Categorical independence and non-commutative distributions for simplicial complexes

September 12, 2019

Contents

Abstract

Graphs provide basic and illustrative examples of *non-commutative* random variables via their adjacency matrices. The main objective of this thesis is to discuss the general question of assigning distributions to graphs and simplicial complexes to consider them as random variables in some probability space.
We split the problem in three parts. First, we need to decide which matrices associated with simplicial complexes are we considering as random variables. Second, we must specify a expectation and/or state to endow our matrices with distributions. Third, we may also want to consider collections of *conditional expectations*, compatible with the expectation chosen, so that we may relate the different distributions for different choices of matrices. Surprisingly, if we restrict the third question, allowing only the simplest types of operator-valued spaces, i.e. projetion-valued spaces, then the joint algebraic distribution of incidence and boundary matrices and their duals contain the distributions of most matrices involved when studying simplicial complexes. This includes adjacency matricesand combinatorial Laplace operators, of all dimensions.
In addition, the simplicity of the *-algebra generated by the boundary matrix and leaves little room for finding self-adjoint operators. Two of the few canonical choices are naturally endowed with *analytic distributions* that encode the *Betti numbers* of the considered simplicial complex.

It is to remark that a notion of independence is required for the task. In our case, the concept of categorical independence studied was developed in [26], where it is intended to be the link between all the independence notions in *quantum* probability spaces. For this

work, the categorical indepence of orthogonal spaces is a valuable fact because it enables the study through orthogonal pieces independently.
A second objective, is to define a *Boolean cumulant* type on higher dimensions, i.e. simplicial complexes. We hope this concept will lead to *simpler* examples on aspects of *non-commutative* probability, as the graphs gave examples of random variables and realizations of independence concepts in *non-commutative* probability.

Carlos Eduardo Díaz Aguilera

Introduction

Chapter 1

Preliminaries

The objective of this chapter is to set the background where the work lies in concepts and results. We are interested in using the language of category theory, focusing on some special categories. Homology and cohomology theories are subjects of interest as well because they provide tools to conceive a topological space as a random variable in different non-commutative probability spaces, with distributions encoding homological invariants.

1.1 Category theory elements

We first introduce some category theory elements that we will use throughout the work to understand the non-commutative notions of stochastic independence. We do not give major details on proofs in this chapter because these not required for the goals in this work. We use [22, 31] as references in this section.

Definition 1.1.1. A category $\mathcal{C}$ consists of a class of *objects*, denoted by $\mathrm{Obj}(\mathcal{C})$, and for each two objects X, Y a set of *morphisms*, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, or simply $\mathrm{Hom}(X, Y)$ when the context is clear, that satisfy:

i. For every object X there is an identity morphism $1_X \in \mathrm{Hom}(X, X)$.

ii. For each three objects X, Y, Z if $f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Y, Z)$ then $g \circ f \in$

$\mathrm{Hom}(X, Z)$, this composition is associative and the identity morphisms act neutrally under this operation.

We will write $X \in \mathcal{C}$ to say that X is an object of $\mathcal{C}$ and $f\colon X \to Y$ for a morphism $f \in \mathrm{Hom}(X, Y)$. A morphism $f\colon X \to Y$ is called a *monomorphism* if it is left cancellative, i.e., if $f \circ g_1 = f \circ g_2$ then $g_1 = g_2$. It is called an *epimorphism* if it is right cancellative, that is if $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$. It is called an *isomorphism* if it is invertible, that is, if there exists a (necessarily unique) morphism $f^{-1}\colon Y \to X$ such that $f \circ f^{-1} = 1_Y$ and $f^{-1} \circ f = 1_X$. In this case we say that X is *isomorphic* to Y, denoted by $X \cong Y$ and we write $f\colon X \cong Y$ to indicate that f is an isomorphism.
In a category an object X is said to be *initial* if for every object Y the set $\mathrm{Hom}(X, Y)$ is a singleton, i.e. cointains precisely one morphism, and *terminal* if for every object Z the set $\mathrm{Hom}(Z, X)$ is a singleton. An object that is both initial and terminal is said to be a *zero object*.
If a category has a zero object then this object is unique, in the sense that if 0 and $0'$ are zero objects then they are isomorphic. Indeed, let $\mathrm{Hom}(0, 0') = \{f\}$ and $\mathrm{Hom}(0', 0) = \{g\}$, then $g \circ f = 1_{0'}$ and $f \circ g = 1_0$. We define the $0_{X,Y}$ morphism for objects X, Y in a category with a zero object as the morphism defined by the composition $X \to 0 \to Y$.
A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is a category whose objects and morphisms are respectively objects and morphisms of $\mathcal{C}$ and the identity morphisms and composition maps agree. A subcategory is called *full* if every morphism of $\mathcal{C}$ between objects of $\mathcal{D}$ is also a morphism of $\mathcal{D}$. It is clear that for every subclass of objects there is a unique full subcategory with exactly those objects.

Example 1.1.2. Now we provide some examples of categories and subcategories by giving the objects and morphisms that define each of them.

1. The category **Set** whose objects are all sets and whose morphisms between two sets X and Y are the functions $f\colon X \to Y$. The composition is the usual map composition and the identity map on a set X is the identity morphism 1_X. In this category the empty set is the only initial object and the singleton sets are the only terminal objects.

2. The category **Gr** of groups as objects and group homomorphisms as morphisms. A subcategory of **Gr** is the category **Ab** whose objects are the abelian groups and their respective morphisms.

3. The category **Top** whose objects are all topological spaces and the morphisms are the continuous maps.

4. The category **Vect** of (complex) vector spaces as objects and morphisms given by all the linear maps. A subcategory of **Vect** is $\mathbf{Vect}^{inj}$ with the same objects, but only injective linear transformations as morphisms. In the category **Vect**, the 0 vector space is the zero object where as in $\mathbf{Vect}^{inj}$ it is only an initial object.

5. The category **Hilb** has as objects all Hilbert spaces and as morphisms all bounded linear maps. In this category one subcategory is $\mathbf{Hilb}^{isom}$ with the same objects, but only isometries as morphisms.

Given two categories $\mathcal{C}$ and $\mathcal{D}$. A *functor* $\mathcal{F}$ is a rule that assings to each object $X \in \mathcal{C}$ and each morphism $f\colon X \to Y$ respectively, an object $\mathcal{F}(X) \in \mathcal{D}$ and a morphism $\mathcal{F}(f)$ in $\mathcal{D}$ that preserves identity morphisms ($\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ for all $X \in \mathcal{C}$) and agrees with the composition maps in both categories, in one of the following ways:
Covariant, if $\mathcal{F}(f)\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ and $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for all $f\colon X \to Y$, $g\colon Y \to Z$ with $X, Y, Z \in \mathcal{C}$.
Contravariant, if $\mathcal{F}(f)\colon \mathcal{F}(Y) \to \mathcal{F}(X)$ and $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for all $f\colon X \to Y$, $g\colon Y \to Z$ with $X, Y, Z \in \mathcal{C}$.
Given two functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{E}$, the composition $\mathcal{G} \circ \mathcal{F}$ can be defined as the composition of the assignation rules that define each one of them is a functor from $\mathcal{C}$ to $\mathcal{E}$.
Given two categories $\mathcal{C}, \mathcal{D}$, we define the *Cartesian product* $\mathcal{C} \times \mathcal{D}$, consisting of ordered pairs (X, Y) with $X \in \mathcal{C}, Y \in \mathcal{D}$ as objects and ordered pairs (f, g) with $f\colon X \to X', g\colon Y \to Y'$ as morphisms from (X, Y) to (X', Y'). This is a category with the entrywise composition rule and identity map $1_{(X,Y)} := (1_X, 1_Y)$ for $(X, Y) \in \mathcal{C} \times \mathcal{D}$. In this sense, we can think about the projections on each component for objects and morphisms. The

projections define functors $\mathcal{P}_1 \colon \mathcal{C} \times \mathcal{D} \to \mathcal{C}$, $\mathcal{P}_2 \colon \mathcal{C} \times \mathcal{D} \to \mathcal{D}$. A functor defined on a cartesian product category is called *bifunctor*. We notice that we can define inductively the cartesian product of a finite collection of categories, $\prod_{i=1}^{n} \mathcal{C}_i$ as the n-tuples of objects and morphisms as described before.

For a pair of functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$ between the same categories, a *natural transformation* is a family $\alpha = (\alpha_A \colon \mathcal{F}(A) \to \mathcal{G}(A))_{A \in \mathcal{C}}$ of morphisms such that the diagram

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\alpha_A} & \mathcal{G}(A) \\ {\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(B) & \xrightarrow{\alpha_B} & \mathcal{G}(B) \end{array}$$

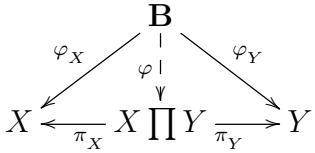
commutes for every morphism $f \colon A \to B$ in $\mathcal{C}$. A natural transformation is usually denoted by $\alpha \colon \mathcal{F} \Rightarrow \mathcal{G}$ and it is also convenient to write $\alpha \colon \mathcal{F}(A) \to \mathcal{G}(A)$ instead of α_A for the single morphism. A natural transformation is called a *natural isomorphism* if all α_A are isomorphisms.

1.1.1 Additive Categories

Now we introduce a type of category where the concepts of homology and cohomology are developed: the additive categories.

Definition 1.1.3. Let $\mathcal{C}$ be a category and $\{X_i : i \in I\} \subset \mathcal{C}$, we define a *product* of the family $\{X_i\}_{i \in I}$ as an object $\mathbf{P} \in \mathcal{C}$ together with a family of morphisms $\{\pi_i \colon \mathbf{P} \to X_i\}_{i \in I}$ such that for any other object $\mathbf{B} \in \mathcal{C}$ and a family of morphisms $\{\varphi_i \colon \mathbf{B} \to X_i\}_{i \in I}$, there is a unique morphism $\varphi \colon \mathbf{B} \to \mathbf{P}$ such that $\pi_i \circ \varphi = \varphi_i$. The object $\mathbf{P}$ is denoted by $\prod_{i \in I} X_i$.

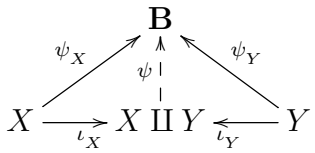
In the case of two objects $X, Y \in \mathcal{C}$ we can give a commutative diagram for a product $X \prod Y$, where the morphism $\varphi \colon \mathbf{B} \to X \prod Y$ is unique:



We notice that the definition of a product for two objects can be extended, in the sense of the general definition, to any finite amount of objects. For example, in the category **Set** the *cartesian product* $\prod_{i=1}^{n} X_i$ of the sets $X_i, i = 1, \ldots, n$ is a product of the sets X_i. This definition of *product* involves only objects and morphisms and not the elements of each object. We give the dual definition; i.e. the *coproduct*, a concept that plays a central role in this thesis, as it allows universal formulations of the different notions of stochastic independence.

Definition 1.1.4. Let $\mathcal{C}$ be a category and $\{X_i : i \in I\} \subset \mathcal{C}$, we define a *coproduct* (or *sum*) of the family $\{X_i\}_{i\in I}$ as an object $\mathbf{S} \in \mathcal{C}$ together with a family of morphisms $\{\iota_i \colon X_i \to \mathbf{S}\}_{i\in I}$ such that for any other object $\mathbf{B} \in \mathcal{C}$ and a family of morphisms $\{\psi_i \colon X_i \to \mathbf{B}\}_{i\in I}$, there is a unique morphism $\psi \colon \mathbf{S} \to \mathbf{B}$ such that $\psi \circ \iota_i = \psi_i$. The object $\mathbf{S}$ is denoted by $\coprod_{i\in I} X_i$.

We give a commutative diagram for $X \amalg Y$ in the case of two objects $X, Y \in \mathcal{C}$: By inverting arrows,



As before, we can extend this definition of coproduct to a finite colection of objects X_i, $\coprod_{i=1}^{n} X_i$. For example, in the category **Set** the *disjoint union* is a coproduct of the sets involved.

Remark 1.1.5. In a category with a zero object, the empty product and the empty coproduct are both the zero object. In the sense of initial and terminal objects, we can say that a product is a terminal object and a coproduct is an initial object, closing the respective type of diagram.

Definition 1.1.6. A *pre-additive* category is a category $\mathcal{C}$ enriched in the category of the abelian groups **Ab** and the composition is bilinear, in the sense that all of its Hom sets are abelian groups and $(f + f') \circ g = f \circ g + f' \circ g$, $f \circ (g + g') = f \circ g + f \circ g'$, for all $f, f' \in \mathrm{Hom}(X, Y)$ and $g, g' \in \mathrm{Hom}(Y, Z)$.

As a consequence, there is a $0_{\mathrm{Hom}(X,Y)}$ morphism for every $X, Y \in \mathcal{C}$ such that $f + 0_{\mathrm{Hom}(X,Y)} = f = 0_{\mathrm{Hom}(X,Y)} + f$. This morphism also satifies that $f \circ 0_{\mathrm{Hom}(X,Y)} = 0_{\mathrm{Hom}(X,Z)}$ and $0_{\mathrm{Hom}(Y,Z)} \circ g = 0_{\mathrm{Hom}(X,Z)}$ for all f, g due to the bilinear behavior of the composition. We will denote the 0 morphism on each Hom set without the subindex. If the category has zero object for any objects X, Y it holds that $0_{XY} = 0_{\mathrm{Hom}(X,Y)}$.

Remark 1.1.7. In a preadditive category an initial object is also a final object and vice versa, so a preadditive category with either initial or final objects have a zero object. We can verify this from the fact that, in either case, for an initial or terminal object X, it holds that $\mathrm{Hom}(X, X) = \{1_X\} = \{0_{\mathrm{Hom}(X,X)}\}$, and thus its Hom set is the zero group and $1_X = 0_{\mathrm{Hom}(X,X)}$. Let Y, Z be any objects and $f\colon X \to Y$ and $g\colon Z \to X$; then $f = f \circ 1_X = f \circ 0_{\mathrm{Hom}(X,X)} = 0_{\mathrm{Hom}(X,Y)}$ and $g = 1_X \circ g = 0_{\mathrm{Hom}(X,X)} \circ g = 0_{\mathrm{Hom}(X,Y)}$. This means that both the Hom sets $\mathrm{Hom}(X, Y)$ and $\mathrm{Hom}(Z, X)$ have only one element, i.e. X is a zero object.

Therefore, as a consequence of the Remarks 1.1.5 and 1.1.7, in a preadditive category, a product is a coproduct and vice versa. This kind of products or coproducts are called *biproducts*. Now we have all the requirements to define the additive categories.

Definition 1.1.8. An *additive category* is a preadditive category which admits a finitary product.

Notice that this is equivalent to say that an additive category is a preadditive category with a zero object in which every pair of objects admit a product.
The foremost example of an additive category is the category of all R-modules for a fixed ring R, the R-module homomorphisms have a bilinear behavior respect to the composition, the direct sum of R-modules is a finitary product and there is the 0 R-module as zero object.

1.1.2 Tensor Categories

Tensor categories are important in this work because we need a categorical framework to develop categorical independence and remark that this notion unifies different notions of independence.

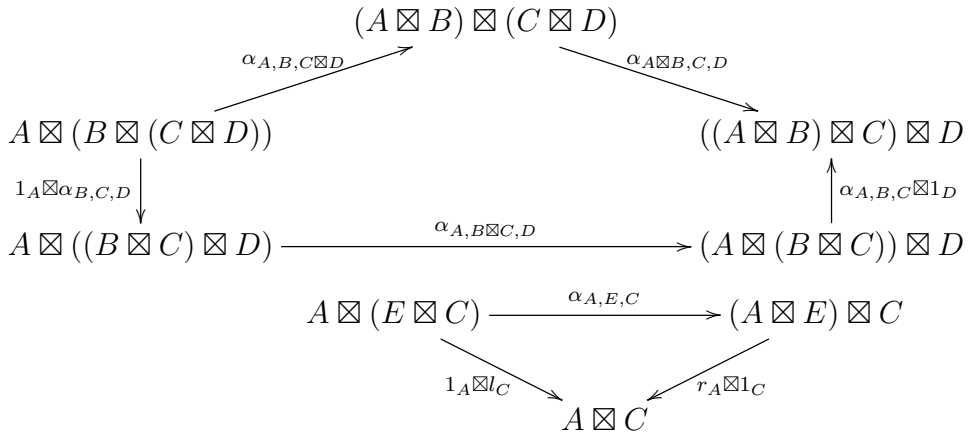
Definition 1.1.9. A *tensor category* is a category $\mathcal{C}$ together with a bifunctor $\boxtimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ which is associative under natural isomorphism with components

$$\alpha_{A,B,C}\colon A\boxtimes(B\boxtimes C)\to(A\boxtimes B)\boxtimes C$$

called *associativity constraint*, and has a unit object $E\in\mathcal{C}$ acting as left and right identity under natural isomorphisms with components

$$l_A\colon E\boxtimes A\to A,\ r_A\colon A\boxtimes E\to A$$

called *left unit constraint* and *right unit constraint*, respectively, such that the *coherence* axioms hold, i.e. the diagrams



commute for all objects $A,B,C,D\in\mathcal{C}$. If the natural transformations α, r and l are all identities, we say that the category is *strict*.

The coherence axioms are called the pentagon and triangle axioms respectively and it is shown that these imply the commutativity of all diagrams which only contain α, r and l.

As examples of tensor categories we have the additive categories.

1.1.3 Chain and cochain complexes and the Homotopy Category

The chain complexes are the backbone of the homology and cohomology theories. In this section we study chain and cochain complexes of an additive category and define the homotopy category.

Definition 1.1.10. Let $\mathcal{C}$ be an additive category. A *complex* $(C^\bullet, d^\bullet)$ in $\mathcal{C}$ is a sequence of objects and morphisms in $\mathcal{C}$, $\{C^n, d^n \colon C^n \to C^{n+1}\}_{n\in\mathbb{Z}}$ such that $d^{n+1} \circ d^n = 0$ for each n. In this case we say that the complex is indexed by cohomology and sometimes we call it a *cochain complex*. In a diagram, we have

$$C^\bullet : \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots .$$

We will use the notation $C^\bullet$ instead of the pair $(C^\bullet, d^\bullet)$ to refer to the complex.
A *chain complex* $(C_\bullet, d_\bullet)$ in $\mathcal{C}$ is a sequence of objects and morphisms in $\mathcal{C}$, $\{C_n, d_n \colon C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ such that $d_n \circ d_{n+1} = 0$ for each n. In this case we say that the complex is indexed by homology. In a diagram we have

$$C_\bullet : \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots .$$

As in the former case, we use $C_\bullet$ instead of $(C_\bullet, d_\bullet)$.

We will always consider complexes unless it is said to be otherwise though the notation should be clear about it. We consider the complexes in $\mathcal{C}$ as objects of some nature, so we wonder if there is a category associated to $\mathcal{C}$ whose objects are the complexes. We have the following definition

Definition 1.1.11. A *morphism of complexes* $f^\bullet \colon C^\bullet \to D^\bullet$ between the complexes $C^\bullet$ and $D^\bullet$ is a sequence of morphisms $\{f^n \colon C^n \to D^n\}_{n\in\mathbb{Z}}$ such that the diagram

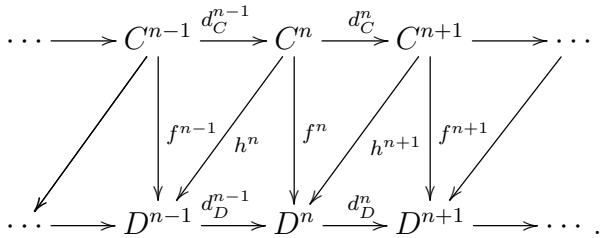
$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^{n-1} & \xrightarrow{d_C^{n-1}} & C^n & \xrightarrow{d_C^n} & C^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & D^{n-1} & \xrightarrow{d_D^{n-1}} & D^n & \xrightarrow{d_D^n} & D^{n+1} & \longrightarrow & \cdots
\end{array}$$

commutes, i.e. $f^{n+1} \circ d_C^n = d_D^n \circ f^n$ for each n.

The identity morfism $1_{C^\bullet}$ is the sequence of identity morphisms 1_{C^n} for each n. From the definition is clear that if $f^\bullet \colon C^\bullet \to D^\bullet$ and $g^\bullet \colon D^\bullet \to E^\bullet$ are morphisms of complexes in $\mathcal{C}$, then the sequence of morphisms $\{g^n \circ f^n\}_{n\in\mathbb{Z}}$ denoted by $g^\bullet \circ f^\bullet$ is a morphism of the complexes $C^\bullet$ and $E^\bullet$. Consequently, we define the category of complexes of $\mathcal{C}$ as the category denoted by $\mathbf{Comp}(\mathcal{C})$ whose objects are the complexes in $\mathcal{C}$ and the morphisms are the morphisms of complexes.

