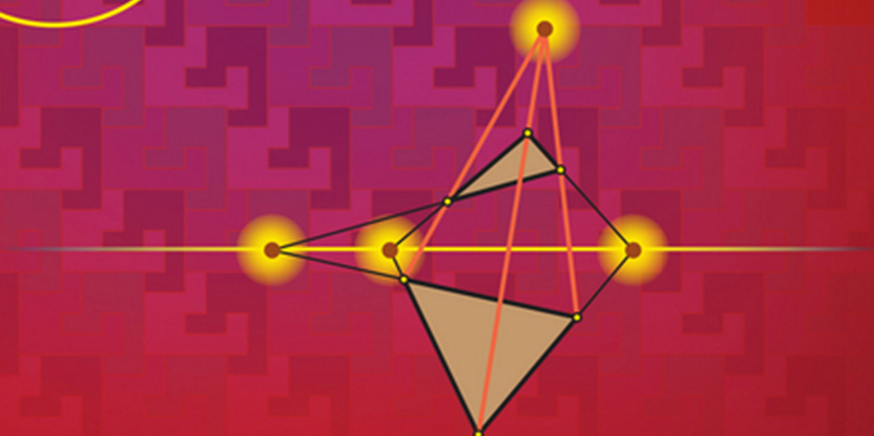
Classical Geometry

Euclidean, Transformational, Inversive, and Projective



I. E. Leonard • J. E. Lewis • A. C. F. Liu • G. W. Tokarsky

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Euclidean, Transformational, Inversive, and Projective

I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky
Department of Mathematical and Statistical Sciences
University of Alberta

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PREFACE

It is sometimes said that geometry should be studied because it is a useful and valuable discipline, but in fact many people study it simply because geometry is a very enjoyable subject. It is filled with problems at every level that are entertaining and elegant, and this enjoyment is what we have attempted to bring to this textbook.

This text is based on class notes that we developed for a three-semester sequence of undergraduate geometry courses that has been taught at the University of Alberta for many years. It is appropriate for students from all disciplines who have previously studied high school algebra, geometry, and trigonometry.

When we first started teaching these courses, our main problem was finding a suitable method for teaching geometry to university students who have had minimal experience with geometry in high school. We experimented with material from high school texts but found it was not challenging enough. We also tried an axiomatic approach, but students often showed little enthusiasm for proving theorems, particularly since the early theorems seemed almost as self-evident as the axioms. We found the most success by starting early with problem solving, and this is the approach we have incorporated throughout the book.

The geometry in this text is synthetic rather than Cartesian or coordinate geometry. We remain close to classical themes in order to encourage the development of geometric intuition, and for the most part we avoid abstract algebra although we do demonstrate its use in the sections on transformational geometry.

Part I is about Euclidean geometry; that is, the study of the properties of points and lines that are invariant under isometries and similarities. As well as many of the usual topics, it includes material that many students will not have seen, for example, the theorems of Ceva and Menelaus and their applications. Part I is the basis for Parts II and III.

Part II discusses the properties of Euclidean transformations or isometries of the plane (translations, reflections, and rotations and their compositions). It also introduces groups and their use in studying transformations.

Part III introduces inversive and projective geometry. These subjects are presented as natural extensions of Euclidean geometry, with no abstract algebra involved.

We would like to acknowledge our late colleagues George Cree and Murray Klamkin, without whose inspiration and encouragement over the years this project would not have been possible.

Finally, we would like to thank our families for their patience and understanding in the preparation of the textbook. In particular, I. E. Leonard would like to thank Sarah for proofreading the manuscript numerous times.

ED, TED, ANDY, AND GEORGE

Edmonton, Alberta, Canada
January, 2014

PART I

EUCLIDEAN GEOMETRY

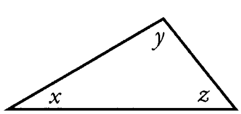
CHAPTER 1

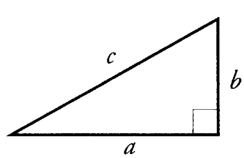
CONGRUENCY

1.1 Introduction

Assumed Knowledge

This text assumes a bit of knowledge on the part of the reader. For example, it assumes that you know that the sum of the angles of a triangle in the plane is 180° ($x + y + z = 180^\circ$ in the figure below), and that in a right triangle with hypotenuse c and sides a and b, the Pythagorean relation holds: $c^2 = a^2 + b^2$.



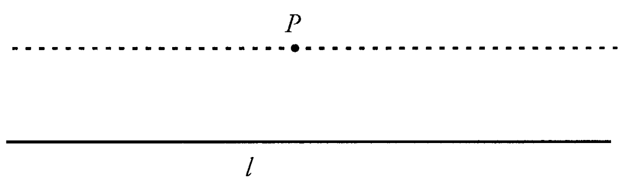


Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

We use the word ***line*** to mean *straight line*, and we assume that you know that two lines either do not intersect, intersect at exactly one point, or completely coincide. Two lines that do not intersect are said to be ***parallel***.

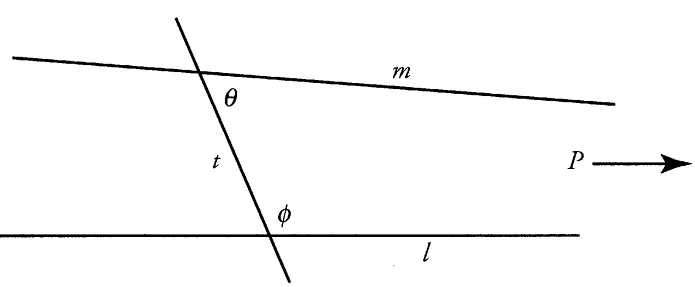
We also assume certain knowledge about parallel lines, namely, that you have seen some form of the ***parallel axiom***:

Given a line l and a point P in the plane, there is exactly one line through P parallel to l.



The preceding version of the parallel axiom is often called ***Playfair's Axiom***. You may even know something equivalent to it that is close to the original version of the ***parallel postulate***:

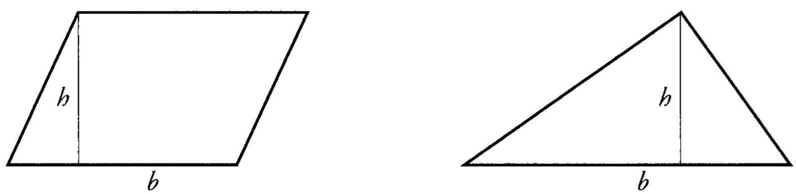
Given two lines l and m, and a third line t cutting both l and m and forming angles ϕ and θ on the same side of t, if $\phi + \theta < 180°$, then l and m meet at a point on the same side of t as the angles.



The subject of this part of the text is Euclidean geometry, and the above-mentioned parallel postulate characterizes Euclidean geometry. Although the postulate may seem to be obvious, there are perfectly good geometries in which it does not hold.

We also assume that you know certain facts about areas. A ***parallelogram*** is a quadrilateral (figure with four sides) such that the opposite sides are parallel.

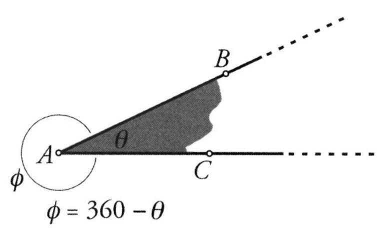
The area of a parallelogram with base b and height h is $b \cdot h$, and the area of a triangle with base b and height h is $b \cdot h/2$.



Notation and Terminology

Throughout this text, we use uppercase Latin letters to denote points and lowercase Latin letters to denote lines and rays. Given two points A and B, there is one and only one line through A and B. A ***ray*** is a half-line, and the notation $\overrightarrow{AB}$ denotes the ray starting at A and passing through B. It consists of the points A and B, all points between A and B, and all points X on the line such that B is between A and X.

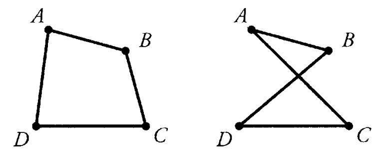
Given rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, we denote by $\angle BAC$ the angle formed by the two rays (the shaded region in the following figure). When no confusion can arise, we sometimes use $\angle A$ instead of $\angle BAC$. We also use lowercase letters, either Greek or Latin, to denote angles.



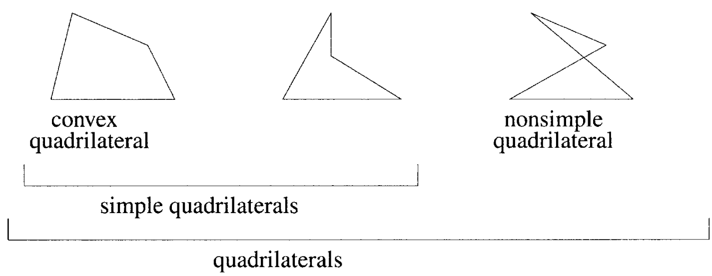
When two rays form an angle other than $180°$, there are actually two angles to talk about: the smaller angle (sometimes called the ***interior angle***) and the larger angle (called the ***reflex angle***). When we refer to $\angle BAC$, we always mean the nonreflex angle.

Note. The angles that we are talking about here are *undirected angles*; that is, they do not have negative values, and so can range in magnitude from $0°$ to $360°$. Some people prefer to use $m(\angle A)$ for the measure of the angle A; however, we will use the same notation for both the angle and the measure of the angle.

When we refer to a quadrilateral as $ABCD$ we mean one whose edges are AB, BC, CD, and DA. Thus, the quadrilateral $ABCD$ and the quadrilateral $ABDC$ are quite different.



There are three classifications of quadrilaterals: convex, simple, and nonsimple, as shown in the following diagram.



1.2 Congruent Figures

Two figures that have exactly the same shape and exactly the same size are said to be ***congruent***. More explicitly:

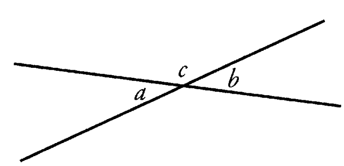
1. Two angles are ***congruent*** if they have the same measure.
2. Two line segments are ***congruent*** if they are the same length.
3. Two circles are ***congruent*** if they have the same radius.
4. Two triangles are ***congruent*** if corresponding sides and angles are the same size.
5. All rays are ***congruent***.
6. All lines are ***congruent***.

Theorem 1.2.1. *Vertically opposite angles are congruent.*

Proof. We want to show that $a = b$. We have

$$a + c = 180 \text{ and } b + c = 180,$$

and it follows from this that $a = b$.



□

Notation. The symbol $\equiv$ denotes congruence. We use the notation $\triangle ABC$ to denote a triangle with vertices A, B, and C, and we use $\mathcal{C}(P, r)$ to denote a circle with center P and radius r.

Thus, $\mathcal{C}(P, r) \equiv \mathcal{C}(Q, s)$ if and only if $r = s$.

We will be mostly concerned with the notion of congruent triangles, and we mention that in the definition, $\triangle ABC \equiv \triangle DEF$ if and only if the following six conditions hold:

$$\begin{aligned}
\angle A &\equiv \angle D \\
\angle B &\equiv \angle E \\
\angle C &\equiv \angle F \\
AB &\equiv DE \\
BC &\equiv EF \\
AC &\equiv DF.
\end{aligned}$$

Note that the two statements $\triangle ABC \equiv \triangle DEF$ and $\triangle ABC \equiv \triangle EFD$ are not the same!

The Basic Congruency Conditions

According to the definition of congruency, two triangles are congruent if and only if six different parts of one are congruent to the six corresponding parts of the other. Do we really need to check all six items? The answer is no.

If you give three straight sticks to one person and three identical sticks to another and ask both to constuct a triangle with the sticks as the sides, you would expect the two triangles to be exactly the same. In other words, you would expect that it is possible to verify congruency by checking that the three corresponding sides are congruent. Indeed this is the case, and, in fact, there are several ways to verify congruency without checking all six conditions.

The three congruency conditions that are used most often are the Side-Angle-Side (**SAS**) condition, the Side-Side-Side (**SSS**) condition, and the Angle-Side-Angle (**ASA**) condition.

Axiom 1.2.2. *(***SAS** *Congruency)*

Two triangles are congruent if two sides and the included angle of one are congruent to two sides and the included angle of the other.

Theorem 1.2.3. *(***SSS** *Congruency)*

Two triangles are congruent if the three sides of one are congruent to the corresponding three sides of the other.

Theorem 1.2.4. *(***ASA** *Congruency)*

Two triangles are congruent if two angles and the included side of one are congruent to two angles and the included side of the other.

You will note that the **SAS** condition is an axiom, and the other two are stated as theorems. We will not prove the theorems but will freely use all three conditions.

Any one of the three conditions could be used as an axiom with the other two then derived as theorems. In case you are wondering why the **SAS** condition is preferred as the basic axiom rather than the **SSS** condition, it is because *it is always possible to construct a triangle given two sides and the included angle*, whereas it is not always possible to construct a triangle given three sides (consider sides of length 3, 1, and 1).

Axiom 1.2.5. *(The Triangle Inequality)*

The sum of the lengths of two sides of a triangle is always greater than the length of the remaining side.

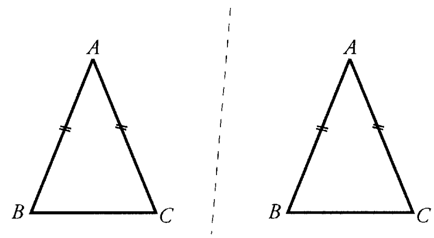
The congruency conditions are useful because they allow us to conclude that certain parts of two triangles are congruent by determining that certain other parts are congruent.

Here is how congruency may be used to prove two well-known theorems about isosceles triangles. (An ***isosceles*** triangle is one that has two equal sides.)

Theorem 1.2.6. *(The Isosceles Triangle Theorem)*

In an isosceles triangle, the angles opposite the equal sides are equal.

Proof. Let us suppose that the triangle is ABC with $AB = AC$.



(same triangle, recopied for reference purposes)

In $\triangle ABC$ and $\triangle ACB$ we have

$$\begin{aligned} AB &= AC, \\ \angle BAC &= \angle CAB, \\ AC &= AB, \end{aligned}$$

so $\triangle ABC \equiv \triangle ACB$ by **SAS**.

Since the triangles are congruent, it follows that all corresponding parts are congruent, so $\angle B$ of $\triangle ABC$ must be congruent to $\angle C$ of $\triangle ACB$.

□

Theorem 1.2.7. *(Converse of the Isosceles Triangle Theorem)*

If in $\triangle ABC$ we have $\angle B = \angle C$, then $AB = AC$.

Proof. In $\triangle ABC$ and $\triangle ACB$ we have

$$\begin{aligned} \angle ABC &= \angle ACB, \\ BC &= CB, \\ \angle ACB &= \angle ABC, \end{aligned}$$

so $\triangle ABC \equiv \triangle ACB$ by **ASA**.

Since $\triangle ABC \equiv \triangle ACB$ it follows that $AB = AC$.

□

Perhaps now is a good time to explain what the converse of a statement is. Many statements in mathematics have the form

If $\mathcal{P}$, then $\mathcal{Q}$,

where $\mathcal{P}$ and $\mathcal{Q}$ are assertions of some sort.

For example:

If $ABCD$ is a square, then angles A, B, C, and D are all right angles.

Here, $\mathcal{P}$ is the assertion "$ABCD$ is a square," and $\mathcal{Q}$ is the assertion "angles A, B, C, and D are all right angles."

The ***converse*** of the statement "If $\mathcal{P}$, then $\mathcal{Q}$" is the statement

If $\mathcal{Q}$, then $\mathcal{P}$.

Thus, the converse of the statement "If $ABCD$ is a square, then angles A, B, C, and D are all right angles" is the statement

If angles A, B, C, and D are all right angles, then $ABCD$ is a square.

A common error in mathematics is to confuse a statement with its converse. Given a statement and its converse, if one of them is true, it does not automatically follow that the other is also true.

Exercise 1.2.8. *For each of the following statements, state the converse and determine whether it is true or false.*

1. *Given triangle ABC, if $\angle ABC$ is a right angle, then $AB^2 + BC^2 = AC^2$.*
2. *If $ABCD$ is a parallelogram, then $AB = CD$ and $AD = BC$.*
3. *If $ABCD$ is a convex quadrilateral, then $ABCD$ is a rectangle.*
4. *Given quadrilateral $ABCD$, if $AC \neq BD$, then $ABCD$ is not a rectangle.*

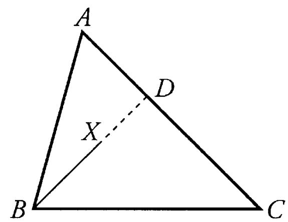
Solutions to the exercises are given at the end of the chapter.

The Isosceles Triangle Theorem and its converse raise questions about how sides are related to unequal angles, and there are useful theorems for this case.

Theorem 1.2.9. *(The Angle-Side Inequality)*

In $\triangle ABC$, if $\angle ABC > \angle ACB$, then $AC > AB$.

Proof. Draw a ray BX so that $\angle CBX \equiv \angle BCA$ with X to the same side of BC as A, as in the figure on the following page.



Since $\angle ABC > \angle CBX$, the point X is interior to $\angle ABC$ and so BX will cut side AC at a point D. Then we have

$$DB = DC$$

by the converse to the Isosceles Triangle Theorem.

By the Triangle Inequality, we have

$$AB < AD + DB,$$

and combining these gives us

$$AB < AD + DC = AC,$$

which is what we wanted to prove.

□

The converse of the Angle-Side Inequality is also true. Note that the proof of the converse uses the statement of the original theorem. This is something that frequently occurs when proving that the converse is true.

Theorem 1.2.10. *In $\triangle ABC$, if $AC > AB$, then $\angle ABC > \angle ACB$.*

Proof. There are three possible cases to consider:

(1) $\angle ABC = \angle ACB$.

(2) $\angle ABC < \angle ACB$.

(3) $\angle ABC > \angle ACB$.

If case (1) arises, then $AC = AB$ by the converse to the Isosceles Triangle Theorem, so case (1) cannot in fact arise. If case (2) arises, then $AC < AB$ by the Angle-Side Inequality, so (2) cannot arise. The only possibility is therefore case (3).

□

The preceding examples, as well as showing how congruency is used, are facts that are themselves very useful. They can be summarized very succinctly: in a triangle,

Equal angles are opposite equal sides.
The larger angle is opposite the larger side.

1.3 Parallel Lines

Two lines in the plane are ***parallel*** if

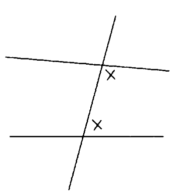
(a) they do not intersect or

(b) they are the same line.

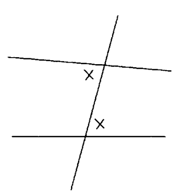
Note that (b) means that a line is parallel to itself.

Notation. We use $l \parallel m$ to denote that the lines l and m are parallel and sometimes use $l \nparallel m$ to denote that they are not parallel. If l and m are not parallel, they meet at precisely one point in the plane.

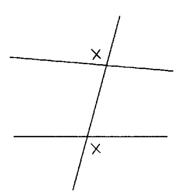
When a transversal crosses two other lines, various pairs of angles are endowed with special names:



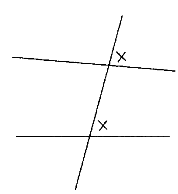
adjacent angles



opposite or alternate angles



opposite or alternate exterior angles



corresponding angles

The proofs of the next two theorems are omitted; however, we mention that the proof of Theorem 1.3.2 requires the parallel postulate, but the proof of Theorem 1.3.1 does not.

Theorem 1.3.1. *If a transversal cuts two lines and any one of the following four conditions holds, then the lines are parallel:*

(1) adjacent angles total $180°$,

(2) alternate angles are equal,

(3) alternate exterior angles are equal,

(4) corresponding angles are equal.

Theorem 1.3.2. *If a transversal cuts two parallel lines, then all four statements of Theorem 1.3.1 hold.*

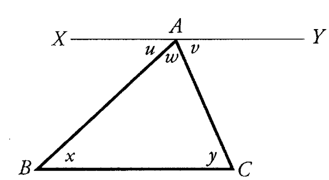
Remark. Theorem 1.3.1 can be proved using the External Angle Inequality, which is described below. The proof of the inequality itself ultimately depends on Theorem 1.3.1, but this would mean that we are using circular reasoning, which is not permitted. However, there is a proof of the External Angle Inequality which does not in any way depend upon Theorem 1.3.1, and so it is possible to avoid circular reasoning.

1.3.1 Angles in a Triangle

The parallel postulate is what distinguishes Euclidean geometry from other geometries, and as we see now, it is also what guarantees that the sum of the angles in a triangle is $180°$.

Theorem 1.3.3. *The sum of the angles of a triangle is* $180°$.

Proof. Given triangle ABC, draw the line XY through A parallel to BC, as shown. Consider AB as a transversal for the parallel lines XY and BC, then $x = u$ and similarly $y = v$. Consequently,



$$x + y + w = u + v + w = 180,$$

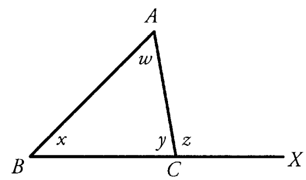
which is what we wanted to prove.

□

Given triangle ABC, extend the side BC beyond C to X. The angle ACX is called an ***exterior angle*** of $\triangle ABC$.

Theorem 1.3.4. *(The Exterior Angle Theorem)*

An exterior angle of a triangle is equal to the sum of the opposite interior angles.



Proof. In the diagram above, we have $y + z = 180 = y + x + w$, so $z = x + w$.

$\square$

The Exterior Angle Theorem has a useful corollary:

Corollary 1.3.5. *(The Exterior Angle Inequality)*

An exterior angle of a triangle is greater than either of the opposite interior angles.

Note. The proof of the Exterior Angle Inequality given above ultimately depends on the fact that the sum of the angles of a triangle is $180°$, which turns out to be equivalent to the parallel postulate. It is possible to prove the Exterior Angle Inequality without using any facts that follow from the parallel postulate, but we will omit that proof here.

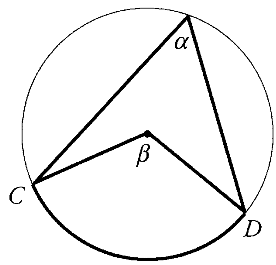
1.3.2 Thales' Theorem

One of the most useful theorems about circles is credited to Thales, who is reported to have sacrificed two oxen after discovering the proof. (In truth, versions of the theorem were known to the Babylonians some one thousand years earlier.)

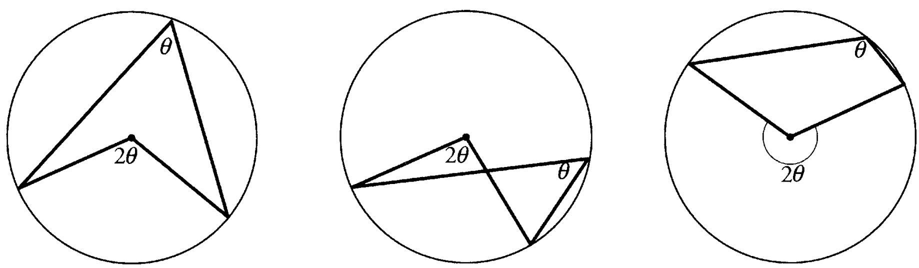
Theorem 1.3.6. *(Thales' Theorem)*

An angle inscribed in a circle is half the angle measure of the intercepted arc.

In the diagram, α is the measure of the inscribed angle, the arc CD is the intercepted arc, and β is the angle measure of the intercepted arc.



The following diagrams illustrate Thales' Theorem.



Proof. As the figures above indicate, there are several separate cases to consider. We will prove the first case and leave the others as exercises.

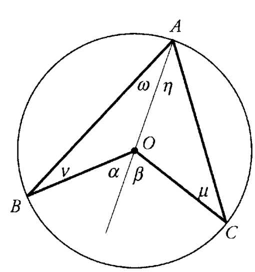
Referring to the diagram below, we have

$$\alpha = \nu + \omega \quad \text{and} \quad \beta = \mu + \eta.$$

But $\nu = \omega$ and $\mu = \eta$ (isosceles triangles). Consequently,

$$\angle BOC = \alpha + \beta = 2\omega + 2\eta = 2\angle BAC,$$

and the theorem follows.



□

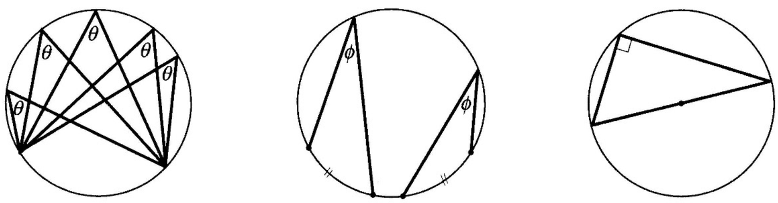
Thales' Theorem has several useful corollaries.

Corollary 1.3.7. *In a given circle:*

(1) All inscribed angles that intercept the same arc are equal in size.

(2) All inscribed angles that intercept congruent arcs are equal in size.

(3) The angle in a semicircle is a right angle.

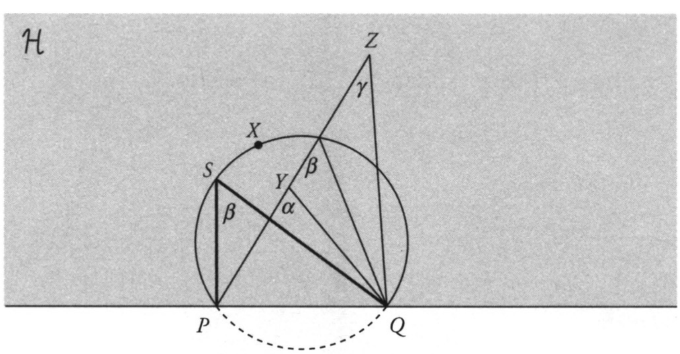


The converse of Thales' Theorem is also very useful.

Theorem 1.3.8. *(Converse of Thales' Theorem)*

Let $\mathcal{H}$ be a halfplane determined by a line PQ. The set of points in $\mathcal{H}$ that form a constant angle β with P and Q is an arc of a circle passing through P and Q.

Furthermore, every point of $\mathcal{H}$ inside the circle makes a larger angle with P and Q and every point of $\mathcal{H}$ outside the circle makes a smaller angle with P and Q.

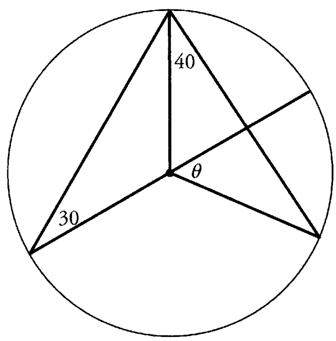


Proof. Let S be a point such that $\angle PSQ = \beta$ and let $\mathcal{C}$ be the circumcircle of $\triangle SPQ$. In the halfplane $\mathcal{H}$, all points X on $\mathcal{C}$ intercept the same arc of $\mathcal{C}$, so by Thales' Theorem, all angles PXQ have measure β.

