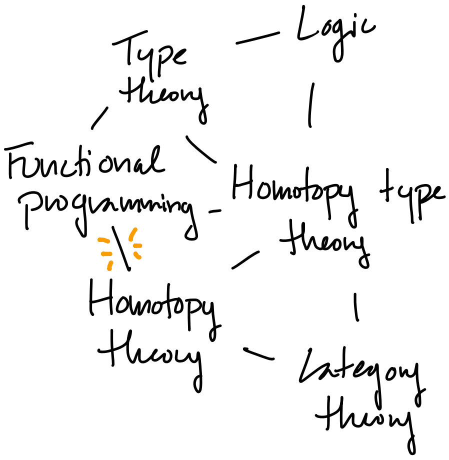
Map:



Logic (natural deduction)

(see *Logic and Proof* by Avigad et al.)

In natural deduction, we prove statements about propositions using <u>proof trees</u> built out of <u>rules</u>.

<u>Ex</u>. We can prove that Q follows from $P \wedge (P \to Q)$ for any propositions P, Q.

$$\dfrac{\dfrac{P \wedge (P \to Q)}{P}\ \wedge\text{-elim-}\ell \qquad \dfrac{P \wedge (P \to Q)}{P \to Q}\ \wedge\text{-elim-r}}{Q}\ \to\text{-elim}$$

Rules for $\wedge$:

$\wedge$-intro: $\dfrac{P \quad Q}{P \wedge Q}$

$\wedge$-elim-l: $\dfrac{P \wedge Q}{P}$ $\qquad$ $\wedge$-elim-r: $\dfrac{P \wedge Q}{Q}$

Rules for $\to$:

$\to$-intro:

$$\dfrac{\begin{array}{c} \overline{P}^{\,\prime} \\ \vdots \\ Q \end{array}}{P \to Q}\,{}^{\prime}$$

$\to$-elim: $\dfrac{P \quad P \to Q}{Q}$

<u>Introduction rules</u> tell you how to prove something.
<u>Elimination rules</u> tell you how to use something.

<u>Ex</u>. $(P \wedge Q) \to P$

$$\dfrac{\dfrac{\overline{P \wedge Q}^{\,\prime}}{P}\ \wedge\text{-elim-l}}{(P \wedge Q) \to P}\,{}^{\prime}\ \to\text{-intro}$$

Simply-typed λ calculus

(Proofs and types by Girard)

If we make natural deduction proof relevant, we get the STΛC.

In ND, we write "P" to mean "P holds".
In STΛC, we write "$p:P$" to mean "p is a proof/witness of P" or "p holds/is inhabited by P".
We call p a proof/witness/term/element of P.
We call P a proposition or type.

Ex. Q follows from $P \wedge (P \to Q)$

$$\dfrac{\land\text{-elim-}\ell\ \dfrac{a : P \wedge (P \to Q)}{pr_1 a : P} \qquad \dfrac{a : P \wedge (P \to Q)}{pr_2 a : P \to Q}\ \land\text{-elim-r}}{(pr_1 a)(pr_2 a) : Q}\ \to\text{-elim}$$

Rules for $\wedge$:

$\wedge$-form: $\dfrac{P \text{ type} \quad Q \text{ type}}{P \wedge Q \text{ type}}$

$\wedge$-intro: $\dfrac{\Gamma \vdash p : P \quad \Gamma \vdash q : Q}{\Gamma \vdash (p,q) : P \wedge Q}$

$\wedge$-elim-ℓ: $\dfrac{\Gamma \vdash a : P \wedge Q}{\Gamma \vdash pr_1 a : P}$ $\qquad$ $\wedge$-elim-r: $\dfrac{\Gamma \vdash a : P \wedge Q}{\Gamma \vdash pr_2 a : Q}$

$\wedge$-comp-β-ℓ: $\dfrac{\Gamma \vdash p : P \quad \Gamma \vdash q : Q}{\Gamma \vdash pr_1 (p,q) \doteq p : P}$ $\qquad$ $\wedge$-comp-β-r: $\dfrac{\Gamma \vdash p : P \quad \Gamma \vdash q : Q}{\Gamma \vdash pr_2 (p,q) \doteq q : Q}$

$\wedge$-comp-η : $\dfrac{\Gamma \vdash a : P \wedge Q}{\Gamma \vdash (pr_1 a, pr_2 a) \doteq a : P \wedge Q}$

Notice: If we think of types as sets and terms as elements, then $P \wedge Q$ behaves like the product $P \times Q$ of sets.

