Determined classically, muons traveling at a speed of 0.98c cover the 2,000 m in $6.8 \times 10^{-6}$ s, and 45 muons should survive the flight from 2,000 m to sea level according to the radioactive decay law. But experimental measurement indicates that 542 muons survive, a factor of 12 more.

This phenomenon must be treated relativistically. The decaying muons are moving at a high speed relative to the experimenters fixed on the earth. We therefore observe the muons' clock to be running slower. In the muons' rest frame, the time period of the muons' flight is not $\Delta t = 6.8 \times 10^{-6}$ s but rather $\Delta t/\gamma$. For $v = 0.98c$, $\gamma = 5$, so we measure the flight time on a clock at rest in the muons' system to be $1.36 \times 10^{-6}$ s. The radioactive decay law predicts that 538 muons survive, much closer to our measurement and within the experimental uncertainties. An experiment similar to this has verified the time dilation prediction.*

**EXAMPLE 14.3**

Examine the muon decay just discussed from the perspective of an observer moving with the muon.

***Solution.*** The half-life of the muon according to its own clock is $1.52 \times 10^{-6}$ s. But an observer moving with the muon would not measure the distance from the top of the mountain to sea level to be 2,000 m. According to that observer, the distance would be only 400 m. At a speed of $0.98c$, it takes the muon only $1.36 \times 10^{-6}$ s to travel the 400 m. An observer in the muon system would predict 538 muons to survive, in agreement with an observer on the earth.

Muon decay is an excellent example of a natural phenomenon that can be described in two systems moving with respect to each other. One observer sees time dilated and the other observer sees length contracted. Each, however, predicts a result in agreement with experiment.

## Atomic Clock Time Measurements

An even more direct confirmation of special relativity was reported by two American physicists, J. C. Hafele and Richard E. Keating, in 1972.† They used four extremely accurate cesium atomic clocks. Two clocks were flown on regularly scheduled commercial jet airplanes around the world, one eastward and one westward; the other two reference clocks stayed fixed on the earth at the U.S. Naval Observatory. A well-defined, hyperfine transition in the ground state of the $^{133}Cs$ atom has a frequency of 9,192,631,770 Hz and can be used as an accurate measurement of a time period.

*The experiment was reported by B. Rossi and D. B. Hall in the *Phys. Rev.*, **59,** 223 (1941). A film entitled "Time Dilation—An Experiment with $\mu$-Mesons" by D. H. Frisch and J. H. Smith is available from the Education Development Center; Newton, Mass. See also D. H. Frisch and J. H. Smith, *Am. J. Phys.*, **31,** 342 (1963).

†See J. C. Hafele and Richard E. Keating, *Science,* **177,** 166–170 (1972).

The time measured on the two moving clocks was compared with that of the two reference clocks. The eastward trip lasted 65.4 hours with 41.2 flight hours. The westward trip, a week later, took 80.3 hours with 48.6 flight hours. The predictions are complicated by the rapid rotation of the earth and by a gravitational effect from the general theory of relativity.

We can gain some insight to the expected effect by neglecting the corrections and calculating the time difference as if the earth were not rotating. The circumference of the earth is about $4 \times 10^7$ m, and a typical jet airplane speed is almost 300 m/s. A clock fixed on the ground measures a flight time $T_0$ of

$$T_0 = \frac{4 \times 10^7\,\text{m}}{300\ \text{m/s}} = 1.33 \times 10^5\,\text{s}(\approx 37\ \text{hr}) \tag{14.22}$$

Because the moving clock runs more slowly, the observer on the earth would say that the moving clock measures only $T = T_0\sqrt{1 - \beta^2}$. The time difference is

$$\begin{aligned} \Delta T &= T_0 - T = T_0(1 - \sqrt{1 - \beta^2}) \\ &\approx \frac{1}{2}\beta^2 T_0 \end{aligned} \tag{14.23}$$

where only the first and second terms of the power series expansion for $\sqrt{1 - \beta^2}$ are kept because $\beta^2$ is so small.

$$\begin{aligned} \Delta T &= \frac{1}{2}\left(\frac{300\ \text{m/s}}{3 \times 10^8\ \text{m/s}}\right)^2 (1.33 \times 10^5\ \text{s}) \\ &= 6.65 \times 10^{-8}\,\text{s} = 66.5\ \text{ns} \end{aligned} \tag{14.24}$$

This time difference is greater than the uncertainty of the measurement. Notice that in this case, the clock left on the earth actually measures more time in seconds than the moving clock. This seems at variance with our earlier comments (see Equation 14.21 and discussion). But the time period referred to in Equation 14.21 is the time between two ticks, in this case, a transition in $^{133}$Cs, which we measure in seconds. It is easy to remember that moving clocks run more slowly, so that in seconds the measured time difference involves fewer ticks and, according to the definition of a second, fewer seconds.

The actual predictions and observations for the time difference are

| **Travel** | **Predicted** | **Observed** |
|---|---|---|
| Eastward | $-40 \pm 23$ ns | $-59 \pm 10$ ns |
| Westward | $275 \pm 21$ ns | $273 \pm 7$ ns |

Again, the special theory of relativity is verified within the experimental uncertainties. A negative sign indicates that the time on the moving clock is less than the earth reference clock. The moving clocks lost time (ran slower) during the eastward trip and gained time (ran faster) during the westward trip. This difference is caused by the rotation of the earth, indicating that the flying clocks actually ticked faster or slower than the reference clocks on the earth. The overall positive time difference is a result of the gravitational potential effect (which we do not discuss here).

We have only briefly described two of the many experiments that have verified the special theory of relativity. There are no known experimental measurements that are inconsistent with the special theory of relativity. Einstein's work in this regard has so far withstood the test of time.

## 14.5 Relativistic Doppler Effect

The Doppler effect in sound is represented by an increased pitch of sound as a source approaches a receiver and a decrease of pitch as the source recedes. The change in frequency of the sound depends on whether the source or receiver is moving. This effect seems to violate Postulate I of the theory of relativity until we realize that there is a special frame for sound waves because there is a medium (e.g., air or water) in which the waves travel. In the case of light, however, there is no such medium. Only relative motion of source and receiver is meaningful in this context, and we should therefore expect some differences in the relativistic Doppler effect for light from the normal Doppler effect of sound.

Consider a source of light (e.g., a star) and a receiver approaching one another with relative speed $v$ (Figure 14-4a). First, consider the receiver fixed in system $K$ and the light source in system $K'$ moving toward the receiver with speed $v$. During time $\Delta t$ as measured by the receiver, the source emits $n$ waves. During that time $\Delta t$, the total distance between the front and rear of the waves is

$$\text{length of wave train} = c\Delta t - v\Delta t \tag{14.25}$$

The wavelength is then

$$\lambda = \frac{c\Delta t - v\Delta t}{n} \tag{14.26}$$

and the frequency is

$$\nu = \frac{c}{\lambda} = \frac{cn}{c\Delta t - v\Delta t} \tag{14.27}$$

According to the source, it emits $n$ waves of frequency $\nu_0$ during the proper time $\Delta t'$:

$$n = \nu_0 \Delta t' \tag{14.28}$$

This proper time $\Delta t'$ measured on a clock in the source system is related to the time $\Delta t$ measured on a clock fixed in system $K$ of the receiver by

$$\Delta t' = \frac{\Delta t}{\gamma} \tag{14.29}$$

The clock moving with the source measures the proper time, because it is present at both the beginning and end of the waves.

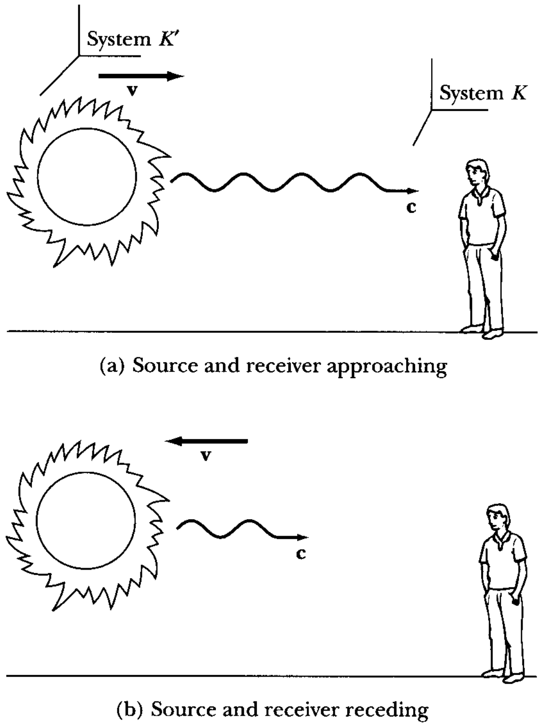



**FIGURE 14-4** (a) An observer in system $K$ sees light coming from a source fixed in system $K'$. System $K'$ is moving toward the observer with speed $v$. The frequency of the light is observed in $K$ to be increased over the value observed in $K'$. (b) When system $K'$ is moving away from the observer, the frequency of the light decreases (the wavelength increases). This is the source of the term *redshifted.*

Substituting Equation 14.29 into Equation 14.28, which in turn is substituted for $n$ in Equation 14.27, gives

$$\nu = \frac{1}{(1 - v/c)} \frac{\nu_0}{\gamma}$$

$$= \frac{\sqrt{1 - v^2/c^2}}{1 - v/c} \nu_0 \tag{14.30}$$

which can be written as

$$\nu = \frac{\sqrt{1 + \beta}}{\sqrt{1 - \beta}} \nu_0 \quad \text{source and receiver approaching} \tag{14.31}$$

It is left for the reader (Problem 14-14) to show that Equation 14.31 is also valid when the source is fixed and the receiver approaches it with speed $v$.

Next, we consider the case in which the source and receiver recede from each other with velocity $v$ (Figure 14-4b). The derivation is similar to the one just presented—with one small exception. In Equation 14.25, the distance between the beginning and end of the waves becomes

$$\text{length of wave train} = c\Delta t + v\Delta t \tag{14.32}$$

This change in sign is propagated through Equations 14.30 and 14.31, giving

$$\nu = \frac{\sqrt{1 - v^2/c^2}}{1 + v/c}\nu_0$$

$$\nu = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\nu_0 \quad \text{source and receiver receding} \tag{14.33}$$

Equations 14.31 and 14.33 can be combined into one equation,

$$\nu = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}}\nu_0 \quad \text{relativistic Doppler effect} \tag{14.34}$$

if we agree to use a + sign for $\beta(+v/c)$ when the source and receiver are approaching each other and a − sign for $\beta$ when they are receding.

The relativistic Doppler effect is important in astronomy. Equation 14.34 indicates that, if the source is receding at high speed from an observer, then a lower frequency (or longer wavelength) is observed for certain spectral lines or characteristic frequencies. This is the origin of the term *red shift*; the wavelengths of visible light are shifted toward longer wavelengths (red) if the source is receding from us. Astronomical observations indicate that the universe is expanding. The farther away a star is, the faster it appears to be moving away (or the greater its red shift). These data are consistent with the "big bang" origin of the universe, which is estimated to have occurred some 13 billion years ago.

## EXAMPLE 14.4

During a spaceflight to a distant star, an astronaut and her twin brother on the earth send radio signals to each other at annual intervals. What is the frequency of the radio signals each twin receives from the other during the flight to the star if the astronaut is moving at $v = 0.8c$? What is the frequency during the return flight at the same speed?

***Solution.*** We use Equation 14.34 to determine the frequency of radio signals that each receives from the other. The frequency $\nu_0 = 1$ signal/year. On the leg of the trip away from the earth, $\beta = -0.8$ and Equation 14.34 gives

$$\nu = \frac{\sqrt{1-0.8}}{\sqrt{1+0.8}}\nu_0 = \frac{\nu_0}{3}$$

The radio signals are received once every 3 years.

On the return trip, however, $\beta = +0.8$ and Equation 14.34 gives $\nu = 3\nu_0$, so the radio signals are received every 4 months. In this way, the twin on the earth can monitor the progress of his astronaut twin.

## 14.6 Twin Paradox

Consider twins who choose different career paths. Mary becomes an astronaut, and Frank decides to be a stockbroker. At age 30, Mary leaves on a mission to a planet in a nearby star's system. Mary will have to travel at a high speed to reach the planet and return. According to Frank, Mary's biological clock will tick more slowly during her trip, so she will age more slowly. He expects Mary to look and appear younger than he does when she returns. According to Mary, however, Frank will appear to be moving rapidly with respect to her system, and she thinks Frank will be younger when she returns. This is the paradox. Which twin, if either, is younger when Mary (the moving twin) returns to the earth where Frank (the fixed twin) has remained? Because the two expectations are so contradictory, doesn't Nature have a way to prove they will be the same age?

This paradox has existed almost since Einstein first published his special theory of relativity. Variations of the argument have been presented many times. The correct answer is that Mary, the astronaut, will return younger than her twin brother, Frank, who remains busy on Wall Street. The correct analysis is as follows. According to Frank, Mary's spaceship blasts off and quickly reaches a coasting speed of $v = 0.8c$, travels a distance of 8 ly (ly = a light year, the distance light travels in 1 year) to the planet, and quickly decelerates for a short visit to the planet. The acceleration and deceleration times are negligible compared with the total travel time of 10 years to the planet. The return trip also takes 10 years, so on Mary's return to Earth, Frank will be $30 + 10 + 10 = 50$ years old. Frank calculates that Mary's clock is ticking slower and that each leg of the trip takes only $10\sqrt{1 - 0.8^2} = 6$ years. Mary therefore is only $30 + 6 + 6 = 42$ years old when she returns. Frank's clock is (almost) in an inertial system.

When Mary performs the time measurements on her clock, they may be invalid according to the special theory because her system is not in an inertial frame of reference moving at a constant speed with respect to the earth. She accelerates and decelerates at both the earth and the planet, and to make valid time measurements to compare with Frank's clock, she must account for this acceleration and deceleration. The instantaneous rate of Mary's clock is still given by Equation 14.20, because the instantaneous rate is determined by the instantaneous speed $v$.* Thus, there is no paradox if we obey the two postulates of the

*See the clock hypothesis of W. Rindler (Ri82, p. 31).

special theory. It is also clear which twin is in the inertial frame of reference. Mary will actually feel the forces of acceleration and deceleration. Frank feels no such forces. When Mary returns home, her twin brother has invested her 20 years of salary, making her a rich woman at the young age of 42. She was paid a 20-year salary for a job that took her only 12 years!

**EXAMPLE 14.5**

Mary and Frank send radio signals to each other at 1-year intervals after she leaves Earth. Analyze the times of receipt of the radio messages.

***Solution.*** In Example 14.4, we calculated that such radio signals are received every 3 years on the trip out and every $\frac{1}{3}$ year on the trip back. First, we examine the signals Mary receives from Frank. During the 6-year trip to the planet, Mary receives only two radio messages, but on the 6-year return trip, she receives eighteen signals, so she correctly concludes that her twin brother Frank has aged 20 years and is now 50 years old.

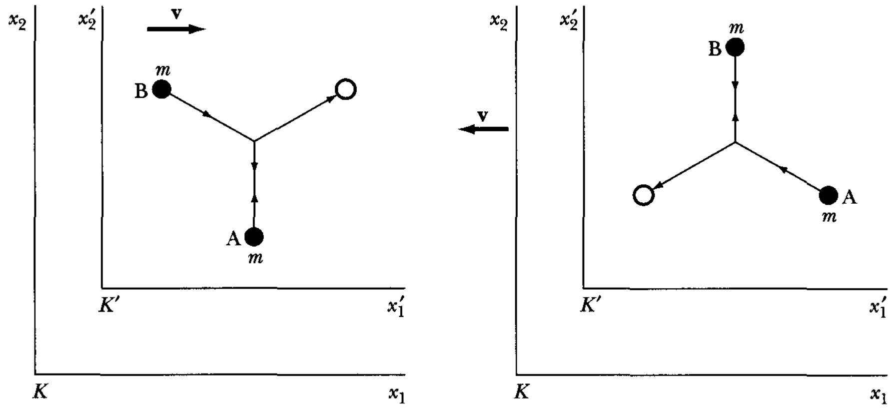
In Frank's system, Mary's trip to the planet takes 10 years. By the time Mary reaches the planet, Frank receives 10/3 signals (i.e., three signals plus one-third of the time to the next one). However, Frank continues to receive a signal every 3 years for the 8 years it takes the last signal Mary sends when she reaches the planet to travel to Frank. Thus, Frank receives signals every 3 years for 8 more years (total of 18 years) for a total of six radio signals from the period of travel to the planet. Frank has no way of knowing that Mary has stopped and turned around until the radio message, which takes 8 years, is received. Of the remaining 2 years of Mary's journey according to Frank ($20 - 18 = 2$), Frank receives signals every $\frac{1}{3}$ year, or six more signals. Frank correctly determines that Mary has aged $6 + 6 = 12$ years during her journey because he receives a total of 12 signals.

Thus, both twins agree about their own ages and about each other's. Mary is 42 and Frank is 50 years old.

## 14.7 Relativistic Momentum

Newton's Second Law, $\mathbf{F} = d\mathbf{p}/dt$, is covariant under a Galilean transformation. Therefore, we do not expect it to keep its form under a Lorentz transformation. We can foresee difficulties with Newton's laws and the conservation laws unless we make some necessary changes. According to Newton's Second Law, for example, an acceleration at high speeds might cause a particle's velocity to exceed $c$, an impossible condition according to the special theory of relativity.

