

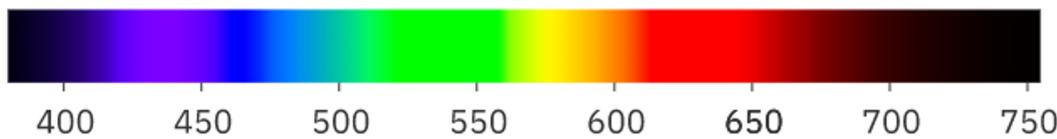
PBRT: Color and Radiometry

Friday, December 4, 2020 10:31 AM

5 Color and radiometry

Radiometry is the study of the propagation of **electromagnetic radiation** in an environment.

Color is a spectrum of visible light, a spectrum that is measured in **nanometers** and that goes from 380 nm to 780 nm. The blues are on the lower end of the spectrum, the greens and yellows in the middle, and the reds on the higher end. What is a **wavelength**?



Electromagnetic radiation is described by 4 radiometric quantities: **flux**, **intensity**, **irradiance**, and **radiance**. These are, in turn, described by their **SPD**, their **spectral power distribution function**.

λ is the variable name used to refer to wavelengths.

5.1 Spectral representation

The **domain** of the spectral power distribution function or SPD is **wavelengths**. The **codomain** is "amount of light". The book gives examples of 2 different SPDs: light emission of a fluorescent light and light reflectance of lemon skin. The ranges of the codomains of these 2 SPDs are different: emission of fluorescent light is $(0, \infty)$ with an absolute maximum of 140 (units unknown) and reflectance of lemon skin is $[0, 1]$ which is clearly the fraction of **each wavelength** of incoming light that gets reflected by the surface.

Functions exist in function spaces (vector spaces of functions). The function space of SPDs is infinite-dimensional. Like every vector space, a function space has a **basis**, a set of linearly independent functions that generate the rest of the function space. Every SPD function of the function space has a **unique representation** as a **linear combination** of the functions of the basis. A **compact** representation of an SPD is then as the **set of coefficients** of the linear combination of the **basis functions** that represents it.

There exist many choices of sets of basis functions for **spectral representation**, each one with its own strengths and weaknesses. PBRT uses 2 different bases for SPDs: RGB (RGBSpectrum) and a set of point samples over a range of wavelengths (SampledSpectrum). These 2 types are concrete implementations of the Spectrum **alias type**.

5.1.1 The Spectrum type

PBRT is coded against the abstract Spectrum **alias type**. The choice of concrete implementation, RGBSpectrum or SampledSpectrum, is made at compile time. RGBSpectrum is more efficient but less accurate.

5.1.2 CoefficientSpectrum implementation

The CoefficientSpectrum type stores a fixed number of samples of the SPD. The number of samples is part of the type (template).

The number of samples is also the number of coefficients of the linear combination of basis SPD functions that represents the SPD.

In some cases it is necessary to be able to iterate over a set of spectral samples that represent an SPD, for example to perform a spectral sample-based table lookup or to evaluate a piecewise function over wavelengths.

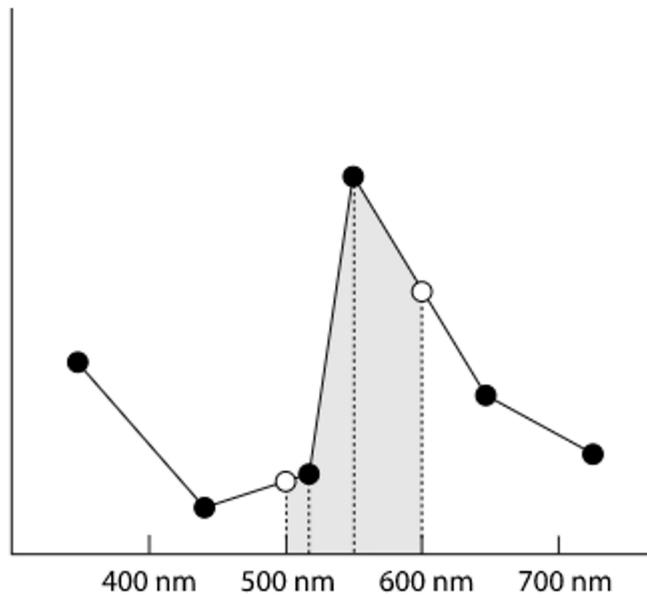
5.2 The SampledSpectrum class

SampledSpectrum represents an SPD using a regular partition of the interval of wavelengths to which the human visual system is most sensitive: [400nm, 700nm]. A value of each of the $n = 60$ subintervals is called a **spectral sample**.

A spectral sample comes in the form (λ_i, v_i) , where λ is its wavelength and v is its value. The meaning of the value depends on the use case.

The SPD is a continuous function of which we only have a number of samples. We reconstruct the function by computing the average of the samples of each of the $n = 60$ subintervals. In this example, the [400nm, 500nm] subinterval only has 1 sample, but [500nm, 600nm] has 2.

In each subinterval, the average is computed as the **area under the curve** (divided by the length of the subinterval) of the **piecewise linear** function defined between the subinterval's left endpoint x_0 , the x_k sample, the x_{k+1} sample, and the right endpoint x_p , where the subinterval has $p - 2$ samples.



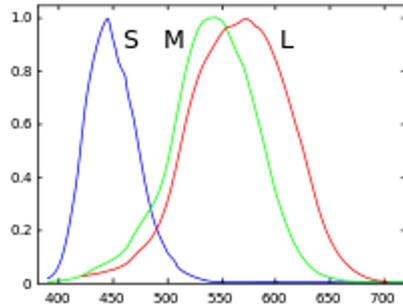
5.2.1 XYZ color

(See also [Colorimetry](#).)

The retina contains **cone cells**, which are light receptors. Different cone cells are sensitive to different intervals of the wavelength spectrum. A certain set of cone cells will send a strong signal to the brain when presented with light of a wavelength interval $[\lambda_i, \lambda_j]$, but a weaker signal when presented with light of another interval.