We remark that $\mathbf{Comp}(\mathcal{C})$ is also an additive category; where the product of complexes $C_i^\bullet, i = 1, \ldots, k$ is the complex $\prod_{i=1}^k C_i^\bullet$ given by the sequence of objects and morphisms $\{\prod_{i=1}^k C_i^n, \prod_{i=1}^k d_{C_i}^n\}_{n\in\mathbb{Z}}$ and $\prod$ is the product in $\mathcal{C}$.
An interesting concept is that of homotopy between two morphisms of complexes of either type chain or cochain. We will develop this later in the work.

Definition 1.1.12. An *homotopy* h between the morphisms of complexes $f^\bullet, g^\bullet \colon C^\bullet \to D^\bullet$ is a sequence of morphisms $\{h^n \colon C^n \to D^{n-1}\}_{n\in\mathbb{Z}}$ such that $f^n - g^n = d_D^{n-1} \circ h^n + h^{n+1} \circ d_C^n$. In a diagram we have:



The concept is analogous for chain complexes. An homotopy h between the morphisms of chain complexes $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ is a sequence of morphisms $\{h_n \colon C^n \to D^{n+1}\}_{n\in\mathbb{Z}}$ such that $f_n - g_n = d_{Dn+1} \circ h_n + h_{n-1} \circ d_{Cn}$. We draw no diagram for this case as it looks exactly as the former one.
In either case, when there is an homotopy between morphisms of complexes we say that the morphisms are *homotopic*

Remark 1.1.13. This concept induces a relation between morphisms of complexes. We mean that $f^\bullet \sim g^\bullet$ if they are homotopic; the same relation holds for morphisms of chain complexes. As a matter of fact, the relation is an equivalence relation.
Indeed, the sequence of 0 morphisms define an homotopy between f and f, if there is an homotopy h between f and g, then $-h$ as the sequence of the additive inverses of the sequence h is an homotopy between g and f. Finally if there are homotopies h and h' between f and g, and g and f' then the sequence $h + h'$ is an homotopy between f and f'. This proves that being homotopic is an equivalence relation. We omited the chain or cochain notation with bullets because the homotopies hold in either case.

Therefore, it makes sense to think of the equivalence classes of morphisms of complexes, $\mathrm{Hom}_{\mathbf{Comp}(\mathcal{C})}/\sim$, as morphisms of a category. We define the *homotopy category* of an

additive category $\mathcal{C}$ as the category $K(\mathcal{C})$ whose objects are the complexes in $\mathcal{C}$ and $\mathrm{Hom}_{\mathbf{Comp}(\mathcal{C})}/\sim$ as its set of morphisms.
When the category $\mathcal{C}$ is made of modules over some ring R, the morphisms between them are linear transformations and $\ker f$, $\mathrm{Im}\ f$ are concepts we have clear, unlike the $\ker f$ in an additive category in general. A complex in $\mathcal{C}$, $C^\bullet$ has the condition $d^{n+1} \circ d^n = 0$, in this context mean that $\mathrm{Im}\ d^n \subseteq \ker d^{n+1}$. The same for a chain complex in $\mathcal{C}$, where the condition means that $\mathrm{Im}\ d_{n+1} \subseteq \ker d_n$.

Definition 1.1.14. For a complex $C^\bullet$ in a category of R-modules over a fixed ring R, we define the nth-*cohomology* as the R-module

$$H^n = \frac{\ker d^{n+1}}{\mathrm{Im}\ d^n}.$$

and for a chain complex $C_\bullet$, the nth-*homology* as the R-module

$$H_n = \frac{\ker d_n}{\mathrm{Im}\ d_{n+1}}.$$

We may use the notation $H^\bullet$ and $H_\bullet$ to denote the sequences of cohomologies and homologies respectively.

These R-modules have meaning according to the nature of the objects where the complexes are taken from and the morphisms that define the complex. We study some cases in the next section.

1.2 Free product of algebras

Despite its similarity to the coproduct, the free product of algebraic structures of the same type is not necessarily a coproduct in the corresponding category where the structures belong. We will use [26, 38, 44] as references.
We think of the free product of a family $\{\mathcal{A}_i\}_{i\in I}$ of unital algebras as an algebra $\mathcal{A}$ made of words with letters taken from each factor, that satisfy an universal property. Let us assume that $\mathcal{A}_i \cap \mathcal{A}_j = \mathbb{C}1$, where 1 is the common unit of all the $\mathcal{A}_i$. If not, we identify $1_i = 1$ for each i and 1 is the unit in $\mathcal{A}$.

Definition 1.2.1. Let $\mathcal{A}_i$ be a unital algebra over $\mathbb{C}$ for each $i \in I$. The *free product with identification of units* of the $\mathcal{A}_i$ is a unital algebra $\mathcal{A}$, given together with a family of unital algebra homomorphisms $\iota_i \colon \mathcal{A}_i \to \mathcal{A}$ indexed in I, such that for every family of unital algebra homomorphisms $\phi_i \colon \mathcal{A}_i \to \mathcal{B}$ indexed in I to a unital algebra $\mathcal{B}$, there exist a unique homomorphism $\Phi \colon \mathcal{A} \to \mathcal{B}$ such that $\Phi \circ \iota_i = \phi_i$, $\forall i \in I$. The free product algebra $\mathcal{A}$ of the $\mathcal{A}_i$ is denoted by $*_{i\in I}\mathcal{A}_i$.

In the definition, it is not explicitly stated (even though suggested by our assumptions), that all the $\mathcal{A}_i$ have nothing in common but the scalar multiples of the unit. It is valid to think of the free product $\mathcal{B} * \mathcal{B}$ for some unital algebra $\mathcal{B}$, as long as we relabel each copy of $\mathcal{B}$, to think them as different algebras, with only $\mathbb{C}1$ in common.
We want to understand the structure of the free product algebra $\mathcal{A}$ of the $\mathcal{A}_i$. Let $\mathcal{A}_i$ be equipped with a unital linear functional $\tau_i \colon \mathcal{A}_i \to \mathbb{C}$ (NCPS in Definition 2.1.1), define $\mathcal{A}_i^o := \ker \tau_i$ and let $\mathcal{W}_{i_1,\dots,i_n} := \operatorname{span}\{a_1 \cdots a_n : a_1 \in \mathcal{A}_{i_1}^o, \dots, a_n \in \mathcal{A}_{i_n}^o\}$.
Thus, we get a direct sum decomposition of the underlying vector space of $\mathcal{A} = *_{i\in I}\mathcal{A}_i$ as

$$\mathcal{A} = \mathbb{C}1 \oplus \left(\bigoplus_{n=1}^{\infty} \bigoplus_{\substack{i_1,\dots,i_n \in I \\ i_j \neq i_{j+1}, j=1,\dots,n-1}} \mathcal{W}_{i_1,\dots,i_n} \right). \tag{1.1}$$

Remark 1.2.2. For $n \geq 1$ and every $i_1, \dots, i_n \in I$ such that $i_j \neq i_{j+1}, j = 1, \dots, n-1$, the vector space $\mathcal{W}_{i_1,\dots,i_n}$ is isomorphic to the tensor product $\mathcal{A}_{i_1}^o \otimes \cdots \otimes \mathcal{A}_{i_n}^o$, via the linear map determined by $a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdots a_n$ for $a_1 \in \mathcal{A}_{i_1}^o, \dots, a_n \in \mathcal{A}_{i_n}^o$. The Equation 1.1 takes the form

$$\mathcal{A} = \mathbb{C}1 \oplus \left(\bigoplus_{n=1}^{\infty} \bigoplus_{\substack{i_1,\dots,i_n \in I \\ i_j \neq i_{j+1}, j=1,\dots,n-1}} \mathcal{A}_{i_1}^o \otimes \cdots \otimes \mathcal{A}_{i_n}^o \right).$$

From the universal properties of the direct sum and the tensor product, it is clear that $\mathcal{A}$ satisfy the universal property stated in the Definition 1.2.1. Yet, it is not clear that $\mathcal{A}$ is an unital algebra because we have not stated a product in $\mathcal{A}$. To define a multiplication in $\mathcal{A}$, let us focus first on factors $a_1 \otimes a_2$ and $b_1 \otimes b_2$. If a_2 and b_1 do not belong to the same vector space $\mathcal{A}_j^o$ then we can think of $(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = a_1 \otimes a_2 \otimes b_1 \otimes b_2$ with no

problem because it is in a summand of $\mathcal{A}$. A problematic situation is, for instance, when $a_1, b_1 \in \mathcal{A}_{i_1}^0$, $a_2, b_2 \in \mathcal{A}_{i_2}^o$ and we need the product $(a_1 \otimes a_2) \cdot (b_2 \otimes b_1)$ in $\mathcal{A}$. A natural candidate for this product is $a_1 \otimes (a_2 b_2) \otimes b_1$ but $a_2 b_2$ may not be in any $\mathcal{A}_j^o$. We consider the *centering* of $a_2 b_2$, $(\dot{a_2 b_2}) = a_2 b_2 - \tau_{i_2}(a_2 b_2) \cdot 1 \in \mathcal{A}_{i_2}$. So the new candidate becomes $a_1 \otimes (a_2 b_2) \otimes b_1 = a_1 \otimes (\dot{a_2 b_2}) \otimes b_1 + \tau_{i_2}(a_2 b_2) \cdot a_1 b_1$. Once again, $a_1 b_1$ may not be an element in any summand of $\mathcal{A}$. By taking the centering of $a_1 b_1$ we get a perfect candidate for the product $(a_1 \otimes a_2) \cdot (b_2 \otimes b_1) = (\tau_{i_1}(a_1 b_1)\tau_{i_2}(a_2 b_2)) \cdot 1 + \tau_{i_2}(a_2 b_2) \cdot (\dot{a_1 b_1}) + a_1 \otimes (\dot{a_2 b_2}) \otimes b_1$ which is an element in the summand $\mathbb{C}1 \oplus \mathcal{A}_{i_1}^0 \oplus (\mathcal{A}_{i_1}^0 \otimes \mathcal{A}_{i_2}^0 \otimes \mathcal{A}_{i_1}^0) \subset \mathcal{A}$.

A problem in this way of defining the product in $\mathcal{A}$ is that it may seem that it depends on the selected functionals τ_i, but the class of isomorphism of $\mathcal{A}$ itself does not depend on the τ_i due the universal property satisfied by $\mathcal{A}$.

Definition 1.2.3. Let $(\mathcal{A}_i, \tau_i)_{i \in I}$ be a family of unital algebras equipped with a unital linear functional $\tau_i \colon \mathcal{A}_i \to \mathbb{C}$. Consider the free product algebra $\mathcal{A} = *_{i \in I} \mathcal{A}_i$ and its direct sum decomposition in Equation 1.1, where the subspaces $\mathcal{A}_i^o = \ker \tau_i$, $i \in I$. The *free product of the functionals* $(\tau_i)_{i \in I}$ is defined as the unique unital linear functional $\tau \colon \mathcal{A} \to \mathbb{C}$ such that $\tau|_{\mathcal{W}_{i_1,\ldots,i_n}} = 0$ for every $n \geq 1$ and $i_1, \ldots, i_n \in I$ with $i_j \neq i_{j+1}, j = 1, \ldots, n-1$. This functional τ is denoted by $*_{i \in I} \tau_i$ and the pair $(\mathcal{A}, \tau)$ is called the *free product of* $(\mathcal{A}_i, \tau_i)_{i \in I}$. We write sometimes $(\mathcal{A}, \tau) = *_{i \in I}(\mathcal{A}_i, \tau_i)$ instead of $(*_{i \in I} \mathcal{A}_i, *_{i \in I} \tau_i)$.

Remark 1.2.4. The inclusion homomorphisms $\iota_i \colon \mathcal{A}_i \to \mathcal{A}$ turn out to be one to one, so we think that $\mathcal{A}_i \subset \mathcal{A}$ for all $i \in I$. The restriction of the free product functional τ to an $\mathcal{A}_i$ is equal to the orignial $\tau_i \colon \mathcal{A}_i \to \mathbb{C}$. Indeed, τ was defined so that $\ker \tau \supset \ker \tau_i$ and both $\tau(1) = 1 = \tau_i(1)$; hence the funcionals coincide on $\ker \tau_i$ and $\mathbb{C}1$. Therefore $\tau|_{\mathcal{A}_i} = \tau_i$. Consequently, we think of $(\mathcal{A}_i, \tau_i)$ a subspace of $(\mathcal{A}, \tau)$.

The free product in Definitions 1.2.1 and 1.2.3 is also called *free product amalgamated by the unit*, because we had the assumption that $\mathcal{A}_i \cap \mathcal{A}_j = \mathbb{C}1$. Now lets assume that we have a family of algebras $\{\mathcal{A}_i\}_{i \in I}$ and an algebra $\mathcal{B}$ not necessarily unital such that $\mathcal{B} \subset \mathcal{A}_i, \forall i$ and $\mathcal{A}_i \cap \mathcal{A}_j = \mathcal{B}, i \neq j$. Once again, we think that the only thing in common among the $\mathcal{A}_i$ is the subalgebra $\mathcal{B}$ relabeling if needed. When $\mathcal{B} \subset \mathcal{A}$ we call $\mathcal{A}$ an algebra over $\mathcal{B}$.

Definition 1.2.5. Let $\mathcal{A}_i$ be an algebra over $\mathcal{B}$ for each $i \in I$. The *free product with*

amalgamation over $\mathcal{B}$ of the $\mathcal{A}_i$ via the monomorphisms $\alpha_i\colon \mathcal{B} \to \mathcal{A}_i$ is an algebra $\mathcal{A}$, together with a family of algebra homomorphisms $\iota_i\colon \mathcal{A}_i \to \mathcal{A}$ indexed in I, such that $\iota_i \circ \alpha_i = \iota_j \circ \alpha_j$ for all $i, j \in I$ and for every family of algebra homomorphisms $\phi_i\colon \mathcal{A}_i \to \mathcal{C}$ indexed in I to an algebra $\mathcal{C}$, there exist a unique homomorphism $\Phi\colon \mathcal{A} \to \mathcal{C}$ such that $\Phi \circ \iota_i = \phi_i$, $\forall i \in I$. The free product algebra $\mathcal{A}$ of the $\mathcal{A}_i$ is denoted by $*_{\substack{\mathcal{B}\\ i\in I}} \mathcal{A}_i$.

Definition 1.2.6. Let $\mathcal{A}$ be an algebra over $\mathcal{B}$. A linear map $\mathbb{F}\colon \mathcal{A} \to \mathcal{B}$ is a $\mathcal{B}$-*functional* (or a *conditional expectation*) if for all $b, b_1, b_2 \in \mathcal{B}, a \in \mathcal{A}$ it holds that $\mathbb{F}(b) = b$ and $\mathbb{F}(b_1 a b_2) = b_1 \mathbb{F}(a) b_2$. We will denote an algebra $\mathcal{A}$ over $\mathcal{B}$ equipped with a $\mathcal{B}$-functional by $(\mathcal{A}, \mathcal{B}, \mathbb{F})$ (OVPS in Definition 2.1.1).

In our case, we consider the set theoretical inclusions $\mathcal{B} \hookrightarrow \mathcal{A}_i$ as the algebra monomorphisms for the amalgamated free product of the $\mathcal{A}_i$. The structure of $\mathcal{A}$ can be understood as the algebra whose underlying vector space is

$$\mathcal{A} = \mathcal{B} \oplus \left(\bigoplus_{n=1}^{\infty} \bigoplus_{\substack{i_1,\ldots,i_n \in I \\ i_j \neq i_{j+1}, j=1,\ldots,n-1}} \mathcal{W}_{i_1,\ldots,i_n} \right),$$

where the $\mathcal{W}_{i_1,\ldots,i_n}$ can be understood as words not containing letters in $\mathcal{B}$. The letters can be taken exactly as in the unital case, let $\mathcal{A}_i$ be equipped with a $\mathcal{B}$-functional $\mathbb{F}_i\colon \mathcal{A}_i \to \mathcal{B}$ and define $\mathcal{A}_i^o := \ker \mathbb{F}_i$.

Definition 1.2.7. Let $(\mathcal{A}_i, \mathcal{B}, \mathbb{F}_i)_{i\in I}$ be a family of algebras over $\mathcal{B}$ equipped with a $\mathcal{B}$-functional $\mathbb{F}_i\colon \mathcal{A}_i \to \mathcal{B}$. The *free product of the* $\mathcal{B}$*-functionals* $(\mathbb{F}_i)_{i\in I}$ is defined as the unique $\mathcal{B}$-functional $\mathbb{F}\colon \mathcal{A} \to \mathcal{B}$ such that $\mathbb{F}|_{\mathcal{W}_{i_1,\ldots,i_n}} = 0$ for every $n \geq 1$ and $i_1, \ldots, i_n \in I$ with $i_j \neq i_{j+1}, j = 1, \ldots, n-1$. This $\mathcal{B}$-functional $\mathbb{F}$ is denoted by $*_{\substack{\mathcal{B}\\ i\in I}} \mathbb{F}_i$ and the tripple $(\mathcal{A}, \mathcal{B}, \mathbb{F})$ is called the $\mathcal{B}$-*amalgamatedfree product of* $(\mathcal{A}_i, \mathcal{B}, \mathbb{F}_i)$. We write $(\mathcal{A}, \mathcal{B}, \mathbb{F}) = *_{\substack{\mathcal{B}\\ i\in I}} (\mathcal{A}_i, \mathcal{B}, \mathbb{F}_i)$.

Remark 1.2.8. The free product of unital algebras is a particular case of the $\mathcal{B}$-amalgamated free product of algebras, where the $\mathcal{A}_i$ are unital and $\mathcal{B} = \mathbb{C}1$.

1.3 Simplicial homology elements

In this section we will provide some elements of algebraic topology to help us exploring simplicial complexes through boundary matrices. As we did in the former section, we will not give major proofs as the content is asumed to be known. We use [17, 28, 46] as references.

1.3.1 Simplicial complexes

One of the main objects of this work are simplicial complexes as they are the main example we resented so far of treating a topological space as a random variable inQ a non-commutative probability space. Furthermore, the way of doing it relates of properties of the simplicial complex itself.

Definition 1.3.1. A set $A = \{a_0, a_1, \ldots, a_k\} \subset \mathbb{R}^n, k \geq 1$ is said to be *geometrically independent* or *affine independent* if the set $\{a_0 - a_1, \ldots, a_0 - a_k\}$ is linearly independent. A set having only one point is considered to be geometrically independent.

We now introduce the notion of *simplex* which is fundamental for the definition of simplicial complexes.

Definition 1.3.2. Let $A = \{a_0, a_1, \ldots, a_k\} \subset \mathbb{R}^n; n \geq k$ be a geometrically independent set of points. The *k-dimensional simplex*, or *k-simplex*, spanned by A, and denoted by σ^k, is the set in $\mathbb{R}^n$ given by

$$\sigma^k = \left\{ x = \sum_{i=0}^{k} \alpha_i a_i : \alpha_i \geq 0 \text{ and } \sum_{i=0}^{k} \alpha_i = 1 \right\},$$

the set A is called the *vertex set*. We will denote sometimes σ^k as A, as it is costumary in computing sciences textooks.

Remark 1.3.3. We are always referring to simplices as geometric objects. This geometric representation is denoted by $|\sigma|$ for a simplex σ, so we are thinking $|\sigma|$ as a topological space (subspace of $\mathbb{R}^n$, n large enough) with the subspace topology.

Definition 1.3.4. Let σ and τ be simplices. A *simplicial map* $f\colon \sigma \to \tau$ is a map f from the vertices of σ to the vertices of τ such that a simplex $\{a_0, \ldots, a_k\}$ of σ is mapped to a simplex $\{f(a_0), \ldots, f(a_k)\}$ of τ. f is a *simplicial isomorphism*, or an *isomorphism*, if it is bijective and $\{a_0, \ldots, a_k\}$ is a simplex of σ if and only if $\{f(a_0), \ldots, f(a_k)\}$ is a simplex of τ. If there is an isomorphism $f\colon \sigma \to \tau$, we say that σ and τ are *isomorphic*.

The 0-simplex is one point, the 1-simplex is the straight line segment defined by two points and which we will always consider of lenght $\sqrt{2}$. The 2-simplex is a triangle and its inside points, considered to be an equilateral triangle with side lenght $\sqrt{2}$. This consideration is known as the *standard k-simplex*, spanned by the canonical basis vectors $\{e_1, \ldots, e_{k+1}\} \subset \mathbb{R}^{k+1}$. Of course each k-simplex is embedded in any euclidean space of dimension $n \geq k+1$. We make this consideration because every k-simplex is isomorphic to the standard k-simplex via the linear map that assigns $a_{i-1} \mapsto e_i, i = 1, \ldots, k+1$.
For example, let σ^k be a k-simplex. A 0-simplex is a *vertex* of σ^k, a 1-simplex is called an *edge* of σ^k, and higher dimensional simplices of σ^k are called faces. For $r \leq k$ it is said that σ^r is an *r-dimensional face* (or an r-simplex) of σ^k if every vertex of σ^r is a vertex of σ^k. If $r < k$ then σ^r is called a *proper r-face* of σ^k.
Simplices are supposed to be building blocks for a class of topological spaces, called simplicial complexes. Spaces in this class are simple enough to find invariants that were later generalized to arbitrary spaces.