We begin by examining the conservation of linear momentum in a force-free (no external forces) collision. There are no accelerations. Observer A at rest in system $K$ holds a ball of mass $m$, as does observer $B$ in system $K'$ moving to




(a) Collision according to system $K$ (b) Collision according to system $K'$

**FIGURE 14-5** Observer A, at rest in fixed system $K$, throws a ball straight up in system $K$. Observer B, at rest in system $K'$, which is moving to the right with velocity **v**, throws a ball straight down so that the two balls collide. (a) The collision according to observer A in system $K$. (b) The collision according to observer B in system $K'$. Each observer measures the speed of his or her ball to be $u_0$. We examine the linear momentum of the ball.

the right with relative speed $v$ with respect to system $K$, as in Figure 14-1. The two observers throw their (identical) balls along their respective $x_2$-axes, which results in a perfectly elastic collision. The collision, according to observers in the two systems, is shown in Figure 14-5. Each observer measures the speed of his or her ball to be $u_0$.

We first examine the conservation of momentum according to system $K$. The velocity of the ball thrown by observer A has components

$$\left.\begin{aligned} u_{A1} &= 0 \\ u_{A2} &= u_0 \end{aligned}\right\} \tag{14.35}$$

The momentum of ball A is in the $x_2$-direction:

$$p_{A2} = mu_0 \tag{14.36}$$

The collision is perfectly elastic, so the ball returns down with speed $u_0$. The change in momentum observed in system $K$ is

$$\Delta p_{A2} = -2mu_0 \tag{14.37}$$

Does Equation 14.37 also represent the change in momentum of the ball thrown by observer B in the moving system $K'$? We use the inverse velocity transformation of Equations 14.17 (i.e., we interchange primes and unprimes and let $v \rightarrow -v$) to determine

$$\left.\begin{aligned} u_{B1} &= v \\ u_{B2} &= -u_0\sqrt{1 - v^2/c^2} \end{aligned}\right\} \tag{14.38}$$

where $u'_{B1} = 0$ and $u'_{B2} = -u_0$. The momentum of ball B and its change in momentum during the collision become

$$p_{B2} = -mu_0\sqrt{1 - v^2/c^2} \tag{14.39}$$

$$\Delta p_{B2} = +2mu_0\sqrt{1 - v^2/c^2} \tag{14.40}$$

Equations 14.37 and 14.40 do not add to zero: *Linear momentum is not conserved according to the special theory if we use the conventions for momentum of classical physics.* Rather than abandoning the law of conservation of momentum, we look for a solution that allows us to retain both it and Newton's Second Law.

As we did for the Lorentz transformation, we assume the simplest possible change. We assume that the classical form of momentum $m\mathbf{u}$ is multiplied by a constant that may depend on speed $k(u)$:

$$\mathbf{p} = k(u)m\mathbf{u} \tag{14.41}$$

In Example 14.6, we show that the value

$$k(u) = \frac{1}{\sqrt{1 - u^2/c^2}} \tag{14.42}$$

allows us to retain the conservation of linear momentum. Notice that the *form* of Equation 14.42 is the same as that found for the Lorentz transformation. In fact, the constant $k(u)$ is given the same label: $\gamma$. However, this $\gamma$ contains the speed of the particle $u$, whereas the Lorentz transformation contains the relative speed $v$ between the two inertial reference frames. This distinction must be kept in mind; it often causes confusion.

We can make a plausible calculation for the relativistic momentum if we use the proper time $\tau$ (see Equation 14.21) rather than the normal time $t$. In this case.

$$\mathbf{p} = m\frac{d\mathbf{x}}{d\tau} = m\frac{d\mathbf{x}}{dt}\frac{dt}{d\tau} \tag{14.43}$$

$$= m\frac{d\mathbf{x}}{dt}\frac{1}{\sqrt{1 - u^2/c^2}} \tag{14.44}$$

$$\boxed{\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}} = \gamma m\mathbf{u}} \quad \text{relativistic momentum} \tag{14.45}$$

where we retain $\mathbf{u} = d\mathbf{x}/dt$ as used classically. Although all observers do not agree as to $d\mathbf{x}/dt$, they do agree as to $d\mathbf{x}/d\tau$, where the proper time $d\tau$ is measured by the moving object itself. The relation $dt/d\tau$ is obtained from Equation 14.21, where the speed $u$ has been used in $\gamma$ to represent the speed of a reference frame fixed in the object that is moving with respect to a fixed frame.

Equation 14.45 is our new definition of momentum, called **relativistic momentum.** Notice that it reduces to the classical result for small values of $u/c$. It was fashionable in past years to call the mass in Equation 14.45 the **rest mass** $m_0$ and to call the term

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}} \quad \textbf{(old-fashioned notation)} \tag{14.46}$$

the **relativistic mass.** The term *rest mass* resulted from Equation 14.46 when $u = 0$, and the classical form of momentum was thus retained: $\mathbf{p} = m\mathbf{u}$. Scientists spoke of the mass increasing at high speeds. We prefer to keep the concept of mass as an invariant, intrinsic property of an object. The use of the two terms *relativistic* and *rest mass* is now considered old-fashioned, although the terms are still sometimes used. *We always refer to the mass m, which is the same as the rest mass.* The use of relativistic mass often leads to mistakes when using classical expressions.

**EXAMPLE 14.6**

Show that linear momentum is conserved in the $x_2$-direction for the collision shown in Figure 14-5 if relativistic momentum is used.

***Solution.*** We can modify the classical expressions for momentum already obtained for the two balls. The momentum for ball A becomes (from Equation 14.36)

$$p_{A2} = \frac{mu_0}{\sqrt{1 - u_0^2/c^2}} \tag{14.47}$$

and

$$\Delta p_{A2} = \frac{-2mu_0}{\sqrt{1 - u_0^2/c^2}} \tag{14.48}$$

Before modifying Equation 14.39 for the momentum of ball B, we must first find the speed of ball B as measured in system *K*. We use Equation 14.38 to determine

$$\begin{aligned} u_B &= \sqrt{u_{B1}^2 + u_{B2}^2} \\ &= \sqrt{v^2 + u_0^2(1 - v^2/c^2)} \end{aligned} \tag{14.49}$$

The momentum $p_{B2}$ is found by modifying Equation 14.39:

$$p_{B2} = -mu_0\gamma\sqrt{1 - v^2/c^2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - u_B^2/c^2}}$$

$$p_{B2} = \frac{-mu_0\sqrt{1 - v^2/c^2}}{\sqrt{1 - u_B^2/c^2}} \tag{14.50}$$

Using $u_B$ from Equation 14.49 gives

$$\begin{aligned} p_{B2} &= \frac{-mu_0\sqrt{1 - v^2/c^2}}{\sqrt{(1 - u_0^2/c^2)(1 - v^2/c^2)}} \\ &= \frac{-mu_0}{\sqrt{1 - u_0^2/c^2}} \end{aligned} \tag{14.51}$$

$$\Delta p_{B2} = \frac{+2mu_0}{\sqrt{1 - u_0^2/c^2}} \tag{14.52}$$

Equations 14.48 and 14.52 add to zero, as required for the conservation of linear momentum.

## 14.8 Energy

With a new definition of linear momentum (Equation 14.45) in hand, we turn our attention to energy and force. We keep our former definition (Equation 2.86) of kinetic energy as being the work done on a particle. The work done is defined in Equation 2.84 to be

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = T_2 - T_1 \tag{14.53}$$

Equation 2.2 for Newton's Second Law is modified to account for the new definition of linear momentum:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m\mathbf{u}) \tag{14.54}$$

If we start from rest, $T_1 = 0$, and the velocity $\mathbf{u}$ is initially along the direction of the force.

$$W = T = \int \frac{d}{dt}(\gamma m\mathbf{u}) \cdot \mathbf{u}\, dt \tag{14.55}$$

$$= m\int_0^u u\, d(\gamma u) \tag{14.56}$$

Equation 14.56 is integrated by parts to obtain

$$T = \gamma mu^2 - m\int_0^u \frac{u du}{\sqrt{1 - u^2/c^2}}$$

$$= \gamma mu^2 + mc^2\sqrt{1 - u^2/c^2}\,\Big|_0^u$$

$$= \gamma mu^2 + mc^2\sqrt{1 - u^2/c^2} - mc^2 \tag{14.57}$$

With algebraic manipulation, Equation 14.57 becomes

$$\boxed{T = \gamma mc^2 - mc^2} \quad \text{relativistic kinetic energy} \tag{14.58}$$

Equation 14.58 seems to resemble in no way our former result for kinetic energy, $T = \frac{1}{2}mu^2$. However, Equation 14.58 must reduce to $\frac{1}{2}mu^2$ for small values of velocity.

**EXAMPLE 14.7**

Show that Equation 14.58 reduces to the classical result for small speeds, $u \ll c$.

***Solution.*** The first term of Equation 14.58 can be expanded in a power series:

$$\begin{aligned} T &= mc^2(1 - u^2/c^2)^{-1/2} - mc^2 \\ &= mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) - mc^2 \end{aligned} \tag{14.59}$$

where all terms of power $(u/c)^4$ or greater are neglected because $u \ll c$.

$$\begin{aligned} T &= mc^2 + \frac{1}{2}mu^2 - mc^2 \\ &= \frac{1}{2}mu^2 \end{aligned} \tag{14.60}$$

which is the classical result.

It is important to note that neither $\frac{1}{2}mu^2$ nor $\frac{1}{2}\gamma mu^2$ gives the correct relativistic value for the kinetic energy.

The term $mc^2$ in Equation 14.58 is called the **rest energy** and is denoted by $E_0$.

$$\boxed{E_0 \equiv mc^2} \quad \text{rest energy} \tag{14.61}$$

Equation 14.58 is rewritten

$$\gamma mc^2 = T + mc^2$$

Thus,

$$E = T + E_0 \tag{14.62}$$

where

$$\boxed{E \equiv \gamma mc^2 = T + E_0} \quad \text{total energy} \tag{14.63}$$

The total energy, $E = \gamma mc^2$, is defined as the sum of kinetic energy and the rest energy. Equations 14.58–14.63 are the origin of Einstein's famous relativistic result of the equivalence of mass and energy (energy $= mc^2$). These equations are consistent with this interpretation. Note that when a body is not in motion ($u = 0 = T$), Equation 14.63 indicates that the total energy is equal to the rest energy.

If mass is simply another form of energy, then we must combine the classical conservation laws of mass and energy into one conservation law of mass-energy represented by Equation 14.63. This law is easily demonstrated in the atomic nucleus, where the mass of constituent particles is converted to the energy that binds the individual particles together.

**EXAMPLE 14.8**

Use the atomic masses of the particles involved to calculate the binding energy of a deuteron.

***Solution.*** A deuteron is composed of a neutron and a proton. We use atomic masses, because the electron masses cancel.

$$\begin{aligned} \text{mass of neutron} &= 1.008665 \text{ u} \\ \text{mass of proton } (^1\text{H}) &= \underline{1.007825 \text{ u}} \\ \text{sum} &= 2.016490 \text{ u} \\ \text{mass of deuteron } (^2\text{H}) &= 2.014102 \text{ u} \\ \text{difference} &= 0.002388 \text{ u} \end{aligned}$$

This difference in mass-energy is equal to the binding energy holding the neutron and proton together as a deuteron. The mass units are atomic mass units (u), which can be converted to kilograms if necessary. However, the conversion of mass to energy is facilitated by the well-known relation between mass and energy:

$$1 \text{ u}c^2 = 931.5 \text{ MeV} \tag{14.64}$$

The binding energy of the deuteron is therefore

$$0.002388 \text{ u}c^2 \times 931.5 \frac{\text{MeV}}{\text{u}c^2} = 2.22 \text{ MeV}$$

Nuclear experiments of the form $\gamma + {}^2\text{H} \rightarrow \text{n} + \text{p}$ indicate that gamma rays of energy just greater than 2.22 MeV are required to break the deuteron apart into a neutron and a proton. Conversely, when a neutron and proton join at rest to form a deuteron, 2.22 MeV of energy is released in the form of kinetic energy of the deuteron and gamma ray.

Because physicists believe that momentum is a more fundamental concept than kinetic energy (for example, there is no general law of conservation of kinetic energy), we would like a relation for mass-energy that includes momentum rather than kinetic energy. We begin with Equation 14.45 for momentum:

$$\begin{aligned} p &= \gamma m u \\ p^2c^2 &= \gamma^2 m^2 u^2 c^2 \\ &= \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right) \end{aligned} \tag{14.65}$$

It is easy to show that

$$\frac{u^2}{c^2} = 1 - \frac{1}{\gamma^2} \tag{14.66}$$

so Equation 14.65 becomes

$$p^2c^2 = \gamma^2 m^2 c^4\left(1 - \frac{1}{\gamma^2}\right)$$

$$= \gamma^2 m^2 c^4 - m^2 c^4$$

$$= E^2 - E_0^2$$

$$\boxed{E^2 = p^2c^2 + E_0^2} \tag{14.67}$$

Equation 14.67 is a very useful kinematic relationship. It relates the total energy of a particle to its momentum and rest energy.

Notice that a photon has no mass, so that Equation 14.67 gives

$$E = pc \quad \text{photon} \tag{14.68}$$

There is no such thing as a photon at rest.

## 14.9 Spacetime and Four-Vectors

In Section 14.3 (Equation 14.5), we noticed that the quantities

$$\left.\begin{aligned} \sum_{j=1}^{3} x_j^2 - c^2t^2 = 0 \\ \sum_{j=1}^{3} x_j'^2 - c^2t'^2 = 0 \end{aligned}\right\}$$

are invariant because the speed of light is the same in all inertial systems in relative motion. Consider two events separated by space and time. In system $K$,

$$\Delta x_i = x_i(\text{event } 2) - x_i(\text{event } 1)$$

$$\Delta t = t(\text{event } 2) - t(\text{event } 1)$$

The interval $\Delta s^2$ is invariant in all inertial systems in relative motion (see Problem 14-34):

$$\Delta s^2 = \sum_{j=1}^{3} (\Delta x_j)^2 - c^2 \Delta t^2 \tag{14.69}$$

$$\Delta s^2 = \Delta s'^2 = \sum_{j=1}^{3} (\Delta x_j')^2 - c^2 \Delta t'^2 \tag{14.70}$$

Equation 14.69 can be written as a differential equation:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2dt^2 \tag{14.71}$$

Consider the system $K'$, where the particle is instantaneously at rest. Because $dx_1' = dx_2' = dx_3' = 0$ in this case, $dt' = d\tau$, the proper time interval discussed

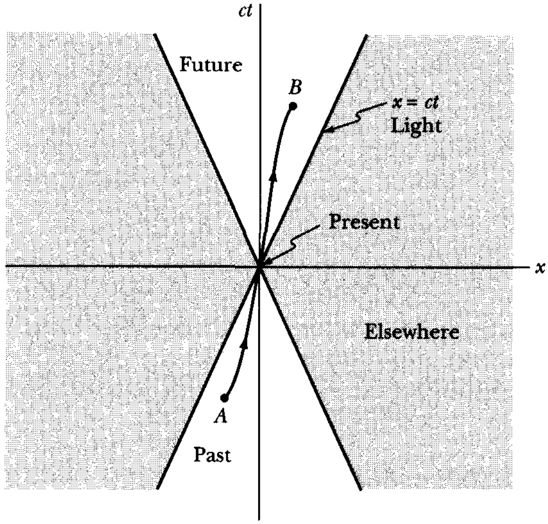



**FIGURE 14-6** The variable $ct$ is plotted versus $x$ with the origin being the present. The heavy solid lines indicate the past and future paths of light and form a *light cone.* To the right and left of these lines is considered "elsewhere," because we cannot reach this region from the present. The path from $A$ to $B$ represents a *worldline,* a path that we can take traveling at speeds less than or equal to light.

above (Equation 14.21). Equation 14.70 becomes

$$-c^2 d\tau^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2 \tag{14.72}$$

Using the Lorentz transformation, Equation 14.72 gives a similar result to Equation 14.21:

$$d\tau = \frac{dt}{\gamma} \tag{14.73}$$

The proper time $\tau$ is, along with the length quantity $\Delta s^2$, another Lorentz invariant quantity.

A useful concept in special relativity is that of the **light cone.** The invariant length $\Delta s^2$ suggests adding $ct$ as a fourth dimension to the three space dimensions $x_1$, $x_2$, and $x_3$. In Figure 14-6, we plot $ct$ versus one of the Euclidean space coordinates. The origin of $(x, ct)$ is the present $(0, 0)$. The solid lines represent the paths taken in the past and in the future by light. A particle traveling the path from $A$ to $B$ is said to be moving along its **worldline.** For time $t < 0$, the particle has been in the lower cone, the past. Similarly, for $t > 0$ the particle will move in the upper cone, the future. It is not possible for us to know about events outside the light cone; this region, called "elsewhere," requires $v > c$.

