MODALIZED T-SCHEMATA

CARLO NICOLAI

King's College London

BA, August 20, 2021

Motivation

I am interested in the claim that the Tr-schema

'A' is true iff A

is (conceptually / logically) necessary.

There are a few desiderata that I wish to rely on. They come from my broadly deflationist view.

1. Some satisfactory way of dealing with paradox.

2. Truth should enable us to express (and, if appropriate, ESTABLISH!) general laws of

appropriate, ESTABLISH!) general laws of broadly logical nature such as:

- COMPOSITIONAL PRINCIPLES
- REFLECTION PRINCIPLES

3. Truth is much simpler and theoretically central than other concepts. It must be ultimately accounted for via RE systems. Their strength is sign of the generalizing power afforded by truth.

Classical Modal Disquotation

- Let $\mathcal{L}_{\mathbb{N}} = \{\bar{0}, \bar{1}, \times, +, \dots\}$ the dots stand for additional syntactic functions.
- $\mathcal{L}_{Tr}^{\Box} = \mathcal{L}_{\mathbb{N}} \cup \{Tr, \Box\}$, where Tr is a unary truth predicate, and $\Box$ a unary necessity predicate. Logical constants are $\neg, \wedge, \forall$.

DEFINITION (MD) Extends PA (in $\mathcal{L}_{Tr}^{\Box}$) with:

1. $\forall \varphi \in \mathcal{L}_{Tr}^{\Box} (Prov_{\emptyset}(\varphi) \rightarrow \Box \varphi)$
2. $\forall \varphi, \psi \in \mathcal{L}_{Tr}^{\Box} (\Box \varphi \wedge \Box(\varphi \rightarrow \psi) \rightarrow \Box \psi)$
3. $\forall \varphi(v) \in \mathcal{L}_{Tr}^{+} : \Box(Tr\, \varphi \dot{x} \leftrightarrow \varphi x)$ (MODAL DISQUOTATION)
4. for $A \in \mathcal{L}_{Tr}^{\Box}$: if $PA + (1)-(4) \vdash A$, then $MD \vdash \Box \ulcorner A \urcorner$ (OPTIONAL)
5. for $Av \in \mathcal{L}_{Tr}$: $\Box \ulcorner A\dot{x} \urcorner \rightarrow Ax$ (FACTIVITY)

LEMMA MD IS CONSISTENT.

PROOF. ONE CAN INTERPRET

$$\Box x \;:\leftrightarrow\; \mathrm{Prov}_{KF}(x),$$

WHERE KF IS THE WELL KNOWN SYSTEM OF KRIPKE-FEFERMAN.

ω-MODELS OF MD ARE EASY TO FIND (E.G. MINIMAL FIXED POINT).

QED.

MD is able to recover some desirable laws of truth, such as principles for compositionality (NB universally quantified!). Not all of them, though...

PROPOSITION (COMPOSITIONALITY)

MD PROVES:

i) $\forall \varphi \in \mathcal{L}^+_{Tr} \, (Tr \neg\neg\varphi \leftrightarrow T\varphi)$

ii) $\forall \varphi, \psi \in \mathcal{L}^+_{Tr} \, (Tr(\varphi \wedge \psi) \leftrightarrow Tr\varphi \wedge Tr\psi)$

iii) $\forall \varphi, \psi \in \mathcal{L}^+_{Tr} \, (Tr \neg(\varphi \wedge \psi) \leftrightarrow Tr\neg\varphi \vee Tr\neg\psi)$

iv) $\forall \varphi v \in \mathcal{L}^+_{Tr} \, (Tr(\forall v \varphi) \leftrightarrow \forall y \, Tr \, \varphi(y/v))$

v) $\forall \varphi v \in \mathcal{L}^+_{Tr} \, (Tr \neg\forall v\varphi \leftrightarrow \exists y \, Tr\neg\varphi(y/v))$

PROOF. SIMILAR FOR ALL CASES. WE TREAT i), REASONING IN MD.

a) $\forall \varphi \in \mathcal{L}^+_{Tr} \; \Box(\neg\neg\varphi \leftrightarrow \varphi)$
b) $\forall \varphi \in \mathcal{L}^+_{Tr} \; \Box(Tr\varphi \leftrightarrow \varphi)$
c) $\forall \varphi \in \mathcal{L}^+_{Tr} \; \Box(Tr\neg\neg\varphi \leftrightarrow \neg\neg\varphi)$

SINCE WE CAN REASON CLASSICALLY INSIDE $\Box$, WE OBTAIN (i). CRUCIALLY, THIS QUANTIFICATION CAN BE EXPRESSED IN THE OBJECT LANGUAGE. WE OBTAIN THE RESULT BY FACTIVITY.

