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Physics applications for finance and their practical implementation

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"What I cannot create,
I do not understand."
Richard Feynman

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Abstract

The interplay between physics and finance has been a subject of mutual attraction since as far back as Jules R. (1863) [4].
The outcomes of this intersection have been remarkable, significantly influencing the approaches of economics and finance professionals.

Examples include Black & Scholes (Ch. III), the ARCH and GARCH models (Ch. I), the Heston model (Ch. IV), and the stochastic jump models (Ch. V), among the others.

The term "Econophysics" itself[1], the genesis of which is outlined in section 1.1.4, has enjoyed remarkable success. Approximately 20% of articles published in the esteemed journal *Physica A* are labelled as "Econophysics" [1][2].
This estimate seems to be conservative nonetheless, as it does not account for more recent publications. The popular aggregator of scientific papers ResearchGate currently[3] has over 27,500 publications tagged "Econophysics - Science topic"[4].

This seems to suggest that "Econophysics" is a well-established and accepted topic.
However, this is not the case (see [1], [2], [3], [27], [28] among the main ones)[5].

Finally, it should be noted that the majority of sponsors in the field of econophysics are physicists, rather than economists (see [1] on p. 91, "*The Difficult Dialogue between Financial Economists and Econophysicists*").

The aim of this thesis is to approach the challenging topic from a neutral point of view, with a foundation that is certainly grounded in the economic context, but which also wants to look at the opportunities offered by a solid quantitative approach, bringing economics closer to a hard science.

For this purpose, Chapter I will introduce a selection of notable historians, including several Nobel Prize winners.

[1] In this article, the term "econophysics" is used as a proxy, without the intention of specifically addressing a subject separate from the simple union of physics and finance.
[2] According to the timetable that was available as of 8 February 2025, eight new publications are scheduled to appear in the journal's upcoming publication. One of these specifically concerns the application of physical models to finance. An empirical sign that research on these issues is currently alive and well.
[3] 19 Jan 2025.
[4] https://www.researchgate.net/topic/Econophysics/publications
[5] The topic is also widely debated in online journals and other less structured sources which are therefore omitted.

Then, as mathematically rigorous as possible, some notable contributions are explored, chosen for their impact and practical relevance:

1. Brownian motion (Chapter II).
2. The Black-Scholes equation (Chapter III).
3. The Heston model (Chapter IV).
4. The jump diffusion models with Poisson processes (Chapter V).

Each of these models has been developed "on the shoulder of giants", thereby raising the economic debate to new standards.

In order to maintain a rigorous yet pragmatic approach, each thematic section is supported by a few code examples.

Finally (Chapter VI), an attempt is made to provide an overview of the phenomenon and the possible benefit of integrating physics and finance (Chapter VII). There are many references to scientific literature, which is abundant and multifaceted, indicative of a vibrant and ongoing debate, even in the present context[6].

The thesis concludes with an appendix that illustrates key terms that are worthy of mention[7].

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Il rapporto tra fisica e finanza è un rapporto di attrazione reciproca, fin da tempi ormai remoti, come Jules R. (1863) [4].

I risultati di questa crasi sono stati impressionanti, hanno letteralmente plasmato il modo nel quale i professionisti di economia e finanza lavorano.

Basti pensare a Black & Scholes (Cap. III), i modelli ARCH e GARCH (Cap. I), il modello di Heston (Cap. IV), i modelli stocastici a salti (Cap. V), tra gli altri.

Il termine stesso, "Econofisica"[8], la cui genesi viene illustrata nella sezione 1.1.4, ha avuto una incredibile fortuna: circa il 20% degli articoli pubblicati sulla celeberrima rivista *Physica A* riguardano il termine "Econofisica" [1][9], e la stima pare conservativa

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[6] This is noteworthy, considering the late 1990s as the peak period for econophysics.

[7] Compliance with the originality of this document was validated using the Compilatio.net tool.

[8] In questo articolo, il termine "econofisica" viene utilizzato come proxy, senza l'intenzione di affrontare specificamente una materia separata dalla semplice unione di fisica e finanza.

[9] Nel numero previsto in uscita a Marzo 2025 sono listati 8 nuovi articoli. Uno di essi riguarda applicazioni di fisica alla finanza. un empirico segnale del fatto che la ricerca sia viva al tempo presente su tali tematiche.

in quanto non vengono prese in considerazione in modo esaustivo altre pubblicazioni in tempi più recenti.

A oggi[10], il popolare aggregatore di paper scientifici ResearchGate propone oltre 27'500 pubblicazioni sotto il tag *"Econophysics - Science topic"*[11].

Questo dovrebbe far pensare ad un contesto ormai consolidato e anzi, addirittura considerabile classico.

Non è così, tutt'altro (si veda [1], [2], [3], [27], [28] tra i principali)[12].

Vale anche la pena citare che la maggior parte degli sponsor del contesto dell'econofisica sono fisici, piuttosto che economisti (si veda in particolare [1] a pagina 91, "*The Difficult Dialogue between Financial Economists and Econophysicists*").

Con questa tesi si vuole tentare di approcciare la sfidante tematica da un punto di vista neutrale, con una base che certamente poggia sul contesto economico, ma che vuole guardare alle opportunità offerte da un approccio quantitativo solido, rendendo l'economia più prossima a una scienza "dura".

Per fare questo si è scelto di partire da un primo capitolo introduttivo (Capitolo I), che enumera un campionario di storici notevolissimi (tra i quali diversi Premi Nobel).
Si esplorano poi, con un piglio per quanto possibile matematicamente rigoroso, alcuni contributi notevoli, scelti per rilevanza ma anche per un loro continuum pratico:

1. Il moto Browniano (Capitolo II).
2. L'equazione di Black & Scholes (Capitolo III).
3. Il modello di Heston (Capitolo IV).
4. I modelli di diffusione a salti con processi di Poisson (Capitolo V).

Ciascuno di questi modelli "poggia sulle spalle dei giganti" elevando il dibattito economico a nuovi standard.

Con la volontà di mantenere un approccio rigoroso, ma pragmatico, si è scelto di corredare ciascuna sezione tematica con alcuni esempi in codice.

Infine (Capitolo VI) si cerca di conciliare questa visione con una panoramica d'insieme, per forza di cose comunque incompleta, anche in virtù del dinamismo della tematica trattata.

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[10] 19 Jan 2025.
[11] https://www.researchgate.net/topic/Econophysics/publications
[12] Il tema trova ampio dibattito anche su riviste online e altre sorgenti meno strutturate e pertanto omesse.

Nelle Conclusioni (Capitolo VII) si va in cerca di possibili evoluzioni, o quanto meno spunti di riflessione ulteriori. La volontà è di redigere un testo supportato da evidenze scientifiche. Le quali, comunque, sono numerose e presentano un vivo dibattito; questo segno di un confronto ancora vivo al tempo presente[13].

La tesi termina con un'appendice che illustra alcuni termini ricorrenti degni di menzione[14].

[13] Questo non è scontato, considerando che l'econofisica ha avuto il suo picco nei tardi anni '90.
[14] Il rispetto del pensiero originale del presente documento è stata validato mediante lo strumento Compilatio.net

Interactive open-source code repository

This thesis is complemented by several Python scripts that aim to illustrate real-life implementations of the methods, theories and formulae described herein in text format.

To preserve the self-consistency of the document, the code is provided after each section, with the printout of the code results.

An interactive Jupyter Notebook is also available, which allows all the proposed code to be executed (and even modified in its main parameters, such as price of assets or volatility).

The reader can access the Notebook here:
tesi_occhiena_physics for finance_2024_unige.ipynb

The decision to use Python for the simulations was based on its versatility, the wide availability of specialised mathematical and quantitative finance libraries (such as NumPy, SciPy and Pandas), and the ability to implement complex algorithms in a clear and efficient manner.

Python's widespread use in academic and professional settings ensures reproducibility, scalability, and a shared language for validating simulation results against a common knowledge background.

Please note that scripts dependent on random processes or from market data may yield different results to those reported in the document. Also, code formatting (due to the strict indentation form of Python code) could be approximate on the document in respect to the rigorous formatting on the IDE.

The code has been created and it is proposed for illustrative purposes only and should not be used as actual code in production environments or used to value financial assets in the real world.

I. Physics and Finance: an unusual association

It is human nature to seek to understand natural phenomena: why they occur, how they reproduce, what variables affect them.

The ultimate fascination is to be able to control such events. Either through increasingly sophisticated and accurate methods of prediction, or by being able to reproduce such episodes having gained sufficient awareness of them.

Is it possible (or, even better, is it reasonable) to compare the economic system to a physical system?

Can "*the economy*" and "*the ecosystem*" be associated?
Can the random movements of electrons within an atom at a certain time be compared to the movement of the random component of a stock price? Can the random movement of a pollen particle in a fluid inspire a pricing model for derivatives?

It is acknowledged that these generalisations may not be palatable to a rigorous economist or a physicist. Physical systems obey universal laws and are perfectly repeatable (e.g. the heat equation, Brownian motion, thermodynamics). By contrast, economic and financial systems are the result of human behaviour and are typically subject to the irrationality and susceptibility of human nature[15] (only the expected behaviour can be inferred).
Nevertheless, it is reasonable to consider that a physical approach to finance allows to capture the dynamics of financial systems and to define statistical structures suitable for modelling financial products, which, although within a somehow subjective framework, remain valid[16].

It is from these premises that econophysics has taken root: a multidisciplinary approach that aims to apply concepts, methods, theorems and formulas born in the field of physics to interpret financial and economics phenomena.

1.1 Genesys of econophysics

The field of physics applied to finance and economics falls fairly broadly within the definition of Econophysics.

[15] See e.g. Rousseau F. et al., 2008. Irrational Financial Markets, Economics Department Working Paper Series n1870108.pdf, Department of Economics, National University of Ireland - Maynooth.
[16] The impact that physicists had on the Nobel Prize in Economics may account as a proxy for the validity of this statement. Aside from historical records, see in recent times Physic World, 11 Oct 2022, Breaking boundaries: physicist bags 2022 economics Nobel.

To date, the field of econophysics is considered unorthodox from an economic point of view and not strictly delimited in terms of practices and adoptions [10].

The definition of econophysics encompasses a variety of approaches that have been traced back over a century and have been formalised through gradual and increasing utilisation, ultimately becoming part of the classical body of knowledge of financial economists.

In the following sections, only a few of those who recognised the potential of a multidisciplinary approach that could merge physics and finance will be mentioned.

Many important names are excluded, including outstanding figures such as Ettore Majorana ("The Value of Statistical Science in Physics and the Social Sciences"[17], published posthumously in 1942)[18].
A historical treatise would be a book in itself[19].

1.1.2 Econophysics pioneers (1860 - 1900)

Jules Regnault

The first case of practical application of physical principles to economics is traced back to Jules Regnault[20].

In "Calcul des Chances et Philosophie de la Bourse" (1863), Regnault is pioneering a scientific approach to the "science of the stock market", particularly in its probabilistic foundation.

On page 50 of his work [4], Regnault stated that:

"*[...]There is a mathematical law that governs the variations and the average deviation of the prices of the Borse, and this law, which seems never to have been suspected until now, is formulated here for the first time:*

The deviation of prices is a direct result of the square root of the times"[21].

This statement may be regarded as a precursor to the mathematical formulation of

[17] Original title is in Italian; translated by the author.
[18] It is said that Adam Smith's work was also influenced by the physics of Isaac Newton. See Diemer A., Guillemin H., 2011.
[19] A good place to start is [1].
[20] Jules Regnault was from a humble background (he was exempted from compulsory military services in order to preserve potential sources of income for his family, his father having passed away in the years prior). However, thanks to his financial trading activities, Regnault bequeathed a considerable fortune upon his death in 1884 (Jovanovic F., 2004).
[21] Original quote is in French. Translation done by the author.

Brownian motion (Robert Brown, 1827).

Leaving the formal definition of Brownian Motion to the subsequent Chapter II, in the following section it is shown how, under specific assumptions, the statement formulated by Jules Regnault can be regarded as a precursor definition of volatility in a stochastic process such as the Brownian motion.

For the sake of this demonstration, let assume a Brownian motion with a deterministic drift, expressed as a Markov process and described by a stochastic integral equation:

$$X_t = X_0 + \int_0^t \mu \, ds + \int_0^t \sigma \, dW_s$$

The above equation provides the explicit solution for X_t, with:

1. X_t: process with drift component.
2. X_0: initial value of the process.
3. μ: drift coefficient (deterministic).
4. σ: diffusion coefficient.
5. W_s: standard Brownian motion (Wiener process).
6. $W_s \sim \mathcal{N}(0, ds)$.
7. $\int_0^t \mu \, s$: deterministic accumulation term, representing growth at a μ rate.
8. $\int_0^t \sigma \, dW_s$: stochastic term, obtained as an integral with respect to Brownian motion W_s.

The deterministic integral with respect to time can be solved as following:

$$\int_0^t \mu \, ds = \mu t$$

The integral with respect to W_s is a stochastic integral, specifically an Itô integral, which extends the Riemann–Stieltjes integral to stochastic processes with finite quadratic variation.[22]
This integral is defined on the properties of Brownian motion, where the stochastic process W_t models the uncertainty of the process.
Given a stochastic process X_t expressed in the differential form as:

$$dX_t = \mu \, dt + \sigma \, dW_t, \;\; X_0 \in \mathbb{R}$$

Applying Itô's formula to a process $Z(t) = f(Xt, t)$ leads to the following stochastic

[22] See 2.2.2.

differential equation:

$$dZ(t) = df(X_t, t) = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dW_t,\ Z_0 = f(X_0, 0)$$

Using the properties of Brownian motion and the definition of quadratic variation, the following relations hold:

$$(dt)^2 = 0,\ dtdW = 0,\ (dW)^2 = dt$$

After appropriate substitution and simplification of terms, the Itô's formula is obtained:

$$df(X_t, t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 dt$$

Assuming a linear function in the form $f(x) = ax + b$, and applying Itô notable integral to Brownian motion W_t:

$$df(W_t) = adW_t,\ f(W_0) = aW_0 + b = b$$

Integrating both sides of the equation on the interval [0, t]:

$$\int_0^t df(W_s) = \int_0^t a\,dW_s$$

That simplifies to:

$$f(W_t) - f(W_0) = a(W_t - W_0).$$

Posing a = 1, b = 0:

$$\int_0^t dW_s = W_t - W_0$$

Hence:

$$\int_0^t \sigma\,dW_s = \sigma W_t$$

After a substitution in the initial equation, the final result is as follows:

$$X_t = X_0 + \mu t + \sigma W_t$$

Where the variance of the stochastic term is:

$$Var(\sigma W_t) = \sigma^2 t$$

That leads to the subsequent standard deviation:

$$\text{Standard Deviation}(X_t) = \sigma\sqrt{t}$$

That finally reconciles with Regnault's exposition.

Louis Bachelier

In 1900, Louis Bachelier approached the problem of stock pricing from a purely mathematical point of view in his doctoral thesis “Théorie de la spéculation” (1900)[23].

In particular, Bachelier focused on a probabilistic approach and assumed the principle of intertemporal price consistency[24].

According to this initial postulate, Bachelier recognised the centrality of a probabilistic model, which he formulated in terms of a stochastic process, comparable to what is now known as Brownian motion.

Bachelier's approach can be considered pioneering[25]. It lays the foundation for what will be milestones such as Efficient Market Hypothesis (Fama, 1970) and the Black-Scholes-Merton model (1973).

1.1.3 In search of mathematical robustness (1960 - 1990)

Norbert Wiener

Bachelier’s pioneering theory, introducing stochastic processes to model financial markets, marked a significant intellectual advance, offering a novel and ambitious insight into the field of financial modeling.

However, Bachelier’s postulate lacked the mathematical rigour that would be provided by the seminal work of Norbert Wiener[26] in 1923. Wiener’s contribution was the

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[23] Notably, his advisor was Henri Poincaré.

[24] Price is consistent across time horizons. A paramount assumption for capital market expectations theories.

[25] See Bachelier’s page on University of York website, as one of the many examples of this affirmation.

[26] Despite the remarkable contribution on Brownian motion, Wiener has gone down in history as “the father of cybernetics”.

formulation of a mathematical model of a Brownian motion, that came to be known as Wiener's process, which would set a new standard of mathematical rigour, specifically within the fields of limit theorem and probability theory[27].

Norbert Wiener's remarkable insight was to prove the existence of a continuous stochastic process in the form of Brownian motion, which possesses its characteristic properties (see Chapter II, "Brownian Motion").
Wiener's challenge was to prove that the stationary and independent increments of Brownian motion and their normal distribution do not preclude its continuity (see Lalley S., "Brownian Motion", Ch. 1-1, University of Chicago).

Wiener process can be also see as a limit of a rescaled simple random walk[28]. Let X_k be "i.i.d." (independent, identically distributed random variables), $X_k \sim N(0, 1)$.
For each $n \geq 1$ a continuous, stochastic process $W_n(t)$, $t \geq 0$, can be defined as:

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{1 \leq j \leq nt} X_k$$

The statistical significance of Brownian motion is remarkable.
Many natural phenomena can be traced back to cases of Brownian motion[29].
In the field of finance, it plays a pivotal role in pricing financial assets characterised by constant volatility and continuous price fluctuations over time[30].

It should be noted, however, that Wiener's work did not originate in a context devoted to financial applications. The connection between Wiener's work and Bachelier was posited posthumously, in 1965, by Paul Samuelson.

Paul Samuelson
Samuelson rediscovered Louis Bachelier's work on modelling price fluctuations as a Brownian motion and recognised its pioneering value in this field.

However, he realised that Bachelier's arithmetic Brownian motion was not appropriate in the financial context, particularly in relation to stock pricing, as it could generate negative prices[31].

Samuelson therefore proposed the use of geometric Brownian motion, which ensures

[27] The literature on the subject is vast. An overview can be found in J. L. Doob, Wiener's work in probability theory, in Bulletin of the American Mathematical Society 72, pp. 69-72.
[28] See e.g [21] at pages 8-9, or Smoluchowski M., 1906.
[29] See e.g. Romanczuk, Pawel; Couzin, Iain D.; and Schimansky-Geier, Lutz. Collective Motion due to Individual Escape and Pursuit Response. Physical Review Letters 102, 010602 (2009).
[30] Ali N. Akansu, Mustafa U. Torun Torun, in A Primer for Financial Engineering, 2015
[31] See however section 2.2.5, "Modelling for negative prices".

that prices remain strictly positive due to the exponential solution of the SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where:

1. S_t: underlying asset price in time t.
2. μ: drift term, representing the deterministic component of the expected rate of return.
3. σ: volatility (diffusion coefficient).
4. dW_t: infinitesimal increment of Wiener process.

Samuelson's work demonstrated that, in an efficient market, prices follow a random walk with a stochastic component modelled by a geometric Brownian motion.

Paul Samuelson was awarded a Nobel Prize in Economics in 1970.

Eugene Fama

In 1965, it was Eugene Fama's turn to build on Bachelier's work, recognising the irrational nature of investors' expectations.

Bachelier had observed that the set of market expectations had $\mathbb{E}[X] = 0$, with bullish and bearish expectations in the long run cancelling each other out.