From the Exterior Angle Inequality, in the figure on the previous page we have $\alpha > \beta > \gamma$. As a consequence, every point Z of $\mathcal{H}$ outside $\mathcal{C}$ must have $\angle PZQ < \beta$, and every point Y of $\mathcal{H}$ inside $\mathcal{C}$ must have $\angle PYQ > \beta$, and this completes the proof.

□

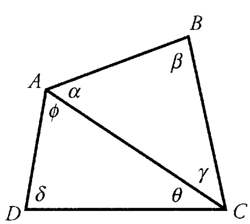
Exercise 1.3.9. *Calculate the size of θ in the following figure.*

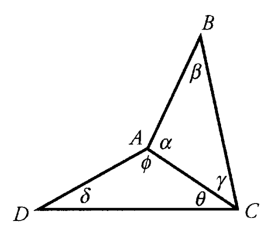


1.3.3 Quadrilaterals

The following theorem uses the fact that a simple quadrilateral always has at least one diagonal that is interior to the quadrilateral.

Theorem 1.3.10. *The sum of the interior angles of a simple quadrilateral is* $360°$.





Proof. Let the quadrilateral have vertices A, B, C, and D, with AC being an internal diagonal. Referring to the diagram, we have

$$\begin{aligned}\angle A + \angle B + \angle C + \angle D &= (\phi + \alpha) + \beta + (\gamma + \theta) + \delta \\ &= (\alpha + \beta + \gamma) + (\theta + \delta + \phi) \\ &= 180° + 180° \\ &= 360°.\end{aligned}$$

□

Note. This theorem is false if the quadrilateral is not simple, in which case the sum of the interior angles is less than 360°.

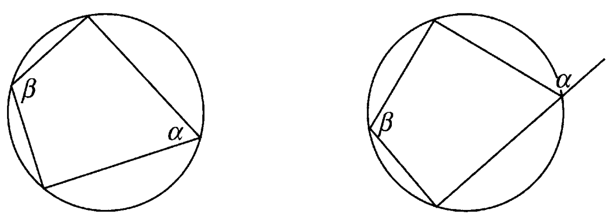
Cyclic Quadrilaterals

A quadrilateral that is inscribed in a circle is called a ***cyclic quadrilateral*** or, equivalently, a ***concyclic quadrilateral***. The circle is called the ***circumcircle*** of the quadrilateral.

Theorem 1.3.11. *Let $ABCD$ be a simple cyclic quadrilateral. Then:*

(1) The opposite angles sum to 180°.

(2) Each exterior angle is congruent to the opposite interior angle.



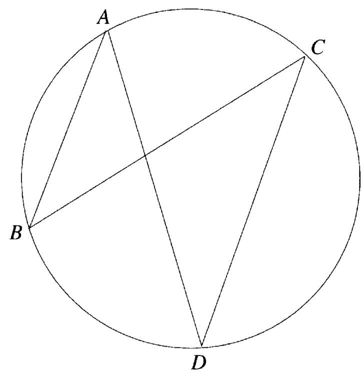
$\alpha+\beta = 180$ $\qquad\qquad$ $\alpha=\beta$

Theorem 1.3.12. *Let $ABCD$ be a simple quadrilateral. If the opposite angles sum to* 180°, *then $ABCD$ is a cyclic quadrilateral.*

We leave the proofs of Theorem 1.3.11 and Theorem 1.3.12 as exercises and give a similar result for nonsimple quadrilaterals.

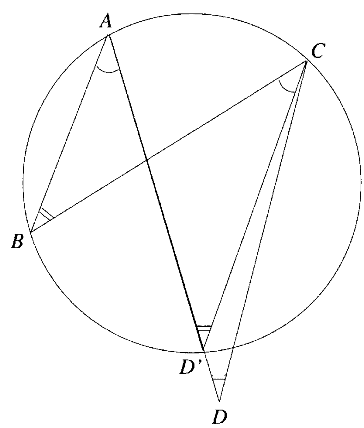
Example 1.3.13. *(Cyclic Nonsimple Quadrilaterals)*

A nonsimple quadrilateral can be inscribed in a circle if and only if opposite angles are equal. For example, in the figure on the following page, the nonsimple quadrilateral $ABCD$ can be inscribed in a circle if and only if $\angle A = \angle C$ and $\angle B = \angle D$.



Solution. Suppose first that the quadrilateral $ABCD$ is cyclic. Then $\angle A = \angle C$ since they are both subtended by the chord BD, while $\angle B = \angle D$ since they are both subtended by the chord AC.

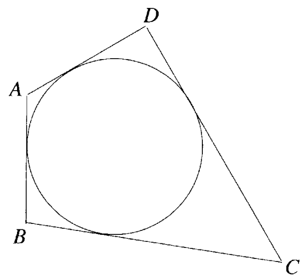
Conversely, suppose that $\angle A = \angle C$ and $\angle B = \angle D$, and let the circle below be the circumcircle of $\triangle ABC$.



If the quadrilateral $ABCD$ is not cyclic, then the point D does not lie on this circumcircle. Assume that it lies outside the circle and let D' be the point where the line segment AD hits the circle. Since $ABCD'$ is a cyclic quadrilateral, then from the first part of the proof, $\angle B = \angle D'$ and therefore $\angle D = \angle D'$, which contradicts the External Angle Inequality in $\triangle CD'D$. Thus, if $\angle A = \angle C$ and $\angle B = \angle D$, then quadrilateral $ABCD$ is cyclic.

□

Exercise 1.3.14. *Show that a quadrilateral has an inscribed circle (that is, a circle tangent to each of its sides) if and only if the sums of the lengths of the two pairs of opposite sides are equal. For example, the quadrilateral $ABCD$ has an inscribed circle if and only if $AB + CD = AD + BC$.*

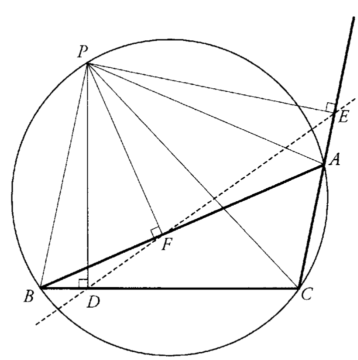


The following example is a result named after Robert Simson (1687–1768), whose *Elements of Euclid* was a standard textbook published in 24 editions from 1756 until 1834 and upon which many modern English versions of Euclid are based. However, in their book *Geometry Revisited,* Coxeter and Greitzer report that the result attributed to Simson was actually discovered later, in 1797, by William Wallace.

Example 1.3.15. *(Simson's Theorem)*

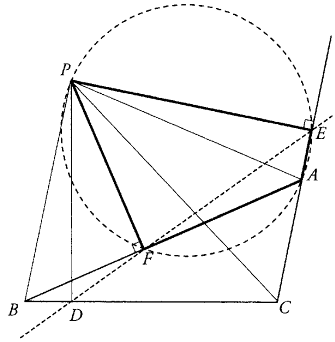
Given $\triangle ABC$ inscribed in a circle and a point P on its circumference, the perpendiculars dropped from P meet the sides of the triangle in three collinear points.

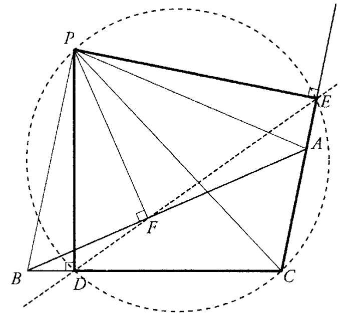
The line is called the **Simson line** *corresponding to P.*



Solution. We will prove Simson's Theorem by showing that $\angle PEF = \angle PED$ (which means that the rays EF and ED coincide).

As well as the cyclic quadrilateral $PACB$, there are two other cyclic quadrilaterals, namely $PEAF$ and $PECD$, which are reproduced in the figure on the following page. (These are cyclic because in each case two of the opposite angles sum to $180°$).





By Thales' Theorem applied to the circumcircle of $PEAF$, we get

$$\angle PEF = \angle PAF = \angle PAB.$$

By Thales' Theorem applied to the circumcircle of $PABC$, we get

$$\angle PAB = \angle PCB = \angle PCD.$$

By Thales' Theorem applied to the circumcircle of $PECD$, we get

$$\angle PCD = \angle PED.$$

Therefore, $\angle PEF = \angle PED$, which completes the proof. □

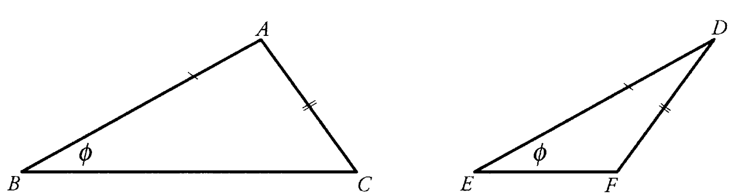
1.4 More About Congruency

The next theorem follows from **ASA** congruency together with the fact that the angle sum in a triangle is $180°$.

Theorem 1.4.1. *(**SAA** Congruency)*

Two triangles are congruent if two angles and a side of one are congruent to two angles and the corresponding side of the other.

In the figure below we have noncongruent triangles ABC and DEF. In these triangles, $AB \equiv DE$, $AC \equiv DF$, and $\angle B \equiv \angle E$. This shows that, in general, **SSA** does not guarantee congruency.



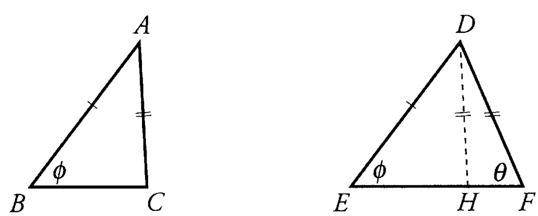
With further conditions we do get congruency:

Theorem 1.4.2. *(**SSA**$^+$ Congruency)*

***SSA** congruency is valid if the length of the side opposite the given angle is greater than or equal to the length of the other side.*

Proof. Suppose that in triangles ABC and DEF we have $AB = DE$, $AC = DF$, and $\angle ABC = \angle DEF$ and that the side opposite the given angle is the larger of the two sides.

We will prove the theorem by contradiction. Assume that the theorem is false, that is, assume that $BC \neq EF$; then we may assume that $BC < EF$. Let H be a point on EF so that $EH = BC$, as in the figure below.



Then, $\triangle ABC \equiv \triangle DEH$ by **SSS** congruency. This means that $DH = DF$, so $\triangle DHF$ is isosceles. Then $\angle DFE = \angle DHF$.

Since we are given that $DF \geq DE$, the Angle-Side Inequality tells us that

$$\angle DEF \geq \angle DFE,$$

and so it follows that $\angle DEF \geq \angle DHF$. However, this contradicts the External Angle Inequality.

We must therefore conclude that the assumption that the theorem is false is incorrect, and so we can conclude that the theorem is true.

□

Since the hypotenuse of a right triangle is always the longest side, there is an immediate corollary:

Corollary 1.4.3. *(**HSR** Congruency)*

If the hypotenuse and one side of a right triangle are congruent to the hypotenuse and one side of another right triangle, the two triangles are congruent.

Counterexamples and Proof by Contradiction

If we were to say that

If $ABCD$ is a rectangle, then $AB = BC$,

you would most likely show us that the statement is false by drawing a rectangle that is not a square. When you do something like this, you are providing what is called a ***counterexample***.

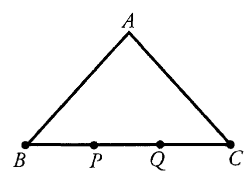
In the assertion "If $\mathcal{P}$, then $\mathcal{Q}$," the statement $\mathcal{P}$ is called the ***hypothesis*** and the statement $\mathcal{Q}$ is called the ***conclusion***. A counterexample to the assertion is any example in which the hypothesis is true and the conclusion is false.

To prove that an assertion is not true, all you need to do is find a single counterexample. (You do not have to show that it is never true, you only have to show that it is not always true!)

Exercise 1.4.4. *For each of the following statements, provide a diagram that is a counterexample to the statement.*

1. *Given triangle ABC, if $\angle ABC = 60°$, then ABC is isosceles.*

2. *Given that ABC is an isosceles triangle with $AB = AC$ and with P and Q on side BC as shown in the picture, if $BP = PQ = QC$, then $\angle BAP = \angle PAQ = \angle QAC$.*



3. *In a quadrilateral $ABCD$, if $AB = CD$ and $\angle BAD = \angle ADC = 90°$, then $ABCD$ is a rectangle.*

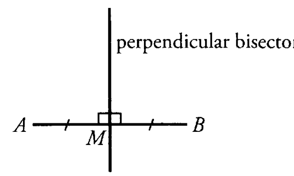
A *proof by contradiction* to verify an assertion of the form "If $\mathcal{P}$, then $\mathcal{Q}$" consists of the following two steps:

(a) Assume that the assertion is false. This amounts to assuming that there is a counterexample to the assertion; that is, we assume that it is possible for the hypothesis $\mathcal{P}$ to be true while the conclusion $\mathcal{Q}$ is false. In other words, *assume that it is possible for the hypothesis **and** the negative of the conclusion to simultaneously be true.*

(b) Show that this leads to a contradiction of a fact that is known to be true. In such circumstances, somewhere along the way an error must have been made. Presuming that the reasoning is correct, the only possibility is that the assumption that the assertion is false must be in error. Thus, we must conclude that the assertion is true.

1.5 Perpendiculars and Angle Bisectors

Two lines that intersect each other at right angles are said to be ***perpendicular*** to each other. The ***right bisector*** or ***perpendicular bisector*** of a line segment AB is a line perpendicular to AB that passes through the midpoint M of AB.



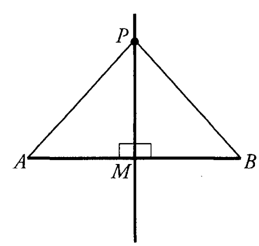
The following theorem is the characterization of the perpendicular bisector.

Theorem 1.5.1. *(Characterization of the Perpendicular Bisector)*

Given different points A and B, the perpendicular bisector of AB consists of all points P that are equidistant from A and B.

Proof. Let P be a point on the right bisector. Then in triangles PMA and PMB we have

$$\begin{aligned} PM &= PM, \\ \angle PMA &= 90^\circ = \angle PMB, \\ MA &= MB, \end{aligned}$$



so triangles PMA and PMB are congruent by **SAS**. It follows that $PA = PB$.

Conversely, suppose that P is some point such that $PA = PB$. Then triangles PMA and PMB are congruent by **SSS**. It follows that $\angle PMA = \angle PMB$, and since the sum of the two angles is 180°, we have $\angle PMA = 90^\circ$. That is, P is on the right bisector of AB. □

Exercise 1.5.2. *If m is the perpendicular bisector of AB, then A and B are on opposite sides of m. Show that if P is on the same side of m as B, then P is closer to B than to A.*

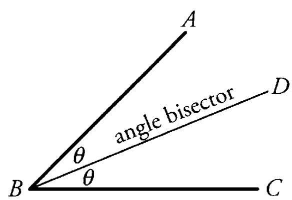
The following exercise follows easily from Pythagoras' Theorem. Try to do it without using Pythagoras' Theorem.

Exercise 1.5.3. *Show that the hypotenuse of a right triangle is its longest side.*

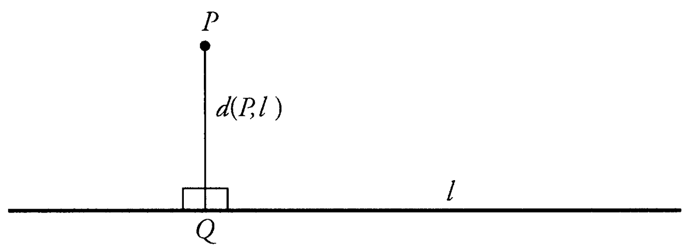
Exercise 1.5.4. *Let l be a line and let P be a point not on l. Let Q be the foot of the perpendicular from P to l. Show that Q is the point on l that is closest to P.*

Exercise 1.5.5. *Let l be a line and let P be a point not on l. Show that there is at most one line through P perpendicular to l.*

Given a nonreflex angle $\angle ABC$, a ray BD such that $\angle ABD = \angle CBD$ is called an ***angle bisector*** of $\angle ABC$.



Given a line l and a point P not on l, the ***distance*** from P to l, denoted $d(P, l)$, is the length of the segment PQ where Q is the foot of the perpendicular from P to l.

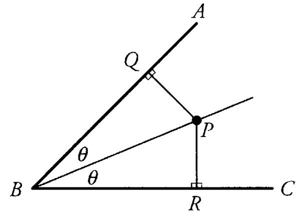


Theorem 1.5.6. *(Characterization of the Angle Bisector)*

The angle bisector of a nonreflex angle consists of all points interior to the angle that are equidistant from the arms of the angle.

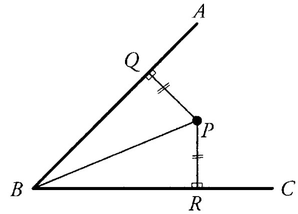
Proof. Let P be a point on the angle bisector. Let Q and R be the feet of the perpendiculars from P to AB and CB, respectively. Triangles PQB and PRB have the side PB in common,

$$\angle PQB = \angle PRB \quad \text{and} \quad \angle PBQ = \angle PBR.$$



Thus, the triangles are congruent by **SAA**, hence $PQ = PR$. Therefore, P is equidistant from AB and CB.

Conversely, let P be a point that is equidistant from BA and BC. Let Q and R be the feet of the perpendiculars from P to AB and CB, respectively, so that $PQ = PR$. Thus, triangles PQB and PRB are congruent by **HSR**, and it follows that



$$\angle PBA = \angle PBQ = \angle PBR = \angle PBC.$$

Hence, P is on the angle bisector of $\angle ABC$. □

Inequalities in Proofs

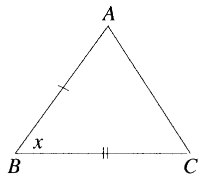
Before turning to construction problems, we list the inequalities that we have used in proofs and add one more to the list.

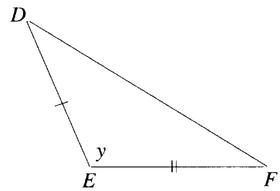
1. Triangle Inequality
2. Exterior Angle Inequality
3. Angle-Side Inequality
4. Open Jaw Inequality

This last inequality is given in the following theorem.

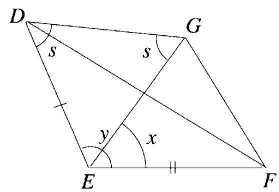
Theorem 1.5.7. *(Open Jaw Inequality)*

Given two triangles $\triangle ABC$ and $\triangle DEF$ with $AB = DE$ and $BC = EF$. Then $\angle ABC < \angle DEF$ if and only if $AC < DF$, as in the figure.





Proof. Suppose that $x < y$. Then we can build x in $\triangle DEF$ so that $EG = AB$. G can be inside or on the triangle. Here, we assume that G is outside $\triangle DEF$, as in the figure below.



Now note that $\triangle ABC \equiv \triangle GEF$ by the **SAS** congruency theorem, and $\triangle EDG$ is isosceles with angles as shown. Also, $s < \angle DGF$, since GE is interior to $\angle G$, and $s > \angle GDF$, since DF is interior to $\angle D$. Therefore,

$$\angle DGF > s > \angle GDF,$$

and by the Angle-Side Inequality, this implies that

$$DF > GF = AC.$$

Now suppose that $AC < DF$. Then exactly one of the following is true:

$$x = y \qquad \text{or} \qquad x > y \qquad \text{or} \qquad x < y$$

(this is called the ***law of trichotomy*** for the real number system).

If $x = y$, then $\triangle ABC \equiv \triangle DEF$ by the **SAS** congruency theorem, which is a contradiction since we are assuming that $AC < DF$.

If $x > y$, then by the first part of the theorem we would have $AC > DF$, which is also a contradiction.

Therefore, the only possibility left is that $x < y$, and we are done.

□

1.6 Construction Problems

Although there are many ways to physically draw a straight line, the image that first comes to mind is a pencil sliding along a ruler. Likewise, the draftsman's compass comes to mind when one thinks of drawing a circle. To most people, the words *straightedge* and *compass* are synonymous with these physical instruments. In geometry, the same words are also used to describe idealized instruments. Unlike their physical counterparts, the geometric straightedge enables us to draw a line of arbitrary length, and the geometric compass allows us to draw arcs and circles of any radius we please. When doing geometry, you should regard the physical straightedge and compass as instruments that mimic the "true" or "idealized" instruments.

There is a reason for dealing with idealized instruments rather than physical ones. Mathematics is motivated by a desire to look at the basic essence of a problem, and to achieve this we have to jettison any unnecessary baggage. For example, we do not want to worry about the problem of the thickness of the pencil line, for this is a drafting problem rather than a geometry problem. However, as we strip away the unnecessary limitations of the draftsman's straightedge and compass, the effect is to create versions of the instruments that behave somewhat differently from their physical counterparts.

The idealized instruments are not "real," nor are the lines and circles that they draw. As a consequence, we cannot appeal to the properties of the physical instruments as verification for whatever we do in geometry. In order to work with idealized instruments, it is important to describe very clearly what they can do. The rules for the abstract instruments closely resemble the properties of the physical ones:

Straightedge Operations

> A straightedge can be used to draw a straight line that passes through two given points.

Compass Operations

> A compass can be used to draw an arc or circle centered at a given point with a given distance as radius. (The given distance is defined by two points.)

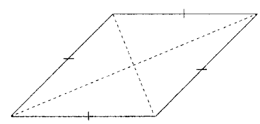
These two statements completely describe how the straightedge and compass operate, and there are no further restrictions, nor any additional properties. For example, the most common physical counterpart of the straightedge is a ruler, and it is a fairly easy matter to place a ruler so that the line to be drawn appears to be tangent to a given physical circle. With the true straightedge, this operation is forbidden. If you wish to draw a tangent line, you must first find two points on the line and then use the straightedge to draw the line through these two points.

A ruler has another property that the straightedge does not. It has a scale that can be used for measuring. A straightedge has no marks on it at all and so cannot be used as a measuring device.

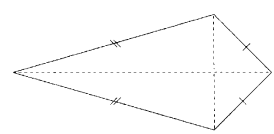
We cannot justify our results by appealing to the physical properties of the instruments. Nevertheless, experimenting with the physical instruments sometimes leads to an understanding of the problem at hand, and if we restrict the physical instruments so that we only use the two operations described above, we are seldom led astray.

Useful Facts in Justifying Constructions

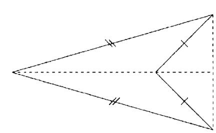
Recall that a ***rhombus*** is a parallelogram whose sides are all congruent, as in the figure on the right.



A ***kite*** is a convex quadrilateral with two pairs of adjacent sides congruent. Note that the diagonals intersect in the interior of the kite.



A ***dart*** is a nonconvex quadrilateral with two pairs of adjacent sides congruent. Note that the diagonals intersect in the exterior of the dart.



Theorem 1.6.1.

(1) The diagonals of a parallelogram bisect each other.

(2) The diagonals of a rhombus bisect each other at right angles.

(3) The diagonals (possibly extended) of a kite or a dart intersect at right angles.

Basic Constructions

The first three basic constructions are left as exercises.

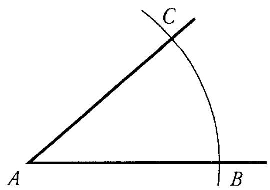
Exercise 1.6.2. *To construct a triangle given two sides and the included angle.*

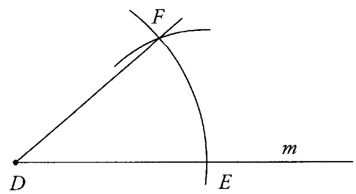
Exercise 1.6.3. *To construct a triangle given two angles and the included side.*

Exercise 1.6.4. *To construct a triangle given three sides.*

Example 1.6.5. *To copy an angle.*

Solution. Given $\angle A$ and a point D, we wish to construct a congruent angle FDE.





Draw a line m through the point D.
With center A, draw an arc cutting the arms of the given angle at B and C.
With center D, draw an arc of the same radius cutting m at E.
With center E and radius BC, draw an arc cutting the previous arc at F.
Then, $\angle FDE \equiv \angle A$.

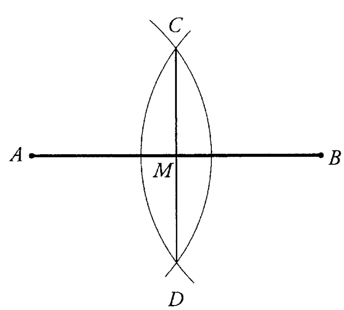
Since triangles BAC and EDF are congruent by **SSS**, $\angle BAC \equiv \angle EDF$.

□

Example 1.6.6. *To construct the right bisector of a segment.*

Solution. Given points A and B, with centers A and B, draw two arcs of the same radius meeting at C and D. Then CD is the right bisector of AB.

To see this, let M be the point where CD meets AB. First, we note that $\triangle ACD \equiv \triangle BCD$ by **SSS**, so $\angle ACD = \angle BCD$. Then in triangles ACM and BCM we have



$$AC = BC,$$
$$\angle ACM = \angle BCM,$$
$$CM \text{ is common,}$$

so $\triangle ACM \equiv \triangle BCM$ by **SAS**. Then

$$AM = BM \quad \text{and} \quad \angle AMC = \angle BMC = 90^\circ,$$

which means that CM is the right bisector of AB.

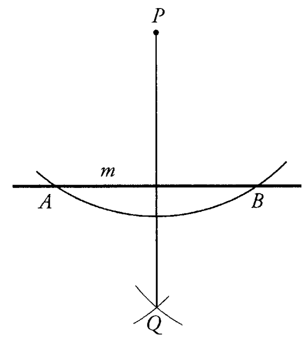
□

Example 1.6.7. *To construct a perpendicular to a line from a point not on the line.*

Solution. Let the point be P and the line be m.

With center P, draw an arc cutting m at A and B. With centers A and B, draw two arcs of the same radius meeting at Q, where $Q \neq P$. Then PQ is perpendicular to m.

Since by construction both P and Q are equidistant from A and B, both are on the right bisector of the segment AB, and hence PQ is perpendicular to m.

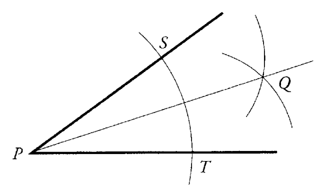


□

Example 1.6.8. *To construct the angle bisector of a given angle.*

Solution. Let P be the vertex of the given angle. With center P, draw an arc cutting the arms of the angle at S and T. With centers S and T, draw arcs of the same radius meeting at Q. Then PQ is the bisector of the given angle.

Since triangles SPQ and TPQ are congruent by **SSS**, $\angle SPQ = \angle TPQ$.



□

Exercise 1.6.9. *To construct a perpendicular to a line from a point on the line.*

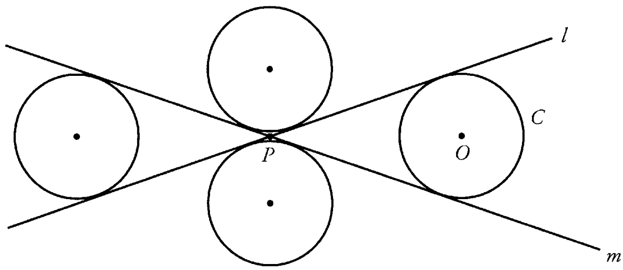
1.6.1 The Method of Loci

The ***locus*** of a point that "moves" according to some condition is the traditional language used to describe the set of points that satisfy a given condition. For example, the locus of a point that is equidistant from two points A and B is the set of all points that are equidistant from A and B — in other words, the right bisector of AB.