In general, cone cells are grouped into 3 sets: those that are sensitive to **short wavelengths** (S cones), those that are sensitive to **medium-sized wavelengths** (M cones), and those that are sensitive to **long wavelengths** (L cones). Of course, the **intervals overlap**.

LMS response curves



Of course, exposure to light is simultaneous for all the cone cells. The brain receives a signal combined from all of the cone cells, each one "weighting" it differently. The weighted stimulus is interpreted by the brain as a particular color.

For that reason, the **tristimulus theory of color perception** says that any SPD over the interval of visible wavelengths can be represented with 3 real numbers (the tristimulus theory exploits the **trichromacy** of the human visual system). The **LMS color space**, for example, maps values of a given SPD $S(\lambda)$ to a 3-tuple of $[0,1]$ real numbers, each one quantifying the level of short, medium, and long wavelength contained in the input spectrum. LMS models how cone cells perceive color.

The **XYZ color space** also maps the value of a given SPD $S(\lambda)$ to 3 non-negative real numbers x_λ , y_λ , and z_λ for every wavelength λ :

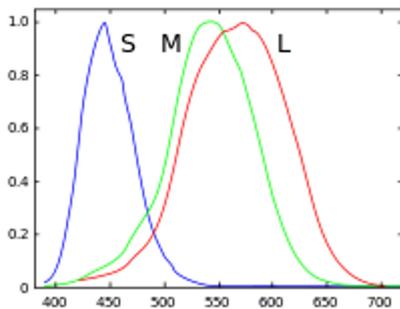
$$x_\lambda = \frac{1}{\int Y(\lambda) d\lambda} \int S(\lambda) X(\lambda) d\lambda$$

$$y_\lambda = \frac{1}{\int Y(\lambda) d\lambda} \int S(\lambda) Y(\lambda) d\lambda$$

$$z_\lambda = \frac{1}{\int Y(\lambda) d\lambda} \int S(\lambda) Z(\lambda) d\lambda$$

where $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$ are 3 standard **spectral matching curves**. (I don't know why the definite integral of the $Y(\lambda)$ curve divides.)

Every color space has a number of **primaries**. In XYZ space, primaries are $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$. Now, look at the LMS response curves: $[1,0,0]$, $[0,1,0]$, and $[0,0,1]$ LMS responses are not possible; there's no LMS response value that doesn't have >0 components. Since LMS models real perception, the XYZ primaries are said to be **imaginary**.



PBRT hardcodes the values of $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$ for increments of 1 nm in the interval [360nm, 830nm].

Then it resamples the XYZ curves over the [360nm, 830nm] interval at subintervals determined by the number of coefficients of the SampledSpectrum. Since the number of coefficients is usually much lower than the total 471 1 nm XYZ curve samples, the SampledSpectrum averages the 1 nm samples contained by its subintervals.

The XYZ curves resampled by the SampledSpectrum are then used to compute the X, Y, and Z tristimulus values of each of the spectrum's lambda/wavelengths. As shown above the tristimulus values that correspond to a wavelength are defined by the **definite integrals** of the products of the spectrum's SPD and the XYZ color-matching curves, and approximated by the corresponding **Riemann sums** over the spectrum's subintervals.

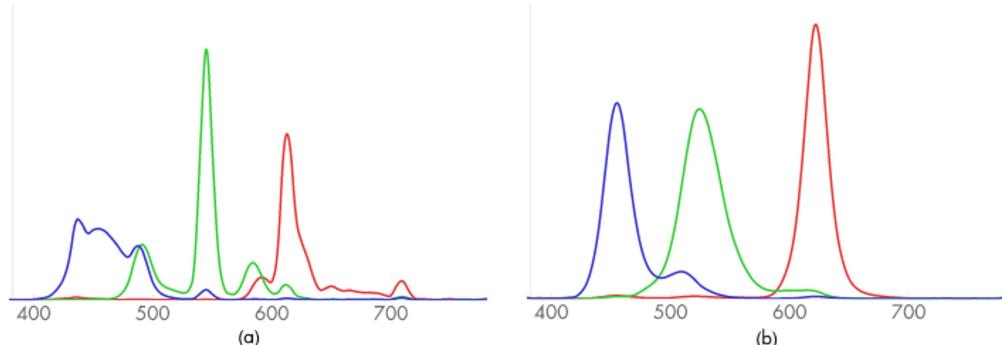
MOST IMPORTANT: An SPD $S(\lambda)$ is represented compactly as the set of coefficients of the linear combination of basis SPD functions that represents it. The CIE XYZ $X(\lambda)$, $Y(\lambda)$, and $Z(\lambda)$ are basis functions (but are they SPDs?) and x_λ , y_λ , and z_λ are the coefficients of the linear combination that represents the SPD:

$$S(\lambda) = x_\lambda X(\lambda) + y_\lambda Y(\lambda) + z_\lambda Z(\lambda)$$

Looks like XYZ is a conversion format. The light carried by a ray is represented by a Spectrum. A Spectrum can transform its SPD into an XYZ Spectrum. The XYZ Spectrum can then be turned into an RGB Spectrum for presentation to the screen or storage in an image file.

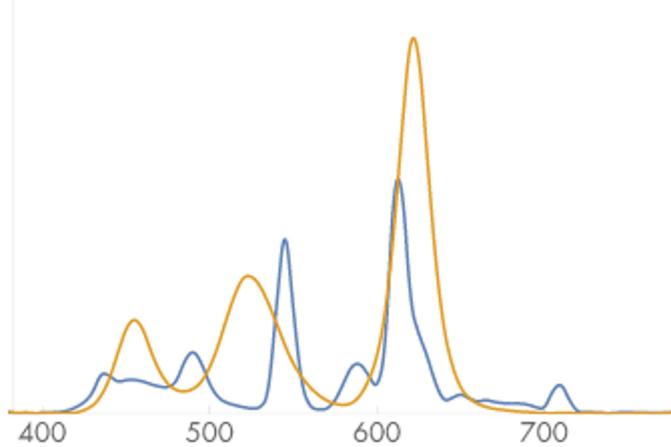
5.2.2 RGB color

The RGB colors shown by a display are the linear combination of 3 **spectral emission response curves**, one for red, one for green, and one for blue. Different display technologies have different spectral response curves; these are LCD's and LED's:



Since different display technologies use different spectral response curves, a given RGB value (0.6, 0.3, 0.2)

will have a different spectral power distribution SPD and, of course, will be perceived differently by the human eye:



We could say that RGB values are "not portable".