There are two possibilities concerning the value of $\Delta s^2$. If $\Delta s^2 > 0$, the two events have a **spacelike interval.** One can always find an inertial frame traveling with $v < c$ such that the two events occur at different space coordinates but at the same time. When $\Delta s^2 < 0$, the two events are said to have a **timelike interval.** One can always find a suitable inertial frame in which the events occur at the same point in space but at different times. In the case $\Delta s^2 = 0$, the two events are separated by a light ray.

Only events separated by a timelike interval can be causally connected. The present event in the light cone can be causally related only to events in the past region of the light cone. Events with a spacelike interval cannot be causally connected. Space and time, although distinct, are nonetheless intricately related.

The previous discussion of space and time suggests using $ct$ as a fourth dimensional parameter. We continue this line of thought by defining $x_4 \equiv ict$ and $x_4' \equiv ict'$. The use of the imaginary number $i(\sqrt{-1})$ does not indicate that this component is imaginary. The imaginary number simply allows us to represent the relations in concise, mathematical form. The rest of this section could just as well be carried out without the use of $i$ (e.g., $x_4 = ct$), but the mathematics would be more cumbersome. The useful results are in terms of real, physical quantities.

Using $x_4 = ict$ and $x_4' = ict'$, we can write Equations 14.5 as*

$$\left.\begin{aligned} \sum_{\mu=1}^{4} x_\mu^2 &= 0 \\ \sum_{\mu=1}^{4} x_\mu'^2 &= 0 \end{aligned}\right\} \tag{14.74}$$

From these equations, it is clear that the two sums must be proportional, and because the motion is symmetrical between the systems, the proportionality constant is unity.† Thus,

$$\sum_\mu x_\mu^2 = \sum_\mu x_\mu'^2 \tag{14.75}$$

This relation is analogous to the three-dimensional, distance-preserving, orthogonal rotations we have studied previously (see Section 1.4) and indicates that the Lorentz transformation corresponds to a rotation in a *four-dimensional* space (called **world space** or **Minkowski space**‡). The Lorentz transformations are then orthogonal transformations in Minkowski space:

$$x_\mu' = \sum_\nu \lambda_{\mu\nu} x_\nu \tag{14.76}$$

*In accordance with standard convention, we use Greek indices (usually $\mu$ or $\nu$) to indicate summations that run from 1 to 4; in relativity theory, Latin indices are usually reserved for summations that run from 1 to 3.

†A "proof" is given in Appendix G.

‡Herman Minkowski (1864–1909) made important contributions to the mathematical theory of relativity and introduced *ict* as a fourth component.

where the $\lambda_{\mu\nu}$ are the elements of the Lorentz transformation matrix. From Equations 14.14, the transformation $\boldsymbol{\lambda}$ is

$$\boldsymbol{\lambda} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{14.77}$$

A quantity is called a **four-vector** if it consists of four components, each of which transforms according to the relation*

$$A'_\mu = \sum_\nu \lambda_{\mu\nu} A_\nu \tag{14.78}$$

where the $\lambda_{\mu\nu}$ define a Lorentz transformation. Such a four-vector† is

$$\mathbb{X} = (x_1, x_2, x_3, ict) \tag{14.79a}$$

or

$$\boxed{\mathbb{X} = (\mathbf{x}, ict)} \tag{14.79b}$$

where the notation of the last line means that the first three (space) components of $\mathbb{X}$ define the ordinary three-dimensional position vector **x** and that the fourth component is *ict*. Similarly, the differential of $\mathbb{X}$ is a four-vector:

$$d\mathbb{X} = (d\mathbf{x}, ic\, dt) \tag{14.80}$$

In Minkowski space, the four-dimensional element of length is invariant. Its magnitude is unaffected by a Lorentz transformation, and such a quantity is called a **four-scalar** or **world scalar.** Equation 14.71 can be written as

$$ds = \sqrt{\sum_\mu dx_\mu^2} \tag{14.81}$$

and Equation 14.72 as

$$d\tau = \frac{i}{c}\sqrt{\sum_\mu dx_\mu^2} = \frac{i}{c}\, ds \tag{14.82}$$

The proper time $d\tau$ is invariant because it is simply $i/c$ times the element of length $ds$. The ratio of the four-vector $d\mathbb{X}$ to the invariant $d\tau$ is therefore also a four-vector, called the four-vector velocity $\mathbb{V}$:

$$\boxed{\mathbb{V} = \frac{d\mathbb{X}}{d\tau} = \left(\frac{d\mathbf{x}}{d\tau}, ic\frac{dt}{d\tau}\right)} \tag{14.83}$$

The components of the ordinary velocity **u** are

$$u_j = \frac{dx_j}{dt}$$

*We do not distinguish here between *covariant* and *contravariant* vector components; see, for example, Bergmann (Be46, Chapter 5).

†Four-vectors are denoted exclusively by openface capital letters.

so, using Equations 14.71 and 14.82, $d\tau$ can be expressed as

$$d\tau = dt\sqrt{1 - \frac{1}{c^2}\sum_j \frac{dx_j^2}{dt^2}}$$

or

$$d\tau = dt\sqrt{1-\beta^2} \tag{14.84}$$

as we found in Equation 14.73. The four-vector velocity can therefore be written as

$$\boxed{\mathbb{V} = \frac{1}{\sqrt{1-\beta^2}}(\mathbf{u}, ic)} \tag{14.85}$$

where $\mathbf{u}$ represents the three space components of ordinary velocity, $u_1$, $u_2$, $u_3$. (Remember that the particle's velocity is now denoted by $\mathbf{u}$ to distinguish it from the moving frame velocity $\mathbf{v}$.) The four-vector momentum is now simply the mass times four-vector velocity,* because mass is invariant:

$$\mathbb{P} = m\mathbb{V} \tag{14.86}$$

$$\boxed{\mathbb{P} = \left(\frac{m\mathbf{u}}{\sqrt{1-\beta^2}}, ip_4\right)} \tag{14.87}$$

where

$$p_4 \equiv \frac{mc}{\sqrt{1-\beta^2}} \tag{14.88}$$

The first three components of the four-vector momentum $\mathbb{P}$ are the components of the relativistic momentum (Equation 14.45):

$$P_j = p_j = \gamma m u_j, \quad j = 1, 2, 3 \tag{14.89}$$

Using Equation 14.63, the fourth component of the momentum is related to the total energy $E$:

$$p_4 = \gamma mc = \frac{E}{c} \tag{14.90}$$

The four-vector momentum can therefore be written as

$$\mathbb{P} = \left(\mathbf{p}, i\frac{E}{c}\right) \tag{14.91}$$

where $\mathbf{p}$ stands for the three space components of momentum. Thus, in relativity theory, momentum and energy are linked in a manner similar to that which joins the concepts of space and time. If we apply the Lorentz transformation matrix

*A four-vector multiplied by a four-scalar is also a four-vector.

(Equation 14.77) to the momentum $\mathbb{P}$, we find

$$\boxed{\begin{aligned} p_1' &= \frac{p_1 - (v/c^2)E}{\sqrt{1-\beta^2}} \\ p_2' &= p_2 \\ p_3' &= p_3 \\ E' &= \frac{E - vp_1}{\sqrt{1-\beta^2}} \end{aligned}} \tag{14.92}$$

## EXAMPLE 14.9

Using the methods of this section, derive Equation 14.67.

***Solution.*** If we place the origin of the moving system $K'$ fixed on the particle, we have $u = v$. The square of the four-vector velocity (Equation 14.85) is invariant:

$$\mathbb{V}^2 = \sum_\mu V_\mu^2 = \frac{v^2 - c^2}{1 - \beta^2} = -c^2 \tag{14.93}$$

Hence, the square of the four-vector momentum is also invariant:

$$\mathbb{P}^2 = \sum_\mu P_\mu^2 = m^2\mathbb{V}^2 = -m^2c^2 \tag{14.94}$$

From Equation 14.91, we also have, using $\mathbf{p} \cdot \mathbf{p} = p^2 = p_1^2 + p_2^2 + p_3^2$,

$$\mathbb{P}^2 = p^2 - \frac{E^2}{c^2} \tag{14.95}$$

Combining the last two equations gives Equation 14.67.

$$E^2 = p^2c^2 + m^2c^4 = p^2c^2 + E_0^2$$

If we define an angle $\phi$ such that $\beta = \sin\phi$, the relativistic relations between velocity, momentum, and energy can be obtained by trigonometric relations involving the so-called "relativistic triangle" (Figure 14-7).

## EXAMPLE 14.10

Derive the velocity addition rule.

***Solution.*** Suppose that there are three inertial reference frames, $K$, $K'$, and $K''$, which are in collinear motion along their respective $x_1$-axes. Let the velocity of $K'$ relative to $K$ be $v_1$ and let the velocity of $K''$ relative to $K'$ be $v_2$. The speed of $K''$ relative to $K$ cannot be $v_1 + v_2$, because it must be possible to propagate a signal between any two inertial frames, and if both $v_1$ and $v_2$ are greater than $c/2$ (but less than $c$), then $v_1 + v_2 > c$. Therefore, the rule for the addition of velocities in relativity must be different from that in Galilean theory. The relativistic

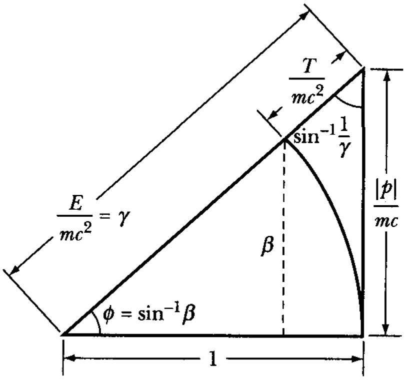



**FIGURE 14-7** The relativistic triangle allows us to find relations between velocity, momentum, and energy by using trigonometric relations.

velocity addition rule can be obtained by considering the Lorentz transformation matrix connecting $K$ and $K''$. The individual transformation matrices are

$$\boldsymbol{\lambda}_{K'\to K} = \begin{pmatrix} \gamma_1 & 0 & 0 & i\beta_1\gamma_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta_1\gamma_1 & 0 & 0 & \gamma_1 \end{pmatrix}$$

$$\boldsymbol{\lambda}_{K''\to K'} = \begin{pmatrix} \gamma_2 & 0 & 0 & i\beta_2\gamma_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta_2\gamma_2 & 0 & 0 & \gamma_2 \end{pmatrix}$$

The transformation from $K''$ to $K$ is just the product of these two transformations:

$$\boldsymbol{\lambda}_{K''\to K} = \boldsymbol{\lambda}_{K''\to K'}\boldsymbol{\lambda}_{K'\to K} = \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 & i\gamma_1\gamma_2(\beta_1+\beta_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 & \gamma_1\gamma_2(1+\beta_1\beta_2) \end{pmatrix}$$

So that the elements of this matrix correspond to those of the normal Lorentz matrix (Equation 14.77), we must identify $\beta$ and $\gamma$ for the $K'' \to K$ transformation as

$$\left.\begin{aligned} \gamma &= \gamma_1\gamma_2(1+\beta_1\beta_2) \\ \beta\gamma &= \gamma_1\gamma_2(\beta_1+\beta_2) \end{aligned}\right\} \tag{14.96}$$

from which we obtain

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \tag{14.97}$$

If we multiply this last expression by $c$, we have the usual form of the velocity (speed) addition rule:

$$\boxed{v = \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)}} \tag{14.98}$$

It follows that if $v_1 < c$ and $v_2 < c$, then $v < c$ also.

---

Even though *signal* velocities can never exceed $c$, there are other types of velocity that can be greater than $c$. For example, the *phase velocity* of a light wave in a medium for which the index of refraction is less than unity is greater than $c$, but the phase velocity does not correspond to the signal velocity in such a medium; the signal velocity is indeed less than $c$. Or consider an electron gun that emits a beam of electrons. If the gun is rotated, then the electron beam describes a certain path on a screen placed at some appropriate distance. If the angular velocity of the gun and the distance to the screen are sufficiently large, then the velocity of the spot traveling across the screen can be *any* velocity, arbitrarily large. Thus, the *writing speed* of an oscilloscope can exceed $c$, but again the writing speed does not correspond to the signal velocity; that is, information cannot be transmitted from one point on the screen to another by means of the electron beam. In such a device, a signal can be transmitted only from the gun to the screen, and this transmission takes place at the velocity of the electrons in the beam (i.e., $< c$).

## EXAMPLE 14.11

Derive the relativistic Doppler effect if the angle between the light source and direction of relative motion of the observer is $\theta$ (Figure 14-8).

***Solution.*** This example can easily be solved using the momentum-energy four-vector by treating the light as a photon with total energy $E = h\nu$. The light source is at rest in system $K$ and emits a single frequency $\nu_0$.

$$E = h\nu_0 \tag{14.99}$$

$$p = \frac{E}{c} = \frac{h\nu_0}{c} \tag{14.100}$$

The observer moving to the right in system $K'$ measures the energy $E'$ for a photon of frequency $\nu'$. From Equation 14.92, we have

$$E' = \gamma(h\nu_0 - vp_1) \tag{14.101}$$

$$h\nu' = \gamma\left(h\nu_0 - \frac{vh\nu_0}{c}\cos\theta\right) \tag{14.102}$$

where $p_1 = p\cos\theta$. Equation 14.102 reduces to

$$\nu' = \gamma\nu_0(1 - \beta\cos\theta) \tag{14.103}$$

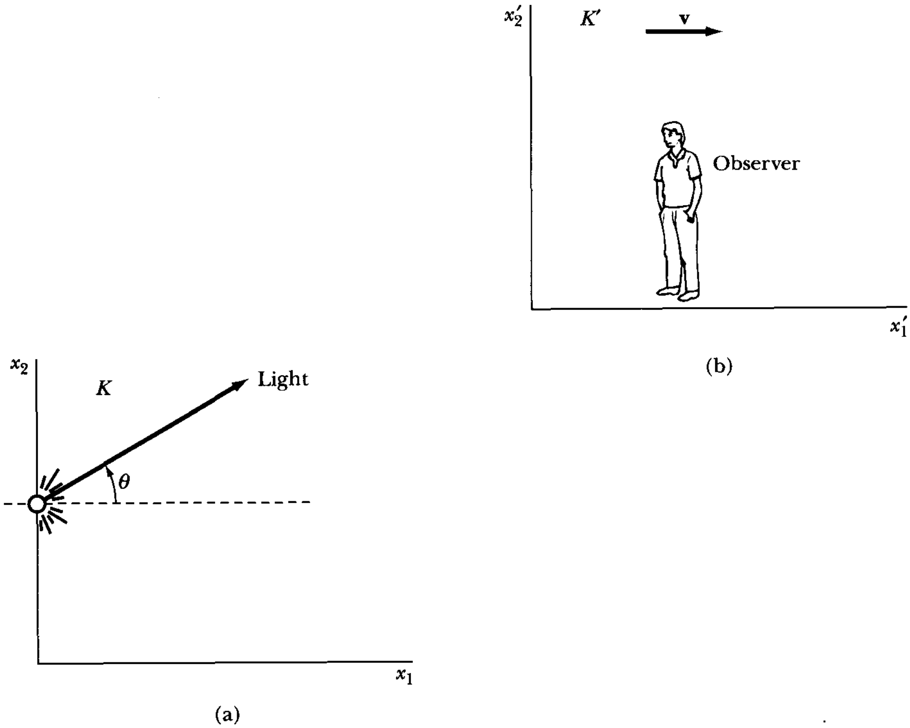



**FIGURE 14-8** A light source fixed in system $K$ emits light at a single frequency $\nu_0$. An observer in system $K'$, moving to the right at velocity **v** with respect to $K$, measures the light frequency to be $\nu'$.

which is equivalent to Equation 14.34, depending on the value of $\theta$. For an early time, the observer is far to the left of the source, and as the observer approaches the source ($\theta = \pi$),

$$\nu' = \nu_0 \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \qquad \text{observer approaching source} \qquad \textbf{(14.104)}$$

as in Equation 14.31. At a much later time, the observer is receding ($\theta = 0$) and

$$\nu' = \nu_0 \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \qquad \text{observer receding from source} \qquad \textbf{(14.105)}$$

as in Equation 14.33. When the observer just passes the source ($\theta = \pi/2$),

$$\nu' = \frac{\nu_0}{\sqrt{1-\beta^2}} \qquad \text{observer passing source} \qquad \textbf{(14.106)}$$

We can also treat the case where the observer is at rest and the source is moving (see Problem 14-18). We still obtain Equations 14.104–14.106 because, according to the principle of relativity, it is not possible to distinguish between the motion of the observer and the motion of the source.