Definition 1.3.5. A *simplicial complex* K is a finite collection of simplices of $\mathbb{R}^n$, n sufficiently large, which satisfies the following conditions:

1. If $\sigma \in K$, then every face of σ is also in K.

2. If σ and τ are in K, then either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau$ is a common face of both σ and τ.

The *dimension* of the simplicial complex K, denoted as $\dim K$, is defined to be -1 if $K = \emptyset$ and $d \geq 0$ if d is the largest integer such that K has a d-simplex.

Remark 1.3.6. What we have defined above is actually a *finite* simplicial complex, which are for now the objects of our interest. Also, a d-dimensional simplicial complex does not

need to lie in $\mathbb{R}^d$. Since a simplicial complex satisfies the two conditions, its geometrical representation can be forced to lie in $\mathbb{R}^m$ where m is greater than d. Actually, it is known that the minimum m for a d-dimensional simplicial complex to lie in $\mathbb{R}^m$ is, in general, $m = 2d + 1$.

Example 1.3.7. Let σ^2 be the 2-standard simplex, then the set K of all faces of σ^2, i.e.

$$K = \{\{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\}$$

is a simplicial complex. This simplicial complex is known as the *closure* of σ^2 and denoted by $\mathrm{Cl}(\sigma^2)$. In general, we have that $\mathrm{Cl}(\sigma^n)$ is the set of all faces of σ^n.

Remark 1.3.8. This way to define simplicial complexes can be thought as gluing simplices together through faces of common dimension lesser than the dimension of the simplicess involved in the process. In the *non-connected case*, just put disjoint simplices in the same euclidean space without making any identifications and, thinking of them as one object. This way of thinking the construction of a simplicial complex makes sense when thinking of simplices as building-blocks and will be useful later in the work.

The Definition 1.3.4 for simplicial maps and isomorphisms are the same for simplicial complexes. We notice that if we have simplicial maps $f\colon K_1 \to K_2$ and $g\colon K_2 \to K_3$, then $g \circ f\colon K_1 \to K_3$ is a simplicial map and for every K, there is an identity map $1_K\colon K \to K$ given by the set theoretic identity, so the simplicial complexes along with simplicial morphisms form a category, which we denote as **SC**.

Definition 1.3.9. Let K be a d-dimensional simplicial complex. For each r, $0 \leq r \leq d$, let K^r denote the set of all those simplices of K which are of dimension $k \leq r$. Then K^r is a simplicial complex which is called the *r-dimensional skeleton*, or the *r-skeleton*, of K. The r-skeleton is a subsimplicial complex of K and naturally a subspace of K as well.

For example, given a a simplicial complex K the 0-skeleton, K^0, is its vertex set. The 1-skeleton, K^1 of K is a graph given by the vertices and edges of K; it is said that K is *connected* if K^1 is a connected graph, i.e. for any pair of vertices a and b in K, there is a sequence of edges $\{a_i, a_{i+1}\}, i = 1, \ldots, m-1$ such that $a_1 = a$ and $b = a_m$ in K.
As we mentioned when studying simplices, the geometric realization $|\sigma|$ for a simplex σ is a compact subspace of $\mathbb{R}^m$ and a simplicial complex K has a geometric realization, $|K|$,

which is also a topological space. As K is built from simplices and each simplex has a topology on its own; we consider the weak topology on $|K|$ defined by the closed simpleces in K. We call this topology on K the *coherent topology*.

1.3.2 Simplicial Homology

In this section we are interested in studying the simplicial complexes via algebraic tools. In other words, we are interested in associating a sequence of abelian groups $\{H_n(K) : n \geq 0\}$ with a given simplicial complex K. These groups, called *homology groups* of K, are invariants of the topological structure of $|K|$ and have interesting functorial properties.

Definition 1.3.10 (Orientation of Simplicial Complexes)**.** Let σ^k be a k-simplex, $\sigma^k = \{v_0, \ldots, v_k\}$. Let us suppose that the vertices are ordered by declaring $v_0 < v_1 < \cdots < v_k$. Then this ordering determines a direction among the vertices of σ^k. Any other ordering of these vertices gives raise to a permutation of vertices, which is either an even permutation or an odd permutation of the given ordering. When we consider the simplex σ^k together with the equivalence class of even permutations of its vertices, we say that σ^k is *positively oriented* and we denote this pair by $+\sigma^k$. Also, when we consider σ^k with the equivalence class of odd permutations of its vertices, we say that σ^k is *negatively oriented* and denote the pair by $-\sigma^k$. The 0-simplex is always considered to be positively oriented.

By an oriented simplex σ^k we mean, the simplex σ^k together with an order of vertices determining a positive orientation. When we change the given order by a transposition, the same simplex σ^k will be positively oriented on the new ordering but negatively oriented with respect to the earlier ordering. Thus, we note that a simplex can be ordered in several ways.
For example, consider the 1-simplex $\sigma^1 = \{v_0, v_1\}$. If we order $v_0 < v_1$ then σ^1 is $\{v_0, v_1\}$ and $-\sigma^1$ is $\{v_1, v_0\}$. These two simplices, $+\sigma^1$ and $-\sigma^1$ are now different 1-simplices, one having the direction from v_0 to v_1 and the other having the direction from v_1 to v_0.
Note that a k-simplex $\sigma^k = \{v_0, \ldots, v_k\}$ with a given order of vertices will have all of its faces ordered by the ordering induced by the ordeirng of σ^k.

Definition 1.3.11. A simplicial complex K is said to be *oriented* if each of its simplices

is assigned an orientation.

As a consequence, we will have several ways to order a simplicial complex K. An order of its vertices induces an ordering for each simplex of K. We can also give an arbitray order to each simplex of K. Thus, if a simplex τ of K is a face of a simplex σ of K, it might happen that the orientation of τ induced by σ may coincide or may be opposite to the original orientation of τ.

Definition 1.3.12. Let K be an oriented simplicial complex and σ^p and σ^{p+1} be two simplices of K whose dimensions differ by 1. For each pair (σ^{p+1}, σ^p) we associate its *incidence number*, denoted by $[\sigma^{p+1}, \sigma^p]$, as follows: If σ^p is not a face of σ^{p+1}, we say that $[\sigma^{p+1}, \sigma^p] = 0$. If σ^p is a face of σ^{p+1}, we label the vertices of σ^p so that $+\sigma^p = \{v_0, \ldots, v_p\}$. Let v be the extra vertex of σ^{p+1}, then $\{v, v_0, \ldots, v_p\}$ might be $+\sigma^{p+1}$ or $-\sigma^{p+1}$. We define

$$[\sigma^{p+1}, \sigma^p] = \begin{cases} 1 & \text{if } +\sigma^{p+1} = \{v, v_0, \ldots, v_p\} \\ -1 & \text{if } -\sigma^{p+1} = \{v, v_0, \ldots, v_p\} \end{cases}.$$

Theorem 1.3.13. *Let K be an oriented simplicial complex. If σ^{p+2} is a $(p-2)$-simplex face of a simplex σ^p of K, then*

$$\sum [\sigma^p, \sigma^{p-1}][\sigma^{p-1}, \sigma^{p-2}] = 0,$$

where the sum is over all $(p-1)$-simplices σ^{p-1} of K.

Let $\tilde{S}_q$ denote the set of oriented q-simplices of K. We will only write σ^q instead $+\sigma^q$ when referring to the elements of $\tilde{S}_q$.

Definition 1.3.14. Let $0 \geq q \geq \dim K$ and $\mathbb{Z}$ be the additive group of the integers. A map $f\colon \tilde{S}_q \to \mathbb{Z}$ such that if $q \geq 1$, then $f(-\sigma^q) = -f(\sigma^q)$ for each $\sigma^q \in \tilde{S}_q$ is called a *q-chain*. For $q = 0$, a 0-chain is just a map from the vertices (the oriented 0-simplices) of K to $\mathbb{Z}$. The set of all the q-chains of K is denoted by $C_q(K)$. If $q < 0$ or $q > \dim K$ we set $C_q(K) = 0$.

Remark 1.3.15. For each $q \in \mathbb{Z}$, the set $C_q(K)$ is an abelian group with the pointwise maps operation. The group $C_q(K)$ is called the *q-dimensional chain group* of K. We have

that $C_q(K)$ is 0 when $q \notin \{0, \dots, \dim K\}$, so the sequence indexed in $\mathbb{Z}$ of chain groups of K is $\dots, 0, C_n(K), C_{n-1}(K), \dots, C_0(K), 0, \dots$, where $n = \dim K$. Also, the group $C_q(K)$ is a free abelian group generated by the *elementary q-chains* for every q-simplex σ^q given by

$$\bar{\sigma}^q(\tau^q) = \begin{cases} 1 & \tau^q = \sigma^q \\ -1 & \tau^q = -\sigma^q \\ 0 & \text{otherwise} \end{cases}.$$

Consequently, $C_q(K)$ is generated by the positively oriented q-simplices of K.

Definition 1.3.16. For each $0 < q \leq \dim K$ we define a map $\partial_q \colon C_q(K) \to C_{q-1}(K)$ on the generators of $C_q(K)$ as follows

$$\partial_q(\sigma^q) = \sum_{\sigma^{q-1} \in K} [\sigma^q, \sigma^{q-1}]\sigma^{q-1}.$$

Then, it is extended over $C_q(K)$ by linearity, i.e. if $f = \sum \eta_q \sigma^q$ then $\partial_q(f) = \sum \eta_q \partial_q(\sigma^q)$. The map ∂_q is called the *boundary morphism*. We note that for $q \leq 0$ or $q > \dim K$ the morphism ∂_q is the zero morphism.

The ∂_q morphisms are well defined because if $\sigma^q = \{v_0, \dots, v_q\}$ is positively oriented by the vertex ordering written in the set, then $\sigma^q = \{v_{\rho(0)}, \dots, v_{\rho(q)}\}$ for every even permutation ρ of $\{0, \dots, q\}$ and $\partial_q(\sigma^q) = \partial_q(\sigma^q_\rho)$ for every ρ even permutation; σ^q_ρ denotes σ^q with the order induced by ρ. For an odd permutation φ of $\{0, \dots, q\}$ we have that $\partial_q(\sigma^q) = -\partial_q(\sigma^q_\varphi)$.

Remark 1.3.17. If $\sigma^q = \{v_0, \dots, v_q\}$ is oriented by the vertex ordering written in the set, then σ^q has $q+1$ faces $\sigma_i^{q-1} = \{v_0, v_1, \dots, \hat{v_i}, \dots, v_q\}$, $i = 0, 1, \dots, q$, where $\hat{v_i}$ denotes that the vertex v_i has been omitted on the list and each of these faces are oriented by the induced ordering. The σ_i^{q-1} also satisfy that $[\sigma^q, \sigma_i^{q-1}] = (-1)^i$ and, therefore, the ∂_q morphism takes the form $\partial_q(\sigma^q) = \sum_{i=0}^q (-1)^i \sigma_i^{q-1}$. To make it more explicit:

$$\partial_q\{v_0, v_1, \dots, v_q\} = \sum_{i=0}^{q} (-1)^i \{v_0, v_1, \dots, \hat{v_i}, \dots, v_q\}. \tag{1.2}$$

This version of the formula for the boundary morphism is used in many textbooks to avoid the concept of incidence number.

Lemma 1.3.18. *For each $q \in \mathbb{Z}$, the composite homomorhism $\partial_q \circ \partial_{q+1} \colon C_{q+1}(K) \to C_{q-1}(K)$ is the zero morphism, i.e. $\partial_q \circ \partial_{q+1} = 0$.*

Remark 1.3.19. The chain groups and the boundary morphisms form a sequence of groups and morphisms $\{C_q(K), \partial_q \colon C_q(K) \to C_{q-1}(K)\}_{q\in\mathbb{Z}}$ which is a chain complex, and we denote it by $C(K)_\bullet$. Thus, we have the sequence of homology groups

$$H_q(K) = \frac{\ker \partial_q}{\operatorname{Im} \partial_{q+1}},$$

which are all zero for $q > \dim K$ or $q < 0$, because $\partial_q = 0$, and possibly non-trivial groups $H_q(K)$ occur when $0 \leq q \leq \dim K$.

Definition 1.3.20. Let K be an oriented simplicial complex. The subgroup $\ker \partial_q$ of $C_q(K)$ is called the *cycle group* and the subgroup $\operatorname{Im} \partial_{q+1}$ of $C_q(K)$ is called the *boundary group*, denoted by $Z_q(K)$ and $B_q(K)$ respectively. The elements of $Z_q(K)$ and $B_q(K)$ are called q-cycles and q-boundaries respectively.

Taking this definition into account, the homology groups of the chain complex $C(K)_\bullet$ are of the form

$$H_q(K) = \frac{Z_q(K)}{B_q(K)}.$$

Two q-cycles b_1 and b_q are *homologous* if they yield the same homology class, i.e. $b_1 - b_2$ is a q-boundary.

We must note that the chain complex $C(K)_\bullet$ is defined depending on the orientation given to the simplicial complex K. Let K be a simplical complex and, K_1 and K_2 be K with two different orientations. We defined the groups $C_q(K)$ as the free abelian group generated by the postively oriented q-simplices of K. In this sense we can think that a positively oriented q-simplex of K_1, as a simplex is the same simplex of K_2 but may be positively oriented in K_2 or not. We mean, $\sigma^q(K_1)$ is either $\sigma^q(K_2)$ or $-\sigma^q(K_2)$. Thus, intuitively, the sets of positively oriented q-simplices of K_1 and the respective set of K_2 must generate the same group, according to that relation. This also implies that the

boundary morphisms yield the same morphisms and the homology groups are essentially the same. An important theorem states that the homology for a simplicial complex K does not depend on the orientation:

Theorem 1.3.21. *Let K_1 and K_2 be the same simplicial complex K equipped with two different orientations. Then, $H_q(K_1) \cong H_q(K_2)$ for all $q \in \mathbb{Z}$.*

Therefore, we can refer to the homology groups of a simplicial complex K independently of the orientation given to K. We notice that each group $C_q(K)$ is free abelian with a finite base then the subgroups $Z_q(K)$ and $B_q(K)$ are free abelian finitely generated groups. As a consequence, $H_0(K)$ is free abelian and $H_q(K)$ are abelian groups that may or not be free but are finitely generated. Then, for each $q \in \mathbb{Z}$ we know that each $H_q(K) \cong F_q(K) \oplus T_q(K)$ where $F_q(K)$ is the direct sum of copies of $\mathbb{Z}$, called the free part of $H_q(K)$ and $T_q(K)$ is the direct sum of finite cyclic groups called the torsion part of $H_q(K)$.

Definition 1.3.22. Let K be a simplicial complex with homology groups $H_q(K)$. If for each q we write $H_q(K) \cong F_q(K) \oplus T_q(K)$, we define the q-th *Betti number* of K as the rank of $F_q(K)$.

We will omit the subindices on the boundary morphisms ∂ and on the sequence of morphisms that defines a morphism of chain complexes from now on in order to avoid cumbersome notation.

Definition 1.3.23. Let K and L be simplicial complexes, and $f\colon K \to L$ a simplicial map. Then for each $q \geq 0$, we define a homomorphism $f_\#\colon C_q(K) \to C_q(L)$ by

$$f_\#(\{v_0, \ldots, v_q\}) = \begin{cases} \{f(v_0), \ldots, f(v_q)\}, & \text{if all the } f(v_i)'\text{s are distinct,} \\ 0, & \text{otherwise,} \end{cases}$$

for each simplex $\{v_0, \ldots, v_q\}$ of K and then extending it linearly on $C_q(K)$.

The map $f_\#$ is a well-defined since any permutation on the left side produces the same permutation on the right side. The collection $\{f_\#\colon C_q(K) \to C_q(L)\}$ of homomorphisms is a morphism of chain complexes, i.e. $f_\#\partial = \partial f_\#$. This is stated in the next result.

Proposition 1.3.24. *If $f\colon K \to L$ is a simplicial map, then the induced sequence $\{f_{\#}\colon C_q(K) \to C_q(L)\}$ of homomorphisms commutes with the boundary homomorphisms ∂ of the chain complexes and hence, $f_{\#}$ induces a group homomorphism $f_*\colon H_q(K) \to H_q(L)$ for each q given by*

$$f_*([z_q]) = [f_{\#}(z_q)].$$

Remark 1.3.25. From the definition of the induced sequence of homomorphisms $\{f_*\colon H_q(K) \to H_q(L)\}$, it follows that

(a) If $1_K\colon K \to K$ is the identity map, then the induced homomorphism $(1_K)_*\colon H_q(K) \to H_q(K)$ is the identity map for each q.

(b) If $f\colon K \to L$ and $g\colon L \to M$ are simplicial maps, then for all q,

$$(g \circ f)_* = g_* \circ f_*\colon H_q(K) \to H_q(M).$$

These two properties mean that the induced homomorphism defines a covariant functor from the category of simplicial complexes **SC** to the category of sequences of abelian groups **SeqAb** whose objects are sequences of abelian groups and morphisms sequences of group homomorphisms.

Taking this into account, the homology groups help us to distinguish when two simplicial complexes are different in the sense that two non-isomorphic simplicial complexes have at least one non-isomorphic homology group.

Theorem 1.3.26. *If K and L are isomorphic simplicial complexes then, $H_q(K) \cong H_q(L)$ for all q.*

The theorem states the homology groups are invariants of the simplicial structure of a simplicial complex K. Now we are going to discuss about the topology of $|K|$. The homology groups are in fact a topological invariant of the *polyhedron* $|K|$.

Definition 1.3.27. A topological space X is said to be a *polyhedron* or *triangulable* if there exist a simplicial complex K and an homeomorphism $f\colon |K| \to X$. The simplicial complex K is called a *triangulation* of X.

We note that we are talking about finite triangulations and considering only finite polyhedra. It is clear from the definition that for a simplicial complex K, $|K|$ is a polyhedron and K a triangulation of $|K|$.

Proposition 1.3.28. *Let $f\colon K \to L$ a simplicial map. Then there is an induced continuous map $|f|\colon |K| \to |L|$ defined as follows: if $x \in |K|$, then x belongs to an unique simplex $\{v_0, \ldots, v_k\}$ of K. We write $x = \sum_{i=0}^{k} \alpha_i v_i$ and define $|f|(x) = \sum_{i=0}^{k} \alpha_i f(v_i)$.*

Remark 1.3.29. If we are considering $|K|$ and $|L|$ as topological spaces, then a continuous map $f\colon |K| \to |L|$ may or may not be induced by a simplicial map. In case it is induced by a simplicial map we will refer to the map as f instead of $|f|$. In this sense we think of the continuous maps from $|K|$ to $|L|$ as simplicial or non-simpicial.

Let $f\colon K \to L$ and $g\colon L \to M$ be simplicial maps. It follows from the definitions that the induced map $|g \circ f|\colon |K| \to |M|$ is the composite map $|g| \circ |f|$. If $1_K\colon K \to K$ is the identity map of the simplicial complex K, then the induced map $|1_K|\colon |K| \to |K|$ is the identity map $1_{|K|}$. The assignation $K \to |K|$ is a covariant functor from **SC** to **Top**.

Definition 1.3.30. Let K be a simplicial complex and v be a vertex of K. The *star* of v and the *open star* of v, denoted respectively by $\mathrm{st}(v)$ and $\mathrm{ost}(v)$ are defined as the following subsets of K and $|K|$:

$$\mathrm{st}(v) = \{\sigma : \sigma \text{ is a simplex of } K \text{ and } v \text{ is a vertex of } \sigma\},$$

$$\mathrm{ost}(v) = \bigcup \{\mathrm{int}(\sigma) : \sigma \text{ is a simplex of } K \text{ and } v \text{ is a vertex of } \sigma\}$$

Given a continuous map $h\colon |K| \to |L|$, a *simplicial approximation* of h is a simplicial map $g\colon K \to L$ if for each vertex v of K, $h(\mathrm{ost}(v)) \subset \mathrm{ost}(g(v))$.

Let $f\colon |K| \to |L|$ and $h\colon |L| \to |M|$ be continuous maps and $g_1\colon K \to L$, $g_2\colon L \to M$ simplicial aproximations of f and h, respectively. Then $g_2 \circ g_1$ is a simplicial aproximation of $h \circ f$.

