The Phonological Conundrum in Formal Theories of Truth and Modalities

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(slides at carlonicolai.github.io)

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$\mathrm{Tr}\,\ulcorner\varphi(0)\urcorner \wedge \forall x(\mathrm{Tr}\,\ulcorner\varphi(\dot{x})\urcorner \to \mathrm{Tr}\,\ulcorner\varphi(\dot{x}+1)\urcorner) \to \forall x\,\mathrm{Tr}\,\ulcorner\varphi(\dot{x})\urcorner$

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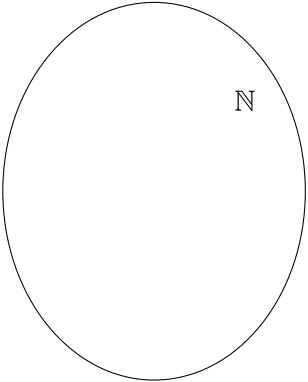

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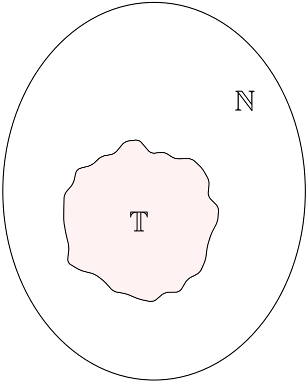

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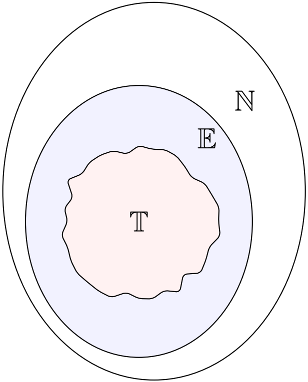

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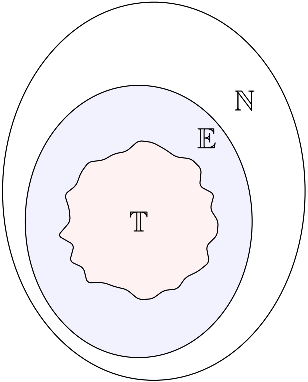
- **One sort of quantification**
- **One kind of induction**



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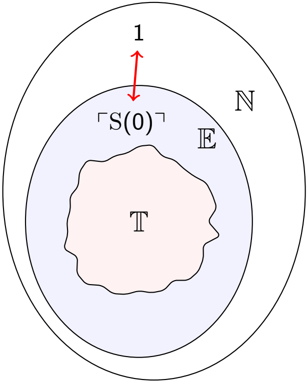
- $\mathcal{L}_{\mathrm{Tr}} := \mathcal{L}_{\mathbb{N}} \cup \{\mathrm{Tr}\}$
- One sort of quantification
- One kind of induction
- **Some implicit bridge principles**



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The language of mereology $\{\circ\}$. Virtually all reasonable, pure mereological theories are decidable. Therefore, there is no hope of formulating a reasonable syntax theory in them.

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In a more general setting, what we learn about truth is also more general (cf. Tarski 1936).

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Analyses of this sort rely essentially on the way in which induction is presented in the standard setting.