The spectral response curves of a given display technology are denoted $R(\lambda)$, $G(\lambda)$, and $B(\lambda)$, and the coefficients of the *linear combination* that represents a given SPD are r, g, b .

The rgb coefficients that correspond to a given $(x_\lambda, y_\lambda, z_\lambda)$ color (i.e. an XYZ SPD) are computed as follows:

$$\begin{aligned} r &= \int R(\lambda) S(\lambda) d\lambda \\ &= \int R(\lambda) (x_\lambda X(\lambda) + y_\lambda Y(\lambda) + z_\lambda Z(\lambda)) d\lambda \\ &= x_\lambda \int R(\lambda) X(\lambda) d\lambda + y_\lambda \int R(\lambda) Y(\lambda) d\lambda + z_\lambda \int R(\lambda) Z(\lambda) d\lambda \end{aligned}$$

where $S(\lambda) = x_\lambda X(\lambda) + y_\lambda Y(\lambda) + z_\lambda Z(\lambda)$ is the linear combination of the XYZ curves that represents a given (in our case, sampled) SPD.

Note that we have a weighted sum of definite integrals. The definite integrals are constant, because the RGB and XYZ curves are fixed. Storing these definite integrals in a 3×3 matrix allows for a compact expression of the rgb coefficients as the vector that results from multiplying the matrix by the $(x_\lambda, y_\lambda, z_\lambda)$ coefficients:

$$\begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} \int R(\lambda) X(\lambda) d\lambda & \int R(\lambda) Y(\lambda) d\lambda & \int R(\lambda) Z(\lambda) d\lambda \\ \int G(\lambda) X(\lambda) d\lambda & \int G(\lambda) Y(\lambda) d\lambda & \int G(\lambda) Z(\lambda) d\lambda \\ \int B(\lambda) X(\lambda) d\lambda & \int B(\lambda) Y(\lambda) d\lambda & \int B(\lambda) Z(\lambda) d\lambda \end{bmatrix} \begin{bmatrix} x_\lambda \\ y_\lambda \\ z_\lambda \end{bmatrix}$$

This is effectively a *matrix transformation* of $\begin{bmatrix} x_\lambda \\ y_\lambda \\ z_\lambda \end{bmatrix}$ to $\begin{bmatrix} r \\ g \\ b \end{bmatrix}$.

This matrix is invertible and its inverse maps $\begin{bmatrix} r \\ g \\ b \end{bmatrix}$ to $\begin{bmatrix} x_\lambda \\ y_\lambda \\ z_\lambda \end{bmatrix}$.

PBRT draws the values of the integrals from those defined for high-definition television.

Metamers are SPDs that have the same $(x_\lambda, y_\lambda, z_\lambda)$, that is, they are all perceived the same by the human eye. This means that a given $(x_\lambda, y_\lambda, z_\lambda)$ may correspond to multiple SPDs (infinitely many, actually).

So, given an SPD, you can obtain their unique *XYZ* and *RGB* representations. But given an *XYZ* or *RGB*, you may obtain infinitely many different SPDs.

PBRT designed its *XYZ* \rightarrow *SPD* and *RGB* \rightarrow *SPD* maps with 2 characteristics for the output SPD: **smooth** (as opposed to spiky) and **constant** when all the 3 input coefficients are equal.

The *RGB* \rightarrow *SPD* map computes the output SPD as a linear combination of 3 SPDs with the input input (r, g, b) as coefficients. These 3 SPDs are 1 for red $S_R(\lambda)$, 1 for green $S_G(\lambda)$, and 1 for blue $S_B(\lambda)$ that were **chosen** for their **smoothness**.

The *RGB* \rightarrow *SPD* map does that mapping for all inputs, **except** for:

- (r, g, b) s with **equal components** $r = g = b$; instead of computing the linear combination of the smooth $S_R(\lambda)$, $S_G(\lambda)$, and $S_B(\lambda)$ (which may not be smooth), a constant SPD is returned.
- (r, g, b) s that are **mixtures of primaries exclusively**; instead of computing them as linear combinations of the smooth $S_R(\lambda)$, $S_G(\lambda)$, and $S_B(\lambda)$, PBRT uses precomputed smooth SPDs for these (r, g, b) s.

5.3 RGBSpectrum implementation

2 displays of different technology will emit different SPDs for the same RGB value. It depends on the primaries that they use.

5.4 Radiometry

The theory of **radiometry** doesn't have its basis in the physics of light (e.g. Maxwell's equations). Instead, radiometry describes an **abstraction of light** in the form of particles that flow through space.

Radiative transfer is the phenomenological study of the transfer of radiant energy. The interaction of light and matter can be studied at the micro, atomic scale, and at the macro, wavelength (nanometer) scale. Quantum mechanics studies the micro scale and **geometric optics and radiometry** study the macro scale.