Fama formalised the Efficient Market Hypothesis (EMH) by theorising that financial instruments traded on the market already reflect all available information in their price[32].

Fama then built on Bachelier's work by formulating a robust construct, which he presented in three different forms[33]:

1. Weak Form: prices reflect all historical price and trading volume information.
   Technical analysis (based on past price charts) cannot generate systematic profits.
2. Semi-Firm Form: prices reflect all available public information (financial statements, company news, etc.).
   Fundamental analysis cannot generate extra profits.
3. Strong Form: prices reflect all information, both public and private (inside information).
   Even those who have access to insider information cannot beat the market.

[32] It is often said that *markets are discount machines*.

[33] The reproducibility of the Strong Form remains a subject of debate, particularly in relation to insider trading restrictions and paid information. It is generally accepted that the semi-firm form is considered established, with the strong form's validity being restricted to specific contexts.

EMH earned Eugene Fama the Nobel Prize in Economics in 2013.

EMH justifies the use of stochastic models in the pricing of financial assets. The significant impact of EMH had also encouraged further studies in the field.

Towards the 1970s, a number of factors led to a strong impetus to structure a new approach to global financial markets, which were increasingly volatile and subject to the influence of a growing number of interrelated variables.

In particular, the end of the Bretton Woods system (1971) (with the end of the Gold Standard), the birth of the Chicago Board Options Exchange (CBOE) (1973), and the rapid financial globalisation, demanded new mathematical tools to manage new and growing opportunities, and especially the associated risks.

Fischer Black, Myron Scholes & Robert Merton

"If options are correctly priced in the market, it should not be possible to make profits by creating portfolios of long and short positions in options and their underlying stocks"[34].

The launch of the CBOE and the publication of the Black & Scholes[35] model were essentially concurrent events. The mathematical rigour provided by the Black & Scholes model, combined with the brilliance of its underlying insights, helped regulators and technicians to see option trading as a legitimate financial asset, crucial to risk management, rather than mere instruments of speculation or even gambling (Mackenzie and Millo, 2003)[36].

The increasing use of derivative financial instruments, and in particular of options on equities, highlighted the pressing need for a fair[37] pricing model for options.

The Black & Scholes[38] model is designed to price European derivatives.
The model works under the following key assumptions[39]:

1. The underlying asset follows a geometric Brownian motion.
2. There are no dividends paid by the underlying asset (in the basic model).

[34] Opening citation in the original Black & Scholes submission paper [7].
[35] The "Black-Scholes Model" name was humbly proposed by Merton himself.
[36] An interesting historical read is Kumiega, A.; Sterijevski, G.; Wills, E. Black–Scholes 50 Years Later: Has the Outperformance of Passive Option Strategies Finally Faded? Int. J. Financial Stud. 2024, 12, 114. https://doi.org/10.3390/ijfs12040114.
[37] Easy to be calculated, easy to be verified.
[38] See Chapter 3, Black & Scholes Equation.
[39] See section 3.1.3, Assumption to the Black & Scholes equation.

3. The interest rate r and volatility σ are constant.
4. The markets are efficient and there are no transaction costs.
5. The option is a European option (exercisable only at maturity).
6. The asset is tradable at any time.

Black & Scholes model is based on the following equation:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} = rP$$

Where:
1. P: option value.
2. S: price of the underlying asset.
3. t: time.
4. r: risk-free rate.
5. σ: volatility.

Interestingly, it compares to the heat-transfer parabolic partial differential equation (PDE) (from physics)[40]:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Where:
1. u: temperature or heat distribution.
2. x: space.
3. t: time.
4. α: thermal diffusion coefficient.

Therefore:
1. Option price P is analogous to the temperature u.
2. The price of the underlying asset S is analogous to the spatial position x.
3. The volatility σ is analogous to the diffusion coefficient α.

The Black & Scholes model has paved the way for new areas of research in finance and asset pricing, such as advanced stochastic process theory, stochastic volatility models (e.g. Heston model, see Chapter IV) and the valuation of complex and exotic derivatives.
Finally, it stimulated the emergence of new professionals, proficient in math and

[40] *"the differential equation is the heat-transfer equation of physics"*, quoted from the original submission paper of Black and Scholes [7].

physics as well in finance and economics: the quantitative analyst (the "quant")[41].

Myron Scholes and Robert Merton received the Nobel Prize in Economics in 1997[42] for their contribution to the development of theory.

Robert Engle
Robert Engle identified a key issue in financial time series analysis: the non-constant volatility of financial asset returns.
Traditional statistical methodologies assumed constant variance (homoscedasticity).
But empirical data clearly showed that volatility changes over time and tends to occur in clusters of high and low volatility (see e.g. Da Cunha, 2021).

In response to this challenge, Engle proposed the ARCH (Autoregressive Conditional Heteroskedasticity) model in 1982.
This model was designed to address the non-constant variance of financial asset returns.

ARCH is based on:

1. Conditional heteroskedasticity: the variance at time *t* is not constant, but depends on previous shocks.
2. Volatility clusters: periods of high volatility tend to follow other periods of high volatility, while periods of low volatility tend to persist.
3. Asymmetric shocks: the basic ARCH model is symmetric and does not distinguish between positive and negative shocks.

An ARCH(q) process assumes that the conditional variance of returns depends on the squares of past shocks up to the lag *q*. Let *rt* be the return[43] on a financial asset at time *t*.

The model can be represented as:

$$r_t = \mu + \epsilon_t$$

$$\epsilon_t = z_t \sqrt{h_t}, \quad z_t \sim N(0,1)$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2$$

Where:

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[41] See Derman, 2004: My Life As A Quant: Reflections On Physics And Finance.
[42] Unluckily Fischer Black died in 1995. However his contributions were cited and recognized during the ceremony.
[43] In simpler terms, the financial gains.

1. r_t: yield at time t.
2. μ: mean value of yields (deterministic component).
3. ϵ_t: shock or residual error at time t.
4. z_t: standardised random variable with standard normal distribution.
5. h_t: conditional variance at time t
6. $\alpha_0 > 0$: constant term of the conditional variance.
7. $\alpha_i \geq 0$: coefficients weighting the effect of the squares of past shocks.
8. q: lags.

ARCH is frequently employed in conjunction with ARMA (Autoregressive Moving Average), in an approach known as the ARMA-ARCH process. It serves as the foundation for the development of GARCH (Generalised ARCH).

The significance of ARCH's contribution to time series analysis methods with variable volatility was recognised by Engle, who was awarded the Nobel Prize in Economics in 2003.

1.1.4 Econophysics (1990 - 2000)

Following the success of the illustrious predecessors and the increasing availability of data and computational resources to analyse them, the use of physical methods to approach financial problems gained traction from the 1980s onwards.

In 1994, the term "Econophysics" was coined by Professor Harry Eugene Stanley, and the multidisciplinary approach was formalised thanks to its diffusion in academia[44].

"[...] the term 'econophysics' was chosen with some care to follow the path of such mergers as 'astrophysics', 'biophysics', and 'geophysics'"[45].

Econophysics as a subject has been formally recognised in the Physics and Astrophysics Classification Scheme (PACS) list, under item 89.65.Gh (Other areas of applied and interdisciplinary physics) as: Economics, econophysics, financial markets, business and management[46].

Since the term "econophysics" was coined, its use has grown exponentially[47], at least in the academic context[48].

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[44] Accordingly to [1], the informal genesis is due to a paper by Professor R. N. Mantegna in 1991. Notably, Professor Mantegna is co-author with Professor H. E. Stanley himself of [10].

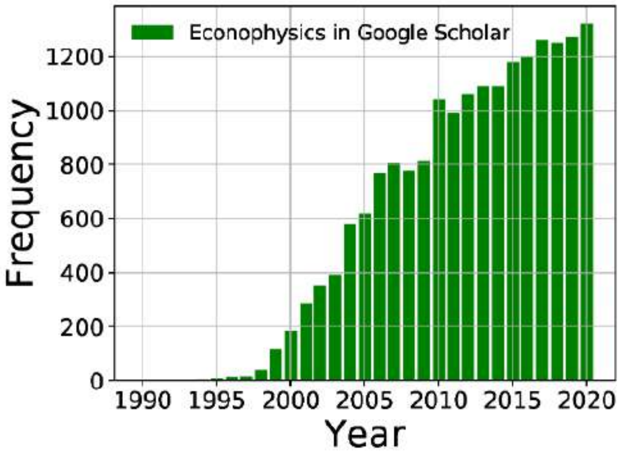
[45] ([Stanley et al., 2006] p. 337).

[46] The PACS list is public and published by many sources; see e.g. (URL retrieved 22 Feb 2025) https://ufn.ru/en/pacs/all/ .

[47] In [1] this phenomenon is shortly described as a colonization approach (see section 4.3.3).

[48] Image 1.1 is from Chakrabarti, B.K., Sinha, A., Development of Econophysics: A Biased Account and Perspective from Kolkata. *Entropy 2021, 23, 254* at URL https://doi.org/10.3390/e23020254 (retrieved 05 Jan 2025). Licensed under Creative Commons Attribution (CC BY).

[Image 1.1]



Notably, within academia, Italy has played a significant role in the field.
The University of Pavia appointed H. E. Stanley as Honorary Professor[49].
Furthermore, Professor Rosario Nunzio Mantegna of the University of Palermo collaborated with Stanley to publish the seminal work, Introduction to Econophysics: Correlations and Complexity in Finance[50] in 1999.

As the historical overview has demonstrated, Bachelier's pioneering insights, subsequently formalised by Wiener and applied in finance by Samuelson, Black, Scholes and Engle, profoundly transformed the understanding of financial markets.

However, to fully comprehend the significance and impact of these contributions, it is crucial to explore the key mathematical formulae that underpin them.

In the following chapters will be proposed a formal analysis of the main equations and stochastic models currently used in financial mathematics:

1. Brownian Motion.
2. Black & Scholes Model.
3. Heston Model.
4. Jump Diffusion Model with Poisson Process.

Each formula will be derived, explained in detail and contextualised in its practical application through interactive code formulation with Python and Jupyter Notebook.

The insights and ideas that revolutionised finance are based on solid mathematical foundations[51].

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[49] https://www.bu.edu/eng/profile/h-eugene-stanley/.
[50] [10].
[51] See Akyıldırım E., Mete Soner H., A brief history of mathematics in finance, Borsa Istanbul Review, Volume 14, Issue 1 (2014), pp 57-63, ISSN 2214-8450, https://doi.org/10.1016/j.bir.2014.01.002.

II. Brownian Motion

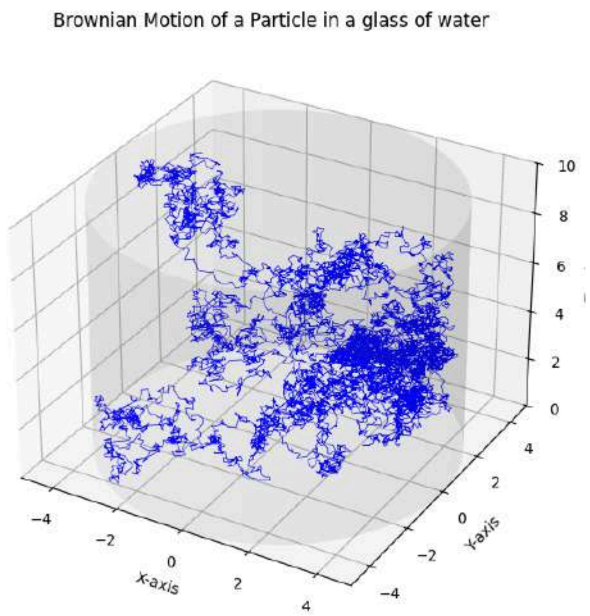
2.1 From grain particles to financial assets

In 1827, Robert Brown, a renowned botanist and palaeobotanist, was engaged in meticulous observation of pollen grains from the *Onagraceae* family under a microscope.

Noticing a constant motion in some of the particles, he initially hypothesised that such motion was derived from some kind of vital processes.

Fascinated by this peculiar observation, he conducted multiple experiments, leading to the conclusion that the observed motion was not due to vital propulsion, but rather resulted from specific, albeit unproven, physical phenomena.

[Image 2.1]



Simulation of particle motion in a 5x10 glass of water,
n = 10'000, σ = 0.1, shaped via a reflection factor of -1.

While Brown did not provide any mathematics or physics-based theories for explaining the phenomenon in his manuscript [16], his empirical observation allowed him to be remembered as the discoverer of the motion that bears his name *ad perpetuam rei memoriam*.

It was not until 1905 that a rigorous mathematical formulation was provided, thanks to the independent yet essentially parallel studies of Albert Einstein and Marian Smoluchowski.

For the analytical definition of the Brownian motion, the reference proposed in [11] has been followed. Note that the diffusion equation is proposed as the convention with diffusion coefficient *D*. It includes the factor *2Dt*, as properly done in mathematics, probability and finance. In Albert Einstein's original formulation (as in physics and statistical physics instead), the factor used is typically *4Dt*.

Einstein's intuition, derived from observation of the physical phenomenon as described by Brown, is based on the assumption that the motion of suspended particles is generated by a continuum of collisions between the particles themselves.
In particular, the distribution of displacements can be described by means of a normal distribution.

Analytically, the process followed by Albert Einstein can be formalized as following:

1. The diffusion equation models the probability density function $p(x,t)$, which represents the probability of finding a particle at position *x* at time *t* in a fluid with diffusion coefficient *D*. This probability density satisfies the Fokker-Planck equation (or the heat equation in the simplest case):[52]

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2}$$

2. In the case of a Brownian motion, the probability density function follows a Gaussian distribution:

$$p(x,t) = \frac{1}{\sqrt{2\pi Dt}} e^{-\frac{x^2}{2Dt}}$$

3. From there, Einstein was able to define the expected value[53] of the displacement of the particle, $\langle x^2 \rangle$. More precisely, the mean square displacement (how far, on average, a particle has moved from its starting point after a certain time)[54].

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \frac{1}{\sqrt{2\pi Dt}} e^{-\frac{x^2}{2Dt}} dx$$

[52] Notice the similarity with Black-Scholes already!
[53] Angle brackets notation is a standard physics notation; economists may read it as "$\mathbb{E}[X]$".
[54] May we call it the "variance" of the particle?

The above integral is a standard case in normal distribution with solution:

$$\int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi}\sigma^3$$

Noted that $\sigma^2 = Dt$, $\sigma = \sqrt{Dt}$. Hence:

$$\langle x^2 \rangle = \frac{1}{\sqrt{2\pi Dt}} \cdot \sqrt{2\pi}(Dt)^{\frac{3}{2}}$$

And, finally, with some (approachable) algebra:

$$\langle x^2 \rangle = (Dt)^{\frac{3}{2}}(Dt)^{-\frac{1}{2}}$$
$$\langle x^2 \rangle = 2Dt$$

The result obtained thus far, although not yet robust, is particularly surprising due to its striking analogy with the financial environment:

[Tab. 2.1]

Particles (Physics)	**Asset price (Finance)**
Particle position at instant t	Asset price at instant t
Diffusion time	Maturity (Investment time horizon)
Diffusion coefficient D	Volatility σ
Mean squared displacement $\langle x^2 \rangle = 2Dt$	Variance of returns $\langle (\Delta S)^2 \rangle$

2.2 Brownian motion - definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space[55], where Ω is a sample space (a set of events), $\mathcal{F}$ is a subset of events in such given space (σ-algebra)[56] and $\mathbb{P}$ is a function defined with a probability outcome in [0,1].

A standard one-dimensional Brownian motion (or Wiener process) is a stochastic process $(W_t)_{t \geq 0}$ with:

1. $\mathbb{P}(W_0 = 0) = 1$.
2. Stationary, independent, increments: W_t - W_s, with $0 \leq s < t$, is an increment independent from every past event occurring within the probability space and

[55] This is essential for the definition of a subset of measurable sets (the collection of all the measurable events that may happen with a knowable probability).

[56] This represents the information available up until time t.

with Normal distribution such as: $(W_t - W_s) \sim N(0, t - s)$, $0 \leq s < t$.

3. Given 1 and 2, under Kolmogorov continuity theorem, the increments, and hence, the paths, will be continuous.

2.2.1 Markov property and Martingale process in the context of stochastic processes

1. Markov property states that, assuming S_t as the price of a financial asset S at time t defined by a stochastic process, the conditional distribution of S_T, $T > t$, depends only on the current value S_t and not on the past value S_s, $s < t$:

$$\mathbb{P}(S_T \mid S_t, S_s \,\forall s < t) = \mathbb{P}(S_T \mid S_t)$$

2. The price of a financial asset S at time t defined by a stochastic process is a martingale[57] with respect to a filtration $\mathcal{F}_t$ if its conditional expectation satisfies the property:

$$\mathbb{E}[S_T \mid \mathcal{F}_t] = S_t, \quad \forall \ T > t$$

   Less formally, the meaning is that the expected future value of the process is equivalent to its present value, given all available information up to present time.

Following the definition given in the section 2.2 above, Brownian motion falls under the Markov property and is also a martingale. The conditional distribution of W_T, $T > t$, depends only on the present state of W_t and not on the past value of W_s, $s < t$. Moreover, its expected value given all available information is equal to its present value, such as $\mathbb{E}[W_T \mid \mathcal{F}_t] = W_t$.

In order to rigorously prove that the process $(W_t)_{t \geq 0}$ is a martingale, start recalling the definition of submartingale:

$$\mathbb{E}[S_T \mid \mathcal{F}_t] \geq S_t, \quad \forall T > t$$

And supermartingale:

$$\mathbb{E}[S_T \mid \mathcal{F}_t] \leq S_t, \quad \forall T > t$$

Submartingales and supermartingales are generalized forms of martingale with less restrictive conditions.
A process being both a submartingale and a supermartingale necessarily satisfies the

[57] See Appendix.

martingale property, as already defined above as:

$$\mathbb{E}[S_T \mid \mathcal{F}_t] = S_t, \quad \forall T > t$$

The process $(W_t)_{t\geq 0}$ is a martingale in respect to its filtration $(\mathcal{F}_t)_{t\geq 0}$ if:

$$\mathbb{E}[W_T \mid \mathcal{F}_t] = W_t, \quad \forall T > t$$

From there, proceed algebraically by subtracting and adding W_t:

$$\mathbb{E}[W_T \mid \mathcal{F}_t] = \mathbb{E}[W_t + (W_T - W_t) \mid \mathcal{F}_t] = \mathbb{E}[W_t \mid \mathcal{F}_t] + \mathbb{E}[W_T - W_t \mid \mathcal{F}_t]$$

Since W_t is measurable in $\mathcal{F}_t$, it follows as:

$$\mathbb{E}[W_t \mid \mathcal{F}_t] = W_t$$

And, by the properties of Brownian motion:

$$\mathbb{E}[W_T - W_t \mid \mathcal{F}_t] = \mathbb{E}[W_T - W_t] = 0$$

Finally, as expected:

$$\mathbb{E}[W_T \mid \mathcal{F}_t] = W_t + 0 = W_t, \quad \forall T > t$$

These assumptions are fundamental to allow the correct pricing of financial instruments, as well as, from a purely practical point of view, to make the calculation process far less burdensome.

