{Tr } \varphi)$$

then there is a sentence A such that PTB with this form of reflection proves $\text{Tr } \ulcorner A \wedge \neg A \urcorner$.

Basics

Paradox(es)

Consistency via

cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- ▶ There seems to be a conceptual problem with this strategy:

If one formulates reflection – as it seems plausible – as:

$$\forall \varphi (\text{Prov}_{\text{PTB}}(\varphi) \rightarrow \text{Tr } \varphi)$$

then there is a sentence A such that PTB with this form of reflection proves $\text{Tr } \ulcorner A \wedge \neg A \urcorner$.

- ▶ A possible way out is to start with a set of ‘biconditionals’ over **FDE**:

$$\text{TS} \upharpoonright \mathcal{L} = \{ \text{Tr } \ulcorner A \urcorner \Leftrightarrow A \mid A \in \mathcal{L}_{\text{Tr}} \}$$

and define reflection as

$$(\text{RR}) \frac{\text{Pr}_S^2(\ulcorner \Gamma \dot{x} \Rightarrow \Delta \dot{x} \urcorner, \ulcorner \Theta \dot{x} \Rightarrow \Lambda \dot{x} \urcorner) \quad \Gamma(x) \Rightarrow \Delta(x)}{\Theta(x) \Rightarrow \Lambda(x)}$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- ▶ There seems to be a conceptual problem with this strategy:

If one formulates reflection – as it seems plausible – as:

$$\forall \varphi (\text{Prov}_{\text{PTB}}(\varphi) \rightarrow \text{Tr } \varphi)$$

then there is a sentence A such that PTB with this form of reflection proves $\text{Tr } \ulcorner A \wedge \neg A \urcorner$.

- ▶ A possible way out is to start with a set of ‘biconditionals’ over FDE:

$$\text{TS} \upharpoonright \mathcal{L} = \{ \text{Tr } \ulcorner A \urcorner \Leftrightarrow A \mid A \in \mathcal{L}_{\text{Tr}} \}$$

and define reflection as

$$(\text{RR}) \frac{\text{Pr}_S^2(\ulcorner \Gamma \dot{x} \Rightarrow \Delta \dot{x} \urcorner, \ulcorner \Theta \dot{x} \Rightarrow \Lambda \dot{x} \urcorner) \quad \Gamma(x) \Rightarrow \Delta(x)}{\Theta(x) \Rightarrow \Lambda(x)}$$

- ▶ Then we can prove:

Proposition (Fischer et al. (2017))

- ▶ $\text{RR}^2(\text{TS} \upharpoonright \mathcal{L}) \supseteq \text{PKF}$.
- ▶ $\text{RR}^\omega(\text{TS} \upharpoonright \mathcal{L}) \vdash \text{TI}_{\mathcal{L}_{\text{Tr}}}^{<\omega^2}$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

An analogy

- ▶ Solovay's theorem tells us that, for all $A \in \mathcal{L}_{\Box}$,

$$\text{GL} \vdash A \text{ iff } \forall \star (\text{PA} \vdash A^{\star})$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

An analogy

- ▶ Solovay's theorem tells us that, for all $A \in \mathcal{L}_\Box$,

$$\text{GL} \vdash A \text{ iff } \forall \star (\text{PA} \vdash A^\star)$$

- ▶ By redefining a realization $\star: \mathcal{L}_\Box \rightarrow \mathcal{L}_{\text{Tr}}$ as:

$$A^\star = \begin{cases} 0 = 1, & \text{if } A = \perp, \\ \text{commutes with connectives} & \\ \text{Tr} \ulcorner B^\star \urcorner & \text{if } A = \Box B \end{cases}$$

one can ask:

Question

$$? \vdash A \Leftrightarrow \forall \star (\text{KF} \vdash A^\star)$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical

Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Nicolai and Stern (2018)

