Semantic closure and the restriction of initial sequents

Axiomatizing metatheory.
Truth, provability, and beyond.

Carlo Nicolai
King's College London
Salzburg, December 7, 2018

semantic closure

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- Proofs of meta-theoretic properties should be replaced by proofs that can are *meaningful* by *object-theoretic means*.

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For any $A \in \mathscr{L}_{\mathrm{Tr}}$:

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \mathrm{Tr}\ulcorner A \urcorner, \Delta} \qquad \frac{A, \Gamma \Rightarrow \Delta}{\mathrm{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow \Delta}$$

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QUESTION

Is cut eliminable in S?

- The standard inductive strategy consists in a multiple induction on the complexity of the cut-formula and the length of the derivation. In the case of the truth rules one needs to reduce the following…

$$\dfrac{\dfrac{\begin{array}{c}\mathscr{D}_0\\ \Gamma \Rightarrow \Delta, A\end{array}}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner A \urcorner} \qquad \dfrac{\begin{array}{c}\mathscr{D}_1\\ A, \Gamma \Rightarrow \Delta\end{array}}{\mathrm{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

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This creates a problem because $\mathrm{Tr}\ulcorner A\urcorner$ is atomic whereas A is of arbitrary (logical) complexity.

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$$\frac{A, \Gamma \Rightarrow \Delta\ [\alpha]}{\mathrm{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow \Delta\ [\alpha+1]} \qquad \frac{\Gamma \Rightarrow \Delta, A\ [\alpha]}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner A \urcorner\ [\alpha+1]}$$

$$\Gamma, \varphi \Rightarrow \varphi, \Delta\ [0] \qquad \frac{\Gamma \Rightarrow \Delta, A\ [\alpha] \qquad \Gamma \Rightarrow \Delta, B\ [\beta]}{\Gamma \Rightarrow \Delta, A \wedge B\ [\mathtt{max}(\alpha, \beta)]}$$

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Fact (Grišin 1982, Cantini 2003)

If one drops contraction, the resulting system of truth admits cut elimination.

One of the $\mathscr{D}_i$'s is obtained by contraction. Then $\mathscr{D}$ may be:

$$\dfrac{\dfrac{\begin{array}{c}\mathscr{D}_{00}\\ \Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner\psi\urcorner, \mathrm{Tr}\ulcorner\psi\urcorner\ [\alpha]\end{array}}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner\psi\urcorner\ [\alpha]} \qquad \begin{array}{c}\mathscr{D}_1\\ \mathrm{Tr}\ulcorner\psi\urcorner, \Theta \Rightarrow \Lambda\ [\beta]\end{array}}{\Gamma, \Theta \Rightarrow \Delta, \Lambda\ [\max(\alpha, \beta)]}$$

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Now the idea here would be that we transform the derivation in

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But this isn't a suitable reduction.

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- If $\mathscr{D}$ ends with
$$\frac{\gamma_1^1, \ldots, \gamma_n^1 \Rightarrow \delta_1^1, \ldots, \delta_m^1, \varphi \qquad \gamma_1^2, \ldots, \gamma_n^2 \Rightarrow \delta_1^2, \ldots, \delta_m^2, \psi}{\gamma_1^3, \ldots, \gamma_n^3 \Rightarrow \delta_1^3, \ldots, \delta_m^3, \varphi \wedge \psi}$$
then
$$\begin{aligned}
&\varkappa_{\mathscr{D}}(\varphi \wedge \psi) = \max(\varkappa_{\mathscr{D}}(\varphi), \varkappa_{\mathscr{D}}(\psi)) \\
&\varkappa_{\mathscr{D}}(\gamma_i^3) = \max(\varkappa_{\mathscr{D}}(\gamma_i^1), \varkappa_{\mathscr{D}}(\gamma_i^2)) \;\; 1 \leq i \leq n \\
&\varkappa_{\mathscr{D}}(\delta_j^3) = \max(\varkappa_{\mathscr{D}}(\delta_j^1), \varkappa_{\mathscr{D}}(\delta_j^2)) \;\; 1 \leq j \leq m
\end{aligned}$$

One of the $\mathscr{D}_i$'s is obtained by contraction. Then $\mathscr{D}$ may be:

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And again, this isn't a suitable reduction.

