A Theory of Implicit Commitment

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(slides at carlonicolai.github.io)

The Project

We defend the following thesis on novel grounds:

> IMPLICIT COMMITMENT THESIS (ICT)*: Anyone who is justified in believing a mathematical formal system S is also implicitly committed to various additional statements which are expressible in the language of S but which are formally independent of its axioms.*

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For a theory τ, we focus on:

$$\mathrm{RFN}(\tau) := \{\forall z \big(\mathrm{Prov}_\tau(\ulcorner \varphi(\dot{z}) \urcorner) \to \varphi(z)\big) \mid \varphi(v) \in \mathcal{L}_{\mathbb{N}}\}$$

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One key contribution is a **direct axiomatization** of implicit commitment.

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Example

Let:

$$\mathrm{PA_I} := \bigcup_{n \in \omega} \mathrm{I}\Sigma_n \qquad\qquad \mathrm{PA_{II}} := \mathrm{Q1\text{-}6} + \mathrm{Ind}_{\mathcal{L}_{\mathbb{N}}}$$

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If $\mathrm{PA} \vdash$ 'every number is an instance of a PA-axiom φ',
then $\mathrm{PA} \vdash \forall x\, \varphi(x)$.

The Formal Theory

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Definition

Suppose that τ and τ' are two theories. We say that τ is **elementarily reducible** to τ', denoted $\tau \leq_{er} \tau'$, iff there exists an EA-provably total elementary function f such that

$$\mathrm{EA} \vdash \mathrm{Proof}_{\tau}(y, x) \to \mathrm{Proof}_{\tau'}(f(y), x).$$

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Reflection

$$\text{if EA} \vdash \forall x\, \tau(\ulcorner \varphi(\dot{x}) \urcorner), \text{ then } \forall x\, \varphi(x) \in \mathcal{I}(\tau).$$

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First, one has (Feferman):

$$\mathrm{EA} \vdash \forall x \, \mathrm{Prov}_\tau(\ulcorner \mathrm{Proof}_\tau(x_1, \ulcorner \varphi(\dot{x}_2) \urcorner) \to \varphi(x_2) \urcorner) \tag{1}$$

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Let

$$\tau'(x) :\leftrightarrow x \in \mathrm{EA} \vee \exists y \, x = \ulcorner \mathrm{Proof}_\tau(y_1, \ulcorner \varphi(\dot{y}_2) \urcorner) \to \varphi(y_2) \urcorner \tag{2}$$

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By (1) and REFLECTION, we get $\mathrm{RFN}(\tau) \subseteq \mathcal{I}(\tau')$. Since (1) also gives us $\tau' \leq_{er} \tau$, INVARIANCE then yields $\mathrm{RFN}(\tau) \subseteq \mathcal{I}(\tau)$. □

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- Let
$$\mathcal{I}_{\mathrm{II}}(\tau) = \{\forall x \varphi \mid \mathrm{EA} \vdash \forall x \, \tau(\ulcorner \varphi(\dot{x}) \urcorner)\}$$
Then $\mathcal{I}_{\mathrm{II}}(\mathrm{PA})$ is deductively equivalent to PA.

Justified Belief, Stability, and Entitlement

Main Claim

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  - Unlike RFN(τ), it involves an elementary property (i.e. membership in τ), not a *recursively enumerable* one.
  - It cannot be iterated.
  - Possibility of error (e.g. hidden ω-inconsistency) is substantially reduced.

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Claim

Once you have the general notion of axiom for a theory τ you're justifiedly believing, you're bound to have justified belief in $\mathrm{RFN}(\tau)$. This leaves untouched weaker versions, in which the 'coherent rationale' is not available to the τ-theorist.

Comparisons

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- We argued that something stronger is true, however...
- Our formal framework can be used to locate the source of the entitlement in principles that are **properly weaker** than Uniform Reflection: INVARIANCE and REFLECTION.

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REF $\forall\varphi\big(\mathrm{Prov}_{\tau B}(\varphi) \to B(\varphi)\big)$;

MP $\forall\varphi,\psi\big(B(\varphi) \land B(\varphi \to \psi) \to B(\psi)\big)$;

ωR $\forall\varphi(v)\big(B(\ulcorner \forall x B(\varphi(\dot{x}))\urcorner) \to B(\forall x \varphi(x))\big)$.

$$\frac{\varphi}{B(\ulcorner \varphi \urcorner)} \qquad (\textsc{nec})$$

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$\mathrm{Int}_{\mathrm{Bel}(\tau)} = \{\varphi \in \mathcal{L}_B \mid \mathrm{Bel}(\tau) \vdash B(\ulcorner\varphi\urcorner)\}$.

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- Moreover, NEC is unrestricted. Unlike our REFLECTION, it applies to Bel(τ)-proofs as well. Why?

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- Moreover, NEC is unrestricted. Unlike our REFLECTION, it applies to $\mathrm{Bel}(\tau)$-proofs as well. Why?

Proposition

Let τ be Σ_1-sound. Then $\mathrm{Int}_{\mathrm{Bel}'(\tau)}$ is conservative over τ.

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- A defence of ICT based on a preservation of justified belief
- A contribution to the debate on entitlement to reflection principles