A model/description/framework of the macro scale suffices for rendering, so PBRT uses the **geometric optics model** for the description of light and light scattering, and makes the following assumptions about the behavior of light per that model

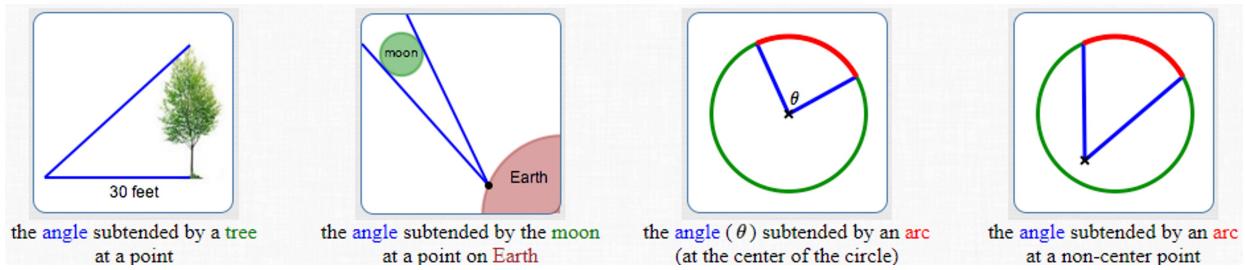
- **Linearity**: the combined effect of two inputs to an optical system is always equal to the sum of the effects of each of the inputs individually. *This is an important assumption made by the [light transport equation](#).*
- **Energy conservation**: *TODO*.
- **No polarization**: *TODO*.

- *No fluorescence or phosphorescence*: this means that no transfer of energy between wavelengths of the spectrum takes place as a result of reflection or transmission (see section 8.1). For example, we can assume that incoming light of a given wavelength is reflected at the same wavelength.
- *Steady state*: the radiance distribution of the scene doesn't change over time because it has reached equilibrium. This is an important assumption made by the [light transport equation](#).

TODO: this section is almost purely theoretical (with the exception of the implementation of spherical coordinate functions). These concepts will be used in later chapters by the rendering algorithms. Let's skip this section for now and come back later.

Solid angle

Intuitive definition of subtended angle:

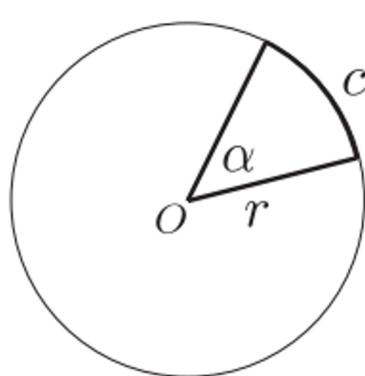


Plane angle α is the ratio of the length of an arc c of the circle to its radius.

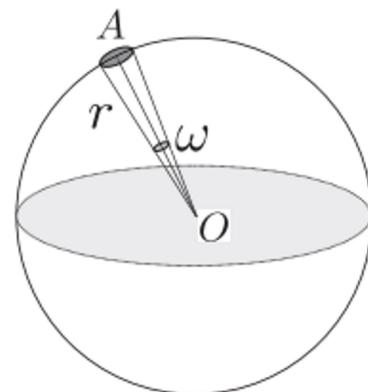
Solid angle ω is the ratio of an area A on the surface of the sphere to the square of its radius. The shape of the area doesn't matter.

If $A = r^2$, then $\omega = \frac{A}{r^2} = \frac{r^2}{r^2} = 1$ steradian.

Since surface area of sphere is $A_S = 4\pi r^2$, the sphere subtends a solid angle of $\omega_S = \frac{A}{r^2} = \frac{4\pi r^2}{r^2} = 4\pi$ steradians.

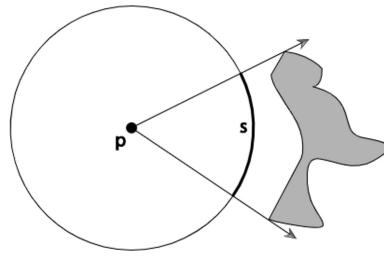


$$\text{plane angle: } \alpha = \frac{c}{r}$$

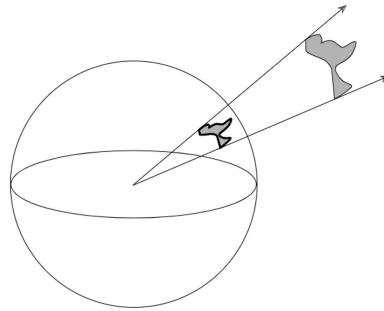


$$\text{solid angle: } \omega = \frac{A}{r^2}$$

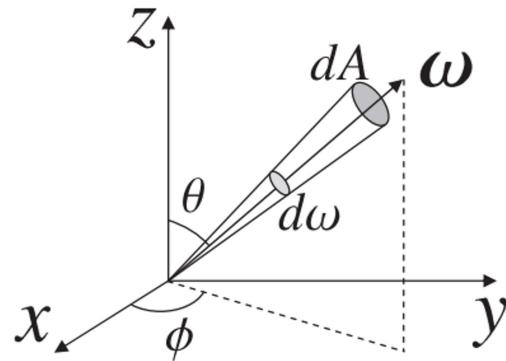
The planar angle subtended by a shape with respect to a point p is the length of the arc that results from projecting the shape onto the circumference of a unit circle centered at p :



The solid angle subtended by an object with respect to a point p is the area of the projection of the object onto the surface of a unit sphere centered at p :



In [spherical coordinates](#), the solid angle ω has components (ρ, ϕ, θ) , that is, (1, azimuth, zenith):



where dA is differential area and $d\omega$ is [differential solid angle](#). $d\omega$ can be regarded as a differential cone of direction.

5.4.1 Basic quantities

- TODO: energy, flux, intensity.

[Irradiance](#) E (arrival) or radiant exitance M (exit) measures flux $\Phi(p)$ per unit area at a point p .

Average irradiance is:

$$E = \frac{\Phi}{A}$$

Irradiance at any point on the surface of a sphere of radius r is:

$$E = \frac{\Phi}{4\pi r^2}$$

because the surface area is $A = 4\pi r^2$. That is how E is measured at a distance r from a point light on any direction.