QED

The previous proposition entails that MD has substantial deductive strength (i.e. $ACA^{<\varepsilon_0}$, the Turing jump hierarchy iterated up to any $\alpha < \varepsilon_0$).

Problems of Classical Modal Disquotation

The logic of $\Box$ is classical, as well as the logic of MD. By contrast, the logic of Tr is not classical.

OBSERVATION FACTIVITY IS EXPRESSED AS THE SCHEMA

$$\Box \ulcorner A\dot{x} \urcorner \rightarrow Ax.$$

ACCORDING TO OUR DESIDERATA ON TRUTH, THIS IS PRECISELY ONE CASE IN WHICH THE TRUTH PREDICATE PROVES TO BE USEFUL.

DEFINITION MD* IS PRECISELY AS MD, BUT WITH THE SCHEMA

$$\Box \ulcorner A\dot{x} \urcorner \longrightarrow Ax$$

REPLACED BY

$$\forall \varphi \in \boxed{\mathcal{L}_{Tr}} (\Box \varphi \rightarrow Tr \varphi) \qquad (*)$$

Remark. MD* is consistent, but cannot prove any of the compositional principles. A proof will be given later.

MD* highligths the strange asymmetry between NECESSITY and TRUTH already implicit in MD.

There are meaningful necessities that are completely opaque for the truth predicate:

EXAMPLE : $MD^* \vdash \Box (\neg Tr(0 \neq 0))$, therefore $MD^* \vdash Tr \neg Tr(0 \neq 0)$.

EXAMPLE: $MD^* \vdash \Box(\neg Tr(0\neq 0))$, therefore $MD^* \vdash Tr \neg Tr(0\neq 0)$.
But Tr is SILENT about obvious sentences such as
$\neg Tr(0\neq 0)$.

An obvious fix is to restore some symmetry between $\Box$ and Tr by means of some BRIDGE PRINCIPLES:

DEFINITION (MD^{**}) OBTAINED BY REPLACING $\forall \varphi \in \mathcal{L}_{Tr} (\Box\varphi \to Tr\varphi)$ with

$$\forall \varphi \in \mathcal{L}_{Tr} \ (\Box\varphi \to Tr\, \varphi^{\tau}) \qquad (**)$$

where:

$$\varphi^{\tau} = \varphi \quad \text{for } \varphi \in \mathcal{L}_{\mathbb{N}}$$
$$(Tr\, t)^{\tau} = Tr\, t^{\tau}$$
$$(\neg Tr\, t)^{\tau} = Tr\, \neg t^{\tau}$$
$$(\varphi \wedge \psi)^{\tau} = \varphi^{\tau} \wedge \psi^{\tau}$$
$$(\neg(\varphi \wedge \psi))^{\tau} = \neg\varphi^{\tau} \vee \neg\psi^{\tau}$$
$$(\neg\neg\varphi)^{\tau} = \neg\neg\varphi^{\tau}$$
$$(\forall x \varphi)^{\tau} = \forall x \varphi^{\tau}$$
$$(\neg\forall x \varphi)^{\tau} = \neg\forall x \varphi^{\tau}$$

(CORE IDEA: TRANSLATE 'untrue' AS 'false')

OBSERVATION: "HARMLESS" SENTENCES ARE NOW HANDLED CORRECTLY.
E.G.: $\Box(\neg Tr\ 0\neq 0)$ entails $Tr\,(\neg Tr\ 0\neq 0)^{\tau}$, which is $Tr\, Tr(\neg\, 0\neq 0)$, which is equivalent to $0=0$, as it should be.

PROPOSITION. MD^{**} IS INTERNALLY INCONSISTENT.

PROPOSITION. MD^{**} IS INTERNALLY INCONSISTENT.

