How long should one stare at the Haar-Ginibre railway?

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Abstract

We study the eigenvalues of a family of matrices

$$R(s,t) = \alpha(s)C + \beta(s)U(t),$$

periodic on the parameter $t \in [0,1]$, with $s \in [0,1]$, determined by a realization of a Ginibre matrix C and a rotating unitary matrix $U(t)$.

The matrices in this family are approximations of R-diagonal elements in free probability, studied by Haagerup-Larsen [?], Kemp-Speicher [?] and P. Zhong [?].

We study the eigenvalue collisions for all values of s and t. To do this, we fix s, and increase the periodic value t, while keeping track of each eigenvalue.

Unless s is too close to 0 or 1, the process leads in general to a non-tivial permutation $\sigma(s)$, with pleasing visualizations of flowing, repellent eigenvalues.

The intricate web of paths that the eigenvalues collectively traverse remains quite stable for small variations in s. However, the actual permutation $\sigma(s + \Delta s)$ itself does present variations, indicating eigenvalue collisions at some intermediate values (s,t), $s \in (s, s+\Delta s)$, which explain the permutation discrepancy.

The eigenvalue 'track-flips' that occur before/after these colissions are the essential differences between consecutive railway systems, as the eigenvalues proceed to team-up to whirl around themselves and then around zero.

We report some first statistics about these processes and their collisions, and we include a simple package to perform/store/display these (parallellizable) computations and visualizations.

1 Introduction

Let C be a realization of an $N \times N$ Ginibre matrix and, for $t \in [0,1]$ and let ω is the first clock-wise non-trivial complex N-th root of 1. Let

$$U(t) = \mathrm{diag}(\omega^{tN}, \omega^{tN+1}, \omega^{tN+2}, \dots, \omega^{tN+N-1}),$$

be a diagonal matrix with equi-distant points along the circle. Notice that $U(0) = U(1)$.

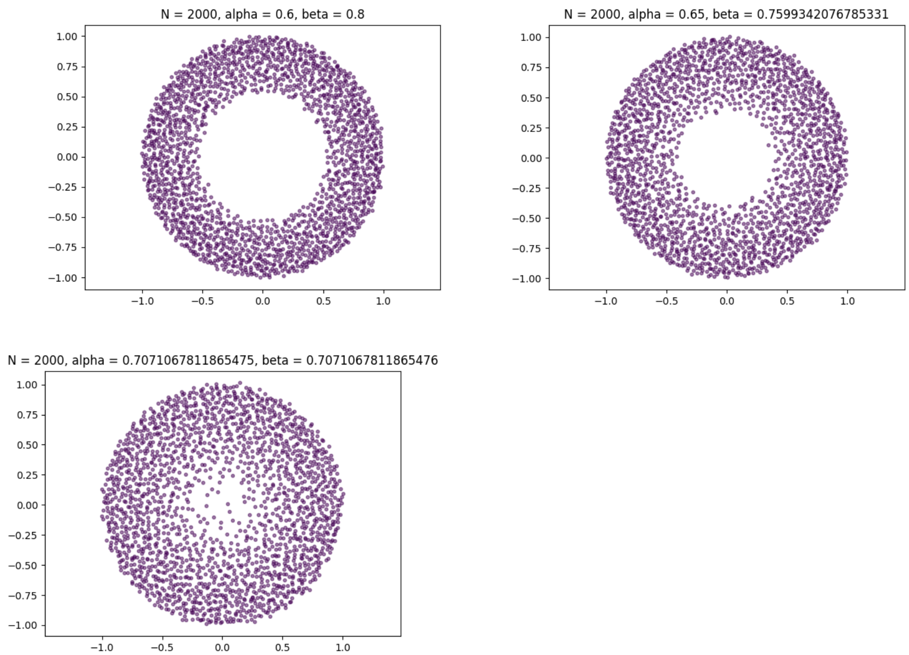
We want to study the eigenvalues of the matrix model:

$$R(s,t) = \alpha(s)C + \beta(s)U(t), \quad \alpha(s) = \cos((s\pi)/2), \beta(s) = \sin((s\pi)/2)U(t),$$

which is periodic on $t \in [0, 1]$ for fixed s.

In general it is quite a task to compute distributions of non-selfadjoint random matrix models. The family of matrices in this model, however, are approximations of R-diagonal elements in free probability, for which some actual computations are possible [?,?].