2.2.2 Itô Formula[58]

Itô’s formula is one of the most significant mathematical results in the field of stochastic processes. The formula, also known as the “change of variable”[59] process, was proven by Itô in 1951.

For the sake of historical accuracy, it should be noted that in 1940 W. Doeblin, a French-German mathematician, had already constructed the stochastic integral and theorised the change of variable, as documented on page 167 of [11].

Assume the price of an asset *S*, defined by a geometric Brownian motion GBM written in its differential form:

[58] In financial applications, it is better known as Itô’s lemma (differential form).
[59] It may be dared to call it the “chain rule” applied to stochastic processes?

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

The goal is to determine the expression of a function $f(t, S_t)$ over time.

The solution to this problem is non-trivial. Intuitively, it appears to be an application of differential calculus (the variation of a value with respect to a variable).
However, in a stochastic context, the variation is random, as is its accumulation process over time.

In this context, following Itô formula[60], given $f(t, S_t)$ is derivable to the second degree with respect to S_t:

$$df(t, S_t) = \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S_t \frac{\partial f}{\partial S} dW_t$$

Where:
1. dWt: infinitesimal increment of Brownian process.
2. $\frac{\partial f}{\partial t} dt$: variation of function in respect to time (partial derivative in respect to time).
3. $\left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2} \right) dt$: drift term.
4. $\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S^2}$: corrective diffusion term.
5. $\sigma S_t \frac{\partial f}{\partial S} dW_t$: linear diffusion term.

This result can be heuristically interpreted as a stochastic version of the Taylor expansion[61], where the key difference is the inclusion of the second-order term due to the property $(dW_t)^2 = dt$.

Unlike a classical Taylor series, this expansion incorporates the stochastic nature of S_t, leading to a correction in the deterministic drift term.
The key concept behind the Itô formula is the Itô stochastic integral, which extends the concept of integration to stochastic processes.
In contrast to a standard Riemann integral, where the integrand is evaluated at arbitrary points within the partition intervals, the Itô integral evaluates the integrand at the beginning of each subinterval.
More formally, given a stochastic process X_t adapted to a filtration $\mathcal{F}_t$, the Itô integral of X_t with respect to a Brownian motion W_t over $[0,T]$ is defined as:

[60] (Kiyoshi Itô, 1951).
[61] The fundamental theorem of *stochastic* calculus.

$$\int_0^T X_t dW_t = \lim_{n\to\infty} \sum_{i=0}^{n-1} X_{t_i}(W_{t_{i+1}} - W_{t_i})$$

This integral is of crucial importance in the field of stochastic calculus, as it facilitates the definition of stochastic differential equations (SDEs), which model the evolution of random processes over time. The key property that distinguishes it from Riemann or Lebesgue Integrals is its ability to satisfy Itô isometry.
This leads to Itô's lemmas, which describes how a function of a stochastic process evolves over time.

2.2.3 Geometric Brownian Motion (GBM)

From section 1.1.3 it's already known that an arithmetic Brownian motion could generate negative prices, hence it is not accurate for modeling financial assets such as stocks[62].

A geometric Brownian motion (GMB) is a stochastic process in which the logarithm of the random variable (e.g., price of a financial asset) follows a Brownian motion. Given the following SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where:

1. S_t: underlying asset price in time t.
2. μ: drift term, representing the deterministic component of the expected rate of return.
3. σ: volatility (diffusion coefficient).
4. dW_t: infinitesimal increment of Wiener process.

The stochastic differential equation (SDE) governing S_t can be rewritten in its integral form as[63]:

$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dW_s$$

The analytical solution for a geometric Brownian motion can be exemplified with the following process:

1. Let a geometric Brownian motion expressed in its differential (SDE) form:

[62] See however Ch 2.2.5, "Modelling for negative prices".
[63] See [11]. Ch. 5.1.4, page 176.

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

2. Posing, for notation purposes, $a = \mu S_t$ and $b = \sigma S_t$, the above equation can be rewritten as:

$$dS_t = a(S_t, t)dt + b(S_t, t)dW_t$$

3. Let $f(S) = ln(S)$. This allows for a significant simplification of the calculations in the following steps. Proceed applying Itô's lemma, thus obtaining:

$$df = d\ln(S_t) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}a + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}b^2\right)dt + b\frac{\partial f}{\partial S}dW_t$$

4. Now, proceed substituting the derivatives and again the functions a and b from 2. Those are standard calculations from calculus, leading to:

$$\frac{\partial f}{\partial t} = \frac{\partial \ln(S_t)}{\partial t} = 0, \ \frac{\partial f}{\partial S} = \frac{\partial \ln(S)}{\partial S} = \frac{1}{S}, \ \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}$$

5. At this point, the equation can be written as:

$$\left(0 + \frac{1}{S}S\mu + \frac{1}{2}\left(-\frac{1}{S^2}\right)\sigma^2 S^2\right)dt + \sigma S\frac{1}{S}dW_t = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

6. Such as:

$$d\ln(S_t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

7. Integrating it in [0, *t*], it leads to:

$$\ln(S_t) - \ln(S_0) = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right)du + \sigma\int_0^t dW_u$$

8. Notably, the integral:

$$\int_0^t dW_u$$

is an integral in the sense of Itô's, equal to W_t - W_0, since it is an integration of the constant with respect to the Brownian motion.

9. Hence:

$$\ln(S_t) = \ln(S_0) + \mu t - \frac{\sigma^2}{2}t + \sigma W_t$$

10. Remembering Brownian motion properties, finally:

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$$

Under GBM:
1. Asset prices follow a log-normal distribution (instead of normal).
2. Asset prices are affected by a deterministic component (drift).
3. The stochastic process is proportional to present asset value S_t.
4. Markov property and martingale process assumptions are valid.

Geometric Brownian motion is a particular solution of a SDE with drift proportional to the value of the process and a stochastic component proportional to the value itself.

2.2.4 Practical applications of GBM
Geometric Brownian motion is a foundational concept in financial mathematics applications. Although a comprehensive review of its use cases is beyond the scope of this paper, at least the following can be enumerated:
1. Modelling the price of a financial asset (see the examples theorized in this section).
2. Pricing of options with the Black & Scholes model (see Chapter III).
3. Monte Carlo Simulations (a model based on the possible outcome of a roulette game that simulates stochastic trajectories faithful to a GBM).
4. Risk management models for calculating value-at-risk (VaR) [21].
5. Specific economics applications (such as wealth accumulation over time) (Goodman J., 2018).

2.2.5 Modelling for negative prices
Although GBM is a realistic model for pricing assets whose minimum price is zero, such as stocks, it is important to note that in light of increased market volatility, there is a growing demand for models that are able to manage negative prices, posing a challenge to stochastic price modeling.

In fact, negative prices are commonplace in specific contexts, such as interest rates and energy derivatives. Negative prices for spot electricity prices are now considered a common occurrence[64].

Consequently, researchers and analysts are revisiting the development of modelling techniques that can manage phenomena that were previously considered rare but are now occurring with greater frequency (see e.g. Schneider, 2011, Brijs, 2015).

2.3 Implementation and plot of a Brownian motion

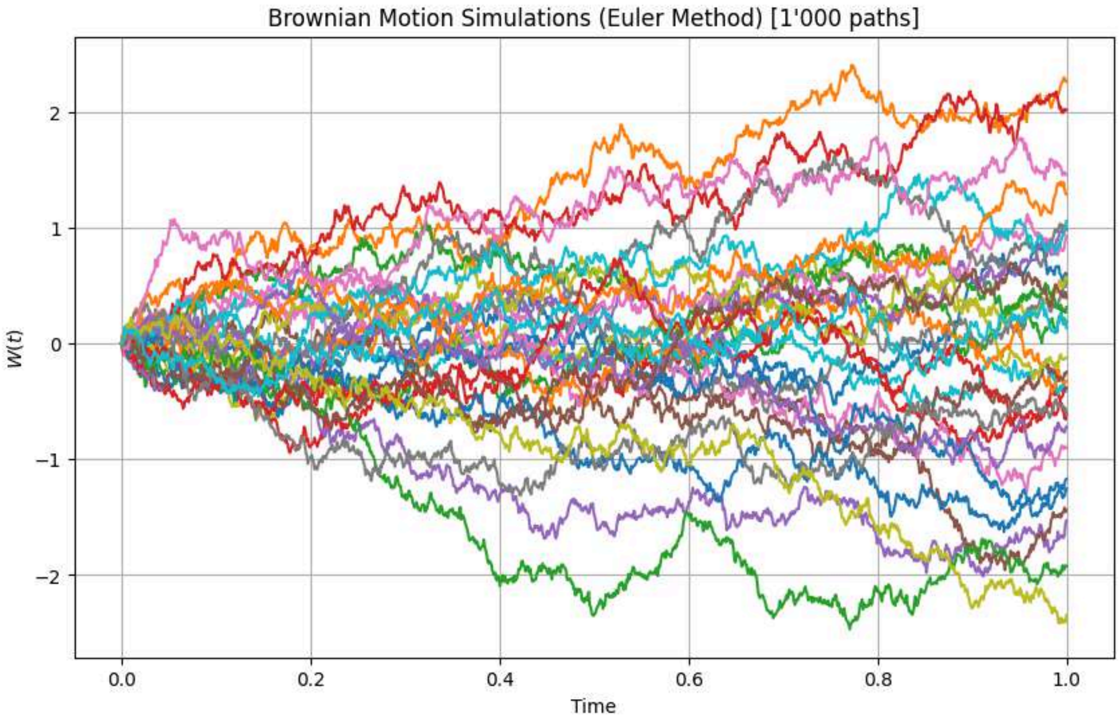
In the following section, Python language is used to implement a Brownian motion process. The Euler-Maruyama method is applied.
The final values of W_T are extracted and their histogram is plotted.
Theoretical normal distribution $N(0, T)$ is overlaid for comparison.

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
num_paths = 1000                  # Number of Brownian motion paths
num_steps = 1000                  # Number of time steps
T = 1.0                           # Time horizon
dt = T / num_steps                # Time step size
t = np.linspace(0, T, num_steps)  # Time intervals

# Euler-Maruyama method for Brownian motion
np.random.seed(55)  # Random seed for reproducibility reasons
dW = np.sqrt(dt) * np.random.randn(num_paths, num_steps)  # Increments
W = np.cumsum(dW, axis=1)  # Cumulative sum to build W(t)

# Add initial value W(0) = 0
W = np.hstack((np.zeros((num_paths, 1)), W))

# Plot a sample of Brownian motion paths
plt.figure(figsize=(10, 6))
for i in range(min(30, num_paths)):  # Show min number of paths
    plt.plot(t, W[i, :-1])
plt.title("Brownian Motion Simulations (Euler Method)")
plt.xlabel("Time")
plt.ylabel("$W(t)$")
plt.grid(True)
plt.show()

# Normal distribution of Brownian motion at t = T
final_values = W[:, -1]
plt.figure(figsize=(8, 6))
```

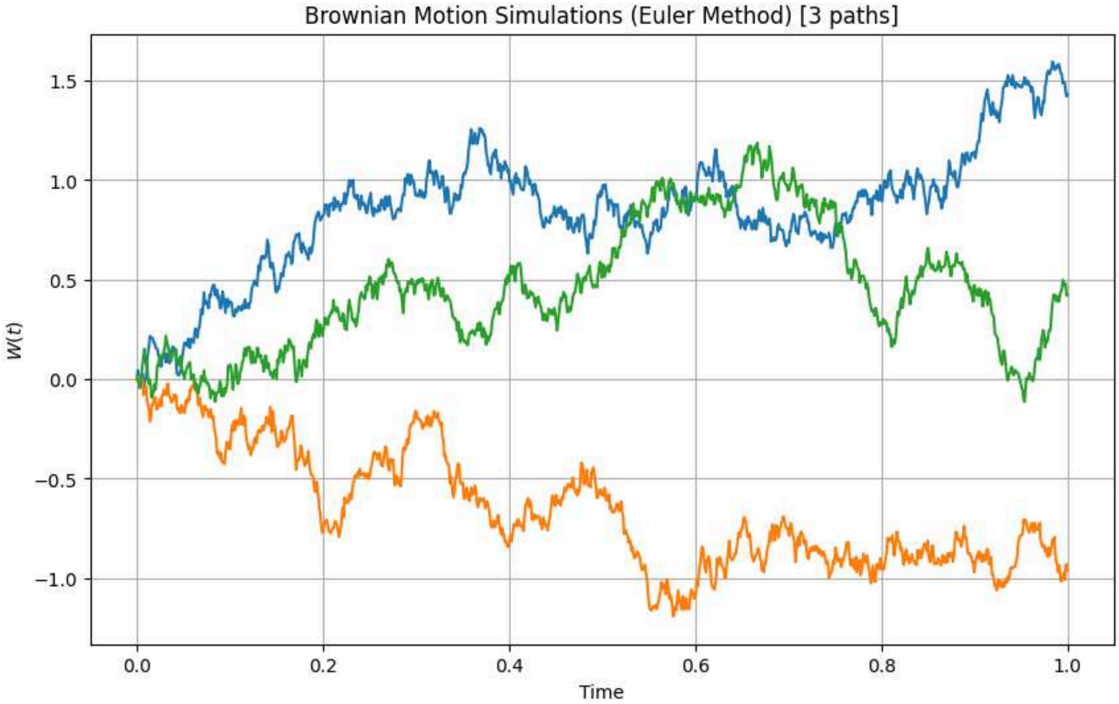
[64] According to data provided by the World Economic Forum, 2024 was a record year for negative energy prices in European markets; at the time of publication of the report, there were recorded more than 7,800 hours with negative prices (Eurozone aggregate).