PROOF. WE HAVE $\Box(\lambda \vee \neg\lambda)$ WHERE $\lambda := \neg Tr\,\ell$, AND

$$PA \vdash \ell = \ulcorner \neg Tr\,\ell \urcorner .$$

THEN: $\Box(\lambda \vee \neg \lambda)$, SO $Tr\ulcorner(\lambda \vee \neg\lambda)\urcorner$, WHICH IS EQUIVALENT TO $Tr\,e^{\tau}$.

BY THE DEF OF τ, THIS ENTAILS $Tr\,e^{\tau} \wedge Tr\,\neg e^{\tau}$.

QED

SUMMING UP: to restore desirable laws of truth, but still not all of them, a necessity closed under logic (classical) is required. To provide a coherent bridge between necessity and truth, one would need to accept dialetheias.

Nonclassical Modal Disquotation

A naive approach would be to work over partial logic (FDE, K3), although keeping our syntax theory PA classical.

WORKING ON A SEQUENT CALCULUS FOR, SAY, FDE:

- $\Rightarrow A$ FOR A A BASIC AXIOM OF PA
- $\Rightarrow s = s$; $s = t,\ A(s) \Rightarrow A(t)$
- $A(x) \Rightarrow Tr\ulcorner A(\dot{x})\urcorner$; $Tr\ulcorner A(\dot{x})\urcorner \Rightarrow A(x)$.

... we call ... "laws" such as

One can recover universally quantified "laws" such as

$$\neg Tr\, \varphi \Rightarrow Tr\, \neg\varphi,$$

(recall that φ is a variable)

$Sent\, x, \neg Tr x \Rightarrow Tr \neg x$

but the rules needed for this are difficult to motivate :

- $\Box(\emptyset,\ Tr\ulcorner A\dot{x}\urcorner \Leftrightarrow Ax)$ (strictly speaking : two principles)

- $$\frac{\Box(\Gamma \Rightarrow \Delta \,/\, \Theta \Rightarrow \Delta) \qquad \Gamma \Rightarrow \Delta}{\Theta \Rightarrow \Delta}$$

(cf. Fischer, N., Horsten, ITERATED REFLECTION OVER FULLY DISQUOTATIONAL TRUTH, JLC '17).

I would like to explore another option to keep both truth and necessity as predicates applying to names of sentences / statements.

HYPE

Essentially : extension of FDE with an intuitionistic conditional.

AXIOMS AND RULES

- FULLY STRUCTURAL RULES

- $$\frac{\Gamma \Rightarrow \neg\Delta}{\Delta \Rightarrow \neg\Gamma} \qquad \frac{\neg\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \Gamma} \qquad \neg X = \{\neg A \mid A \in X\}.$$

- $$\frac{\Gamma \Rightarrow \Delta\, A \qquad \Gamma \Rightarrow \Delta\, B}{\Gamma \Rightarrow \Delta\, A \wedge B} \qquad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

- $$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}$$ $$\boxed{\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B, \Delta}}$$

- $$\frac{A(t)\ \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}$$ $$\frac{\Gamma \Rightarrow \Delta\ A(y)}{\Gamma \Rightarrow \Delta, \forall x A}$$ } usual restrictions