Definition (LPT)

(REF$^-$) $\Gamma, \varphi \Rightarrow \varphi, \Delta$ with $\varphi \in \mathrm{AtFml}_{\mathscr{L}}$

(CUT) $$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

(Tr L) $$\frac{A, \Gamma \Rightarrow \Delta}{\mathrm{Tr}\ulcorner A \urcorner, \Gamma \Rightarrow \Delta}$$

(Tr R) $$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner A \urcorner}$$

($\neg$L) $$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}$$

($\neg$R) $$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}$$

($\wedge$L) $$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$$

($\wedge$R) $$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

($\forall$L) $$\frac{\Gamma, \forall x\varphi, \varphi(s) \Rightarrow \Delta}{\Gamma, \forall x\varphi \Rightarrow \Delta}$$

($\forall$R) $$\frac{\Gamma \Rightarrow \varphi(y), \Delta}{\Gamma \Rightarrow \Delta, \forall x\varphi}\ y \notin \mathrm{FV}(\Gamma, \Delta, \forall x\varphi)$$

Lemma (Weakening)

Weakening is $\varkappa$-admissible in LPT*. That is, if* $\vdash_{\mathsf{LPT}} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathsf{LPT}} \Gamma \Rightarrow \varphi, \Delta$ *(or* $\vdash_{\mathsf{LPT}} \Gamma, \varphi \Rightarrow \Delta$*) so that* $\varkappa(\varphi)$ *is no higher than the maximal truth complexity of the formulae in* Γ, Δ*. In particular, we can set* $\varkappa(\varphi) = 0$*.*

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Lemma (Inversion)

All rules of LPT *are* $\varkappa$*-invertible (i.e. the truth complexity of the complex formulae is* ***no greater*** *than the one of their constituents). Crucially:*

1. *If* $\vdash_{\mathsf{LPT}} \Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner\varphi\urcorner$*, then* $\vdash_{\mathsf{LPT}} \Gamma \Rightarrow \Delta, \varphi$ *with*
$$\begin{array}{ll} \varkappa(\varphi) \leq \varkappa(\mathrm{Tr}\ulcorner\varphi\urcorner) & \text{if } \varkappa(\mathrm{Tr}\ulcorner\varphi\urcorner) = 0, \\ \varkappa(\varphi) < \varkappa(\mathrm{Tr}\ulcorner\varphi\urcorner) & \text{otherwise} \end{array}$$
2. *If* $\vdash_{\mathsf{LPT}} \mathrm{Tr}\ulcorner\varphi\urcorner, \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathsf{LPT}} \varphi, \Gamma \Rightarrow \Delta, \varphi$ *with*
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- By induction on the length of the proof. Suppose $\vdash_{\mathsf{LPT}} \mathrm{Tr}\ulcorner\varphi\urcorner, \Gamma \Rightarrow \Delta$.

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    - If

$$\text{(R)}\ \frac{\begin{array}{c}\mathscr{D}_0\\ \mathrm{Tr}\ulcorner\varphi\urcorner, \Gamma_0 \Rightarrow \Delta_0\end{array} \qquad \begin{array}{c}\mathscr{D}_1\\ \mathrm{Tr}\ulcorner\varphi\urcorner, \Gamma_1 \Rightarrow \Delta_1\end{array}}{\mathrm{Tr}\ulcorner\varphi\urcorner, \Gamma \Rightarrow \Delta}$$

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    then $\varkappa(\mathrm{Tr}\ulcorner\varphi\urcorner) > 0$ and $\mathscr{D}_0$ is the required derivation.

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    then $\varkappa(\mathrm{Tr}\ulcorner\varphi\urcorner) > 0$ and $\mathscr{D}_0$ is the required derivation.
  - Notice that, if initial sequents can be for the form $\mathrm{Tr}\ulcorner\varphi\urcorner, \Gamma \Rightarrow \Delta, \mathrm{Tr}\ulcorner\varphi\urcorner$, then there nothing that leads us to a derivation of, say, $\mathrm{Tr}\ulcorner\varphi\urcorner, \Gamma \Rightarrow \Delta, \varphi$ in which $\varkappa(\varphi)$ is as required.