```
plt.hist(final_values, bins=30, density=True, alpha=0.7, label="Simulated
distribution")
mean, std_dev = 0, np.sqrt(T)
x = np.linspace(-4 * std_dev, 4 * std_dev, 1000)
plt.plot(x, (1 / (np.sqrt(2 * np.pi) * std_dev)) * np.exp(-x**2 / (2 *
std_dev**2)),
         label="Theoretical normal distribution")
plt.title("Distribution of Final Values of Brownian Motion ($t = T$)")
plt.xlabel("$W(T)$")
plt.ylabel("Probability Density")
plt.legend()
plt.grid(True)
plt.show()
```

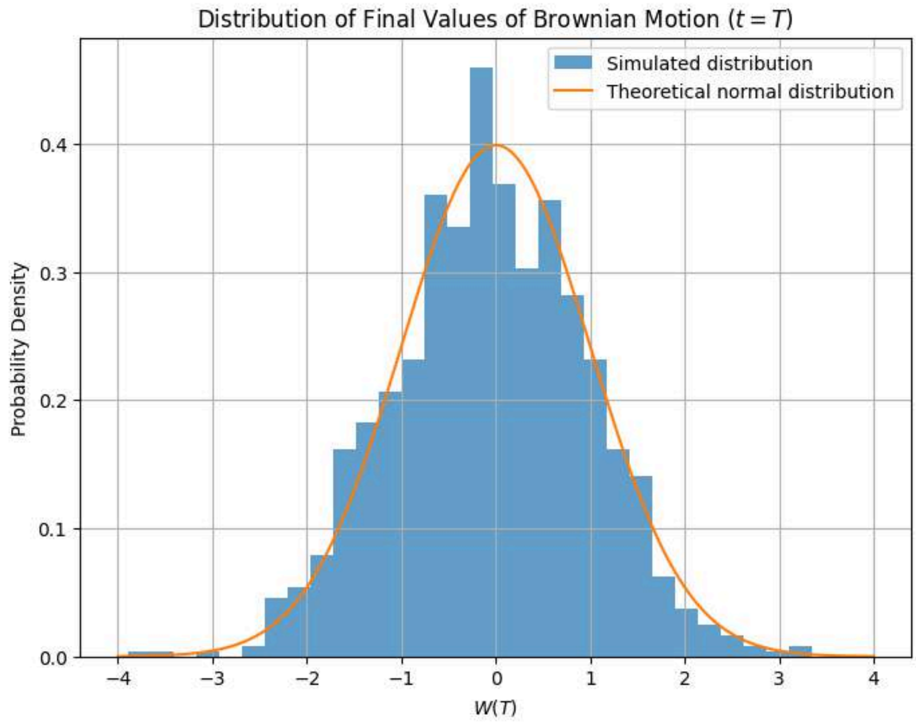
[Image 2.2]



[Image 2.3]



[Image 2.4]



III. Black & Scholes Equation

The Black & Scholes equation is fundamental in the pricing of financial derivatives, specifically pricing of European options. The equation is derived using parabolic partial differential equations (PDEs) from the Black & Scholes model.
The Black & Scholes model has had a significant impact on the finance sector, and it is estimated to be within the ten most cited papers in economics and finance research to date (Merigo et al., 2016).

The model was developed between 1968 and 1973, while the well-known paper [7] was submitted by Fischer Black and Myron Scholes.

As outlined in Chapter I, the connection between physics and finance strengthened during the second half of 1900.
The historical context in which the Black & Scholes model was developed owes greatly to the work of their predecessors. The paper that Robert Merton published in the same year, Theory of Rational Option Pricing, is worth a citation.

A significant rise in derivative trading (specifically options) contributed to the success and dissemination of pricing models, and, at the same time, the availability of a robust mathematical model led to a surge in the use of options hedge strategies (particularly continuous delta hedging).

The Black & Scholes equation is one of the most significant contributions to the formalization of this association, and it is based on a few key insights:[65]

1. The dynamics of particles in a fluid can be likened to the random motion of share prices (Brownian Motion).
2. From the diffusion of heat in a body over time, to the relationship between an option price and volatility (Heat Equation).
3. From a deterministic approach, to a rigorous method for handling continuous stochastic processes (Itô's Formula).

3.1 Black & Scholes equation, put and call formulas

Black & Scholes formula (BS) is a second order partial differential equation that describes the evolution of the value of an (European) option over time.

BS is considered a closed formula (closed-form solution equation) since it provides a

[65] *"Standing on the shoulders of giants."*

semi-explicit and analytical solution[66].

The Black & Scholes equation can be expressed as:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} = rP$$

Where:

6. P: option value.
7. S: price of underlying asset.
8. t: time.
9. r: risk-free rate.
10. σ: volatility.

3.1.1 Derivation of the Black & Scholes equation

There are a number of approaches to derive the Black & Scholes equation (such as by a hedging argument, by a replicating portfolio, using CAPM, deriving it analytically from heat equation, etc.).
In the context of finance, one of the most widely used concepts, and one that will also be covered in the following formulation, concerns the construction of a replicating portfolio within the full market assumption.

1. A replicating portfolio is defined as a combination of financial assets that perfectly replicates the payoff of an option in every possible scenario.
2. The full market assumption refers to the ability to replicate any derivative instrument using plain vanilla instruments, such as stocks and bonds.
3. The change in the replicating portfolio must be equal to the risk-free rate return.
4. The constraints outlined in section 3.1.3 must hold.

Assume an asset S, with a price that follows a geometric Brownian motion (GBM), described by the following SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where:

1. S_t: price of the asset at time t.
2. μ: expected instantaneous rate of return of the asset (deterministic drift).
3. σ: volatility
4. dW_t: infinitesimal increment of Wiener process (Brownian motion).

Then assume a financial plain-vanilla derivative having S as underlying asset (e.g., a

[66] Notably, explicit Black & Scholes problems can be solved with a handheld scientific calculator.

European option), having *P* price at t time defined as:

$$P(S(t), t)$$

And, with maturity time as *T*, a price at maturity as:

$$P(S(T), T)$$

The payoff for the (call) option is hence:

$$P(S(T), T) = max(0, S(T) - K), \text{ with K = strike price}^{67}$$

From Itô's lemma[68], the process followed by the price of the option *P*, *dP*, can be expressed in the form[69]:

$$dP = \left(\frac{\partial P}{\partial t} + \mu S\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}\right)dt + \frac{\partial P}{\partial S}\sigma S dW$$

Proceed by implementing the replicating portfolio Π.
Π is a risk-free portfolio, based on the premise that both the option and the underlying asset are subject to the same source of uncertainty.
By taking an opposite position on these two financial instruments (long position in option *P* and short position of a Δ quantity of asset *S*), investors effectively offset any potential negative changes in value from one asset with the positive changes from the other.
Π at time *t*, (Π(*t*)), is defined as:

$$\Pi(t) = P(S(t), t) - \Delta S(t)$$

Where $\Delta = \partial P/\partial S$ is the quantity of *S* in the portfolio at time *t*, and with:

$$d\Pi(t) = dP - \Delta dS$$

Hence, substituting *dP* and *dS* and grouping the terms:

$$d\Pi(t) = \left(\frac{\partial P}{\partial t} + \mu S\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}\right)dt + \left(\frac{\partial P}{\partial S}\sigma S - \Delta\sigma S\right)dW_t$$

---

[67] An OTM option will not be exercised.
[68] See 2.2.2.
[69] Posing S(t) ≡ S for readability's sake.

Then, in order to remove the stochastic component dW_t, it must be assumed $\Delta = \partial P/\partial S$. Surprisingly:

$$d\Pi(t) = \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2}\right) dt$$

This assumption is central to the Black & Scholes model and is based on the absence of arbitrage. It effectively removes the stochastic component, otherwise known as the risk term W_t. The result is a deterministic portfolio.
Since it is known that Π is a risk-free portfolio, its return $\Pi(t)$ must be equal to the risk-free expected return:

$$d\Pi(t) = r\Pi(t)dt$$

There, since:

$$\Pi(t) = P(S(t), t) - \Delta S(t)$$

and:

$$\Delta = \partial P/\partial S$$

It is consequently possible to rewrite the equation as:

$$\left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2}\right) dt = r(P - S\frac{\partial P}{\partial S})dt$$

Rearranging and dropping the derivation terms dt, finally:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} = rP$$

That is, the Black & Scholes equation.

3.1.2 Solution of the Black & Scholes equation

In order to get the closed Black & Scholes formula, namely the explicit solution for the price of a European derivative, it is necessary to solve the Black & Scholes partial differential equation (PDE).

The intuition for solving the Black & Scholes PDE is that it can be traced back to a diffusion equation[70] (remember from section 1.1.3) by applying a change of variables.

[70] Cauchy–Euler equation.

Posing[71]:

$$\tau = T - t$$

and[72]:

$$x = ln\left(\frac{S}{K}\right)$$

the Black & Scholes equation can be expressed as:

$$\frac{\partial f}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial f}{\partial x} - rf$$

The above is a form of the heat equation (inverse heat equation) that leads to the application of a form of the Cauchy problem for heat equations [19].

Applying a cumulative normal distribution function in the form:

$$f(x,\tau) = AN(d_1) - BN(d_2)$$

Where:

1. $N(\cdot)$: cumulative normal distribution function.
2. d_1, d_2: variables derived from model parameters.

Solving the heat equation[73], $N(\cdot)$ offers the cumulative probabilities for d_1, d_2 terms. Then, reversing the transformation ($x = ln(S/K)$ and $\tau = T - t$), the resulting formula depends again from the original equation parameters (S, t, T, K, r, σ).

Finally, the above leads to:

Price of a European Call Option:

$$C = S_0N(d_1) - Ke^{-rT}N(d_2)$$

Price of a European Put Option:

[71] Time to maturity.
[72] Log-moneyness.
[73] Details omitted for the scope of the document.

$$P = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

Where:

1. C: call option price.
2. P: put option price
3. S_0: current price of the underlying asset.
4. $S_0N(d_1)$: delta weighted discounted expected value of gains from the exercise of the option.
5. $Ke^{-rT}N(d_2)$: discounted value of the exercise price weighted by the probability that the option will be exercised (in the money).
6. K: strike price (exercise price).
7. r: risk-free rate.
8. T: time to maturity.
9. $N(\cdot)$: distribution function of the standard normal distribution
10. d_1 and d_2 are defined as:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Black & Scholes in practice: option pricing exercise[74]:

Evaluate by employing the Black & Scholes model the fair price of a call option on the stock Premium Software Houses with the following characteristics:

1. Market price of the stock: 100 USD.
2. Exercise price: 95 USD.
3. Risk-free interest rate: 10%.
4. Remaining life of the option: 3 months (1/4 year).
5. Standard deviation of the returns of the underlying asset: 50%
6. Dividend: no dividend.

Solution:

Pricing the call option based on the Black and Scholes model first requires identifying the values of d1 and d2.

$$d_1 = \frac{ln(100/95) + (0.10 + 0.5^2/2) \cdot 0.25}{0.5 \cdot \sqrt{0.25}} = 0.43$$

$$d_2 = 0.43 - 0.5 \cdot \sqrt{0.25} = 0.18$$

[74] Adapted from [20].

Through the use of statistical tables or the appropriate Excel function[75], it is necessary to find the values of $N(d_1)$ and $N(d_2)$. This function is commonly referred to as Φ(z).

From the tables, the following values can be obtained:
N(0,43) = 0,6664
N(0,18) = 0,5714

Therefore, the value of the call option is equal to:

$$C = 100 \cdot 0.6664 - 95 \cdot e^{-0.10 \cdot 0.25} \cdot 0.5714 = 66.64 - 52.94 = 13.70$$

3.1.3 Assumptions to the Black & Scholes equation

The Black & Scholes equation is verified if the following assumptions hold [20]:

1. The derivative subject to pricing can be only exercised at maturity and not before (European Option).
2. No transaction costs, market fees, or taxes.
3. Zero probability of default for market parties (no credit risk).
4. Strong form Efficient Market Hypothesis (see Ch. I).
5. Markovian markets (the market satisfies the Markovian hypothesis, where the future price of the asset depends only on the current value and not on past history of assets).
6. The underlying asset follows a geometric Brownian motion (see Ch. II).
7. Variance, drift, interest rate and risk-free rate are constant.
8. No dividends.
9. No risk-free arbitrage opportunities.
10. No minimum trading lot. The underlying asset is arbitrarily divisible, allowing infinitesimal fractions of the security to be traded.
11. No restriction to short selling.
12. Perfect liquidity. Any amount of the asset can be bought and sold without impacting the market price.
13. Every asset or underlying is tradable.

However, the validity of these assumptions does not seem trivial. In fact, the verifiability of these assumptions within their respective contexts appears to be a highly theoretical proposition.

Consequently, the Black & Scholes model has been subject to substantial criticism (see [18] for a critique essay and a collection of key academic sources on the same topic).

[75] NORM.S.DIST(z, cumulative)

It should be noted that this is not intended as a derogatory statement; rather, it highlights the existence of more sophisticated models that are built on Black & Scholes premises.
A development of the BS model, of particular interest within the scope of applicability of physics models to finance, concerns the validity of the Black & Scholes model for non-tradable assets.
The need to price derivatives written on physical underlyings having no market, such as weather data (strictly dependent from physical laws), propelled new research areas:

1. Statistical patterns (such as historical seasonality data).
2. Proxies (such as retail gas prices as a proxy for weather forecasts).
3. Stochastic simulations (such as the Monte Carlo approach).

For a specific in-depth study, see e.g. Stewart R. T., Derivative instruments written on non-tradable assets: The case of weather derivatives, (2002) and Ankirchner et al., Pricing and hedging of derivatives based on non-tradable underlyings, (2018).

Finally, it is not uncommon in the business world to observe the application of Black & Scholes even by analysts who have the necessary mathematical and professional capabilities to manage more sophisticated models: the preference for BS is often rooted in its explainability, making it a popular choice among professionals[76].

A solid understanding of Black & Scholes model is crucial for grasping the intricacies of financial markets and derivative instruments.

3.1.4 The concept of risk-free rate in financial evaluations

In the not always easy task of specifying the parameters needed to calculate pricing models such as Black & Scholes, the determination of a risk-free rate seems to play a marginal role among analysts. Perhaps mistakenly believing such a task is redundant or obvious.
However, as is often the case, the reality is much more complex (see e.g. Damodaran, What is the Risk-free Rate? A Search for the Basic Building Block, 2008).

Intuitively, the risk-free rate of return represents a “safe and predictable rate”:

1. The expected return is a known parameter for $t = 0$, and
2. The variance of the return (or volatility) is also zero (for fixed maturities, since the return is deterministic).

However, the risk-free rate should be reinvested at the same, known, rate. Further than that, it should have a defined time horizon (e.g. spot or forward rates).
What at first seemed to be a simple, obvious task, actually lends itself to non-trivial considerations regarding term structure and reinvestment strategies.

[76] It’s noteworthy [26], page 1088.

The literature tends to follow canonical conventions[77]:

1. For the American market the choice is the zero-coupon Treasury bond (T-bond or T-bills). See e.g. [20], Ch. 1.4: *"T-bill returns are effectively risk free"*.
2. For the Italian market, the typical risk-free rate is the Treasury Bond. This approach is confirmed, for example, by the documentation of the Italian Stock Exchange (Borsa Italiana S.p.A.)[78] and the guidelines of the Italian Institute of Chartered Accountants[79] (Ordine dei Dottori Commercialisti e degli Esperti Contabili)[80].
3. Evaluating derivatives, such as options, it is usually preferred to proxy risk-free rates with the rates offered on the interbank lending market, e.g. LIBOR/SOFR[81] or EURIBOR. Such rates are published with different maturities[82], from overnight to monthly, quarterly or yearly.

Although the use of the above-mentioned "risk-free" rates appears to be unproblematic, the subject is constantly being debated (also due to the fact that long interbank rates tend to be illiquid and their publication cannot be considered indefinitely[83]).
Some of the "risk-free" rates proposed for the pricing of derivatives are ESTER (Euro Short-Term Rate) (also known with its ticker, "€STR") and SONIA (Sterling Overnight Index Average).

For a thoughtful discussion of interbank rates as risk-free rates, a good starting point is Schrimpf, A., Sushko, V., Beyond LIBOR: a primer on the new benchmark rates, (2019)[84].

3.2 Black & Scholes Greeks

Black & Scholes Greeks (Delta, Gamma, Vega, Theta, Rho) are partial derivatives of the Black-Scholes formula with respect to its key parameters.

BS Greeks provide information on the sensitivity of the option value to changes in the underlying variables.

The Greeks are fundamental tools in risk management and trading optimization models [21].

---

[77] Not always providing further evidence for the choice of such "risk-free" rates.
[78] https://www.borsaitaliana.it/borsa/glossario/tasso-risk-free.html
[79] The institute has no English naming convention; translated by the author.
See https://commercialisti.it/about-cndcec/.
[80] https://www.odcecpadova.it/wp-content/uploads/2019/01/Fonti-dei-Parametri-Finanziari-e-di-Mercato-utilizzabili-nelle-Valutazioni_02.pdf
[81] LIBOR was discontinued in 2022, and the U.S. Congress passed the LIBOR Act, to establish SOFR as a default replacement rate for LIBOR.
[82] https://www.adcb.com/en/personal/general/libor
[83] See i.e. the so-called "LIBOR Scandal".
[84] The paper is available online at the following URL (in its Italian translation) (retrieved 14 Jan 2025) https://www.bis.org/publ/qtrpdf/r_qt1903e_it.pdf.

Delta (Δ)
The change in the option price with respect to an infinitesimal change in the price of the underlying asset (the derivative of the option price in respect to the underlying price).

$$\Delta = \frac{\partial P}{\partial S}$$

$$\Delta_{\text{Call}} = N(d_1), \ \Delta_{\text{Put}} = N(d_1) - 1$$

The concept of delta is closely related to the core assumption of the replicating portfolio in the Black & Scholes model. Recalling from section 3.1.1, the replicating portfolio is defined as:

$$\Pi(t) = P(S(t), t) - \Delta S(t)$$

Where:

1. $\Pi(t)$: value of the replicating portfolio at time t.
2. $P(S(t),t)$: option price at time t.
3. $\Delta = \partial P/\partial S$: the quantity of S in the portfolio at time t, adjusted in order to replicate the option.

In the replicating portfolio, an option can be replicated perfectly by a continuous delta hedging process. The concept of delta-hedging involves continuously rebalancing the portfolio of the option and stock to ensure a total delta of zero after rebalancing[85].
When S_t changes by an infinitesimal value dS, the value of the option changes by a dP factor, defined as:

$$dP = \Delta dS + \varepsilon, \text{ with } \varepsilon = \text{terms related to time and volatility}$$

Heuristically, the delta indicates the amount of exposure to the underlying asset required to replicate the option payoff.

1. Δ = 1: the portfolio must hold an amount of the underlying asset equal to the value of the option (typical of a deeply “in the money” (ITM) option).
2. Δ = 0: the option has no value and no underlying asset is required (typical of an “out of the money” (OTM) option).
3. When the delta is a fractional value, the portfolio holds a fraction of the underlying to balance the stochastic component of the option’s value.

Delta hedge: an example

[85] During this *infinitely small* time interval, the risk of the option is equal to the risk-free rate.

Assume an empirical case of delta hedging on the following asset:

1. Price of the underlying asset at time 0 (S_0): 90
2. Strike price (K): 105
3. Risk-free rate (r): 0.05
4. Volatility (σ): 0,2
5. Time to maturity (T): 200 days (0.548 years).

Delta (Δ) can be calculated following the formula exposed in section 3.1.2:

$$\Delta N(d_1) = N\left(\frac{\ln(90/105) + \left(0.05 + \frac{0.2^2}{2}\right)\cdot 0.548}{0.2\sqrt{0.548}}\right) = N(-0.78208) = 0.218$$

With Δ = 0.218, the trader must purchase 21,800 stocks (since each unit of the underlying has a delta of 1). This ensures that the change in the value of the option is approximately offset by the change in the value of the position on the underlying asset.
Dynamism in delta hedging, otherwise known as continuous delta hedging, occurs precisely by adjusting the calculation for different values of t: t_1, t_2, t_3, $t_{\dots}$, t_n.
While theoretically this is a perfectly replicating process, in practice, as is the case with many other financial processes (e.g. bond duration hedges), a continuous process would not be viable due to transaction costs and other constraints, such as execution times or market liquidity. Consequently, periodic or discrete hedging can result in replication error and risk factors.

Gamma (Γ)
The change in Δ from the price of the underlying asset (the second derivative of the option price to the underlying price).

$$\Gamma = \frac{\partial \Delta}{\partial S_t} = \frac{\partial^2 P}{\partial S_t^2} = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

Theta (Θ)
The change in option value with respect to the passage of time (time decay).

$$\Theta_{\text{Call}} = \frac{\partial P}{\partial t} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

Vega ($\mathcal{V}$)[86]
The sensitivity of option value to volatility.

[86] Not a letter of the Greek alphabet.

$$\mathcal{V} = \frac{\partial P}{\partial \sigma} = S_0 N'(d_1)\sqrt{T}$$

Rho (ρ)
The sensitivity of the option value to the interest rate.

$$\rho_{Call} = \frac{\partial P}{\partial r} = KTe^{-rT}N(d_2)$$

3.2.1 Other relevant Black & Scholes Greeks

While Delta, Gamma, Vega, Theta, Rho are the most used, and common, Greeks related to the model sensitivity, there are less-known Greeks used for models fine-tuning or to hedge specific risks in option trading.
For the purposes of this paper, only a brief mention of the most significant additional Greeks in the Black & Scholes model is made.

Lambda (Λ)
Elasticity of option value in respect to underlying price.

$$\Lambda = \Delta \frac{S}{P}$$

Epsilon (ε)
Sensitivity of option value in respect to dividend return (used in Black & Scholes with dividend, see [20]).

$$\epsilon = -S_0 T e^{-\delta T} N(d_1)$$

Charm
Evolution of Δ in respect to time.

$$\text{Charm} = \frac{\partial \Delta}{\partial t}$$

Vanna
Evolution of Δ in respect to σ.

$$\text{Vanna} = \frac{\partial \Delta}{\partial \sigma}$$

3.3 Implementation and plot of implied volatility surface for a Call Option

With regard to the implementation of the code related to Black & Scholes, the pricing algorithm implementation would not be covered in detail, since this is a solvable problem in a few lines of code using common libraries[87].

Instead, it is shown the procedure for graphically representing the implied volatility of month-ahead call options on Microsoft stocks, using real market data (computing date: 30/12/2024).

Market data were obtained through the Yahoo Finance API, and for the sake of brevity, strikes and vol values were also drawn from the finance library.
This procedure simulates the actual work of a quant during a typical business day.

The implied volatility σ_{imp} is the volatility value σ that equals the Black-Scholes value of the option to the option's trading price.

It should be noted that in the Black & Scholes model, the volatility parameter σ is the only parameter that cannot be directly observed. All other parameters can be determined through market data.

Implied volatility is a core concept in evaluation of financial options. It allows to:

1. Compare options with different expirations and/or strike prices, by having a common reference parameter regardless.
2. Easily adopt volatility-based trading strategies (such as straddles[88] or strangles[89]).
3. Have an option value that is directly linked to the riskiness of the derivative, without being influenced by other exogenous variables such as interest rates or exchange rates.
4. Monitor the market's expected volatility surface to detect anomalies in market expectations.

Plain vanilla options are listed in terms of implied volatility instead of their prices [11].

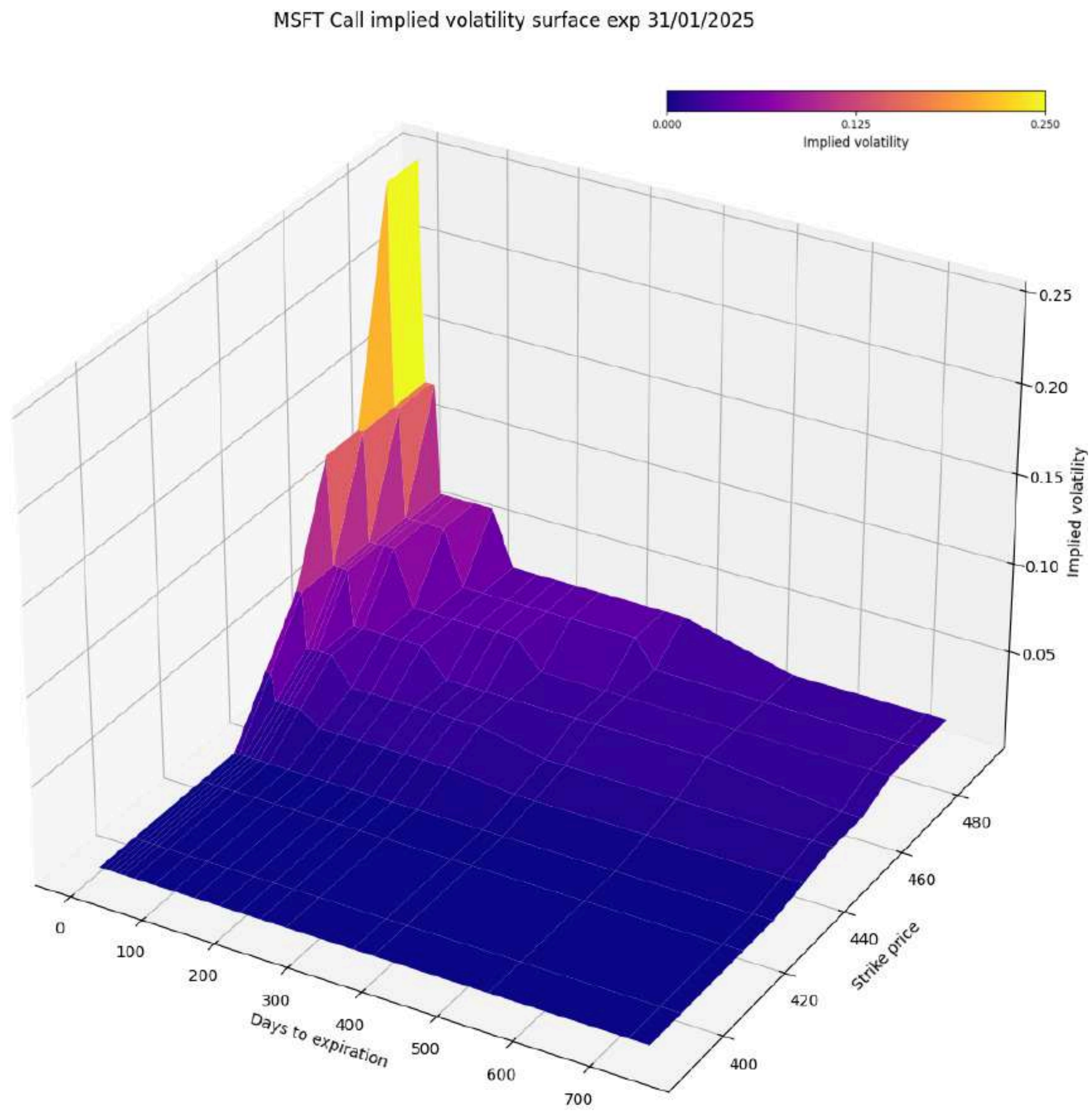
```
import numpy as np
import pandas as pd
```

[87] However, a sample is provided in the Jupyter Notebook.
[88] A strategy involving a call and a put option with same strike and maturity, "betting" on the volatility of the underlying but having no bearish or bullish expectations.
[89] A strategy involving a call and a put option, both out of the money, but with the same maturity, in order to benefit from bearish or bullish expectations on the volatility of the underlying asset, with more risks than a straddle, but with less upfront costs (since options are OTM).