Semantics

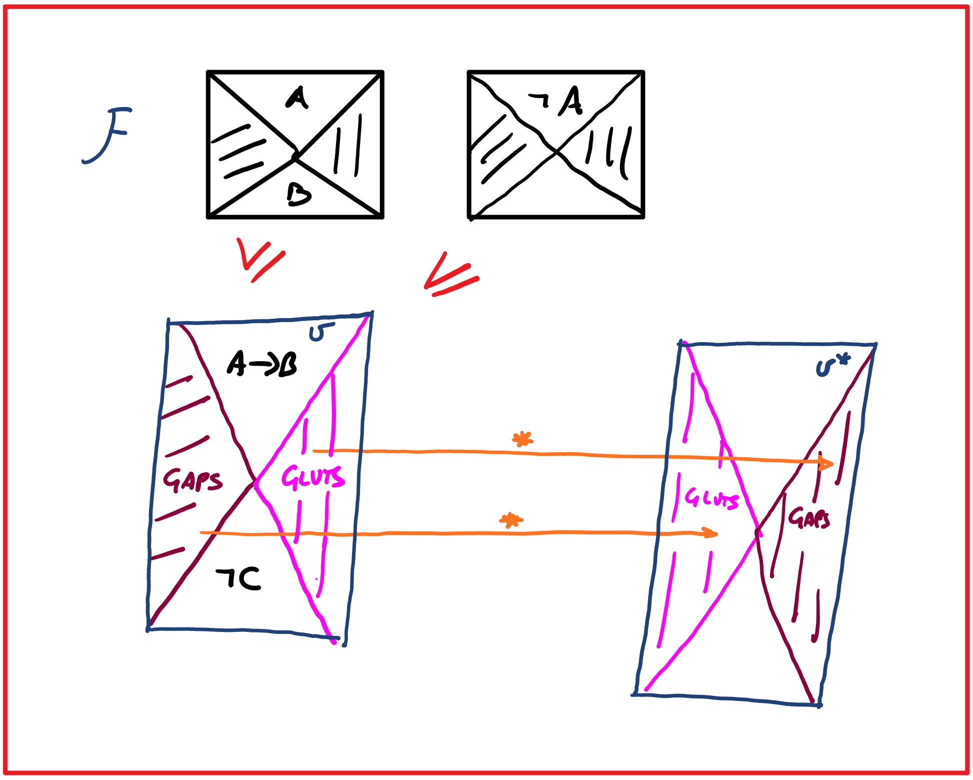
- ROUTLEY FRAME $\mathcal{F} = (W, \leq, *)$, WITH:

  $W \neq \emptyset$, $\leq$ A PREORDER, $*$ ANTIMONOTONE AND INVOLUTIVE.

- CONSTANT DOMAIN MODEL: $\mathcal{M} = (\mathcal{F}, D, I)$, WITH:

  - $I(c) \in D$
  - $I_w(P^n) \subseteq D^n$.
  - $w, v \in W$: $v \leq w \Rightarrow I_v(P) \subseteq I_w(P)$.

- SATISFACTION:
  - $\mathcal{M}, w \models P(\bar{x})\ [\alpha]$ IFF $\overline{\alpha(x)} \in I_w(P)$
  - $\mathcal{M}, w \models \neg A\ [\alpha]$ IFF $\mathcal{M}, w^* \not\models A\ [\alpha]$
  - $\mathcal{M}, w \models A \wedge B\ [\alpha]$ IFF $\mathcal{M}, w \models A\ [\alpha]$ AND $\mathcal{M}, w \models B\ [\alpha]$
  - $\mathcal{M}, w \models A \to B\ [\alpha]$ IFF FOR ALL $v \geq w$: IF $\mathcal{M}, v \models A\ [\alpha]$, THEN $\mathcal{M}, B \models B\ [\alpha]$.
  - $\mathcal{M}, w \models \forall x A\ [\alpha]$ IFF FOR ALL $\beta \overset{x}{\sim} \alpha$: $\mathcal{M}, w \models A(x)\ [\beta]$.



Syntax / Arithmetic

AS CUSTOMARY, WE TAKE IDENTITY TO BE PART OF SYNTAX/MATH, AND THEREFORE CLASSICAL:

- $\Rightarrow t = t$,
- $s = t,\ A(s) \Rightarrow A(t)$.

$PA^{\to}$ IS OBTAINED BY EXTENDING HYPE WITH THE IDENTITY SEQUENTS ABOVE, RECURSION EQUATIONS FOR PRIMITIVE CONCEPTS OF $\mathcal{L}_N^{\to}$, AND

$$\Rightarrow A(0) \wedge \forall x \left(A(x) \underline{\to} A(x+1)\right) \underline{\to} \forall x\, A(x).$$

FOR $A(v) \in \mathcal{L}_N^{\to}$.

OBSERVATION. PA IN $\mathcal{L}_N$ AND $PA^{\to}$ ARE THE SAME THEORY, WHEN INDUCTION IS RESTRICTED TO $\mathcal{L}_N^{\to}$, BECAUSE $\to$ COLLAPSES INTO THE MATERIAL CONDITIONAL.