The most basic method used to solve geometric construction problems is to locate important points by using the intersection of loci, which is usually referred to as the ***method of loci***. We illustrate with the following:

Example 1.6.10. *Given two intersecting lines l and m and a fixed radius r, construct a circle of radius r that is tangent to the two given lines.*

Solution. It is often useful to sketch the expected solution. We refer to this sketch as an ***analysis figure***. The more accurate the sketch, the more useful the figure. In the analysis figure you should attempt to include all possible solutions. The analysis figure for Example 1.6.10 is as follows, where l and m are the given lines intersecting at P.



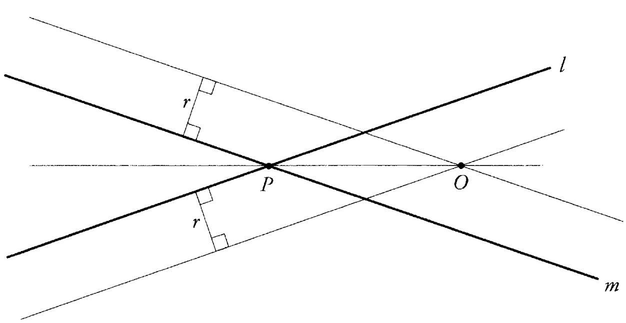
The analysis figure indicates that there are four solutions. The constructions of all four solutions are basically the same, so in this case it suffices to show how to construct one of the four circles.

Since we are given the radius of the circle, it is enough to construct O, the center of circle $\mathcal{C}$. Since we only have a straightedge and a compass, there are only three ways to construct a point, namely, as the intersection of

- two lines,
- two circles, or
- a line and a circle.

The center O of circle $\mathcal{C}$ is equidistant from both l and m and therefore lies on the following constructible loci:

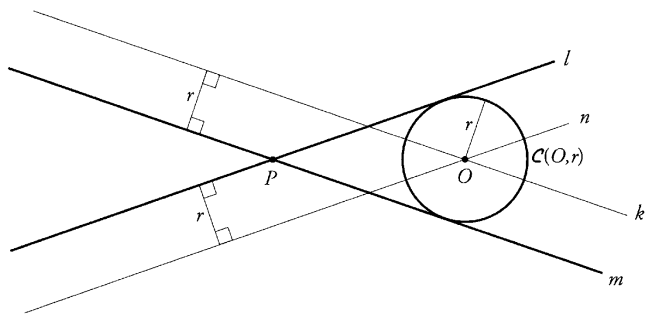
1. an angle bisector,
2. a line parallel to l at distance r from l,
3. a line parallel to m at distance r from m.



Any two of these loci determine the point O.

Having done the analysis, now write up the solution:

1. Construct line n parallel to l at distance r from l.
2. Construct line k parallel to m at distance r from m.
3. Let $O = n \cap k$.
4. With center O and radius r, draw the circle $\mathcal{C}(O, r)$.



□

1.7 Solutions to Selected Exercises

Solution to Exercise 1.2.8

1. *Statement*: Given triangle ABC, if $\angle ABC$ is a right angle, then

$$AB^2 + BC^2 = AC^2.$$

Converse: Given triangle ABC, if

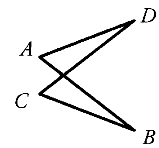
$$AB^2 + BC^2 = AC^2,$$

then $\angle ABC$ is a right angle.

Both the statement and its converse are true.

2. *Statement*: If $ABCD$ is a parallelogram, then $AB = CD$ and $AD = BC$.
   Converse: If $AB = CD$ and $AD = BC$, then $ABCD$ is a parallelogram.

   The statement is true and the converse is false.

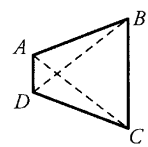
   

3. *Statement*: If $ABCD$ is a convex quadrilateral, then $ABCD$ is a rectangle.
   Converse: If $ABCD$ is a rectangle, then $ABCD$ is a convex quadrilateral.

   The statement is false and the converse is true.

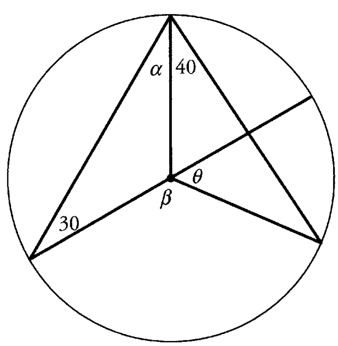
4. *Statement*: Given quadrilateral $ABCD$, if $AC \neq BD$, then $ABCD$ is not a rectangle.
   Converse: Given quadrilateral $ABCD$, if $ABCD$ is not a rectangle, then $AC \neq BD$.

   The statement is true and its converse is false.

Solution to Exercise 1.3.9

In the figure below, we have $\alpha = 30$ (isosceles triangle).



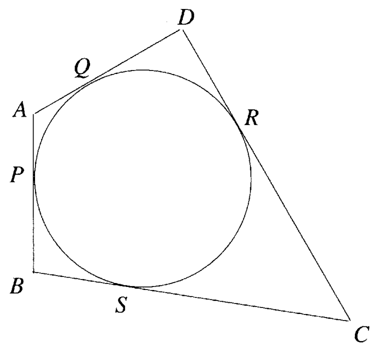
Thus, by Thales' Theorem,

$$\beta = 2(\alpha + 40) = 140,$$

so that $\theta = 180 - \beta = 40$.

Solution to Exercise 1.3.14

Suppose that the quadrilateral $ABCD$ has an inscribed circle that is tangent to the sides at points P, Q, R, and S, as shown in the figure.



Since the tangents to the circle from an external point have the same length, then

$$AP = AQ, \quad PB = BS$$

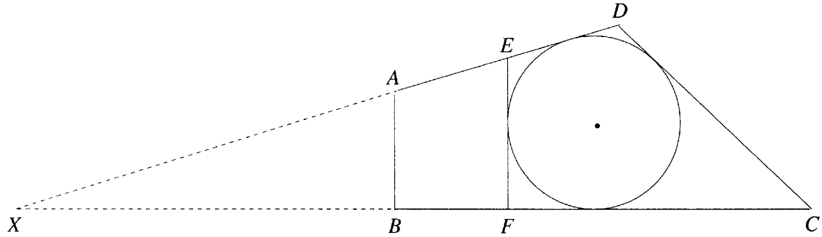
and

$$SC = CR, \quad RD = QD,$$

so that

$$\begin{aligned} AB + CD &= AP + PB + CR + RD \\ &= AQ + BS + SC + QD \\ &= (AQ + QD) + (BS + SC) \\ &= AD + BC. \end{aligned}$$

Conversely, suppose that $AB + CD = AD + BC$, and suppose that the extended sides AD and BC meet at X. Introduce the incircle of $\triangle DXC$, that is, the circle internally tangent to each of the sides of the triangle, and suppose that it is not tangent to AB. Let E and F be on sides AD and BC, respectively, such that EF is parallel to AB and tangent to the incircle, as in the figure.



Note that since $\triangle XEF \sim \triangle XAB$ with proportionality constant $k > 1$, $EF > AB$, and since the quadrilateral $DEFC$ has an inscribed circle, then by the first part of the proof we must have

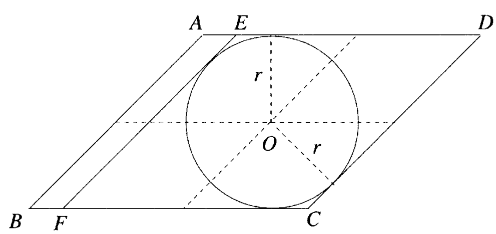
$$AB + CD < EF + CD = DE + CF < AD + BC,$$

which is a contradiction. When the side AB intersects the circle twice, a similar argument also leads to a contradiction. Therefore, if the condition

$$AB + CD = AD + BC$$

holds, then the incircle of $\triangle DXC$ must also be tangent to AB, and $ABCD$ has an inscribed circle.

The case when $ABCD$ is a parallelogram follows in the same way. First, we construct a circle that is tangent to three sides of the parallelogram. The center of the circle must lie on the line parallel to the sides AD and BC and midway between them. Let $2r$ be the perpendicular distance between AD and BC. Then the center must also lie on the line parallel to the side CD and at a perpendicular distance r from CD, as in the figure.



As before, suppose that the circle is not tangent to AB, and let E and F be on sides AD and BC, respectively, such that EF is parallel to AB and tangent to the circle, as in the figure above.

Now, $AB = EF$, and since the quadrilateral $DEFC$ has an inscribed circle, by the first part of the proof we must have

$$AB + CD = EF + CD = DE + CF < AD + BC,$$

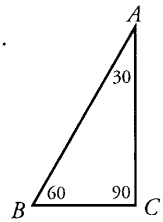
which is a contradiction. When the side AB intersects the circle twice, a similar argument also leads to a contradiction. Therefore, if the condition

$$AB + CD = AD + BC$$

holds, then the circle must also be tangent to AB, and the parallelogram $ABCD$ has an inscribed circle.

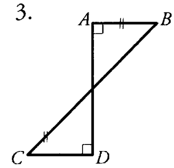
Solution to Exercise 1.4.4

1.



2. The assertion is always false.

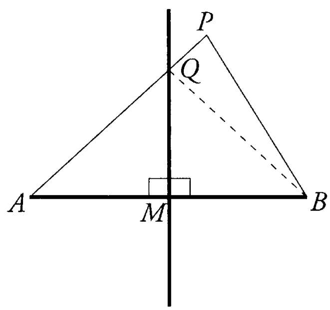
3.



Solution to Exercise 1.5.2

In the figure below, we have

$$PB < PQ + QB = PQ + QA = PA.$$

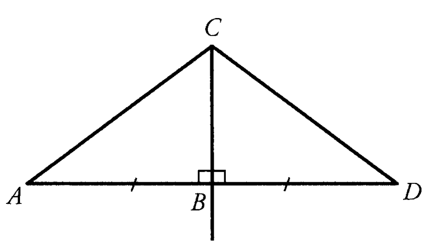


Solution to Exercise 1.5.3

Here are two different solutions.

1. The right angle is the largest angle in the triangle (otherwise the sum of the three angles of the triangle would be larger than $180°$). Since the hypotenuse is opposite the largest angle, it must be the longest side.

2. In the figure below, suppose that B is the right angle.



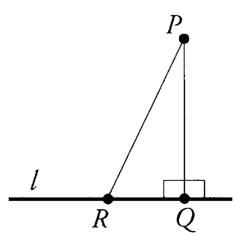
Extend AB beyond B to D so that $BD = AB$. Then C is on the right bisector of AD, so

$$AC = \frac{1}{2}(AC + CD) > \frac{1}{2}AD = AB.$$

This shows that $AC > AB$, and in a similar fashion it can be shown that $AC > CB$.

Solution to Exercise 1.5.4

Let Q be the foot of the perpendicular from P to the line l, and let R be any other point on l. Then $\triangle PQR$ is a right triangle, and by Exercise 1.5.3, $PR > PQ$.



1.8 Problems

1. Prove that the internal and external bisectors of the angles of a triangle are perpendicular.

2. Let P be a point inside $\mathcal{C}(O, r)$ with $P \neq O$. Let Q be the point where the ray $\overrightarrow{OP}$ meets the circle. Use the Triangle Inequality to show that Q is the point on the circle that is closest to P.

3. Let P be a point inside $\triangle ABC$. Use the Triangle Inequality to prove that $AB + BC > AP + PC$.

4. Each of the following statements is true. State the converse of each statement, and if it is false, provide a figure as a counterexample.

   (a) If $\triangle ABC \equiv \triangle DEF$, then $\angle A = \angle D$ and $\angle B = \angle E$.

   (b) If $ABCD$ is a rectangle, then $\angle A = \angle C = 90°$.

   (c) If $ABCD$ is a rectangle, then $\angle A = \angle B = \angle C = 90°$.

5. Given the isosceles triangle ABC with $AB = AC$, let D be the foot of the perpendicular from A to BC. Prove that AD bisects $\angle BAC$.

6. Show that if the perpendicular from A to BC bisects $\angle BAC$, then $\triangle ABC$ is isosceles.

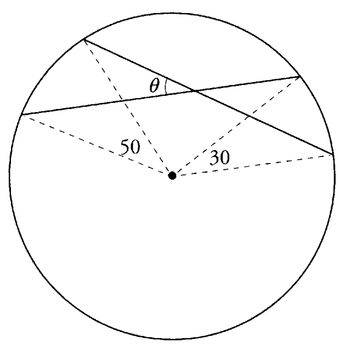
7. D is a point on BC such that AD is the bisector of $\angle A$. Show that

$$\angle ADC = 90 + \frac{\angle B - \angle C}{2}.$$

8. Construct an isosceles triangle ABC, given the unequal angle $\angle A$ and the length of the side BC.

9. Construct a right triangle given the hypotenuse and one side.

10. Calculate θ in the following figure.



(not drawn to scale)

11. Let Q be the foot of the perpendicular from a point P to a line l. Show that Q is the point on l that is closest to P.

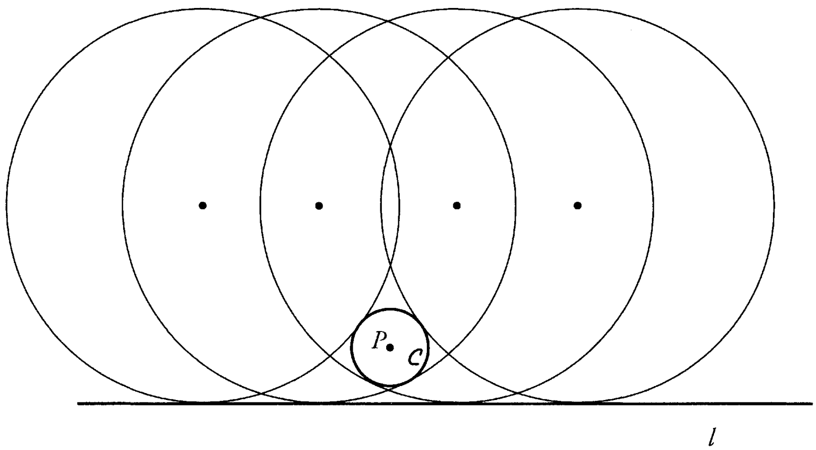
12. Let P be a point inside $\mathcal{C}(O, r)$ with $P \neq O$. Let Q be the point where the ray $\overrightarrow{PO}$ meets the circle. Show that Q is the point of the circle that is farthest from P.

13. Let $ABCD$ be a simple quadrilateral. Show that $ABCD$ is cyclic if and only if the opposite angles sum to $180°$.

14. Draw the locus of a point whose sum of distances from two fixed perpendicular lines is constant.

15. Given a circle $\mathcal{C}(P, s)$, a line l disjoint from $\mathcal{C}(P, s)$, and a radius r $(r > s)$, construct a circle of radius r tangent to both $\mathcal{C}(P, s)$ and l.

Note: The analysis figure indicates that there are four solutions.



CHAPTER 2

CONCURRENCY

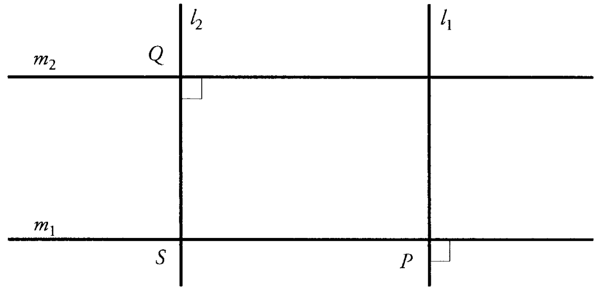
2.1 Perpendicular Bisectors

This chapter is concerned with concurrent lines associated with a triangle. A family of lines is ***concurrent*** at a point P if all members of the family pass through P.

In preparation, we need a few additional facts about parallel lines.

Theorem 2.1.1. *Let l_1 and l_2 be parallel lines, and suppose that m_1 and m_2 are lines with $m_1 \perp l_1$ and $m_2 \perp l_2$. Then m_1 and m_2 are parallel.*

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky



Proof. Let P be the point $l_1 \cap m_1$ and let Q be the point $l_2 \cap m_2$. By the parallel postulate, l_1 is the only line through P parallel to l_2, and so m_1 is not parallel to l_2 and consequently must meet l_2 at some point S. In other words, m_1 is a transversal for the parallel lines l_1 and l_2. Since the sum of the adjacent interior angles is $180°$, it follows that l_2 must be perpendicular to m_1.

But l_2 is also perpendicular to m_2 and so m_1 and m_2 must be parallel.

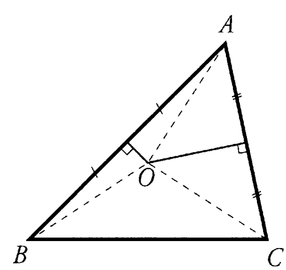
□

In the above proof, if we interchange l_1 and m_1 and l_2 and m_2, we have:

Corollary 2.1.2. *Suppose that $l_1 \perp m_1$ and $l_2 \perp m_2$. Then l_1 and l_2 are parallel if and only if m_1 and m_2 are parallel.*

One of the consequences of this corollary is that if two line segments intersect, then their perpendicular bisectors must also intersect. This fact is crucial in the following theorem, the proof of which also uses the fact that the right bisector of a segment can be characterized as being the set of all points that are equidistant from the endpoints of the segment.

Theorem 2.1.3. *The perpendicular bisectors of the sides of a triangle are concurrent.*



Proof. According to the comments preceding the theorem, the perpendicular bisectors of AB and AC meet at some point O, as shown in the figure.

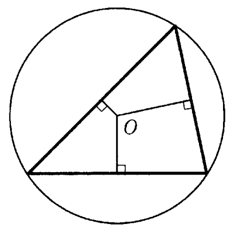
It is enough to show that O lies on the perpendicular bisector of BC. We have

$$OB = OA \text{ (since } O \text{ is on the perpendicular bisector of } AB\text{)},$$
$$OA = OC \text{ (since } O \text{ is on the perpendicular bisector of } AC\text{)}.$$

Therefore $OB = OC$, which means that O is on the perpendicular bisector of BC.

□

The proof shows that the point O is equidistant from the three vertices, so with center O we can circumscribe a circle about the triangle. The circle is called the ***circumcircle***, and the point O is called the ***circumcenter*** of the triangle.



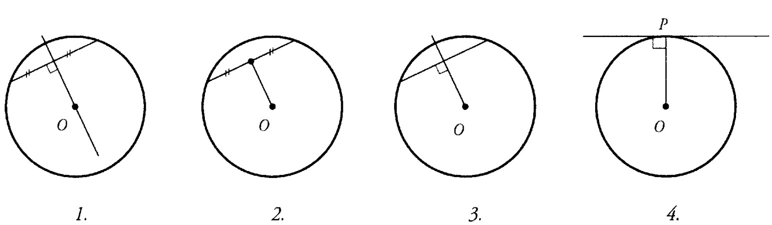
Exercise 2.1.4. *In the figure above, the circumcenter is interior to the triangle. In what sort of a triangle is the circumcenter*

1. *on one of the edges of the triangle?*
2. *outside the triangle?*

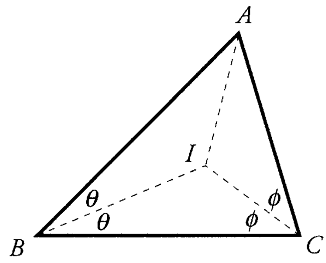
2.2 Angle Bisectors

Theorem 2.2.1. *For a circle $C(O, r)$:*

1. *The right bisector of a chord of $C(O, r)$ passes through O.*
2. *If a given chord is not a diameter, the line joining O to the midpoint of the chord is the right bisector of the chord.*
3. *The line from O that is perpendicular to a given chord is the right bisector of the chord.*
4. *A line is tangent to $C(O, r)$ at a point P if and only if it is perpendicular to the radius OP.*



Theorem 2.2.2. *The internal bisectors of the angles of a triangle are concurrent.*



Proof. In $\triangle ABC$, let the internal bisectors of $\angle B$ and $\angle C$ meet at I, as shown above.

We will show that I lies on the internal bisector of $\angle A$. We have

$$d(I, AB) = d(I, BC),$$

since I lies on the internal bisector of $\angle B$, and

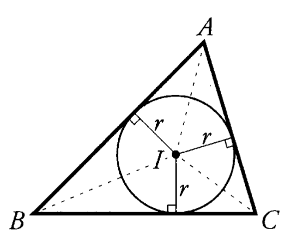
$$d(I, BC) = d(I, AC),$$

since I lies on the internal bisector of $\angle C$.

Hence, $d(I, AB) = d(I, AC)$, which means that I is on the internal bisector of $\angle A$.

□

The proof of Theorem 2.2.2 shows that if we drop the perpendiculars from I to the three sides of the triangle, the lengths of those perpendiculars will all be the same, say r. Then the circle $\mathcal{C}(I, r)$ will be tangent to all three sides by Theorem 2.2.1. That is, I is the center of a circle inscribed in the triangle. This inscribed circle is called the ***incircle*** of the triangle, and I is called the ***incenter*** of the triangle.

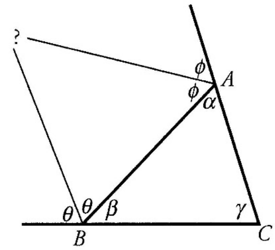


We will next prove a theorem about the external angle bisectors. First, we need to show that two external angle bisectors can never be parallel.

Proposition 2.2.3. *Each pair of external angle bisectors of a triangle intersect.*

Proof. Referring to the diagram,

$$2\theta = \alpha + \gamma,$$
$$2\phi = \beta + \gamma.$$



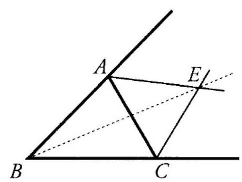
Adding and rearranging:

$$2(\theta + \phi) = 180 + \gamma.$$

Since $\gamma < 180$, it follows that $\theta + \phi < 180$, so the external angle bisectors cannot be parallel.

□

Theorem 2.2.4. *The external bisectors of two of the angles of a triangle and the internal bisector of the third angle are concurrent.*



Proof. Let the two external bisectors meet at E, as shown in the figure. It is enough to show that E lies on the internal bisector of $\angle B$.

Now,

$$d(E, AB) = d(E, AC),$$

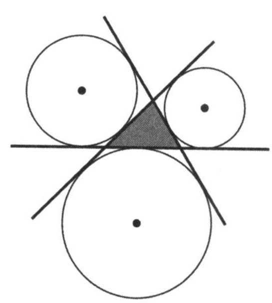
since E lies on the external bisector of $\angle A$, and

$$d(E, AC) = d(E, BC),$$

since E lies on the external bisector of $\angle C$. Hence, $d(E, AB) = d(E, BC)$, which means that E is on the internal bisector of $\angle B$.

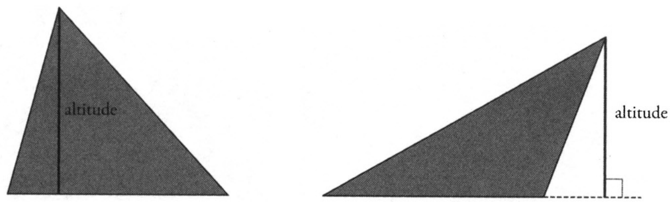
□

The point of concurrency E is called an ***excenter*** of the triangle. Since E is equidistant from all three sides (some extended) of the triangle, we can draw an ***excircle*** tangent to these sides. Every triangle has three excenters and three excircles.



2.3 Altitudes

A line passing through a vertex of a triangle perpendicular to the opposite side is called an ***altitude*** of the triangle.



To prove that the altitudes of a triangle are concurrent, we need some facts about parallelograms.

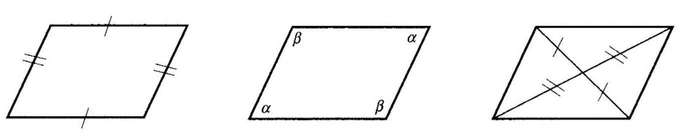
A ***parallelogram*** is a quadrilateral whose opposite sides are parallel. A parallelogram whose sides are equal in length is called a ***rhombus***. Squares and rectangles are special types of parallelograms.

Theorem 2.3.1. *In a parallelogram:*

(1) Opposite sides are congruent.

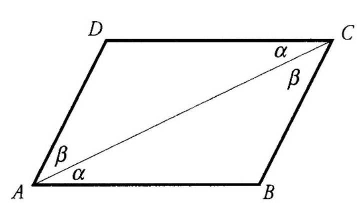
(2) Opposite angles are congruent.

(3) The diagonals bisect each other.



Proof. We will prove (1) and (2). Given parallelogram $ABCD$, as in the figure below, we will show that $\triangle ABC \equiv \triangle CDA$.

Diagonal AC is a transversal for parallel lines AB and CD, and so $\angle BAC$ and $\angle DCA$ are opposite interior angles for the parallel lines. So we have



$$\angle BAC = \angle DCA.$$

Treating AC as a transversal for AD and CB, we have

$$\angle BCA = \angle DAC.$$

Since AC is common to $\triangle BAC$ and $\triangle DCA$, **ASA** implies that they are congruent. Consequently, $AB = CD$ and $BC = DA$, proving (1).

Also, $\angle BAD = \alpha + \beta = \angle BCD$ and $\angle ADC = \angle ABC$ because triangles ADC and ABC are congruent, proving (2).

□

Exercise 2.3.2. *Prove statement (3) of Theorem 2.3.1.*

Theorem 2.3.3. *If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.*

Exercise 2.3.4. *Prove Theorem 2.3.3.*

Theorem 2.3.3 tells us that statement (3) of Theorem 2.3.1 will guarantee that the quadrilateral is a parallelogram. However, neither statement (1) nor statement (2) of Theorem 2.3.1 is by itself enough to guarantee that a quadrilateral is a parallelogram. For example, a nonsimple quadrilateral whose opposite sides are congruent is not a parallelogram. However, if the quadrilateral is a simple polygon, then either statement (1) or (2) is sufficient. As well, there is another useful condition that can help determine if a simple quadrilateral is a parallelogram:

Theorem 2.3.5. *A simple quadrilateral is a parallelogram if any of the following statements are true:*

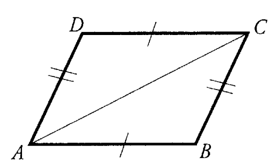
(1) Opposite sides are congruent.

(2) Opposite angles are congruent.

(3) One pair of opposite sides is congruent and parallel.

Proof. We will justify case (1), leaving the others to the reader.

Suppose that $ABCD$ is the quadrilateral, as in the figure on the right. Since $ABCD$ is simple, we may suppose that the diagonal AC is interior to the quadrilateral. Since the opposite sides of the quadrilateral are congruent, the **SSS** congruency condition implies that $\triangle ABC \equiv \triangle CDA$.