```
import yfinance as yf
import datetime as dt
import matplotlib.pyplot as plt

def option_chains(ticker):
    asset = yf.Ticker(ticker)
    expirations = asset.options

    chains = pd.DataFrame()

    for expiration in expirations:
        # tuple of two dataframes
        opt = asset.option_chain(expiration)

        calls = opt.calls
        calls['optionType'] = "call"

        puts = opt.puts
        puts['optionType'] = "put"

        chain = pd.concat([calls, puts])
        chain['expiration'] = pd.to_datetime(expiration) + pd.DateOffset(hours=23,
minutes=59, seconds=59)

        chains = pd.concat([chains, chain])

    chains["daysToExpiration"] = (chains.expiration - dt.datetime.today()).dt.days
+ 1

    return chains

# yfinance provides an estimate of implied volatility. We pick Microsoft
options = option_chains("MSFT")

calls = options[options["optionType"] == "call"]

# print the expirations
set(calls.expiration)

# select an expiration to plot
calls_at_expiry = calls[calls["expiration"] == "2025-01-31 23:59:59"]

# filter out low vols
filtered_calls_at_expiry = calls_at_expiry[calls_at_expiry.impliedVolatility >=
0.001]

# set the strike as the index for graphical purposes
filtered_calls_at_expiry[["strike", "impliedVolatility"]].set_index("strike").plot(
    title="Implied Volatility Skew", figsize=(7, 4)
)

# select an expiration to plot
calls_at_strike = options[options["strike"] == 490.0]
```

```
# filter out low vols
filtered_calls_at_strike = calls_at_strike[calls_at_strike.impliedVolatility >=
0.001]

# set the strike as the index for graphical purposes
filtered_calls_at_strike[["expiration",
"impliedVolatility"]].set_index("expiration").plot(
    title="Implied Volatility Term Structure", figsize=(7, 4)
)

# pivot the dataframe
surface = (
    calls[['daysToExpiration', 'strike', 'impliedVolatility']]
    .pivot_table(values='impliedVolatility', index='strike',
columns='daysToExpiration')
    .dropna()
)

# create the figure object
fig = plt.figure(figsize=(13, 13))

# add the subplot with projection argument
ax = fig.add_subplot(111, projection='3d')

# get the 1d values from the pivoted dataframe
x, y, z = surface.columns.values, surface.index.values, surface.values

# return coordinate matrices from coordinate vectors
X, Y = np.meshgrid(x, y)

# set labels
ax.set_xlabel('Days to expiration')
ax.set_ylabel('Strike price')
ax.set_zlabel('Implied volatility')
ax.set_title('MSFT Call implied volatility surface exp 31/01/2025')

# plot
ax.plot_surface(X, Y, z, cmap='plasma', edgecolor='none')
```

[Image 3.1]



Implied volatility is a *de facto* "implicit value" that cannot be directly observed in the market. Instead, it must be calculated using an options pricing model like Black & Scholes. The current price of the option is used as a starting point, with the level of volatility that would justify that price being determined by working backwards.

The chart illustrates how implied volatility is low (or, lower) far from expiry date and for strikes far from the actual stock value (at the simulation date).

The yellow peak in the chart can be attributed to various factors, including gamma squeeze[90], proximity to the EOFY period, market expectations, and increased trading activity on both the option and the underlying.

[90] Briefly, a sudden major purchase of call options, caused by shock events or unexpected news. This happened for example in the memestock case "Game Stop Corp." {GME}.

IV. Heston Model

The Heston Model is a stochastic model for the dynamics of the price of a financial asset and its volatility, the latter following a stochastic process. It was proposed by Steven Heston[91] in 1993 with the aim of overcoming the constant volatility assumption of the Black & Scholes model.

Indeed, some may assume that stochastic quantitative models were developed to understand the empirical properties of asset pricing. But in fact, the driving force behind the emergence and spread of these models was precisely the constant improvement of option pricing models (see [24], Ch. 1.2, p. 22).

As outlined in [22], Steven Heston, while emphasizing the strengths of the Black & Scholes model, was also keen to point out its limitations (see section 3.1.2).
In particular, citing prominent supporters of his theory, including Rubinstein (1985) and Knoch (1992), he asserted that the normal distribution of returns assumed by Black & Scholes was notably "quite strong"[92].

Building on this premise, significant contributions have been made, primarily by Scott (1987), Hull and White (1987), and Wiggins (1987). However, these models do not offer closed solutions (see section 3.1) and instead they involve computationally extensive numerical solutions.

In light of this, Heston proposes a stochastic volatility model that allows the pricing of options with stochastic volatility and arbitrary correlation between volatility and spot asset returns.
This model ("The Heston Model") allows a closed solution for pricing a European call option, with the addition of stochastic interest rates. This makes the model adaptable to bond and currency options.

The popularity[93] of Heston model relies on:

1. Fast and closed solution method.
2. Explicit formula for the function of the asset log-price.
3. A good proxy shape and dynamic for the implied volatility surface.

---

[91] At the time of this writing, Steve Heston is reported to be employed as a professor at the University of Maryland.
[92] To be read as "very optimistic".
[93] The "Heston Model" as a topic has more than 2,500 publications on average over the last 10 years, with 1.9 millions of citations (as per 01 Feb 2025, data from https://app.dimensions.ai/).

4.1 Heston model definition

The Heston model can be defined by a system of stochastic differential equations (SDE).

Equation for the price of the asset:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S$$

Equation for the volatility (Cox-Ingersoll Ross model):

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v$$

Where:

1. S_t: price of financial asset S at time t.
2. v_t: instant volatility at time t.
3. μ: expected instantaneous rate of return of the asset (deterministic drift).
4. k: mean reversion of volatility; the rate at which v_t reverts to the mean variance.
5. θ: long run variance.
6. σ: volatility of volatility parameter.
7. W_t^S, W_t^v: standard Brownian motion processes with correlation ρ.

And, therefore:

1. $\mu S_t dt$: is the deterministic value of the expected return of the asset.
2. $\sqrt{v_t} S_t dW_t^S$: is the stochastic component of the price of the asset, with a dynamic volatility v_t.
3. $k - (\theta - v_t)$: is the mean return process (the expected value of v_t tends to θ for an infinite value of v_t).
4. $\sigma\sqrt{v_t} dW_t^v$: diffusion component that handles the stochastic volatility.
5. $dW_t^S dW_t^v = \rho dt$: noted that the standard Brownian motion processes are correlated with a ρ correlation coefficient:
    a. If $\rho < 0$: leverage (when price of the asset decreases, the vol increases).
    b. If $\rho > 0$: positive correlation between price and vol.

Notably, in the Heston model, the price process S_t is not a Markovian process. But the joint process of price and volatility (S_t, v_t) is instead Markovian. The two state variables combined make it suitable for numerical methods solutions such as Monte Carlo simulations.

Logarithm of asset's return

Applying Itô's formula to the equation for the price of the asset:

$$df(S_t) = f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2$$

And, with $f(S_t) = \ln S_t$:

$$\frac{\partial \ln S_t}{\partial S_t} = \frac{1}{S_t}, \quad \frac{\partial^2 \ln S_t}{\partial S_t^2} = -\frac{1}{S_t^2}$$

Then, with a substitution in Itô's formula, the logarithmic return is explicit as:

$$d\ln S_t = \left(\mu - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_t^S$$

Cox-Ingersoll-Ross model

The Cox-Ingersoll-Ross (CIR) model describes the evolution of interest rates, as driven only by a source of market risk. It assumes that the short-term interest rate follows a mean-reverting stochastic process. CIR is an extension of the Vasicek model[94], itself an Ornstein-Uhlenbeck process[95].

CIR does not allow negative interest rates while preserving analytical solutions for bond pricings. This is formally done via the Feller condition[96] :

$$2\kappa\theta > \sigma^2$$

Feller condition ensure that:

$$\sigma\sqrt{v_t} \leq k(\theta - v_t)$$

Should the volatility of volatility[97] be excessively high in comparison to the rate of return, there is a possibility that the process may reach zero, invalidating the model (see the following section 4.1.1).

4.1.1 Volatility skew in the Heston model

The Black & Scholes model assumes that volatility is constant over time and independent of the underlying asset price.

This assumption implies that implied volatility (the volatility inferred from option

[94] A model specifically built for interest rates modeling with stochastic processes.
[95] See Appendix.
[96] Feller, W., (1951).
[97] As in [22], page 336.

market prices using Black & Scholes model[98]) should be the same for all strike prices and maturities[99].

This is a remarkable simplification of the real phenomenon, where it is commonly said that volatility has a shape smile ("volatility smile" or "volatility skew"[100]).

The Heston model is specifically designed to address this problem by assuming stochastic volatility (based on standard Brownian motion). The volatility skew, in turn, is not static, but is influenced by the strike price and the time to maturity.

Empirically, the typical convex shape of volatility suggests the following

1. Investors are risk averse: investors pay a higher premium to protect themselves against extreme price movements.
2. Extreme events (fat tails): the distribution of real returns does not follow a perfect normal distribution, but it has thicker (fat) tails.
3. Leverage effect: volatility tends to increase when asset prices fall.

Although in a different context (bonds vs. options), the volatility skew is evocative of the relationship between duration and convexity in the bond market.

4.1.2 Assumptions to the Heston model

The assumptions underpinning Heston's model primarily concern the non-negativity of volatility and the preservation of the probability measure.

I. Non-negative volatility

Given:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v$$

It must be ensured that $v_t \geq 0$.

Recalling Feller's condition from section 4.1:

$$2\kappa\theta > \sigma^2$$

Typically, the condition is ensured with:

1. Larger $\kappa\theta$ values: high mean reversion rate and a sufficiently high mean variance value.
2. Small σ^2 values: small variance volatility.

---

98 See section 3.3.

99 Simply put, the Black & Scholes expectation is that volatility is constant over time.

100 See section 4.3.

However, Feller's condition is not sufficient to guarantee a variance value (quadratic volatility) other than or greater than zero. Even if Feller's condition is satisfied, the process v_t may be negative due to the approximations introduced by the use of discrete-time methods such as Euler-Maruyama.
Since the Euler-Maruyama method is a finite discretisation of a continuous process, it is possible to introduce discretisation errors where v_t can become negative for specific values[101].

If the non-negative volatility condition is not matched, volatility can assume zero or negative value and the model could extend outside of the probability space. The most common approaches in this case are:

1. Mirroring: the straightforward solution is to reflect the negative values of v_t to ensure non-negativity. However, it is immediately evident that this approach can introduce bias in the model (as it alters the statistical behaviour of the original process). Furthermore, if v_t is sufficiently small, v_t can repetitively be close to zero or negative, leading to the needs of multiple corrections.
2. Truncated Variance: truncate the variance to an arbitrarily small minimum value $\epsilon > 0$, such as $v_t = \max(v_{t,} \epsilon)$. This approach avoids $v_t = 0$ without having to reflect negative values, but it is not entirely unbiased.
3. Exact Simulation (Broadie-Kaya): a more rigorous method is to use exact simulations, which avoid direct SDE discretisation and calculate the exact values of v_t. See Broadie, M., Exact Simulation of Stochastic Volatility and Other Affine Jump Diffusion Processes, (2006) for further references.
4. Operational research methods (smaller discretisation steps, moment matching techniques, Fourier approximation, drift interpolation).

However, there is currently no consensus on the best approach to solving negative or zero variance problems in the Heston model[102]. Each approach has its own advantages and limitations, and the most appropriate choice depends on the specific context.

II. Trajectory loss

The Heston model incurs a trajectory loss if the process extends outside of the probability space.
In order to avoid this loss of trajectory, it is necessary to define the model under a well-specified risk-neutral measure (equivalent martingale measure):

1. The parameters must satisfy non-arbitrage conditions.
    a. The process must be consistent with a risk-neutral measure ($\mathbb{Q}$).
    b. Typically, this necessitates that the discount rate and model parameters are clearly defined.

[101] This is a direct consequence of the numerical approximation.
[102] This is an empirical finding derived from the large number of papers consulted on the subject, each of which advocates different approaches from the previous ones.

2. The distribution of trajectories remains confined within a mathematically valid space.
3. The numerical methods employed must meet the specified boundary conditions.

III. Boundary conditions

To guarantee the existence and uniqueness of the solutions, it is necessary that the parameters satisfy the following constraints:

1. $k > 0$: the rate of regression to the mean must be positive.
2. $\theta > 0$: the long-term average value of volatility must be positive.
3. $\sigma > 0$: the volatility of volatility must be positive.
4. $-1 \leq \varrho \leq 1$: the correlation coefficient must fall within a valid range.
5. Ensured convergence of numerical solutions, especially for pricing methods based on the inverse Fourier transform.

4.2 Risk neutral measure and calibration in a Heston model

4.2.1 Risk neutral measure

The Heston model differs from the Black & Scholes model in that it considers the market to be incomplete due to the presence of two sources of risk (asset price and asset variance) and only one tradable financial asset (asset price).

Variance risk cannot be hedged directly with existing instruments in the market (at least without specific derivative instruments, such as volatility options or variance swaps). This discrepancy makes the market incomplete under the Heston model assumption (there is no unique risk-neutral measure).

In such an incomplete market, a whole family of equivalent martingale measures exists, rather than a single neutral risk measure $\mathbb{Q}$[103].

Different neutral measures may produce different prices for derivatives (each one of them will, more precisely, generate different pricings).

According to a recent approach (see Davison, Escobar-Anel et al., 2022) and with the aim to avoid numerical solution methods, the concept of using derivative instruments which depend directly on volatility (variance swaps or VIX futures) could be introduced. These instruments make the market complete even under initial Heston model assumptions.

4.2.2 Heston model calibration

Calibration is a common mathematical process that aims to improve the estimated results of a model reducing the gap between estimated and actual (observed) data (in the context of financial markets).

[103] See Appendix.

Calibration is frequently mentioned in the context of the Heston model due to:

1. The presence of two stochastic and interrelated sources of risk.
2. The presence of a semi-closed solution (hence, not solvable just by an analytical approach).
3. The correlation and high sensitivity of the parameters of the model.

That can lead to significant variations in the results for even minimal changes in the inputs, undermining the significance of the model itself.

Heston model calibration focus on five main parameters:

1. k: mean reversion of volatility; the rate at which v_t reverts to the mean variance.
2. θ: long run variance.
3. σ: volatility of volatility parameter.
4. v_0: instant volatility at time 0 (initial value of volatility).
5. ρ: correlation between the standard Brownian motion W_t^S, W_t^v.

The calibration process is a classic RMSE (Root Mean Squared Error) optimization problem. The objective is to minimise the squared difference between the calculated model prices (“mod”) and the observed market prices (“mkt”):

$$\min_{\kappa,\theta,\sigma,v_0,\rho} \sum_{i=1}^{N} \left(C_{mod,i} - C_{mkt,i}\right)^2$$

The most commonly used calibration techniques for the Heston model are:

1. Direct numerical methods (optimisation algorithms).
    a. Levenberg-Marquardt algorithm: widely used in financial model calibration.
    b. Differential Evolution (DE): genetic algorithm used for global optimisation.
2. Fourier transform based calibration (Heston Semi-Closed Formula) (Carr P., Madan, D. 1999).
3. Bayesian techniques.

Calibration is typically carried out on a subset of vanilla option prices, implied volatility surfaces for different maturities and different strike prices, and historical volatility data. Following the standard practice, the validity of the calibration is checked by backtesting on real data against estimated data.

4.3 Implementation and plot of Heston model with different correlation and volatility values

In this chapter, it is presented the implementations of the Heston model from the ground up, demonstrating its behavior with positive and negative correlation values.
Volatility mirroring is also implemented in practice, and a graphic representation is given.
The chapter concludes by showing the distribution of asset prices under different assumptions and finally plotting the implied volatility chart for put and call options.

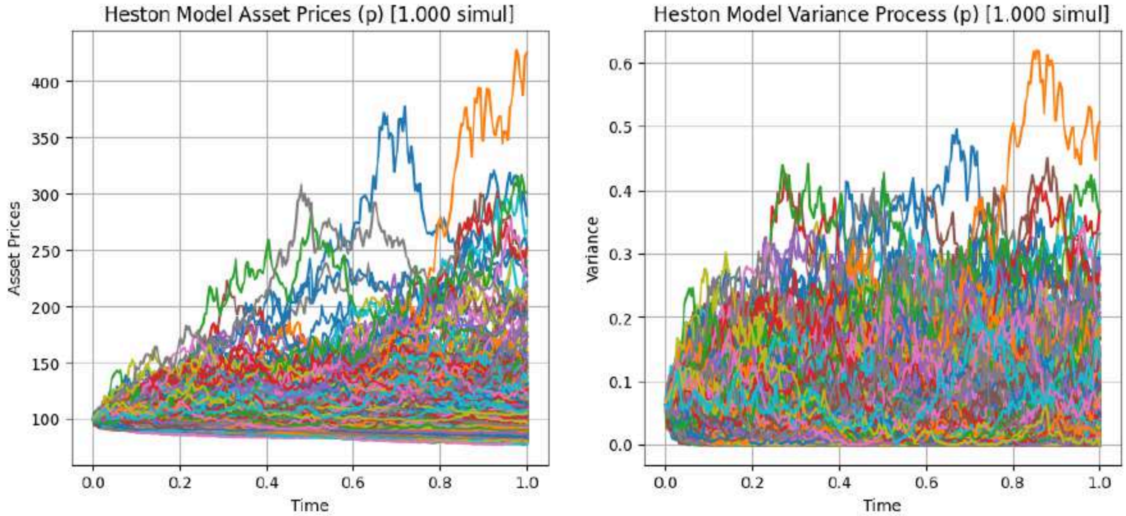
```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
from py_vollib_vectorized import vectorized_implied_volatility as implied_vol

# Model Parameters

# Market parameters
S0 = 100.0     # initial asset price
T  = 1.0       # time to maturity in years
r  = 0.05      # risk-free rate

# Simulation parameters
N = 252        # number of time steps
M = 1000       # number of simulation paths

# Heston model parameters
kappa = 2.7              # rate of mean reversion for the variance process
theta = (0.20)**2        # long-term variance level
v0    = (0.25)**2        # initial variance
sigma = 0.6              # volatility of volatility



# Heston Model Simulation Using Log-Price Transformation

def simulate_heston_log(S0, v0, r, kappa, theta, sigma, rho, T, N, M):
    """
    Simulates the Heston model using the Euler-Maruyama method on the log asset
price.