## 14.10 Lagrangian Function in Special Relativity

Lagrangian and Hamiltonian dynamics (discussed in Chapter 7) must be adjusted in light of the new concepts presented here. We can extend the Lagrangian formalism into the realm of special relativity in the following way. For a single (nonrelativistic) particle moving in a velocity-independent potential, the rectangular momentum components (see Equation 7.150) may be written as

$$p_i = \frac{\partial L}{\partial u_i} \tag{14.107}$$

According to Equation 14.87, the relativistic expression for the ordinary (i.e., space) momentum component is

$$p_i = \frac{mu_i}{\sqrt{1-\beta^2}} \tag{14.108}$$

We now require that the *relativistic* Lagrangian, when differentiated with respect to $u_i$ as in Equation 14.107, yield the momentum components given by Equation 14.108:

$$\frac{\partial L}{\partial u_i} = \frac{mu_i}{\sqrt{1-\beta^2}} \tag{14.109}$$

This requirement involves only the *velocity* of the particle, so we expect that the velocity-*independent* part of the relativistic Lagrangian is unchanged from the nonrelativistic case. The velocity-*dependent* part, however, may no longer be equal to the kinetic energy. We therefore write

$$L = T^* - U \tag{14.110}$$

where $U = U(x_i)$ and $T^* = T^*(u_i)$. The function $T^*$ must satisfy the relation

$$\frac{\partial T^*}{\partial u_i} = \frac{mu_i}{\sqrt{1-\beta^2}} \tag{14.111}$$

It can be easily verified that a suitable expression for $T^*$ (apart from a possible constant of integration that can be suppressed) is

$$T^* = -mc^2\sqrt{1-\beta^2} \tag{14.112}$$

Hence, the relativistic Lagrangian can be written as

$$\boxed{L = -mc^2\sqrt{1-\beta^2} - U} \tag{14.113}$$

and the equations of motion are obtained in the standard way from Lagrange's equations.

Notice that the Lagrangian is *not* given by $T - U$, because the relativistic expression for the kinetic energy (Equation 14.58) is

$$T = \frac{mc^2}{\sqrt{1-\beta^2}} - mc^2 \tag{14.114}$$

The Hamiltonian (see Equation 7.153) can be calculated from

$$\begin{aligned} H &= \sum_i u_i p_i - L \\ &= \sum_i \frac{p_i^2 c^2}{\gamma m c^2} + \frac{mc^2}{\gamma} + U \end{aligned}$$

where we have used Equations 14.108 and 14.113 and changed $\sqrt{1-\beta^2}$ to $\gamma^{-1}$. Thus,

$$\begin{aligned} H &= \frac{p^2c^2}{\gamma mc^2} + \frac{mc^2}{\gamma} + U = \frac{1}{\gamma mc^2}(p^2c^2 + m^2c^4) + U \\ &= \frac{E^2}{\gamma mc^2} + U \\ &= E + U = T + U + E_0 \end{aligned} \tag{14.115}$$

The relativistic Hamiltonian is equal to the total energy defined in Section 14.8 *plus* the potential energy. It differs from the total energy used previously in Chapter 7 by now including the rest energy.

## 14.11 Relativistic Kinematics

In the event that the velocities in a collision process are not negligible with respect to the velocity of light, it becomes necessary to use *relativistic* kinematics. In the discussion in Chapter 9, we took advantage of the properties of the center-of-mass coordinate system in deriving many of the kinematic relations. Because mass and energy are interrelated in relativity theory, it no longer is meaningful to speak of a "center-of-mass" system; in relativistic kinematics, one uses a "center-of-momentum" coordinate system instead. Such a system possesses the same essential property as the previously used center-of-mass system—the total linear momentum in the system is zero. Therefore, if a particle of mass $m_1$ collides elastically with a particle of mass $m_2$, then in the center-of-momentum system we have

$$p_1' = p_2' \tag{14.116}$$

Using Equation 14.87, the space components of the momentum four-vector can be written as

$$m_1 u_1' \gamma_1' = m_2 u_2' \gamma_2' \tag{14.117}$$

where, as before, $\gamma \equiv 1/\sqrt{1-\beta^2}$ and $\beta \equiv u/c$.

In a collision problem, it is convenient to associate the laboratory coordinate system with the inertial system $K$ and the center-of-momentum system with $K'$ (see Figure 14-9). A simple Lorentz transformation then connects the two systems. To derive the relativistic kinematic expressions, the procedure is

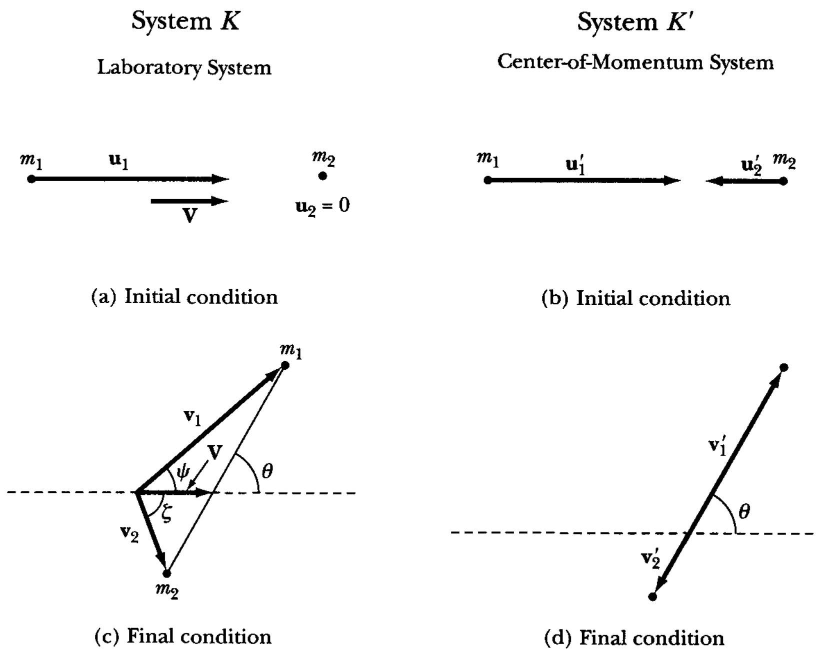



**FIGURE 14-9** The elastic collision schematic of Figure 9-10 is redisplayed with systems $K$ and $K'$ indicated.

to obtain the center-of-momentum relations and then perform a Lorentz transformation back to the laboratory system. We choose the coordinate axes so that $m_1$ moves along the $x$-axis in $K$ with speed $u_1$. Because $m_2$ is initially at rest in $K$, $u_2 = 0$. In $K'$, $m_2$ moves with speed $u_2'$ and so $K'$ moves with respect to $K$ also with speed $u_2'$ and in the same direction as the initial motion of $m_1$.

Using the fact that $\beta\gamma = \sqrt{\gamma^2 - 1}$, we have

$$\begin{aligned} p_1' &= m_1 u_1' \gamma_1' = m_1 c \beta_1' \gamma_1' \\ &= m_1 c \sqrt{\gamma_1'^2 - 1} = m_2 c \sqrt{\gamma_2'^2 - 1} \\ &= p_2' \end{aligned} \tag{14.118}$$

which expresses the equality of the momenta in the center-of-momentum system.

According to Equation 14.92, the transformation of the momentum $p_1$ (from $K$ to $K'$) is

$$p_1' = \left(p_1 - \frac{u_2'}{c^2} E_1\right)\gamma_2' \tag{14.119}$$

We also have

$$\left.\begin{aligned} p_1 &= m_1 u_1 \gamma_1 \\ E_1 &= m_1 c^2 \gamma_1 \end{aligned}\right\} \tag{14.120}$$

so Equation 14.118 can be used to obtain

$$\begin{aligned} m_1 c\sqrt{\gamma_1'^2 - 1} &= (m_1 c\beta_1\gamma_1 - \beta_2' m_1 c\gamma_1)\gamma_2' \\ &= m_1 c\left(\gamma_2'\sqrt{\gamma_1^2 - 1} - \gamma_1\sqrt{\gamma_2'^2 - 1}\right) \\ &= m_2 c\sqrt{\gamma_2'^2 - 1} \end{aligned} \tag{14.121}$$

These equations can be solved for $\gamma_1'$ and $\gamma_2'$ in terms of $\gamma_1$:

$$\gamma_1' = \frac{\gamma_1 + \dfrac{m_1}{m_2}}{\sqrt{1 + 2\gamma_1\left(\dfrac{m_1}{m_2}\right) + \left(\dfrac{m_1}{m_2}\right)^2}} \tag{14.122a}$$

$$\gamma_2' = \frac{\gamma_1 + \dfrac{m_2}{m_1}}{\sqrt{1 + 2\gamma_1\left(\dfrac{m_2}{m_1}\right) + \left(\dfrac{m_2}{m_1}\right)^2}} \tag{14.122b}$$

Next, we write the equations of the transformation of the momentum components from $K'$ back to $K$ after the scattering. We now have both $x$- and $y$-components:

$$\begin{aligned} p_{1,x} &= \left(p_{1,x}' + \frac{u_2'}{c^2}E_1'\right)\gamma_2' \\ &= (m_1 c\beta_1'\gamma_1' \cos\theta + m_1 c\beta_2'\gamma_1')\gamma_2' \\ &= m_1 c\gamma_1'\gamma_2'(\beta_1' \cos\theta + \beta_2') \end{aligned} \tag{14.123a}$$

(Note that, because the transformation is from $K'$ to $K$, a plus sign occurs before the second term, in contrast to Equation 14.119.) Also,

$$p_{1,y} = m_1 c\beta_1'\gamma_1' \sin\theta \tag{14.123b}$$

The tangent of the laboratory scattering angle $\psi$ is given by $p_{1,y}/p_{1,x}$; therefore, dividing Equation 14.123b by Equation 14.123a, we obtain

$$\tan\psi = \frac{1}{\gamma_2'}\,\frac{\sin\theta}{\cos\theta + (\beta_2'/\beta_1')}$$

Using Equation 14.117 to express $\beta_2'/\beta_1'$, the result is

$$\tan\psi = \frac{1}{\gamma_2'}\,\frac{\sin\theta}{\cos\theta + (m_1\gamma_1'/m_2\gamma_2')} \tag{14.124}$$

For the recoil particle, we have

$$\begin{aligned} p_{2,x} &= \left(p_{2,x}' + \frac{u_2'}{c^2}E_2'\right)\gamma_2' \\ &= (-m_2 c\beta_2'\gamma_2' \cos\theta + m_2 c\beta_2'\gamma_2')\gamma_2' \\ &= m_2 c\beta_2'\gamma_2'^2(1 - \cos\theta) \end{aligned} \tag{14.125a}$$

where a minus sign occurs in the first term because $p'_{2,x}$ is directed opposite to $p_{1,x}$. Also,

$$p_{2,y} = -m_2 c\beta'_2\gamma'_2 \sin\theta \tag{14.125b}$$

As before, the tangent of the laboratory recoil angle $\zeta$ is given by $p_{2,y}/p_{2,x}$:

$$\tan\zeta = -\frac{1}{\gamma'_2}\frac{\sin\theta}{1-\cos\theta} \tag{14.126}$$

The overall minus sign indicates that if $m_1$ is scattered toward positive values of $\psi$, then $m_2$ recoils in the negative $\zeta$-direction.

A case of special interest is that in which $m_1 = m_2$. From Equations 14.122, we find

$$\gamma'_1 = \gamma'_2 = \sqrt{\frac{1+\gamma_1}{2}}, \qquad m_1 = m_2 \tag{14.127}$$

The tangents of the scattering angles become

$$\tan\psi = \sqrt{\frac{2}{1+\gamma_1}}\cdot\frac{\sin\theta}{1+\cos\theta} \tag{14.128}$$

$$\tan\zeta = -\sqrt{\frac{2}{1+\gamma_1}}\cdot\frac{\sin\theta}{1-\cos\theta} \tag{14.129}$$

The product is therefore

$$\tan\psi\tan\zeta = -\frac{2}{1+\gamma_1}, \qquad m_1 = m_2 \tag{14.130}$$

(The minus sign is of no essential importance; it only indicates that $\psi$ and $\zeta$ are measured in opposite directions.)

We previously found that in the nonrelativistic limit there was always a right angle between the final velocity vectors in the scattering of particles of equal mass. Indeed, in the limit $\gamma_1 \to 1$, Equations 14.128 and 14.129 become equal to Equations 9.69 and 9.73, respectively, and so $\psi + \zeta = \pi/2$. Equation 14.130, however, shows that in the relativistic case $\psi + \zeta < \pi/2$; thus, the included angle in the scattering is always smaller than in the nonrelativistic limit. For equal scattering and recoil angles ($\psi = \zeta$), Equation 14.130 becomes

$$\tan\psi = \left(\frac{2}{1+\gamma_1}\right)^{1/2}, \qquad m_1 = m_2$$

and the included angle between the directions of the scattered and recoil particles is

$$\begin{aligned}\phi &= \psi + \zeta = 2\psi \\ &= 2\tan^{-1}\left(\frac{2}{1+\gamma_1}\right)^{1/2}, \qquad m_1 = m_2\end{aligned} \tag{14.131}$$

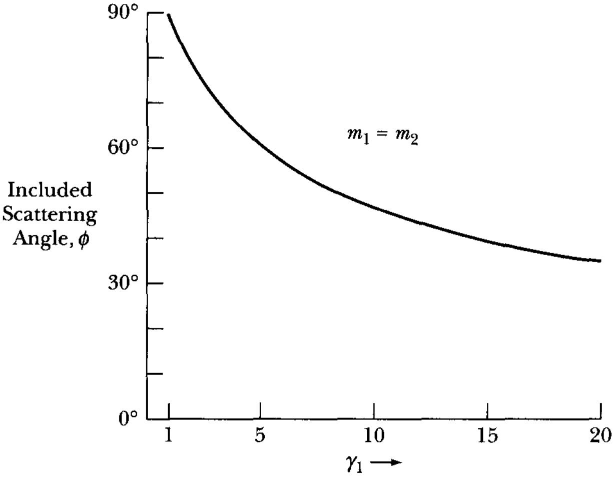



**FIGURE 14-10** The included scattering angle, $\phi = \psi + \zeta$, is shown as a function of the relativistic parameter $\gamma_1$ for $m_1 = m_2$. For nonrelativistic scattering ($\gamma_1 = 1$), this angle is always 90°.

Figure 14-10 shows $\phi$ as a function of $\gamma_1$ up to $\gamma_1 = 20$. At $\gamma_1 = 10$, the included angle is approximately 46°. This value of $\gamma_1$ corresponds to an initial velocity that is 99.5% of the velocity of light. According to Equation 14.58, the kinetic energy is given by $T_1 = m_1c^2(\gamma_1 - 1)$; therefore, a proton with $\gamma_1 = 10$ would have a kinetic energy of approximately 8.4 GeV, whereas an electron with the same velocity would have $T_1 \cong 4.6$ MeV.*

By using the transformation properties of the fourth component of the momentum four-vector (i.e., the total energy), it is possible to obtain the relativistic analogs of all the energy equations we have previously derived in the nonrelativistic limit.

## PROBLEMS

**14-1.** Prove Equation 14.13 by using Equations 14.9–14.12.

**14-2.** Show that the transformation equations connecting the $K'$ and $K$ systems (Equations 14.14) can be expressed as

$$x_1' = x_1 \cosh \alpha - ct \sinh \alpha$$

$$x_2' = x_2, \quad x_3' = x_3$$

$$t' = t \cosh \alpha - \frac{x_1}{c} \sinh \alpha$$

where $\tanh \alpha = v/c$. Show that the Lorentz transformation corresponds to a rotation through an angle $i\alpha$ in four-dimensional space.

*These units of energy are defined in Problem 14-39: 1 GeV = $10^3$ MeV = $10^9$ eV = $1.602 \times 10^{-3}$ erg = $1.602 \times 10^{-10}$ J.

**14-3.** Show that the equation

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = 0$$

is invariant under a Lorentz transformation but not under a Galilean transformation. (This is the wave equation that describes the propagation of light waves in free space.)

**14-4.** Show that the expression for the FitzGerald-Lorentz contraction (Equation 14.19) can also be obtained if the observer in the $K'$ system measures the time necessary for the rod to pass a fixed point in that system and then multiplies the result by $v$.

**14-5.** What is the apparent shape of a cube moving with a uniform velocity directly *toward* or *away from* an observer?

**14-6.** Consider two events that take place at different points in the $K$ system at the same instant $t$. If these two points are separated by a distance $\Delta x$, show that in the $K'$ system the events are not simultaneous but are separated by a time interval $\Delta t' = -v\gamma\, \Delta x/c^2$.

**14-7.** Two clocks located at the origins of the $K$ and $K'$ systems (which have a relative speed $v$) are synchronized when the origins coincide. After a time $t$, an observer at the origin of the $K$ system observes the $K'$ clock by means of a telescope. What does the $K'$ clock read?

**14-8.** In his 1905 paper (see the translation in Lo23), Einstein states: "We conclude that a balance-clock at the equator must go more slowly, by a very small amount, than a precisely similar clock situated at one of the poles under otherwise identical conditions." Neglect the fact that the equator clock does not undergo uniform motion and show that after a century the clocks will differ by approximately 0.0038 s.

**14-9.** Consider a relativistic rocket whose velocity with respect to a certain inertial frame is $v$ and whose exhaust gases are emitted with a constant velocity $V$ with respect to the rocket. Show that the equation of motion is

$$m\frac{dv}{dt} + V\frac{dm}{dt}(1 - \beta^2) = 0$$

where $m = m(t)$ is the mass of the rocket in its rest frame and $\beta = v/c$.

**14-10.** Show by algebraic methods that Equations 14.15 follow from Equations 14.14.

**14-11.** A stick of length $l$ is fixed at an angle $\theta$ from its $x_1$-axis in its own rest system $K$. What is the length and orientation of the stick as measured by an observer moving along $x_1$ with speed $v$?