The logic L_{\Box}

$$\Box \top$$

$$\Box A \leftrightarrow \Box \neg \neg A$$

$$\Box (A \wedge B) \leftrightarrow \Box A \wedge \Box B$$

$$\Box A \leftrightarrow \Box \Box A$$

$$\Box \perp$$

$$\Box A \wedge \neg \Box \neg A \rightarrow A$$

$$\Box \neg (A \wedge B) \leftrightarrow \Box \neg A \vee \Box \neg B$$

$$\Box \neg A \leftrightarrow \Box \neg \Box A$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Nicolai and Stern (2018)

The logic L_{\Box}

$$\Box T$$

$$\Box A \leftrightarrow \Box \neg \neg A$$

$$\Box (A \wedge B) \leftrightarrow \Box A \wedge \Box B$$

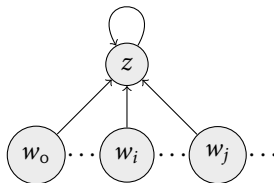
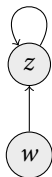
$$\Box A \leftrightarrow \Box \Box A$$

$$\Box \perp$$

$$\Box A \wedge \neg \Box \neg A \rightarrow A$$

$$\Box \neg (A \wedge B) \leftrightarrow \Box \neg A \vee \Box \neg B$$

$$\Box \neg A \leftrightarrow \Box \neg \Box A$$



Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Nicolai and Stern (2018)

The logic L_{\Box}

$$\Box \top$$

$$\Box A \leftrightarrow \Box \neg \neg A$$

$$\Box (A \wedge B) \leftrightarrow \Box A \wedge \Box B$$

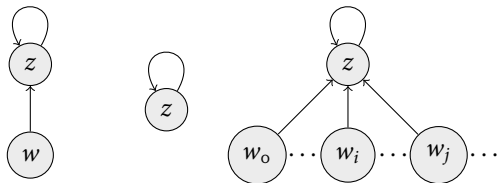
$$\Box A \leftrightarrow \Box \Box A$$

$$\Box \perp$$

$$\Box A \wedge \neg \Box \neg A \rightarrow A$$

$$\Box \neg (A \wedge B) \leftrightarrow \Box \neg A \vee \Box \neg B$$

$$\Box \neg A \leftrightarrow \Box \neg \Box A$$



Proposition

$$L_{\Box} \vdash A \text{ iff } \forall \star (KF \vdash A^{\star})$$

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Basics

Paradox(es)

Consistency via cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- Beckmann, A., Pollett, C., and Buss, S. R. (2003). Ordinal notations and well-orderings in bounded arithmetic. *Annals of Pure and Applied Logic*, 120(1-3):197–223.
- Buss, S. (1986). *Bounded arithmetic*. Bibliopolis, Napoli.
- Cantini, A. (1989). Notes on formal theories of truth. *Zeitschrift für Logik und Grundlagen der Mathematik*, 35:97–130.
- Cantini, A. (2003). The undecidability of Grisin's set theory. *Studia Logica*, 74(3):345–368.
- Cieliski, C. (2017). *The Epistemic Lightness of Truth: Deflationism and its Logic*. Cambridge University Press.
- Davidson, D. (1984). *Inquiries Into Truth And Interpretation*. Oxford University Press.
- Enayat, A. and Pakhomov, F. (2018). Truth, disjunction, and induction. arXiv:1805.09890 [math.LO].
- Feferman, S. (1984). Towards useful type-free theories I. *Journal of Symbolic Logic*, 49(1):75–111.
- Feferman, S. (1991). Reflecting on incompleteness. *Journal of Symbolic Logic*, 56: 1–49.
- Field, H. (1994). Deflationist views of meaning and content. *Mind*, 103(411):249–285.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Field, H. (2008). *Saving truth from paradox*. Oxford University Press, Oxford.

Fischer, M., Nicolai, C., and Horsten, L. (2017). Iterated reflection over full disquotational truth. *Journal of Logic and Computation*, 27(8):2631–2651.

Ganea, M. (2009). Arithmetic on semigroups. *Journal of Symbolic Logic*, 74(1):265–278.

Grišin, V. (1982). Predicate and set-theoretic calculi based on logic without contraction. *Math. Izvestija*, 18:41–59. (English Translation).