Lemma ($\varkappa$-admissibility of contraction)

1. *If* $\mathscr{D} \vdash_{\mathrm{LPT}} \Gamma^1, \varphi^1, \varphi^2 \Rightarrow \Delta^1$, *then there is a* $\mathscr{D}' \vdash_{\mathrm{LPT}} \Gamma, \varphi \Rightarrow \Delta$ *with* $\varkappa(\Gamma^1) \leq \varkappa(\Gamma)$, $\varkappa(\Delta^1) \leq \varkappa(\Delta)$, *and* $\varkappa(\varphi) \leq max(\varkappa(\varphi^1), \varkappa(\varphi^2))$;
2. *If* $\mathscr{D} \vdash_{\mathrm{LPT}} \Gamma^1 \Rightarrow \varphi^1, \varphi^2, \Delta^1$, *then there is a* $\mathscr{D}' \vdash_{\mathrm{LPT}} \Gamma \Rightarrow \varphi, \Delta$ *with* $\varkappa(\Gamma^1) \leq \varkappa(\Gamma)$, $\varkappa(\Delta^1) \leq \varkappa(\Delta)$, *and* $\varkappa(\varphi) \leq max(\varkappa(\varphi^1), \varkappa(\varphi^2))$.

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Lemma (cut-elimination)

If $\mathscr{D}_0$ *is a cut-free proof of* $\Gamma^1 \Rightarrow \Delta^1, \varphi^1$ *in* LPT, *and* $\mathscr{D}_1$ *is a cut-free* LPT*-proof of* $\varphi^2, \Gamma^2 \Rightarrow \Delta^2$, *then there is a cut-free proof* $\mathscr{D}$ *of* $\Gamma^3 \Rightarrow \Delta^3$ *in* LPT *with* $\varkappa(\Gamma^3) \leq max(\varkappa(\Gamma^1), \varkappa(\Gamma^2))$ *and* $\varkappa(\Delta^3) \leq max(\varkappa(\Delta^1), \varkappa(\Delta^2))$.

Lemma ($\varkappa$-admissibility of contraction)

1. *If* $\mathscr{D} \vdash_{\mathrm{LPT}} \Gamma^1, \varphi^1, \varphi^2 \Rightarrow \Delta^1$*, then there is a* $\mathscr{D}' \vdash_{\mathrm{LPT}} \Gamma, \varphi \Rightarrow \Delta$ *with* $\varkappa(\Gamma^1) \leq \varkappa(\Gamma)$, $\varkappa(\Delta^1) \leq \varkappa(\Delta)$, *and* $\varkappa(\varphi) \leq max(\varkappa(\varphi^1), \varkappa(\varphi^2))$;
2. *If* $\mathscr{D} \vdash_{\mathrm{LPT}} \Gamma^1 \Rightarrow \varphi^1, \varphi^2, \Delta^1$*, then there is a* $\mathscr{D}' \vdash_{\mathrm{LPT}} \Gamma \Rightarrow \varphi, \Delta$ *with* $\varkappa(\Gamma^1) \leq \varkappa(\Gamma)$, $\varkappa(\Delta^1) \leq \varkappa(\Delta)$, *and* $\varkappa(\varphi) \leq max(\varkappa(\varphi^1), \varkappa(\varphi^2))$.

Moreover, in both claims the length of the original derivation is preserved.

Lemma (cut-elimination)

If $\mathscr{D}_0$ *is a cut-free proof of* $\Gamma^1 \Rightarrow \Delta^1, \varphi^1$ *in* LPT*, and* $\mathscr{D}_1$ *is a cut-free* LPT*-proof of* $\varphi^2, \Gamma^2 \Rightarrow \Delta^2$*, then there is a cut-free proof* $\mathscr{D}$ *of* $\Gamma^3 \Rightarrow \Delta^3$ *in* LPT *with* $\varkappa(\Gamma^3) \leq max(\varkappa(\Gamma^1), \varkappa(\Gamma^2))$ *and* $\varkappa(\Delta^3) \leq max(\varkappa(\Delta^1), \varkappa(\Delta^2))$.