**14-12.** A racer attempting to break the land speed record rockets by two markers spaced 100 m apart on the ground in a time of 0.4 $\mu$s as measured by an observer on the ground. How far apart do the two markers appear to the racer? What elapsed time does the racer measure? What speeds do the racer and ground observer measure?

**14-13.** A muon is moving with speed $v = 0.999c$ vertically down through the atmosphere. If its half-life in its own rest frame is 1.5 $\mu$s, what is its half-life as measured by an observer on Earth?

**14-14.** Show that Equation 14.31 is valid when a receiver approaches a fixed light source with speed $v$.

**14-15.** A star is known to be moving away from Earth at a speed of $4 \times 10^4$ m/s. This speed is determined by measuring the shift of the $H_\alpha$ line ($\lambda = 656.3$ nm). By how much and in what direction is the shift of the wavelength of the $H_\alpha$ line?

**14-16.** A photon is emitted at an angle $\theta'$ by a star (system $K'$) and then received at an angle $\theta$ on Earth (system $K$). The angles are measured from a line between the star and Earth. The star is receding at speed $v$ with respect to Earth. Find the relation between $\theta$ and $\theta'$; this effect is called the *aberration of light.*

**14-17.** The wavelength of a spectral line measured to be $\lambda$ on Earth is found to increase by 50% on a far distant galaxy. What is the speed of the galaxy relative to Earth?

**14-18.** Solve Example 14.11 for the case of the observer at rest and the source moving. Show that the results are the same as those given in Example 14.11.

**14-19.** Equation 14.34 indicates that a red (blue) shift occurs when a source and observer are receding (approaching) with respect to one another in purely radial motion (i.e., $\beta = \beta_r$). Show that, if there is also a relative tangential speed $\beta_t$, Equation 14.34 becomes

$$\frac{\lambda_0}{\lambda} = \frac{\nu}{\nu_0} = \frac{\sqrt{1 - \beta_r^2 - \beta_t^2}}{1 - \beta_r}$$

and that the condition for always having a red shift (i.e., no blue shift), $\lambda > \lambda_0$ or $\nu < \nu_0$, is*

$$\beta_t^2 > 2\beta_r(1 - \beta_r)$$

**14-20.** An astronaut travels to the nearest star system, 4 light years away, and returns at speed $0.3c$. How much has the astronaut aged relative to those people remaining on Earth?

**14-21.** The expression for the ordinary force is

$$\mathbf{F} = \frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1 - \beta^2}}\right)$$

Take $\mathbf{u}$ to be in the $x_1$-direction and compute the components of the force. Show that

$$F_1 = m_l \dot{u}_1, \quad F_2 = m_t \dot{u}_2, \quad F_3 = m_t \dot{u}_3$$

where $m_l$ and $m_t$ are, respectively, the *longitudinal mass* and the *transverse mass*:

$$m_l = \frac{m}{(1 - \beta^2)^{3/2}}, \qquad m_t = \frac{m}{\sqrt{1 - \beta^2}}$$

*See J. J. Dykla, *Am. J. Phys.* **47**, 381 (1979).

**14-22.** The average rate at which solar radiant energy reaches Earth is approximately $1.4 \times 10^3$ W/m$^2$. Assume that all this energy results from the conversion of mass to energy. Calculate the rate at which the solar mass is being lost. If this rate is maintained, calculate the remaining lifetime of the Sun. (Pertinent numerical data can be found in Table 8-1.)

**14-23.** Show that the momentum and the kinetic energy of a particle are related by $p^2c^2 = 2Tmc^2 + T^2$.

**14-24.** What is the minimum proton energy needed in an accelerator to produce antiprotons $\bar{p}$ by the reaction

$$p + p \rightarrow p + p + (p + \bar{p})$$

The mass of a proton and antiproton is $m_p$.

**14-25.** A particle of mass $m$, kinetic energy $T$, and charge $q$ is moving perpendicular to a magnetic field $B$ as in a cyclotron. Find the relation for the radius $r$ of the particle's path in terms of $m$, $T$, $q$, and $B$.

**14-26.** Show that an isolated photon cannot be converted into an electron-positron pair, $\gamma \rightarrow e^- + e^+$. (The conservation laws allow this to happen only near another object.)

**14-27.** Electrons and positrons collide from opposite directions head-on with equal energies in a storage ring to produce protons by the reaction

$$e^- + e^+ \rightarrow p + \bar{p}$$

The rest energy of a proton and antiproton is 938 MeV. What is the minimum kinetic energy for each particle to produce this reaction?

**14-28.** Calculate the range of speeds for a particle of mass $m$ in which the classical relation for kinetic energy, $\frac{1}{2}mv^2$, is within one percent of the correct relativistic value. Find the values for an electron and a proton.

**14-29.** The 2-mile long Stanford Linear Accelerator accelerates electrons to 50 GeV ($50 \times 10^9$ eV). What is the speed of the electrons at the end?

**14-30.** A free neutron is unstable and decays into a proton and an electron. How much energy other than the rest energies of the proton and electron is available if a neutron at rest decays? (This is an example of nuclear beta decay. Another particle, called a neutrino—actually an antineutrino $\bar{\nu}$ is also produced.)

**14-31.** A neutral pion $\pi^0$ moving at speed $v = 0.98c$ decays in flight into two photons. If the two photons emerge on each side of the pion's direction with equal angles $\theta$, find the angle $\theta$ and energies of the photons. The rest energy of $\pi^0$ is 135 MeV.

**14-32.** In nuclear and particle physics, momentum is usually quoted in MeV/$c$ to facilitate calculations. Calculate the kinetic energy of an electron and proton if each has a momentum of 1000 MeV/$c$.

**14-33.** A neutron ($m_n = 939.6$ MeV/$c^2$) at rest decays into a proton ($m_p = 938.3$ MeV/$c^2$), an electron ($m_e = 0.5$ MeV/$c^2$), and an antineutrino ($m_{\bar{\nu}} \approx 0$). The three particles emerge at symmetrical angles in a plane, 120° apart. Find the momentum and kinetic energy of each particle.

**14-34.** Show that $\Delta s^2$ is invariant in all inertial systems moving at relative velocities to each other.

**14-35.** A spacecraft passes Saturn with a speed of $0.9c$ relative to Saturn. A second spacecraft is observed to pass the first one (going in the same direction) at relative speed of $0.2c$. What is the speed of the second spacecraft relative to Saturn?

**14-36.** We define the four-vector force $\mathbb{F}$ (called the Minkowski force) by differentiating the four-vector momentum with respect to proper time.

$$\mathbb{F} = \frac{d\mathbb{P}}{d\tau}$$

Show that the four-vector force transformation is

$$\begin{aligned} F_1' &= \gamma(F_1 + i\beta F_4) \\ F_2' &= F_2 \\ F_3' &= F_3 \\ F_4' &= \gamma(F_4 - i\beta F_1) \end{aligned}$$

**14-37.** Consider a one-dimensional, relativistic harmonic oscillator for which the Lagrangian is

$$L = mc^2\left(1 - \sqrt{1-\beta^2}\right) - \frac{1}{2}kx^2$$

Obtain the Lagrange equation of motion and show that it can be integrated to yield

$$E = mc^2 + \frac{1}{2}ka^2$$

where $a$ is the maximum excursion from equilibrium of the oscillating particle. Show that the period

$$\tau = 4\int_{x=0}^{x=a} dt$$

can be expressed as

$$\tau = \frac{2a}{\kappa c}\int_0^{\pi/2} \frac{1 + 2\kappa^2\cos^2\phi}{\sqrt{1+\kappa^2\cos^2\phi}}\,d\phi$$

Expand the integrand in powers of $\kappa \equiv (a/2)\sqrt{k/mc^2}$ and show that, to first order in $\kappa$,

$$\tau \cong \tau_0\left(1 + \frac{3}{16}\frac{ka^2}{mc^2}\right)$$

where $\tau_0$ is the nonrelativistic period for small oscillations, $2\pi\sqrt{m/k}$.

**14-38.** Show that the relativistic form of Newton's Second Law becomes

$$F = m\frac{du}{dt}\left(1 - \frac{u^2}{c^2}\right)^{-3/2}$$

**14-39.** A common unit of energy used in atomic and nuclear physics is the electron volt (eV), the energy acquired by an electron in falling through a potential difference of one volt: 1 MeV = $10^6$ eV = $1.602 \times 10^{-13}$ J. In these units, the mass of an electron is $m_e c^2 = 0.511$ MeV and that of a proton is $m_p c^2 = 938$ MeV. Calculate the kinetic energy and the quantities $\beta$ and $\gamma$ for an electron and for a proton each having a momentum of 100 MeV/$c$. Show that the electron is "relativistic" whereas the proton is "nonrelativistic."

**14-40.** Consider an inertial frame $K$ that contains a number of particles with masses $m_\alpha$, ordinary momentum components $p_{\alpha,j}$, and total energies $E_\alpha$. The center-of-mass system of such a group of particles is defined to be that system in which the net ordinary momentum is zero. Show that the velocity components of the center-of-mass system with respect to $K$ are given by

$$\frac{v_j}{c} = \frac{\sum_\alpha p_{\alpha,j}c}{\sum_\alpha E_\alpha}$$

**14-41.** Show that the relativistic expression for the kinetic energy of a particle scattered through an angle $\psi$ by a target particle of equal mass is

$$\frac{T_1}{T_0} = \frac{2\cos^2\psi}{(\gamma_1 + 1) - (\gamma_1 - 1)\cos^2\psi}$$

The expression evidently reduces to Equation 9.89a in the nonrelativistic limit $\gamma_1 \to 1$. Sketch $T_1(\psi)$ for neutron-proton scattering for incident neutron energies of 100 MeV, 1 GeV, and 10 GeV.

**14-42.** The energy of a light quantum (or photon) is expressed by $E = h\nu$, where $h$ is Planck's constant and $\nu$ is the frequency of the photon. The momentum of the photon is $h\nu/c$. Show that, if the photon scatters from a free electron (of mass $m_e$), the scattered photon has an energy

$$E' = E\left[1 + \frac{E}{m_e c^2}(1 - \cos\theta)\right]^{-1}$$

where $\theta$ is the angle through which the photon scatters. Show also that the electron acquires a kinetic energy

$$T = \frac{E^2}{m_e c^2}\left[\frac{1 - \cos\theta}{1 + \frac{E}{m_e c^2}(1 - \cos\theta)}\right]$$

"*Better is the end of a thing than the beginning thereof.*"—Ecclesiastes

# APPENDIX A

# *Taylor's Theorem*

A theorem of considerable importance in mathematical physics is **Taylor's theorem,*** which relates to the expansion of an arbitrary function in a power series. In many instances, it is necessary to use this theorem to simplify a problem to a tractable form.

Consider a function $f(x)$ with continuous derivatives of all orders within a certain interval of the independent variable $x$. If this interval includes $x_0 \leq x \leq x_0 + h$, we may write

$$I \equiv \int_{x_0}^{x_0+h} f'(x)\,dx = f(x_0 + h) - f(x_0) \tag{A.1}$$

where $f'(x)$ is the derivative of $f(x)$ with respect to $x$. If we make the change of variable

$$x = x_0 + h - t \tag{A.2}$$

we have

$$I = \int_0^h f'(x_0 + h - t)\,dt \tag{A.3}$$

Integrating by parts

$$\begin{aligned} I &= tf'(x_0 + h - t)\Big|_0^h + \int_0^h tf''(x_0 + h - t)\,dt \\ &= hf'(x_0) + \int_0^h tf''(x_0 + h - t)\,dt \end{aligned} \tag{A.4}$$

*First published in 1715 by the English mathematician Brook Taylor (1685–1731).

Integrating the second term by parts, we find

$$I = hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \int_0^h \frac{t^2}{2!}f'''(x_0 + h - t)\,dt \tag{A.5}$$

Continuing this process, we generate an infinite series for $I$. From the definition of $I$, we then have

$$\boxed{f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots} \tag{A.6}$$

This is the Taylor series expansion* of the function $f(x_0 + h)$. A more common form of the series results if we set $x_0 = 0$ and $h = x$ [i.e., the function $f(x)$ is expanded about the origin]:

$$\boxed{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots} \tag{A.7}$$

where

$$f^{(n)}(0) \equiv \left.\frac{d^n}{dx^n}f(x)\right|_{x=0} \tag{A.8}$$

Equation A.7 is usually called the **Maclaurin's series**† for the function $f(x)$.

The series expansions given in Equations A.6 and A.7 possess two important properties. Under very general conditions, they may be differentiated or integrated term by term, and the resulting series converge to the derivative or integral of the original function.

### EXAMPLE A.1

Find the Taylor series expansion of $e^x$.

***Solution.*** Because the derivative of $\exp(x)$ of any order is just $\exp(x)$, the exponential series is

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots} \tag{A.9}$$

This result is of considerable importance and will be used often.

*The *remainder* term of a series that is terminated after a finite number of terms is discussed, for example, by Kaplan (Ka84).

†Discovered by James Stirling in 1717 and published by Colin Maclaurin in 1742.

### EXAMPLE A.4

Taylor's series can be used to restructure a function as well as to approximate it. For some applications, such a restructuring may be more useful to work with. We may, for example, want to expand the polynomial $f(x) = 4 + 6x + 3x^2 + 2x^3 + x^4$ about $x = 2$ rather than $x = 0$.

***Solution.*** First, we compute the various derivatives and evaluate them at $x = 2$:

$$
\begin{aligned}
f(2) &= 60 \\
f'(2) &= (6 + 6x + 6x^2 + 4x^3)\big|_{x=2} = 74 \\
f''(2) &= (6 + 12x + 12x^2)\big|_{x=2} = 78 \\
f'''(2) &= (12 + 24x)\big|_{x=2} = 60 \\
f^{\text{iv}}(2) &= 24 \\
f^{\text{v}}(2) &= 0
\end{aligned}
$$

Using Equation A.6 with $h = (x - 2)$

$$f(x) = 60 + 74(x-2) + 39(x-2)^2 + 10(x-2)^3 + (x-2)^4 \tag{A.15}$$

### EXAMPLE A.5

There are a great many important integrals arising in physics that cannot be integrated in closed form, that is, in terms of elementary functions (polynomials, exponentials, logarithms, trigonometric functions, and their inverses). Integrals with integrands

$$e^{-x^2}, \quad \frac{e^{-x}}{x}, \quad x \tan x, \quad \sin x^2, \quad 1/\ln x, \quad (\sin x)/x, \quad \text{or} \quad 1/\sqrt{1 - x^3}$$

are a few such examples. Nevertheless, the values of the integrals or good approximations of their values are needed. A Taylor series expansion of all or part of the integrand followed by a term-by-term integration of the resulting series produces an answer as precise as is wished. As an example, solve the following integral:

$$\int_1^x \frac{e^t}{t}\,dt \tag{A.16}$$

***Solution.*** Using Equation A.9,

$$\int_1^x \frac{e^t}{t}\,dt = \int_1^x \frac{\left(1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \cdots\right)dt}{t} \tag{A.17}$$

**EXAMPLE A.2**

Find the Taylor series expansion of $\sin x$.

***Solution.*** To expand $f(x) = \sin x$, we need

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0 \\ f'(x) &= \cos x, & f'(0) &= 1 \\ f''(x) &= -\sin x, & f''(0) &= 0 \\ f'''(x) &= -\cos x, & f'''(0) &= -1 \end{aligned}$$

Therefore,

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots} \tag{A.10}$$

Similarly,

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots} \tag{A.11}$$

**EXAMPLE A.3**

Use the Taylor series expansion of $(1 + t)^{-1}$ to integrate

$$\int_0^x \frac{dt}{1 + t}$$

***Solution.*** A series expansion can often be profitably used in the evaluation of a definite integral. (This is particularly true for those cases in which the indefinite integral cannot be found in closed form.)

$$\int_0^x \frac{dt}{1 + t} = \int_0^x (1 - t^2 + t^3 - \cdots)\, dt, \quad |t| < 1$$

Integrating term by term, we find

$$\int_0^x \frac{dt}{1 + t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{A.12}$$

Because

$$\frac{d}{dx} \ln(1 + x) = \frac{1}{1 + x} \tag{A.13}$$

We also have the result

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{A.14}$$

$$= \int_1^x \frac{dt}{t} + \int_1^x dt + \int_1^x \frac{t}{2!}dt + \int_1^x \frac{t^2}{3!}dt + \cdots$$

$$= \ln x - (x-1) + \frac{1}{4}(x^2-1) + \frac{1}{18}(x^3-1) + \cdots \qquad \textbf{(A.18)}$$

## PROBLEMS

**A-1.** Show by division and by direct expansion in a Taylor series that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

For what range of $x$ is the series valid?

**A-2.** Expand $\cos x$ about the point $x = \pi/4$.

**A-3.** Use a series expansion to show that

$$\int_0^1 \frac{e^x - e^{-x}}{x}\,dx = 2.1145\ldots$$

**A-4.** Use a Taylor series to expand $\sin^{-1} x$. Verify the result by expanding the integral in the relation

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

**A-5.** Evaluate to three decimal places:

$$\int_0^1 \exp(-x^2/2)\,dx$$

Compare the result with that determined from tables of the probability integral.

**A-6.** Show that if $f(x) = (1+x)^n$ (with $|x| < 1$) is expanded in a Taylor series, the result is the same as a binomial expansion.