At a point p of a surface of area A , irradiance is:

$$E(p) = \lim_{\Delta A \rightarrow 0} \frac{\Delta\Phi(p)}{\Delta A} = \frac{d\Phi(p)}{dA} \cos \theta$$

The surface we are talking about is the region of an object that is incident to the light of a light source of area A . Since the area of incidence changes with the angle of incidence θ , $E(p)$ is dependent on that angle, per Lambert's law:

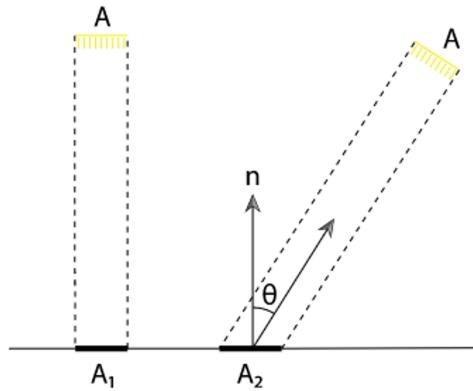
$$E = \frac{\Phi}{A} \cos \theta$$

because the area of incidence is now $\frac{A}{\cos \theta}$.

Given irradiance E over some area A , total flux is:

$$\Phi = \int_A E(p) dA$$

where $p \in A$ is a point in A .



Radiance measures flux per unit area (irradiance) per unit solid angle at a point p of a surface perpendicular to ω with area dA^\perp :

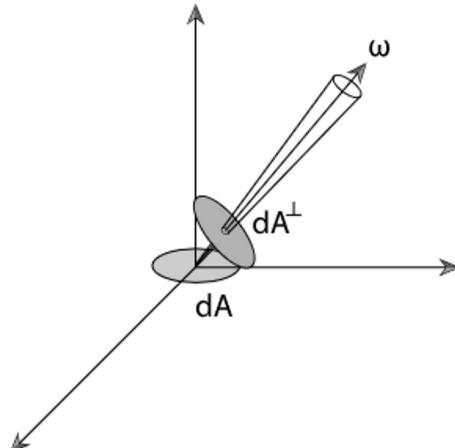
$$L(p, \omega) = \lim_{\Delta\omega \rightarrow 0} \frac{\Delta E_\omega(p)}{\Delta\omega} = \frac{dE_\omega(p)}{d\omega}$$

or equivalently:

$$L(p, \omega) = \lim_{\Delta\omega \rightarrow 0} \frac{\frac{\Delta\Phi(p)}{\Delta A}}{\Delta\omega} = \frac{d\Phi(p)}{dA^\perp d\omega}$$

where the solid angle ω is a cone of directions; when the radius of its base approaches 0, it becomes a differential solid angle.

Note that dA^\perp and dA are not the same. dA^\perp is perpendicular to ω . Also note that the $\cos \theta$ factor of E is not there either: its is 1 because $d\omega$ is the direction of incidence.



Intensity

Angular distribution of emitted power (flux), power per solid angle, directional distribution of light.

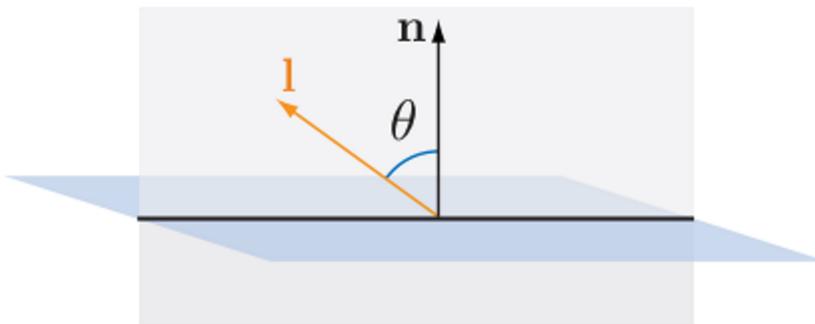
$$I = \lim_{\Delta\omega \rightarrow 0} \frac{\Delta\Phi}{\Delta\omega} = \frac{d\Phi}{d\omega}$$

$$\Phi = \int_{\Omega} I(\omega) d\omega$$

where Ω is a set of solid angles.

5.4.2 Incident and exitant radiance

A surface differential looks like this:



The radiance function L is generally not continuous across a surface's boundary. Consider the value of L at a point p in the direction ω : $L(p, \omega)$. Now consider it at a point $p_a = p + tn_p$, $t > 0$ above point p and at a point $p_b = p + tn_p$, $t < 0$ below p . In general:

$$L(p_a, \omega) \neq L(p_b, \omega)$$

L has a **jump discontinuity** at p .

~~That is not to say that $L(p, \omega)$ is undefined at p . It is defined at p . (Is this true? It looks like L is not defined at p .)~~

p , only at the one-sided limits.)

To compute $L(p, \omega)$ at p , we just need to evaluate it at p . To compute $L(p_a, \omega)$ and $L(p_b, \omega)$ for points p_a and p_b very near p , we need one-sided limits:

$$L^+(p, \omega) = \lim_{t \rightarrow 0^+} L(p + tn_p)$$

$$L^-(p, \omega) = \lim_{t \rightarrow 0^-} L(p + tn_p)$$

So,

$$L(p, \omega) = \begin{cases} L^+(p, \omega) \\ L^-(p, \omega) \end{cases}$$

We also distinguish between radiance that arrives at p (from some direction ω) and radiance that leaves p (in some direction ω):

Incident radiance:

$$L_i(p, \omega) = \begin{cases} L^+(p, -\omega), & \omega \cdot n_p > 0, \quad (\text{above } p) \\ L^-(p, -\omega), & \omega \cdot n_p < 0, \quad (\text{below } p) \end{cases}$$

Exitant radiance:

$$L_o(p, \omega) = \begin{cases} L^+(p, \omega), & \omega \cdot n_p > 0, \quad (\text{above } p) \\ L^-(p, \omega), & \omega \cdot n_p < 0, \quad (\text{below } p) \end{cases}$$

Note that the only difference is the sign of ω : the convention is that ω is the direction of exitance, and $-\omega$ is the corresponding direction of incidence.

IMPORTANT: incidence and exitance for the direction ω describes the reflection of the ray of light.

At a point in free space, L is continuous.

5.4.3 Luminance and photometry

Photometry is the study of visible electromagnetic radiation as perceived by the human eye.

Every radiometric quantity has an equivalent photometric counterpart.

Recall that X and Z in *CIE XYZ* are chromaticity and Y is luminance.