    Parameters:
      S0    : initial asset price
      v0    : initial variance
      r     : risk-free rate
      kappa : mean-reversion speed of the variance process
      theta : long-run mean variance
      sigma : volatility of volatility
      rho   : correlation between asset price and variance Brownian motions
      T     : total time (in years)
      N     : number of time steps
```

```
      M     : number of simulation paths

    Returns:
      S       : simulated asset prices (array of shape (N+1, M))
      variance: simulated variance process (array of shape (N+1, M))
    """
    dt = T / N  # time increment

    # Pre-allocate arrays for log asset prices and variance
    log_S    = np.zeros((N+1, M))
    log_S[0] = np.log(S0)
    variance = np.zeros((N+1, M))
    variance[0] = v0

    # Generate independent normal increments scaled by sqrt(dt)
    dW1 = np.random.normal(0, np.sqrt(dt), size=(N, M))
    dZ  = np.random.normal(0, np.sqrt(dt), size=(N, M))
    # Construct the second Brownian increment to have correlation rho with dW1
    dW2 = rho * dW1 + np.sqrt(1 - rho**2) * dZ

    # Time stepping loop using Euler-Maruyama discretization
    for i in range(1, N+1):
        # Update variance; ensure non-negativity by applying np.maximum
        variance[i] = variance[i-1] + kappa * (theta - variance[i-1]) * dt \
                      + sigma * np.sqrt(np.maximum(variance[i-1], 0)) * dW2[i-1]
        variance[i] = np.maximum(variance[i], 0)

        # Update log asset price using the previous variance value
        log_S[i] = log_S[i-1] + (r - 0.5 * variance[i-1]) * dt \
                   + np.sqrt(np.maximum(variance[i-1], 0)) * dW1[i-1]

    # Convert log asset prices back to actual asset prices
    S = np.exp(log_S)
    return S, variance


# Simulations for Different Correlation Scenarios

# Simulate for a high positive correlation
rho_pos = 0.97
S_pos, var_pos = simulate_heston_log(S0, v0, r, kappa, theta, sigma, rho_pos, T, N,
M)

# Simulate for a high negative correlation
rho_neg = -0.97
S_neg, var_neg = simulate_heston_log(S0, v0, r, kappa, theta, sigma, rho_neg, T, N,
M)

# Mirrored Volatility Scenario
mirrored_variance = np.maximum(-var_pos, 0)
S_mirrored, _ = simulate_heston_log(S0, mirrored_variance[0, 0], r, kappa, theta,
sigma, rho_pos, T, N, M)

# Plotting the Simulation Results
```

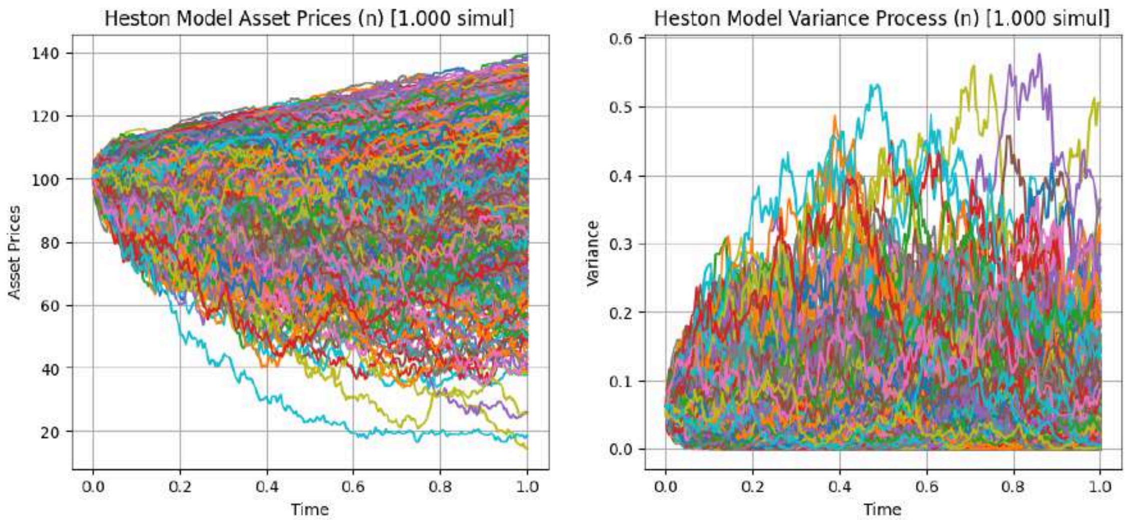
```
time_grid = np.linspace(0, T, N+1)

# positive correlation
fig1, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 5))
ax1.plot(time_grid, S_pos, lw=0.5)
ax1.set_title('Asset Price Trajectories (ρ = 0.97)')
ax1.set_xlabel('Time (years)')
ax1.set_ylabel('Asset Price')
ax1.grid(True)

ax2.plot(time_grid, var_pos, lw=0.5)
ax2.set_title('Variance Process (ρ = 0.97)')
ax2.set_xlabel('Time (years)')
ax2.set_ylabel('Variance')
ax2.grid(True)
plt.tight_layout()
plt.show()

# negative correlation
fig2, (ax3, ax4) = plt.subplots(1, 2, figsize=(12, 5))
ax3.plot(time_grid, S_neg, lw=0.5)
ax3.set_title('Asset Price Trajectories (ρ = -0.97)')
ax3.set_xlabel('Time (years)')
ax3.set_ylabel('Asset Price')
ax3.grid(True)

ax4.plot(time_grid, var_neg, lw=0.5)
ax4.set_title('Variance Process (ρ = -0.97)')
ax4.set_xlabel('Time (years)')
ax4.set_ylabel('Variance')
ax4.grid(True)
plt.tight_layout()
plt.show()

# og vs. mirr
fig3, ax5 = plt.subplots(figsize=(10, 6))
ax5.plot(time_grid, S_pos[:, 0], label='Original Trajectory', lw=2)
ax5.plot(time_grid, S_mirrored[:, 0], label='Mirrored Trajectory', lw=2,
linestyle='--')
ax5.set_title('Comparison of Original and Mirrored Asset Price Trajectories')
ax5.set_xlabel('Time (years)')
ax5.set_ylabel('Asset Price')
ax5.legend()
ax5.grid(True)
plt.show()

# distr comparison
# GBM generation for analogy
gbm_final = S0 * np.exp((r - 0.5 * theta) * T + np.sqrt(theta * T) *
np.random.normal(0, 1, M))
final_prices_pos = S_pos[-1, :]

fig4, ax6 = plt.subplots(figsize=(10, 6))
```

```
sns.kdeplot(final_prices_pos, label=r"Heston (ρ = 0.97)", ax=ax6)
sns.kdeplot(S_neg[-1, :], label=r"Heston (ρ = -0.97)", ax=ax6)
sns.kdeplot(gbm_final, label="GBM", ax=ax6)
ax6.set_title('Distribution of Final Asset Prices')
ax6.set_xlabel('Final Asset Price ($S_T$)')
ax6.set_ylabel('Density')
ax6.grid(True)
ax6.legend()
plt.xlim(20, 180)
plt.show()


# Option Pricing and Implied Volatility Smile

# For this section, we use the negative correlation simulation's final asset
prices.
S_final_neg = S_neg[-1, :]

# Define strikes for option pricing
strikes = np.arange(20, 180, 2)

# Compute prices for European calls and puts (discounted)
call_prices = np.array([np.exp(-r * T) * np.mean(np.maximum(S_final_neg - k, 0))
for k in strikes])
put_prices  = np.array([np.exp(-r * T) * np.mean(np.maximum(k - S_final_neg, 0))
for k in strikes])

# Calculate the implied volatilities using py_vollib_vectorized
call_iv = implied_vol(call_prices, S0, strikes, T, r, flag='c', q=0,
return_as='numpy')
put_iv  = implied_vol(put_prices, S0, strikes, T, r, flag='p', q=0,
return_as='numpy', on_error='ignore')