# APPENDIX B

# Elliptic Integrals

There is a large and important class of integrals called **elliptic integrals** that cannot be evaluated in closed form in terms of elementary functions. Elliptic integrals occur in many physical situations; for example, see the exact solution to the plane pendulum in Section 4.4. Any integral of the form

$$\int (a \sin\theta + b\cos\theta + c)^{\pm 1/2}\, d\theta, \quad \text{or} \quad \int R\left(x, \sqrt{y}\right) dx \tag{B.1}$$

where $R$ is a rational function, $y = ax^4 + bx^3 + cx^2 + dx + e$, with distinct linear factors and $a$, $b$, $c$, $d$, and $e$ constants (with not both $a$, $b$ zero) is an elliptic integral. It is customary, however, to transform all elliptic integrals into one or more of three standard forms. These standard forms have been much studied and tabulated. Several handbooks are available with tables of values for them*

## B.1 Elliptic Integrals of the First Kind

$$F(k,\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \quad k^2 < 1 \tag{B.2a}$$

or if $z = \sin\theta$

$$\overline{F}(k, x) = \int_0^{x} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}, \quad k^2 < 1 \tag{B.2b}$$

*One of the best of these is Abramowitz and Stegun (Ab65). See also extensive numerical tables in Adams and Hippisley (Ad22) and short tables in Dwight (Dw61).

## B.2 Elliptic Integrals of the Second Kind

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta, \quad k^2 < 1 \tag{B.3a}$$

or if $z = \sin \theta$

$$\overline{E}(k, x) = \int_0^{x} \sqrt{\frac{1 - k^2 z^2}{1 - z^2}}\, dz, \quad k^2 < 1 \tag{B.3b}$$

## B.3 Elliptic Integrals of the Third Kind

$$\Pi(n, k, \phi) = \int_0^{\phi} \frac{d\theta}{(1 + n \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} \tag{B.4a}$$

or if $z = \sin \theta$

$$\overline{\Pi}(n, k, x) = \int_0^{x} \frac{dz}{(1 + nz^2)\sqrt{(1 - z^2)(1 - k^2 z^2)}} \tag{B.4b}$$

These standard forms obey the following identities, which are often helpful:

$$\left.\begin{aligned} F(k, \phi) &= F(k, \pi) - F(k, \pi - \phi) \\ E(k, \phi) &= E(k, \pi) - E(k, \pi - \phi) \end{aligned}\right\} \tag{B.5}$$

and

$$\left.\begin{aligned} F(k, m\pi + \phi) &= mF(k, \pi) + F(k, \phi) \\ E(k, m\pi + \phi) &= mE(k, \pi) + E(k, \phi) \end{aligned}\right\} \tag{B.6}$$

where $m$ is an integer.

If tables are not handy or if $\phi$ or $x$ is needed as a variable, the standard integrals may be approximated by expanding the integrand in an infinite series and integrating term by term. For example, consider

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta$$

Using the binomial theorem on the integrand

$$(1 - k^2 \sin^2 \theta)^{1/2} = 1 - \frac{1}{2} k^2 \sin^2 \theta - \frac{1}{8} k^4 \sin^4 \theta - \cdots$$

so

$$E(k, \phi) = \int_0^\phi \left[ 1 - \frac{1}{2} k^2 \sin^2 \theta - \frac{1}{8} k^4 \sin^4 \theta - \cdots \right.$$
$$\left. - \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n)} k^{2n} \sin^{2n} \theta - \cdots \right] d\theta$$
$$= \phi - \frac{k^2}{2} \int_0^\phi \sin^2 \theta d\theta - \cdots - \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n)} k^{2n}$$
$$\times \int_0^\phi \sin^{2n} \theta \, d\theta - \cdots \qquad \textbf{(B.7)}$$

Similarly, the binomial theorem can be used to expand $(1 - k^2 \sin^2 \theta)^{-1/2}$ to yield

$$F(k, \phi) = \phi + \frac{1}{2} k^2 \int_0^\phi \sin^2 \theta d\theta + \frac{3}{8} k^4 \int_0^\phi \sin^4 \theta d\theta + \cdots$$
$$+ \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} k^{2n} \int_0^\phi \sin^{2n} \theta d\theta + \cdots \qquad \textbf{(B.8)}$$

**EXAMPLE B.1**

**Put the integral $\int_{\phi_1}^{\phi_2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$ into standard form.**

***Solution.*** Recall from calculus that for any integral $\int_a^b f(x)\, dx$ it is possible to write

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

so

$$\int_{\phi_1}^{\phi_2} \sqrt{1 - k \sin^2 \theta} \, d\theta = \int_{\phi_1}^{0} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta + \int_0^{\phi_2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$$

But there is another property of integrals:

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

so

$$\int_{\phi_1}^{\phi_2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta = \int_0^{\phi_2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta - \int_0^{\phi_1} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$$

or

$$\int_{\phi_1}^{\phi_2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta = E(k, \phi_2) - E(k, \phi_1) \tag{B.9}$$

The terms on the right can be looked up in a handbook.

---

**EXAMPLE B.2**

Transform the elliptic integral

$$\int_0^{\phi} \frac{d\theta}{\sqrt{1 - n^2 \sin^2 \theta}}, \quad \text{where } n^2 > 1$$

into a standard form.

***Solution.*** To reduce this integral to standard form, the radical must be transformed to $\sqrt{1 - k^2 \sin^2 \theta}$, with $k^2 < 1$. To do this, consider the transformation $n \sin \theta = \sin \beta$. Differentiating, we have

$$n \cos \theta\, d\theta = \cos \beta\, d\beta$$

so

$$d\theta = \frac{\cos \beta\, d\beta}{n \cos \theta}$$

Using the identity $\sin^2 \theta + \cos^2 \theta = 1$ leads to

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left(\frac{\sin \beta}{n}\right)^2}$$

Also, $\cos \beta = \sqrt{1 - \sin^2 \beta}$, and $\sqrt{1 - n^2 \sin^2 \theta} = \sqrt{1 - \sin^2 \beta}$. Hence the integral becomes

$$\int_0^{\phi} \frac{d\theta}{\sqrt{1 - n^2 \sin^2 \theta}} = \int_0^{\sin^{-1}(n \sin \phi)} \frac{\sqrt{1 - \sin^2 \beta}}{n\sqrt{1 - \left(\dfrac{\sin \beta}{n}\right)^2}\sqrt{1 - \sin^2 \beta}}\, d\beta$$

$$= \frac{1}{n} \int_0^{\sin^{-1}(n \sin \phi)} \frac{d\beta}{\sqrt{1 - \left(\dfrac{1}{n^2}\right)\sin^2 \beta}}$$

so

$$\int_0^{\phi} \frac{d\theta}{\sqrt{1 - n^2 \sin^2 \theta}} = \frac{1}{n} \int_0^{\sin^{-1}(n \sin \phi)} \frac{d\beta}{\sqrt{1 - \left(\frac{1}{n^2}\right) \sin^2 \beta}} \tag{B.10}$$

where $1/n^2 < 1$. The integral on the right is now in standard form.

### EXAMPLE B.3

Transform the elliptic integral

$$\int_0^{\phi} \frac{d\theta}{\sqrt{\cos 2\theta}}$$

into a standard form.

***Solution.*** Let $\mu = \sin \theta$; then $d\mu = \cos \theta \, d\theta$. Because $\cos^2 \theta + \sin^2 \theta = 1$, $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \mu^2}$, so $d\theta = d\mu/\sqrt{1 - \mu^2}$. By another trigonometric identity, $\cos 2\theta = 1 - 2 \sin^2 \theta = 1 - 2\mu^2$. Thus $\sqrt{\cos 2\theta} = \sqrt{1 - 2\mu^2}$, and

$$\int_0^{\phi} \frac{d\theta}{\sqrt{\cos 2\theta}} = \int_0^{\sin \phi} \frac{d\mu}{\sqrt{1 - \mu^2}\sqrt{1 - 2\mu^2}}$$

Let $z = \sqrt{2}\,\mu$; then $dz = \sqrt{2}\, d\mu$, so

$$\int_0^{\phi} \frac{d\theta}{\sqrt{\cos 2\theta}} = \frac{1}{\sqrt{2}} \int_0^{\sqrt{2}\sin \phi} \frac{dz}{\sqrt{(1 - z^2)\left(1 - \frac{1}{2}z^2\right)}} \tag{B.11}$$

The integral on the right is in standard form.

## PROBLEMS

**B-1.** Evaluate the following integrals using a set of tables.

**(a)** $F(0.27, \pi/3)$ **(b)** $E(0.27, \pi/3)$

**(c)** $F(0.27, 7\pi/4)$ **(d)** $E(0.27, 7\pi/4)$

**B-2.** Reduce to standard form:

**(a)** $\displaystyle\int_0^{\pi/6} \frac{d\theta}{\sqrt{1 - 4\sin^2\theta}}$ **(b)** $\displaystyle\int_{-1/4}^{3/4} \sqrt{\frac{25 - 4z^2}{1 - z^2}}\, dz$

**B-3.** Find the binomial expansion of $(1 - k^2 \sin^2 \theta)^{-1/2}$ and then derive Equation B.8.

# APPENDIX C

# *Ordinary Differential Equations of Second Order**

## C.1 Linear Homogeneous Equations

By far, the most important type of ordinary differential equation encountered in problems in mathematical physics is the second-order linear equation with constant coefficients. Equations of this type have the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \tag{C.1a}$$

or, denoting derivatives by primes,

$$y'' + ay' + by = f(x) \tag{C.1b}$$

A particularly important class of such equations are those for which $f(x) = 0$. These equations (called **homogeneous equations**) are important not only in themselves but also as *reduced* equations in the solution of the more general type of equation (Equation C.1).

We consider the linear homogeneous second-order equation with constant coefficients first.†

$$y'' + ay' + by = 0 \tag{C.2}$$

*A standard treatise on differential equations is that of Ince (In27). A listing of many types of equations and their solutions is given by Murphy (Mu60). A modern viewpoint is contained in the book by Hochstadt (Ho64).

†The first published solution of an equation of this type was by Euler in 1743, but the solution appears to have been known to Daniel and Johann Bernoulli in 1739.

These equations have the following important properties:

a. If $y_1(x)$ is a solution of Equation C.2, then $c_1y_1(x)$ is also a solution.
b. If $y_1(x)$ and $y_2(x)$ are solutions, then $y_1(x) + y_2(x)$ is also a solution (principle of *superposition*).
c. If $y_1(x)$ and $y_2(x)$ are *linearly* independent solutions, then the *general* solution to the equation is given by $c_1y_1(x) + c_2y_2(x)$. (The general solution always contains two arbitrary constants.)

The functions $y_1(x)$ and $y_2(x)$ are **linearly independent** if and only if the equation

$$\lambda y_1(x) + \mu y_2(x) \equiv 0 \tag{C.3}$$

is satisfied only by $\lambda = \mu = 0$. If Equation C.3 can be satisfied with $\lambda$ and $\mu$ different from zero, then $y_1(x)$ and $y_2(x)$ are said to be **linearly dependent.**

The general condition (i.e., the necessary and sufficient condition) that a set of functions $y_1, y_2, y_3, \ldots$ be linearly dependent is that the **Wronskian determinant** of these functions vanish identically:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = 0 \tag{C.4}$$

where $y^{(n)}$ is the $n$th derivative of $y$ with respect to $x$.

The properties (a) and (b) above can be verified by direction substitution, but (c) is only asserted here to yield the general solution. These properties apply *only* to the homogeneous equation (Equation C.2) and *not* to the general equation (Equation C.1).

Equations of the type C.2 are reducible through the substitution

$$y = e^{rx} \tag{C.5}$$

Now

$$y' = re^{rx}, \quad y^n = r^2e^{rx} \tag{C.6}$$

Using these expressions for $y'$ and $y''$ in Equation C.2, we find an algebraic equation called the **auxiliary equation:**

$$r^2 + ar + b = 0 \tag{C.7}$$

The solution of this quadratic in $r$ is

$$r = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b} \tag{C.8}$$

We first assume that the two roots, denoted by $r_1$ and $r_2$, are not identical and write the solution as

$$y = e^{r_1x} + e^{r_2x} \tag{C.9}$$

Because the Wronskian determinant of $\exp(r_1 x)$ and $\exp(r_2 x)$ does not vanish, these functions are linearly independent. Thus, the general solution is

$$\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad r_1 \neq r_2} \tag{C.10}$$

If it happens that $r_1 = r_2 = r$, then it can be verified by direct substitution that $x \exp(rx)$ is also a solution, and because $\exp(rx)$ and $x \exp(rx)$ are linearly independent, the general solution for identical roots is given by

$$\boxed{y = c_1 e^{rx} + c_2 x e^{rx}, \quad r_1 = r_2 \equiv r} \tag{C.11}$$

**EXAMPLE C.1**

Solve the equation

$$y'' - 2y' - 3y = 0 \tag{C.12}$$

***Solution.*** The auxiliary equation is

$$r^2 - 2r - 3 = (r - 3)(r + 1) = 0 \tag{C.13}$$

The roots are

$$r_1 = 3, \quad r_2 = -1 \tag{C.14}$$

The general solution is therefore

$$y = c_1 e^{3x} + c_2 e^{-x} \tag{C.15}$$

**EXAMPLE C.2**

Solve the equation

$$y'' + 4y' + 4y = 0 \tag{C.16}$$

***Solution.*** The auxiliary equation is

$$r^2 + 4r + 4 = (r + 2)^2 = 0 \tag{C.17}$$

The roots are equal, are $r = -2$. The general solution is therefore

$$y = c_1 e^{-2x} + c_2 x e^{-2x} \tag{C.18}$$

If the roots $r_1$ and $r_2$ of the auxiliary equation are imaginary, the solutions given by $c_1 \exp(r_1 x)$ and $c_2 \exp(r_2 x)$ are still correct.

To give the solutions entirely in terms of real quantities, we use the Euler relations to express the exponentials. Then,

$$\left.\begin{aligned} e^{r_1 x} &= e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x) \\ e^{r_2 x} &= e^{\alpha x} e^{-i\beta x} = e^{\alpha x}(\cos\beta x - i\sin\beta x) \end{aligned}\right\} \tag{C.19}$$

and the general solution is

$$\begin{aligned} y &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ &= e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x] \end{aligned} \tag{C.20}$$

Now $c_1$ and $c_2$ are arbitrary, but these constants may be complex. However, not all four elements can be independent (because there would be *four* arbitrary constants rather than *two*). The number of independent elements can be reduced to the required *two* by making $c_1$ and $c_2$ complex conjugates. Then the combinations $A \equiv c_1 + c_2$ and $B \equiv i(c_1 - c_2)$ become a pair of arbitrary, real constants. Using these quantities in the solution, we have

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x) \tag{C.21}$$

Equation C.21 may be put into a form that is sometimes more convenient by multiplying and dividing by $\mu = \sqrt{A^2 + B^2}$:

$$y = \mu e^{\alpha x}[(A/\mu)\cos\beta x + (B/\mu)\sin\beta x] \tag{C.22}$$

Next, we define an angle $\delta$ (see Figure C-1) such that

$$\sin\delta = A/\mu, \quad \cos\delta = B/\mu, \quad \tan\delta = A/B \tag{C.23}$$

Then, the solution becomes

$$\begin{aligned} y &= \mu e^{\alpha x}(\sin\delta\cos\beta x + \cos\delta\sin\beta x) \\ &= \mu e^{\alpha x}\sin(\beta x + \delta) \end{aligned}$$

Depending on the exact definition of the phase $\delta$, we may write the solution alternatively as

$$\boxed{\begin{aligned} y &= \mu e^{\alpha x}\sin(\beta x + \delta) && \text{(C.24a)} \\ y &= \mu e^{\alpha x}\cos(\beta x + \delta) && \text{(C.24b)} \end{aligned}}$$

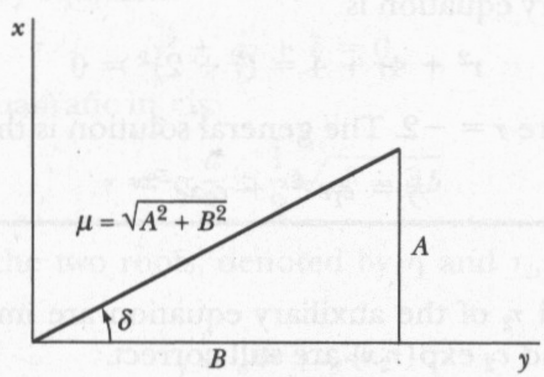



**FIGURE C-1**

**EXAMPLE C.3**

Solve the equation

$$y'' + 2y' + 4y = 0 \tag{C.25}$$

***Solution.*** The auxiliary equation is

$$r^2 + 2r + 4 = 0 \tag{C.26}$$

with

$$r = \frac{-2 \pm \sqrt{4 - 16}}{2} = -1 \pm i\sqrt{3} \tag{C.27}$$

Hence,

$$\alpha = -1, \quad \beta = \sqrt{3} \tag{C.28}$$

and the general solution is

$$y = e^{-x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) \tag{C.29}$$

or

$$y = \mu e^{-x} \sin[(\sqrt{3}x + \delta)] \tag{C.30}$$

Summarizing, then, there are three possible types of general solutions to homogeneous second-order linear differential equations, as indicated in Table C-1.