<u>Thm</u> (Lambek 1985). There is an interpretation of the STΛC into Set, the category of sets. (Actually there is an equivalence between STΛC and CCC.)

Rules for $\to$:

$\to$-form : $\dfrac{\Gamma \vdash P \text{ type} \quad \Gamma \vdash Q \text{ type}}{P \to Q \text{ type}}$

$\to$-intro : $\dfrac{\begin{matrix} \overline{P}^{\,1} \\ \vdots \\ Q \end{matrix}}{P \to Q}\,{}^{1}$ $\left.\right\}$ a proof of Q from P $\rightsquigarrow$ $\dfrac{\Gamma, x : P \vdash q : Q}{\Gamma \vdash \lambda x . q : P \to Q}$

$\to$-elim : $\dfrac{\Gamma \vdash p : P \quad f : P \to Q}{\Gamma \vdash fp : Q}$

...

Ex. How do we prove $P \wedge Q \to P$?

$$\dfrac{\dfrac{\dfrac{}{a : P\wedge Q \vdash a : P \wedge Q}}{a : P \wedge Q \vdash pr_1 a : P}\ \wedge\text{-elim-}\ell}{\lambda a . pr_1 a : (P \wedge Q) \to P}\ \to\text{-intro}$$

generic element / projection / variable (see Riehl)

→ add contexts everywhere

So an expression like $a : P\wedge Q \vdash pr_1 a : P$ corresponds to a proof tree with a hypothesis that $P \wedge Q$ holds and which concludes that P holds. The term $pr_1 a$ records the shape of the proof tree.

Ex. $x : P \wedge Q \vdash (pr_2 x, pr_1 x) : Q \wedge P$

corresponds to

$$\dfrac{\wedge\text{-elim-}r\ \dfrac{P \wedge Q}{Q} \qquad \dfrac{P\wedge Q}{P}\ \wedge\text{-elim-}\ell}{Q \wedge P}\ \wedge\text{-intro}$$

<u>Thm</u> (Howard 1969) (Falls under the umbrella of the 'Curry-Howard correspondence')
The proof trees of natural deduction are in 1-to-1 correspondence with terms of STλC.

Computation rules for $\to$.

$\to$-comp-β:
$$\frac{\Gamma \vdash x : P \vdash q : Q \qquad \Gamma \vdash p : P}{\Gamma \vdash (\lambda x.\, q)\, p \doteq q[p/x] : Q}$$
substitution

$\to$-comp-η:
$$\frac{\Gamma \vdash f : P \to Q}{\Gamma \vdash \lambda x.\, f x \doteq f : P \to Q}$$

Under the Howard (logical) interpretation, $\to$ corresponds to implication.
Under the Lambek (set) interpretation, $\to$ corresponds to functions.

We can also interpret types as program specifications
- ex. A type $P \to P$ specifies a program that takes a term of type P as an input and returns a term of type P as an output.

and terms as programs meeting the specification
- ex. We can construct the identity $id_P : P \to P$.

<u>Dependent type theory</u>
- In natural deduction, we have no terms.
- In STλC, terms can depend on terms.
  - ex. $a : P \wedge Q \vdash pr_1\, a : P$
- In dependent type theory, not only terms but <u>types</u> can depend on terms.

- ex. $n:\mathbb{N} \vdash \mathrm{Vect}(n)$ type

  $n:\mathbb{N} \vdash \mathrm{isEven}(n)$ type

If we interpret types as

- propositions: dependent types are predicates
- sets: dependent types are indexed families of sets
- programs: dependent types are program specifications with a parameter

We have the same rules as before, except the formation rules can also have a context.