INDUCTION IS RESTRICTED TO $\mathcal{L}_{\mathbb{N}}$, BECAUSE $\rightarrow$ COLLAPSES INTO THE MATERIAL CONDITIONAL.

The two theories are <u>of course different</u> when additional vocabulary in addition to $\mathcal{L}_{\mathbb{N}}^{\rightarrow}$, e.g. Tr and TS, are added. However, the following is very good news:

LEMMA. LET $\mathcal{L}^+ \supseteq \mathcal{L}_{\mathbb{N}}^{\rightarrow}$ CONTAIN FINITELY MANY ADDITIONAL PREDICATES. THEN $PA^{\rightarrow}$ IN $\mathcal{L}^+$ PROVES:

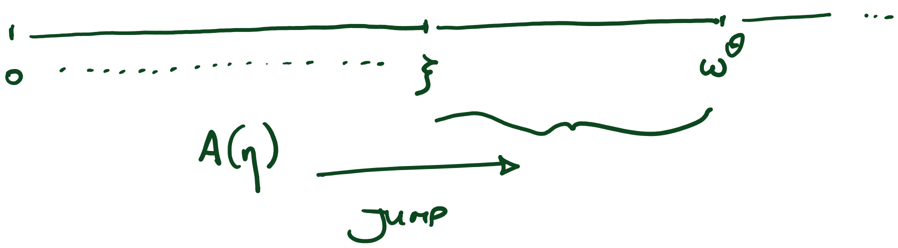
$$\forall \xi \left(\forall \eta < \xi \, A(\eta) \rightarrow A(\xi) \right) \longrightarrow \forall \eta < \underbrace{\alpha}_{\alpha < \varepsilon_0} A(\eta) \qquad (\Pi_{\mathcal{L}^+}(<\varepsilon_0))$$

[THIS CONTRAST WITH PA in, SAY, K3 OR FDS, WHICH PROVES ONLY $\Pi_{\mathcal{L}^+}(<\omega^\omega)$]

PROOF IDEA. THE PROOF IS GENTZEN'S (ESSENTIALLY):

THE GENTZEN JUMP IS:

$$B(\theta) := \forall \eta < \xi \, A(\eta) \;\underline{\underline{\longrightarrow}}\; \forall \eta < \xi + \omega^{\theta} A(\eta)$$



ONE SHOWS THAT STANDARD INDUCTION CAN BE APPLIED TO $A(\theta)$ TO OBTAIN THAT: $\Pi_{\mathcal{L}^+}(\theta) \Longrightarrow \Pi_{\mathcal{L}^+}(\omega^{\theta})$.

SO EACH $\omega^{\omega^{\cdot^{\cdot^{\omega}}}}$ CAN BE REACHED BY FINITELY MANY APPLICATIONS OF THIS JUMP.

QED.

Truth and Necessity

DEFINITION. WE CALL BT THE EXTENSION OF $PA^{\rightarrow}$ in $\mathcal{L}_{Tr}^{\rightarrow}$ WITH :

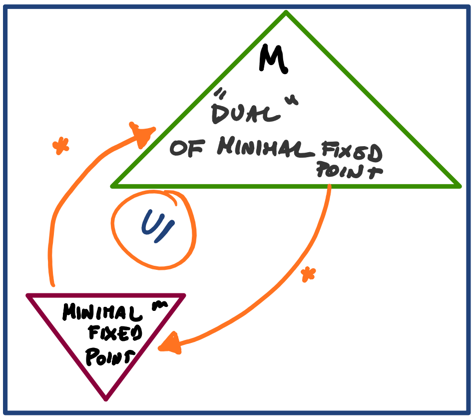
$$Tr\ulcorner A(\dot{x})\urcorner \leftrightarrow Ax,$$

WHERE CRUCIALLY $A(\dot{x}) \in \mathcal{L}_{Tr}$ (NO CONDITIONAL).

Remark. Unlike classical positive disquotation, BT deals with the full "Kripkean" language.

LEMMA (SEMANTICS). BT IS CONSISTENT AND HAS "NICE" MODELS.

PROOF. THE MINIMAL MODEL OF BT LOOKS LIKE :



$\mathcal{F} = \{M, m, \subseteq, *\}$ WHERE

$X^* = Sent_{\mathcal{L}_{Tr}} \setminus \{\neg\varphi \mid \varphi \in X\}$.