**TABLE C-1**

| Roots of the auxiliary equations | General solution |
|---|---|
| Real, unequal ($r_1 \neq r_2$) | $c_1 e^{r_1 x} + c_2 e^{r_2 x}$ |
| Real, equal ($r_1 = r_2 \equiv r$) | $c_1 e^{rx} + c_2 x e^{rx}$ |
| Imaginary ($\alpha \pm i\beta$) | $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ |
| | or |
| | $\mu e^{\alpha x} \sin(\beta x + \delta)$ |

## C.2 Linear Inhomogeneous Equations

To solve the general (i.e., inhomogeneous) second-order linear differential equation, consider the following. Let $y = u$ be the *general solution* of

$$y'' + ay' + by = 0 \tag{C.31}$$

and let $y = v$ be *any* solution of

$$y'' + ay' + by = f(x) \tag{C.32}$$

Then, $y = u + v$ is a solution of Equation C.32, because

$$y'' + ay' + by = (u'' + au' + bu) + (v'' + av' + bv)$$
$$= 0 + f(x)$$

Because $u$ contains the two arbitrary constants $c_1$ and $c_2$, the combinations $u + v$ satisfies all the requirements of the general solution to Equation C.32. The function $u$ is the **complementary function** and $v$ is the **particular integral** of the equation. Because a general method of finding $u$ has been given above, it only remains to find, by inspection or by trial, some function $v$ that satisfies

$$v'' + av' + bv = f(x) \tag{C.33}$$

## EXAMPLE C.4

Solve the equation

$$y'' + 5y' + 6y = x^2 + 2x \tag{C.34}$$

***Solution.*** The auxiliary equation is

$$r^2 + 5r + 6 = (r + 3)(r + 2) = 0 \tag{C.35}$$

$$r_1 = -3, \quad r_2 = -2 \tag{C.36}$$

so the complementary function is

$$u = c_1e^{-3x} + c_2e^{-2x} \tag{C.37}$$

Because the right-hand side of the original equation is a second-degree polynomial, we guess a particular integral of the form

$$v = Ax^2 + Bx + C \tag{C.38}$$

Then,

$$v' = 2Ax + B \tag{C.39}$$

$$v'' = 2A \tag{C.40}$$

Substituting into the differential equation, we have

$$2A + 5(2Ax + B) + 6(Ax^2 + Bx + C) = x^2 + 2x \tag{C.41}$$

or

$$(6A)x^2 + (10A + 6B)x + (2A + 5B + 6C) = x^2 + 2x \tag{C.42}$$

Equation coefficients of like powers of $x$:

$$\left.\begin{aligned} 6A &= 1 \\ 10A + 6B &= 2 \\ 2A + 5B + 6C &= 0 \end{aligned}\right\} \tag{C.43}$$

Solving,

$$A = \frac{1}{6}, \quad B = \frac{1}{18}, \quad C = -\frac{11}{108} \tag{C.44}$$

Hence,

$$\begin{aligned} v &= \frac{1}{6}x^2 + \frac{1}{18}x - \frac{11}{108} \\ &= \frac{18x^2 + 6x - 11}{108} \end{aligned} \tag{C.45}$$

The general solution is therefore

$$y = u + v = c_1 e^{-3x} + c_2 e^{-2x} + \frac{18x^2 + 6x - 11}{108} \tag{C.46}$$

The type of solution illustrated in this example is called the **method of undetermined coefficients.**

---

## EXAMPLE C.5

Solve the equation

$$y'' + 4y = 3x \cos x \tag{C.47}$$

***Solution.*** The auxiliary equation is

$$r^2 + 4 = (r + 2i)(r - 2i) = 0 \tag{C.48}$$

with roots

$$\left.\begin{aligned} r_1 &= \alpha + i\beta = 0 + 2i \\ r_2 &= \alpha - i\beta = 0 - 2i \end{aligned}\right\} \tag{C.49}$$

so

$$\alpha = 0, \quad \beta = 2 \tag{C.50}$$

and the complementary function is

$$\begin{aligned} u &= e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \\ &= c_1 \cos 2x + c_2 \sin 2x \end{aligned} \tag{C.51}$$

To find a particular integral, we note that from $x \cos x$ and its derivatives it is possible to generate only terms involving the following functions:

$$x \cos x, \quad x \sin x, \quad \cos x, \quad \sin x$$

Therefore, because these functions are linearly independent, the trial particular integral is

$$v = Ax \cos x + Bx \sin x + C \cos x + D \sin x \tag{C.52}$$

$$\begin{aligned} v' &= A(\cos x - x \sin x) + B(\sin x + x \cos x) \\ &\quad - C \sin x + D \cos x \end{aligned} \tag{C.53}$$

$$\begin{aligned} v'' &= -A(2 \sin x + x \cos x) + B(2 \cos x - x \sin x) \\ &\quad - C \cos x - D \sin x \end{aligned} \tag{C.54}$$

Substituting into the original differential equation,

$$(3D - 2A)\sin x + (2B + 3C)\cos x + 3(A - 1)x \cos x + (3B)x \sin x = 0 \tag{C.55}$$

The coefficient of each term must vanish (because of the linear independence of the terms):

$$3D = 2A, \quad 2B = -3C, \quad A = 1, \quad 3B = 0 \tag{C.56}$$

from which

$$A = 1, \quad B = 0, \quad C = 0, \quad D = \frac{2}{3} \tag{C.57}$$

The general solution is therefore

$$y = c_1 \sin 2x + c_2 \cos 2x + x \cos x + \frac{2}{3} \sin x \tag{C.58}$$

---

If the right-hand side, $f(x)$, of the general equation (Equation C.1 or C.32) is such that $f(x)$ and its first two derivatives (only second-order equations are being considered) contain only linearly independent functions, then a linear combination of these functions constitutes the trial particular integral. In the event that the trial function contains a term that already appears in the complementary function, use the term multiplied by $x$; if this combination also appears in the complementary function, use the term multiplied by $x^2$. No higher powers are needed because only second-order equations are being considered and only $\exp(rx)$ or $x \exp(rx)$ occur as solutions to the reduced equation; $(x^2) \exp(rx)$ never occurs.

## PROBLEMS

**C-1.** Solve the following homogeneous second-order equations:

**(a)** $y'' + 2y' - 3y = 0$ **(b)** $y'' + y = 0$

**(c)** $y'' - 2y' + 2y = 0$ **(d)** $y'' - 2y' + 5y = 0$

**C-2.** Solve the following inhomogeneous equations by the method of undetermined coefficients:

**(a)** $y'' + 2y' - 8y = 16x$ **(b)** $y'' - 2y' + y = 2e^{2x}$

**(c)** $y'' + y = \sin x$ **(d)** $y'' - 2y' + y = 3xe^{x}$

**(e)** $y'' - 4y' + 5y = e^{2x} + 4\sin x$

**C-3.** Use a Taylor series expansion to obtain the solution of

$$y'' + y^2 = x^2$$

that obeys the conditions $y(0) = 1$ and $y'(0) = 0$. (Differentiate the equation successively to obtain the derivatives that occur in the Taylor series.)

# APPENDIX D

# *Useful Formulas**

## D.1 Binomial Expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

$$+ \cdots + \binom{n}{r} x^r + \cdots, \qquad |x| < 1 \tag{D.1}$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3$$

$$+ \cdots + (-1)^r \binom{n}{r} x^r + \cdots, \qquad |x| < 1 \tag{D.2}$$

where the **binomial coefficient** is

$$\binom{n}{r} \equiv \frac{n!}{(n-r)!\,r!} \tag{D.3}$$

Some particularly useful cases of the above are

$$(1 \pm x)^{1/2} = 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{1}{16}x^3 - \cdots \tag{D.4}$$

$$(1 \pm x)^{1/3} = 1 \pm \frac{1}{3}x - \frac{1}{9}x^2 \pm \frac{5}{81}x^3 - \cdots \tag{D.5}$$

*An extensive list may be found, for example, in Dwight (Dw61).

$$(1 \pm x)^{-1/2} = 1 \mp \frac{1}{2}x + \frac{3}{8}x^2 \mp \frac{5}{16}x^3 + \cdots \tag{D.6}$$

$$(1 \pm x)^{-1/3} = 1 \mp \frac{1}{3}x + \frac{2}{9}x^2 \mp \frac{14}{81}x^3 + \cdots \tag{D.7}$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + \cdots \tag{D.8}$$

$$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + \cdots \tag{D.9}$$

$$(1 \pm x)^{-3} = 1 \mp 3x + 6x^2 \mp 10x^3 + \cdots \tag{D.10}$$

For convergence of *all* the above series, we must have $|x| < 1$.

## D.2 Trigonometric Relations

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{D.11}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{D.12}$$

$$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A} \tag{D.13}$$

$$\cos 2A = 2 \cos^2 A - 1 \tag{D.14}$$

$$\sin^2 \frac{A}{2} = \frac{1}{2}(1 - \cos A) \tag{D.15}$$

$$\cos^2 \frac{A}{2} = \frac{1}{2}(1 + \cos A) \tag{D.16}$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A) \tag{D.17}$$

$$\sin^3 A = \frac{1}{4}(3 \sin A - \sin 3A) \tag{D.18}$$

$$\sin^4 A = \frac{1}{8}(3 - 4 \cos 2A + \cos 4A) \tag{D.19}$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A) \tag{D.20}$$

$$\cos^3 A = \frac{1}{4}(3 \cos A + \cos 3A) \tag{D.21}$$

$$\cos^4 A = \frac{1}{8}(3 + 4 \cos 2A + \cos 4A) \tag{D.22}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{D.23}$$

$$\tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A} \tag{D.24}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{D.25}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{D.26}$$

$$e^{ix} = \cos x + i \sin x \tag{D.27}$$

## D.3 Trigonometric Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{D.28}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{D.29}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots, \qquad |x| < \pi/2 \tag{D.30}$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots, \qquad \begin{cases} |x| < 1 \\ |\sin^{-1} x| < \pi/2 \end{cases} \tag{D.31}$$

$$\cos^{-1} x = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3}{40}x^5 - \cdots, \qquad \begin{cases} |x| < 1 \\ 0 < \cos^{-1} x < \pi \end{cases} \tag{D.32}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad |x| < 1 \tag{D.33}$$

## D.4 Exponential and Logarithmic Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{D.34}$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad |x| < 1, \quad x = 1 \tag{D.35}$$

$$\ln\left[\sqrt{(x^2/a^2) + 1} + (x/a)\right] = \sinh^{-1} x/a \tag{D.36}$$

$$= -\ln\left[\sqrt{(x^2/a^2) + 1} - (x/a)\right] \tag{D.37}$$

## D.5 Complex Quantities

Cartesian form: $z = x + iy$, complex conjugate $z^* = x - iy$, $i = \sqrt{-1}$ (D.38)

Polar form:
$$z = |z|\, e^{i\theta} \tag{D.39}$$

$$z^* = |z|\, e^{-i\theta} \tag{D.40}$$

$$zz^* = |z|^2 = x^2 + y^2 \tag{D.41}$$

Real part of $z$:
$$\mathrm{Re}\, z = \frac{1}{2}(z + z^*) = x \tag{D.42}$$

Imaginary part of $z$:
$$\mathrm{Im}\, z = -\frac{1}{2}(z - z^*) = y \tag{D.43}$$

Euler's formula:
$$e^{i\theta} = \cos\theta + i\sin\theta \tag{D.44}$$

## D.6 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{D.45}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{D.46}$$

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \tag{D.47}$$

$$\sin ix = i \sinh x \tag{D.48}$$

$$\cos ix = \cosh x \tag{D.49}$$

$$\sinh ix = i \sin x \tag{D.50}$$

$$\cosh ix = \cos x \tag{D.51}$$

$$\sinh^{-1} x = \tanh^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \tag{D.52}$$

$$= \ln(x + \sqrt{x^2+1}) \tag{D.53}$$

$$= \cosh^{-1}(\sqrt{x^2+1}), \quad \begin{cases} > 0, & x > 0 \\ < 0, & x < 0 \end{cases} \tag{D.54}$$

$$\cosh^{-1} x = \pm \tanh^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right), \quad x > 1 \tag{D.55}$$

$$= \pm \ln(x + \sqrt{x^2-1}), \quad x > 1 \tag{D.56}$$

$$\cosh^{-1} x = \pm \sinh^{-1}(\sqrt{x^2 - 1}), \qquad x > 1 \tag{D.57}$$

$$\frac{d}{dy} \sinh y = \cosh y \tag{D.58}$$

$$\frac{d}{dy} \cosh y = \sinh y \tag{D.59}$$

$$\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2 \tag{D.60}$$

$$\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2 \tag{D.61}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{D.62}$$

## PROBLEMS

**D-1.** Is it possible to ascribe a meaning to the inequality $z_1 < z_2$? Explain. Does the inequality $|z_1| < |z_2|$ have a different meaning?

**D-2.** Solve the following equations:
**(a)** $z^2 + 2z + 2 = 0$ **(b)** $2z^2 + z + 2 = 0$

**D-3.** Express the following in polar form:
**(a)** $z_1 = i$ **(b)** $z_2 = -1$
**(c)** $z_3 = 1 + i\sqrt{3}$ **(d)** $z_4 = 1 + 2i$
**(e)** Find the product $z_1 z_2$ **(f)** Find the product $z_1 z_3$
**(g)** Find the product $z_3 z_4$

**D-4.** Express $(z^2 - 1)^{-1/2}$ in polar form.

**D-5.** If the function $w = \sin^{-1} z$ is defined as the inverse of $z = \sin w$, then use the Euler relation for $\sin w$ to find an equation for $\exp(iw)$. Solve this equation and obtain the result

$$w = \sin^{-1} z = -i \ln\left(iz + \sqrt{1 - z^2}\right)$$

**D-6.** Show that

$$y = Ae^{ix} + Be^{-ix}$$

can be written as

$$y = C \cos(x - \delta)$$

where $A$ and $B$ are *complex* but where $C$ and $\delta$ are *real*.

**D-7.** Show that
**(a)** $\sinh(x_1 + x_2) = \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2$
**(b)** $\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2$

# APPENDIX E

# *Useful Integrals**

## E.1 Algebraic Functions

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \left|\tan^{-1}\left(\frac{x}{a}\right)\right| < \frac{\pi}{2} \tag{E.1}$$

$$\int \frac{x\,dx}{a^2 + x^2} = \frac{1}{2}\ln(a^2 + x^2) \tag{E.2}$$

$$\int \frac{dx}{x(a^2 + x^2)} = \frac{1}{2a^2}\ln\left(\frac{x^2}{a^2 + x^2}\right) \tag{E.3}$$

$$\int \frac{dx}{a^2x^2 - b^2} = \frac{1}{2ab}\ln\left(\frac{ax - b}{ax + b}\right) \tag{E.4a}$$

$$= -\frac{1}{ab}\coth^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 > b^2 \tag{E.4b}$$

$$= -\frac{1}{ab}\tanh^{-1}\left(\frac{ax}{b}\right), \qquad a^2x^2 < b^2 \tag{E.4c}$$

$$\int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b}\sqrt{a + bx} \tag{E.5}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) \tag{E.6}$$

---

*This list is confined to those (nontrivial) integrals that arise in the text and in the problems. Extremely useful compilations are, for example, Pierce and Foster (Pi57) and Dwight (Dw61).

$$\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \tag{E.7}$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}}\ln(2\sqrt{a}\sqrt{ax^2 + bx + c} + 2ax + b), \quad a > 0 \tag{E.8a}$$

$$= \frac{1}{\sqrt{a}}\sinh^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right), \quad \begin{cases} a > 0 \\ 4ac > b^2 \end{cases} \tag{E.8b}$$

$$= -\frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right), \quad \begin{cases} a < 0 \\ b^2 > 4ac \\ |2ax + b| < \sqrt{b^2 - 4ac} \end{cases} \tag{E.8c}$$

$$\int \frac{x\,dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{a}\sqrt{ax^2 + bx + c} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2 + bx + c}} \tag{E.9}$$

$$\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{c}}\sinh^{-1}\left(\frac{bx + 2c}{|x|\sqrt{4ac - b^2}}\right), \quad \begin{cases} c > 0 \\ 4ac > b^2 \end{cases} \tag{E.10a}$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}\right), \quad \begin{cases} c < 0 \\ b^2 > 4ac \end{cases} \tag{E.10b}$$

$$= -\frac{1}{\sqrt{c}}\ln\left(\frac{2\sqrt{c}}{x}\sqrt{ax^2 + bx + c} + \frac{2c}{x} + b\right), \quad c > 0 \tag{E.10c}$$

$$\int \sqrt{ax^2 + bx + c}\,dx = \frac{2ax + b}{4a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a}\int \frac{dx}{\sqrt{ax^2 + bx + c}} \tag{E.11}$$

## E.2 Trigonometric Functions

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x \tag{E.12}$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x \tag{E.13}$$

$$\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left[\frac{a\tan(x/2) + b}{\sqrt{a^2 - b^2}}\right], \quad a^2 > b^2 \tag{E.14}$$

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\frac{(a-b)\tan(x/2)}{\sqrt{a^2 - b^2}}\right], \quad a^2 > b^2 \tag{E.15}$$