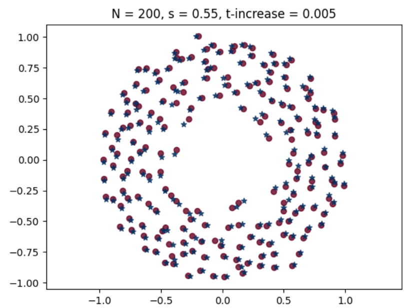
Depending on the values of α and β, the asymptotic distributions are supported on a centered annulus or disk, (see Figure).

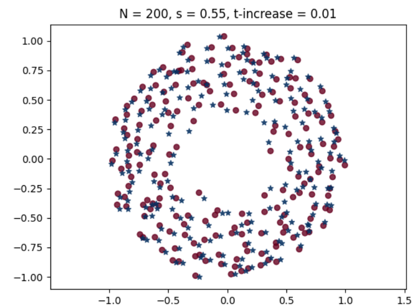


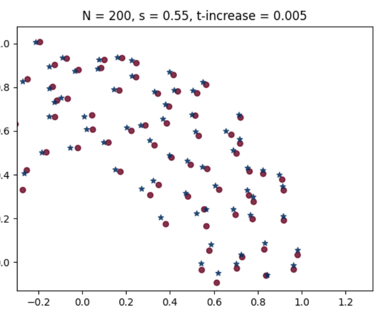
We choose $\alpha(s) = \cos(s\pi/2)$ and $\beta(s) = \sin(s\pi/2)$ specificaly to make the outer radious of the model equal to one. The eigenvalues for any other positive pairs (α, β) can be obtained by scaling values within our parametrization.

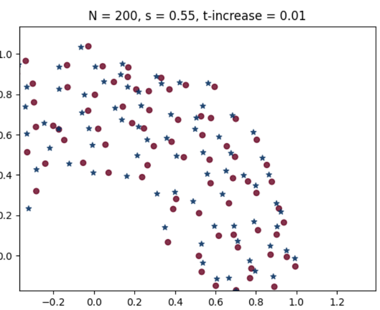
In this work we mainly want to draw attention to the effect of increasing t, which 'turns' the eigenvalues of the model clockwise.

In the figure, we show the eigenvalues (red circles) tracking the eigenvalues at after small increases of t (blue stars)


N = 200, s = 0.55, t-increase = 0.005


N = 200, s = 0.55, t-increase = 0.01


N = 200, s = 0.55, t-increase = 0.005


N = 200, s = 0.55, t-increase = 0.01

2 Permutation $\sigma(s)$

Unless s is too close to 0 or 1, the process of increasing the periodic parameter t leads in general to a non-trivial permutation $\sigma(s)$ after reaching $t = 1$. Indeed, since eigenvalues at the outer part of the annulus are moving more slowly, they wont be able to perform a complete turn, thus falling into some other eigenvalue's original position at $t = 0$.

Let us consider first the eigenvalues of $R(0, t) = R(0, 0)$. These are just the eigenvalues of a complex Gaussian Ginibre matrix C, with explicit joint distribution

$$(insertformula)$$

As $N \to \infty$ this distribution converges to uniform distribution on the unit disc.

To be able to store the relevant data properly, we will first label the eigenvalues increasingly by norm at $R(0, 0)$.

Then we consider all the values $R(s, 0)$, (with $t = 0$) and increase s slowly, keeping track of the labels continuosly until we reach $R(1, 0)$.

In case of ambiguity of eigenvalue tracking, a refinement is performent on the partition on $s \in [0, 1]$.

After this is performed and a continous ordering of the eigenvalues for all values of s and $t = 0$ has been achieved, now we consider, for each $s \in [0, 1]$ (in the s-partition) the process $R(s, t)$ when increasing t from 0 to 1, keeping track of the eigenvalues.

For some s relatively small we may observe some differences on the permutations, while noticing very little discrepancies on the collective trails traveled.

In the Figure $s = 0.0500, 0.0505, 0.0510, 0.0515$. The eigenvalue tracks are colored according to cycle length

(yellow = singletons, orange = 2-cycles, ... , darker cycles of greater length).

Notice first the large, 26-element purple cycle $[1, 7, 19, 51, 25, \ldots, 16, 8]$.

The collective paths remain almost unaltered, but there are some few eigenvalue collisions in between each frame:

The corresponding permutations $\sigma(s_0)$, $\sigma(s_1)$, $\sigma(s_2)$, $\sigma(s_3)$ all present small differences. These are witness to *eigenvalue collisions* at some values (s, t), $s \in (s_0, s_3)$.