Proof strategy. The proof is by main induction on $\varkappa(\varphi)$, and side inductions on the complexity of φ and on the level of the cut (i.e. $d_0 + d_1$). This yields the usual superexponential bound.

Let $\mathscr{L}_{\mathbb{N}}$ be the language of arithmetic and $\mathscr{L}_{\mathrm{Tr}}^{\mathbb{N}} := \mathscr{L}_{\mathbb{N}} \cup \{\mathrm{Tr}\}$.

Definition (LPT^∞)

LPT^∞ is obtained from LPT by omitting free variables and:

- adding axioms $\Gamma \Rightarrow r = s, \Delta$ and $\Gamma, r = s \Rightarrow \Delta$; where r, s are closed terms of $\mathscr{L}_{\mathrm{Tr}}^{\mathbb{N}}$ and, respectively, $r^{\mathbb{N}} = s^{\mathbb{N}}$ and $r^{\mathbb{N}} \neq s^{\mathbb{N}}$.
- Replacing ($\forall$R) with:

$$\frac{\Gamma \Rightarrow \varphi(s), \Delta \qquad \text{for any closed term } s}{\Gamma \Rightarrow \forall x\, \varphi(x), \Delta}\ (\omega)$$

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Proposition

Let $\mathscr{I}$ be the minimal fixed point of the (Strong-Kleene version of the) Kripke-jump. Then $\varphi \in \mathscr{I}_\alpha$ iff $\vdash^{\alpha}_{\mathsf{LPT}^\infty} \Rightarrow \varphi$ for $\varphi \in \mathscr{L}_{\mathrm{Tr}}^{\mathbb{N}}$.

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- one considers the language $\mathscr{L}_{\mathrm{C}} := \mathscr{L} \cup \{\mathrm{C}\}$ and replaces the truth rules in LPT with:

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(\mathrm{C}\textsc{l})\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\Gamma, \mathrm{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta} \qquad (\mathrm{C}\textsc{r})\ \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \mathrm{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta}
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- In Nicolai Rossi 2018 we defined a (consistent) fixed-point model $\mathscr{I}_{\mathrm{C}}$ for $\mathscr{L}_{\mathbb{N}} \cup \{\mathrm{C}\}$ that delivers:

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\varphi \Rightarrow \psi \in \mathscr{I}_{\mathrm{C}} \ \textbf{iff}\ (\text{either } \neg\varphi \in \mathscr{I}_{\mathrm{C}} \text{ or } \psi \in \mathscr{I}_{\mathrm{C}}) \ \textbf{iff} \Rightarrow \mathrm{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \in \mathscr{I}_{\mathrm{C}}.
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- As before, the infinitary version of this 'logic' of consequence, call it LPC^{∞}, is such that:

$$\varphi \Rightarrow \psi \in \mathscr{I}_{\mathrm{C}}^{\alpha} \quad \text{iff} \quad \vdash^{\alpha}_{\mathsf{LPC}^{\infty}} \Rightarrow \mathrm{C}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner).$$

The theory RKF$\restriction$ in $\mathscr{L}_{\mathrm{Tr}}$ has the following components:

1. Initial sequents and rules of LPT with identity except (TrL) and (TrR)
2. Initial sequents $\Gamma \Rightarrow \Delta, \varphi$ for φ an axiom of PA, including instances of the induction schema for all formulae $\varphi(v)$ of $\mathscr{L}_{\mathbb{N}}$
3. Truth rules:

$$(=\mathrm{R})\ \frac{\Gamma \Rightarrow s^{\circ} = t^{\circ}, \Delta}{\Gamma \Rightarrow \mathrm{Tr}(s = t), \Delta} \qquad (=\mathrm{L})\ \frac{\Gamma, s^{\circ} = t^{\circ} \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(s = t) \Rightarrow \Delta}$$

$$(\mathrm{Tr}\,1)\ \frac{\Gamma \Rightarrow \mathrm{Tr}(x), \Delta}{\Gamma \Rightarrow \mathrm{Tr}\ulcorner \mathrm{Tr}(\dot{x})\urcorner} \qquad (\mathrm{Tr}\,2)\ \frac{\Gamma \Rightarrow \mathrm{Tr}(x), \Delta}{\Gamma \Rightarrow \mathrm{Tr}\ulcorner \mathrm{Tr}(\dot{x})\urcorner}$$

