

# PROBLEM SET III: APPROXIMATING SUMS

*October 20, 2025*

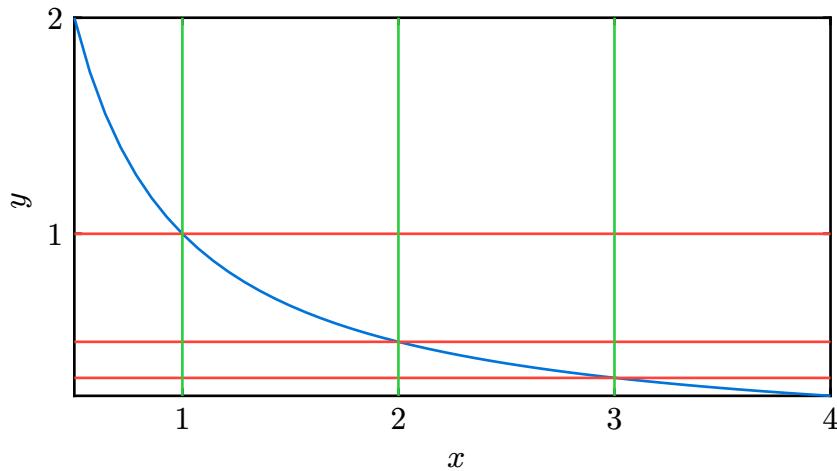
Very often we may find it very useful to find a closed form expression for a sequence of the form

$$s_n = \sum_{k=1}^n f(k)$$

Where  $f : \mathbb{N} \rightarrow \mathbb{R}^+$  happens to be a monotonically decreasing function, and always positive.

One such example is the sequence of harmonic numbers  $H_n := \sum_{k=1}^n \frac{1}{k}$ , but there are many. Mathematicians have tried to find closed form expressions for some of these sequences without any success.

But as it turns out, most of the time we care about such a sequence, we actually do not care about its *precise* value, and just about its *asymptotic behavior*: in computer science we don't care that much about  $s_n$  and more about  $\Theta(s_n)$ , and even when solving problems in mathematics, often it happens that as long as we are able to find good lower and upper bounds for  $s_n$ , the proofs work (and the exact value is of no relevance if what we care about is studying the convergence of the sequence).



The method outlined last week in class helps with this. To recap: suppose we want to study such a sequence  $s_n := \sum_{k=1}^n f(k)$ , with  $f$  monotonically decreasing, and *it so happens that we know the antiderivative  $F$  of  $f$* . As the diagram above suggests, we can relate the sum with an integral, by noting that for each  $n \in \mathbb{N}$ :

$$f(n+1) \leq \int_n^{n+1} f(t) dt \leq f(n) \quad (1)$$

This holds because  $f$  is decreasing and positive, and  $f(n+1)$  is the area of the lower rectangle of width 1, while  $f(n)$  is the area of the rectangle of width 1.

We can now add (1) for each  $k = 1, \dots, n$  to obtain:

$$f(2) + \dots + f(n+1) \leq \int_1^{n+1} f(t)dt \leq f(1) + \dots + f(n) \quad (2)$$

If we know that the antiderivative of  $f$  is  $F$ , we can apply the Fundamental Theorem of Calculus to obtain

$$s_n + f(n+1) - f(1) \leq F(n+1) - F(1) \leq s_n \quad (3)$$

The left inequality gives us  $s_n \leq F(n+1) - F(1) + f(1) - f(n+1)$ . Using this and the right one, we get the following bound for  $s_n$ :

$$F(n+1) - F(1) \leq s_n \leq F(n+1) - F(1) + (f(1) - f(n+1)) \quad (4)$$

**INSTRUCTIONS:** No need to solve them fully, just ponder them. We'll probably go over the solutions of a few of them during class. Have fun!

### Problem 1. (Asymptotics for free)

Recalling that  $f$  is decreasing, and under the reasonable assumption that  $f$  is positive, transform (4) into a slightly looser, but easier to handle, inequality. What does this say about  $\Theta(s_n)$ ? Why is this such a good bound?

### Problem 2. (Harmonic numbers)

Recall that an antiderivative of  $\frac{1}{x}$  is  $\ln x$ . Use the method to obtain a bound for the harmonic numbers  $H_n := \sum_{k=1}^n \frac{1}{k}$ . How far away does it get from the true value of  $H_n$ ? What does this tell us about the convergence or divergence of the harmonic series (i.e.  $\lim_{n \rightarrow \infty} H_n$ )?

### Problem 3. (The Riemann Zeta Function)

Prove that the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^t} \quad (5)$$

converges for all real  $t > 1$ .

### Problem 4. (Prime numbers form a large set)

Let  $p_n$  be the  $n$ th prime number. An equivalent formulation of the Prime Number Theorem tells us that  $p_n \sim n \ln n$ , that is,  $\lim_{n \rightarrow \infty} \frac{p_n}{n \ln n} = 1$ .

Use this to find a bound for  $p_n$  that holds eventually: that is, for some  $N$ , we have that for all  $n > N$  this bound holds. Then, use this bound to prove that the infinite

series  $\sum_{k=1}^{\infty} \frac{1}{p_k}$  diverges. This means that, not only are there infinitely many primes, there are enough primes that the sum of their reciprocals diverges!