THE INTERPRETATION OF $\mathcal{L}_{\mathbb{N}}$ IS GIVEN BY $\mathbb{N}$.

QED

FACT. BT SUFFICES TO RECOVER Π^0_1-CA$^{<\varepsilon_0}$.

PROOF IDEA. THE DIAGONAL LEMMA GIVES US A PREDICATE

$$\Theta(x) \leftrightarrow K(\ulcorner\Theta\dot{x}\urcorner),$$

WHERE $K(x)$ IS THE INDUCTIVE DEFINITION OF Tr FOR $\mathcal{L}_{Tr}$ GIVEN IN $\mathcal{L}_{Tr}^{\rightarrow}$.

QED

For future reference, it's worth noticing:

OBSERVATION. GIVEN THE POWER OF $\to$, CURRY'S PARADOX IMMEDIATELY STRIKES IF ONE WANTS TO EXTEND BT TO SENTENCES CONTAINING Tr AND $\to$.

We move to the extension of BT with necessity, following the blueprint of MD.

DEFINITION (BASIC $\Box$) WORKING IN $\mathcal{L}^{\to}_{Tr,\Box} = \mathcal{L}^{\to}_{Tr} \cup \{\Box\}$, WE EXTEND BT WITH THE PRINCIPLES:

$$\text{(M0)} \quad \forall \varphi \in \mathcal{L}^{\to}_{Tr} \; (\mathrm{Prov}_{HYPE}(\varphi) \to \Box\varphi)$$

$$\text{(M1)} \quad \forall \varphi \in \mathcal{L}_{Tr} \; \Box(Tr\,\dot{\varphi} \leftrightarrow \varphi)$$

$$\text{(M2)} \quad \forall \varphi, \psi \in \mathcal{L}_{Tr} \; [\Box(\varphi \underset{HYPE}{\underline{\underline{\to}}} \psi) \wedge \Box\varphi \to \Box\psi]$$

There are some (misleading) promising results for the formulation of a coherent theory of necessity and truth.

OBSERVATION. OVER BT + (M1) + (M2), THE FACTIVITY PRINCIPLES

(i) $\Box\ulcorner A\dot{x}\urcorner \to Ax$ [NB. THIS IS A SCHEMA, FOR $Av \in \mathcal{L}_{Tr}$]

(ii) $\forall \varphi \in \boxed{\mathcal{L}_{Tr}} \; (\Box\varphi \to Tr\,\varphi)$

ARE EQUIVALENT.

PROOF.

PROOF.

(i) ⇒ (ii) : SUPPOSE $\Box\varphi$, SINCE $\Box(Tr\dot{\varphi} \leftrightarrow \varphi)$ BY (M1), VIA (M2) WE HAVE $\Box(Tr\dot{\varphi})$. BY (i), $Tr\varphi$.

(ii) ⇒ (i) : SUPPOSE $\Box \ulcorner A\dot{x} \urcorner$. SINCE $\forall\varphi$ ... QUANTIFIES OVER SENTENCES OF $\mathcal{L}_{Tr}$, AND $PA \vdash Sent_{\mathcal{L}_{Tr}}(\ulcorner A\dot{x} \urcorner)$, WE HAVE $Tr \ulcorner A\dot{x} \urcorner$. BY BT, Ax.

QED.

However, in order to recover interesting (universally quantified) laws, the principle

$$\Box \ulcorner A\dot{x} \urcorner \to Ax, \qquad \text{for } A(v) \in \mathcal{L}_{Tr}$$

is not sufficient, because these laws feature the conditional of HYPE.

DEFINITION. MBT IS OBTAINED BY EXTENDING PA IN $\mathcal{L}^{\to}_{Tr,\Box}$ WITH

(M0) $\forall\varphi \in \mathcal{L}^{\to}_{Tr} : Prov_{HYPE}(\varphi) \to \Box\varphi$

(M1) $\forall\varphi v \in \mathcal{L}_{Tr}\ \forall x\ \Box(Tr\,\varphi\dot{x} \leftrightarrow \varphi x)$

(M2) $\forall\varphi,\psi \in \mathcal{L}^{\to}_{Tr} : \Box\varphi \wedge \Box(\varphi \to \psi) \to \Box\psi$

(M3) $\Box \ulcorner A\dot{x} \urcorner \to Ax$, FOR ALL $A(v) \in \mathcal{L}^{\to}_{Tr}$

Notice that BT is a subtheory of MBT by M1 and M3.