In photometry, luminance is also denoted Y , but its value is not exactly the same to *CIE Y*, although it is proportional:

$$Y = 683 \int_{\lambda} L(\lambda) Y(\lambda) d\lambda$$

where Y is photometric luminance and $Y(\lambda)$ is the *CIE Y* response curve.

These are the correspondences between radiometric and photometric quantities:

Radiometric	Unit	Photometric	Unit
Radiant energy	joule (J)	Luminous energy	talbot (T)
Radiant flux	watt (W)	Luminous flux	lumen (lm)
Intensity	W/sr	Luminous intensity	lm/sr = candela (cd)
Irradiance	W/m ²	Illuminance	lm/m ² = lux (lx)
Radiance	W/(m ² sr)	Luminance	lm/(m ² sr) = cd/m ² = nit

Integrating a radiometric quantity against the $V(\lambda)$ response curve converts it to the corresponding photometric quantity. For example, radiance to luminance:

$$Y = \int_{\lambda} L(\lambda) V(\lambda) d\lambda$$

$V(\lambda)$ describes the relative sensitivity of the human eye to various wavelengths. The brightest color the human eye can perceive is 555nm, where V has its absolute maximum and which is a bright green (no SPD with energy concentrated in some blue or red wavelengths will be perceived as brightly as one with the same amount of energy concentrated at 555nm).

5.5 Working with radiometric integrals

Spheres, cones, and cylinders have awkward descriptions in Cartesian coordinates. They lend themselves to be described in cylindrical coordinates or in spherical coordinates.

Spherical coordinates is a coordinate system in R^3 that represents a point p with 3 coordinates: (ρ, ϕ, θ) , that is, rho, phi, and theta:

- ρ is the distance from the origin to p , the length of the line OP .

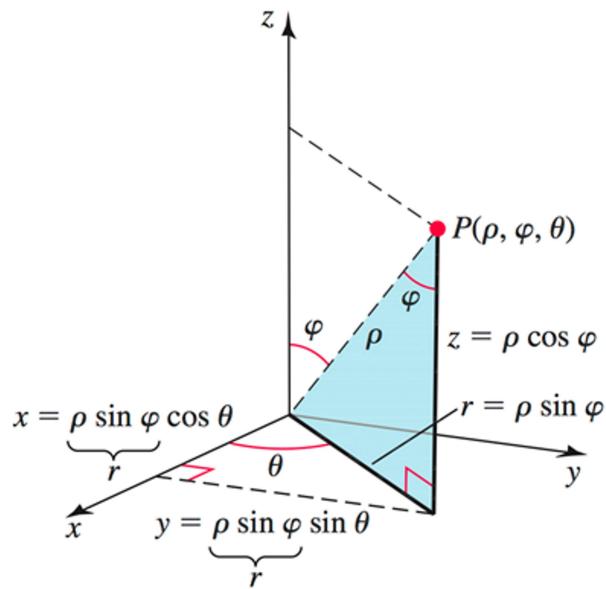
$$0 \leq \rho < \infty$$

- ϕ is the elevation angle between the positive z-axis and the line OP .

$$0 \leq \phi \leq \pi$$

- θ measures rotation about the z-axis relative to the positive x-axis. Azimuth angle.

$$0 \leq \theta \leq 2\pi$$



Transformations between spherical and rectangular (Cartesian) coordinates:

Rectangular → Spherical

$$\rho^2 = x^2 + y^2 + z^2$$

Use trigonometry to find φ and θ

Spherical → Rectangular

$$x = \rho \sin \varphi \cos \theta$$

$$y = \rho \sin \varphi \sin \theta$$

$$z = \rho \cos \varphi$$

- TODO: until after studying double integrals and spherical coordinates.

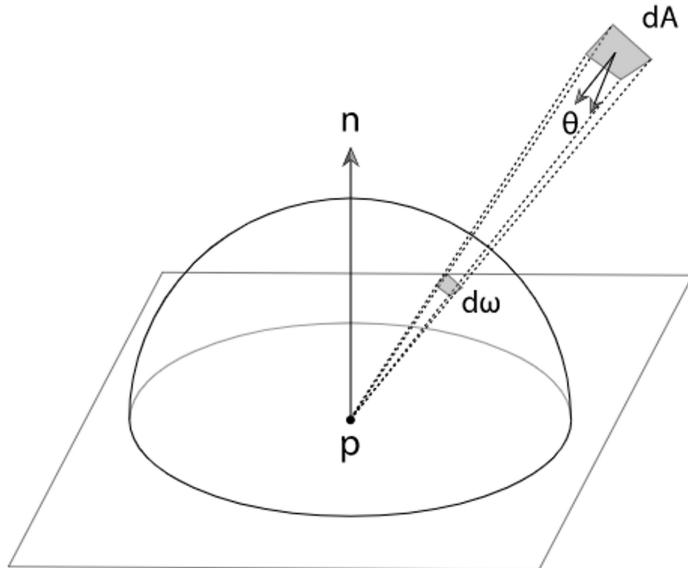
5.5.3 Integrals over area

Transform integrals over directions into integrals over area.

(It helps here to think of the area differential as one on the surface of an area light.) Differential area is related to differential solid angle (as viewed from a point p) by:

$$d\omega = \frac{dA \cos \theta}{r^2}$$

where θ is the angle between the normal of the area differential dA (of the light source) and the direction from the area differential to the point p on the other surface, and r^2 is the distance between dA and p :



We say that dA subtends the differential solid angle $d\omega$.