$\wedge$-form: $\dfrac{\Gamma \vdash P \text{ type} \quad \Gamma \vdash Q \text{ type}}{\Gamma \vdash P \wedge Q \text{ type}}$ $\quad\rightarrow$-form: $\dfrac{\Gamma \vdash P \text{ type} \quad \Gamma \vdash Q \text{ type}}{\Gamma \vdash P \rightarrow Q \text{ type}}$

Ex. $$\dfrac{n:\mathbb{N} \vdash \mathrm{isEven}(n) \qquad n:\mathbb{N} \vdash \mathrm{isDivFour}(n)}{n:\mathbb{N} \vdash \mathrm{isEven}(n) \wedge \mathrm{isDivFour}(n)}$$

$$n:\mathbb{N} \vdash \mathrm{isDivFour}(n) \rightarrow \mathrm{isEven}(n)$$

Dependent functions

Ex. (informal) Let Vect be the set of all vectors (of any length) (i.e., finite lists) in $\mathbb{N}$.

Define $0:\mathbb{N} \rightarrow \mathrm{Vect}$ which takes n to the vector of length n whose components are all 0.

<u>But</u> $O(n)$ actually lives in $\mathrm{Vect}(n)$, the set of vectors of length n. We can encode this by considering O as a <u>dependent function</u>

$$O : \prod_{n:\mathbb{N}} \mathrm{Vect}(n) \quad (\text{sometimes written } O : (n:\mathbb{N}) \to \mathrm{Vect}(n))$$

The elimination rule gives us $O(n) : \mathrm{Vect}(n)$ for any $n:\mathbb{N}$.

<u>Ex.</u> Consider

$$n:\mathbb{N} \vdash \mathrm{isDivFour}(n) \to \mathrm{isEven}(n).$$

To show this for all n, we construct a term

$$n:\mathbb{N} \vdash t(n) : \mathrm{isDivFour}(n) \to \mathrm{isEven}(n)$$

The introduction rule for dependent functions gives us

$$\vdash \lambda n.t(n) : \prod_{n:\mathbb{N}} \mathrm{isDivFour}(n) \to \mathrm{isEven}(n).$$

In the logical interpretation, we interpret Π as $\forall$.

<u>Rem.</u>
- $\to$ is a special case of Π.

  ex. $\prod_{n:\mathbb{N}} \mathrm{Vect}$ is the same as $\mathbb{N} \to \mathrm{Vect}$ (the rules become the same)

- $\wedge$ is a special case of Π, if we have $\mathbb{B}$ (the type with 2 elements)

  <u>ex.</u> $\mathbb{B} \to \mathrm{Vect}$ (i.e. $\prod_{b:\mathbb{B}} \mathrm{Vect}$) is the same as $\mathrm{Vect} \wedge \mathrm{Vect}$.

  $\prod_{b:\mathbb{B}} \mathrm{Vect}_b$ is the same as $\mathrm{Vect}_0 \wedge \mathrm{Vect}_1$.

Rules for Π-types.

Π-form:
$$\frac{\Gamma, x:P \vdash Q \text{ type}}{\Gamma \vdash \prod_{x:P} Q \text{ type}}$$

Π-intro:
$$\frac{\Gamma, x:P \vdash q : Q}{\Gamma \vdash \lambda x.q : \prod_{x:P} Q}$$

Π-elim:
$$\frac{\Gamma \vdash f : \prod_{x:P} Q \qquad \Gamma \vdash p : P}{\Gamma \vdash fp : Q[p/x]}$$

Π-comp-β:
$$\frac{\Gamma, x:P \vdash q : Q \qquad \Gamma \vdash p : P}{\Gamma \vdash (\lambda x.q)p \doteq q[p/x] : Q[p/x]}$$

Π-comp-η:
$$\frac{\Gamma \vdash f : \prod_{x:P} Q}{\Gamma \vdash \lambda x.fx \doteq f : \prod_{x:P} Q}$$