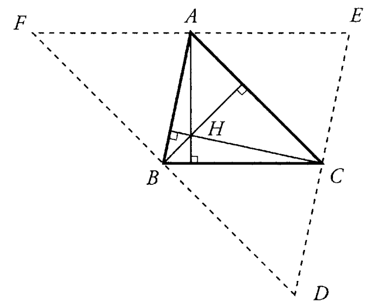
This in turn implies that the alternate interior angles $\angle BAC$ and $\angle DCA$ are congruent and also that the alternate interior angles $\angle BCA$ and $\angle DAC$ are congruent. The fact that the edges are parallel now follows from the well-known facts about parallel lines.

□

The next theorem uses a clever trick. It embeds the given triangle in a larger one in such a way that the altitudes of the given triangle are right bisectors of the sides of the larger one.

Theorem 2.3.6. *The altitudes of a triangle are concurrent.*

Proof. Given $\triangle ABC$, we embed it in a larger triangle $\triangle DEF$ by drawing lines through the vertices of $\triangle ABC$ that are parallel to the opposite sides so that $ABCE$, $ACBF$, and $CABD$ are parallelograms, as in the diagram on the right. Clearly, A, B, and C are midpoints of the sides of $\triangle DEF$, and an altitude of $\triangle ABC$ is a perpendicular bisector of a side of $\triangle DEF$. However, since the perpendicular bisectors of $\triangle DEF$ are concurrent, so are the altitudes of $\triangle ABC$.



□

The point of concurrency of the altitudes is called the ***orthocenter*** of the triangle, and it is usually denoted by the letter H. In the proof, the orthocenter of $\triangle ABC$ is the circumcenter of $\triangle DEF$.

Note that the orthocenter can lie outside the triangle (for obtuse-angled triangles) or on the triangle (for right-angled triangles). The same proof also works for these types of triangles.

Exercise 2.3.7. *The figure above shows a triangle whose orthocenter is interior to the triangle. Give examples of triangles where:*

1. *The orthocenter is on a side of the triangle.*
2. *The orthocenter is exterior to the triangle.*

2.4 Medians

A ***median*** of a triangle is a line passing through a vertex and the midpoint of the opposite side.

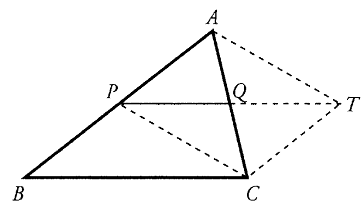
Exercise 2.4.1. *Show that in an equilateral triangle ABC the following are all the same:*

1. *The perpendicular bisector of BC.*
2. *The bisector of $\angle A$.*
3. *The altitude from vertex A.*
4. *The median passing through vertex A.*

The next theorem, which is useful on many occasions, is also proved by using the properties of a parallelogram.

Theorem 2.4.2. *(The Midline Theorem)*

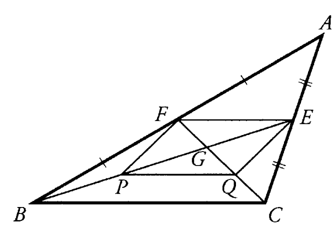
If P and Q are the respective midpoints of sides AB and AC of triangle ABC, then PQ is parallel to BC and $PQ = BC/2$.



Proof. First, extend PQ to T so that $PQ = QT$, as in the figure. Then $ATCP$ is a parallelogram, because the diagonals bisect each other. This means that TC is parallel to and congruent to AP. Since P is the midpoint of AB, it follows that TC is parallel to and congruent to BP, and so $TCBP$ is also a parallelogram. From this we can conclude that PT is parallel to and congruent to BC; that is, PQ is parallel to BC and half the length of BC.

□

Lemma 2.4.3. *Any two medians of a triangle trisect each other at their point of intersection.*



Proof. Let the medians BE and CF intersect at G, as shown in the figure. Draw FE. By the Midline Theorem, FE is parallel to BC and half its length.

Let P be the midpoint of GB and let Q be the midpoint of GC. Then, again by the Midline Theorem, PQ is parallel to BC and half its length. Since FE and PQ are parallel and equal in length, $EFPQ$ is a parallelogram, and so the diagonals of $EFPQ$ bisect each other. It follows that $FG = GQ = QC$ and $EG = GP = PB$, which proves the lemma.

□

There are two different points that trisect a given line segment. The point of intersection of the two medians is the trisection point of each that is farthest from the vertex.

Theorem 2.4.4. *The medians of a triangle are concurrent.*

Proof. Let the medians be AD, BE, and CF. Then BE and CF meet at a point G for which $EG/EB = 1/3$. Also, AD and BE meet at a point G' for which $EG'/EB = 1/3$. Since both G and G' are between B and E, we must have $G = G'$, which means that the three medians are concurrent.

□

The point of concurrency of the three medians is called the ***centroid***. The centroid always lies inside the triangle. A thin triangular plate can be balanced at its centroid on the point of a needle, so physically the centroid corresponds to the center of gravity.

The partial converses of the Midline Theorem are useful:

Theorem 2.4.5. *Let P be the midpoint of side AB of triangle ABC, and let Q be a point on AC such that PQ is parallel to BC. Then Q is the midpoint of AC.*

Proof. Let Q' be the midpoint of AC, then $PQ' \parallel BC$. Since there is only one line through P parallel to BC, the lines PQ and PQ' must be the same, so the points Q and Q' are also the same.

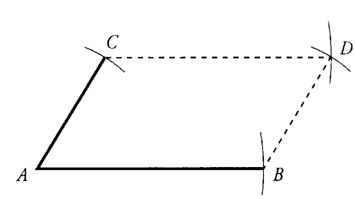
□

2.5 Construction Problems

The two basic construction problems associated with parallel lines are constructing a parallelogram given two of its adjacent edges (that is, completing a parallelogram) and constructing a line through a given point parallel to a given line. Once we have solved the first problem, the second one is straightforward.

Example 2.5.1. *To construct a parallelogram given two adjacent edges.*

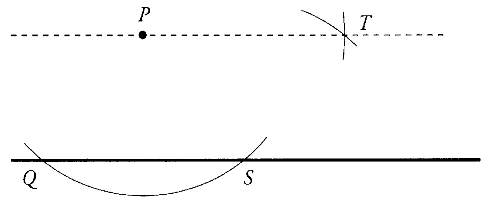
Solution. Given edges AB and AC, with center B and radius AC, draw an arc. With center C and radius AB, draw a second arc cutting the first at D on the same side of AC as B. Then $ABDC$ is the desired parallelogram.



□

Example 2.5.2. *To construct a line parallel to a given line through a point not on the line.*

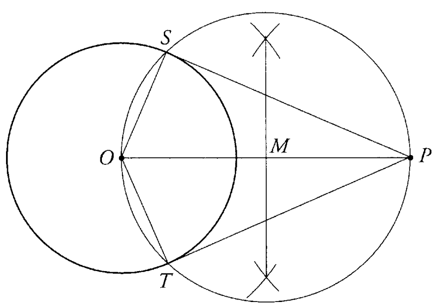
Solution. With center P, draw an arc cutting the given line at Q and S. With the same radius and center S, draw a second arc. With center P and radius QS, draw a third arc cutting the second at T. Then PT is the desired line, because $PQST$ is a parallelogram.



□

Example 2.5.3. *Given a circle with center O and a point P outside the circle, construct the lines through P tangent to the circle.*

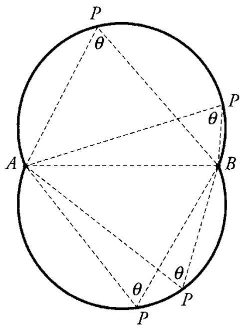
Solution. Draw the line OP and construct the right bisector of OP, obtaining the midpoint M of OP. With center M and radius MP, draw a circle cutting the given circle at S and T. Then PS and PT are the desired tangent lines.



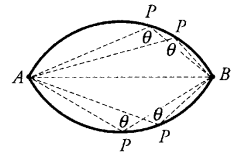
Note that angles OSP and OTP are angles in a semicircle, so by Thales' Theorem both are right angles. Thus, PS and PT are perpendicular to radii OS and OT, respectively, and so must be tangent lines by Theorem 2.2.1. □

Thales' Locus

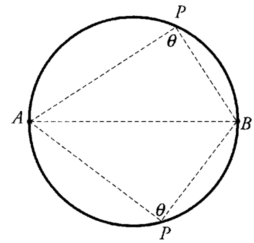
Given a segment AB and an angle θ, the set of all points P such that $\angle APB = \theta$ forms the union of two arcs of a circle, which we shall call ***Thales' Locus***, shown in the figure on the following page.



θ acute



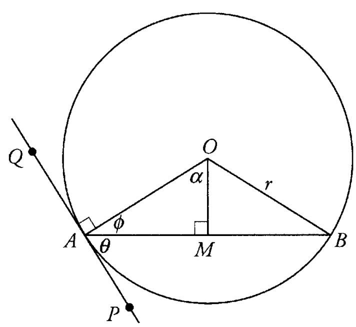
θ obtuse



$\theta = 90$

To help us construct Thales' Locus, we need the following useful theorem.

Theorem 2.5.4. *Let AB be a chord of the circle $\mathcal{C}(O, r)$ and let P be a point on the line tangent to the circle, as shown in the diagram below. Then $\angle AOB = 2\angle PAB$.*



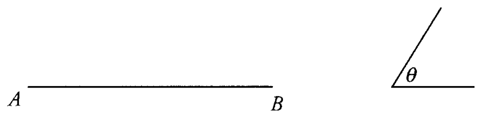
Proof. Referring to the diagram, drop the perpendicular OM from O to AB. Then $\angle AOB = 2\angle AOM = 2\alpha$. Now, $\theta = 90 - \phi = \alpha$, and it follows that $\angle AOB = 2\angle PAB$.

□

Note. In a similar way, it can be shown that the reflex angle AOB is twice the size of $\angle QAB$.

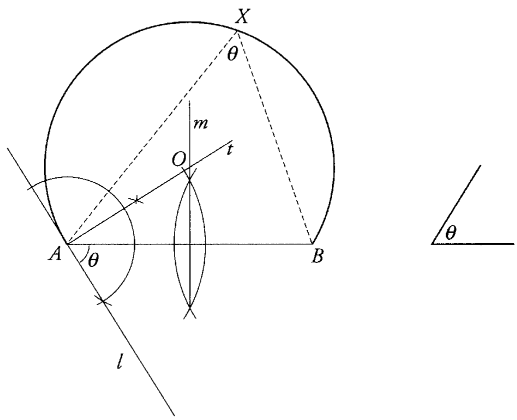
Example 2.5.5. *Given a segment AB and an angle $\theta < 90°$, construct Thales' Locus for the given data:*

Data:



Solution. Here is one method of doing this construction.

1. Copy angle θ to point A so that AB is one arm of the angle, and l is the line containing the other arm.
2. Construct the right bisector m of AB.
3. Construct the line t through A perpendicular to l. Let O be the point $t \cap m$.
4. Draw $\mathcal{C}(O, OA)$. Then one of the arcs determined by $\mathcal{C}(O, OA)$ is part of Thales' Locus for the given data.



By Theorem 2.5.4, $\angle AOB = 2\theta$, so $\angle AXB$ at the circumference is of size θ.

□

Example 2.5.6. *Construct a triangle ABC given the size θ of $\angle A$, the vertices B and C, and the length of the altitude h from A.*

Solution. We give a quick outline of the solution.

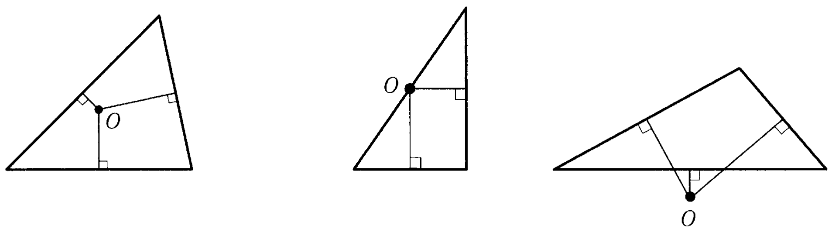
1. Construct Thales' Locus for BC and θ.
2. Construct the perpendicular BX to BC so that $BX = h$.
3. Through X, draw the line l parallel to BC cutting the locus at a point A.

Then ABC is the desired triangle.

□

2.6 Solutions to the Exercises

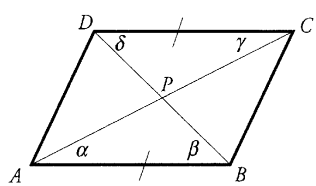
Solution to Exercise 2.1.4



In an acute-angled triangle, the circumcenter is always interior to the triangle.

1. In a right triangle, the circumcenter is always on the hypotenuse.
2. In an obtuse-angled triangle, the circumcenter is always outside the triangle.

Solution to Exercise 2.3.2



In triangles ABP and CDP we have

$$\alpha = \gamma,$$
$$\beta = \delta,$$

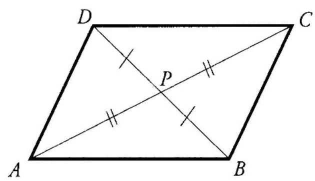
because $AB \parallel CD$, and

$$AB = CD,$$

by statement (1) of Theorem 2.3.1.

So, $\triangle ABP \equiv \triangle CDP$ by **ASA**, and it follows that $AP = CP$ and $BP = DP$.

Solution to Exercise 2.3.4



In triangles ABP and CDP we have

$$AP = CP,$$
$$\angle APB = \angle CPD$$

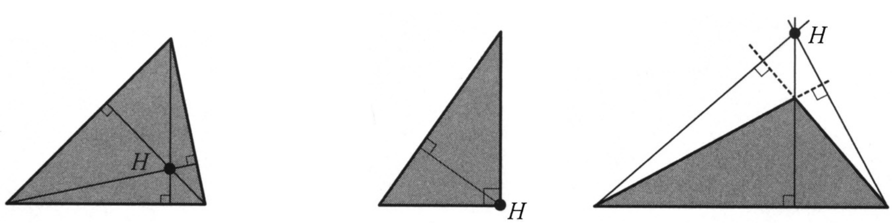
since they are vertically opposite angles, and

$$BP = DP.$$

So triangles ABP and CDP are congruent, and thus $\angle PAB = \angle PCD$, which implies that $AB \parallel CD$.

Similarly, $AD \parallel CB$, which shows that $ABCD$ is a parallelogram.

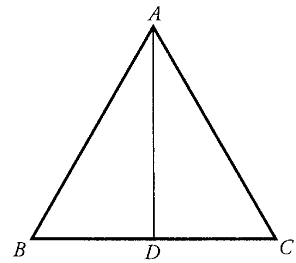
Solution to Exercise 2.3.7



In an acute-angled triangle, the orthocenter is always interior to the triangle.

1. In a right triangle, the orthocenter is always the vertex of the right angle.
2. In an obtuse-angled triangle, the orthocenter is always outside the triangle.

Solution to Exercise 2.4.1



We will show that the line through the vertex A and the midpoint D of BC is simultaneously the perpendicular bisector of BC, the bisector of $\angle A$, the altitude from A, and the median from A.

1. Since $AB = AC$, the point A is equidistant from B and C and so A is on the right bisector of BC. It follows that AD is the right bisector of BC.
2. Triangles ADB and ADC are congruent by **SSS**, so $\angle DAB = \angle DAC$; that is, AD is the bisector of $\angle A$.
3. Since $AD \perp BC$, by statement 1 above, AD is an altitude of $\triangle ABC$.
4. Since D is the midpoint of BC, AD is a median of $\triangle ABC$.

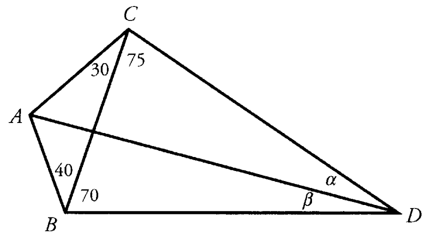
2.7 Problems

1. BE and CF are altitudes of $\triangle ABC$, and M is the midpoint of BC. Show that $ME \equiv MF$.
2. BE and CF are altitudes of $\triangle ABC$, and EF is parallel to BC. Prove that $\triangle ABC$ is isosceles.
3. The perpendicular bisector of side BC of $\triangle ABC$ meets the circumcircle at D on the opposite side of BC from A. Prove that AD bisects $\angle BAC$.
4. Given $\triangle ABC$ with incenter I, prove that

$$\angle BIC = 90 + \tfrac{1}{2}\angle BAC.$$

Note. This is an important property of the incenter that will prove useful later.

5. In the given figure, calculate the sizes of the angles marked α and β.

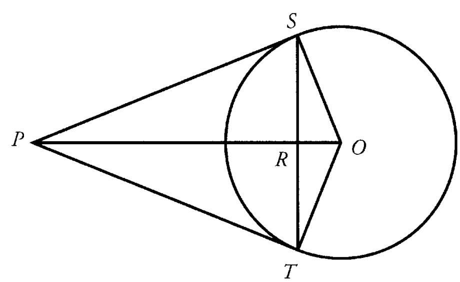


6. In $\triangle ABC$, $\angle BAC = 100°$ and $\angle ABC = 50°$. AD is an altitude and BE is a median. Find $\angle CDE$.

7. Segments PS and PT are tangent to the circle at S and T. Show that

   (a) $PS \equiv PT$ and

   (b) $ST \perp OP$.



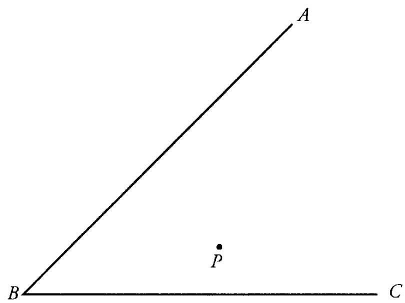
8. Construct $\triangle ABC$ given the side BC and the lengths h_b and h_c of the altitudes from B and C, respectively.

9. Construct $\triangle ABC$ given the side BC, the length h_b of the altitude from B, and the length m_a of the median from A.

10. Construct $\triangle ABC$ given the length of the altitude h from A and the length of the sides b and c. Here, b is the side opposite $\angle B$, and c is the side opposite $\angle C$.

11. Construct triangle ABC given BC, an angle β congruent to $\angle B$, and the length t of the median from B.

12. Given a point P inside angle ABC as shown below, construct a segment XY with endpoints in AB and CB such that P is the midpoint of XY.



13. Given segments AB and CD, which meet at a point P off the page, construct the bisector of $\angle P$. All constructions must take place within the page.

14. Let AB be a diameter of a circle. Show that the points where the right bisector of AB meet the circle are the points of the circle that are farthest from AB.

15. M is the midpoint of the chord AB of a circle $\mathcal{C}(O, r)$. Show that if a different chord CD contains M, then $AB < CD$. (You may use Pythagoras' Theorem.)

16. $ABCD$ is a nonsimple quadrilateral. P, Q, R, and S are the midpoints of AB, BC, CD, and DA, respectively. Show that $PQRS$ is a parallelogram.

17. A ***regular polygon*** is one in which all sides are equal and all angles are equal. Show that the vertices of a regular convex polygon lie on a common circle.

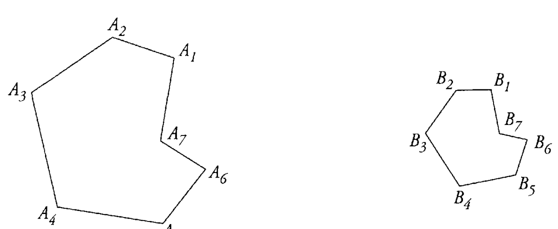
CHAPTER 3

SIMILARITY

3.1 Similar Triangles

The word *similar* is used in geometry to describe two figures that have identical shapes but are not necessarily the same size. A working definition of similarity can be obtained in terms of angles and ratios of distances.

Two polygons are ***similar*** if corresponding angles are congruent and the ratios of corresponding sides are equal.



Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

Notation. We use the symbol $\sim$ to denote similarity. Thus, we write

$$ABCDE \sim QRSTU$$

to denote that the polygons $ABCDE$ and $QRSTU$ are similar. As with congruency, the order of the letters is important.

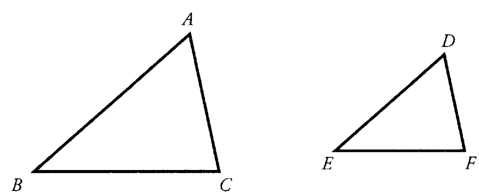
To say that $\triangle ABC \sim \triangle DEF$ means that

$$\angle A \equiv \angle D, \qquad \angle B \equiv \angle E, \qquad \angle C \equiv \angle F,$$

and

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = k\,,$$

where k is a positive real number.



The constant k is called the ***proportionality constant*** or the ***magnification factor***. If $k > 1$, triangle ABC is larger than triangle DEF; if $0 < k < 1$, triangle ABC is smaller than triangle DEF; and if $k = 1$, the triangles are congruent.

Note that congruent figures are necessarily similar, but similar figures do not need to be congruent.

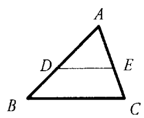
3.2 Parallel Lines and Similarity

There is a close relationship between parallel lines and similarity.

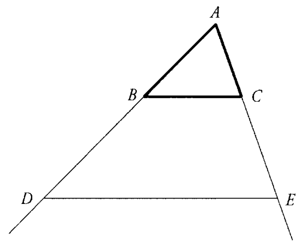
Lemma 3.2.1. *In $\triangle ABC$, suppose that D and E are points of AB and AC, respectively, and that DE is parallel to BC. Then*

$$\frac{AD}{DB} = \frac{AE}{EC}\,.$$

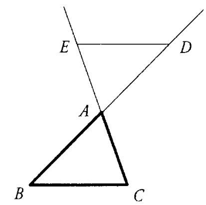
Figures (i), (ii), and (iii) on the following page illustrate the three different possibilities that can occur.



(i)



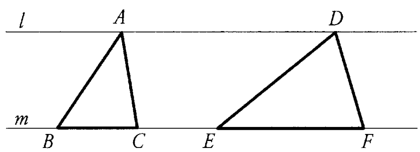
(ii)



(iii)

The proof of this lemma uses some simple facts about areas:

Theorem 3.2.2. *Given parallel lines l and m and two triangles each with its base on one line and its remaining vertex on the other, the ratio of the areas of the triangles is the ratio of the lengths of their bases.*

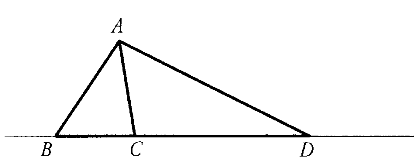


For the diagram above, the theorem says that

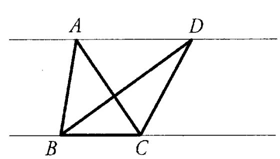
$$\frac{[ABC]}{[DEF]} = \frac{BC}{EF}.$$

Note that we use square brackets to denote area; that is, $[XYZ]$ is the area of $\triangle XYZ$.

There are two special cases worth mentioning, and these are illustrated by the figures below:



$$\frac{[ABC]}{[ACD]} = \frac{BC}{CD}$$

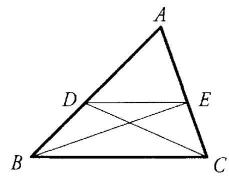


$$[ABC] = [DBC]$$

We sketch the proof of Lemma 3.2.1 for case (i). The proof of the other cases is very much the same.

Proof. In case (i), the line DE enters the triangle ABC through side AB and so must exit either through vertex C or through one of the other two sides.[1] Since DE is parallel to BC, the line DE cannot pass through vertex C or any other point on side BC. It follows that DE must exit the triangle through side AC.

Insert segments BE and CD and use the previously cited facts about areas of triangles to get



$$\frac{[ADE]}{[BDE]} = \frac{AD}{BD} \qquad \text{and} \qquad \frac{[ADE]}{[CDE]} = \frac{AE}{CE}.$$

Since $[BDE] = [CDE]$, we must have

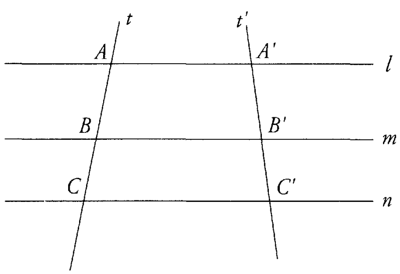
$$\frac{AD}{DB} = \frac{AE}{EC}.$$

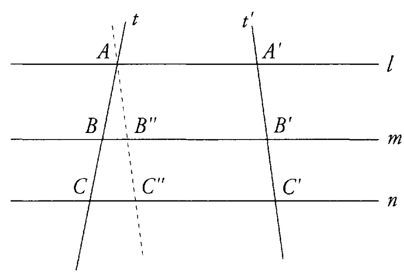
□

A useful extension of the lemma is the following:

Theorem 3.2.3. *Parallel projections preserve ratios. Suppose that l, m, and n are parallel lines that are met by transversals t and t' at points A, B, C and A', B', C', respectively. Then*

$$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$





Proof. Draw a line through A parallel to t' meeting m at B'' and n at C''. Then $AA'B'B''$ and $B''B'C'C''$ are parallelograms, so $AB'' = A'B'$ and $B''C'' = B'C'$. Applying Lemma 3.2.1 to triangle ACC'', we have

$$\frac{AB}{BC} = \frac{AB''}{B''C''},$$

and the theorem follows. □

[1]This statement is known as Pasch's Axiom.

Next is the basic similarity theorem for triangles:

Theorem 3.2.4. *In $\triangle ABC$, suppose that DE is parallel to BC. If D and E are points of AB and AC, with D being neither A nor B nor C, then*

$$\triangle ABC \sim \triangle ADE.$$

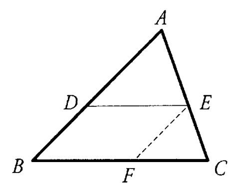
Proof. As in the case of Lemma 3.2.1, there are three cases to consider:

(1) D is between A and B.

(2) D is on the ray $\overrightarrow{AB}$ beyond B.

(3) D is on the ray $\overrightarrow{BA}$ beyond A.

We will prove the theorem for case (1). The proofs for the other cases are almost identical. Because of the properties of parallel lines, the corresponding angles of ABC and ADE are equal, so it remains to show that the ratios of the corresponding sides are equal.



By Lemma 3.2.1, we have

$$\frac{AD}{DB} = \frac{AE}{EC},$$

which implies that

$$\frac{DB}{AD} = \frac{EC}{AE},$$

and this in turn implies that

$$\frac{DB}{AD} + 1 = \frac{EC}{AE} + 1,$$

so that

$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE},$$

and therefore

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

Now, through E, draw EF parallel to AB, with F on BC. Using an argument similar to the one above, we get

$$\frac{AC}{AE} = \frac{BC}{BF}.$$

Since $DE = BF$ (because $BDEF$ is a parallelogram), we get

$$\frac{AC}{AE} = \frac{BC}{DE}.$$

Thus,

$$\frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE},$$

which completes the proof that triangles ABC and ADE are similar.