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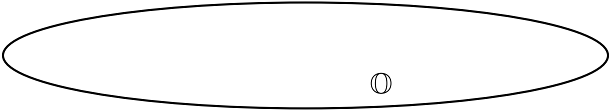
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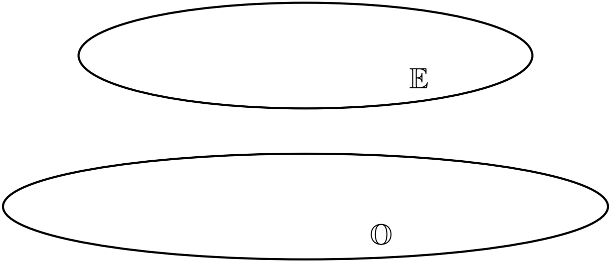
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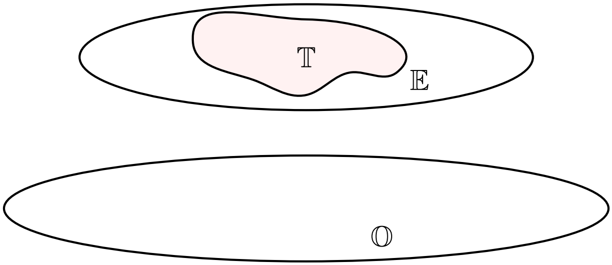
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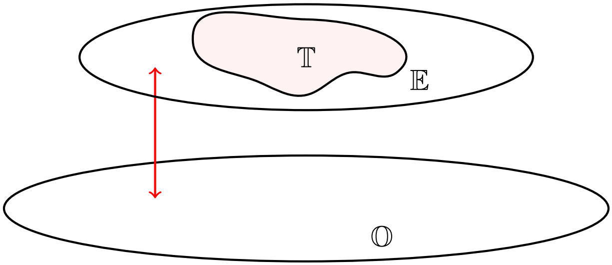
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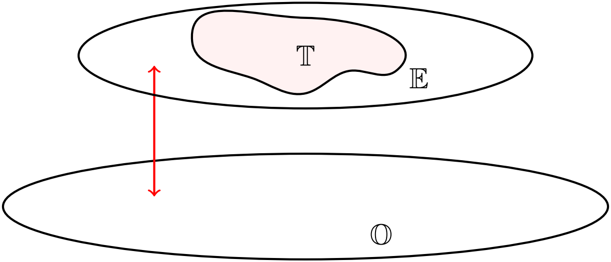
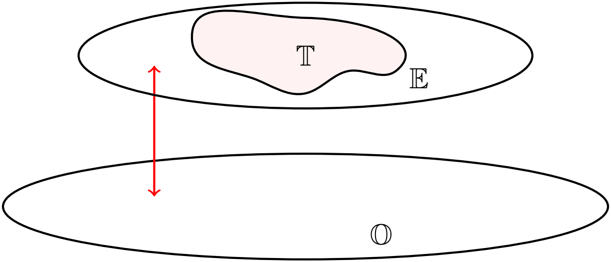


Axioms



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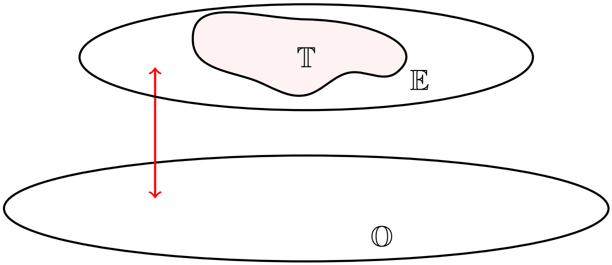

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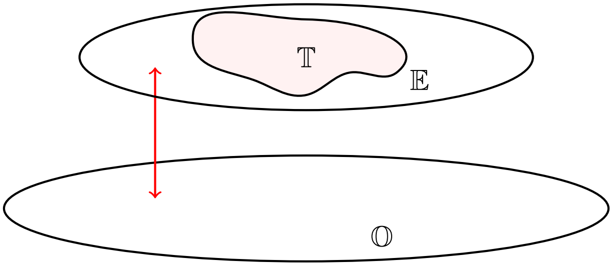

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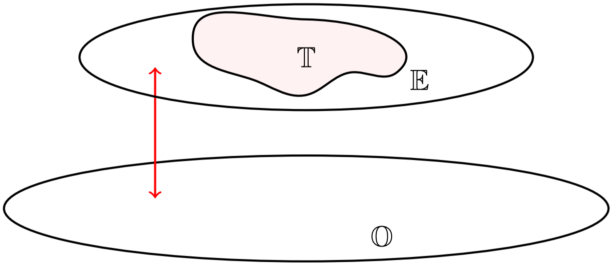

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- the smash function $x, y \mapsto 2^{|x|\cdot|y|}$
- the Σ^1_1-PIND induction schema:

$$\varphi(0) \wedge \forall x(\varphi(\lfloor \frac{1}{2}x \rfloor) \to \varphi(x)) \to \forall x \varphi(x) \text{ for } \varphi \in \Sigma^b_1$$

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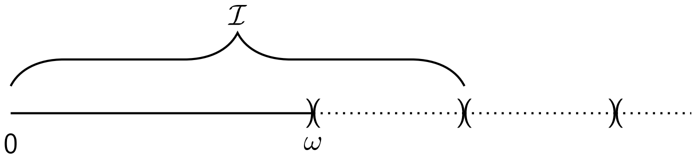

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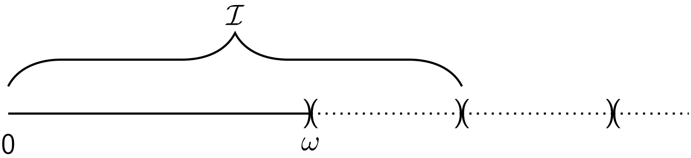

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- It is finitely axiomatisable.

