Univalence for logic and sets

Def. Given a type T and a predicate $P : T \to \mathrm{Prop}$, the <u>subtype</u> of T given by P is
$$\sum_{t:T} P(t).$$
The projection
$$\pi_1 : \sum_{t:T} P(t) \longrightarrow T$$
gives us the 'inclusion'.

Ex. isContr, isProp, is of hlevel n are all predicates $\mathcal{U} \to \mathrm{Prop}$.

Def. $\mathrm{Prop} := \sum_{P:\mathrm{Type}} \mathrm{isProp}(P)$

Ex. The univalence axiom implies
$$(P \underset{\mathrm{Prop}}{=} Q) \simeq (P \leftrightarrow Q).$$

Lem. $(P \leftrightarrow Q)$ is a proposition.

Cor. Prop is a set.

Def. $\mathrm{Set} := \sum_{S:\mathrm{Type}} \mathrm{isSet}(S)$

Fact. The univalence axiom implies
$$(P \underset{\mathrm{Set}}{=} Q) \simeq (P \cong Q).$$

Lem. $P \simeq Q$ is a set.

Cor. Set is a groupoid.

Def.
$$\mathrm{Grp} := \sum_{G:\mathrm{Set}} \sum_{e:G} \sum_{m:G\to G\to G} \sum_{i:G\to G} \prod_{x:G} (m(e,x)=x)\times(m(x,e)=x) \times \prod_{x,y,z:G} ((xy)z = x(yz)) \times \prod_{x:G} (m(ix,x)=e) \times (m(x,ix)=e).$$

Q. Why do we ask G to be a set?

Fact. The univalence axiom implies

$$(G =_{\mathrm{Grp}} H) \simeq (G \cong H)$$

Cor. Grp is a groupoid.

Fact. We have the same univalence principle for any algebraic structure on a set.

Moral: univalence allows us to do mathematics <u>up to the appropriate notion of</u> sameness in a type (in these examples).

→ 'Structure identity principle' (Aczel, Coquand)

→ 'identity of indiscernables' (Leibniz)

Higher inductive types.

Homotopy type theory = MLTT + UA + higher inductive types

Recall: inductive types are generated by their constructors (terms) / canonical terms

Since we now consider types as having

- terms
- equalities
- equalities between equalities

We can consider *higher* inductive types, whose constructors can be terms, equalities, equalities between equalities, etc.

Ex. $S^0 := \text{bool}$ has constructors

- $\text{true} : \text{bool}$
- $\text{false} : \text{bool}$

Def. D^1 (the interval) has constructors

- $\text{true} : D^1$
- $\text{false} : D^1$
- $p : \text{true} =_{D^1} \text{false}$

We could define D^1 with four rules:

$$\frac{}{D' \text{ type}} \quad D'\text{-form}$$

$$\frac{}{\text{true} : D'} \qquad \frac{}{\text{false} : D'} \qquad \frac{}{p : \text{true} =_{D'} \text{false}} \qquad D'\text{-intro}$$

$$\frac{\begin{array}{l} d : D' \vdash E(d) \text{ Type} \\ \quad \vdash t : E(\text{true}) \\ \quad \vdash f : E(\text{false}) \\ \quad \vdash \pi : p_* t = f : E(\text{false}) \end{array}}{d : D' \vdash \text{ind}_{D',t,f,\pi}(d) : E(d)} \quad \} \; D'\text{-elim}$$

$$\left.\begin{array}{l} \vdash \text{ind}_{D',t,f,\pi}(\text{true}) \doteq t : E(\text{true}) \\ \vdash \text{ind}_{D',t,f,\pi}(\text{false}) \doteq f : E(\text{false}) \\ \vdash \text{ind}_{D',t,f,\pi}(p) \doteq \pi : p_* t = f : E(\text{false}) \end{array}\right\} \; D'\text{-comp}$$

<u>Def</u>. S^1 (the circle) has constructors
- $\text{base} : S^1$
- $\text{loop} : \text{base} = \text{base}$

<u>Thm</u>. $\pi_1(S^1) = \mathbb{Z}$ where

$\pi_1 : T \to \text{Set}$ is defined as

$S^1 \to T \; \dots \; ?$

- We want to make this a set.
- We also want to make thing into propositions.

<u>Ex</u>. Given $P, Q : Prop$

$P + Q$ is not a proposition in general

<u>Thm</u>. • If $P, Q : Prop$, $P \times Q : Prop$ (write $P \wedge Q$)

• If $P : Prop$, $E : P \to Prop$,

$$\sum_{p:P} e(p) \quad \prod_{p:P} e(p) \quad \left(\text{write } \exists_{p:P} e(p), \ \forall_{p:P} e(p)\right)$$

are in Prop.

<u>Def</u>. Given a type T, the <u>propositional truncation $\|T\|_{1}$ of T</u> is the higher inductive type with constructors

- $|-|_1 : T \to \|T\|_1$
- $\prod_{x,y:T} |x|_1 =_{\|T\|_1} |y|_1$

<u>Ex</u>. Show that for any type T, $\|T\|_1$ is a proposition.

<u>Def</u>. Define $P \vee Q := \|P + Q\|_1$ for $P, Q : Prop$.

<u>Ex</u>. Functions $(T \to P) \simeq (\|T\|_1 \to P)$ for any type T, proposition P.

Def. Given a type T, the set truncation $\|T\|_2$ of T is the higher inductive type with constructor

- $|-|_2 : T \to \|T\|_2$
- $\prod_{\substack{x,y:T \\ p,q:x=y}} |p|_2 = |q|_2$.

Ex. Show that for any type T, $\|T\|_2$ is a set.

Def. $\pi_1 : U \to \mathrm{Set}$

$:= \lambda T.\ \|S^1 \to T\|_2$.

Thm. $\pi_1(S^1) = \mathbb{Z}$.