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{b \sin x}{(b^2 - a^2)(a + b\cos x)} - \frac{a}{b^2 - a^2}\int \frac{dx}{a + b\cos x} \tag{E.16}$$

$$\int \tan x \, dx = -\ln|\cos x| \tag{E.17a}$$

$$\int \tanh x \, dx = \ln \cosh x \tag{E.17b}$$

$$\int e^{ax} \sin x \, dx = \frac{e^{ax}}{a^2 + 1}(a \sin x - \cos x) \tag{E.18a}$$

$$\int e^{ax} \sin^2 x \, dx = \frac{e^{ax}}{a^2 + 4}\left(a \sin^2 x - 2 \sin x \cos x + \frac{2}{a}\right) \tag{E.18b}$$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a} \tag{E.18c}$$

## E.3 Gamma Functions

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx \tag{E.19a}$$

$$= \int_0^1 [\ln(1/x)]^{n-1} dx \tag{E.19b}$$

$$\Gamma(n) = (n-1)!, \quad \text{for } n = \text{positive integer} \tag{E.19c}$$

$$n\Gamma(n) = \Gamma(n+1) \tag{E.20}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{E.21}$$

$$\Gamma(1) = 1 \tag{E.22}$$

$$\Gamma\left(1\frac{1}{4}\right) = 0.906 \tag{E.23}$$

$$\Gamma\left(1\frac{3}{4}\right) = 0.919 \tag{E.24}$$

$$\Gamma(2) = 1 \tag{E.25}$$

$$\int_0^1 \frac{dx}{\sqrt{1 - x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} \tag{E.26}$$

$$\int_0^1 x^m (1 - x^2)^n \, dx = \frac{\Gamma(n + 1)\Gamma\left(\frac{m + 1}{2}\right)}{2\Gamma\left(n + \frac{m + 3}{2}\right)} \tag{E.27a}$$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n + 1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad n > -1 \tag{E.27b}$$

APPENDIX F

# Differential Relations in Different Coordinate Systems

## F.1 Rectangular Coordinates

$$\mathbf{grad}\, U = \nabla U = \sum_i \mathbf{e}_i \frac{\partial U}{\partial x_i} \tag{F.1}$$

$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \sum_i \frac{\partial A_i}{\partial x_i} \tag{F.2}$$

$$\mathbf{curl}\, \mathbf{A} = \nabla \times \mathbf{A} = \sum_{i,j,k} \varepsilon_{ijk} \frac{\partial A_k}{\partial x_j} \mathbf{e}_i \tag{F.3}$$

$$\nabla^2 U = \nabla \cdot \nabla U = \sum_i \frac{\partial^2 U}{\partial x_i^2} \tag{F.4}$$

## F.2 Cylindrical Coordinates

Refer to Figures F-1 and F-2.

$$x_1 = r \cos \phi, \quad x_2 = r \sin \phi, \quad x_3 = z \tag{F.5}$$

$$r = \sqrt{x_1^2 + x_2^2}, \quad \phi = \tan^{-1} \frac{x_2}{x_1}, \quad z = x_3 \tag{F.6}$$

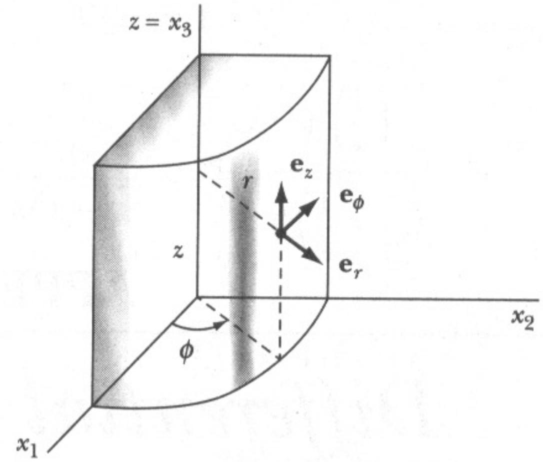



**FIGURE F-1**

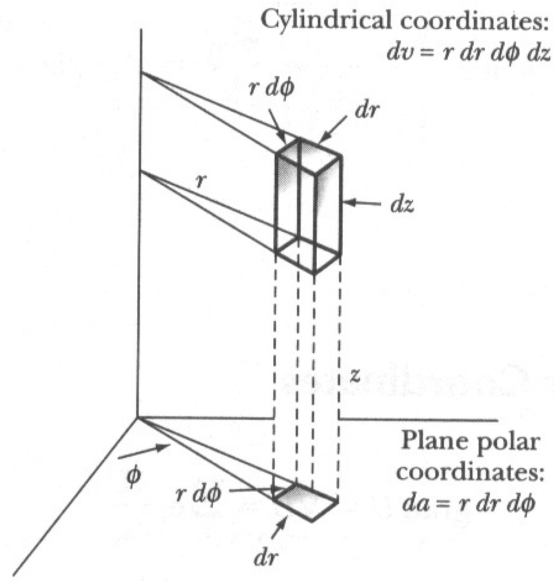



**FIGURE F-2**

$$ds^2 = dr^2 + r^2\,d\phi^2 + dz^2 \tag{F.7}$$

$$dv = r\,dr\,d\phi\,dz \tag{F.8}$$

$$\mathbf{grad}\;\psi = \nabla\psi = \mathbf{e}_r \frac{\partial\psi}{\partial r} + \mathbf{e}_\phi \frac{1}{r}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z \frac{\partial\psi}{\partial z} \tag{F.9}$$

$$\text{div}\;\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{F.10}$$

$$\mathbf{curl}\;\mathbf{A} = \mathbf{e}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{e}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{e}_z\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial\phi}\right) \tag{F.11}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{F.12}$$

## F.3 Spherical Coordinates

Refer to Figures F-3 and F-4

$$x_1 = r \sin\theta \cos\phi, \quad x_2 = r \sin\theta \sin\phi, \quad x_3 = r\cos\theta \tag{F.13}$$

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \cos^{-1}\frac{x_3}{r}, \quad \phi = \tan^{-1}\frac{x_2}{x_1} \tag{F.14}$$

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{F.15}$$

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{F.16}$$

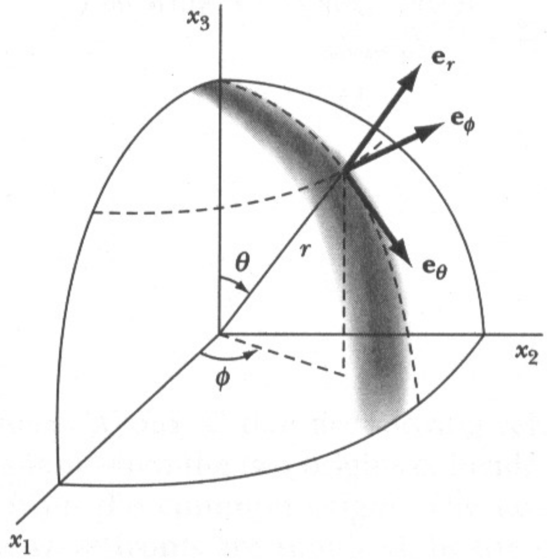



**FIGURE F-3**

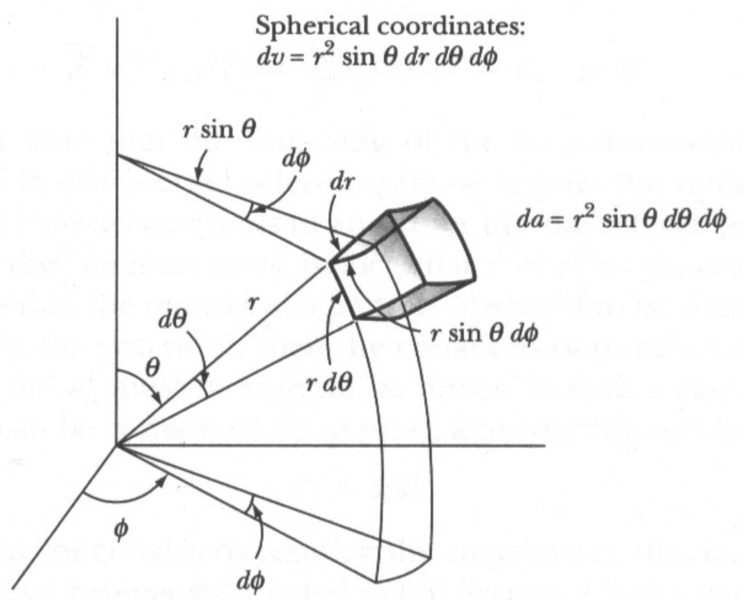



**FIGURE F-4**

$$\mathbf{grad}\,\psi = \nabla\psi = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{F.17}$$

$$\mathrm{div}\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial \mathbf{r}}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta} + \frac{\partial A_\phi}{\partial\phi} \tag{F.18}$$

$$\mathbf{curl}\,\mathbf{A} = \mathbf{e}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]$$
$$+ \mathbf{e}_\theta\frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}(rA_\phi)\right] + \mathbf{e}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \tag{F.19}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{F.20}$$

# APPENDIX G

# A "Proof" of the Relation $\sum_\mu x_\mu^2 = \sum_\mu x_\mu'^2$

Consider the two inertial systems $K$ and $K'$ that are moving relative to one another with a speed $v$. At the instant when the two origins coincide ($t = 0$, $t' = 0$), let a light pulse be emitted from the common origin. The equations that describe the propagation of the wave fronts are required, by the second Einstein postulate, to be of the same form in the two systems:

$$\sum_j x_j^2 - c^2t^2 = \sum_\mu x_\mu^2 \equiv s^2 = 0, \quad \text{in } K \tag{G.1a}$$

$$\sum_j x_j'^2 - c^2t'^2 = \sum_\mu x_\mu'^2 \equiv s'^2 = 0, \quad \text{in } K' \tag{G.1b}$$

These equations state that the vanishing of the four-dimensional interval between two events in one inertial reference frame implies the vanishing of the interval between the same two events in any other inertial reference frame. But we need more than this; we must show, in fact, that $s^2 = s'^2$ in general.

If we require that the motion of a particle observed to be *linear* in the system $K$ also be linear in the system $K'$, then the equations of transformation that connect the $x_\mu$ and the $x_\mu'$ must themselves be linear. In such a case, the quadratic forms $s^2$ and $s'^2$ can be connected by, at most, a proportionality factor:

$$s'^2 = \kappa s^2 \tag{G.2a}$$

The factor $\kappa$ could conceivably depend on the coordinates, the time, and the relative speed of the two systems. As pointed out in Section 2.3, the space and time associated with an inertial reference frame are *homogeneous*, so the relation between

$s^2$ and $s'^2$ cannot be different at different points in space nor at different instants of time. Therefore, the factor $\kappa$ cannot depend on either the coordinates or the time. A dependence on $v$ is still allowed, however, but the *isotropy* of space forbids a dependence on the *direction* of $v$. We have therefore reduced the possible dependence of $s'^2$ on $s^2$ to a factor that involves at most the magnitude of the speed $v$; that is, we have

$$s'^2 = \kappa(v)\,s^2 \tag{G.2b}$$

If we make the transformation from $K'$ back to $K$, we have the result

$$s^2 = \kappa(-v)\,s'^2$$

where $-v$ occurs because the velocity of $K$ relative to $K'$ is the negative of the velocity of $K'$ relative to $K$. But we have already argued that the factor $\kappa$ can depend only on the *magnitude* of $v$. We therefore have the two equations

$$\left.\begin{aligned} s'^2 &= \kappa(v)\,s^2 \\ s^2 &= \kappa(v)\,s'^2 \end{aligned}\right\} \tag{G.3}$$

Combining these equations, we conclude that $\kappa^2 = 1$, or $\kappa(v) = \pm 1$. The value of $\kappa(v)$ must not be a discontinuous function of $v$; that is, if we change $v$ at some rate, $\kappa$ cannot suddenly jump from $+1$ to $-1$. In the limit of zero velocity, the systems $K$ and $K'$ become identical, so that $\kappa(v = 0) = +1$. Hence,

$$\kappa = +1 \tag{G.4}$$

for all values of the velocity, and we have, finally,

$$s^2 = s'^2 \tag{G.5}$$

This important result states that the four-dimensional interval between two events is the same in all inertial reference frames.

APPENDIX H

# *Numerical Solution for Example 2.7*

In this appendix, we show the MathCad solution that produced Figures 2-8 and 2-9 for Example 2.7. This program was written for MathCad for Windows, version 4.0.

| | |
|---|---|
| $g := 9.8$ | acceleration of gravity |
| $th := 60 \cdot \left(\frac{\pi}{180}\right)$ | initial angle |
| $vo := 600$ | |
| $u := vo \cdot \cos(th)$ | initial velocity |
| $v := vo \cdot \sin(th)$ | initial horizontal velocity |
| $i := 1 .. 6$ | initial vertical velocity |

$k_i :=$

| |
|---|
| 0.0000001 |
| 0.01 |
| 0.02 |
| 0.04 |
| 0.08 |
| 0.005 |

table of drag coefficients

| | |
|---|---|
| $t := 0, 1 .. 130$ | range of time values |
| $x(t, K) := \left(\left(\frac{u}{K}\right)\right) \cdot (1 - \exp(-K \cdot t))$ | calculate horizontal position |

$$y(t, K) := -g\cdot\frac{t}{K} + \frac{K\cdot v + g}{(K)^2}\cdot(1 - \exp(-K\cdot t))$$ calculate vertical position

*[Now plot $y(t,k_j)$ versus $x(t,k_j)$ to produce Figure 2-8.]*

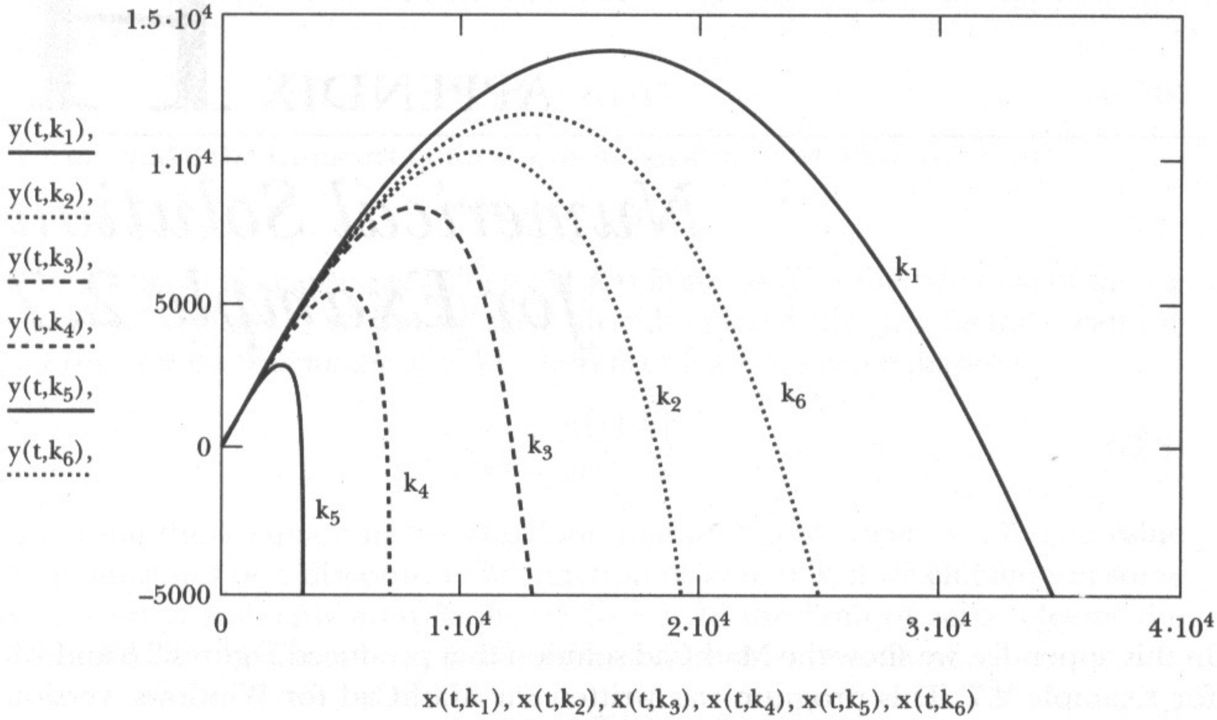



**FIGURE 2-8**

Now set up an equation to solve Equation 2.45 for *T* for any value of the retarding force constant k.

$$f(k, T) := \mathrm{root}\left[T - \frac{k\cdot v + g}{g\cdot k}\cdot(1 - \exp(-k\cdot T)), T\right] \qquad \mathbf{(2.45)}$$

$j := 1, 2 .. 81$ — Set up range of values to calculate; 80 values.

$K_j := -0.001 + 0.001\cdot j + 0.00000001$ — This will allow us to calculate over a range of k values from 0 to 0.08.

$Tr_0 := 100$ — The time value for k = 0 is 106 s. This is a guess to get the calculation started.

$Tr_j := f(K_j, Tr_{j-1})$ — We now determine the solution for the time T for all the values of k. Solve Equation 2.45.

$Tr_1 := 106.074$ — This is the value of T for k1. We do not bother to calculate all the others here.