□

There are other conditions that allow us to conclude that two triangles are similar, but there are many occasions where Theorem 3.2.4 is the most appropriate one to use.

Congruency versus Similarity

Congruency and similarity are both ***equivalence relations*** — both relations are reflexive, symmetric, and transitive.

Both are ***reflexive***:
$\triangle ABC \equiv \triangle ABC.$
$\triangle ABC \sim \triangle ABC.$

Both are ***symmetric***:
If $\triangle ABC \equiv \triangle DEF$, then $\triangle DEF \equiv \triangle ABC.$
If $\triangle ABC \sim \triangle DEF$, then $\triangle DEF \sim \triangle ABC.$

Both are ***transitive***:
If $\triangle ABC \equiv \triangle DEF$ and $\triangle DEF \equiv \triangle GHI$, then $\triangle ABC \equiv \triangle GHI.$
If $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle GHI$, then $\triangle ABC \sim \triangle GHI.$

3.3 Other Conditions Implying Similarity

According to the definition, two triangles are similar if and only if the three angles are congruent and the three ratios of the corresponding sides are equal. As with congruent triangles, it is not necessary to check all six items. Here are some of the conditions that will allow us to verify that triangles are similar without checking all six.

Theorem 3.3.1. *(***AAA** *or Angle-Angle-Angle Similarity)*

Two triangles are similar if and only if all three corresponding angles are congruent.

Exercise 3.3.2. *Show that two quadrilaterals need not be similar even if all their corresponding angles are congruent.*

Theorem 3.3.3. *(***sAs** *or side-Angle-side Similarity)*

If in $\triangle ABC$ and $\triangle DEF$ we have

$$\frac{AB}{DE} = \frac{AC}{DF} \quad \text{and} \quad \angle A \equiv \angle D,$$

then $\triangle ABC$ and $\triangle DEF$ are similar.

Theorem 3.3.4. *(***sss** *or side-side-side Similarity)*

Triangles ABC and DEF are similar if and only if

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

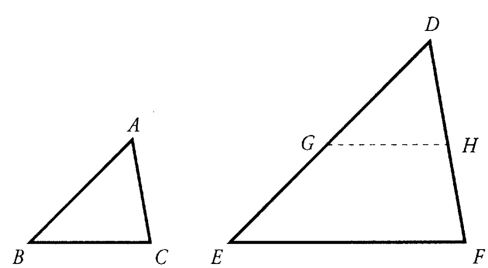
The lowercase letter **s** in **sAs** and **sss** is to remind us that the sides need only be *in proportion* rather than congruent, while the uppercase letter **A** is to remind us that the angles must be *congruent.*

It is worth mentioning that since there are 180° in a triangle, the **AAA** similarity condition is equivalent to:

Theorem 3.3.5. *(***AA** *Similarity)*

Two triangles are similar if and only if two of the three corresponding angles are congruent.

Proof of the **AAA** *similarity condition.* Make a congruent copy of one triangle so that it shares an angle with the other, that is, so that two sides of the one triangle fall upon two sides of the other. The congruency of the angles then guarantees that the third sides are parallel, and the proof is completed by applying Theorem 3.2.4.



□

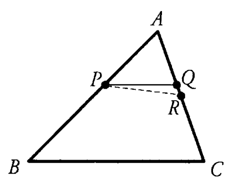
The proof for the **sAs** similarity condition uses the converse of Lemma 3.2.1.

Lemma 3.3.6. *Let P and Q be points on AB and AC with P between A and B and Q between A and C. If*

$$\frac{AP}{PB} = \frac{AQ}{QC},$$

then PQ is parallel to BC.

Proof. Through P, draw a line PR parallel to BC with R on AC. Then, since PR enters triangle ABC through side AB, it must exit the triangle through side AC or side BC. Since PR is parallel to BC, it follows that R is between A and C.



By Lemma 3.2.1,

$$\frac{AP}{PB} = \frac{AR}{RC},$$

so it follows that

$$\frac{AR}{RC} = \frac{AQ}{QC} = k\,.$$

However, given the positive number k, there is only one point X between A and C such that

$$\frac{AX}{XC} = k,$$

and so it follows that $R = Q$, and so $PQ = PR$, showing that PQ is parallel to BC.

□

Proof of the **sAs** *similarity condition.*

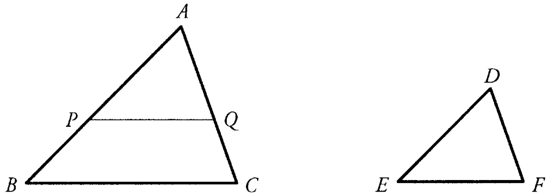
Suppose that in triangles ABC and DEF we have $\angle A \equiv \angle D$ and that

$$\frac{AB}{DE} = \frac{AC}{DF} = k\,.$$

We may assume that $k > 1$, that is, that $AB > DE$ and $AC > DF$. Cut off a segment AP on AB so that $AP = DE$. Cut off a segment AQ on AC so that $AQ = DF$.

Then it follows that

$$\frac{AB}{AP} = \frac{AC}{AQ},$$



and subtracting 1 from both sides of the equation gives us

$$\frac{PB}{AP} = \frac{QC}{AQ}.$$

It now follows from Lemma 3.3.6 that PQ is parallel to BC, and Theorem 3.2.4 implies that $\triangle ABC \sim \triangle APQ$. Since $\triangle APQ \equiv \triangle DEF$ (by the **SAS** congruency condition), it follows that $\triangle ABC \sim \triangle DEF$.

□

Exercise 3.3.7. *Prove that two triangles are similar if they satisfy the* **sss** *similarity condition.*

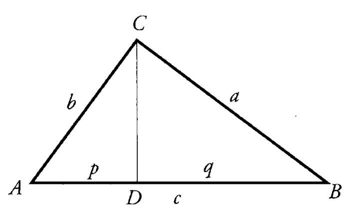
3.4 Examples

Pythagoras' Theorem

Theorem 3.4.1. *(Pythagoras' Theorem)*

If two sides of a right triangle have lengths a and b and the hypotenuse has length c, then $a^2 + b^2 = c^2$.

Proof. In the figure below, drop the perpendicular CD to the hypotenuse, and let $AD = p$ and $BD = q$.



From the **AA** similarity condition,

$$\triangle CBD \sim \triangle ABC \qquad \text{and} \qquad \triangle ACD \sim \triangle ABC,$$

implying that

$$\frac{a}{q} = \frac{c}{a} \qquad \text{and} \qquad \frac{b}{p} = \frac{c}{b},$$

or, equivalently, that

$$a^2 = cq \qquad \text{and} \qquad b^2 = cp\,,$$

from which it follows that

$$a^2 + b^2 = cq + cp = c(q + p)\,,$$

and since $q + p = c$ we have $a^2 + b^2 = c^2$. □

Theorem 3.4.2. *(Converse of Pythagoras' Theorem)*

Let ABC be a triangle. If

$$AB^2 = BC^2 + CA^2,$$

then $\angle C$ is a right angle.

Exercise 3.4.3. *Prove the converse of Pythagoras' Theorem.*

Apollonius' Theorem

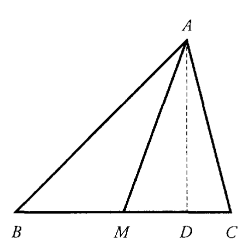
As an application of Pythagoras' Theorem, we prove the following result.

Theorem 3.4.4. *(Apollonius' Theorem)*

Let M be the midpoint of the side BC of triangle ABC. Then

$$AB^2 + AC^2 = 2AM^2 + 2BM^2.$$

Proof. Let AD be the altitude on the base BC, and assume that D lies between M and C.



By Pythagoras' Theorem,

$$\begin{aligned} AB^2 + AC^2 &= BD^2 + 2AD^2 + CD^2 \\ &= (BM + MD)^2 + (BM - MD)^2 + 2AD^2 \\ &= 2BM^2 + 2MD^2 + 2AD^2 \\ &= 2BM^2 + 2AM^2. \end{aligned}$$

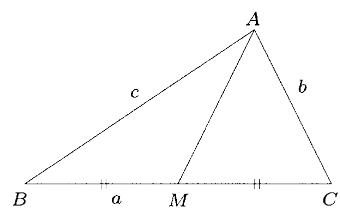
For other positions of D, the argument is essentially the same.

□

Example 3.4.5. *Use Apollonius' Theorem to prove that in $\triangle ABC$, if $m_a = AM$ is the median from the vertex A, then*

$$m_a = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2}$$

where $a = BC$, $b = AC$, and $c = AB$, as in the figure.



Solution. From Apollonius' Theorem, we have

$$AB^2 + AC^2 = 2BM^2 + 2AM^2,$$

that is,

$$c^2 + b^2 = 2(a/2)^2 + 2m_a^2,$$

so that

$$4m_a^2 = 2(b^2 + c^2) - a^2.$$

Therefore,

$$m_a = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2}.$$

□

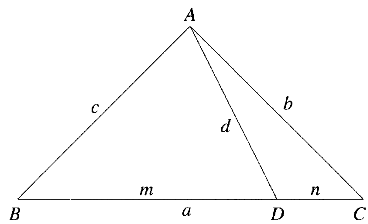
Stewart's Theorem

A related theorem is the following, interesting in its own right, which allows us to express the lengths of the internal angle bisectors of a triangle in terms of the lengths of the sides.

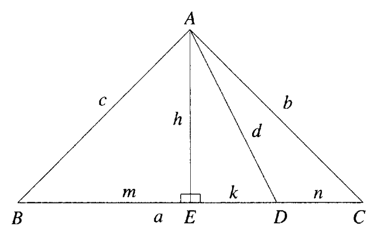
Theorem 3.4.6. *(Stewart's Theorem)*

In $\triangle ABC$, if D is any point internal to the segment $\overline{BC}$, $d = AD$, $m = BD$, $n = DC$, $b = AC$, $c = AB$, and $a = BC$, as in the figure, then

$$c^2 \cdot n + b^2 \cdot m = a \cdot (d^2 + m \cdot n).$$



Proof. Drop a perpendicular from A to BC, hitting BC at E, and let $h = AE$ and $k = DE$, as in the figure below.



From Pythagoras' Theorem, we have

$$h^2 + k^2 = d^2,$$

so that

$$c^2 = (m - k)^2 + h^2 = m^2 - 2mk + k^2 + h^2 = m^2 - 2mk + d^2$$

and

$$b^2 = (n + k)^2 + h^2 = n^2 + 2nk + k^2 + h^2 = n^2 + 2nk + d^2.$$

Multiplying the equation for c^2 by n and the equation for b^2 by m, we have

$$n \cdot c^2 = n \cdot m^2 - 2mnk + n \cdot d^2,$$
$$m \cdot b^2 = m \cdot n^2 + 2mnk + m \cdot d^2,$$

so that

$$n \cdot c^2 + m \cdot b^2 = m \cdot n(m + n) + (m + n) \cdot d^2,$$

and since $m + n = a$, then

$$c^2 \cdot n + b^2 \cdot m = a \cdot (d^2 + m \cdot n).$$

□

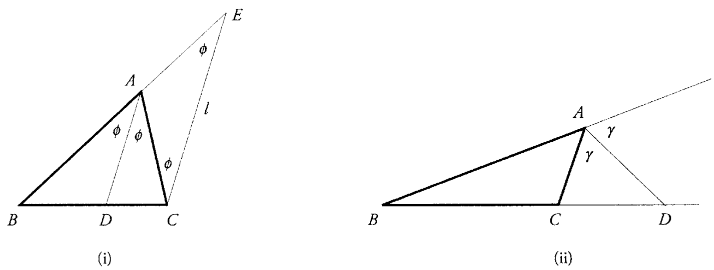
Angle Bisector Theorem

Theorem 3.4.7. *(The Angle Bisector Theorem)*

Let D be a point on side BC of triangle ABC.

(1) If AD is the internal bisector of $\angle BAC$, then $AB/AC = DB/DC$.

(2) If AD is the external bisector of $\angle BAC$, then $AB/AC = DB/DC$.



Proof. (1) Let l be a line parallel to AD through C meeting AB at E. Then

$$\angle BAD \equiv \angle BEC$$

and

$$\angle CAD \equiv \angle ACE,$$

and so $\angle AEC \equiv \angle ACE$; that is, $\triangle ACE$ is isosceles. Thus,

$$AC = AE,$$

and, again using the fact that $CE \parallel DA$,

$$\triangle ABD \sim \triangle EBC.$$

Hence,

$$\frac{DB}{DC} = \frac{AB}{AE} = \frac{AB}{AC}.$$

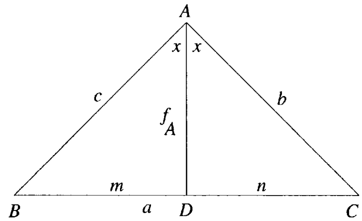
The proof of (2) using similarity is left as an exercise.

□

Example 3.4.8. *Use Stewart's Theorem to prove that in $\triangle ABC$, if $AD = f_A$ is the internal angle bisector at $\angle A$, then*

$$f_A^2 = bc\left[1 - \left(\frac{a}{b+c}\right)^2\right]$$

where $a = BC$, $b = AC$, and $c = AB$, as in the figure.



Solution. From the Internal Angle Bisector Theorem, we have

$$\frac{m}{n} = \frac{DB}{DC} = \frac{AB}{AC} = \frac{c}{b},$$

so that

$$m = \frac{c \cdot n}{b},$$

and since $a = m + n$, then

$$a = m + n = \frac{c \cdot n}{b} + n$$

and

$$n\left(1 + \frac{c}{b}\right) = a.$$

That is,

$$n = \frac{ab}{b+c} \qquad \text{and} \qquad m = \frac{ac}{b+c}.$$

From Stewart's Theorem, we have

$$\frac{c^2ab}{b+c} + \frac{b^2ac}{b+c} = a\left(f_A^2 + \frac{a^2bc}{(b+c)^2}\right);$$

that is,

$$abc\left(\frac{c}{b+c} + \frac{b}{b+c}\right) = a\left(f_A^2 + \frac{a^2bc}{(b+c)^2}\right).$$

Therefore,

$$bc = f_A^2 + \frac{a^2bc}{(b+c)^2};$$

that is,

$$f_A^2 = bc\left[1 - \left(\frac{a}{b+c}\right)^2\right].$$

□

Corollary 3.4.9. *In $\triangle ABC$, if f_A, f_B, and f_C denote the lengths of the internal angle bisectors at $\angle A$, $\angle B$, and $\angle C$, respectively, then*

$$f_A^2 = bc\left[1 - \left(\frac{a}{b+c}\right)^2\right],$$

$$f_B^2 = ac\left[1 - \left(\frac{b}{a+c}\right)^2\right],$$

$$f_C^2 = ab\left[1 - \left(\frac{c}{a+b}\right)^2\right].$$

Exercise 3.4.10. *In $\triangle ABC$, let AD, BE, and CF be the angle bisectors of $\angle BAC$, $\angle ABC$, and $\angle ACB$, respectively. Show that if*

$$\angle BAC < \angle ABC < \angle ACB,$$

then

$$|AD| > |BE| > |CF|.$$

Medians

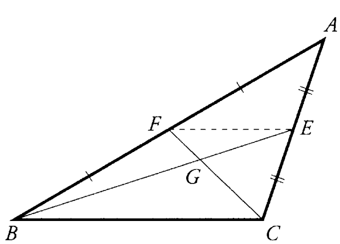
In Chapter 2, we used area to show that the medians of a triangle trisect each other and are concurrent. Here is how similar triangles can be used to prove the same thing:

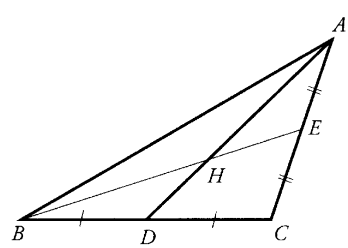
Example 3.4.11. *Let BE and CF be two medians of a triangle meeting each other at G. Show that*

$$\frac{EG}{GB} = \frac{FG}{GC} = \frac{1}{2}$$

and deduce that all three medians are concurrent at G.

Solution. In the figure below,





by the **sAs** similarity condition we have $\triangle ABC \sim \triangle AFE$, with proportionality constant 1/2. This means that $\angle ABC \equiv \angle AFE$, from which it follows that FE is parallel to BC with $FE = BC/2$. This means that $\triangle BCG \sim \triangle EFG$ with a proportionality constant of $1/2$. Hence,

$$GE = \frac{1}{2}GB \qquad \text{and} \qquad GF = \frac{1}{2}GC,$$

which means that $EG/GB = FG/GC = 1/2$.

Applying the same reasoning, medians BE and AD intersect at a point H for which $EH/HB = HD/AH = 1/2$. But this means that

$$\frac{EG}{GB} = \frac{EH}{HB} = \frac{1}{2},$$

and since both G and H are between B and E, we must have $H = G$. This shows that the three medians are concurrent at G.

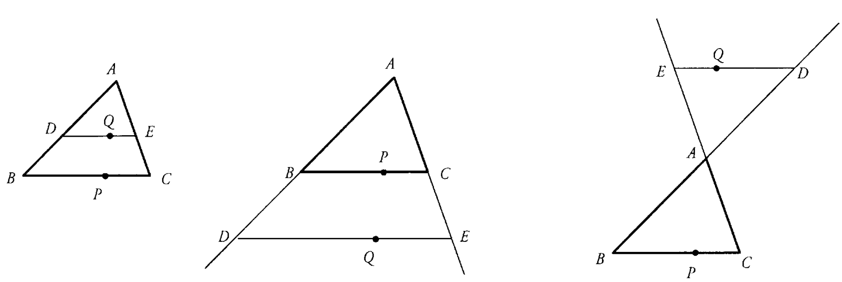
□

We will end this section with a theorem that is sometimes useful in construction problems.

Theorem 3.4.12. *In $\triangle ABC$, suppose that D and E are points on AB and AC, respectively, such that $DE \parallel BC$. Let P be a point on BC between B and C, and let Q be a point on DE between D and E. Then A, P, and Q are collinear if and only if*

$$\frac{BP}{PC} = \frac{DQ}{QE}.$$

The figures below illustrate the three cases that can arise.



Proof.
(i) Suppose A, P, and Q are collinear. Since DE is parallel to BC, we have

$$\triangle APB \sim \triangle AQD \quad \text{and} \quad \triangle APC \sim \triangle AQE,$$

and so

$$\frac{BP}{DQ} = \frac{AP}{AQ} \quad \text{and} \quad \frac{AP}{AQ} = \frac{PC}{QE},$$

from which it follows that

$$\frac{BP}{DQ} = \frac{PC}{QE},$$

or, equivalently, that

$$\frac{BP}{PC} = \frac{DQ}{QE}.$$

(ii) Conversely, suppose that

$$\frac{BP}{PC} = \frac{DQ}{QE}.$$

Draw the line AP meeting DE at Q'. We will show that $Q = Q'$.

Since A, P, and Q' are collinear, the first part of the proof shows that

$$\frac{BP}{PC} = \frac{DQ'}{Q'E}.$$

We are given that

$$\frac{BP}{PC} = \frac{DQ}{QE},$$

so that

$$\frac{DQ'}{Q'E} = \frac{DQ}{QE},$$

from which it follows that $Q' = Q$, and this completes the proof.

□

3.5 Construction Problems

If you were asked to bisect a given line segment, you would probably construct the right bisector. How would you solve the following problem?

Example 3.5.1. *To divide a given line segment into three equal parts.*

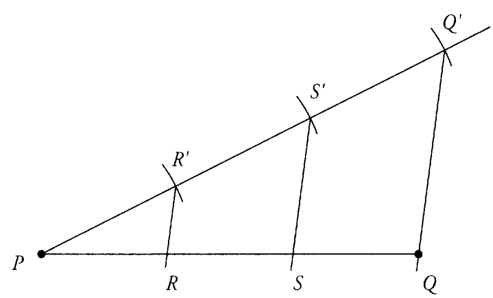
Many construction problems involve similarity. The idea is to create a figure similar to the desired one and then, by means of parallel lines or some other device, transform the similar figure into the desired figure.

Creating a similar figure effectively removes size restrictions, and this can make a difficult problem seem almost trivial. Without size restrictions, Example 3.5.1 becomes:

To construct any line segment that is divided into three equal parts.

This is an easy task: using any line, fix the compass at any radius, and strike off three segments of equal length AB, BC, and CD. Then AD is a line segment that has been divided into three equal parts. What follows shows how this can be used to solve the original problem.

Solution. We are given the line segment PQ, which is to be divided into three equal parts. First, draw a ray from P making an angle with PQ. With the compass set at a convenient radius, strike off congruent segments PR', $R'S'$, and $S'Q'$ along the ray.



Join Q' and Q. Through R' and S', draw lines parallel to $Q'Q$ that meet PQ at R and S. Since parallel projections preserve ratios, we have

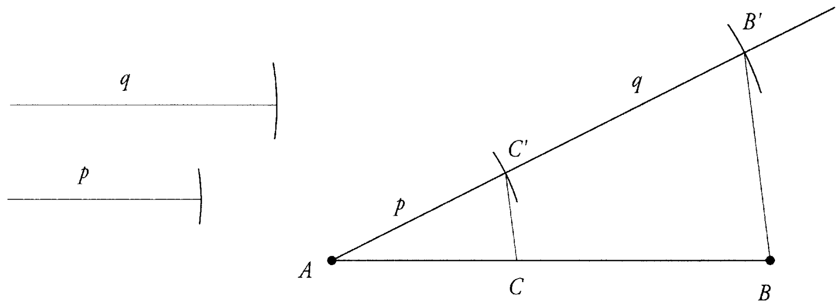
$$\frac{RS}{SQ} = \frac{R'S'}{S'Q'} = 1,$$

and so $RS = SQ$. Similarly, $PR = RS$.

□

The preceding construction can be modified to divide a line segment into given proportions. That is, it can be used to solve the following: given AB and line segments of length p and q, construct the point C between A and B so that

$$\frac{AC}{CB} = \frac{p}{q}.$$

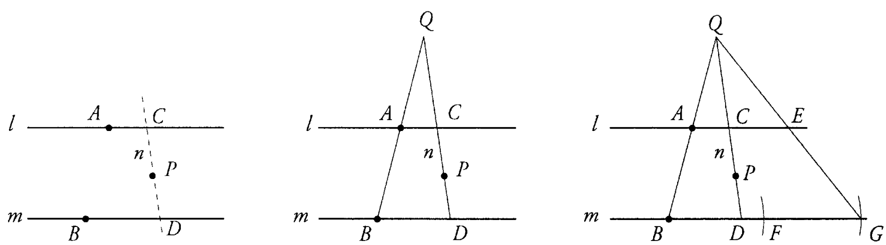


There is frequently more than one way to use similarity to solve a construction problem.

Example 3.5.2. *Given two parallel lines l and m and given points A on l, B on m, and P between l and m, construct a line n through P that meets l at C and m at D so that*

$$AC = \frac{1}{2}BD,$$

as in figure (a) below.



(a) (b) (c)

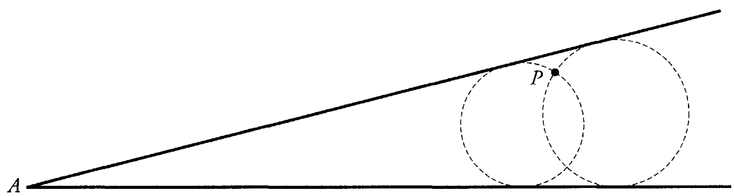
Solution.

(1) As in figure (b), draw the segment AB and extend it beyond A to Q so that $BA = AQ$. Draw the line PQ, meeting l at C and m at D, and then by similar triangles, $AC = BD/2$.

(2) As in figure (c), on line l, strike off a segment AE. On line m, strike off segments BF and FG of the same length as AE. Let Q be the point where the lines AB and EG meet. Draw the line PQ, meeting l at C and m at D, and then by similar triangles, $AC = BD/2$.

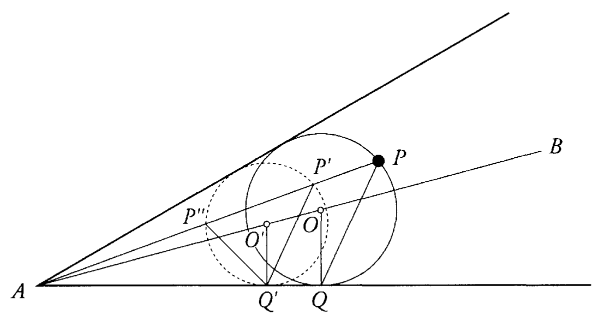
□

Example 3.5.3. *Given a point P inside an angle A, construct the two circles passing through P that are tangent to the arms of the angle.*



Solution. In the diagram, several construction arcs have been omitted for clarity.

Analysis Figure:



Construction:

(1) Construct the angle bisector AB of $\angle A$.

(2) Choose a point O' on AB, and drop the perpendicular $O'Q'$ to one of the arms of the angle.

(3) Draw the circle $\mathcal{C}(O', O'Q')$. This is the dotted circle in the diagram and it is tangent to both arms of the angle because O' is on the angle bisector.

(4) Draw the ray AP cutting the circle at P' and P''.

(5) Through P, construct the line PQ parallel to $P'Q'$, with Q on AQ'.

(6) Through Q, construct the line perpendicular to AQ meeting AO' at O.

(7) Draw the circle $\mathcal{C}(O, OQ)$. This is one of the desired circles.

(8) Through P, construct the line PQ'' parallel to $P''Q'$, with Q'' on AQ' (not shown).

(9) Through Q'', construct the line perpendicular to AQ'' meeting AO' at O'' (not shown).

(10) Draw the circle $\mathcal{C}(O'', O''Q'')$ (not shown). This is the second desired circle.

Justification:

To show that $\mathcal{C}(O, OQ)$ is one of the desired circles, we know that it is tangent to both arms of the angle because O is on the angle bisector and the radius OQ is perpendicular to an arm of the angle. It remains to show that the circle passes through P; that is, that $OP = OQ$, the radius of the circle.

Since $O'Q' \parallel OQ$, it follows that

$$\frac{OQ}{O'Q'} = \frac{AQ}{AQ'}.$$

Since $Q'P' \parallel QP$, it follows that

$$\frac{QP}{Q'P'} = \frac{AQ}{AQ'}$$

and, consequently, that

$$\frac{OQ}{O'Q'} = \frac{QP}{Q'P'}.$$

Again, since $Q'P' \parallel QP$, it follows that $\angle AQ'P' = \angle AQP$, and so

$$\angle O'Q'P' = \angle OQP.$$

Then, by the **sAs** criteria, $\triangle O'Q'P' \sim \triangle OQP$, and so

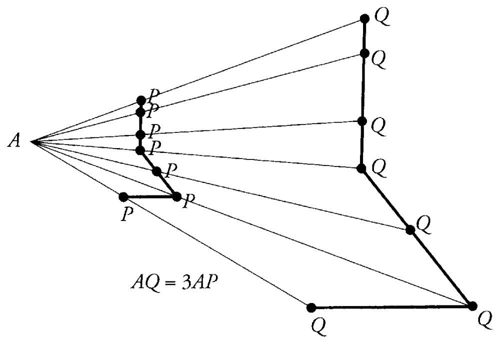
$$\frac{OP}{O'P'} = \frac{OQ}{O'Q'} = \frac{OQ}{O'P'}.$$

Thus $OP = OQ$, and this completes the proof.