OBSERVATION. MBT IS CONSISTENT. AGAIN, WE CAN INTERPRET THE BOX $\Box$ AS FORMAL PROVABILITY IN BT. CONSISTENCY IS THEN GUARANTEED BECAUSE

$$(\mathbb{N}, \{m, M\}) \models BT + \text{"Uniform Reflection"}$$
$$[\forall x\,(\mathrm{Prov}_{BT}(\dot{A x}) \to Ax),\ \text{WITH } A \in \mathcal{L}_{Tr}^{\to}]$$

The good news is now that **all** compositional principles are derivable in MBT.

PROPOSITION. THE THEORY KFL:

- $\forall s, t\,(Tr\,(s = t) \leftrightarrow val(s) = val(t))$
- $\forall x\,(Tr\,(\ulcorner Tr\,\dot{x} \urcorner) \leftrightarrow Tr\,x)$
- $\forall \varphi \in \mathcal{L}_{Tr}\,(Tr\,\neg\varphi \leftrightarrow \neg Tr\,\varphi)$
- $\forall \varphi, \psi \in \mathcal{L}_{Tr}\,(Tr\,(\varphi \wedge \psi) \leftrightarrow Tr\,\varphi \wedge Tr\,\psi)$
- $\forall \varphi\, v \in \mathcal{L}_{Tr}\,(Tr\,(\forall x \varphi) \leftrightarrow \forall t\, Tr\, \varphi(t/v))$

IS A SUBTHEORY OF MBT.

PROOF. SAME ARGUMENT AS ABOVE: E.G. SINCE

- $\forall \varphi, \psi \in \mathcal{L}_{Tr} : \Box\,(Tr\,(\varphi \wedge \psi) \leftrightarrow \varphi \wedge \psi)$
- $\forall \varphi, \psi \in \mathcal{L}_{Tr} : \Box\,(Tr\,(\varphi) \leftrightarrow \varphi) \wedge \Box\,(Tr\,(\psi) \leftrightarrow \psi)$

CLOSURE OF $\Box$ UNDER HYPE AND (M3) GIVES US

$$\forall \varphi, \psi \in \mathcal{L}_{Tr}\ (Tr\,(\varphi \wedge \psi) \leftrightarrow Tr\,\varphi \wedge Tr\,\psi).$$

QED.

Remark. This is another way to surpass the strength of $\Pi_1^0\text{-}CA^{<\varepsilon_0}$.

It's worth emphasising that, unlike classical modal disquotation, "MBT" gives us $\boxed{\text{FULL COMPOSITIONALITY}}$ for the "kripkean" language $\mathcal{L}_{Tr}$.

But there are also problems. The equivalence of the formulations of factivity for $\mathcal{L}_{Tr}$ seen above is not enough. One would require the equivalence of

$$(M3) \quad \Box \ulcorner A\dot{x} \urcorner \to Ax, \quad \text{for } Ax \in \mathcal{L}_{Tr}^{\to}$$

AND

$$(M3^*) \quad \forall \varphi \in \mathcal{L}_{Tr}^{\to} : \Box\varphi \to Tr\,\varphi .$$

LEMMA. PA (IN $\mathcal{L}^{\to}_{Tr,\Box}$) + (M0)–(M2), (M3*) IS CONSISTENT.