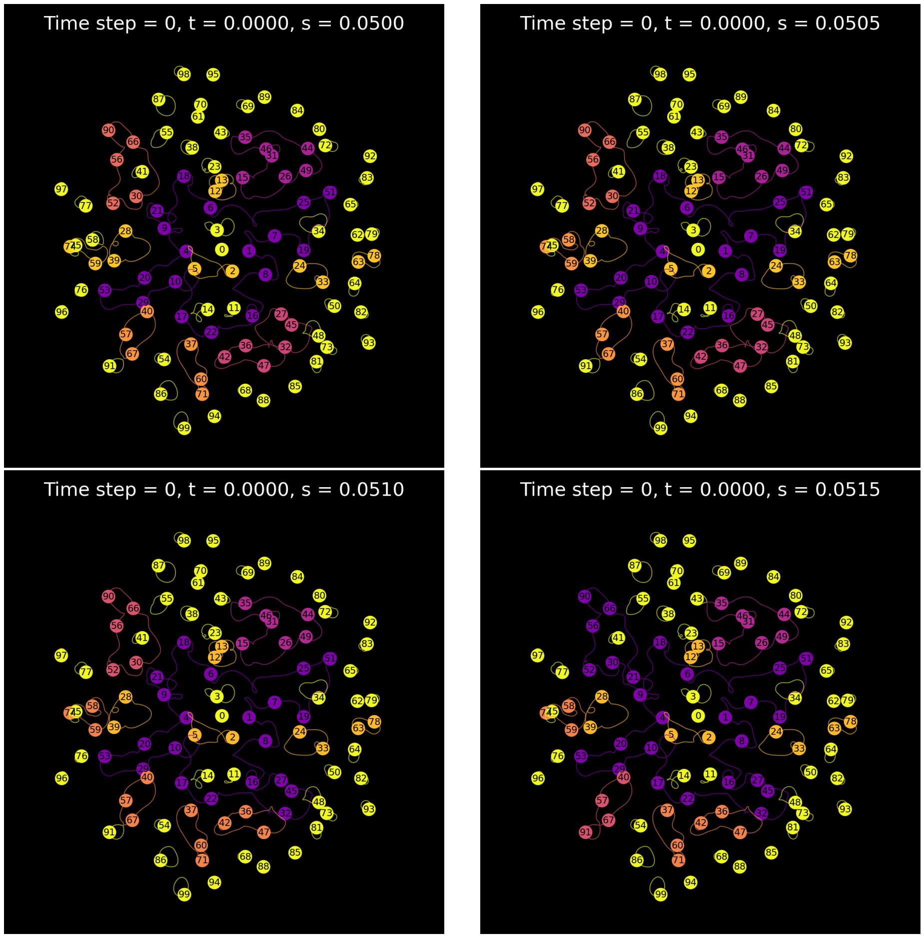
The collissions explain the permutation discrepancies in the following way: if the eigenvalues i and j collided, and they were originally pointing to k and l, they will point after the collision to l and k instead.

The eigenvalues keep their initial tracks, but switch paths after the value of t where the collision occurred.

In the figure, there is a collision of the singleton [58] and 59, from the cycle $[59, 74]$ between s_0 and s_1 that makes produces the cycle [58, 59, 74] in the next frames.

The next permutation configuration $\sigma(s_2)$ is explained by two collisions:

One collision (27 vs 36) splits the 6-cycle $[32, 45, 27, 47, 42, 36]$ into two cycles $[32, 45, 27]$, $[47, 42, 36]$.



The second collision 27 vs 22 (or the first, we are not sure of the order of the collisions at the current refinement) joins the cycle $[47, 42, 36]$ with the large purple cycle.

Finally, the last permutation is explained by two collisions: One involving 91 and 67 which joins the singleton $[91]$ from the cycle $[67, 57, 40]$, to produce the 4-cycle $[67, 91, 57, 40]$

The second is crash involving 21 and 30 which results in incorporating the 5-cycle ok 30 to the big purple cycle.

Thus, we use these permutation discrepancies to detect eigenvalue collisions. From computations for small N we conjecture that there are exactly $N(N+1)$ collisions as s goes from 0 to 1.

We report some first statistics about these processes and their collisions.

There is an interesting aspect about counting collisions: if a collision occurs between two eigenvalues which are both not currently singletons, then it is ambiguous to simply report that a given pair of eigenvalues crashed.

The pairs of eigenvalues that acually crash depends on how we reach $R(s_0, t_0)$. Did we go in a straight line from $(0, 0)$ to $(s_0, 0)$ and then rotated?, or were we allowed to rotate at intermediate values s, $0 < s < s_0$?

We include a simple package for these computations and visualizations.

From the t-processes we may extract different types of eigenvalue collision data for visualtization and/or statistics.