For simplicity, a minimal theory of 'mixed' sequences $\alpha, \beta, \gamma \ldots$:

$$\forall \alpha \forall o \forall n \exists \alpha \forall m \big((m \neq n \rightarrow \alpha(n) = \beta(n)) \wedge \beta(m) = o \big) \qquad \text{(SEQ)}$$

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what we know so far

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Lemma

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Lemma

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Proof. For the k-many relevant formulas appearing in instances of disquotation, define:

$$\mathrm{Sat}(x, y) :\leftrightarrow \bigvee_{n=1}^{k} \left(y = \overline{\varphi_n(v_i)} \wedge \varphi_n(x(i))\right)$$

$\mathrm{ct}[\mathsf{O}]^-$

One restricts Σ_1^b-PIND to $\mathcal{L}_{\mathbb{N}}$, and adds the axioms:

$$\mathrm{Sat}(\alpha, \ulcorner R(v_1, \ldots, v_n)\urcorner) \leftrightarrow R(\alpha(1), \ldots, \alpha(n)) \text{ for } R \in \mathcal{L}_O$$
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The extended Σ^b_1-PIND is replaced with $\exists\Delta^b_1$-induction

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- *If O is finitely axiomatised, $\mathrm{ct}[O]^+$ proves the Global Reflection Principle:*

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Corollary

With the conditions above, $\mathrm{ct}[O]^+$ proves $\mathrm{Con}(O)$.

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Proof. I consider the first claim:

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- Second, one employs a miniaturised version of the Henkin-construction: in $\mathrm{Q} + \mathrm{Con}(O)$, one constructs a term model for O coming with a satisfaction predicate that satisfies $\mathrm{ct}[O]^-$.

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Proof. If it did, O would interpret $\mathrm{S}^1_2 + \mathrm{Con}(O)$, contradicting Pudlák's version of Gödel's second incompleteness theorem.

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- Granted: an adequate theory of truth over a fairly arbitrary object theory O either proves or interprets a consistency statement for O.
- However, in the present setting it is fairly clear that what is proved or interpreted is a syntactic claim that does not belong to the subject matter we are reasoning about.
- This seems compatible with the kinds of metatheoretic explanation that truth should provide.

doing without typing

The setting just presented is essentially typed. Is there a type-free version of it?

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Perhaps a Kripkean construction will do. For instance:

- $X_0 = \{(f, \varphi) \mid \varphi \in \mathcal{L}_O, f \colon \mathrm{Var}^{\mathbb{E}}_{\mathcal{L}_O} \to \mathbb{O}, \mathbb{O} \vDash^{f}_{k3} \varphi\}$

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- $(\alpha, \mathrm{Sat}(v_0, v_1)) \in X_{\alpha+1}$ if $(\alpha(v_0), \alpha(v_1)) \in X_\alpha$
- and so on for propositional (positive!) connectives.

The setting just presented is essentially typed. Is there a type-free version of it?

Perhaps a Kripkean construction will do. For instance:

- $X_0 = \{(f, \varphi) \mid \varphi \in \mathcal{L}_O, f : \mathrm{Var}^{\mathbb{E}}_{\mathcal{L}_O} \to \mathbb{O}, \mathbb{O} \vDash^{f}_{k3} \varphi\}$
- $(\alpha, \mathrm{Sat}(v_0, v_1)) \in X_{\alpha+1}$ if $(\alpha(v_0), \alpha(v_1)) \in X_\alpha$
- and so on for propositional (positive!) connectives.

This toy construction reaches a fixed point at ω, but is is crucial that now sequences are mappings $\mathbb{E} \cup \mathrm{SEQ} \to \mathbb{E} \cup \mathrm{SEQ} \cup \mathbb{O}$!