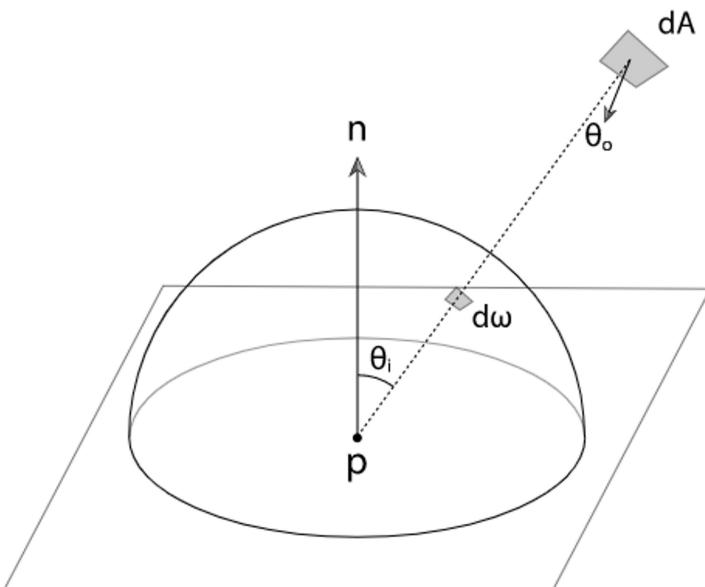
The irradiance equation that integrates solid angle:

$$E(p, n) = \int_{\Omega} L_i(p, \omega) |\cos \theta| d\omega$$

can be rewritten in terms of area using the relation between $d\omega$ and dA described above:

$$E(p, n) = \int_A L \cos \theta_i \frac{\cos \theta_o}{r^2} dA$$

Note that L is no longer a function of solid angle, but the (total?) emitted radiance from the light source. θ_i and θ_o are understood as follows:



5.6 Surface reflection

Light reflection is described by 2 abstractions: the **BRDF** and the **BSSRDF**.

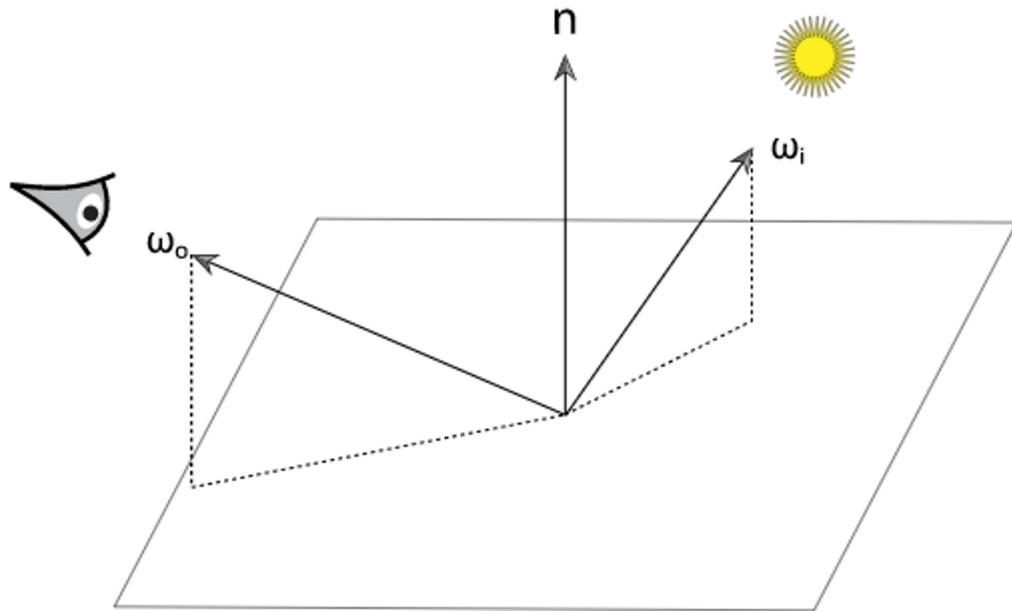
The BSSRDF is a generalization of the BRDF that also models **subsurface light transport**: light that enters a surface at one point exits it some distance away. Most materials exhibit subsurface light transport to varying degrees. For a great many materials, subsurface light transport has such a **minimal effect** on how light reflects off of the surface, that it's better to **ignore it**. The BRDF does just that.

What the BRDF and BSSRDF model are **spectral distribution** of the reflected light (i.e. radiance? at each wavelength) and its **directional distribution**. The SPD of incident light is not the same as the SPD of reflected light: the radiance at some wavelengths is absorbed before light gets reflected, changing its spectral distribution; the features of the surface also change its direction.

5.6.1 The BRDF

The **BRDF** (bidirectional reflectance distribution function) formally describes reflection from a surface.

Given **incident radiance** $L_i(p, \omega_i)$ arriving from ω_i at p , the BRDF **helps** to compute **exitant radiance** $L_o(p, \omega_o)$ leaving p in the direction ω_o of the viewer. (The BRDF does not give $L_o(p, \omega_o)$, but a **ratio** of differential exitance to differential irradiance, as explained later.)



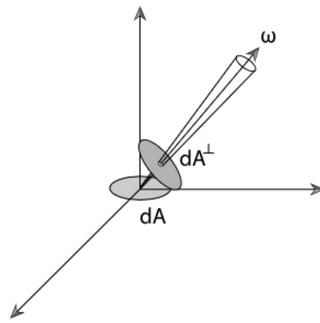
Recall that solid angle ω_i is a cone of directions.

Now, take incident radiance $L_i(p, \omega_i)$ and solve for differential irradiance:

$$L_i(p, \omega_i) = \frac{dE(p, \omega_i)}{d\omega_i}$$

$$dE(p, \omega_i) = L_i(p, \omega_i) d\omega_i$$

This is irradiance at a point on a plane perpendicular to ω_i (where dA^\perp lies in this picture):



That's because that's where radiance is measured. But we are interested in the surface (where dA lies in the picture), so we multiply by the cosine of the angle of incidence θ :

$$dE(p, \omega_i) = L_i(p, \omega_i) d\omega_i \cos \theta_i$$

Geometric optics establishes a linear proportional relation between incident irradiance and exitant radiance:

$$dL_o(p, \omega_o) \propto dE(p, \omega_i)$$

For a pair of incident and exitant directions (ω_i, ω_o) and a point, the constant of proportionality is given by the BRDF function:

$$f_r(p, \omega_o, \omega_i) = \frac{dL_o(p, \omega_o)}{dE(p, \omega_i)} = \frac{dL_o(p, \omega_o)}{L_i(p, \omega_i) \cos \theta_i d\omega_i}$$