Theorem 1.3.31. *Let $f\colon |K| \to |L|$ be a continuous map. Then there is a barycentric subdivision $K^{(r)}$ of K and a simplicial map $g\colon K^{(r)} \to L$ such that g is a simplicial aproximation of f. Also, $|g|\colon |K^{(r)}| = |K| \to |L|$ is homotopic to f.*

On the road to the topological invariance of the homology groups, we want to know wether or not there is an induced homomorphism $f_*\colon H_q(K) \to H_q(L)$ for a continuous map $f\colon |K| \to |L|$. In the case that the map is induced by a simplicial map, the answer is yes because f_* is the group homomorphism induced by the simplicial map. If the map f is not induced by a simplicial map, we know there is a simpicial aproximiation $g\colon K^{(r)} \to L$ of h so that the induced continuous map $|g|\colon |K^{(r)}| = |K| \to |L|$ is homotopic to h. As a consequence, we have an induced homomorphism $g_*\colon H_q(K^r) \to H_q(L)$. There is an isomorphism $\mu\colon H_q(K) \to H_q(K^{(r)})$ for any integer r, and if we have two simplicial aproximations of h, $g\colon K^{(r)} \to L$ and $g_1\colon K^{(m)} \to L$, then $g_* \circ \mu = (g_1)_* \circ \mu$. With these two conditions, which can be proved, the induced homomorphism $h_*\colon H_q(K) \to H_q(L)$ given by $h_* = g_* \circ \mu$ is well-defined.

Remark 1.3.32. It follows that for continuous maps $f\colon |K| \to |L|$ the sequence of induced homomorphisms $\{f_*\colon H_q(K) \to H_q(L)\}$ satisfies

(a) The identity map $1_{|K|}\colon |K| \to |K|$ induces the identity homomorphism $1_{H_q(K)}\colon H_q(K) \to H_q(K)$ for each q.

(b) If $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$ are continuous maps, then the induced homomorphism $(g \circ f)\colon H_q(K) \to H_q(M)$ is the homomorphism $g_* \circ f_*$

This means that we have a functorial relation between polyhedra and sequences of Abelian groups and hence the homology groups are a topological invariant.

Theorem 1.3.33. *If $|K|$ is homeomorphic to $|L|$ then, $H_q(K) \cong H_q(L)$ for all q.*

We note that for a polyhedron there is not a unique triangulation, for example a subdivision of a triangulation for a polyhedron is also a triangulation because $|K^{(r)}| = |K|$. But, this is not the only way to have different triangulations. If K and L are triangulations of a polyhedron X then, there are homeomorphisms $f\colon |K| \to X$ and $g\colon |L| \to X$. This means that $h = g^{-1} \circ f$ is an homeomorphism between $|K|$ and $|L|$. By the topological invariance of the homology groups, $H_q(K) \cong H_q(L)$. Taking this into account, we can compute the homolgy of a polyhedron by choosing a suitable triangulation as it does not

depend on the triangulation. We may take one further step to compute the homology of a polyhedron by homotopy.

Definition 1.3.34. A *homotopy* between two continuous maps $f, g\colon X \to Y$ is a continuous map $H\colon X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1)$ for all $x \in X$. H can be considered as a family of maps $H_t\colon X \to Y$, $t \in I$ where each map is jointly continuous. If there is a homotopy between two maps f and g, we say that they are *homotopic*, and we denote it by $f \simeq g$.
We say that a continuous map $f\colon X \to Y$ is a *homotopy equivalence* if there is a continuous map $g\colon Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. In this case we say that the spaces X and Y are *homotopically equivalent* and we write $X \simeq Y$.

Theorem 1.3.35. *If $f, g\colon |K| \to |L|$ are homotopic maps, then they induce the same homomorphisms between homology groups, i.e. $f_* = g_*\colon H_q(K) \to H_q(L)$. In particular $|K|$ and $|L|$ have the same homology.*

As a consequence of the previous Theorem, if $|K| \simeq |L|$ then $H_q(K) \cong H_q(L)$ for all q. We conclude that the homology groups are a homotopic invariant.

Chapter 2

Non-commutative Probability and categorical independence

Non-commutative, or quantum probability (NCP) is an area of Mathematics which started in the 70's with the works of Cushen, Giri, Hudson and von Waldenfels [16, 27, 30, 51]. The main idea of NCP is to re-interpret fundamental notions of probability theory (in particular, the notions of distributions and independence, as well as the derived limit theorems and stochastic processes) in terms of algebraic, analytic and (some years later) combinatorial structures.

This abstract setting allows to understand certain special relations between operators as "new notions" of independence. An important boost on the interest in the theory happened when Voiculescu observed that his recently defined notion of free independence [47] actually explained the global collective behaviour of prominent random matrix ensembles in his striking work [48]. Ever since, new notions of independence have emerged, trying to better describe relations between operators.

In this work, our first goal is to present the ideas and concepts of the recent manuscript [18] which relates the concepts of Betti numbers, Boolean cumulants and non-commutative distributions, from a more categorical point of view. Our second goal is to develop and consolidate those ideas by considering the relevant operators in more general algebraic-probabilistic frameworks. As pointed out in [18], one already needs a (very basic) notion

of operator-valued independence to provide elementary examples of "topological" random variables.

2.1 Main definitions and examples

A non-commutative probability space is an abstract setting where spectral measures related to operators and probability measures associated to random variables can be studied in a unified way.

Definition 2.1.1. We define some concepts.

i. A *non-commutative probability probability space* (NCPS) is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a complex algebra with multiplicative unit $1_{\mathcal{A}}$ and $\tau\colon \mathcal{A} \to \mathbb{C}$ is a unit-preserving linear functional, i.e. $\tau(1_{\mathcal{A}}) = 1$. The elements $a \in \mathcal{A}$ are called *non-commutative random variables* in $(\mathcal{A}, \tau)$ and we are referring to them as random variables, ommiting the non-commutative part or nc-random variable if the context needs.

We call the value $\tau(a^k)$ the *k-th moment* of the random variable $a \in \mathcal{A}$ and for $a_1, \ldots, a_k \in \mathcal{A}$ the *mixed moment* is the collection of values $\tau(a_{i(1)}a_{i(2)} \cdots a_{i(k)})$; $i\colon [m] \to [k], m \in \mathbb{N}$. These collections of values are known respectively as the *algebraic distribution* of the random variable a, and the *algebraic joint distribution* of $(a_1, \ldots, a_k) \in \mathcal{A}^k$ or the *multivariate distribution* of $(a_1, \ldots, a_k)$.

A *NCPS morphism* between two NCPS $(\mathcal{A}, \tau)$ and $(\mathcal{B}, \psi)$ is a unital homomorphism of algebras $\Phi\colon \mathcal{A} \to \mathcal{B}$ such that $\psi \circ \Phi = \tau$.

ii. An *operator-valued probability space* (OVPS) is a tripple $(\mathcal{A}, \mathcal{B}, \mathbb{F})$, where, $\mathcal{B} \subseteq \mathcal{A}$ is a subalgebra of the unital algebra $\mathcal{A}$ (which may or may not contain the unit), and $\mathbb{F}$ is a $\mathcal{B}$-linear *conditional expectation*. This means that for all $a \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ we have

$$\mathbb{F}(bab') = b\mathbb{F}(a)b' \quad , \mathbb{F}(b) = b.$$

We will say that the OVPS $(\mathcal{A}, \mathcal{B}, \mathbb{F})$ is a $\mathcal{B}$-valued probability space or just a $\mathcal{B}$-probability space.

A *OVPS morphism* between two OVPS $(\mathcal{A}_1, \mathcal{B}_1, \mathbb{F}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2, \mathbb{F}_2)$ is a homomorphism of algebras $\Phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ such that $\Phi(B_1) \subset \mathcal{B}_2$ and $\mathbb{F}_2 \circ \Phi = \Phi \circ \mathbb{F}_1$. When the morphism is between two $\mathcal{B}$-probability spaces, the $\mathcal{B}$-VPS *morphism* satisfy that $\mathbb{F}_2 \circ \Phi = \mathbb{F}_1$ because $\mathbb{F}_2(\mathcal{B}) = \mathcal{B} = \mathbb{F}_1(\mathcal{B})$.

Typically it is asked that $\mathcal{A}$ is a $*$-algebra, i.e. $\mathcal{A}$ is endowed with an anti-linear involution $^*\colon \mathcal{A} \to \mathcal{A}$. The functional τ is often required to be positive, in the sense that $\tau(a^*a) \geq 0$ for all $a \in \mathcal{A}$. A random variable a is said to have an *analytic distribution* if there is a measure μ_a such that the moments are encoded by the measure, as seen in the appendix. This is why we are particulary interested in *selfadjoint* ($a^* = a$), *unitary* ($a^*a = aa^* = 1$), or *normal* ($a^*a = aa^*$) random variables in a C^*-algebra or a W^*-algebra: As a consequence of the Riesz-Markov representation theorem there is always a probability measure with those moments $\tau(a^{m_1}(a^*)^{m_2})$, either supported on the real line, on S^1, or on the complex plane respectively, and thus, these special random variables have analytic distribution. It is remarkable that outside the single random variable normal situation, it is a rare fact that random variables have an analytic distribution.

We say that two operator-valued probability spaces $(\mathcal{A}, \mathcal{B}_1, \mathbb{F}_1)$ and $(\mathcal{A}, \mathcal{B}_2, \mathbb{F}_2)$ are *compatible* if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathbb{F}_1 \circ \mathbb{F}_2 = \mathbb{F}_1$. If $\mathcal{B} \cong \mathbb{C}$ we usually write τ instead of $\mathbb{F}$ and we say that the OVPS is scalar. In particular, scalar OVPS are NCPS.

Remark 2.1.2. From definition 2.1.1 we see that the OVPS define a category whose objects are the OVPS and the morphisms are the OVPS morphisms. Indeed, the identity morphism for each OVPS $(\mathcal{A}, \mathcal{B}, \mathbb{F})$ is the homomorphism of algebras given by the set theoretical identity and morphisms hold the composition law, i.e. if $\Phi\colon \mathcal{A} \to \mathcal{B}$ and $\Psi\colon \mathcal{B} \to \mathcal{C}$ are OVPS morphisms then $\Psi \circ \Phi\colon \mathcal{A} \to \mathcal{C}$ is an OVPS morphism. Thus, the NCPS form a subcategory of the OVPS category whose morphisms are the NCPS morphisms. Also, the $\mathcal{B}$-VPS form a subcategory of the OVPS category.

Let us introduce the basic examples of classical probability spaces as NCPS and deterministic matrices. We start by considering these first as scalar probability spaces and later as operator-valued spaces.

Example 2.1.3. We will be working on these basic examples of NCPS and OVPS examples in this work.

1. Let $\mathcal{A}_{\mathcal{F}} := L^{\infty}(\Omega)$ be the algebra of bounded complex random variables on a probability space Ω.

   Involution is given by pointwise complex conjugation of random variables and $\tau := \mathbb{E}$ is the usual expectation.

   The probabilistic terminology used in non-commutative probability theory comes from this example. We call $a = (a_1, a_2, \ldots, a_k) \in \mathcal{A}_{\mathcal{F}}^k$ a random variable, and we refer to the (ordered collection of) values

   $$(\tau(a_{i(1)}a_{i(2)}\cdots a_{i(m)}), m \geq 1, i\colon [m] \to [k])$$

   as the moments of a, or more formally, as its algebraic non-commuative distribution.

2. Now consider $\mathcal{A} := M_n(\mathbb{C})$, with involution given by the Hermitian transpose. There are several candidates for the linear functional τ. Let us mention some of them.

   The normalized trace $\tau = \frac{1}{n}\mathrm{tr}$. Note that, for a normal matrix A, i.e. $A^*A = AA^*$, the moments of (A, A^*) are determined by the values of $\frac{1}{n}\mathrm{tr}(A^m(A^*)^l)$ for all $m, l \geq 1$. Let U be an orthonormal eigenvector basis matrix, i.e. each column of U is a unit length eigenvector u, and the u are orthogonal. Then,

   $$\frac{1}{n}\mathrm{tr}(A^m(A^*)^l) = \frac{1}{n}\mathrm{tr}((U\Lambda U^*)^m(U\Lambda^* U^*)^l) = \frac{1}{n}\mathrm{tr}(\Lambda^m(\Lambda^*)^l) = \frac{1}{n}\sum_{i\leq n}(\lambda_i)^m(\overline{\lambda_i})^l,$$

   which are exactly the moments of the unique complex random variable with distribution μ_A that assigns a weight of $\frac{1}{n}$ to each eigenvalue λ_i of A, counting multiplicity.

   Canonical conditional expectations compatible with $\frac{1}{n}\mathrm{tr}\colon M_n(\mathbb{C}) \to \langle I_n\rangle$ are the identity $\mathrm{id}\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and the projection to the diagonal

   $$\mathbb{F}\colon (a_{ij})_{ij} \mapsto (a_{ij}\delta_{ij})_{ij}.$$

   If $\langle , \rangle$ is a sesquilinear product on $\mathbb{C}^n$ and $v \in \mathbb{C}^n$ is a unit vector, we may define the *vector state* $\tau_v(A) = \langle v, Av\rangle$. In particular, for the vector $v = e_1 = (1, 0, 0, \ldots, 0)$, this expectation yields just the $(1,1)$-entry of the matrix, $\tau_v(A) = A_{11}$. We denote $\tau_i = \tau_{e_i}$.

Any convex combination of unit-preserving linear functionals is a unit-preserving linear funcional. In particular, the normalized trace is the convex combination of vector states $\frac{1}{n}\mathrm{tr} = \frac{1}{n}\sum \tau_{e_i}$.

Associated probability measures for normal elements $A \in M_n(\mathbb{C})$ to $\frac{1}{n}\mathrm{tr}$ is $\mu = \frac{1}{n}\sum \delta_{\lambda_i(A)}$, and associated to the vector state τ_v is $\mu_v = \sum_j \omega_j^{(i)} \delta_{\lambda_j(A)}$, with weights $\omega_j^{(i)}$ adding up to 1, i.e. $\sum_j \omega_j^{(i)} = 1 = \sum_i \omega_j^{(i)}$. Indeed, if we express $v = \sum_{j\leq n} \alpha_j v_j$ in terms of an orthonormal basis of eigenvectors $\{v_1, \ldots, v_n\}$ for A, then

$$\tau_v(A^k(A^*)^l) = \langle v, A^k(A^*)^l v\rangle = \langle \sum_{j\leq n} \alpha_j v_j, \sum_{j\leq n} \lambda_j^k \bar{\lambda}_j^l \alpha_j v_j\rangle = \sum_{j\leq n} |\alpha_j|^2 \lambda_j^k \bar{\lambda}_j^l$$

are the moments of the discrete probability measure with atoms at the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding weights $w_j^{(i)} = |\alpha_j|^2$, and the sum of them is 1 because v is a unit vector.

An interesting case is when A is an adjacency matrix of a graph. We label the vertices in certain order $v_1, \ldots, v_n$ and identify the vertex v_i with the vector e_i of the standar basis in $\mathbb{R}^n$. The moments $\tau_i(A^k)$ are the number of cycles in v_i of lenght k and the distribution μ_i has that weight $\omega_i^{(j)} = 0$ if v_i and v_j are in different connected components of the graph.

The measure μ_i can be used to extend Voiculescu's results on adjoint distributions of Ginibre and deterministic matrices, from the global case $\tau = \frac{1}{n}\mathrm{tr} \otimes \mathbb{E}$, to the local case $\tau = \tau_{e_i} \otimes \mathbb{E}$.

3. *Projection-valued (or rectagular) spaces* [6, 7].

Let $(\mathcal{A}, \tau)$ be a NCPS equipped with pairwise orthogonal, self-adjoint projections $p_0, \ldots, p_d \in \mathcal{A}$, with $\tau(p_i) > 0, p_i^2 = p_i$ and $p_0 + p_1 + \cdots + p_d = 1_{\mathcal{A}}$. If $\mathcal{B} = \langle p_0, \ldots, p_d\rangle$ is the $(d+1)$-dimensional commutative algebra generated by these projections, there is a unique conditional expectation $\mathbb{F}\colon \mathcal{A} \to \mathcal{B}$ compatible with τ, given by

$$\mathbb{F}(a) = \sum_{i=0}^{d} \tau(p_i)^{-1} \tau(p_i a p_i) p_i.$$

For block-diagonal elements in a rectangular space (i.e. elements $a \in \mathcal{A}$ of the form $a = \sum_{i=0}^{d} p_i a p_i$), the study of the distribution of (a, a^*) is essentially the separated study of the pairwise orthogonal pieces (a_i, a_i^*) where $a_i := p_i a p_i$. In fact orthogonality is a categorical independence and pieces in different blocks are $\mathcal{B}$-independent with respect to classical, free, boolean and monotone independence.

Thus, it is natural to associate a normal block-diagonal element $a \in \mathcal{A}$ in a rectangular space with the tuple of probability measures $\mu = (\mu_0, \mu_1, \ldots, \mu_d)$, where μ_i is the analytic distribution of the normal element $p_i a p_i$ in the compressed NCPS $(p_i \mathcal{A} p_i, \tau(p_i)^{-1}\tau|_{p_i \mathcal{A} p_i})$. So once again normal block-diagonal elements have an analytic distribution. In $M_N(\mathbb{C})$ we have a conditional expectation on the algebra of diagonal matrices $\mathbb{F}\colon M_N(\mathbb{C}) \to D_N(\mathbb{C})$ given by the assignation $(a_{ij})_{ij} \mapsto (\delta_{ij} a_{ij})_{ij}$ is compatible with $\frac{1}{N}\mathrm{tr}$. For block-diagonal matrices $A = \mathrm{diag}(B_0, \ldots, B_d) \in M_N(\mathbb{C})$, where $N = n_0 + \cdots + n_d$ and $B_i \in M_{n_i}(\mathbb{C})$, we have the $\langle P_0, \ldots, P_d \rangle$-valued probability space and P_i is the projection to the i-th block such that $P_0 + \cdots + P_d = I_N$.

4. *Tensor product of OVPS.*

   The algebraic tensor product can be used to combine examples of OVPS. If $(\mathcal{A}_1, \mathcal{B}_1, \mathbb{F}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2, \mathbb{F}_2)$ are OVPS, then so is $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{B}_1 \otimes \mathcal{B}_2, \mathbb{F}_1 \otimes \mathbb{F}_2)$. Let us mention two important contructions from prototypical examples.

   First, the tensor product $(M_n(\mathbb{C}) \otimes \mathcal{A}_{\mathcal{F}}, \frac{1}{N}\mathrm{tr} \otimes \mathbb{E})$ which can be identified with the space of matrices with random entries, equipped with the normalized expected trace. The moments of a normal random matrix with respect to $\frac{1}{n}\mathrm{tr} \otimes \mathbb{E}$ are now the moments of the *averaged* uniform spectral probability distribution. Voiculescu's seminal work [48] is framed in these kind of NCPS, where he showed that, as the size of the matrices n becomes larger, important random matrices behave increassingly as free operators (see next subsection for definition of free independence and appendix B for more details).

   Second, consider the tensor product of two algebras of deterministic matrices. Recall that the algebra $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ can be identified with $M_{nm}(\mathbb{C})$ and $\frac{1}{n}\mathrm{tr} \otimes \frac{1}{m}\mathrm{tr}$ corresponds to the normalized trace $\frac{1}{nm}\mathrm{tr}$. Indeed, if $A \in M_n(\mathbb{C}), B \in M_m(\mathbb{C})$ and

their inclusions in the tensor product $A \mapsto A \otimes I_m =: \tilde{A}, B \mapsto I_n \otimes B =: \tilde{B}$, we have

$$\frac{1}{nm}\mathrm{tr}(A \otimes B) = \frac{1}{nm}\mathrm{tr}(\tilde{A})\frac{1}{nm}\mathrm{tr}(\tilde{B}) = \frac{m}{nm}\mathrm{tr}(A)\frac{n}{nm}\mathrm{tr}(B) = \frac{1}{n}\mathrm{tr}(A)\frac{1}{m}\mathrm{tr}(B).$$

If we consider the same spaces but change the normalized trace for the vector state τ_i. We can move from the global case $\tau = \frac{1}{n}\mathrm{tr} \otimes \mathbb{E}$ to the local case $\tau = \tau_i \otimes \mathbb{E}$ and to $\tau_i \otimes \tau_j$ respectively. In the first NCPS the functional τ is clear. In the second case, the operator $\tau_i \otimes \tau_j$ corresponds to the operator $\tau_{e_i \otimes e'_j}$. Indeed,

$$\tau_{e_i \otimes e'_j}(A \otimes B) = \langle e_i \otimes e'_j, A \otimes B(e_i \otimes e'_j)\rangle = \langle e_i, Ae_i\rangle\langle e'_j, Be'_j\rangle = \tau_i(A)\tau_j(B).$$

The idea of $\mathcal{B}$-valued probability is to enlarge the algebra of scalars to $\mathcal{B} \subseteq \mathcal{A}$ instead of $\mathbb{C}\langle 1_{\mathcal{A}}\rangle$.