$$(\mathrm{Tr}\neg 1)\ \frac{\Gamma \Rightarrow \neg\mathrm{Tr}(\sigma), \Delta}{\Gamma \Rightarrow \mathrm{Tr}(\neg\sigma), \Delta} \qquad (\mathrm{Tr}\neg 2)\ \frac{\Gamma, \neg\mathrm{Tr}(\sigma) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(\neg\sigma), \Delta}$$

$$(\mathrm{Tr}\wedge 1)\ \frac{\Gamma \Rightarrow \mathrm{Tr}(\sigma), \mathrm{Tr}(\tau), \Delta}{\Gamma \Rightarrow \mathrm{Tr}(\sigma \wedge \tau) \Rightarrow \Delta} \qquad (\mathrm{Tr}\wedge 2)\ \frac{\Gamma, \mathrm{Tr}(\sigma), \mathrm{Tr}(\tau) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(\sigma \wedge \tau) \Rightarrow \Delta}$$

$$(\mathrm{Tr}\forall 1)\ \frac{\Gamma \Rightarrow \forall x\, \mathrm{Tr}(\sigma(\dot{x}/v)), \Delta}{\Gamma \Rightarrow \mathrm{Tr}(\forall v \sigma), \Delta} \qquad (\mathrm{Tr}\forall 2)\ \frac{\Gamma, \forall x\, \mathrm{Tr}(\sigma(\dot{x}/v)) \Rightarrow \Delta}{\Gamma, \mathrm{Tr}(\forall v \sigma) \Rightarrow \Delta}$$

Lemma

1. ***Weakening*** *is $\varkappa$-admissible in* $\mathsf{RKF}\restriction$.
2. *All* ***truth rules*** *of* $\mathsf{RKF}\restriction$ *are $\varkappa$-invertible – again, with a strict inequality if the active formula has* Tr*-complexity* $> \mathsf{0}$.

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A Tr **-cut** is a cut on formulae of the form $\mathrm{Tr}\, t$.

Lemma (Tr -cut elimination)

Let $\mathscr{D}_0$, $\mathscr{D}_1$ be cut-free derivations, in $\mathsf{RKF}{\upharpoonright}$, *of* $\Gamma^1 \Rightarrow \Delta^1, \mathrm{Tr}\,(t)$ *and* $\mathrm{Tr}\,(t), \Gamma^2 \Rightarrow \Delta^2$ *respectively. Then we can find a derivation $\mathscr{D}$ of $\Gamma \Rightarrow \Delta$ with* $\varkappa(\Gamma^k) \leq \varkappa(\Gamma)$ *and* $\varkappa(\Delta^k) \leq \tau(\Delta)$ *and* $k \in \{1, 2\}$.

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Corollary ($\mathsf{I}\Delta_0(\mathsf{superexp})$)

$\mathsf{RKF}\!\restriction$ is **conservative** over PA: if $\Rightarrow \varphi$ is derivable in $\mathsf{RKF}\!\restriction$ and $\varphi \in \mathscr{L}_{\mathbb{N}}$, then there is a derivation of $\Rightarrow \varphi$ without Tr-cuts, i.e. a PA-derivation of $\Rightarrow \varphi$. A fortiori, $\mathsf{RKF}\!\restriction$ is consistent if PA is.

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- The version of $\mathrm{CT}\restriction$ with restricted initial sequents – a subtheory of $\mathsf{RKF}\restriction$ – admits, by contrast, a smooth cut-elimination procedure.
- What I presented for truth can be extended to a *compositional theory of consequence* in the style of $\mathsf{RKF}\restriction$.

- The idea of axiomatising one's metatheory calls (at least in principle) for **alternative means** to establish standard metatheoretic properties such as consistency and/or conservativity.

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- Although consistency proofs are of course beyond what can be directly established in one's axiomatic system, they should at least be **meaningful** from such a point of view.
- In the context of core semantical notions such as truth and consequence and unrestricted principles for them, *the restriction of initial sequents yields syntactic consistency and conservativity proofs that are formalisable by finitary means*. This does not seem to be available for alternative (classical and) nonclassical options.