□

Remark. Example 3.5.3 uses the idea that given a point A and a figure $\mathcal{F}$, we can construct a similar figure $\mathcal{G}$ by constructing all points Q such that $AQ = k \cdot AP$ for points P in $\mathcal{F}$, where k is a fixed nonzero constant. In the example, we constructed a circle centered at a point O' and then constructed a magnified version of this circle centered at point O.

The transformation that maps each point P to the corresponding point Q so that $AQ = k \cdot AP$ is called a ***homothety*** and is denoted $H(A, k)$. The number k is the magnification constant. The figure on the following page illustrates the effect of $H(A, 3)$.

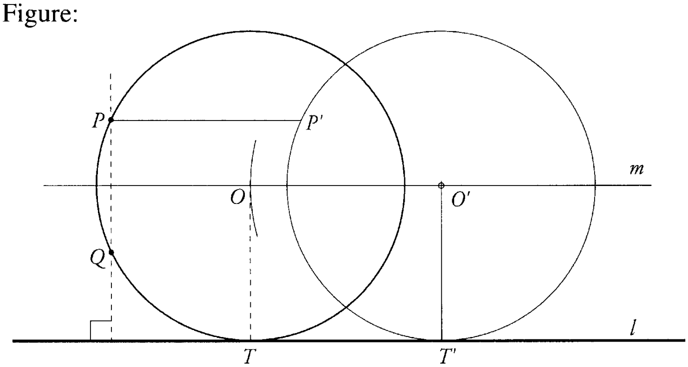


Here are two problems that are solved by translations rather than a homothety. The strategy is very similar: we construct a circle that fulfills part of the criteria and then use a translation to move it to the correct place.

Example 3.5.4. *Given a line l and points P and Q such that P and Q are on the same side of l and such that $PQ \perp l$, construct the circle tangent to l that passes through P and Q.*

Solution. In the figure, we have omitted construction lines for the standard constructions (like dropping a perpendicular from a point to a line).

Analysis Figure:



Construction:

(1) Construct the right bisector m of PQ.

(2) Choose any point O' on m and drop the perpendicular $O'T'$ from O' to l.

(3) Construct $\mathcal{C}(O', r)$ where $r = O'T'$. Note that $\mathcal{C}(O', r)$ is tangent to l.

(4) Construct a line through P perpendicular to PQ that meets $\mathcal{C}(O', r)$ at P'.

(5) With center O' and radius PP', draw an arc cutting m at O.

(6) Draw $\mathcal{C}(O, r)$. This is the desired circle.

Justification:

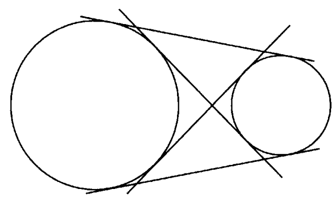
$PP'O'O$ is a parallelogram because PP' is congruent to and parallel to OO'. Then $OP = O'P' = r$, showing that P is on the circle $\mathcal{C}(O, r)$. Since O is on the right bisector of PQ, it follows that $OQ = OP = r$, showing that Q is on $\mathcal{C}(O, r)$.

Finally, since m and l are parallel, the perpendicular distance from O to l is the same as the perpendicular distance from O' to l, and it follows that $\mathcal{C}(O, r)$ is tangent to l.

□

Example 3.5.5. *Given two disjoint circles of radii R and r, construct the four tangents to the circle.*

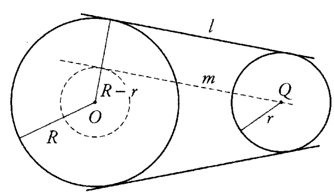
The diagram on the right illustrates that two nonoverlapping circles have four common tangent lines. The problem is to construct those lines. We will show how to construct the "external" tangents and leave the construction of the "internal" ones as an exercise.



Solution.

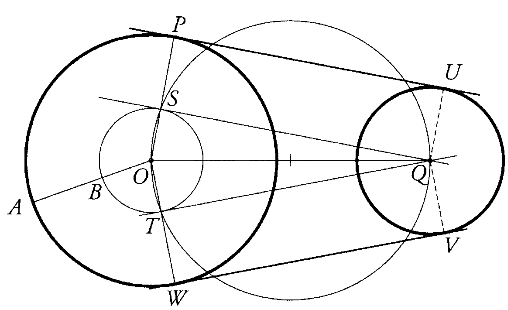
Analysis Figure:

Let us suppose that the radii of the circles are R and r, as shown on the right. If we draw a line m through the center Q of the smaller circle parallel to a tangent line l as shown, then its distance from the center O of the larger circle will be $R - r$, and it will be tangent to $\mathcal{C}(O, R - r)$.



Construction:

Here is the step-by-step construction.



(1) Draw a radius OA of the larger circle. With the compass, cut off a point B so that $AB = r$. Then $OB = R - r$. Draw the circle $\mathcal{C}(O, OB)$.

(2) Through the point Q, draw the tangents QS and QT to $\mathcal{C}(O, OB)$.

(3) Draw the rays OS and OT, cutting the large circle at P and W, respectively.

(4) Through P, draw a line parallel to QS. Through W, draw a line parallel to QT. These are the desired tangent lines.

Justification:

Let QU be parallel to SP so that $QSPU$ is a rectangle. Then $QU = PS = r$ where r is the radius of the smaller circle. Since $PU \perp QU$, then PU is tangent to the smaller circle.

Since OP is a radius of the larger circle and since $PU \perp OP$, then PU is tangent to the larger circle.

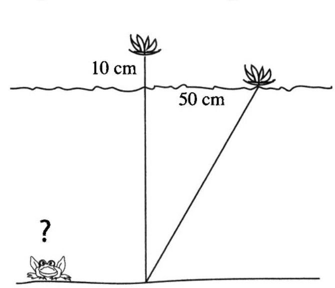
Let QV be parallel to TW so that $QTWV$ is a rectangle. Reasoning as before, WV is tangent to both circles.

□

3.6 The Power of a Point

The following problem is taken from *AHA! Insight*, a delightful book written by Martin Gardner. Gardner attributes the problem to Henry Wadsworth Longfellow.

A lily pad floats on the surface of a pond as far as possible from where its root is attached to the bottom. If it is pulled out of the water vertically, until its stem is taut, it can be lifted 10 cm out of the water. The stem enters the water at a point 50 cm from where the lily pad was originally floating. What is the depth of the pond ?

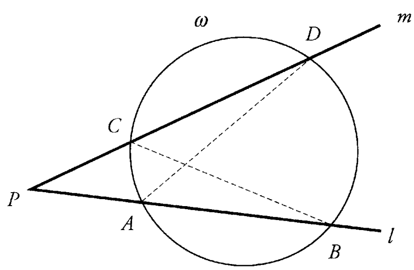


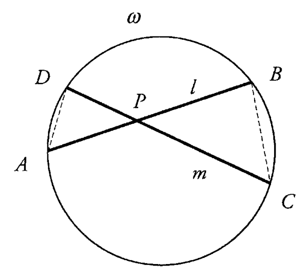
Most people attack this problem by using Pythagoras' Theorem together with the fact that a chord is perpendicular to the radius that bisects it. At the end of this section we will see that there is a much more elegant approach. We begin with a valuable fact about intersecting secants and chords.

Theorem 3.6.1. *Let ω be a circle, let P be any point in the plane, and let l and m be two lines through P meeting the circle at A and B and C and D, respectively. Then*

$$PA \cdot PB = PC \cdot PD.$$

Proof. If P is outside the circle, suppose that A is between P and B and that C is between P and D. Insert the line segments AD and BC, and consider triangles PAD and PCB.





By Thales' Theorem,

$$\angle PDA \equiv \angle PBC \qquad \text{and} \qquad \angle P \text{ is common,}$$

so by the **AA** similarity criteria, $\triangle PAD \sim \triangle PCB$.

Consequently,

$$\frac{PA}{PC} = \frac{PD}{PB},$$

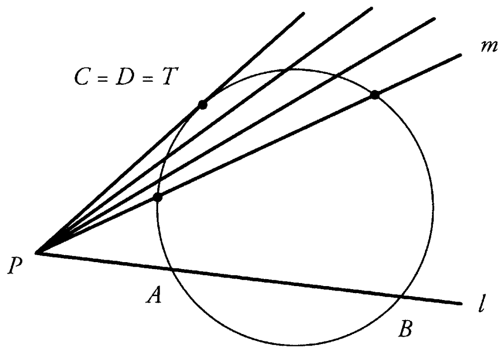
and so

$$PA \cdot PB = PC \cdot PD.$$

If P is inside the circle, again insert line segments AD and BC. Then, triangles PDA and PBC are similar, and so we again have $PA \cdot PB = PC \cdot PD$.

□

When P is outside the circle, an interesting thing happens if we swing the secant line PCD to a tangent position, as in the figure on the following page. In this case, the points C and D approach each other and coalesce at the point T, and so the product $PC \cdot PD$ approaches PT^2.



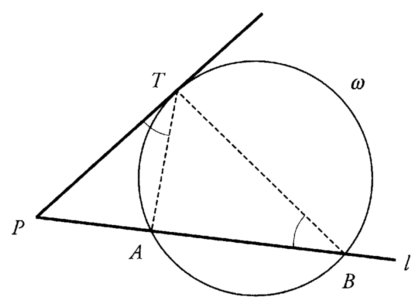
As a consequence:

Theorem 3.6.2. *Let P be a point in the plane outside a circle ω. Let PT be tangent to the circle at T, and let l be a line through P meeting the circle at A and B. Then*

$$PT^2 = PA \cdot PB.$$

Proof. Here is a proof that does not involve limits.

We may assume that A is between P and B. Insert the segments TA and TB, and we have the figure below.



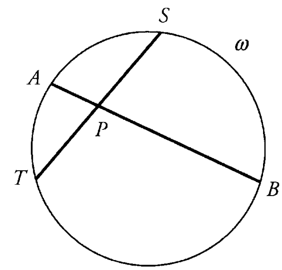
A consequence of Thales' Theorem is that $\angle PTA$ and $\angle PBT$ are equal in size. It follows that $\triangle PTA \sim \triangle PBT$, and so

$$PT/PB = PA/PT.$$

It follows immediately that $PT^2 = PA \cdot PB$.

□

If P is inside the circle, then, of course, no tangent to the circle passes through P. However, there is a result that looks somewhat like the previous theorem when P is inside the circle:

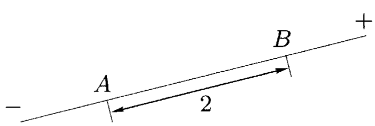


Theorem 3.6.3. *Let P be a point in the plane inside a circle ω. Let TS be a chord whose midpoint is P and let AB be any other chord containing P. Then*

$$PT^2 = PA \cdot PB.$$

Now let l be a line, and assign a direction to the line. For two points A and B on the line, with $A \neq B$, let AB be the distance between A and B. The ***directed distance*** or ***signed distance*** from A to B, denoted $\overline{AB}$, is defined as follows:

$$\overline{AB} = \begin{cases} AB & \text{if } A \text{ is before } B \text{ in the direction along } l, \\ 0 & \text{if } A = B, \\ -AB & \text{if } B \text{ is before } A \text{ in the direction along } l. \end{cases}$$



By using directed distances, the theorems above can be combined and extended as follows:

Theorem 3.6.4. *Let P be any point in the plane, let ω be a given circle, and let l be a line through P meeting the circle at A and B. Then the value of the product $\overline{PA} \cdot \overline{PB}$ is independent of the line l, and*

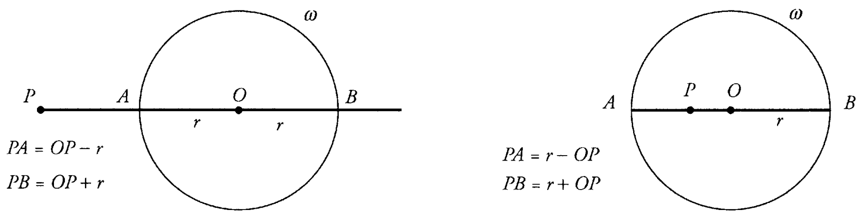
(1) $\overline{PA} \cdot \overline{PB} > 0$ *if P is outside the circle,*

(2) $\overline{PA} \cdot \overline{PB} = 0$ *if P is on the circle,*

(3) $\overline{PA} \cdot \overline{PB} < 0$ *if P is inside the circle.*

The value $\overline{PA} \cdot \overline{PB}$ is called the ***power*** of the point P with respect to the circle ω.

Suppose we are given a circle ω with center O and radius r, and suppose that we are given a point P and that we know the distance OP. It is fairly obvious how we could experimentally determine the power of P with respect to ω: draw any line through P meeting the circle at A and B, and measure PA and PB. Of course, if one chooses a line passing through O, then PA and PB are readily determined without recourse to measurement (see the figure on the following page).



By doing this, we find:

Corollary 3.6.5. *The power of a point P with respect to a circle with center O and radius r is* $OP^2 - r^2$.

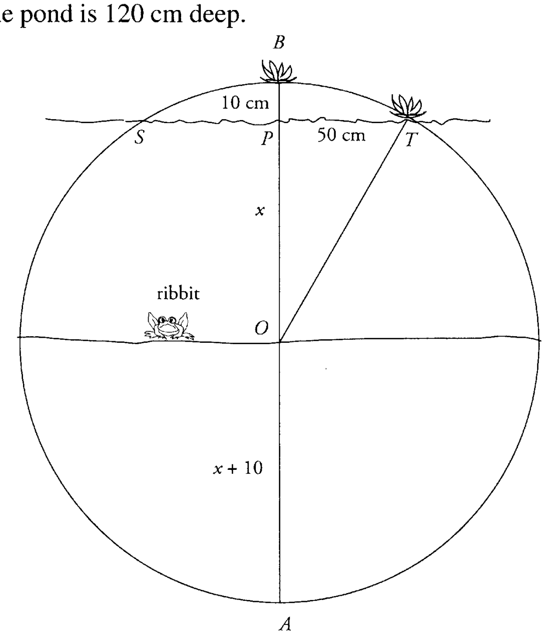
The Lily Pad Problem

The figure below illustrates the geometry of the lily pad problem. The depth of the pond is $OP = x$, and the length of the stem of the lily pad is the radius of a circle centered at O.

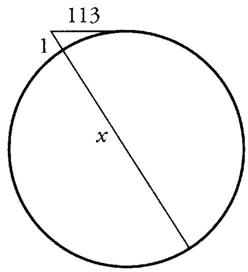
We know now that $PA \cdot PB = PT^2$, so referring to the diagram,

$$(2x + 10)(10) = 50^2,$$

and therefore the pond is 120 cm deep.



Example 3.6.6. *A person whose eyes are exactly 1 km above sea level looks towards the horizon at sea. The point on the horizon is 113 km from the observer's eyes. Use this to estimate the diameter of the earth.*



Solution. The line of sight to the horizon is tangent to the earth. If x is the diameter of the earth, we have

$$(x+1)1 = 113^2,$$

so

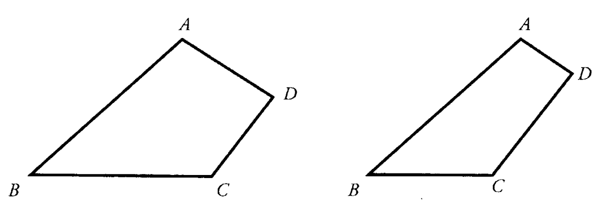
$$x = 113^2 - 1 = 12768\,\text{km}.$$

□

3.7 Solutions to the Exercises

Solution to Exercise 3.3.2

There are many examples, as the following figures indicate:

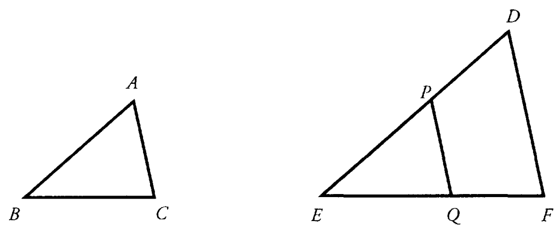


Solution to Exercise 3.3.7

We are given that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = k,$$

and so we have to prove that the angles are equal.



We may assume that $k < 1$, that is, that $AB < DE$.

Let P be the point on ED such that $EP = BA$. Draw PQ parallel to DF with Q on EF. By Theorem 3.2.4, $\triangle EPQ \sim \triangle EDF$, and so

$$\frac{EQ}{EF} = \frac{EP}{ED} = \frac{AB}{ED} = k = \frac{BC}{EF}.$$

This implies that $EQ = BC$.

Similarly, $PQ = AC$, so $\triangle EPQ \equiv \triangle BAC$, and it follows that

$$\angle DEF = \angle PEQ = \angle ABC.$$

Since $PQ \parallel DF$ we also have

$$\angle EDF = \angle EPQ = \angle BAC \quad \text{and} \quad \angle EFD = \angle EQP = \angle BCA,$$

and this completes the proof.

Solution to Exercise 3.4.3

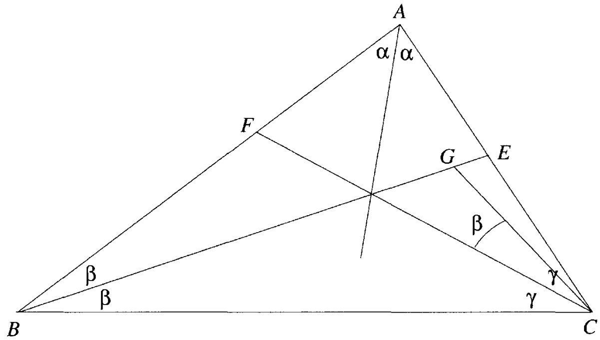
Given triangle ABC with $AB^2 = BC^2 + CA^2$, let PQR be a triangle with a right angle at R, such that $QR = BC$ and $RP = CA$. Then

$$PQ^2 = QR^2 + RP^2 = BC^2 + CA^2 = AB^2.$$

Hence, $PQ = AB$, so ABC and PQR are congruent triangles by **SSS**. It follows that $\angle C$ is a right angle.

Solution to Exercise 3.4.10

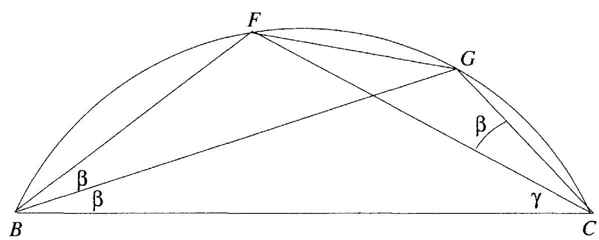
Let $\beta = \angle CBE$ and let $\gamma = \angle BCF$. First, we note that if $\alpha = \beta$, then $\triangle ABC$ is isosceles, so that $BE = CF$. Thus, we may assume, without loss of generality, that $\beta < \gamma$.



Let G be the point on BE between B and E such that $\angle FCG = \beta$. Then

$$\angle FBG = \angle FCG = \beta,$$

so by Thales' Theorem, $F,\ G,\ B,$ and C lie on a circle.



Now,

$$\angle B = 2\beta < \beta + \gamma = \frac{1}{2}\big(2\beta + 2\gamma\big) < \frac{1}{2}\big(2\alpha + 2\beta + 2\gamma\big) = 90,$$

so that $\angle CBF < \angle BCG < 90$, and therefore $CF < BG < BE$.

These last inequalities follow from the fact that given two chords in a circle, one is smaller if and only if it is further from the center, and this is true if and only if it subtends a smaller central angle (the **SAS** inequality and its converse) and therefore a smaller acute angle at the circumference.

So, if a triangle has two different angles, the smaller angle has the longer internal bisector, which is what we wanted to show.

3.8 Problems

1. P and Q are points on the side BC of $\triangle ABC$ with $BP = PQ = QC$. The line through P parallel to AC meets AB at X, and the line through Q parallel to AB meets AC at Y. Show that $\triangle ABC \sim \triangle AXY$.

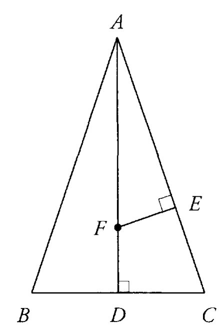
2. BE and CF are altitudes of triangle ABC that meet at H. Prove that

$$BH \cdot HE = CH \cdot HF.$$

3. In the figure on the right, $\triangle ABC$ is an isosceles triangle with altitude AD,

$$AB = \tfrac{3}{2}BC, \quad AF = 4FD, \quad \text{and} \quad FD = 1.$$

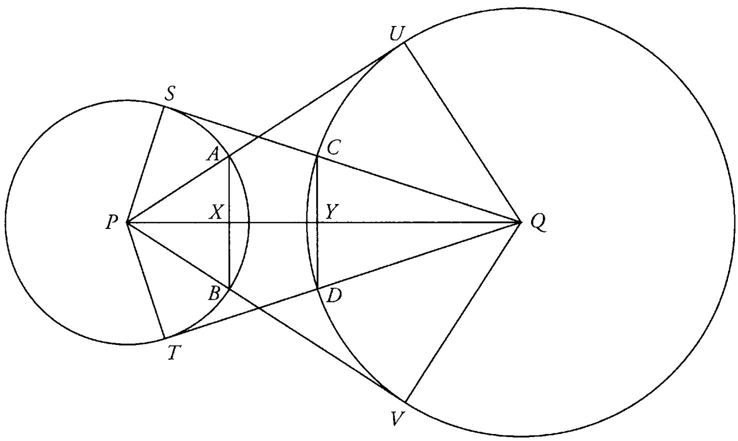
FE is a perpendicular from F to AC. Find the length of FE.



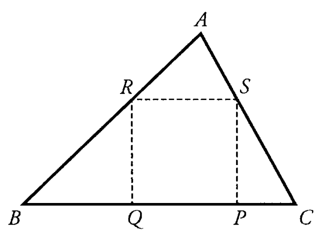
4. Triangle ABC is isosceles with $AB = AC$, and D is the midpoint of BC. The point K is on the line through C parallel to AB and $K \neq C$. The line KD intersects the sides AB and AC (or the extensions of the sides) at P and Q, respectively. Show that $AP \cdot QK = BP \cdot QP$.

5. The bisector of $\angle BAC$ meets BC at D. The circle with center C passing through D meets AD at X. Prove that $AB \cdot AX = AC \cdot AD$.

6. (*The Eyeball Theorem*) In the diagram on the following page, P and Q are centers of circles, and tangent lines are drawn from the centers to the other circle, intersecting the circles at A, B, C, and D, as shown. Prove that the chords AB and CD are equal in length.

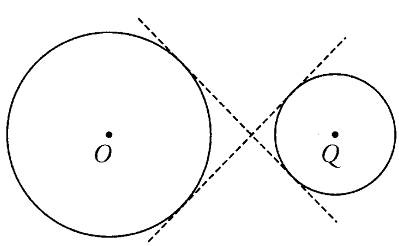


7. Given a triangle ABC with acute angles B and C, construct a square $PQRS$ with PQ in BC and vertices R and S in AB and AC, respectively.

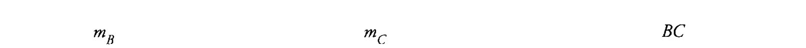


8. Given a line l and two points P and Q on the same side of l but with PQ not parallel to l, construct a circle tangent to l passing through P and Q.

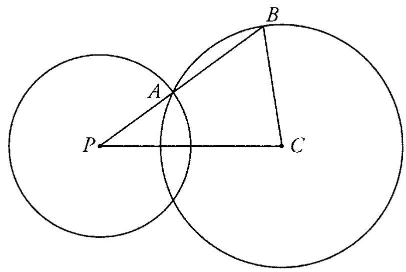
9. Given two disjoint circles $\mathcal{C}(O, R)$ and $\mathcal{C}(Q, r)$, with $R > r$, construct the two "internal" tangents:



10. Construct a triangle ABC given the length of BC and the lengths of the medians m_B and m_C from B and C, respectively.

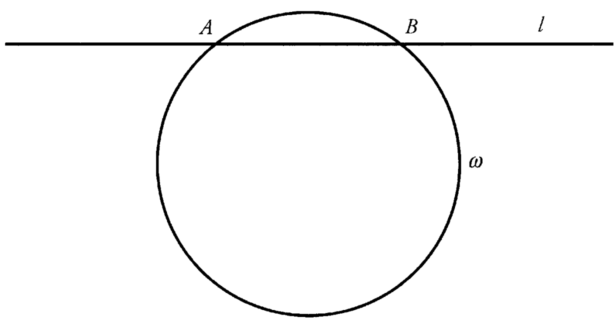


11. In the diagram, $PA = 3$, $BC = 4$, and $PC = 6$. Find the length of the segment AB.



12. Given a circle of radius 1, find all points P such that the power of P with respect to the circle is 3.

13. In the following diagram, the segment AB is of length 3. Construct all points on the line AB whose power with respect to ω is 4.



14. Prove the following partial converse of Theorem 3.6.4. Note the use of directed distances.

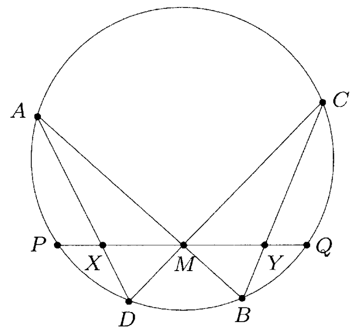
Let l and m be two different lines intersecting at P. Let A and B be points on l. Let C and D be points on m. If

$$\overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD},$$

then $ABCD$ is a cyclic quadrilateral.

15. Two different circles intersect at two points A and B. Find all points P such that the power of P is the same with respect to both circles.

16. M is the midpoint of the chord PQ of a circle. AB and CD are two other chords through M. PQ meets AD at X and BC at Y. Prove that $MX = MY$. This result is called the ***Butterfly Theorem***.



Hint. Drop perpendiculars from X and Y to AB and CD. Find four pairs of similar triangles.

CHAPTER 4

THEOREMS OF CEVA AND MENELAUS

4.1 Directed Distances, Directed Ratios

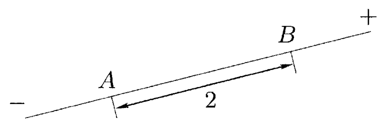
This chapter provides a more algebraic approach to concurrency and collinearity through the theorems of Ceva and Menelaus. The theorems are best understood using the notions of directed distances and directed ratios. We repeat the definition given in the previous chapter.

Let l be a line and assign a direction to the line. For two points A and B on the line, with $A \neq B$, let AB be the distance between A and B. The ***directed distance*** or ***signed distance*** from A to B, denoted $\overline{AB}$, is defined as follows:

$$\overline{AB} = \begin{cases} AB & \text{if } A \text{ is before } B \text{ in the direction along } l, \\ 0 & \text{if } A = B, \\ -AB & \text{if } B \text{ is before } A \text{ in the direction along } l. \end{cases}$$

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

The directed distances $\overline{AB}$ and $\overline{BA}$ for the given direction along l are shown in the figure below, where $AB = 2$.