The $\mathcal{B}$-valued analogue of an algebraic distribution requires a bit more care. It is no longer sufficient to consider the collection of moments

$$(\mathbb{F}(a_{i(1)}a_{i(2)} \cdots a_{i(m)}), m \geq 1, i\colon [m] \to [k]).$$

Since the notions of independence are concepts relative to the generated algebras of the corresponding random variables (see discusion on independence in next section), we will be interested not only in those moments, but also more general expressions, such as

$$(\mathbb{F}(a_{i(1)}b_1a_{i(2)} \cdots b_{m-1}a_{i(m)}), m \geq 1, i\colon [m] \to [k], b_1, \ldots, b_{m-1} \in \mathcal{B}).$$

Obseve that, in the scalar case, no new information is provided since we can just factor the b_i's out of the expectation, since these commute with the rest of the algebra.

The following theorem guarantees the existense of unique conditional expectations which are compatible with a given state.

Theorem 2.1.4. *(Umegaki). Let $\mathcal{A}$ be a W^*-algebra with a regular tracial state $\tau\colon \mathcal{A} \to \mathbb{C}$ and let $\mathcal{B} \subseteq \mathcal{A}$ be a W^*-sub-algebra. Then there exists a unique conditional expectation $\mathbb{F}\colon \mathcal{A} \to \mathcal{B}$, compatible with τ.*

2.2 Independence notions in probability

The first work on classification of notions of independence was formulated in a combinatorial-algebraic way by Speicher [42] and it included the Boolean independence (recently shown to be explained by combinatorial-algebraic structures [43]). The combinatorial-algebraic axioms of Speicher were later reformulated by Ben-Ghorbal and Schürmann [8] in terms of universal properties of co-products on different categories of algebras (see also Franz [23]).

The BGS classification is hierarchical. It uses the same set of axioms (and varies only on the considered category).

At the first level of the classification (on the category of unital, commutative algebras) the only notion of independence is the classical tensor independence. At the second level, of unital, non-commutative algebras, there are two: the classical independence and Voiculescu's free independence. At the third level, of non-unital, non-commutative algebras, there are three notions of independence that meet the axioms.

Muraki observed that, at the third level, a simmetry axiom from the BGS classification only prevented two new notions of independence meeting the rest of the axioms. These are called the monotone, and anti-monotone notions of independence.

Definition 2.2.1. Let $(\mathcal{A}, \mathcal{B}, \mathbb{F})$ be a $\mathcal{B}$-probability space. Let $A_1, \ldots, A_k \subseteq \mathcal{A}$ be not necessarilly unital sub-algebras containing $\mathcal{B}$.

i. The algebras $A_1, \ldots, A_k$ are $\mathcal{B}$-*classically independent* if they commute and

$$\mathbb{F}(a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}) = \mathbb{F}(a_1^{n_1})\mathbb{F}(a_2^{n_2}) \cdots \mathbb{F}(a_k^{n_k}),$$

for any $a_i \in A_i$, $n_i \geq 0$, $i \leq k$. This independence is also called *tensorial independence*.

ii. For any $a \in \mathcal{A}$, let $\dot{a} := a - \mathbb{F}(a)$. The algebras $A_1, \ldots, A_k$ are $\mathcal{B}$-*free* if

$$\mathbb{F}(\dot{a}_1 \dot{a}_2 \ldots \dot{a}_m) = 0,$$

for any $m \geq 1$ and any $a_i \in A_{j(i)}$, $i \leq m$, such that $j(1) \neq j(2) \neq \cdots \neq j(m)$.

iii. The algebras $A_1, \ldots, A_k$ are $\mathcal{B}$-*Boolean independent* if

$$\mathbb{F}(a_1 a_2 \cdots a_m) = \mathbb{F}(a_1)\mathbb{F}(a_2)\cdots\mathbb{F}(a_m),$$

for any $m \geq 1$ and any $a_i \in A_{j(i)}$, $i \leq m$, such that $j(1) \neq j(2) \neq \cdots \neq j(m)$.

iv. The algebras (A_1, A_2) are $\mathcal{B}$-*monotone independent* (in that order), if

$$\mathbb{F}(a_0 a_1 \cdots a_{2m}) = \mathbb{F}(a_1)\mathbb{F}(a_3)\cdots\mathbb{F}(a_{2m+1})\mathbb{F}(a_0 a_2 \cdots a_{2m-2}a_{2m}),$$

for any $m \geq 1$ and any $a_1, a_3, \ldots, a_{2m-1} \in A_1$, $a_2, a_4, \ldots, a_{2m-2} \in A_2$, and $a_0, a_{2m} \in A_2 \cup \langle 1_{\mathcal{A}} \rangle$.

v. The algebras (A_1, A_2) are $\mathcal{B}$-*anti-monotone independent* (in that order), if

$$\mathbb{F}(a_0 a_1 \cdots a_{2m}) = \mathbb{F}(a_1)\mathbb{F}(a_3)\cdots\mathbb{F}(a_{2m+1})\mathbb{F}(a_0 a_2 \cdots a_{2m-2}a_{2m}),$$

for any $m \geq 1$ and any $a_1, a_3, \ldots, a_{2m-1} \in A_2$, $a_2, a_4, \ldots, a_{2m-2} \in A_1$, and $a_0, a_{2m} \in A_1 \cup \langle 1_{\mathcal{A}} \rangle$.

Remark 2.2.2. Note that all the independence notions mentioned in the definition depend on the conditional expectation $\mathbb{F}$ that defines the OVPS. Two algebras $\mathcal{A}_1, \mathcal{A}_2$ can be $\mathcal{B}$-independent (in some sense) respect to a conditional expectation $\mathbb{F}_1$ but not respect to another conditional expectation $\mathbb{F}_2$.

We point out now that our realization $A \otimes I_m, I_n \otimes B$ of classical independent random variables works not only for the normalized traces but also for vector states $\tau_{e_i}, \tau_{e'_i}$ in each of the factors. We only need to observe that, just as $\frac{1}{nm}\mathrm{tr}(A^k \otimes B^l) = \frac{1}{n}\mathrm{tr}(A^k)\frac{1}{m}\mathrm{tr}(B^l)$ for the normalized traces, $\tau_{e_i \otimes e'_j}(A^k \otimes B^l) = \tau_{e_i}(A^k)\tau_{e'_j}(B^l)$ is the $(i \otimes j)$-th entry in the diagonal of the tensor product. Thus, again, $A \otimes I_m$ and $I_n \otimes B$ behave as independent random variables with respect to those vector states and the sum yields the additive convolution of the local distributions $\mu_A^{(i)}, \mu_B^{(j)}$.

Boolean, monotone and anti-monotone independence also find simple realizations in terms of tensor products. Indeed, if P_i and P'_j are the projections to the subspace generated by the elementary vectors e_i and e'_j in each of the factors, then, with respect to $\tau_{e_i \otimes e'_j}$, the

matrices $A \otimes P_i$, and $P'_j \otimes B$ are Boolean independent, $A \otimes P_i$, and $I_n \otimes B$ are monotone independent and $A \otimes I_m$, and $P'_j \otimes B$ are anti-monotone independent. When A, B are adjacency matrices of graphs, the additive convolutions correspond to graph products. For their realizations, free random variables actually require infinite dimensional spaces (or infinite graph products). We discuss about these products in Section 3.2.

Remark 2.2.3. In [38], it is exhibited that in the scalar case $\mathcal{B} = \langle 1_{\mathcal{A}} \rangle \cong \mathbb{C}$ if a and b are classical and free independent, then at least one of them is constant. Thus, in the scalar case the notions of independence exclude each other. These notions become less exclusive as the algebra $\mathcal{B}$ grows larger, as we saw in Example 2.1.3 with the OVPS which was independent in three senses. The extreme case when $\mathcal{B} = \mathcal{A}$ implies that $\mathbb{F} = 1_{\mathcal{A}}$ and makes all random variables in $\mathcal{A}$ independent in all five senses. If we want to find more meaningful independence relations and more useful conditional expectation $\mathbb{F}$, we need to find a smaller subalgebra $\mathcal{B}$.

2.2.1 Categorical independence

In order to unify the different notions of independence in quantum probability, Franz introduced a categorical definition of independence [23]. For this subsection, we follow [26]. We are going to highlight that linear independence and orthogonality fit into this general notion of independence as well. Let $\mathcal{C}$ be a tensor category and $\mathcal{P}_i \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for $i \in \{1, 2\}$ be the projection functors of the respective component.

Definition 2.2.4. Let $\mathcal{C}$ be a tensor category. A natural transformation $\iota^1 \colon \mathcal{P}_1 \Rightarrow \boxtimes$ is called *left inclusion* and a natural transformation $\iota^2 \colon \mathcal{P}_2 \Rightarrow \boxtimes$ is called *right inclusion*. A tensor category together with a left and a right inclusion is called an *inclusive tensor category* or a *tensor category with inclusions*. It is denoted by $(\mathcal{C}, \boxtimes, \iota^1, \iota^2)$ or simply $\mathcal{C}$ when the context is clear.

In the previous definition, inclusions for a tensor category are two collections of morphisms

$\iota^i_{B_1,B_2}\colon B_i \to B_1 \boxtimes B_2$ for $i \in \{1,2\}, B_1, B_2 \in \mathcal{C}$ such that

$$\begin{array}{ccccc} A_1 & \xrightarrow{\iota^1_{A_1,A_2}} & A_1 \boxtimes A_2 & \xleftarrow{\iota^2_{A_1,A_2}} & A_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_1 \boxtimes f_2} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\iota^1_{B_1,B_2}} & B_1 \boxtimes B_2 & \xleftarrow{\iota^2_{B_1,B_2}} & B_2 \end{array}$$

commutes for all $f_i\colon A_i \to B_i, i \in \{1,2\}$. We shall omit the objects on the subindex to simplify notation.

We will give some examples of tensor categories with inclusions later. For now, let us introduce the categorical notion of independence.

Definition 2.2.5. Let $\mathcal{C}$ be a tensor category with inclusions. Two morphisms $f_1, f_2\colon B_i \to A$ are *independent* if there exists a morphism $h\colon B_1 \boxtimes B_2 \to A$ such that the diagram

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle f_1}\nearrow & \uparrow{\scriptstyle h} & \nwarrow{\scriptstyle f_2} & \\ B_1 & \xrightarrow[\iota^1]{} & B_1 \boxtimes B_2 & \xleftarrow[\iota^2]{} & B_2 \end{array}$$

commutes. Such a morphism h is called *independence morphism* for f_1 and f_2.

Example 2.2.6. We give examples of tensor categories with inclusions and independent morphisms on them.

1. In the category **Vect**, the direct sum of vector spaces V_1 and V_2 is a coproduct. Therefore, given two linear maps $f_i\colon V_i \to W$ to a vector space W, there exist a unique linear map $h\colon V_1 \oplus V_2 \to W$ with $h(v_i) = f_i(v_i)$ for all $v_i \in V_i$, $i \in \{1,2\}$, namely $h = f_1 + f_2$. Here we are identifying V_i with the corresponding subspace of $V_1 \oplus V_2, i \in \{1,2\}$. This means that the direct sum of vector spaces is a coproduct in the category and hence, **Vect** is a tensor category. The canonical inclusions on the direct sum define the inclusions for **Vect** to be a tensor category with inclusions. In conclusion, the morphisms f_1 and f_2 are independent with independence morphism $f_1 + f_2$.

   We remark that a category with a coproduct is a tensor category with inclusions and any two morphisms $f_i\colon B_i \to A, i \in \{1,2\}$ are independent with independence morphism $h\colon B_1 \amalg B_2 \to A$ given by the universal property of the coproduct.

2. With the previous example in mind, let us consider the category $\mathbf{Vect}^{inj}$ of vector spaces with injective linear transformations. The direct sum turns this into a tensor category with inclusions respect to the canonical inclusions $V_i \hookrightarrow V_1 \oplus V_2$. Then, two morphisms $f_i \colon V_i \to W$, $i \in \{1, 2\}$ are independent only if $h = f_1 + f_2$ is injective, i.e. f_1 and f_2 are independent if and only if they have linearly independent ranges. If h is injective then $0 = f_1(v_1) + f_2(v_2)$ implies that $f_1(v_1) = 0$ and $f_2(v_2) = 0$, conlcuding that the ranges are linearly independent. Conversely, if the ranges are linearly independent and $h(v_1 \oplus v_2) = 0$, then $f_1(v_1) + f_2(v_2) = 0$ and we conclude that $f_i(v_i) = 0, i \in \{1, 2\}$. Since f_i are injections, it follows that $v_1 = 0$ and $v_2 = 0$, so h is injective.

   In this sense, if A and B are subspaces of a vector space V, they are linearly independent if and only if their canonical inclusions in V are categorically independent, as the ranges of the inclusions are A and B respectively.

3. Similarly, $\mathbf{Hilb}^{isom}$ is a tensor category with inclusions as Hilbert spaces are vector spaces. In this case, two isometries $f_i \colon H_i \to G$ are independent if and only if their ranges are orthogonal. As before, the only choice for an independence morphism is $h = f_1 + f_2$. h is an isometry if and only if

$$\begin{aligned} 0 &= \langle f_1(x_1) + f_2(x_2), f_1(x_1) + f_2(x_2)\rangle - \langle x_1 \oplus x_2, x_1 \oplus x_2\rangle \\ &= \langle f_1(x_1), f_2(x_2)\rangle + \langle f_2(x_2), f_1(x_1)\rangle \\ &= 2\mathrm{Re}\langle f_1(x_1), f_2(x_2)\rangle \end{aligned}$$

   for all $x_1 \in H_1, x_2 \in H_2$, which means $f_1(H_1) \perp f_2(H_2)$.

   We remark that if A and B are subspaces of a Hilbert space H. Then, they are orthogonal if and only if their canonical inclusions in H are categorically independent.

4. In Definition 2.2.1, the five $\mathcal{B}$-independence notions for algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are recipes on their free product $\mathcal{A}_1 * \mathcal{A}_2$, and each recipe comes from an universal product defined on $\mathcal{A}_1 * \mathcal{A}_2$ that turns the category of $\mathcal{B}$-probability spaces into a tensor category with natural inclusions $\mathcal{A}_i \hookrightarrow \mathcal{A}_1 * \mathcal{A}_2$. The simplest example is the

classical independence where the recipe is the tensor product of $\mathcal{B}$-linear functionals:

$$\mathbb{F}_1 \otimes \mathbb{F}_2(a_1 \cdots a_n) = \mathbb{F}_1\left(\overrightarrow{\prod}_{a_i \in \mathcal{A}_1} a_i\right) \mathbb{F}_2\left(\overrightarrow{\prod}_{a_i \in \mathcal{A}_2} a_i\right)$$

and $\overrightarrow{\prod}$ denotes the product of the elements a_i in the same order they appear in $a_1, \ldots, a_n$. This is studied in detail on the Appendix (see Section A.2 for the particular case of classical probability).

2.2.2 Free independence

We want to show with more detail that free independence of algebras $\mathcal{A}_i$ is a categorical notion independence, as mentioned in Example 2.2.6, 4.
We notice that Definition 1.2.7 is in fact, the definition for the free product of $\mathcal{B}$-VPS and 1.2.3 for the free product of NCPS. In [38], the next proposition is proven in the scalar case, and the demonstration for the operator-valued case is analogue.

Proposition 2.2.7. *Let $(\mathcal{A}_i, \mathcal{B}, \mathbb{F}_i)_{i\in I}$ be a family of OVPS, and let $(\mathcal{A}, \mathcal{B}, \mathbb{F})$ be their $\mathcal{B}$-amalgamated free product. Then the following statements hold.*

(1) The subalgebras $\mathcal{A}_i$, $i \in I$ are $\mathcal{B}$-freely independent in $\mathcal{A}$.

(2) $(\mathcal{A}, \mathcal{B}, \mathbb{F})$ has the following universal property. Let $(\mathcal{C}, \mathcal{B}, \mathbb{F}_\mathcal{C})$ be an OVPS, suppose that for every $i \in I$ we have a $\mathcal{B}$-VPS morphism $\Phi_i \colon (\mathcal{A}_i, \mathcal{B}, \mathbb{F}_i) \to (\mathcal{C}, \mathcal{B}, \mathbb{F}_\mathcal{C})$ such that $(\Phi_i(\mathcal{A}_i))_{i\in I}$ are freely independent in $(\mathcal{C}, \mathcal{B}, \mathbb{F}_\mathcal{C})$. Then there exist a unique $\mathcal{B}$-VPS morphism Φ between $(\mathcal{A}, \mathcal{B}, \mathbb{F})$ and $(\mathcal{C}, \mathbb{F}_\mathcal{C})$ such that $\Phi|_{\mathcal{A}_i} = \Phi_i$ for every $i \in I$.

Proposition 2.2.7 states that a family of subalgebras $\mathcal{A}_i$, $i \in I$, are $\mathcal{B}$-freely independent if and only if for any $\mathcal{B}$-VPS $(\mathcal{C}, \mathcal{B}, \mathbb{F}_\mathcal{C})$ such that any $\mathcal{B}$-VPS morphisms Φ_i between $(\mathcal{A}_i, \mathcal{B}, \mathbb{F}_i)_{i\in I}$ and $(\mathcal{C}, \mathcal{B}, \mathbb{F}_\mathcal{C})$ are categorically independent in the category of $\mathcal{B}$-VPS.

Remark 2.2.8. Nica and Speicher prove in [38] that if all the $\mathcal{A}_i$ are $*$-algebras (or W^*-algebras) and the functionals τ_i are all positive (or normal, or self-adjoint). Then, $\mathcal{A}$ and τ are of the same type as the $\mathcal{A}_i$ and τ_i. Thus, $(\mathcal{A}, \tau)$ is the same type of NCPS as $(\mathcal{A}_i, \tau_i)$ for all i, i.e. $(\mathcal{A}, \tau)$ is an object in the category of the NCPS we are working with.

Remark 2.2.9. We must remark that in the commutative case, the free product of algebras is not a commutative algebra. Consequently, if $\mathcal{A}_i \subset \mathcal{A}$ are subalgebras of a commutative algebra $\mathcal{A}$, then the only notion of independence that they support is the classic one (tensor notion). This is due to the fact that there are no free products as objects in the category. This is why in Remark 2.2.3 we say that in the scalar case the tensor and free independence notions are exclusive to each other.

Chapter 3

Non-commutative distributions for graphs and simplicial complexes

From the point of view of Ben Ghorbal - Schuermann [8], there are only three notions of independence. An additional, non-symmetric notion of independence, called monotone independence, is included in the classification if one removes the commutativity axiom, as observed first by Muraki [36].
Graphs have been important for NCP-theory, as adjacency matrices associated to graph products have provided important/simple examples of realizations of convolutions, for each notion of independence.
In this chapter we show that the boolean probabilistic structures show-up when considering processes of simplicial complexes. For this, we need only quite standard NCPS with their usual expectations.
The operators considered in [18] are concrete examples of non-commutative random variables living in graded algebras (as in [33]) where non-commutative analytic and algebraic distributions encode topological data of the associated topological space.

3.1 Graphs as random variables

Definition 3.1.1. Let $G = (V, E)$ be an undirected graph, where V is the vertex set and E is the edge set. The adjacency matrix of G is the matrix $A_G = (a_{ij}) \in M_n(\mathbb{C})$ with entries

$$a_{ij} = a_{ji} = \begin{cases} 1, & (v_i, v_j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the choice of an adjacency matrix is determined by an order of the vertices of G (the spectrum of A_G is invariant to this choice). The graphs in this work are always understood to be finite.

A *rooted graph* is a graph G with a highlighted vertex, called *root*, and labelled r. A rooted graph is denoted by (G, r). The adjacency matrix for a rooted graph is the usual adjacency matrix of G. The root is often selected in a special position (first or last) for convenience.

According to example 2.1.3, we have already several ways to study the adjacency matrix of a graph A_G as a random variable: either in $(\mathcal{A}, \tau)$ or in $(\mathcal{A}, \tau_{e_i})$, where $\mathcal{A} = \langle A_G \rangle_{\mathbb{C}}$, $\tau = \frac{1}{n}\text{tr}$ and $\tau_{e_i} = \langle e_i, Ae_i \rangle$.

For the first case, the k-th *moment* of G is

$$\frac{1}{n}\text{tr}(A_G^k) = \frac{1}{n} \sum_{i:\ [k]\to[n]} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

Each non-zero product $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$ correspods to a k-vertex cycle (consisting of k edges, connecting a sequence of $k+1$ adjacent vertices in the graph, starting (and ending) at the vertex i_1). Thus, the k-th moment is just the average number of such k-cycles.

The *analityc distribution* of G is the measure $\mu_{A_G} = \frac{1}{n} \sum \delta_{\lambda_i(A)}$ that assigns a weight of $\frac{1}{n}$ to each eigenvalue λ_i of A, counting multiplicity.

In the same NCPS, we may choose to represent a graph as a non-commutative random variable, through the graph Laplacian, rather than the adjacency matrix. The graph Laplacian is defined as $L_G = D_G - A_G \in M_n(\mathbb{C})$, where D_G is the *degree matrix* of G (the diagonal matrix with the vertex degrees).