# Plot the implied volatility smile
fig5, ax7 = plt.subplots(figsize=(10, 6))
ax7.plot(strikes, call_iv, label='Call Implied Volatility', marker='o')
ax7.plot(strikes, put_iv, label='Put Implied Volatility', marker='o')
ax7.set_title('Implied Volatility Smile from Heston Model')
ax7.set_xlabel('Strike Price')
ax7.set_ylabel('Implied Volatility')
ax7.grid(True)
ax7.legend()
plt.show()
```

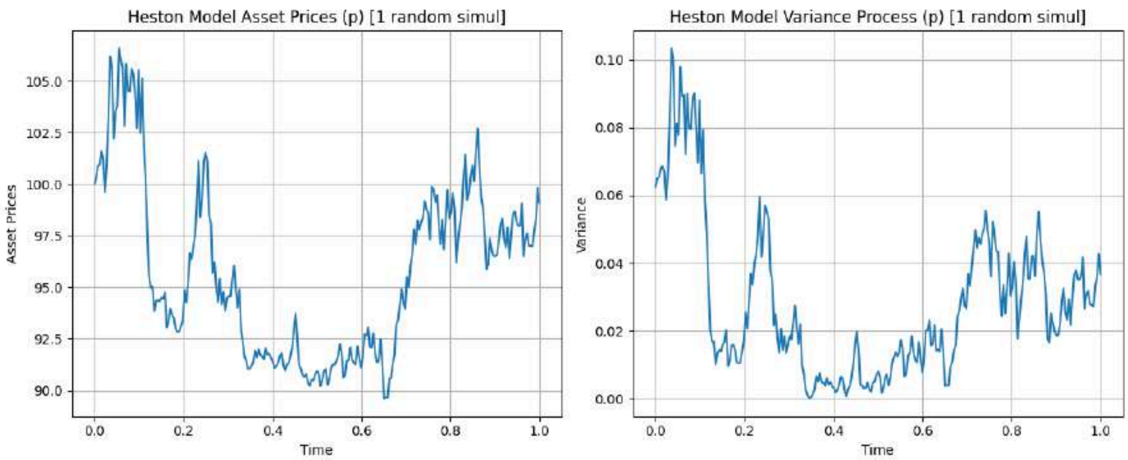
[Image 4.1]



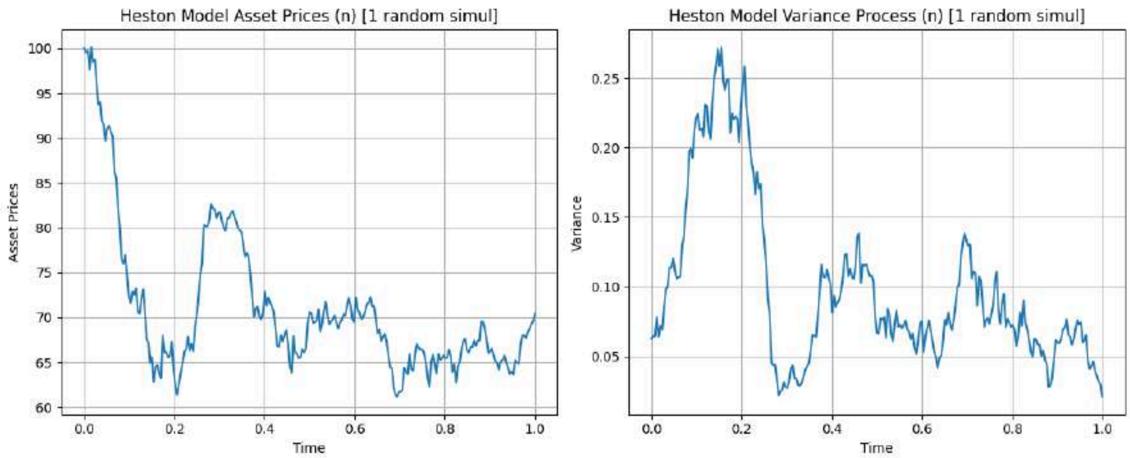
[Image 4.2]



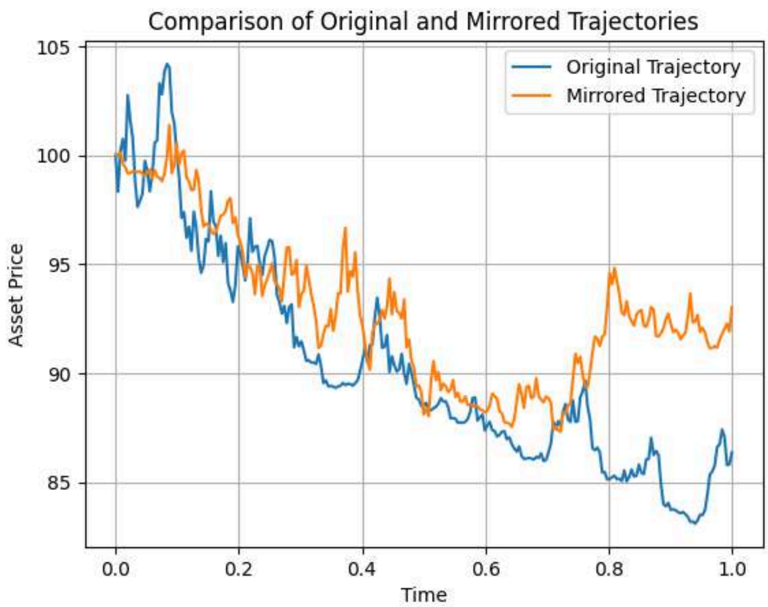
[Image 4.3]



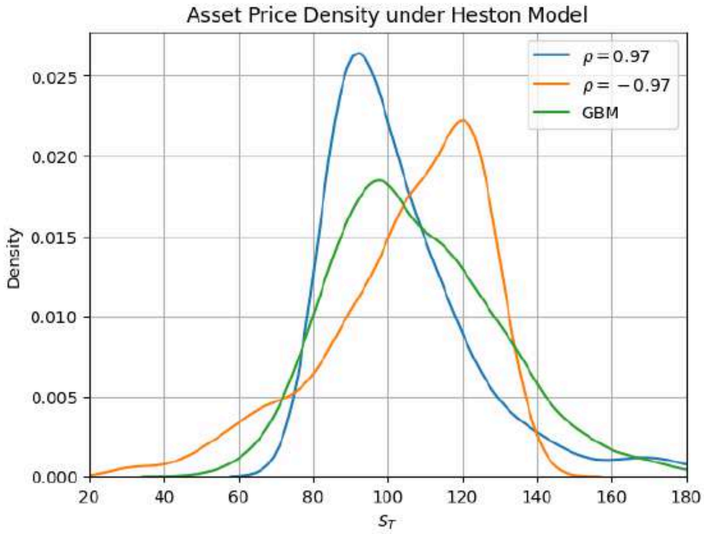
[Image 4.4]



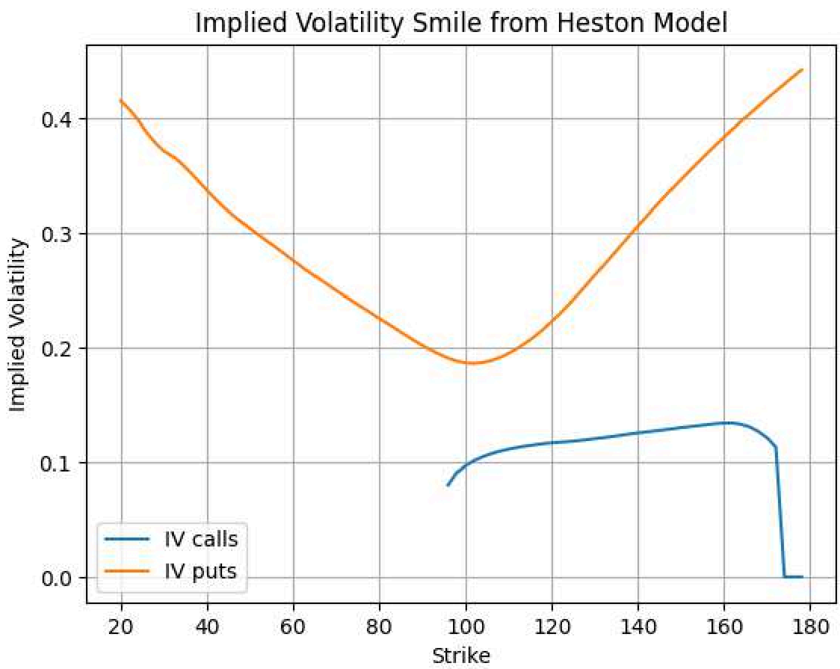
[Image 4.5]



[Image 4.6]



[Image 4.6]



V. Jump Diffusion Models & Poisson Processes

After thoroughly analysing financial derivative pricing models of unparalleled historical relevance (Black & Scholes) or that are mathematically refined (Heston Model), it may seem reasonable to conclude that a sufficient set of tools are available to price financial instruments with an appropriate degree of confidence (and *peace of mind*).

However, empirical observations of the behaviour of such instruments in financial markets would unfortunately refute this sense of accomplishment.
The presence of discontinuities in the price trajectories of financial assets (commonly referred to as "jumps") is well documented, as is the solution proposed [25]: adopt jump diffusion models.

Jump diffusion models can be outlined as a combination of Brownian motions and Poisson processes, which are independent of each other [11]. Under specific conditions, they also exhibit closed form solutions.

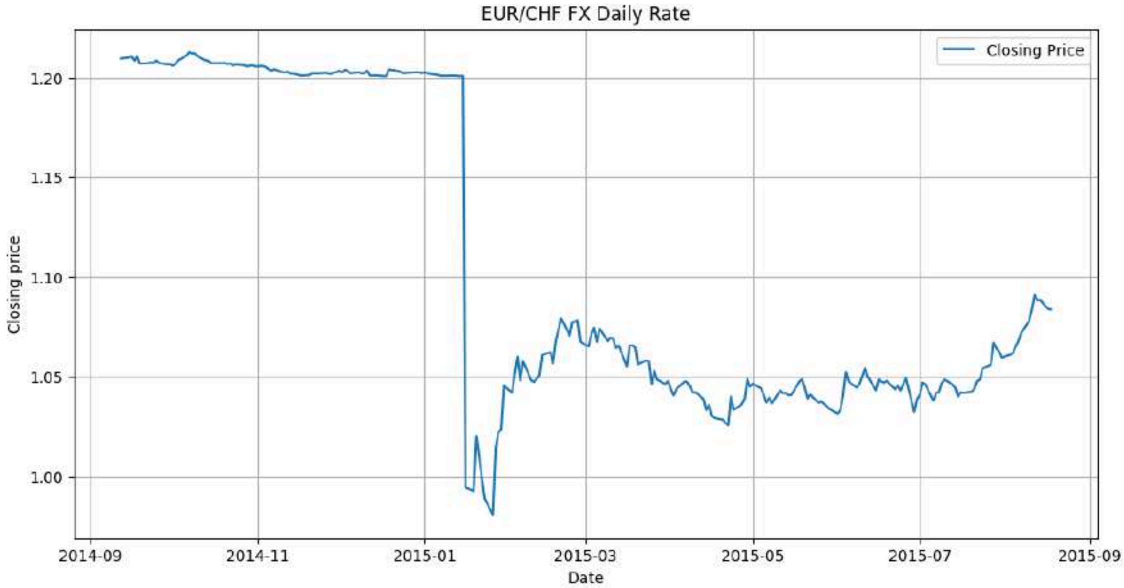
The relevance of jump models becomes even more significant in the context of risk management, which in continuous models' results are consistently underestimated [11], [24]. The presence of extreme events (fat tails) in price distributions defeats *de facto* Gaussian distribution assumptions (see Mandelbrot 1963, pp. 392, 417).

It is evident that it is not feasible to formulate continuous price trajectories in the presence of disruptive events, such as those experienced in 2001 (Twin Towers attack, with 1.4 usd trillion loss in market value and S&P500 dropping about 15% in a week), 2008 (subprime crisis, with S&P500 losing 21.6% in ten days) and 2020 (Covid outbreak, with S&P500 losing 34% of its value).

Even in the absence of such extreme socio-economics events, an event such as a central bank announcement can have a similar effect on the markets (see Image 5.1)[104].
Consequently, it is essential to develop methodologies that can accommodate discontinuous trajectories, incorporating stochastic models with jumps.

[104] See e.g. Bouchaud J.P., Economics needs a scientific revolution, NATURE Vol 455 30 October 2008

[Image 5.1]



15 January 2015: the Central Bank of Switzerland removed the cap on the Swiss Franc. The EUR/CHF FX rate dropped 40% in a matter of seconds, with an average SAEPS (Sell Amount Executed per Second) of 25.3 million euros[105]. Own chart from ECB data.

5.1 Jump diffusion model definition

The price of a financial asset S_t under a jump diffusion model in the physical probability measure $\mathbb{P}$ can be modeled with the following stochastic differential equation:

$$\frac{dS_t}{S_{t^-}} = \mu\, dt + \sigma\, dW_t + d\left(\sum_{i=1}^{N_t}(J_i - 1)\right)$$

Or, in its explicit form, as:

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]\prod_{i=1}^{N_t} J_i$$

Where:

1. S_{t^-}: value of the underlying asset immediately before any jump at time t.
2. μ: instantaneous rate of return on the underlying asset.
3. σ: volatility of the diffusion process (Brownian component).
4. W_t: standard Wiener process, representing the stochastic component of the diffusion process.

[105] An interesting analysis on the event can be read here (URLs retrieved 09 Feb 2025) https://www.kaggle.com/code/shaunoilund/flash-crash-2015 and here https://www.globenewswire.com/news-release/2015/03/12/714464/33730/en/FXCM-Releases-Detailed-Data-on-the-SNB-Flash-Crash.html

5. N_t: Poisson process with λ intensity, representing the number of jumps up to time t.
6. J_i: jump multiplicative factor for the *i*-th jump, a random variable describing the size of the jump.

The interpretation of the SDE within the financial context can be expressed as:

1. μdt: deterministic expected proportional change in price over an infinitesimal time interval (drift).
2. σdW_t: ordinary random fluctuations in prices (continuous and stochastic)[106].
3. $(J_i - 1)dN_t$: sudden (and unexpected) jumps of prices (discontinuous and stochastic), where $J_i - 1$ represents the relative size of the *i*-th jump triggered by the Poisson process N_t.

Building on the mathematical definition, the heuristic meaning of the model is an expectation of continuous movement at any given moment ("price volatility"), with notable deviations occurring at specific times ("shocks"), typically coinciding with significant socio-economic events such as macroeconomic announcements, geopolitical events, rate cuts, etc.

5.1.1 Poisson process

As defined in section 5.1, the Poisson process is understood to be the fundamental interpretive key in a jump diffusion model. It has been outlined why financial assets are subject to jumps, and why this is important. The next step is to understand their analytical source through a Poisson process.

A stochastic process N_t with $t \geq 0$ is a Poisson process with intensity $\lambda > 0$ if [11]:

1. $N_0 = 0$: events at time 0 are null.
2. Increments $N(t + \Delta t) - N(t)$ are independent.
3. The probability distribution of the increments is independent of the starting point and is only dependent on the length of the interval ("memorylessness"):

$$P[N_{(t+\Delta t)} - N_t = k] = \frac{(\lambda \Delta t)^k}{k!} e^{-\lambda \Delta t}, \quad k = 0, 1, 2, \ldots$$

4. The number of events in any interval of length t is a Poisson random variable with parameter (or mean) λt:

$$P[N_t = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, \ldots$$

[106] Remember from the Blacn & Scholes model.

Properties of a Poisson process

1. Expected value: $\mathbb{E}[N_t] = \lambda t$
2. Variance: $\text{Var}(N_t) = \lambda t$
3. Exponential increments: $f_{\Delta T}(t) = \lambda e^{-\lambda t}, \quad t > 0$.
4. Additivity: $N_{t1+t2} = N_{t1} + (N_{t1+t2} - N_{t1})$
5. Jumps trajectories in respect to time N_t are right continuous with finite left limits for all t (càdlàg, as for "*continue à droite et limité à gauche*").

The empirical interpretation can be expressed as:

1. The jumps are instantaneous and unpredictable, but their frequency can be estimated statistically.
2. Expectation is to have N_t jumps up to time t.
3. Jumps are expected to have λ intensity.

5.1.2 Compound Poisson process

A compound Poisson process is defined as:

$$X_t = \sum_{i=1}^{N_t} Y_i$$

With:

$$N_t = 0 \Rightarrow X_t = 0$$

Where:

1. N_t: Poisson process with $\lambda > 0$ intensity up to time t ("when prices jump").
2. Y_i: independent and identically distributed random variables ("i.i.d."), representing the magnitude of the jumps ("how big are the jumps").
3. X_t: sum of jump size up to time t.

And with:

$$\mathbb{E}[X_t] = \lambda t \mathbb{E}[Y]$$

$$Var(X_t) = \lambda t \mathbb{E}[Y^2]$$

5.1.3 Cox process (doubly stochastic Poisson process)

The Cox process, or doubly stochastic Poisson process, is a noteworthy generalization of the Poisson process in which the intensity of the process λ is no longer a constant but is modelled by a stochastic process (e.g., Ornstein-Uhlenbeck process[107] or CIR). Given the stochastic intensity λt, the number of events over times N_t follows a Poisson process with a time-dependent intensity:

$$P\left(N_{(t+\Delta t)} - N_t = k \mid \lambda t\right) = \frac{(\lambda t \Delta t)^k}{k!} e^{-\lambda t \Delta t}, \quad k = 0, 1, 2, \ldots$$

Under the following assumption:

1. Unidimensional temporal domain.
2. Intensity λt is a non-negative, continuous process.
3. N_t follows a time-dependent Poisson process.

The expected number of events up to time t, conditional on the trajectory of λt, is given by:

$$\mathbb{E}[N_t \mid \{\lambda s, s \in [0, t]\}] = \int_0^t \lambda s ds$$

Cox Process is specifically used in a financial context where the credit risk is significant[108].

5.2 Analytical solution of a jump diffusion model with Poisson processes.

Proceeding from section 5.1, with the aim of find the explicit solution for S_t:

$$\frac{dS_t}{S_{t^-}} = \mu\, dt + \sigma\, dW_t + d\left(\sum_{i=1}^{N_t} (J_i - 1)\right)$$

Approach the continuous part of the SDE:

$$\frac{dS_{ct}}{S_{ct}} = \mu\, dt + \sigma\, dW_t$$

Having explicit solution:

[107] See Appendix.

[108] See e.g. Chiarella, C., Musti, S., Modelling the evolution of credit spreads using the Cox process within the HJM framework; A CDS option pricing model (2011).

$$S_{ct} = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$$

Then, the discrete part of the SDE:

$$d\left(\sum_{i=1}^{N_t}(J_i - 1)\right)$$

Can be rewritten as:

$$S_{dt} = \prod_{i=1}^{N_t}$$

Combining the factors, such as ($S_t = S_0 \cdot S_{ct} \cdot S_{dt}$):

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] \prod_{i=1}^{N_t} J_i$$

That is, the explicit form for S_t.
However, it should be noted that in real-world applications, numerical methods (e.g. Monte Carlo) are frequently employed to obtain solutions through the use of computational resources, thereby bypassing the complexities inherent in the underlying distributions.

5.3 Implementation of a jump diffusion model

Merton's paper [25] was selected for the code implementation of this section.

By means of a simple modification to the variables, any number of patterns can be generated, simulating a stochastic movement with different jump price scenarios. However, for clarity's sake, here the `Npaths` component has been set equal to 1.

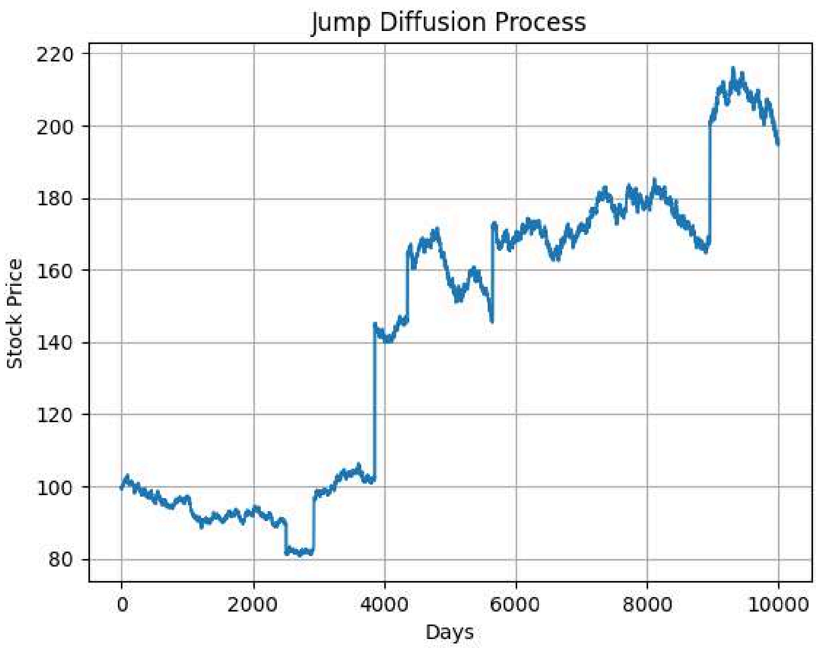
As the graph illustrates, the trend is immediately more similar to that of a financial asset in a context of increased volatility.

```
import numpy as np
import matplotlib.pyplot as plt

# Simulation Parameters

S0 = 100.0       # Initial stock price
T = 1.0          # Time to maturity in years
r = 0.02         # Risk-free interest rate
```

```
sigma = 0.2      # Annual volatility of the continuous diffusion component
lam = 5          # Intensity (jumps per year)
m = 0            # Mean of the jump size (in logarithmic terms)
v = 0.3          # Standard deviation of the jump size
steps = 10000    # Number of time steps in the simulation
Npaths = 1       # Number of simulation paths

# Function: simulate_jump_diffusion

def simulate_jump_diffusion(S0, T, r, sigma, lam, m, v, steps, Npaths):
    """
    Simulates a jump diffusion process using the Merton jump diffusion model.

    The stock price process is modeled as:

     S(t) = S0 * exp{ (r - 0.5*sigma^2 - λ*(E[J]-1)) * t + sigma*W(t) +Σ_{i=1}^{N(t)}
Y_i }

    where:
      - W(t) is a Wiener process,
      - N(t) is a Poisson process with intensity λ,
      - Y_i ~ N(m, v^2) represents the logarithmic jump sizes, and
      - E[J] = exp(m + 0.5*v^2) is the expected jump multiplier.

    The term λ*(E[J]-1) adjusts the drift to compensate for the average jump effect.

    Parameters:
      S0     : Initial stock price
      T      : Time to maturity (years)
      r      : Risk-free rate
      sigma  : Volatility of the continuous diffusion component
      lam    : Intensity of the jump process (jumps per year)
      m      : Mean of the jump size (in log terms)
      v      : Standard deviation of the jump size
      steps  : Number of time steps in the simulation
      Npaths : Number of simulation paths

    Returns:
      S_paths: Simulated stock price paths as a numpy array of shape (steps+1,
Npaths)
    """
    dt = T / steps  # Time increment per step

    # Continuous Diffusion Component
    # Generate increments of the Wiener process (Brownian motion)
    dW = np.random.normal(0, np.sqrt(dt), size=(steps, Npaths))

    # Compute the drift adjustment for the continuous part.
    # The compensation term is: λ*(E[J]-1) = λ*(exp(m + 0.5*v^2) - 1)
    drift_compensation = lam * (np.exp(m + 0.5 * v**2) - 1)
    continuous_drift = (r - 0.5 * sigma**2 - drift_compensation) * dt
    continuous_component = continuous_drift + sigma * dW

    # Jump Component
    jump_counts = np.random.poisson(lam * dt, size=(steps, Npaths))
```

```
    jump_component = jump_counts * np.random.normal(m, v, size=(steps, Npaths))

    # Combine Both Components
    # The total log-return is the sum of the continuous and jump components.
    log_returns = continuous_component + jump_component

    # Compute the cumulative sum to get the log-price path.
    log_S = np.vstack([np.zeros((1, Npaths)), np.cumsum(log_returns, axis=0)])

    S_paths = S0 * np.exp(log_S)
    return S_paths

# Generate and Plot the Jump Diffusion Process

jump_paths = simulate_jump_diffusion(S0, T, r, sigma, lam, m, v, steps, Npaths)