$AB = 2$
$\overline{AB} = 2$
$\overline{BA} = -2$

Properties of Directed Distance

(1) $\overline{AB} = -\overline{BA}$.

(2) For points A, B, and C on a line, $\overline{AB} + \overline{BC} = \overline{AC}$.

(3) If $\overline{AB} = \overline{AC}$, then $B = C$.

Sometimes property (2) is stated as $\overline{AB} + \overline{BC} + \overline{CA} = 0$. Note that properties (2) and (3) do not hold for unsigned distances.

Ratios

In the theorems of Ceva and Menelaus, given three points A, B, and C on a line, the following ***directed ratio*** occurs frequently:

$$\frac{\overline{AC}}{\overline{CB}}.$$

Note that if C is between A and B, then

$$\frac{\overline{AC}}{\overline{CB}} = +\frac{AC}{CB},$$

while if C is external to the segment AB, then

$$\frac{\overline{AC}}{\overline{CB}} = -\frac{AC}{CB}.$$

4.2 The Theorems

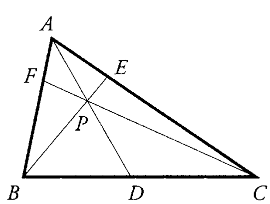
The two theorems are considered as companions of each other, although their discoveries were separated by many centuries. Menelaus proved his theorem around the year 100 CE. It languished in obscurity until 1678, when it was uncovered by Giovanni Ceva, who published it along with the theorem that bears his name. The two theorems are strikingly similar, and it is surprising that there was such a time span between the two discoveries.

Theorem 4.2.1. *(Ceva's Theorem)*

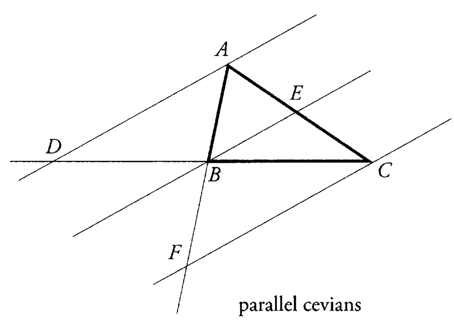
Let AD, BE, and CF be lines from the vertices A, B, and C of a triangle to nonvertex points D, E, and F of the opposite sides. The lines are either concurrent or parallel if and only if

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = +1 .$$

Given a triangle, any line that passes through a vertex of the triangle and also through a nonvertex point of the opposite side is a called a ***cevian line*** or ***cevian*** of the triangle.[2] Medians, angle bisectors, and altitudes are cevians, but the right bisector of a side of a nonisosceles triangle is not. Also, a line through a vertex parallel to the opposite side of a triangle is not a cevian, although we will later incorporate this when we discuss the extended Euclidean plane.



concurrent cevians



parallel cevians

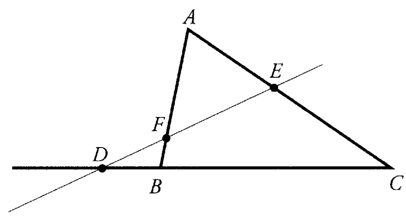
Given a triangle, a nonvertex point in a side or in an extended side of the triangle is called a ***menelaus point***. A line that passes through each of the three edges of a triangle, but not through any of the vertices, is called a ***transversal line*** or a ***transversal***. Menelaus' Theorem tells us when three menelaus points lie on a transversal.

[2]The word "cevian" is a combination of Ceva's name and the word "median."

Theorem 4.2.2. *(Menelaus' Theorem)*

Let D, E, and F be menelaus points on the (extended) sides BC, CA, and AB of a triangle ABC. The points are collinear if and only if

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1 .$$



To use the two theorems, simply assign a direction to each side of the triangle, and proceed around the triangle in either a clockwise or counterclockwise direction. The product of the ratios in the theorems is sometimes called the ***cevian product***.

Note that the cevian product

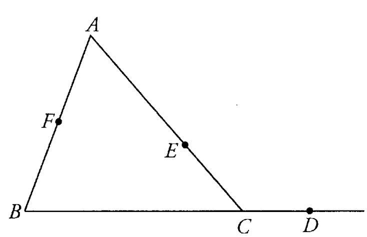
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}}$$

involves the directed ratios

$$\frac{\overline{AF}}{\overline{FB}} \qquad \text{(from } A \text{ to } F \text{ to } B \text{ along side } AB\text{),}$$

$$\frac{\overline{BD}}{\overline{DC}} \qquad \text{(from } B \text{ to } D \text{ to } C \text{ along side } BC\text{),}$$

$$\frac{\overline{CE}}{\overline{EA}} \qquad \text{(from } C \text{ to } E \text{ to } A \text{ along side } CA\text{).}$$



These ratios are meaningful because points F, D, and E are in the (perhaps extended) sides AB, BC, and CA of the triangle, respectively.

Note also that changing the direction assigned to the sides of the triangle does not affect the sign of the directed ratio. It should also be noted that if you compute the cevian product by proceeding clockwise rather than counterclockwise, the two cevian products will be reciprocals of each other. This does not alter the theorems because $+1$ and -1 are the only two real numbers that are their own reciprocals!

The original versions of the theorems used undirected distances, and, although the theorems are often stated that way, the modern versions are easier to use.

Ceva's Theorem will be the object of study in the next section, and Menelaus' Theorem will be discussed in the section following that.

4.3 Applications of Ceva's Theorem

The theorems of Ceva and Menelaus involve directed ratios of the form

$$\frac{\overline{AX}}{\overline{XB}}$$

where X is a point on the line AB other than A or B. Consequently, either X divides AB *internally* (meaning that X is between A and B) or else X divides AB *externally* (X is not between A and B). It is useful to recall that in these cases,

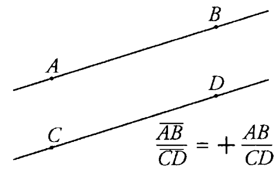
$$\frac{\overline{AX}}{\overline{XB}} = \begin{cases} +\dfrac{AX}{XB} & \text{if } X \text{ divides } AB \text{ internally,} \\ \\ -\dfrac{AX}{XB} & \text{if } X \text{ divides } AB \text{ externally.} \end{cases}$$

As well, it is convenient to define directed ratios for segments that belong to *different but parallel* lines. Thus,

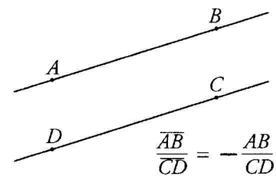
$$\frac{\overline{AB}}{\overline{CD}} = \begin{cases} +\dfrac{AB}{CD} & \text{if } AB \parallel CD \text{ and } ABDC \text{ is convex,} \\ \\ -\dfrac{AB}{CD} & \text{if } AB \parallel CD \text{ and } ABDC \text{ is nonconvex.} \end{cases}$$

This is illustrated in the figure on the following page.

$ABDC$ is convex:



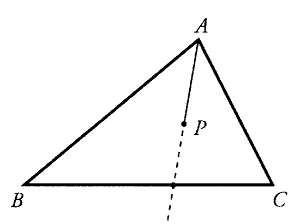
$ABDC$ is nonconvex:



Before proving Ceva's Theorem, we will illustrate how it can be used to show that the most familar cevians are concurrent. Some of the results use the so-called ***Crossbar Theorem***, which we state without proof:

Theorem 4.3.1. *(Crossbar Theorem)*

If P is an interior point of triangle ABC, then the ray $\overrightarrow{AP}$ meets side B at some point between B and C.



A frequently used consequence, that we again state without proof, is:

Theorem 4.3.2. *Two cevians that pass through the interior of a triangle are not parallel.*

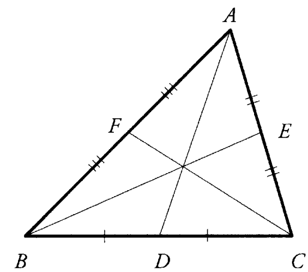
Here is how Ceva's Theorem can be used to prove that medians, angle bisectors, and altitudes are concurrent.

Example 4.3.3. *The medians of a triangle are concurrent.*

Solution. We have

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1 \cdot 1 \cdot 1 = +1,$$

since $\overline{AF} = \overline{FB}$, $\overline{BD} = \overline{DC}$, and $\overline{CE} = \overline{EA}$.

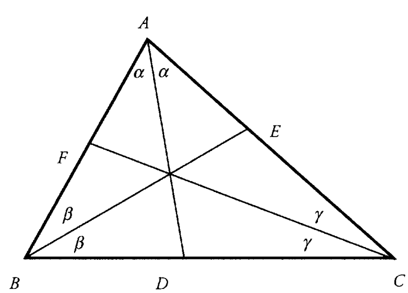


By Ceva's Theorem, the medians are either concurrent or parallel. Since they cannot be parallel by Theorem 4.3.2, they must be concurrent.

□

Example 4.3.4. *The internal bisectors of the angles of a triangle are concurrent.*

Solution.



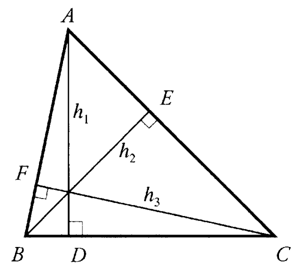
Recall from the Angle Bisector Theorem that if the interior angle bisector of $\angle A$ meets side BC at D, then $BD/DC = AB/AC$. Therefore,

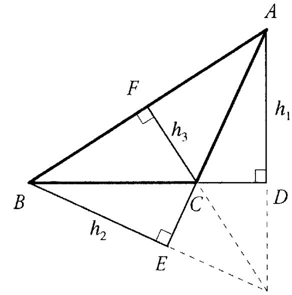
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{CA}{CB} \cdot \frac{AB}{AC} \cdot \frac{BC}{BA} = +1.$$

By Theorem 4.3.2, the cevians in this case cannot be parallel, so by Ceva's Theorem, they are concurrent. □

Example 4.3.5. *The altitudes of a triangle are concurrent.*

Solution.





For a right triangle, the three altitudes are concurrent at the vertex of the right angle. For an acute-angled triangle, the three altitudes divide the sides internally, and for an obtuse-angled triangle, the altitudes divide exactly two of the sides externally. (A proof of this may be obtained from the External Angle Inequality and is left as an exercise.)

Therefore, for a triangle ABC with altitudes AD, BE, and CF, we have

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA},$$

since either all ratios are internal or else exactly two are external.

Using similar triangles to relate the ratios to the length of altitudes,

$$\begin{aligned}
\triangle BEC &\sim \triangle ADC, &\quad \text{so} \quad \frac{CE}{DC} &= \frac{h_2}{h_1}, \\
\triangle ABE &\sim \triangle ACF, &\quad \text{so} \quad \frac{AF}{AE} &= \frac{h_3}{h_2}, \\
\triangle ABD &\sim \triangle CBF, &\quad \text{so} \quad \frac{BD}{BF} &= \frac{h_1}{h_3}.
\end{aligned}$$

Hence,

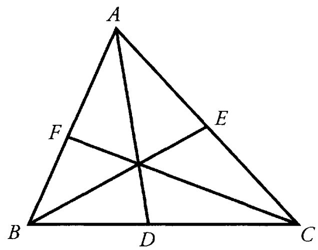
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{h_2}{h_1} \cdot \frac{h_3}{h_2} \cdot \frac{h_1}{h_3} = 1,$$

and Ceva's Theorem shows that the altitudes are concurrent.

□

The next example illustrates how Ceva's Theorem can be used to compute a ratio or a distance rather than to conclude that certain cevians are concurrent.

Example 4.3.6. *In the figure, $AB = 4$, $BC = 5$, and $AC = 6$. AD is an angle bisector, and BE is a median. Find the length of AF.*



Solution. By Ceva's Theorem,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

By the Angle Bisector Theorem, $BD/DC = 4/6$, and since BE is a median, $CE/EA = 1$, so that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{AF}{4 - AF} \cdot \frac{4}{6} \cdot 1 = 1,$$

from which we get $AF = 12/5$.

□

4.4 Applications of Menelaus' Theorem

Angle Bisectors

There are three internal bisectors and three external bisectors of the angles of a triangle. If we take one bisector, either internal or external, from each vertex, then there are four possible combinations:

(i) All three are internal bisectors.

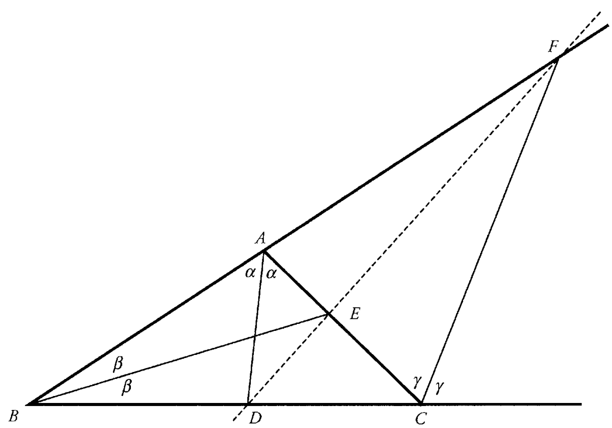
(ii) Two are external bisectors and one is an internal bisector.

(iii) One is an external bisector and two are internal bisectors.

(iv) All three are external bisectors.

For (i) and (ii), the bisectors are concurrent, and so it is probably no surprise to the reader that there is something significant about (iii) and (iv).

Example 4.4.1. *The internal bisectors of two angles of a triangle and the external bisector of the third meet the opposite sides in three collinear points.*



Solution. Let us consider the internal bisectors of $\angle A$ and $\angle B$ and the external bisector of $\angle C$. Using the Internal and External Bisector Theorems,

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \left(-\frac{CA}{CB}\right) \cdot \frac{AB}{AC} \cdot \frac{BC}{BA} = -1,$$

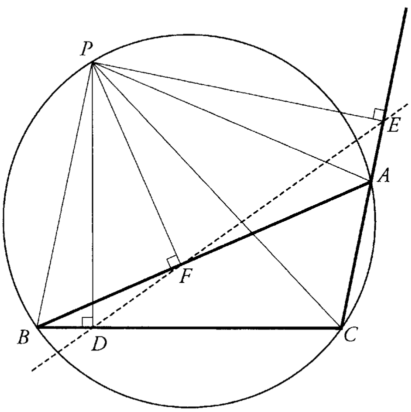
and so by Menelaus' Theorem, the points are collinear.

□

Simson's Theorem

The following proof of Simson's Theorem uses Menelaus' Theorem.

Example 4.4.2. *Given $\triangle ABC$ and a point P on its circumcircle, the perpendiculars dropped from P meet the sides of the triangle in three collinear points.*



Solution. Let D, E, and F be the feet of the perpendiculars on BC, CA, and AB, respectively. Since D, E, and F are three menelaus points of $\triangle ABC$, it is enough to show that the cevian product is -1.

Introducing PA, PB, and PC, Thales' Theorem reveals that $\angle PAF \equiv \angle PCD$. Since both $\triangle PAF$ and $\triangle PCD$ are right triangles, they are similar. In fact, we find that

$$\triangle PAF \sim \triangle PCD, \quad \triangle PCE \sim \triangle PBF, \quad \text{and} \quad \triangle PBD \sim \triangle PAE,$$

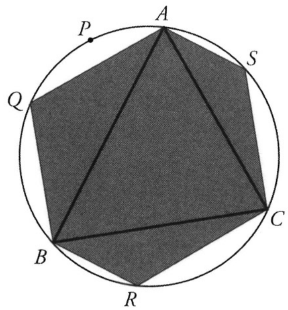
with the last similarity following from Theorem 1.3.11. It follows that

$$\frac{AF}{CD} = \frac{PF}{PD},$$
$$\frac{EC}{FB} = \frac{PE}{PF},$$
$$\frac{BD}{AE} = \frac{PD}{PE},$$

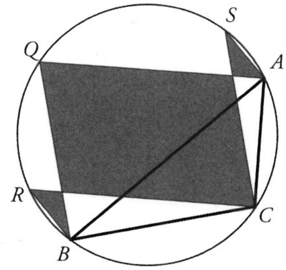
and the result is that

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \pm 1.$$

To check that the sign is actually negative, we have to show that an odd number of the points D, E, and F divide the sides externally. This may be accomplished by constructing a hexagon $AQBRCS$ whose sides are perpendicular to the sides of $\triangle ABC$, as in figure (a) on the following page.



(a)



(b)

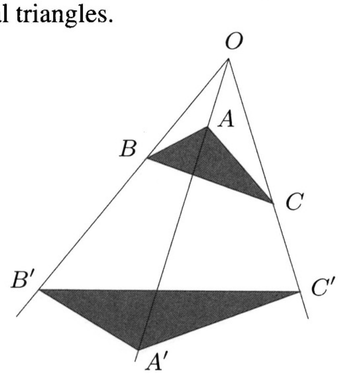
Then, the vertices Q, R, and S are on the circumcircle of $\triangle ABC$, since the quadrilaterals $AQBC$, $BRCA$, and $CSAB$ are cyclic.

Referring to figure (a), it is not difficult to see that if P is strictly between A and Q on the small arc AQ, then the perpendicular PE from P to AC divides AC externally. (The reason is that in order to divide AC internally, the point P would have to be between the lines AQ and CR.) On the other hand, the perpendiculars from P to AB and BC divide those sides internally.

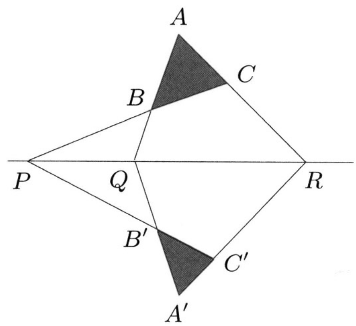
Similar reasoning applies when P is one of the other six arcs of the circumcircle and also to the case where the hexagon is nonconvex, as in figure (b) above. □

Desargues' Two Triangle Theorem

In this section we will prove Desargues' Two Triangle Theorem, but first the definitions of copolar and coaxial triangles.



Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be ***copolar*** from a point O if and only if the joins of corresponding vertices are concurrent at the point O; that is, if and only if the lines AA', BB', and CC' are concurrent at O, as in the figure above. The point O is called the ***pole***.



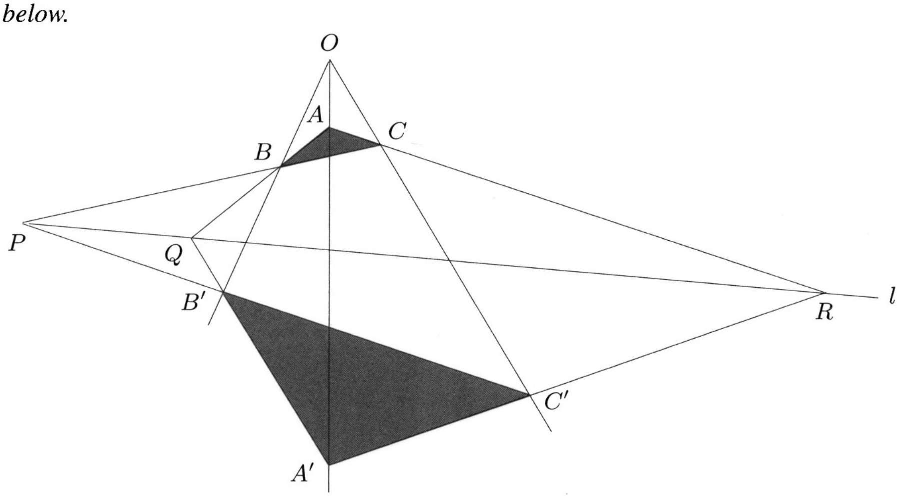
Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are ***coaxial*** from a line l if and only if the points of intersection of corresponding sides

$$P = \overline{BC} \cap \overline{B'C'}, \qquad Q = \overline{AB} \cap \overline{A'B'}, \qquad R = \overline{AC} \cap \overline{A'C'}$$

are collinear, as in the figure above.

Theorem 4.4.3. *(Desargues' Two Triangle Theorem)*

Two triangles in the plane are copolar if and only if they are coaxial, as in the figure below.



The line l is called the ***Desargues line***.

Proof. *Copolar implies Coaxial.*

Given that $\triangle ABC$ and $\triangle A'B'C'$ are copolar from O, we need to show that

$$P = \overline{BC} \cap \overline{B'C'}, \qquad Q = \overline{AB} \cap \overline{A'B'}, \qquad R = \overline{AC} \cap \overline{A'C'}$$

are collinear, and from Menelaus' Theorem we only have to show that

$$\frac{\overline{AQ}}{\overline{QB}} \cdot \frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CR}}{\overline{RA}} = -1.$$

We will use a decomposition type argument. We decompose $\triangle BOC$ into three triangles $\triangle AOB$, $\triangle AOC$, and $\triangle ABC$ and apply Menelaus' Theorem to three transversals as follows:

For $\triangle AOB$ with transversal $A'B'Q$, we have

$$\frac{\overline{AQ}}{\overline{QB}} \cdot \frac{\overline{BB'}}{\overline{B'O}} \cdot \frac{\overline{OA'}}{\overline{A'A}} = -1.$$

For $\triangle BOC$ with transversal $PB'C'$, we have

$$\frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CC'}}{\overline{C'O}} \cdot \frac{\overline{OB'}}{\overline{B'B}} = -1.$$

For $\triangle AOC$ with transversal $A'C'R$, we have

$$\frac{\overline{CR}}{\overline{RA}} \cdot \frac{\overline{AA'}}{\overline{A'O}} \cdot \frac{\overline{OC'}}{\overline{C'C}} = -1.$$

Multiplying these three expressions together, we have

$$\frac{\overline{AQ}}{\overline{QB}} \cdot \frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CR}}{\overline{RA}} \cdot X = (-1)^3 = -1,$$

where

$$X = \frac{\overline{BB'}}{\overline{B'O}} \cdot \frac{\overline{OA'}}{\overline{A'A}} \cdot \frac{\overline{CC'}}{\overline{C'O}} \cdot \frac{\overline{OB'}}{\overline{B'B}} \cdot \frac{\overline{AA'}}{\overline{A'O}} \cdot \frac{\overline{OC'}}{\overline{C'C}}.$$

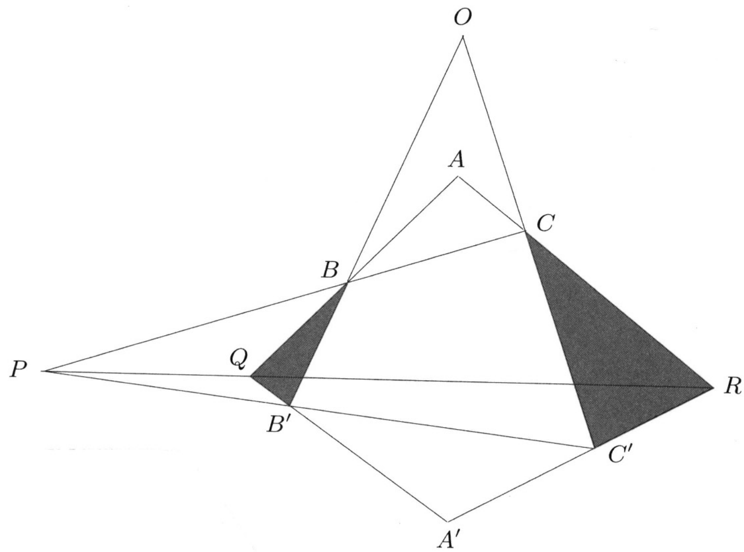
Thus, $X = 1$, so that

$$\frac{\overline{AQ}}{\overline{QB}} \cdot \frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CR}}{\overline{RA}} = -1,$$

and P, Q, and R are collinear.

Coaxial implies Copolar.

Given a pair of coaxial triangles $\triangle ABC$ and $\triangle A'B'C'$, as in the figure on the following page, we want to show that they are copolar; that is, we want to show that AA', BB', and CC' are concurrent.



Let O be the intersection of BB' and CC'. Then we have to show that AA' also goes through O.

Observe that $\triangle QBB'$ and $\triangle RCC'$ are copolar from P. From the first half of the proof, we know that $\triangle QBB'$ and $\triangle RCC'$ are also coaxial, so that

$$QB \cap RC = A, \qquad QB' \cap RC' = A', \qquad \text{and} \qquad BB' \cap CC' = O$$

are collinear, and so AA' passes through O. Therefore, $\triangle ABC$ and $\triangle A'B'C'$ are copolar from O.

□

Pascal's Mystic Hexagon Theorem

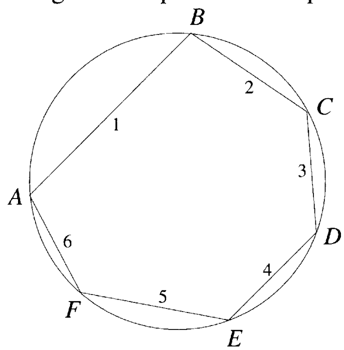
In this section we will prove Pascal's Mystic Hexagon Theorem, but first we give a convention about labeling the sides of a hexagon, simple or nonsimple.

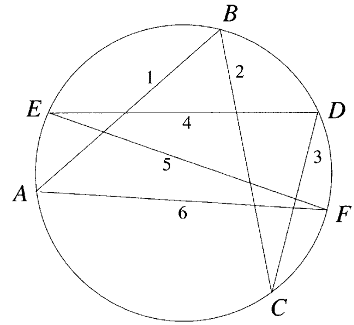
Given a hexagon $ABCDEF$, simple or nonsimple, inscribed in a circle, we label the sides with the positive integers so that

$$1 \leftrightarrow AB, \quad 2 \leftrightarrow BC, \quad 3 \leftrightarrow CD, \quad 4 \leftrightarrow DE, \quad 5 \leftrightarrow EF, \quad \text{and} \quad 6 \leftrightarrow FA,$$

as in the figure on the following page.

We say that sides 1 and 4, sides 2 and 5, and sides 3 and 6 are ***opposite sides***, whether the hexagon is simple or nonsimple.

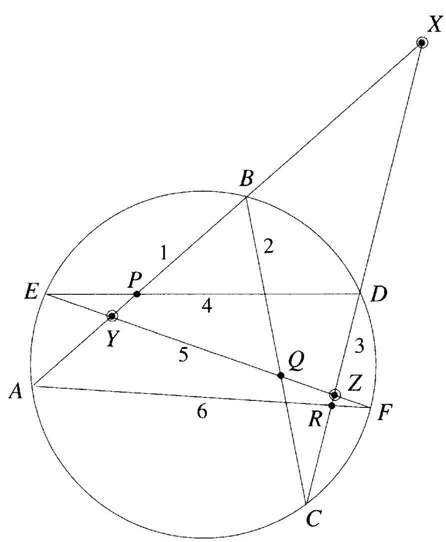




Theorem 4.4.4. *(Pascal's Mystic Hexagon Theorem)*

Given a hexagon (simple or nonsimple) inscribed in a circle, the points of intersection of opposite sides are collinear and form the ***Pascal line****.*

Proof. Given an inscribed hexagon as shown, we want to show that P, Q, and R are collinear.