Halbach, V. (2014). *Axiomatic theories of truth. Revised edition*. Cambridge University Press.

Halbach, V. and Horsten, L. (2006). Axiomatizing Kripke's theory of truth in partial logic. *Journal of Symbolic Logic*, 71: 677–712.

Halbach, V. and Nicolai, C. (2018). On the costs of nonclassical logic. *Journal of Philosophical Logic*, 47:227–257.

Horsten, L. and Leigh, G. E. (2017). Truth is simple. *Mind*, 126(501):195–232.

Horwich, P. (1998). *Truth*. Clarendon Press.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- Jäger, G. and Stärk, R. F. (1998). A proof-theoretic framework for logic programming. In Buss, S. R., editor, *Handbook of Proof-Theory*, pages 639–682. Elsevier.
- Jago, M. (2018). *What Truth Is*. Oxford: Oxford University Press.
- Kaye, R. and Wong, T. (2006). On interpretations of arithmetic and set theory. Online Draft.
- Ketland, J. (1999). Deflationism and tarski's paradise. *Mind*, 108(429):69–94.
- Kotlarski, H., Krajewski, S., and Lachlan, A. H. (1981). Construction of satisfaction classes for nonstandard models. *Canadian Mathematical Bulletin*, 24(1):283–93.
- Kreuger, P. (1994). Axioms in definitional calculi. In Dyckhoff, R., editor, *Extensions of Logic Programming, 4th International Workshop, ELP'93, St Andrews, UK*, number 798 in Lecture Notes in Computer Science, pages 196–205.
- Kripke, S. (1975). Outline of a theory of truth. *Journal of Philosophy*, 72:690–712.
- Leigh, G. (2015). Conservativity for theories of compositional truth via cut elimination. *The Journal of Symbolic Logic*, 80.
- Martin, R. L. and Woodruff, P. W. (1975). On representing 'true-in-l' in l. *Philosophia*, 5(3):213–217.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

- Nicolai, C. (2015). Deflationary truth and the ontology of expressions. *Synthese*, 192(12):4031–4055.
- Nicolai, C. (2016). A note on typed truth and consistency assertions. *Journal of Philosophical Logic*, 45(1):89–119.
- Nicolai, C. (2018a). Cut-elimination for systems of naïve consequence and truth with restricted initial sequents. Draft.
- Nicolai, C. (2018b). Provably true sentences across axiomatizations of Kripke's theory of truth. *Studia Logica*, 106:101–130.
- Nicolai, C. and Stern, J. (2018). The modal completeness of Kripke-Feferman truth. Draft.
- Petersen, U. (2000). Logic without contraction as based on inclusion and unrestricted abstraction. *Studia Logica*, 64(3):365–403.
- Quine, W. V. (1970). *Philosophy of Logic*. Harvard University Press.
- Schroeder-Heister, P. (2016). Restricting initial sequents: the trade-off between identity, contraction, cut. In *Advances in Proof Theory*, volume 28 of *Progress in Computer Science and Applied Logic*. Springer.
- Shapiro, S. (1998). Proof and truth: Through thick and thin. *Journal of Philosophy*, 95(10):493–521.
- Soames, S. (1998). *Understanding Truth*. Oxford University Press USA.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References

Świerczkowski, S. (2003). Finite sets and gödel's incompleteness theorems. *Dissertationes Mathematicae*.

Visser, A. (2010). What is the right notion of sequentiality? *Logic Group Preprint Series*, 288:1–24.

Visser, A. and Enayat, A. (2015). New constructions of satisfaction classes. In Fujimoto, K., Fernández, J. M., Galinon, H., and Achourioti, T., editors, *Unifying the Philosophy of Truth*. Springer Verlag.

Williamson, T. (2000). *Knowledge and its Limits*. Oxford University Press.

Basics

Paradox(es)

Consistency via
cut-elimination

Objects of Truth

Systems of Truth

Deflation and Conservation

Classical v Nonclassical
Kripkean truth

Logical Pluralism

Extensions

Reflection

Modal Logic

Modal Predicates

References