plt.figure(figsize=(10, 6))
plt.plot(jump_paths, lw=1.5)
plt.xlabel('Time Steps')
plt.ylabel('Stock Price')
plt.title('Jump Diffusion Process (Merton Model)')
plt.grid(True)
plt.show()
```

[Image 5.2]



VI. Physics and Finance: *raison d'être*

"Has econophysics really degenerated into irrelevance?
It doesn't seem that way to me"[109].

The relationship between physics and finance can be observed from two distinct perspectives[110]: a formal approach or a pragmatic approach.

1. The formal approach (top down) is marked by semantics, domain, roles. It focuses on the conceptual and structural aspects of the interplay between physics and finance. The debate follows a philosophical approach, with the aim to come down to define "*Who contributes to what, and how*"[111].
   While this approach may foster theoretical discussions, it appears somewhat arbitrary and subjective. Looking at the two subjects, physics and finance, in a rigorous manner, it is inevitable to note that not only do one and the other not benefit from a complete overlap. Economics and finance will always have to interact and deal with a degree of uncertainty and subjectivity, given the behavioural component of human nature. Physicists would have little interest in discussing the details of economic and financial models such as IS-LM or supply and demand. As things stand, a strictly formal analysis seems to sanction an incompatibility between the two worlds. (for a detailed overview: introduction and chapter 4 of [1], followed by [3], [10], [27], [28] and the introduction to [Py3])[112].

2. The pragmatic approach (bottom-up) is built over effective and methodological cross-domain synergy. It advocates for a collaborative environment where physicists and economists work together to pursue common goals.
   This approach places greater emphasis, and prioritizes, quantitative techniques and empirical validation over conceptual classification. The impact of models in real-world application becomes paramount. It is conceivable that in this context even cross-collaboration between even more diverse fields will be encouraged, for example by incorporating contributions from computer scientists, mathematicians, behavioural psychologists, etc.[113]

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109 [28]
110 Formulated upon Ausloos M., Jovanovic F., Schinckus C., On the 'usual' misunderstandings between econophysics and finance: Some clarifications on modelling approaches and efficient market hypothesis, International Review of Financial Analysis, Volume 47 (2016) Pages 7-14, and on [32].
111 And where will it be published then!
112 Interestingly, the debate is hot even on some social media platforms. See Reddit https://www.reddit.com/r/Physics/comments/ev7s1g/what_is_going_on_in_the_econophysics_world/
113 An interesting view is offered by Marin G., Vona F., Finance and the reallocation of scientific,

In [32], the authors propose an even more progressive view: the contributions to economics and finance from physical models are so relevant and remarkable that they should be considered a subject in their own right: Econophysics. They argue that references to classical economics are no longer necessary.
To support this thesis, they cite the creations of standalone BSc, MSc, and PhD programs devoted to the study and development of econophysics (e.g. Houston University, Ulm University, Dublin and Wroclaw universities). According to [1], the first full-time degree program labeled "Econophysics" is from Silesia university in Poland, during 2009-2010. Econophysics has reached its maturity having its own academic courses, events and conferences, journals and publications.

To explore this approach in more detail, it is recommended to examine the remarkable historical achievements these exceptional minds have accomplished working together.
Some of these achievements have been presented in this thesis, and many others would have deserved acknowledgement.

It would be very limiting to enumerate the contributions physics has made to economics, not least because in many areas the intersections are now so spurious that dividing one subject from the other would be arbitrary.

An inventory might be as follows[114]:

1. Financial market modelling in complex scenarios and high volatility (e.g. Seemann et al., 2011).
2. Stochastic processes for the evolution of financial phenomena, as well as the mathematical tools to work with stochastics processes (e.g. Lux T., 2009).
3. Complex networks theory to understand highly connected networks (e.g. Namatame A., Kaizouji T., et al., 2006).
4. Complex systems: nonlinear dynamics and the impact of physical figures in the financial environment (fractal, chaos theory, Brownian motion, heat equation, power-law relations, etc.) (e.g. [8]).
5. Non-Gaussian probability distributions, to overcome the simplification of a *Normal* world (e.g. McCauley J.L., 2004).
6. Risk management techniques and approaches (e.g. [17] and [21]).
7. Financial application of machine learning techniques and neural networks (e.g. Jaouadi R., 2025).
8. Development and application of Agent-Based Models for economics[115] and

engineering and mathematical talent, Research Policy, Volume 52, Issue 5 (2023).

[114] The following list is based on a personal reading list & research. A good overview is otherwise offered in [1], Chapter 5.

[115] Interestingly, in 2011 Michael Pickhardt and Goetz Seibold published a paper where they discussed the use of an agent-based econophysics model derived from ferromagnetism to model the dynamics of tax

finance (ABMs) (e.g. Patriarca M., Chakraborti A., et al., 2011).

9. Pricing of derivatives and other structured financial assets (e.g. [17]).

evasion! See (URL retrieved 22 Feb 2025) https://arxiv.org/pdf/1112.0233.

VII. Conclusions

"Classical economics has no framework through which to understand 'wild' markets"[116].

Opposing perspectives on the role of physics in finance will continue to emerge, along with the constant growth of both domains and related *cross-pollinations*[117].

The debate over the legitimacy of the term "econophysics" is likely to evolve or transform. As an example, the recent consolidations that have taken place in the field of computer science applied to finance domain, and the interest in the world of data, represented by Data Science[118] and its consequent applications, suggest a possible new area of intersection between different but complementary fields: *physics, finance, computer science*.

Physics, finance and the Academia

Notably, navigating this complex field is not trivial. The approach to the subject should be rigorous and formal, through a path built in academic contexts.
STEM pathways can certainly offer valuable insights[119] (indeed, as already noted in the first part of this work, great contributions to the subject of econophysics have come from physicists).
However, it seems relevant that even study courses that originated in the fields of economics and finance could explore the field of econophysics with due rigour[120], while still allowing students to build the necessary skills along the way[121].

Just as the economist needs the rigour and formality provided by advanced mathematical models, the physicist or financial engineer needs the knowledge of

[116] Quote from an article of Jean-Philippe Bouchaud PhD. JPB is currently a physics professor at École Polytechnique in France, and a professional in finance (Asset Management) as well.
[117] A reader may ask about Quantum Computing, Blockchain, Artificial Intelligence…
[118] Notably, Università degli Studi di Genova, "Economics and Data Science" https://corsi.unige.it/corsi/11267 (Retrieved 09 Feb 2025). This is similar to the offers in Bocconi University as well https://www.unibocconi.it/it/corsi-di-studio/lauree-magistrali/data-science-and-business-analytics (Retrieved 17 Feb 2025).
[119] See Università di Torino, Università di Pavia, where econophysics courses are within the Theoretical Physics Department.
[120] See e.g. Bocconi University, Econofisica, fondamenti di una nuova scienza, at the URL https://matematica.unibocconi.eu/articoli/econofisica-fondamenti-di-una-nuova-scienza (Retrieved 02 Feb 2025).
[121] A promising path seems to be the one proposed by the Institut Polytechnique de Paris, where the *Econophysics & complex system* chair is held by M. Benzaquen, a professor from the economics department.

classical economic and financial theories and the main economic models as well[122].

It is important to note that the opportunity of the adoption of rigorous mathematical formalism in economics courses is still a subject of debate (see e.g. [29][30][31]).
It is acknowledged that accurately replicating the behaviour of (only theoretically rational) individuals by using algorithms and models is not always reliable, if not distorsive[123].
On this topic, in [10] on page 88 (noting that both authors are physics professors), the appropriateness of using physical models in the economic context is discussed. The equation of motion governing the underlying process is not known.
For context, it can be cited Newton's second law of motion for classical mechanics, or Schrödinger's equation for quantum mechanics. In finance, however, there is no descriptive, defined equation that governs or models the dynamics of financial assets.
Nevertheless, the authors of [10] seem to opt for a more flexible view, recalling that other physical processes (e.g. dissipative friction) also do not have simple Hamiltonian functions[124]. Financial markets can be seen as specific cases of turbulence[125]. Financial crises share similar statistical characteristics with earthquakes[126]. Scientific truth can be found searching for mathematical analogies (Israel G., 1996).

Furthermore, it is reasonable to believe that an individual's skill should grow independently of the range of its possible applications. This assumption has been backed by scientific studies, contextualised in the study of STEM disciplines[127].
A quantitative economist, proficient in analysis, calculus, probability, computation and programming, will eventually acquire a comprehensive skill set that they may, and justifiably so, be able to apply or not. However, the converse is not necessarily true; a classical economist might encounter challenges when dealing with complex models based on physics studies or rigorous math models such as stochastic processes involving numerical or analytical solution methods.

It seems advisable to maintain a pragmatic approach, based on solid physical foundations and tangible contributions to the field of research (e.g. publications, books, study courses), not influenced by trends or excessively tied to titles or blazons. In this sense, the use of the term "Econophysics" should be handled with caution, avoiding

[122] Notably, [1] has a different advice: since physics have no urge to encapsulate economics phenomena within existing conceptual framework, they could be more prone to make groundbreaking discoveries.
[123] However, it is worth reading Lipton A., Hydrodynamics of Markets: Hidden Links Between Physics and Finance, arXiv:2403.09761 [q-fin.MF] (2024)
[124] Briefly, Hamiltonian function is the function describing the total energy of a physical system.
[125] See again [10], p. 89 for a deep dive into the topic.
[126] [1] at section 3.1.2.
[127] See e.g. Yeager, D. S., Hanselman, P., Walton, G. M., Murray, J. S., Crosnoe, R., Muller, C., … Dweck, C. S. (2019). *A national experiment reveals where a growth mindset improves achievement*. Nature, 573, 364–369 and Canning, E. A., Muenks, K., Green, D. J., & Murphy, M. C. (2019). *STEM faculty who believe ability is fixed have larger racial achievement gaps and inspire less student motivation in their classes*. Science Advances, 5(2)

making it a tag to aspire to without the support of the appropriate formalisms and rigour.

Physics, finance and computer science[128]

It is important to emphasise that the increasing sophistication of financial markets, coupled with the widespread use of advanced computer and computational tools, has made it necessary to resort to advanced simulation techniques even in strictly economic and financial fields.

1. In the context of derivatives trading, it is common to come across models that use high-frequency trading (HFT) by exploiting infinitesimal price movements over extremely small time deviations.
2. The use of numerical solution forms such as Monte Carlo simulations requires the computation of thousands, tens of thousands or more computed scenarios, even on a daily basis.
3. Advanced pricing models, such as the Heston models, Fourier transform (such as Fast Fourier Transform for option pricing), Jump diffusion models, require to possess advanced programming skills, or at the very least, above-average proficiency, to ensure a proper development of such models and their maintenance in the dynamic and evolving financial markets.

Finance has evolved from being based on simple economic models to being closely linked to technological developments, the physics of complex systems and advanced mathematical models. In this context, the combination of physics and finance is not only desirable; it is also a strict, inescapable constraint, a prerequisite for operating in these markets.

Physics, finance and the job market

Another aspect that seems worthy of discussion concerns the consolidation of roles such as the Quant (see section 1.1.3), and, more broadly, of figures with a financial background that is based on strong quantitative foundations (mathematics, physics, computer science).

Quant job searches on the popular financial recruiting site eFinancialCareers have recorded an average annual growth of over 125% in 2022 alone (although this data probably needs to be better contextualized due to the ubiquity of the "quant" role compared to other more specific ones such as "Trader" (+8%) or "M&A" (+77%)).

According to the World Economic Forum's Future of Jobs Report[129], roles combining finance, computer science and mathematical skills (defined as FinTech Engineers in the report) will be among the most sought after up to 2030 (after the Big Data Specialist, another job role that can be seen as connected to the quant). This appears to be a very

[128] [Py3] is a prime illustration of the practical application of this combination.

[129] See https://www.weforum.org/publications/the-future-of-jobs-report-2025/digest/ (URL retrieved 09 Feb 2025).

suitable opportunity for the context and applications discussed thus far.

Physics and finance: next steps

To strengthen the collaboration between physicists and economists, the first step would be to define a shared framework. This framework should highlight a unified working methodology, encompassing shared and accepted theoretical and practical approaches, conventions, expertise and even a common language and notation.

Common standards for the experimental acceptance of economic and financial phenomena should help to accelerate the diffusion of new models, with due rigor and in continuity with the scientific background of the field.

The identification of successful use cases, where these methodologies have been applied, would help define best practices and channel future developments.

In order to foster the growth of the joint field of physics and finance, the presence of academic pathways that take into account the connection of physics and finance as well as the different pool of expertise between physicists and economists should be encouraged. Students could be supported in exploring this interdisciplinary approach within formal educational pathways.

Finally, raising awareness among companies about these competencies and roles would promote their adoption, ensuring a concrete impact on the financial industry.

It remains clear that researchers, students and professionals committed to applying rigorous mathematical and physical methods to financial systems will find fertile ground for their works. The future looks promising for those willing to embrace the quantitative challenges posed by econophysics[130].

This work aims to provide a small yet meaningful contribution to the ongoing refinement of financial and economic methodologies, encouraging openness to cross-disciplinary approaches and innovations.

[130] Interestingly, the 2025 Econophysics Symposium will take place in Rome, Italy. See https://www.econophysics-colloquium.org/ (Retrieved 2 Feb 2025).

Appendix

Stochastic Differential Equation (SDE)

A stochastic differential equation (SDE) is a differential equation (e.g., an equation involving an unknown function and one or more of its derivatives) in which one or more terms are stochastic processes, thus introducing an element of randomness into the evolution of the solution.
SDEs are closely associated with stochastic processes of extreme relevance in the context of physical applications to finance. The stochastic differential is typically expressed in forms such as Wiener white noise, a Poisson jump process or a Lévy process.
The canonical use of SDEs is related to growth models (e.g. population growth), financial models (e.g. stock prices modeled with Brownian motion), and physical processes (e.g. electric charges).

SDE can be expressed in differential form:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Where:

1. X_t: stochastic variable representing the solution of the equation at time t.
2. $\mu(t, X_t)$: drift (deterministic).
3. $\sigma(t, X_t)$: noise coefficient (the amplitude of the random variation introduced by the stochastic term).
4. W_t: Wiener process (standard Brownian motion), a stochastic process with independent, stationary and normally distributed increments with zero mean and variance.
5. dt: deterministic infinitesimal of time.
6. dW_t: infinitesimal increment of the Wiener process.

If the noise coefficient is equal to zero (or, in a process of decay, it reaches zero), the SDE can be treated as an ordinary differential equation (ODE), in the form:

$$dX_t = \mu(t, X_t)dt$$

or, more commonly:

$$\frac{d}{dt}X(t) = \mu(X(t), t)$$

SDE can also be expressed in integral form, such as:

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

Where drift and noise coefficient are integrated in respect to time.

More specifically:

$$\int_0^t \sigma(s, X_s)dW_s$$

is the Itô integral of a stochastic (Wiener) process σ(t, X_t) with respect to the Brownian motion dW_s on the interval [t, 0].

SDE in integral form are often associated with numerical solutions to SDE (such as Euler-Maruyama method).

Notably, SDE of Black & Scholes equation are expressed in integral form.
Due to their stochastic component (the Wiener process is notoriously non-differentiable), SDEs generally do not admit explicit analytical solutions.

Solution approaches typically focus on:

1. Semi-analytical (or analytical in specific and even rare cases) resolutions.
2. Numerical resolutions.

The most common approaches in the case of possible analytical solutions are:

1. Itô lemma.
2. State function method (search for an auxiliary function that simplifies the SDE).
3. Asymptotic expansions.

When it comes to numerical solutions, the following approaches are typically used:

1. Euler & Maruyama scheme.
2. Monte Carlo simulation.
3. Markov Chain Monte Carlo (MCMC).

The aim is to trade-off the accuracy of the solution and the computational cost of simulating solutions over a large number of scenarios (e.g., n = 10,000).

Martingale Measure

Following notation from [12], a martingale measure can be defined as a probability measure Q in a space $(\Omega, \mathcal{F})$ such that:

1. Q is equivalent to P (e.g., with the same zero probability events).
2. For each $n = 1, ..., N$ holds:

$$\bar{S}_{n-1} = E^Q[\tilde{S}_n|\mathcal{F}_{n-1}]$$

That is, $\tilde{S}$ is a Q-martingale.

From this definition it follows that: a discrete-time market is arbitrage free if and only if there exists at least one martingale measure (First Fundamental Theorem of Valuation). Martingale measure can also be defined as a risk-neutral measure.

Given all available information, the expected future value of a stochastic process is equivalent to its present value.
An intuitive analogy is given by assuming gamblers betting over the outcome of a fair coin[131]. In fact, the concept of martingale is strictly related to the expectations of a fair game.

In derivative pricing models (such as Black & Scholes), the martingale assumption guarantees the absence of risk-free profit opportunities (no arbitrage).

Martingale is also close-linked with Efficient Market Hypothesis (EMH): given that prices already reflect all available information, it is not possible to (systematically) predict the outcome of the market based on past movements[132].

Risk-neutral Measure

A risk-neutral measure $\mathbb{Q}$ is a probability assumption under which the present value of financial securities can be calculated discounting their expected future payments at the risk-free rate (under the assumption of linear utility).

While the definition aligns with financial context expectations, the performance of this calculation is not straightforward when applied to derivative instruments without the necessary simplifications:

1. It is assumed that all investors are risk neutral.
2. The expected return on a risky security is equal to the risk-free rate.
3. The existence of the aforementioned assumptions necessarily implies the absence of market arbitrage (Fundamental Theorem of Asset Pricing) [11].

While the concept of risk-neutral measure allows the application of mathematical models for derivative valuation, it should be noted that it is an archetypal construction

[131] In fact, a "martingale strategy" was first defined in gambling clubs during 1600.
[132] Remember this definition the next time you're approaching technical analysis!

that should not be used to infer real distributions of returns of risky financial instruments.

Under the risk-neutral measure:

1. The discounted payoff of a financial asset is a martingale (Girsanov, 1960):

$$e^{-rt}S_t$$

2. The valuation of financial derivatives can be performed in accordance with this measure, which calculates the expected present value of their future payments.

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[V_T]$$

Where:

1. V_0: present value of the derivative financial asset.
2. V_T: maturity value of the derivative financial asset.
3. $\mathbb{E}^{\mathbb{Q}}$: expected value under the assumption of risk-neutral measure.

Lévy Processes

A Lévy process is a stochastic process with continuous-time trajectories that generalises Wiener (Brownian motion) and Poisson processes.
Less formally can be defined as a continuous time random walk.

The name is due to Paul Lévy (1886 - 1971), one of the most prolific authors in the field of stochastic processes.

Lévy processes have wide applications in finance, in particular to model stock returns with jumps (e.g. Merton process with jumps) or stochastic volatility (e.g. extended Heston-type models).

A Lévy process is characterised by four fundamental properties:

1. $X_0 = 0$.
2. Independent increments: the increments of the process over disjoint time intervals are independent, hence the infinite divisibility of distributions.
3. Stationary increments: the distribution of increments depends only on the length of the interval and not on its position in time.
4. Stochastic continuity:

$$\forall \epsilon > 0, \lim_{h \to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$$

Note that the fourth condition does not imply in any way that the sample paths are continuous (e.g., Poisson process). More precisely, the scope of that assumption is to exclude nonrandom effects (such as calendar events) [24].

The distribution of a Lévy process can be described by the Lévy-Khintchine formula, which expresses the characteristic function of the process as:

$$\mathbb{E}[e^{iuX_t}] = e^{t\Psi(u)}$$

Where $\Psi(u)$ is the Lévy characteristic and has the form:

$$\Psi(u) = i\gamma u - \tfrac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|<1}\right) \nu(dx)$$

Where:

1. $\gamma \in R$: deterministic drift. If only this term were present, the process would show linear growth over time.
2. $\sigma^2 \geq 0$: variance of Brownian component. It is a Brownian motion with variance σ^2. It models the smooth (continuous) part of the underlying path.
3. v: Lévy measure. It describes the frequency and intensity of jumps. The jump term introduces sudden discontinuities in the trajectory, modelling rare but significant events.

Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process is a stochastic process that describes the time evolution of a random variable, which tends to return towards an average value over time, with a random noise component.
It is a process that is particularly relevant for pricing derivative instruments, and is frequently used in the context of energy commodities.

The Ornstein-Uhlenbeck process is is described by the stochastic differential equation:

$$dX_t = \theta\left(\mu - X_t\right) dt + \sigma dW_t$$

Where:

1. Xt: stochastic variable at time t.
2. $\theta > 0$: mean-reversion speed.
3. μ: long-term mean value towards which the process tends to converge.
4. σ: volatility intensity (amplitude of random fluctuations).
5. W_t: standard Wiener process (white noise).

The term $\theta(\mu - X_t)$ acts as a recall force that pushes the process towards the mean value μ.

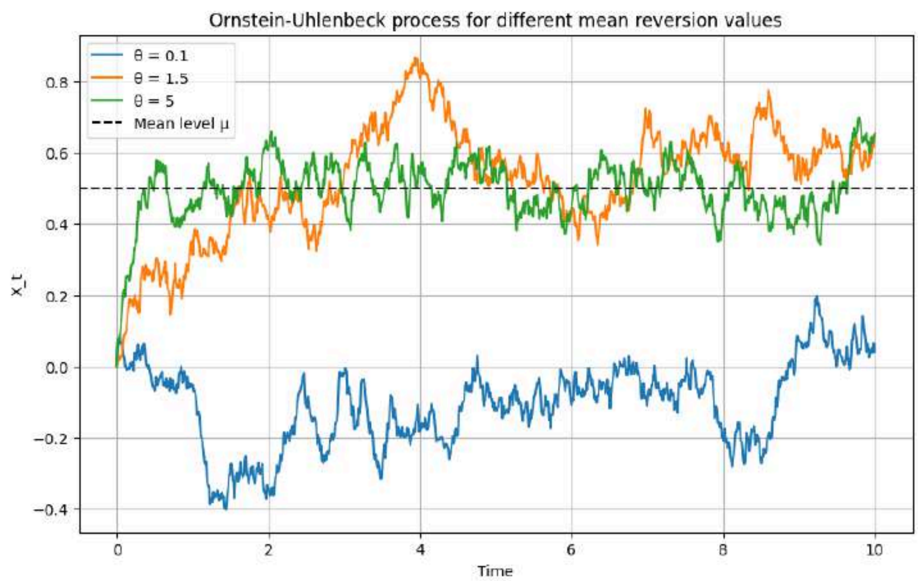
1. If $X_t > \mu$, the term is negative, pushing X_t downwards.
2. If $X_t < \mu$, the term is positive, pushing X_t upwards.

With $t \to \infty$, the process converges to its mean[133]:

$$\mathbb{E}[X_t] = \mu$$

The Ornstein-Uhlenbeck process is a Markov process (its future value depends only on current value, not on its past history).

[Image A.1]



[133] To understand this process empirically consider the movement of a marble rotating in a bowl. Regardless of the amplitude of its oscillations, the marble will always and constantly tend towards the centre of the bowl.

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List of tables and figures

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