PROOF. AGAIN WE EMPLOY THE PROVABILITY INTERPRETATION, BUT IN A SUBTLER WAY. WE SHOW:

STEP (a) BT CAN BE CONSISTENTLY CLOSED UNDER

$$\text{NEC}: \frac{A}{Tr\ulcorner A \urcorner}, \quad \text{FOR } A \in \mathcal{L}_{Tr}^{\to}.$$

BY INDUCTION ON THE NUMBER OF APPLICATIONS OF NEC IN PROOFS IN BT+NEC: FOR EACH PROOF $\mathcal{D}$: BT + NEC $\vdash Tr\ulcorner A \urcorner$, WITH $A \in \mathcal{L}_{Tr}^{\to} \setminus \mathcal{L}_{Tr}$, WE CONSTRUCT A FIXED POINT MODEL

WE CONSTRUCT A FIXED POINT MODEL

$$\langle \mathbb{N}, M\_\{A \in \mathcal{L}_{Tr}^{\rightarrow} \setminus \mathcal{L}_{Tr} \mid D \text{ HAS } A \text{ AS SUBTHEOREM}\}, m\_\{A \in \mathcal{L}_{Tr}^{\rightarrow} \setminus \mathcal{L}_{Tr} \mid D \text{ HAS } A \text{ AS SUBTHEOREM}\} \rangle.$$

[BONUS : THIS GIVES CLOSURE UNDER CONEC $\frac{\vdash \ulcorner A \urcorner}{A}$].

STEP (b) : LET

$$S = \{ A \in \mathcal{L}_{Tr}^{\rightarrow} \setminus \mathcal{L}_{Tr} \mid BT + NEC \vdash A \},$$

AND

$$\mathcal{M} = \langle \mathbb{N}, M_S, m_S \rangle .$$

$\mathcal{M} \models BT$. IF $\mathcal{M} \models Prov_{BT}(\ulcorner A \urcorner)$, THEN EITHER $A \in \mathcal{L}_{Tr}$, SO $\mathcal{M} \models Tr \ulcorner A \urcorner$, OR $A \in \mathcal{L}_{Tr}^{\rightarrow} \setminus \mathcal{L}_{Tr}$, AND BY DEF. OF $\mathcal{M}$, $A \in S$.

QED.

OBSERVATION. THE TWO FACTIVITY PRINCIPLES

(M3) $\quad \Box \ulcorner A\dot{x} \urcorner \rightarrow Ax$, for $Ax \in \mathcal{L}_{Tr}^{\rightarrow}$

AND

(M3*) $\quad \forall \varphi \in \mathcal{L}_{Tr}^{\rightarrow} : \Box \varphi \rightarrow Tr\, \varphi$

ARE NOT EQUIVALENT OVER PA (IN $\mathcal{L}_{Tr}^{\rightarrow}$) + (M1) + (M2), ALTHOUGH CONSISTENT.

PROOF. THIS FOLLOWS SIMPLY FROM THE FACT THAT THEY DON'T

PROOF. THIS FOLLOWS SIMPLY FROM THE FACT THAT THEY DON'T HAVE THE SAME MODELS.
IN PARTICULAR, THAT (M3*) RULES OUT THE MINIMAL MODEL $(\mathbb{N}, m, \mathcal{M})$.

QED

But the situation is slightly worse:

PROPOSITION. MBT* — i.e. WHEN WE FORMULATE FACTIVITY AS $\forall \varphi \in \mathcal{L}_{Tr}^{\rightarrow} (\Box \varphi \rightarrow Tr \varphi)$ —, CANNOT PROVE ANY OF THE COMPOSITIONAL LAWS. THIS HOLDS ALSO IF WE KEEP THE SCHEMA BT IN THE BACKGROUND — AND NOT ONLY UNDER $\Box$.

PROOF IDEA. ONE CAN SHOW THAT

$$MBT^* \vdash A \quad \Rightarrow \quad BT + NEC_0 \vdash A, \quad \boxed{A \in \mathcal{L}_{Tr}}$$

WITH $NEC_0 :=$ THE RULE $\dfrac{B}{Tr \ulcorner B \urcorner}$ FOR $A \in \mathcal{L}_{Tr}^{\rightarrow}$ APPLIED ONCE.

SINCE $BT + NEC_0$ CANNOT PROVE ANY COMPOSITIONAL AXIOM, WE ARE DONE.

QED

Remark. Of course, if one extends the T-SCHEMA to $\mathcal{L}_{Tr}^{\rightarrow}$, Curry's Paradox strikes back.

Classical vs Nonclassical

	MD (classical)	MBT (Hype)
Deductive strength	✓	✓
Compositionality (Truth-functionality)	Partial	✓
Factivity (Coherence)	✗	✗
Paradox	?	? ✓
Unification	✓	Partial