We are interested in the graph Laplacian because it encodes the Betti number $\beta_0(G)$, i.e. the number of connected components of G, in its analytic distribution, as the weight $\mu_{L_G}(0) = n^{-1}\beta_0(G)$.
Indeed, assume that $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are the vertices of a connected component C in G and let $v_C \in \mathbb{C}^n$ be the indicator vector for C whose i-th component is one if $i = i_j, j \leq k$ and zero otherwise. Then we observe that, applied to the indicator vector v_C, the adjacency matrix $A_G(v_C) = D_G(v_C)$ yields exactly the degree vector of the component, and thus, for each component, the vector v_C is a null-eigenvector of L_G. This shows that each component contributes with a weight of $\frac{1}{n}$ to the zero eigenvalue in μ_{L_G} and one can show that there are no additional null eigenvalues, and thus the weight is exactly $n^{-1}\beta_0(G)$.

Remark 3.1.2. It was shown that L_G encodes topological information from G which happens to be homological, the zero-th Betti number. Also, from the relation $L_G = D_G - A_G$, the algebraic joint distribution of any two elements from $\{A_G, L_G, D_G\}$ contain the same information (that is, $\langle A_G, D_G\rangle = \langle A_G, D_G\rangle = \langle A_G, L_G\rangle$). We conclude that the nc-distributions of A_G and L_G are quite related.
In particular, the unital algebras $\langle A_G\rangle, \langle L_G\rangle$ coincide for d-regular graphs, as $D_G = dI_n$ becomes a multiple of the identity.

If the distribution is computed with respect to the vector state $\tau_i = \tau_{e_i}$, the moments simply count the k-cycles that start at the selected vertex v_i (that may be thought as a root of the graph).
Note that, if the graph is not connected the τ_i-moments, and thus the distribution $\mu^{(i)}_{A_G}$, depend only on the connected component of G that contains the root v_i, as observed in Example 2.1.3. At the level of probablity distributions, the weight of the local distribution λ_j in $\mu^{(i)}_{A_G}$ is non-zero if and only if v_j is in the same connected component as the root v_i. Consequently, topological properties of G are encoded in a collection of nc-distributions of A_G. The number of connected components is information available when taking this collection of local distributions, but in a different way than when the Laplacian matrix is used; in this case we can focus on graph properties of each component and, in particular, on the properties of one special vertex.
Taking this into account, the question of assigning an nc-distribution to a graph becomes

a bit ambiguous, with the two main candidates being the adjacency matrix and the graph Laplacian, and the answer depending on the specific feature of the graph to be deduced from the distribution.
There are non-square matrices associated to graphs that are relevant, the *incidence* and the *boundary* matrices.

Definition 3.1.3. Let G be a graph with vertices $v_1, \ldots, v_{n_0}$ and edges $e_1, \ldots, e_{n_1}$. The incidence matrix $I_G \in M_{n_0 \times n_1}(\mathbb{C})$ with entries

$$a_{ij} = \begin{cases} 1, & v_i \in e_j, \\ 0, & \text{otherwise.} \end{cases}$$

We will denote the incidence matrix as I_1 instead of I_G, even if the graph G referred is not explicit.

Remark 3.1.4. The boundary matrix is obtained from the boundary operator in the simplicial homology applied to the graph. We mean, let G be a graph as in the definition. The boundary operator, $\partial_1 \colon C_1 \to C_0$, formula as in 1.2 for simplicial homology for the graph, is defined in the edges $e = (v_i, v_j)$ by

$$\partial_1 e = v_j - v_i.$$

In consequence, the boundary matrix is a signed version of the incidence matri and this signation depend on the order chosen to compute the simplicial homology of the graph.

For a graph G with n_0 vertices and n_1 edges, we will consider these matrices in a rectangular space by inculding them in $M_N(\mathbb{C}), N = n_0 + n_1$ by

$$I_G = \begin{pmatrix} 0 & I_1 \\ 0 & 0 \end{pmatrix}, \; J_G = \begin{pmatrix} 0 & \partial_1 \\ 0 & 0 \end{pmatrix}.$$

We will not be able to properly study these matrices with the functionals that we have considered so far, such as the normalized trace or the vector states. Before fixing states, let us first explore some known identities obtained from these matrices obtained directly by computing the products.

If $I_G \in M_N(\mathbb{C})$ is to be viewed as a nc-random variable, we observe first that $I_G^2 = 0$ and thus for a non-trivial distribution with respect to the normalized trace (or the τ_i's) we need its adjoint. The nc-distribution of $\langle I_G, I_G^* \rangle$ is then completely determined by the individual distributions of the (mutually orthogonal) positive (hence normal) operators

$$I_G I_G^* = \begin{pmatrix} I_1 I_1^* & 0 \\ 0 & 0 \end{pmatrix}, \quad I_G^* I_G = \begin{pmatrix} 0 & 0 \\ 0 & I_1^* I_1 \end{pmatrix}.$$

Since we wish to study these orthogonal operators separately, we should consider the $\langle P_0, P_1 \rangle$-valued distribution of $\langle I_G, I_G^* \rangle$, where P_0 and P_1 are the projections to the diagonal blocks, so that $P_0 + P_1$ is the identity matrix.
The same can be said about $J := J_G$. Thus, it is natural to consider I_G, I_G^*, J_G, J_G^* as elements of a rectangular probability space $(M_N(\mathbb{C}), \langle P_0, P_1 \rangle)$.
The first operator $I_1 I_1^* \in M_{n_0}(\mathbb{C})$ can be expressed in terms of A_G and D_G as $I_1 I_1^* = A_G + D_G$. In the same way, $I_1^* I_1 = A_{G'} + 2P_1$, where G' is the dual graph of G (also called line graph, where edges of G are the vertices of G' and two edges are adjacent if they share one vertex).
Similar considerations hold true for the boundary matrix J_G, in particular,

$$\partial_1 \partial_1^* = D_G - A_G = L_G, \quad \partial_1^* \partial_1 = 2P_1 - \operatorname{sgn}(A_{G'}),$$

where $\operatorname{sgn}(A_{G'})$ is a signed version of the dual adjacency matrix, which depend on the signation of the boundary operator. The operators $JJ^* + J^*J$, JJ^* and J^*J are called *combinatorial*, *up* and *down Laplacians* respectively.
The above identities say that the scalar distributions of adjacency matrices, dual adjacency matrices, degree matrices and graph Laplacian are related to each other and, as we see in in Theorem 3.3.1, these relations may be expressed entirely in terms of distributions of incidence and boundary matrices.

Example 3.1.5. Let K_3 be the complete graph of three vertices. Let us write the blocks of the matrices meentioned, i.e. adjacency, incidence and boundary and the betty numbers encoded in the (global) spectral meassure of the combinatorial Laplacian.

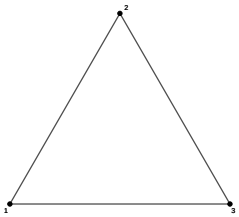


Figure 3.1: The complete graph of three vertices, K_3.

3.2 Boolean cumulants and products of graphs

In terms of graphs, the monotone independence was first treated in [2] and the free case in [1] (where also the increasingly important concepts of subordination and free deconvolution are formulated in terms of graphs).
For any two points they are isomorphic, in particular, if the points are vertices in different graphs. We can think of glueing graphs by vertices if we think on the previous idea.

Definition 3.2.1. Let (G, r) and (G', r') be rooted graphs and let $\phi\colon r \to r'$ be the vertex homeomorphism. We define the *star product* of (G, r) and (G', r') by the rooted graph $G \star G' = G \sqcup_\phi G'$ with root r where r is the glueing vertex of G and G'. We mean, (G, r) and (G', r') is the quotient topological space in $G \sqcup G'$ given by the relation $r \sim r'$ and choosing r as the root in the new graph.

Example 3.2.2. We give two graphs and their star product in the next figure.

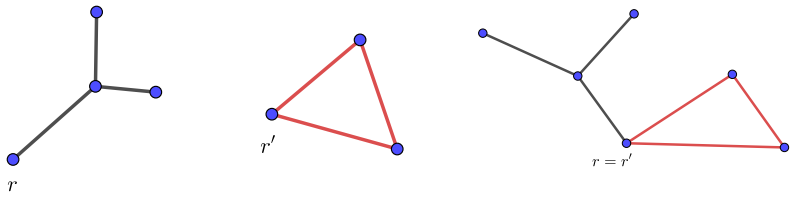


Figure 3.2: Rooted graphs and their star product.

We notice that if we think on a random walk in a rooted graph starting in the root, then the star product of rooted graphs will behave as walking in either on the two pieces if we are not walking through the root because the only way to have a walk in both pieces is taking a step in the root. So it could be said that it is *independent* on which graph we walk. This is going to have an interesting consequence, let us recall about *Boolean cumulants*.

In the framework of an NCPS $(\mathcal{A}, \tau)$, the Boolean cumulants $(b_m(x, \dots, x))_{m\geq 1}$ and the multivariate boolean cumulants $b_m(x_{i(1)}, x_{i(2)}, \dots, x_{i(m)}$, $m \geq 1$, $i\colon [m] \to [k])$ are polynomials on the (mixed) moments. The multivariate boolean cumulants for $(a_1, \dots a_k) \in \mathcal{A}^k$ may be defined inductively, by solving the formlulas:

$$\tau(a_{i(1)}a_{i(2)} \dots a_{i(m)}) = \sum_{1\leq j\leq m} b_j(a_{i(1)}, a_{i(2)}, \dots, a_{i(j)})\tau(a_{i(j+1)}a_{i(j+2)} \cdots a_{im}), m \geq 1, i\colon [m] \to [k]$$

The boolean cumulants $(b_m)_{m\geq 1}$, $b_m\colon \mathcal{A}^m \to \mathbb{C}$ are thus homogeneous polynomials on the moments. For example, the first cumulants are the mean $b_1(a_{i(1)}) = \tau(a_{i(1)})$ and the covariance $b_2(a_1, a_2) = \tau(a_1a_2) - \tau(a_1)\tau(a_2)$.

Recall that the local moments in $M_n(\mathbb{C})$ of a self-adjoint matrix A are $m_k^{(j)} = \tau_j(A^k) = \sum_{i\colon [k-1]\to[n]} A_{ji(1)}A_{i(1)i(2)} \dots A_{i(k-1)j}$ which, in the case of an adjacency matrix of a graph, count the number of k-cycles from the root vertex. It can be shown by induction that the corresponding Boolean cumulants count only those *irreducible* k-cycles, which revisit the root at the very last step.

Hence, it is immediate from the observation to deduce that Boolean cumulants are additive for star products of rooted graphs, that is, with respect to $\tau_{e_i\otimes e'_j}, \tau_{(e_i)}, \tau_{(e'_j)}$, we have

$$b_k(A_{G_1\star G_2}, A_{G_1\star G_2}, \dots, A_{G_1\star G_2}) = b_k(A_{G_1}, \dots, A_{G_1}) + b_k(A_{G_2}, \dots, A_{G_2}).$$

As the Boolean cumulants, the classical, free, and monotone cumulants are also homogeneous polynomials on the mixed moments, [41, 43]. Let us define other rooted graph products to show what we mean about *graph matrices realizing additive convolutions*.

Definition 3.2.3. Let (G, r) and (G', r') be rooted graphs. We define

i. The *cartesian product* of G and G' is the graph consisting of as many copies of (G, r) as vertices in G' and copies of (G', r') as vertices in G. On each vertex of

G glue by the root a copy of (G', r'), then fix the copy on the root of (G, r) and glue on each vertex of that copy, a copy of (G, r); then, glue the remaining vertices of those glued copies to the copies of (G', r') on the corresponding vertices in the copies where (G, r) was glued. This product we denote it as the graph $G \square G'$ which is a simple graph and observe that this product is commutative. This description is a bit cumbersome but we will illustrate it in an example (Figure 3.2).

ii. The *comb product* of (G, r) and (G', r') is the graph obtained by gluing on each vertex of G a copy of (G', r') by its root. The comb product of (G, r) and (G', r') is a rooted graph with root r, the root of G. We notice this product is not commutative.

iii. The *opposite comb product* of (G, r) and (G', r') is the comb product of (G', r') and (G, r).

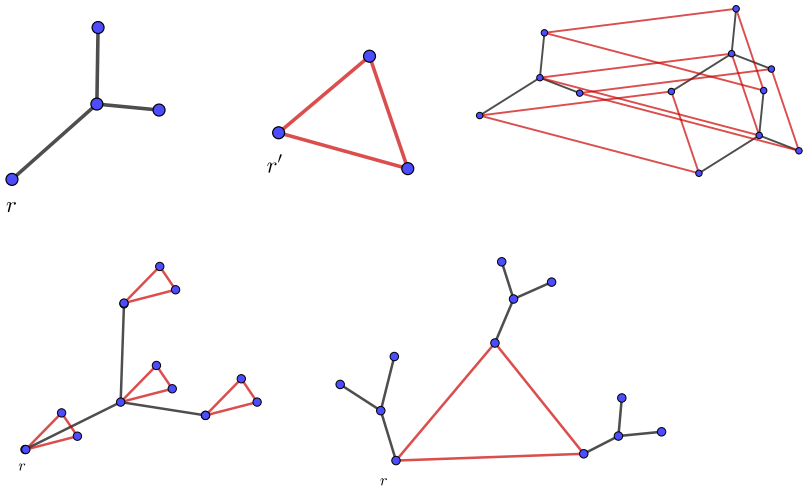


Figure 3.3: Rooted graphs and their cartesian, comb and opposite comb products.

As observed in [2, 1], if the matrices $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ are adjacency matrices of graphs, the same can be said about the elementary nc-independent constructions $A \otimes I_m$, $I_n \otimes B$, $A \otimes P_{e'_j}$, $P_{e_j} \otimes B$ mentioned before. The additive convolutions therefore turn into graph products: cartesian product (tensor), star product of rooted graphs (Boolean),

comb product of rooted graphs (monotone), and the opposite comb product of rooted graphs (anti-monotone).

The case of the Boolean convolution $A \otimes P_{e'_j} + P_{e_j} \otimes B$, realized by the star product of rooted graphs is of special interest for us. Boolean cumulants, defined with respect to a local state τ_j, count the number of *irreducible* k-cycles (i.e. k-cycles not visiting the root at some intermediate step).

The cases of the tensor, monotone and anti-monotone convolutions are realized respectively by the matrices of graph productos $A \otimes I_m + I_n \otimes B$ (cartesian product), $A \otimes P_{e'_j} + I_n \otimes B$ (comb product) and $A \otimes I_m + P_{e_j} \otimes B$ (opposite comb product).

There is also a free product of rooted graphs [1] (see also [53]) which is in fact an infinite graph (Figure 3.2). It can be constructed inductively. First we take the *orthogonal product* of the rooted graphs. For each vertex which is not the root on the first graph, we glue a new copy of the other graph by its root. Then, for each *new* vertex added in the last step, a new copy of the first graph shall be glued to the new vertex, by its root, and we repeat this last step inductively; this is one branch which turns out to be infinite. The second branch is the same process but starting with the other graph and alternate glueings with the first. The star product of these two (infinite) branches is the free product of two rooted graphs. We say a step on the construction is doing a step on the construction on each branch and do the star product of the graphs obtained in that step.

The resulting infinite graph is known as the free product of rooted graphs, and it realizes free additive convolution of the local probability distributions of the factors.

From the perspective of analytic transforms (see [43]), the so-called self-energy transform $K_\mu \colon z \mapsto F_\mu(z) - z$ linearizes the Boolean additive convolution $\mu \uplus \nu$. Indeed, $K_{\mu \uplus \nu}(z) = K_\mu(z) + K_\nu(z))$, where $F_\mu(z) = (G_\mu(z))^{-1}$ is the $\mathbb{C}^+$-analytic self-map, defined as the reciprocal of the Cauchy transform

$$G_{\mu_x}(z) = \tau((z-x)^{-1}) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_x(t)}{z-t}.$$

In turn, the monotone $\mu \triangleright \nu$ and antimonotone $\mu \triangleleft \nu$ additive convolutions are linearized by compositions of the F-transforms: $F_{\mu \triangleright \nu}(z) = F_\mu(F_\nu(z))$ and $F_{\mu \triangleleft \nu}(z) = F_\nu(F_\mu(z))$

Thus, as observed in [1], the free graph product can be thought as a star product of two

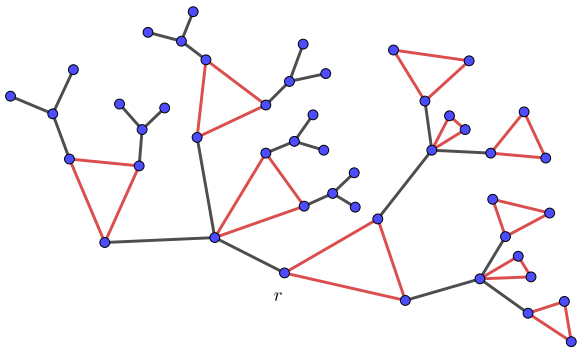


Figure 3.4: A third step of the construction of the free product of rooted graphs on the root.

infinite branches, where each branch is closely related to infinite alternating iterations of comb and opposite-comb products of the factor graphs. This provides some simple insight into the analytic subordination phenomenon for computing free additive convolutions. The method of analytic subordination has been very useful (together with its $\mathcal{B}$-valued generalization) for practical applications in large random matrix theory [4, 5].

The asymptotic semicircularity of Wigner random matrices [52] is *universal*. This means that, under mild moment conditions, the limit eigenvalue distributions of Wigner matrices do not depend on the specific distribution of the entries, as long as these are centered and have a fixed variance.

Thus, it can be shown that the eigenvalue distributions of normalized adjacency matrices from homogeneous Erdös-Rényi random graph models converge to the semicircular distribution [10]. A few years later, Ding and Jiang used asymptotic free independence to show that the spectrum of the corresponding normalized graph Laplacian converges to the free convolution of a normal distribution and the semicircle distribution [19]. Recently, the Laplacian and adjacency matrices of inhomogeneous Erdös-Rényi random graphs have been shown to converge in spectral distributions to operators built-up by free and classically independent random variables [15].

These results suggest that it should be possible to describe spectra of Laplace operators and adjacency matrices of more sophisticated random graph models (likely using more

advanced construccions, such as $\mathcal{B}$-independence or traffics)
Thus, in general, it seems relevant to consider not only adjacency matrices, but also graph Laplacians, for a broader study of nc-random variables associated with graphs.

3.3 Simplicial complexes as random variables

A (finite, abstract) simplicial complex $X = \bigcup_{r\leq d} \cup_{i\leq n_r} \sigma_i^{(r)}$ is a collection of simplices, with the property that any subsimplex of any simplex $\sigma_i^{(r)} \in X$ also belongs to X, and the intersection of any two different simplices of X (possibly of different dimensions), is either empty or equal to some common subsimplex. We write $N = n_0 + n_1 + \cdots + n_d$. Let us always suppose that the simplicial complex X we are talking about is of dimension d. We label the n_0 vertices of X with the numbers $1, 2, \ldots, n_0$ and denote an r-dimensional face $\sigma_j^{(r)} = (i(0)i(1)\ldots i(r))$ by listing its vertices in increasing order.
For example, we may denote the abstract hollow tetrahedron by the collection of faces

$$X = \{1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234\},$$

or, for short, we may list only the faces of maximal degree $\langle 123, 124, 134, 234\rangle$ (when they determine the simplicial complex).
When it comes to higher-dimensional simplicial complexes, the question of selecting a scalar-valued distribution includes as obvious candidates the generalizations of the adjacency matrix to higher dimensions $r \leq d-1$, $A_X^{(r\uparrow)} \in M_{n_r}(\mathbb{C})$, where the NCPS $M_{n_r}(\mathbb{C})$ may be endowed with the normalized trace or a vector-state. Of course we may also consider the corresponding degree matrices $D_X^{(r\uparrow)} \in M_{n_r}(\mathbb{C})$.
Here, the arrow $\uparrow$ indicates that an r-face is adjacent to another r-face iff they belong to an $(r+1)$-face of X. The moments of $A_X^{(r\uparrow)} \in M_{n_r}(\mathbb{C})$ count k-cycles of neighboring r-faces. In fact, $A_X^{(r\uparrow)}$ coincides with the adjacency matrix of the graph where the r-faces of X are the vertices, sharing an edge if the faces belong to the same $(r+1)$-face. We shall discuss these graphs in more detail at the end of the section.
We may also consider *dual* adjacency matrices $(A_X^{(r\downarrow)})_{1\leq r\leq d}$, where adjacency between r-faces is determined by sharing an $(r-1)$-face. Dual adjacency matrices are more

representative to our first undestandings of growing processes of simplicial complexes. When two tetrahedra are glued by a 2-face, we normally think that the two tetrahedra are the main objects, as in $A^{3\downarrow}$, and not the 8 faces, two of which to be identified, as in $A^{2\uparrow}$.