Solving for differential exitant radiance:

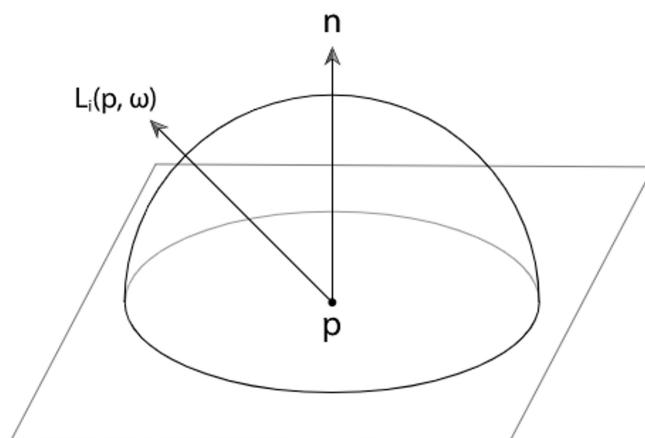
$$dL_o(p, \omega_o) = f_r(p, \omega_o, \omega_i) L_i(p, \omega_i) |\cos \theta_i| d\omega_i$$

(See the book for why the absolute value of the cosine is taken.)

That was differential exitant radiance corresponding to incident direction ω_i . But the point p receives incident radiance from all incident directions, so the total exitant radiance leaving p in the direction ω_o is:

$$L_o(p, \omega_o) = \int_{\Omega=H^2(n)} f_r(p, \omega_o, \omega_i) L_i(p, \omega_i) |\cos \theta_i| d\omega_i$$

where Ω is a set of directions and $H^2(n)$ denotes the hemisphere centered at p on the side of the surface where the normal points. That is the scattering equation.



(From section 8.1.1) **Reflectance** is the spectral distribution that describes the reflective or scattering behavior of the surface at a point. Reflectance alone does not compute radiance, but it dictates the spectral distribution of it.

(Confirmation from *Real-Time Rendering, section 9.3: the amount of light reflected can vary based on wavelength, which can be modeled in one of two ways: either the wavelength is treated as an additional input variable to the BRDF, or the BRDF is treated as returning a spectrally distributed value.*)

(See [energy conservation property](#) for additional confirmation that the reflectance function is a distribution function, decoupled from radiance.)

If the $L_i(p, \omega_i)$ factor is removed from the definition of $L_o(p, \omega_o)$, we get the **hemispherical-directional reflectance** equation:

$$\rho_{hd}(\omega_o) = \int_{\Omega=H^2(\mathbf{n}): \{\omega_i\}} f_r(p, \omega_o, \omega_i) |\cos \theta_i| d\omega_i$$

where Ω is the set of directions in the hemisphere above the surface and centered at p . (In *Real-Time Rendering, section 9.3*, $\cos \theta_i$ is the dot product of the normal and the incident direction vector, both normalized.)

Note the name **hemispherical-directional**, which can be read as "entire hemisphere to single direction": it indicates that it integrates over the entire *hemisphere* of incident directions to compute the $0 \leq \rho(\omega_o) \leq 1$ fraction of total incident light that is reflected in that particular outgoing *direction* ω_o (distributed also across the wavelength spectrum).

By plugging $L_i(p, \omega_i)$ back in, we get to understand that $L_o(p, \omega_o)$ is determined by that reflective behahior.

(From *Real-Time Rendering, section 9.3.*) The unit hemisphere $\Omega = H^2(\mathbf{n})$ is parameterized using spherical coordinates θ (elevation) and ϕ (azimuth). In spherical coordinates, the differential solid angle $d\omega$ is:

$$d\omega = \sin \theta \, d\theta \, d\phi$$

So, exitant radiance becomes:

$$L_o(p, \omega_o) = \int_{\phi_i=0}^{2\pi} \int_{\theta_i=0}^{\pi/2} f_r(p, \theta_o, \phi_o, \theta_i, \phi_i) |\cos \theta_i| \sin \theta_i \, d\theta_i \, d\phi_i$$

Note that the BRDF function f_r receives ω_o and ω_i as pairs of spherical angles.

The **directional-hemispherical reflectance** equation is (see *Real-Time Rendering, section 9.3*):

$$\rho_{dh}(\omega_i) = \int_{\Omega=H^2(\mathbf{n}): \{\omega_o\}} f_r(p, \omega_i, \omega_o) |\cos \theta_o| \, d\omega_o$$

Note the name **directional-hemispherical**, which can be read as "single direction to entire hemisphere": it indicates that the fraction of the total energy coming from a given incident *direction* that is reflected over the entire *hemisphere* of outgoing directions is $\rho_{dh}(\omega_i)$.

Directional-hemispherical reflectance, being a fraction of incident energy, can be used to measure energy

loss (by absorption, etc.).

- TODO: hemispherical-hemispherical reflectance.

BRDFs have 2 important **properties**:

Helmholtz reciprocity, for all pairs of directions ω_i and ω_o , $f_r(p, \omega_o, \omega_i) = f_r(p, \omega_i, \omega_o)$. Since the BRDF is reciprocal, hemispherical-directional reflectance and directional-hemispherical reflectance are equal.

Due to reciprocity, $\rho_{hd}(\omega_o) = \rho_{dh}(\omega_i)$, for every pair of directions.

And **energy conservation**:

$$\int_{\Omega=H^2(\mathbf{n})} f_r(p, \omega_o, \omega_i) |\cos \theta_i| d\omega_i \leq 1$$

1 is the totality of incident energy from all directions. A fraction of this energy comes in each differential incident direction $d\omega_i$; the sum is 1. A fraction of the energy is absorbed at the point p . The remaining energy (≤ 1) leaves p , a fraction of which leaves in each differential outgoing direction $d\omega_o$ according to the distribution function.

5.6.2 The BSSRDF

- TODO