However, while the usual adjacency matrices $(A^{r\uparrow})_{0\leq r\leq d-1}$ completely determine the abstract simplicial complexes, the dual matrices may forget some information. For example, the hollow tethrahedron without the base $X_1 = \langle 123, 134, 124\rangle$ and the notebook with three triangular sheets $X_2 = \langle 123, 124, 125\rangle$ have the same matrix $A^{2\downarrow}$, but the $A^{1\uparrow}$ are different.

For Laplacian matrices, unlike the graphs case, reasonable higher-dimensional generalizations are not simply obtained by substracting degree and adjacency matrices. This will push us to consider *non-square matrices*, such as the incidence or boundary matrices. These demand to be treated as elements of a rectangular (or projection-valued) space [6, 7].

Before leaving the scalar case for $\mathcal{B}$-valued situations, we want to discuss a few combinatorial properties of the square matrices $(A_X^{(r\uparrow)})_{0\leq r\leq d-1}$

In general, $A_X^{(r\uparrow)}$ is the adjacency matrix of a graph, consisting of a collection of n_{r+1} cliques (or complete graphs) $(K_{r+1}^{(i)})_{i\leq n_r}$, each corresponding to an $(r+1)$-face in X. Vertices of such cliques are the r-faces of the $(r+1)$-simplices. If two $(r+1)$-simplices share an r-face, this corresponds at the level of graphs to the identification of the two vertices of the two cliques associated to the indentified r-faces.

The existence of each $(r+1)$-face in X implies the mutual adjacency of all its subfaces of dimension r. Thus, adjacency in general comes with multiplicity.

Now let us go back to our main question of associating matrices and distributions with simplicial complexes.

As we shall see in this section, boundary and incident matrices require to be considered as projection-valued distributions. Projection-valued random variables are very tame instances of $\mathcal{B}$-valued random variables. Both the questions of organizing/assigning algebraic or analytic distributions of simplicial complexes can be better understood in this framework as we saw earlier in the graph case.

For a simplicial complex X, the *incidence matrix* $I_X = I = (I_{ij})_N \in M_N(\mathbb{C})$ consists entirely of zeros and ones. The entry I_{j_1,j_2} is 1 if and only if $\dim(\sigma_{j_1}) + 1 = \dim(\sigma_{j_2})$ and $\sigma_{j_1} \subset \sigma_{j_2}$. We usually consider a convenient order of the N faces, by increasing dimensions (sometimes, when X consists of several disconnected components, one may order faces first by blocks according to components and then each block by dimension).

So this matrix is a upper shifted $(n_0, \ldots, n_d)$-block matrix, where there are matrices $I_r(X), 1 \leq r \leq d$ and I_X is of the form

$$M_N(\mathbb{C}) \ni I = \begin{pmatrix} 0 & I_1(X) & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_2(X) & \cdots & 0 & 0 \\ & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 0 & I_d(X) \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The $I_r(X)$ are the incidence matrices of the graphs associated to the adjacency matrices $A_X^{(r-1)\uparrow}$, so we have an adjacency matrix for the simplicial complex, $A_X = \mathrm{diag}(A_X^{0\uparrow}, \ldots, A_X^{(d-1)\uparrow}, 0)$. Thus, A_X is a block diagonal matrix in $M_N(\mathbb{C})$.

The *boundary matrix* $J_X \in M_N(\mathbb{C})$ is obtained from the simplicial homology applied to the simplicial complex X. This time we have d boundary operators, $\partial_r \colon C_r \to C_{r-1}$, $1 \leq r \leq d$, defined on r-faces $\sigma^{(r)} = (i(0)i(1)\cdots i(k-1)\cdots i(r))$ by the Equation 1.2 as

$$\partial_r(\sigma^{(r)}) = \sum_{k \leq r} (-1)^k (i(0)i(1)\cdots i(k-1)\widehat{i(k)}i(k+1)\cdots i(r)),$$

where the notation $\widehat{i(k)}$ indicates that the vertex $i(k)$ has been removed from $\sigma^{(r)}$ to obtain an $(r-1)$-dimensional face.

Consequently, δ_r is a signed version of $I_r(X)$ for each r. Similar to the incidence matrix, the boundary matrix is a shifted $(n_0, \ldots, n_d)$-block diagonal matrix of the form

$$J_X = J = \begin{pmatrix} 0 & \partial_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \partial_2 & \cdots & 0 & 0 \\ & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 0 & \partial_d \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

A fundamental fact from simplicial homology is that $\partial_{r-1}\partial_r = 0$ for all $1 \leq r \leq d$, and thus $J^2 = 0 = (J^*)^2$. Also, J is a signed version of I, just as in the graph case.
We notice that, as in each dimension of X we have a graph (and a dual version) then the matrices I, I^*, J, J^* hold the identities we exposed for graphs. These relations for each $1 \leq r \leq d$ are:

$$I_r I_r^* = A^{(r-1)\uparrow} + \frac{D^{(r-1)\uparrow}}{r}, \quad I_r^* I_r = A^{(r)\downarrow} + (r+1)P_r, \tag{3.1}$$

$$\partial_r \partial_r^* = \frac{D^{(r-1)\uparrow}}{r} - \operatorname{sgn}(A^{(r-1)\uparrow}), \quad \partial_r^* \partial_r = (r+1)P_r - \operatorname{sgn}(A^{(r)\downarrow}) \tag{3.2}$$

The different dimensions of the faces induce a block-structure on $M_N(\mathbb{C})$ and thus a natural decomposition of the identity matrix into $d+1$ pairwise orthogonal projections $I_N = P_0 + P_1 + \cdots + P_d$, where P_r is the projection to the subspace $C_r \subset \mathbb{C}^N$, $C_r \cong \mathbb{C}^{n_r}$, called *chain group*. In particular, $\operatorname{tr}(P_i) = n_i$.
Consider X with its faces ordered by increasing dimension and let $\mathbb{F}_{\lambda_0} \colon M_N \to \langle P_0, P_1 \dots P_k \rangle$ be the conditional expectation onto the algebra of projections to each chain group. The projection-valued expectation is of course compatible with the diagonal expectation $\mathbb{F}_{\lambda_1} \colon M_N(\mathbb{C}) \to D_N(\mathbb{C})$. Then the previously considered scalar-valued distributions, both local and global, for simplicial complexes, and some more, are compressions of some projection-valued distribution of $\langle I_X, I_X, J_X, J_X^* \rangle$, sitting between (and compatible with) the expectations $\mathbb{F}_{\lambda_0}$ and $\mathbb{F}_{\lambda_1}$.
Recall that we are thinking of the adjacency, degree, incidence and boundary matrices, taking into account the block structure that suggest such conditional expectations which are compatible with the studied expectations, $\frac{1}{n}\operatorname{tr}$ and τ_i. The next result states that all these distributions are obtained by studying a special *-algebra endowed with the conditional expectations before.

Theorem 3.3.1. *All the distributions for simplicial complexes above mentioned, local and global, are included in the joint $*$-distribution of I_X, J_X with respect to a tower of compatible projection-valued expectations, sitting between $\mathbb{F}_{\lambda_0} \colon M_N(\mathbb{C}) \to \langle P_0, P_1, \dots, P_k \rangle$ and $\mathbb{F}_{\lambda_1} \colon M_N(\mathbb{C}) \to D_N(\mathbb{C})$.*

Proof. We first observe that the positive (thus normal) operators $I_X I_X^*$, $I_X^* I_X$, $J_X J_X^*$, $J_X^* J_X$ in $\langle I_X, I_X^*, J_X, J_X^* \rangle$ are $P_0, P_1, \ldots, P_k$ block-diagonal matrices and hence, by compressing, their algebraic distribution can be recovered from the individual distributions of the $d+1$ orthogonal pieces.

We will state the matrix relations (all of them well-known and elementary, expressed in Equations 3.1 and 3.2) in terms of conditional expectations simultaneously for all dimensions. One has access to matrices for each different dimension by considering the corresponding compressed space.

1. $\mathbb{F}_{\lambda_1}(I_X I_X^*) = \mathbb{F}_{\lambda_1}(J_X J_X^*)$ is a block-diagonal matrix, with each block being the r-dimensional (diagonal) degree matrix $D_X^{(r\uparrow)} \in M_{n_r}$, $0 \leq r \leq d-1$. The compressed piece corresponding to P_d is the zero matrix.

2. $I_X I_X^* - \mathbb{F}_{\lambda_1}(I_X I_X^*)$ is a block-diagonal matrix containing each r-dimensional adjacency matrix $A_X^{(r\uparrow)} \in M_{n_r}$, $0 \leq r \leq d-1$. The compressed piece corresponding to P_d is the zero matrix. With J_X we obtain signed versions of each adjacecncy matrix.

3. $\mathbb{F}_{\lambda_1}(I_X^* I_X) = \mathbb{F}_{\lambda_1}(J_X^* J_X) = 0P_0 + 2P_1 + 3P_2 + \cdots + (d+1)P_d$

4. $I_X^* I_X - \mathbb{F}_{\lambda_1}(I_X^* I_X)$ is a block-diagonal matrix containing each dual r-dimensional adjacency matrix $A_X^{(r\downarrow)} \in M_{n_r}(\mathbb{C})$, $1 \leq r \leq d$. The compressed piece corresponding to P_0 is the zero matrix. With J_X we obtain a signed version of each dual adjacency matrix.

5. $J_X J_X^* + J_X^* J_X$ is a block-diagonal matrix containing all combinatorial Laplace operators, defined as $L_r := \partial_{r+1}\partial_{r+1}^* + \partial_r^* \partial_r \in M_{n_r}(\mathbb{C})$, for $0 \leq r \leq d$, with $\partial_0 = \partial_{d+1} = 0$.

6. $\mathbb{F}_{\lambda_1}((J_X^* J_X - \mathbb{F}_{\lambda_1}(J_X^* J_X))^2) = \mathbb{F}_{\lambda_1}((I_X^* I_X - \mathbb{F}_{\lambda_1}(I_X^* I_X))^2)$ is a block-diagonal matrix containing all dual degree matrices $D_X^{(r\downarrow)} \in M_{n_r}(\mathbb{C})$, for $1 \leq r \leq d$.

□

It is worth to mention that the positive operators in the block diagonal of $J_X J_X^*$ and $J_X^* J_X$ are respectively called the *up* and *down Laplacians*, and their spectra have been studied in [29]. The operator $J_X J_X^* + J_X^* J_X$ is called the *combinatorial Laplacian*.

We now recall that block-diagonal normal operators may be equipped with a $(d+1)$-tuple of probability distributions. The choice for a good normal block-diagonal operator is then important for associating X with a tuple of analytic distributions, which is intended to be informative in some sense.

From our previous theorem, we have seen that the algebra $\langle I_X, I_X^*, J_X, J_X^* \rangle$ equipped with projections to the diagonal includes most algebraic distributions associated with simplicial complexes. Now, we turn back to the question of associating analytic distributions.

In the scalar case, this amounts to select a state and a normal operator in $\mathcal{A}$. In the $\mathcal{B}$-valued case, there is in general no analytic distribution unless our operator is of a very special kind. For example, the case of normal, block-diagonal operators in a projection-valued space.

So let us first describe normal block-diagonal operators in the algebra $\langle I_X, I_X^*, J_X, J_X^* \rangle$. In the algebra $\langle I_X, I_X^* \rangle$ there are several choices for such operators. For example: $I_X^k (I_X^*)^k$ or $(I_X^*)^k I_X^k$, for $k \leq d$ or, more generally, the block-diagonal operators TT^* or T^*T, where $T = I_X^{k_1} (I_X^*)^{l_1} I_X^{k_2} (I_X^*)^{l_2} \cdots (I_X^*)^{l_m}$.

The operator T is non-zero, then the block-diagonal operators TT^* and T^*T have non-trivial analytic distributions. if

$$-d \leq \sum_{i \leq r} k_i - \sum_{i \leq r} l_i \leq \sum_{i \leq r} k_i - \sum_{i \leq r-1} l_i \leq d,$$

for all $r \leq m$. Thus, there are plenty of potential analytic distributions if we include I_X in our analysis.

If, instead, we restrict our search to the much better behaved algebra $\langle J_X, J_X^* \rangle$, where we have the convenient identity $0 = J_X^2 = (J_X^*)^2$, the algebraic moments of $\langle J_X, J_X^* \rangle$ are completely determined by the analytic distributions of the mutually orthogonal, positive, block-diagonal operators JJ^* and J^*J. Thus, there are not so many possibilities for block-diagonal normal operators.

Two canonical choices are $\sqrt{L} := J_X + J_X^*$, and its square $L = (J_X + J_X^*)^2 = J_X J_X^* + J_X J_X^* = \sum_{0 \leq r \leq d} P_r L_r P_r$. Both may be equipped with a $(d+1)$-tuple of probability distributions $(\mu_0, \mu_1, \ldots, \mu_d)$. The block-diagonal pieces $L_r = \partial_{r+1}^* \partial_{r+1} + \partial_r \partial_r^*$, $0 \leq r \leq d$ are the so-called higher-dimensional combinatorial Laplace operators (see Eckmann [20])

from Hodge theory.
The diagonal blocks of the orthogonal operators $L^{\uparrow} := J_X J_X^*$ and $L^{\downarrow} := J_X J_X^*$ are respectively, the up and down-Laplacians. We highlight that the analytic distribution of $J_X J_X^*$ and $J_X J_X^*$ completely determines the algebraic $\mathbb{F}_0$-distribution of (J_X, J_X).
By the properties of $(\partial_r)_r$, one can show that $\ker(L) = \mathrm{Im}(L^{\uparrow}) \cap \mathrm{Im}(L^{\downarrow})$ and thus, each Betti number of X is just $\dim \ker(L_r)$. Consequently, Betti numbers can be read from the weight $\mu_r(0) = n_r^{-1}\beta_r(X)$.

Remark 3.3.2. We encountered Betti numbers in analytic nc-distributions quite directly, guided by the natural question in NCP of assigning analytic distributions to simplicial complexes.

3.4 Star product and additive Boolean cumulants of simplicial complexes

Thinking about the random walks in graphs as we talked about in Section 3.2, we could think about random walks for higher dimension simplicial complexes. As in graphs, we walk from a vertex to another through edges, i.e. adjacent vertices; in simplicial complexes we are only allowed to move from a face to another if the faces are adjacent. This restrict the walks to be made in one dimension of faces in the simplicial complex in either of the two ways, upside ($A_X^{\uparrow}$) or downside ($A_X^{\downarrow}$). Let us say that the walks in dimension r are called *r-walks*. To take more general walks in a simplicial complex we would need different notions of adjacency (and their duals) to make it happen.
On the other hand, if we think of simplicial complexes as in Remark 1.3.8; where they are made of simplices properly glued, we think of the *star product* of simplices (or simplicial complexes) and walks in the resulting simplicial complex.
In the Section 1.3.1 we considered all the simplices to be standard and two of r-simplices σ_1 and σ_2 are homeomorphic via a piece-wise linear bijection described in the case where one of them was no standard (it is obvious for standard).

Definition 3.4.1. An *r-rooted simplicial complex* is a simplicial complex X with a high-

lighted r-simplex (often called *root*, and labelled f_r), $r \leq \dim X$. An r-rooted simplicial complex is denoted by (X, f_r). The adjacency matrices of a r-rooted simplicial complex (X, f_r) are the usual adjacency matrices of X.

Definition 3.4.2. Let (X, f_r) and (X', f'_r) be rooted graps, $r \leq \min\{\dim X, \dim X'\}$, and let $\phi\colon f_r \to f'_r$ the homeomorphism between the chosen r-simplices. The *r-star product* of (X, f_r) and (X', f'_r) is the r-rooted simplicial complex $X \star_r X' = X \sqcup_\phi X'$ with root f_r, the glueing r-simplex.

When we think of a random r-walk in an r-rooted simplicial complexes starting at the root, we are interested in a specific type of walk that is compatible with the r-star product. We mean, there are random r-walks in $X \star_r X'$ starting at the root. Then, the r-star product of these simplicial complexes behave as each walk is taken in either of the two pieces if we are not walking through the root.

Let us state a generalization of Boolean cumulants. Let X be a simplicial complex and let $\mathcal{A}^\uparrow = \langle A_X^\uparrow \rangle_{\mathbb{C}}$. $(\mathcal{A}^\uparrow, \tau_i)$ is the NCPS where τ_i is the vector state corresponding to an r-face.

Given an r-rooted simplicial complex (X, f_r), we define a collection of d r-Boolean cumulants for X. The k-th *r-up Boolean cumulant* denoted by $b_k^{(r)\uparrow}$ as the k-th Boolean cumulant of the rooted graph associated to $A_X^{(r)\uparrow}$, $0 \leq r \leq d-1$.

From the fact that $A_X^\uparrow$ is a block diagonal matrix then $(A_X^\uparrow)^k$ is also block diagonal matrix and each block is the k-th power of the corresponding block and thus, the local moments in $M_n(\mathbb{C})$ of the self-adjoint matrix $A_X^\uparrow$ are $m_k^{(j)} = \tau_j((A^\uparrow)^k) = \sum_{i\colon [k-1]\to[n]} A^\uparrow_{ji(1)} A^\uparrow_{i(1)i(2)} \cdots A^\uparrow_{i(k-1)j}$ which count the number of k-r-cycles from the root simplex; A_X denotes the up adjacency matrix. It can be shown by induction that the up Boolean cumulants count only those *irreducible* k-r-cycles, which revisit the root at the very last step, just as in the graph case (the simplicial complex on each dimension is looked like a graph).

Consequently, the boolean cumulants are additive for r-star products of r-rooted simplicial complexes, that is, with respect to vector states, we have

$$b_k^{(r)\uparrow}(A^\uparrow_{X_1\star_r X_2}, A^\uparrow_{X_1\star_r X_2}, \ldots, A^\uparrow_{X_1\star_r X_2}) = b_k^{(r)\uparrow}(A^\uparrow_{X_1}, \ldots, A^\uparrow_{X_1}) + b_k^{(r)\uparrow}(A^\uparrow_{X_2}, \ldots, A^\uparrow_{X_2}).$$

Remark 3.4.3. We talked about the upside and downside adjacencies for a simplicial

complex, but we only define Boolean cumulants for the upside notion. This is due to the associated rooted graphs to simplicial complexes and the star product of them. We mean, for rooted simplicial complexes $(X_1, f_r), (X_2, f'_r)$, the upside adjacency notion and the r-star product, $X_1 \star_r X_2$, on the r-th level. If X_1 and X_2 have n and m r-simplices respectively, it holds that

$$A^{(r)\uparrow}_{X_1 \star_r X_2} = A^{(r)\uparrow}_{X_1} \otimes I_m + I_n \otimes A^{(r)\uparrow}_{X_2}.$$

Thus, the associated graph to $X_1 \star_r X_2$ on the r-th level is the star product of the rooted graphs associated on the r-th level to X_1 and X_2.

On the other hand, on the downside notion of adjacency the star product phenomena does not hold, consequently, if we define the k-th *r-down Boolean cumulant* for the downside notion as the k-th Boolean cumulant of the rooted graph associated to $A^{(r)\downarrow}_X$, $0 \leq r \leq d-1$, this cumulants are not additive respect to the star product of rooted simplicial complexes because the associated graph to the star product simplicial complex is not the star product of the graphs associated to each simplicial complex. For instance, if we consider two copies of the 2-simplex σ_2, rooted on an edge, we have:

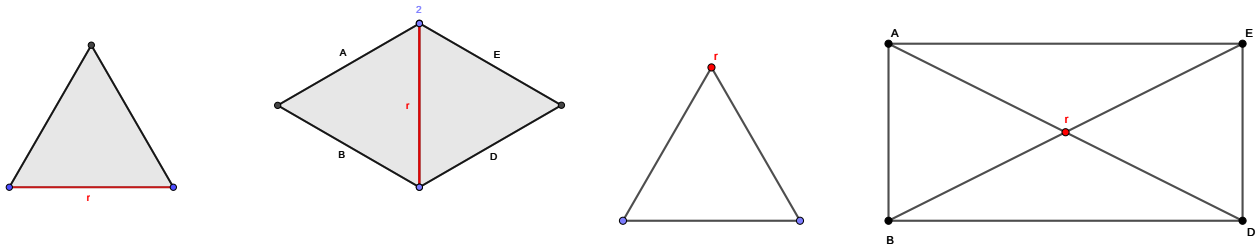


Figure 3.5: σ_2, their rooted star product and their associated downside adjacency graphs respectively on the first level.

We observed in the Figure 3.4.3, the associated graph of $\sigma_2 \star_r \sigma_2$ have cycles that does not exist in either graph associted to the respective copy of σ_2 that are counted in the boolean cumulant for $\sigma_2 \star_r \sigma_2$, not counted in either copy.

Appendix A

Classical probability theory

The purpose of this section is to give a short background to understand the classical notion of independence of random variables and other important topics in classical probability theory such as the central limit theorem and the law of the large numbers.
We will develope this section without major proofs as we did in the preliminaries. We rather give some remarks or do some discussions if necessary. We will follow [25] and [45] as references for this section.

A.1 Foundations

We would say that probability theory is the study of measure spaces with total measure one, so the concepts studied must have the same meaning in every measure space that models the events or random variables. In this sense, the core concepts of probability theory, such as *random variables*, can be defined without explicit mention of a measure space. So we take a functional analysis approach in this section to keep the language and notation used in the work.

Let Ω be a set that we will call the *sample space*, which is the set of the immediate outcomes of all sources of randomness that one is studying. Ω defines a *probability space* $\Omega = (\Omega, \mathcal{F}, \mathbf{P})$ where $\mathcal{F}$ is a σ-algebra of subsets of Ω also called the *family of events* and

a *probability measure* $\mathbf{P}$ on the family of events, i.e. an assignment $E \mapsto \mathbf{P}(E)$ of a real number in $[0,1]$ known as the *probability of the event* E such that $\mathbf{P}(\Omega) = 1$ and $\mathbf{P}$ is countably additive. We usually denote that Ω is a probability space instead of $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition A.1.1. Let Ω be a probability space and $R = (R, \mathcal{R})$ a measurable space. A map $X\colon \Omega \to R$ is called a *random variable* if X is a measurable map. We say X is a *real random variable* when $R = \mathbb{R}$ and is a *complex random variable* when $R = \mathbb{C}$, both equipped with the σ-algebra of the Lebesgue measurable sets.

Example A.1.2. Let X and Y be real or complex random variables over a probability space Ω. Then $X + Y$ and XY given by the pointwise operation in each case are also random variables. In fact, the collection of complex(real) random variables forms an algebra of functions and each element of the algebra is also a random variable.

For now the concept of random variable involves the sample space and we remarked before that it should not. So we need to introduce an object that carries with all the properties of the random variable and does not involve, explicitly at least, the sample space. For now let us take a look to the folowwing example:

Example A.1.3. Let Ω be a probability space, R and R' be measurable spaces and $X\colon \Omega \to R$ be an R-valued random variable. If $f\colon R \to R'$ is a measurable function then $f(X)$ is an R'-valued random variable that can be defined without reference of the sample space as the unique random variable given by the identity of predicates $f(X) \in S$ and $X \in f^{-1}(S)$ which holds for every R'- measurable set S.

In this sense, we could think that the random variable X as the unique random variable given by the identity of events $X^{-1}(S)$ and $X \in S$ for any R-measurable set, which does not require explicit mention of the sample space but of the measurable space R. So it makes sense to think of the probaility of the event $X \in S$, $\mathbf{P}(X \in S)$. Notice that this defines the measure probability $\mathbf{P}$ without explicit mention of the sample space. We denote $p_x := \mathbf{P}(X = x)$.

Taking the discussion into account, we notice that $\mathbf{P} \circ X^{-1}$ is a measure which is in fact a probabilty measure on R. We denote as $\mu_X := \mathbf{P} \circ X^{-1}$, i.e. $\mu_X(S) = \mathbf{P}(X^{-1}(S))$ for every R-measurable set S. μ_X is called the *distribution* of X or the *law* of X. We

remark that μ_X carries out every interesting property of the random variable X and is a pure measure-theoretic concept as it does not mention the sample space but only the measurable spaces involved.

As consequence, we sometimes abuse the notation by writing X instead of μ_X and for random variables X and Y we write $X \equiv Y$ for $\mu_X = \mu_Y$.

So far, we know that an R-valued random variable induces a probability measure; the converse is also true. Given a probability measure μ on a measurable space R there is an R-valued random variable X with distribution is μ.

Now we study an important concept in classical probability that also allows to conceive a classical probability space as a non-commutative probability space. This is the concept of *expectation* or *mean* of an integrable random variable.

Definition A.1.4. Let X be an absolutely integrable R valued random variable on a probability space Ω. We define the *expectation* or *mean* $\mathbb{E}[X]$ as the integral

$$\mathbb{E}[X] = \int_\Omega X(\omega) dP(\omega) = \int_R z d\mu_X(z).$$

An important property of expectation is that it is a linear operator, i.e. $\mathbb{E}[X + \alpha Y] = \mathbb{E}[X] + \alpha\mathbb{E}[Y]$ for absolutely integrable complex or real variables random X, Y and α a scalar. When $\sum_{i=1}^{\infty} |\alpha_i||\mathbb{E}[X_i]|$ is finite, then the linear behavior of $\mathbb{E}$ applies to infinite sums.

Given a measurable map $f\colon R \to R$, we recall R-valued the random variable $f(X)$ and thus we would like to relate about the expectation of $f(X)$ in relation to the expectation of X. We find it as follows

$$\mathbb{E}[f(X)] = \int_R z d\mu_{f(X)}(z) = \int_\Omega f \circ X(\omega) dP(\omega) = \int_R f(z) d\mu_X(z).$$

An important example of expressions such as $\mathbb{E}[f(X)]$ for complex or real random variables are the *moments* of X namely $E[|X|^k]$ for $k \in \mathbb{N}$.

For an absolutely integrable real random valuable we have that $\mathbb{E}[|X|] \geq 0$ and the *Markov inequality*, which states that for any $\lambda > 0$ holds

$$\mathbf{P}(|X| \geq \lambda) \leq \frac{1}{\lambda}\mathbb{E}[|X|].$$

Definition A.1.5. Let X be a complex or real random variable with finite second moment, i,e, $\mathbb{E}[X^2] < \infty$. We define the *variance* of X

$$\mathbf{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Now, for a random variable with finite second moment. The Markov inequality implies the *Chebyshev inequality*, which states that for any $\lambda > 0$ holds

$$P(|X - \mathbb{E}[X]| \geq \lambda) \leq \frac{\mathbf{Var}(X)}{\lambda^2}.$$

In consequence, having knowledge of the first and second moments we could describe better the distribution of the random variable X respect to the mean value.

A.2 Classical independence is tensorial

Now we are interested in studying a family of random variables that interact with each other rather than studying each one of them. Once one studies families $(X_\alpha)_{\alpha \in A}$ of random variables X_α taking values in measurable spaces R_α on a single sample space Ω, the distrbution of the individual random variables X_α is no longer sufficient to describe all the probabilistic statistics of interest, the *joint distribution* of the random variables becomes relevant. Let us recall the product of σ-algebras before introducing the mentioned concept.

Definition A.2.1. Let $(\Omega_j, \mathcal{R}_j)$ be a measurable space for each $j \in J$. Then, the σ-algebra on the cartesian product $\prod_{j \in J} \Omega_j$ generated by the sets $A \times \prod_{j \neq i} \Omega_j, A \in R_\alpha$. We denote this measurable space as $\otimes_{j \in J} \Omega_j$ and is called the *product measurable space*. In case that $(\Omega_j, \mathcal{R}_j)$ are the same measurable space, we denote the product measurable space as Ω^J. Notice that in the product $A \times \prod_{j \neq i} \Omega_j$ we mean that A takes the i-th place and each Ω_j in their corresponding place as in the cartesian product $\prod_{j \in J} \Omega_j$.

Given a family of random variables $(X_\alpha)_\alpha$ taking values in measurable spaces R_α on a single sample space Ω, we can think of them interacting as a single random variable taking values in the product measurable space $\otimes_{j \in J} R_j$.

Definition A.2.2. Let $(X_\alpha : \Omega \to R_\alpha)_{\alpha\in A}$ a family of random variables. The *joint distribution* of the family (X_α) is the probability measure in $\otimes_{\alpha\in A}R_\alpha$ such that for each event $S \in \otimes_{\alpha\in A}R_\alpha$ indicates the probability which the tuple $(X_\alpha)_\alpha$ is in S.

Remark A.2.3. Notice that in the former definition the joint distribution is not given in terms of the distributions of each random variable X_α. There is an important case in which the joint distribution is determined by the individual distributions, this is called *independence*.

The Lebesgue-Fubini-Tonelli theorem states that given two σ-finite measure spaces $(A, \mathcal{A}, \mu)$ and $(B, \mathcal{B}, \nu)$, there is an unique measure $\mu \otimes \nu$ on $A \otimes B$ such that

$$\mu \otimes \nu(S \times T) = \mu(S)\nu(T), \text{ for all } S \in \mathcal{A}, T \in \mathcal{B}.$$

Further more, for every non-negative measurable function $f\colon A \times B \to R$ such that $\int |f| d\mu \otimes \nu < \infty$. Then,

$$\int f d\mu \otimes \nu = \int d\mu(a) \int f(a,b) d\nu(b) = \int d\nu(b) \int f(a,b) d\mu(a).$$

This measure is called *the product measure* of μ and ν on $A \otimes B$.

Definition A.2.4. A family $(X_\alpha)_{\alpha\in A}$ of random variables is said to be *jointly independent* or that the X_α are *independent* if the joint distribution of $(X_\alpha)_{\alpha\in A}$ is the product measure of the individual distributions of each X_α. And we say that the family $(X_\alpha)_\alpha$ is *independently and identically distributed*, or *iid*, if they are jointly independent and all the X_α have the same distribution.

Let us recall that for Hilbert spaces over measure spaces we have the isomorphism

$$L^2(S, d\mu) \otimes L^2(T, d\nu) \cong L^2(S \otimes T, d\mu \otimes d\nu).$$

Where the left side is the tensor product of Hilbert spaces and the right side is the product of measure spaces with the product measure. The isomorphism is given explicitly by $f \otimes g \to fg$.

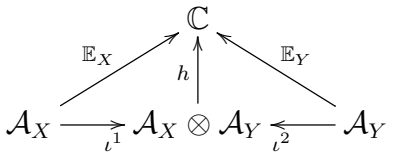
Let $\mathcal{A}_S = L^2(S, d\mu)$ and $\mathcal{A}_T = L^2(T, d\nu)$ the spaces of random variables with finite second moment on their respective measure spaces. Then if $X \in \mathcal{A}_S, Y \in \mathcal{A}_T$ are independent,

the expectation of the joint distribution can be factorized as the product of the expectation of each random variable, we can see this as:

$$\begin{aligned}\mathbb{E}[XY] &= \int xyd\mu_{X,Y} = \int xyd\mu(x) \otimes d\nu(y) = \int d\mu(x) \int xyd\nu(y) \\ &= \int xd\mu(x)\mathbb{E}[Y] = \mathbb{E}[X]\mathbb{E}[Y] \cong \mathbb{E}_X \otimes \mathbb{E}_Y.\end{aligned}$$

Remark A.2.5. If X and Y are independent then the expectation of (X, Y) is the tensor product of the expectation of X and Y respectively. This can be understood the same way for a family $(X_\alpha)_\alpha$ of random variables due to the definition of product measure and independence. We observe here that the expectation of a random vector $(X_\alpha)_\alpha$ can be realized as the product of the expectation of each entry.

In a diagram, For two random variables with all finite moments and $\mathcal{A}_X, \mathcal{A}_Y$ be the algebras spanned by X and Y respectively, we have that



where $\iota^1 = 1_{\mathcal{A}_X} \otimes 1, \iota^2 = 1 \otimes 1_{\mathcal{A}_Y}$, 1 is the constant random variable with value 1 in $\mathbb{C}$ and $h = \mathbb{E}_X \otimes \mathbb{E}_Y$ if and only if X and Y are independent.

A.3 Fundamental theorems

Here, we state the two fundamental theorems about asymptotic distributions due to their importance on the theoretical development of the area.

Definition A.3.1. Let $(X_n)_{n=0}^{\infty}$ be a sequence of random variables taking values in R.

1. We say that the sequence converges in *distribution* to a random variable X, if for every continuous and bounded function $f\colon R \to R$ holds that

$$\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)].$$

2. We say that the sequence converges in *probability* to a random variable X, if for every $\epsilon > 0$ holds that

$$\lim_{n\to\infty} P(|X(\omega) - X_n(\omega)| > \epsilon) = 0.$$

One can see that convergence in probability implies convergence in distribution. Now we can state the *law of large numbers* and the *central limit theorem*.

Theorem A.3.2 (Law of large numbers)**.** *Let $(X_n)_{n=0}^{\infty}$ be a sequence of iid scalar random variables. If we denote for each n as*

$$Y_n := \frac{X_1 + \cdots + X_n}{n}.$$

Then, the sequence of random variables $(Y_n)_n$ converges in probability to $Y = \mathbb{E}[X_1]$.

So the law of large numbers is stating that if we make a histogram for each n of Y_n then the histogram is constant with value $\mathbb{E}[X_1]$.

Theorem A.3.3 (Central limit theorem)**.** *Let $(X_n)_{n=0}^{\infty}$ be a sequence of iid centered real random variables with all finite moments. If we denote for each n as*

$$Z_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}}.$$

Then, the sequence of random variables $(Z_n)_n$ converges in distribution to a Gaussian random variable with $\mu = 0$ and $\sigma^2 = \mathbf{Var}(X_1)$.

Bibliography

[1] L. Accardi, R. Lenczewski and R. Salapata. *Decompositions of the free product of graphs*, Infin. Dimens. Anal. Quantum. Probab. Relat. Top. **10**(03):303–334, 2007.

[2] L. Accardi, A. Ben-Ghorbal and N. Obata. *Monotone independence, comb graphs and Bose-Einstein condensation*, Infin. Dimens. Anal. Quantum. Probab. Relat. Top. **07**, 419, (2004).

[3] G. Anderson, A. Guionnet and O. Zeituni. *An Introduction to Random Matrices*. Cambridge University Press, Cambridge, 2010.

[4] S. Belinschi and H. Bercovici. *A new approach to subordination results in free probability*, Journal dAnalyse Mathematique **101**, 357–365, (2007).

[5] S. Belinschi, T Mai and R. Speicher. *Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem*, Journal fuer die reine und angewandte Mathematik (Crelles Journal), ISSN (Online) 1435-5345, ISSN (Print) 0075-4102.

[6] F. Benaych-Georges. *Rectangular random matrices, related convolution*, Prob. Theory Rel. Field **144** (2009), 471515.

[7] F. Benaych-Georges. *Rectangular random matrices, related free entropy and free Fishers information*, J. Operator Th. **62** (2009), no. 2, 371–419.

[8] A. Ben-Ghorbal and M. Schuermann. *Non-commutative notions of stochastic independence*, Math. Proc. Camb. Phil. Soc. **133** (2002), 531561.

[9] P. Bubenik. *Statistical Topological Data Analysis using Persistence Landscapes*, Journal of Machine Learning Research **16** (2015) 77–102.

[10] W. Bryc, A. Dembo and T. Jiang. *Spectral measure of large random Hankel, Markov and Toeplitz matrices*. Ann. Probab., **34**, 1–38 (2006).

[11] P. Bubenik and J. Scott. *Categorification of persistent homology*, Discrete and Computational Geometry, **51**(3) 600–627, 2014.

[12] G. Carlsson. *Topology and data*, Bull. Amer. Math. Soc. (N.S.), **46**(2):255–308, 2009.

[13] G. Carlsson, A. Zomorodian, A. Collins and L. Guibas. *Persistence Barcodes for Shapes*, Eurographics Symposium on Geometry Processing (2004)

[14] G. Cebron, A. Dahlqvist and C. Male. *Universal constructions for spaces of traffics*, pre-print (2016).

[15] A. Chakrabarty, R.S. Hazra, F. den Hollander and M. Sfragara. *Spectra of Laplacian and adjacency matrices of inhomogeneous Erdös-Rényi random graphs*. Preprint (2018).

[16] C. D. Cushen and R. L. Hudson. *A quantum central limit theorem*, J. Appl. Prob. **8**,454–469, (1971).

[17] S. Deo, *Algebraic Topology*, Springer, (2003).

[18] C. Díaz and C. Vargas. *Non-commutative distributions for graphs and simplicial complexes*, preprint, 2019.

[19] X. Ding and T. Jiang. *Spectral distributions of adjacency and Laplacian matrices of random graphs*. Ann. Appl. Probab. **20**,2086–2117, (2010).

[20] B. Eckmann. *Harmonische Funktionen und Randwertaufgaben in einem Komplex*, Comment. Math. Helv., **17** (1) (1944), pp. 240–255.

[21] H. Edelsbrunner, D. Letscher and A. Zomorodian. *Topological persistence and simplification*, Discrete Comput. Geom. **28**(2002), 511–533.

[22] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik. *Tensor Categories*, American Mathematical Soc. **205**(20016).

[23] U. Franz. *Lévy processes on quantum groups and dual groups*, Quantum independent increment processes. II, Lecture Notes in Math., vol. 1866, Springer, Berlin (2006), 161–267.

[24] J. Friedman. *Computing Betti numbers via combinatorial Laplacians*. Algorithmica, **21** 331–346, 1998.

[25] J. Garza. *Espacios de tráficos. Un nuevo método para el estudio de matrices aleatorias*. Tesis de licenciatura, Universidad Nacional Autónoma de México, 2017, 1–12.

[26] M. Gerhold. *On Several Problems in the Theory of Comonoidal Systems and Subproduct Systems*. Dissertation, 2014.

[27] N. Giri and W. von Waldenfels. *An algebraic version of the central limit theorem*, Z. Wahrsch. verw. Gebiete. **42**, 129–134, (1978).

[28] A. Hatcher, *Algebraic topology*, Notes (2005).

[29] D. Horak and J. Jost. *Spectra of combinatorial Laplace operators on simplicial complexes*, Adv. in Math. **244**, 10 September 2013, Pages 303–336.

[30] R. L. Hudson. *A quantum mechanical central limit theorem for anticommuting observables*. J. Appl. Prob. **10** (1973), 502–509.

[31] S. Mac Lane. *Categories for the working mathematician*, Graduate Texts in Mathematics **5**, (1998).

[32] C. Male. *The distribution of traffics and their free product*, arXiv: 1111.4662, 2011.

[33] S. Manzel and M. Schrmann, Non commutative notions of independence and cumulants. *Infin. Dimens. Anal. Quantum. Probab. Relat. Top.* **20**, 2017.

[34] V. Marčenko and L. Pastur. *Distribution of eigenvalues for some sets of random matrices*, Math. USSR-Sbornik **1** (1967), 457–483.

[35] J. Mingo and R. Speicher *Free Probability and Random Matrices*, Fields Institute Monographs, Springer, 2017.

[36] N. Muraki. *Monotonic independence, monotonic central limit theorem and monotonic law of small numbers*, Infin. Dimens. Anal. Quantum. Probab. Relat. Top. **4**, no.**1**, 39–58, (2001).

[37] N. Muraki. *The five independence as natural products*, Infin. Dimens. Anal. Quantum. Probab. Relat. Top. **06**, 337, (2003).

[38] A. Nica and, R. Speicher *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series, vol. **335**, Cambridge University Press, vol.**13** Cambridge (2006).

[39] A. Patel. *Generalized Persistence Diagrams*, pre-print (2016).

[40] D. Shlyakhtenko, *Random Gaussian band matrices and freeness with amalgamation*, Internat. Math. Res. Notices **20** (1996), 1013–1025.

[41] R. Speicher. *Multiplicative functions on the lattice of noncrossing partitions and free convolution*, Math. Ann. **298** (1994), no. 4, 611–628.

[42] R. Speicher. *On universal products* Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., vol. **12**, Amer. Math. Soc., Providence, RI, 1997, pp. 257–266.

[43] R. Speicher and R. Woroudi. *Boolean convolution* Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., vol. **12**, Amer. Math. Soc., Providence, RI, 1997, pp. 267–279.

[44] R. Speicher. *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory* Memoirs of the American Math. Society, vol. **132**, 1998.

[45] T. Tao. *Topics in random matrix theory*, American Mathematical Society, 2012, Ch. 1.

[46] J. W. Vick, *Homology theory: an introduction to algebraic topology*, Springer Science & Business Media. **145**(2012)

[47] D. Voiculescu. *Symmetries of some reduced free product C^*-algebras*, Operator algebras and their connections with topology and ergodic theory (Busteni, 1983), Lecture Notes in Math., vol. **1132**, Springer, Berlin, 1985, pp. 556–588.

[48] D. Voiculescu. *Limit laws for random matrices and free products*, Invent. Math. **104** (1991), 201–220.

[49] D. Voiculescu. *Operations on certain non-commutative operator-valued random variables. Recent advances in operator algebras (orleans, 1992)* Asterisque **232** (1995).

[50] D. Voiculescu, K. Dykema and A. Nica. *Free Random Variables*, CRM Monograph Series, Vol. **1**, AMS 1992.

[51] W. von Waldenfels. *An algebraic central limit theorem in the anti-commuting case*, Z. Wahrsch. verw. Gebiete **42**, 135–140, (1978).

[52] E. Wigner. *On the distribution of the roots of certain symmetric matrices*, Ann. of Math. **67** (1958), 325–327.

[53] W. Wöss. *Nearest neighbour random walks on free products of discrete groups*, Boll. Unione Mat. Ital. B (6), **5** (3), 961–982, (1986).