Create $\triangle XYZ$ by taking every second side of the hexagon so that

$$X = \text{side } 1 \cap \text{ side } 3, \quad Y = \text{side } 1 \cap \text{ side } 5, \quad \text{and} \quad Z = \text{side } 3 \cap \text{ side } 5.$$

Note that

$$P \text{ is on side } 1, \quad Q \text{ is on side } 5, \quad \text{and} \quad R \text{ is on side 3 (extended),}$$

so that P, Q, and R are menelaus points of $\triangle XYZ$.

In order to show that P, Q, and R are collinear, by Menelaus' Theorem we need only show that

$$\frac{\overline{XP}}{\overline{PY}} \cdot \frac{\overline{YQ}}{\overline{QZ}} \cdot \frac{\overline{ZR}}{\overline{RX}} = -1.$$

Applying Menelaus' Theorem to the following *labeled* transversals of $\triangle XYZ$, we have

$$\begin{aligned}
\overleftrightarrow{EPD}: &\quad \frac{\overline{XP}}{\overline{PY}} \cdot \frac{\overline{YE}}{\overline{EZ}} \cdot \frac{\overline{ZD}}{\overline{DX}} = -1, \\
\overleftrightarrow{BQC}: &\quad \frac{\overline{XB}}{\overline{BY}} \cdot \frac{\overline{YQ}}{\overline{QZ}} \cdot \frac{\overline{ZC}}{\overline{CX}} = -1, \\
\overleftrightarrow{ARF}: &\quad \frac{\overline{XA}}{\overline{AY}} \cdot \frac{\overline{YF}}{\overline{FZ}} \cdot \frac{\overline{ZR}}{\overline{RX}} = -1,
\end{aligned}$$

and multiplying these together we get

$$\begin{aligned}
&\frac{\overline{XP}}{\overline{PY}} \cdot \frac{\overline{YQ}}{\overline{QZ}} \cdot \frac{\overline{ZR}}{\overline{RX}} \cdot \frac{\overline{YE}}{\overline{EZ}} \cdot \frac{\overline{ZD}}{\overline{DX}} \cdot \frac{\overline{XB}}{\overline{BY}} \cdot \frac{\overline{ZC}}{\overline{CX}} \cdot \frac{\overline{XA}}{\overline{AY}} \cdot \frac{\overline{YF}}{\overline{FZ}} \\
&\quad = \frac{\overline{XP}}{\overline{PY}} \cdot \frac{\overline{YQ}}{\overline{QZ}} \cdot \frac{\overline{ZR}}{\overline{RX}} \cdot \left(\frac{\overline{XA} \cdot \overline{XB}}{\overline{XC} \cdot \overline{XD}}\right) \cdot \left(\frac{\overline{YE} \cdot \overline{YF}}{\overline{YA} \cdot \overline{YB}}\right) \cdot \left(\frac{\overline{ZC} \cdot \overline{ZD}}{\overline{ZE} \cdot \overline{ZF}}\right) \\
&\quad = (-1)^3 = -1.
\end{aligned}$$

Now, the power of the point X with respect to the circle is

$$\overline{XA} \cdot \overline{XB} = \overline{XC} \cdot \overline{XD},$$

the power of the point Y with respect to the circle is

$$\overline{YA} \cdot \overline{YB} = \overline{YE} \cdot \overline{YF},$$

while the power of the point Z with respect to the circle is

$$\overline{ZC} \cdot \overline{ZD} = \overline{ZE} \cdot \overline{ZF},$$

and therefore,

$$\left(\frac{\overline{XA} \cdot \overline{XB}}{\overline{XC} \cdot \overline{XD}}\right) \cdot \left(\frac{\overline{YE} \cdot \overline{YF}}{\overline{YA} \cdot \overline{YB}}\right) \cdot \left(\frac{\overline{ZC} \cdot \overline{ZD}}{\overline{ZE} \cdot \overline{ZF}}\right) = 1 \cdot 1 \cdot 1 = 1,$$

so that

$$\frac{\overline{XP}}{\overline{PY}} \cdot \frac{\overline{YQ}}{\overline{QZ}} \cdot \frac{\overline{ZR}}{\overline{RX}} = -1,$$

and the points P, Q, and R are collinear.

□

Pappus' Theorem

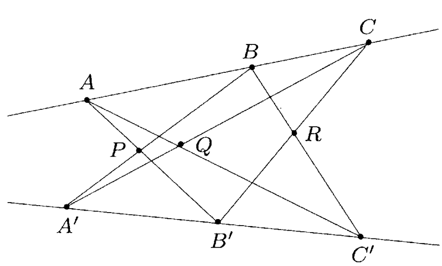
In this section, we will prove Pappus' Theorem. Although it belongs to the realm of projective geometry, we can give an elementary proof using Euclidean geometry via Menelaus' Theorem.

Theorem 4.4.5. *(Pappus' Theorem)*

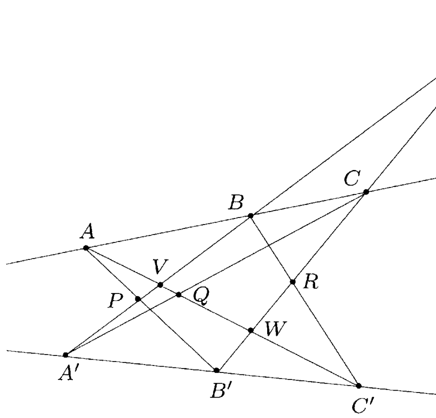
Given points A, B, and C on a line l and points A', B', and C' on a line l', the points of intersection

$$P = AB' \cap A'B, \qquad Q = AC' \cap A'C, \qquad R = BC' \cap B'C$$

are collinear, as in the figure.



Proof. Extending the sides of the triangle formed by the sides $A'B$, $B'C$, and $C'A$, we have the figure on the following page.



We will apply Menelaus' Theorem to each of the five transversals of the triangle $\triangle UVW$

$$APB', \qquad CQA', \qquad BRC', \qquad ABC, \qquad A'B'C'$$

in turn.

For the transversal APB', we have

$$\frac{\overline{UP}}{\overline{PV}} \cdot \frac{\overline{VA}}{\overline{AW}} \cdot \frac{\overline{WB'}}{\overline{B'U}} = -1$$

by Menelaus' Theorem.

For the transversal CQA', we have

$$\frac{\overline{UA'}}{\overline{A'V}} \cdot \frac{\overline{VQ}}{\overline{QW}} \cdot \frac{\overline{WC}}{\overline{CU}} = -1$$

by Menelaus' Theorem.

For the transversal BRC', we have

$$\frac{\overline{UB}}{\overline{BV}} \cdot \frac{\overline{VC'}}{\overline{C'W}} \cdot \frac{\overline{WR}}{\overline{RU}} = -1$$

by Menelaus' Theorem.

For the transversal ABC, we have

$$\frac{\overline{UB}}{\overline{BV}} \cdot \frac{\overline{VA}}{\overline{AW}} \cdot \frac{\overline{WC}}{\overline{CU}} = -1$$

by Menelaus' Theorem.

For the transversal $A'B'C'$, we have

$$\frac{\overline{UA'}}{\overline{A'V}} \cdot \frac{\overline{VC'}}{\overline{C'W}} \cdot \frac{\overline{WB'}}{\overline{B'U}} = -1$$

by Menelaus' Theorem.

Multiplying the first three expressions together, we get

$$\frac{\overline{UP}}{\overline{PV}} \cdot \frac{\overline{VQ}}{\overline{QW}} \cdot \frac{\overline{WR}}{\overline{RU}} \cdot X = -1,$$

where

$$X = \frac{\overline{VA}}{\overline{AW}} \cdot \frac{\overline{WB'}}{\overline{B'U}} \cdot \frac{\overline{UA'}}{\overline{A'V}} \cdot \frac{\overline{WC}}{\overline{CU}} \cdot \frac{\overline{UB}}{\overline{BV}} \cdot \frac{\overline{VC'}}{\overline{C'W}}.$$

Rearranging the terms in X, we get

$$X = \left(\frac{\overline{UB}}{\overline{BV}} \cdot \frac{\overline{VA}}{\overline{AW}} \cdot \frac{\overline{WC}}{\overline{CU}}\right) \cdot \left(\frac{\overline{UA'}}{\overline{A'V}} \cdot \frac{\overline{VC'}}{\overline{C'W}} \cdot \frac{\overline{WB'}}{\overline{B'U}}\right) = (-1)(-1) = 1$$

from the last two expressions, and therefore,

$$\frac{\overline{UP}}{\overline{PV}} \cdot \frac{\overline{VQ}}{\overline{QW}} \cdot \frac{\overline{WR}}{\overline{RU}} = -1,$$

so that P, Q, and R are collinear by Menelaus' Theorem. □

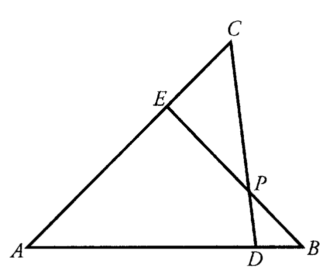
Numerical Applications

In addition to using Menelaus' Theorem to determine when menelaus points are collinear, we can also use it to calculate distances when three points are known to be collinear.

Such problems often present us with several different ways of viewing the same diagram.

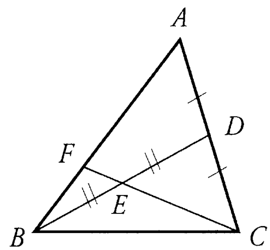
In the figure on the right, there are several "menelaus patterns":

- AB is a transversal for $\triangle PEC$.
- AC is a transversal for $\triangle PDB$.
- BE is a transversal for $\triangle ADC$.
- CD is a transversal for $\triangle ABE$.



In solving a problem, it is sometimes a matter of trying various combinations until we find one that works.

Example 4.4.6. *BD is a median of $\triangle ABC$, and E is the midpoint of BD. Line CE extended meets AB at F. If $AB = 6$, find the length of FB.*



Solution. We wish to find a menelaus pattern whose triangle has two of its edges divided into known ratios. Referring to the figure, note that if we consider CF as a transversal for $\triangle ABD$, then CF divides the sides BD and AD into known ratios. So, let $FB = x$ and apply Menelaus' Theorem to $\triangle ABD$ with transversal CF:

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BE}}{\overline{ED}} \cdot \frac{\overline{DC}}{\overline{CA}} = -1,$$

which implies that

$$\frac{AF}{x} \cdot 1 \cdot \left(-\frac{1}{2}\right) = -1,$$

which in turn implies that

$$\frac{AF}{x} = 2.$$

Thus, $AF = 2x$, so $AB = 3x$, and it follows that $x = AB/3 = 2$.

□

4.5 Proofs of the Theorems

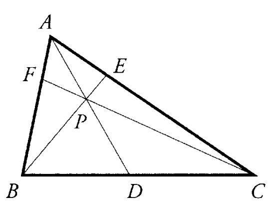
Both Ceva's and Menelaus' Theorems are of the "if and only if" variety, and so in each case there are two things to prove.

Proof of Ceva's Theorem

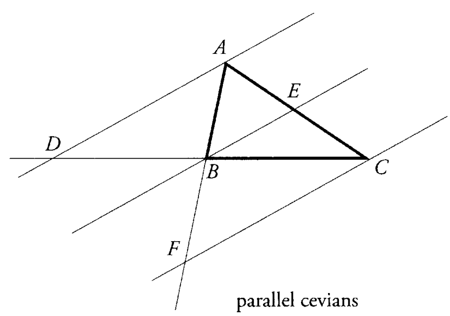
We will first prove the necessity; that is, we will prove the following:

Theorem 4.5.1. *(Ceva's Theorem: Necessity)*
Let AD, BE, and CF be cevians of triangle ABC, with D on BC, E on CA, and F on AB. If AD, BE, and CF are concurrent or parallel, then

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

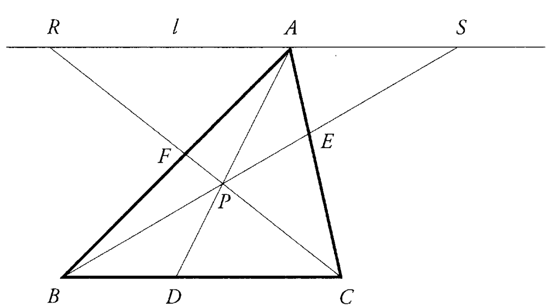


concurrent cevians



parallel cevians

Proof. We will prove the theorem for the case where AD, BE, and CF are concurrent. The case where the cevians are parallel is left as an exercise. Suppose the cevians AD, BE, and CF are concurrent at a point P. Let l be a line through A parallel to BC, let S be the point where BE meets l, and let R be the point where CF meets l.



Then

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AR}}{\overline{BC}} \quad \text{(because } \triangle RAF \sim \triangle CBF\text{)},$$
$$\frac{\overline{CE}}{\overline{EA}} = \frac{\overline{BC}}{\overline{AS}} \quad \text{(because } \triangle ASE \sim \triangle CBE\text{)},$$
$$\frac{\overline{BD}}{\overline{DC}} = \frac{\overline{AS}}{\overline{AR}},$$

with the last equality arising from the fact that

$$\frac{\overline{BD}}{\overline{AS}} = \frac{\overline{DP}}{\overline{PA}} = \frac{\overline{DC}}{\overline{AR}}$$

because $\triangle BPD \sim \triangle SPA$ and $\triangle DPC \sim \triangle APR$. It follows that

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{AR}}{\overline{BC}} \cdot \frac{\overline{AS}}{\overline{AR}} \cdot \frac{\overline{BC}}{\overline{AS}} = 1,$$

which completes the proof of necessity.

□

Next, we wish to prove the sufficiency, that is:

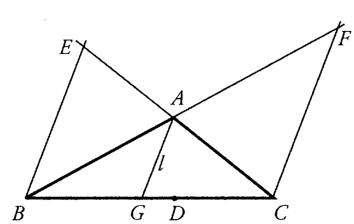
Theorem 4.5.2. *(Ceva's Theorem: Sufficiency)*
Let AD, BE, and CF be cevians of triangle ABC, with D on BC, E on CA, and F on AB. If

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1,$$

then AD, BE, and CF are either concurrent or parallel.

Proof. There are two cases to consider: either (i) BE and CF are parallel or else (ii) BE and CF meet at a single point P.

Case (i). Let l be a line through A parallel to BE. Since BE is not parallel to BC (by the definition of a cevian), it follows that l is not parallel to BC, and so l must meet the line BC at some point G. Then, the cevians AG, BE, and CF are parallel, so by the "necessary" part of the theorem we must have



$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BG}}{\overline{GC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

By hypothesis, we also have

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1,$$

so that

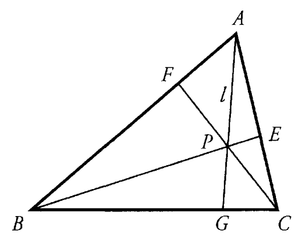
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BG}}{\overline{GC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}}.$$

Thus,

$$\frac{\overline{BG}}{\overline{GC}} = \frac{\overline{BD}}{\overline{DC}},$$

and this implies that $G = D$. This shows that the cevians AD, BE, and CF are parallel.

Case (ii). Let l be a line through A and P. We will first show that if l is not parallel to BC, then l is actually the cevian AD. To see why, let us suppose that l meets BC at G. Then the cevians AG, BE, and CF are concurrent at P, so by the first part of the theorem we must have



$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BG}}{\overline{GC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

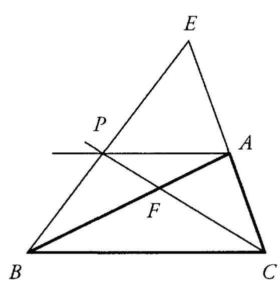
Since we also have

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1,$$

when l is known not to be parallel to BC, the proof may be completed as in case (i).

To completely finish case (ii), we must show that AP cannot be parallel to BC, and we will prove this by contradiction.

Let us suppose that AP is parallel to BC. It then follows that



$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AP}}{\overline{CB}}, \quad \text{because } \triangle AFP \sim \triangle BFC,$$
$$\frac{\overline{CE}}{\overline{EA}} = \frac{\overline{CB}}{\overline{PA}}, \quad \text{because } \triangle CEB \sim \triangle AEP,$$

and, consequently, that

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{AP}}{\overline{CB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CB}}{\overline{PA}} = -\frac{\overline{BD}}{\overline{DC}}.$$

Since

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1,$$

it follows that

$$\frac{\overline{BD}}{\overline{DC}} = -1.$$

But there is no point D on the line BC for which

$$\frac{\overline{BD}}{\overline{DC}} = -1.$$

This shows that AP cannot be parallel to BC and completes the proof of case (ii).

□

Proof of Menelaus' Theorem

As with Ceva's Theorem, we will treat the "necessary" and "sufficient" parts separately. There are a variety of different proofs for the first part, and we will give additional proofs later in this chapter.

Theorem 4.5.3. *(Menelaus' Theorem: Necessity)*
If the menelaus points D, E, and F on the (extended) sides BC, CA, and AB of a triangle are collinear, then

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1.$$

Proof. The proof uses the fact that a transversal meets either two sides internally or no sides internally. Thus, either exactly one or all three of the ratios

$$\frac{\overline{AF}}{\overline{FB}}, \quad \frac{\overline{BD}}{\overline{DC}}, \quad \text{and} \quad \frac{\overline{CE}}{\overline{EA}}$$

are numerically negative, and consequently the sign of

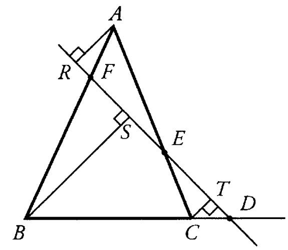
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}}$$

is guaranteed to be negative. So, to prove the "necessary" part, we need only show that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Drop perpendiculars AR, BS, and CT from the vertices to the transversal. Then,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{AR}{BS} \cdot \frac{BS}{CT} \cdot \frac{CT}{AR} = 1\,.$$



The reasons are

$$\frac{AF}{FB} = \frac{AR}{BS} \quad (\triangle AFR \sim \triangle BFS),$$
$$\frac{BD}{DC} = \frac{BS}{CT} \quad (\triangle BSD \sim \triangle CTD),$$

and

$$\frac{CE}{EA} = \frac{CT}{AR} \quad (\triangle ARE \sim \triangle CTE).$$

This completes the proof of the necessity.

□

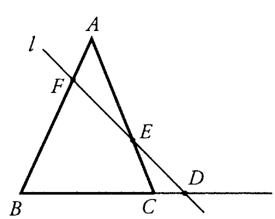
For the sufficiency part of Menelaus' Theorem, we need to prove the following:

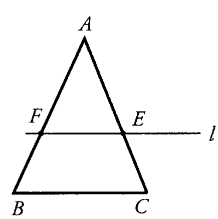
Theorem 4.5.4. *(Menelaus' Theorem: Sufficiency)*

Let D, E, and F be three menelaus points on sides BC, CA, and AB of a triangle. If

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1,$$

then D, E, and F are collinear.





Proof. Suppose that

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1.$$

Let l be the line through F and E. Then there are two possibilities:

(i) l meets BC at some point D' or

(ii) l is parallel to BC.

Case (i). l meets BC at some point P.

In this case, it suffices to show that $P = D$. By the "necessary" part of Menelaus' Theorem,

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1.$$

This, together with the hypothesis, means that

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BP}}{\overline{PC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}},$$

and so

$$\frac{\overline{BP}}{\overline{PC}} = \frac{\overline{BD}}{\overline{DC}},$$

implying that $P = D$ and completing case (i).

Case (ii). l is parallel to BC.

We will show that this case cannot arise, for if l is parallel to BC, then

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AE}}{\overline{EC}}.$$

Now, by the hypothesis,

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1,$$

and since

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AE}}{\overline{EC}},$$

it follows that

$$\frac{\overline{BD}}{\overline{DC}} = -1.$$

But it is impossible for any point D on the line BC to satisfy

$$\frac{\overline{BD}}{\overline{DC}} = -1.$$

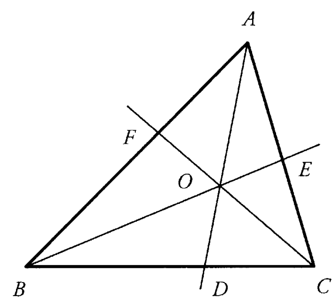
This finishes the proof of the "sufficient" part of the theorem and completes the proof of Menelaus' Theorem.

$\square$

Additional Proofs

The similarity between Menelaus' Theorem and Ceva's Theorem is not just in the language. Here is how Menelaus' Theorem can be used to prove the first part of Ceva's Theorem.

Example 4.5.5. *Menelaus' Theorem implies Ceva's Theorem.*



Solution. In the figure on the right, assuming that the cevians are concurrent, our objective is to show that

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = +1\,.$$

The idea is to use the cevian AD of triangle ABC to decompose the triangle ABC into two "subtriangles" ABD and ADC. We then use the facts that CF is a transversal for $\triangle ABD$ and BE is a transversal for $\triangle ADC$.

Applying Menelaus' Theorem to $\triangle ABD$ and transversal COF,

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BC}}{\overline{CD}} \cdot \frac{\overline{DO}}{\overline{OA}} = -1\,,$$

and also applying Menelaus' Theorem to $\triangle ADC$ and transversal BOE,

$$\frac{\overline{AO}}{\overline{OD}} \cdot \frac{\overline{DB}}{\overline{BC}} \cdot \frac{\overline{CE}}{\overline{EA}} = -1\,.$$

Multiplying the two equations together and cancelling a few terms, we get

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{DB}}{\overline{CD}} \cdot \frac{\overline{CE}}{\overline{EA}} = (-1)^2 = +1.$$

Thus,

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = +1,$$

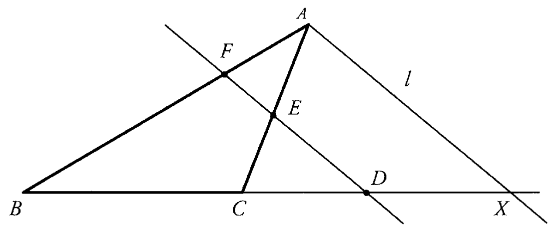
which completes the proof.

□

The following two proofs of Menelaus' Theorem are from a 1937 letter that the physicist Albert Einstein wrote to the psychologist Max Wertheimer.

Einstein's Ugly Proof

Einstein called this his "ugly proof." It is based on similar triangles but uses a single parallel line rather than three perpendicular lines to create them.



Given $\triangle ABC$ and a transversal, draw l through A parallel to the transversal hitting BC at X. Then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{DX}{DB} \cdot \frac{BD}{DC} \cdot \frac{CD}{DX} = 1,$$

and the theorem follows.

□

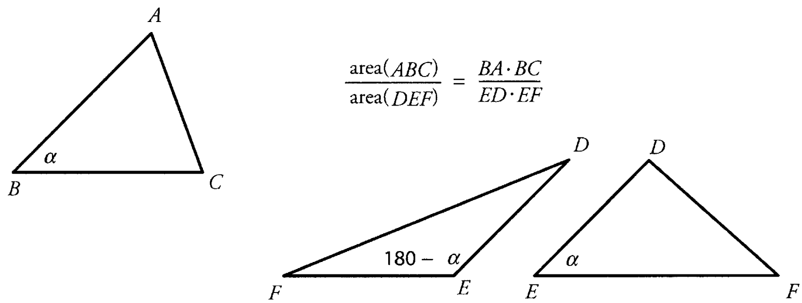
Einstein's Area Proof

Einstein called this next proof his "area proof." It uses the following lemma.

Lemma 4.5.6. *The ratio of the areas of two triangles that have a common or supplementary angle is equal to the ratio of the corresponding products of their adjacent sides.*

Proof. As in the figure on the following page,

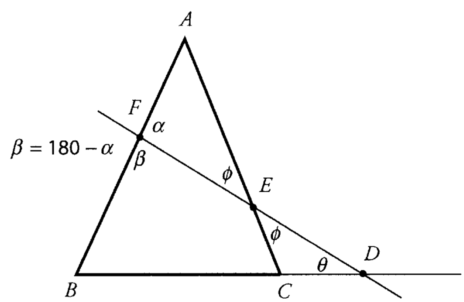
$$\frac{\text{area}(ABC)}{\text{area}(DEF)} = \frac{\frac{1}{2}BC \cdot BA \sin\alpha}{\frac{1}{2}EF \cdot ED \sin\alpha} = \frac{BA \cdot BC}{ED \cdot EF}.$$



Since $\sin(180 - \alpha) = \sin \alpha$, the same result holds for supplementary angles.

□

Einstein's Area Proof.



Given $\triangle ABC$ and a transversal, then by the lemma,

$$\frac{\text{area}(AFE)}{\text{area}(BFD)} = \frac{AF \cdot FE}{BF \cdot FD},$$
$$\frac{\text{area}(BFD)}{\text{area}(CDE)} = \frac{BD \cdot FD}{CD \cdot DE},$$
$$\frac{\text{area}(CDE)}{\text{area}(AFE)} = \frac{CE \cdot ED}{AE \cdot FE},$$

from which it follows that

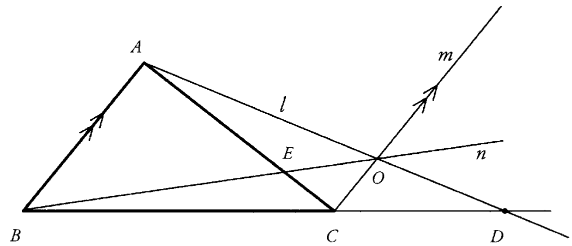
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{FD \cdot \text{area}(AFE)}{FE \cdot \text{area}(BFD)} \cdot \frac{DE \cdot \text{area}(BFD)}{FD \cdot \text{area}(CDE)} \cdot \frac{FE \cdot \text{area}(CDE)}{DE \cdot \text{area}(AFE)}$$
$$= 1,$$

and the theorem follows.

□

4.6 Extended Versions of the Theorems

Example 4.6.1. *(A Cevian Parallel to a Side)*



In the figure above there are three concurrent cevians, l, m, and n, so the cevian product

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}}$$

should be $+1$. But we have a problem: where is F? It should be the point where m meets the extended side AB, but m and AB are parallel and so that point does not exist. □

We have dealt with the problem by defining the word "cevian" so that it excludes lines like m that are parallel to a side. This approach is unsatisfactory. It seems like we are playing with semantics rather than doing geometry. After all, the "cevians" in the example are concurrent, and it is a legitimate source of concern that Ceva's Theorem cannot handle this case.

In this section, we will show that the problem can be remedied by introducing certain ideal elements called "points at infinity" or "ideal points" at which two parallel lines meet. (If you have ever looked down a long straight prairie railway track, you may have used exactly the same language and said that the rails "meet at infinity.")

When we introduce ideal points, we must be careful that we do not create more problems than we remove. The following conventions and definitions will help us avoid complications.

We append to the plane a collection of ***ideal points*** or ***points at infinity***. This collection of ideal points is called the ***ideal line*** or the ***line at infinity***. Points that are not ideal points are called ***ordinary points***; lines that are not ideal lines are called ***ordinary lines***. (The words "point" and "line" by themselves do not distinguish between the ordinary and the ideal versions.)

We postulate that:

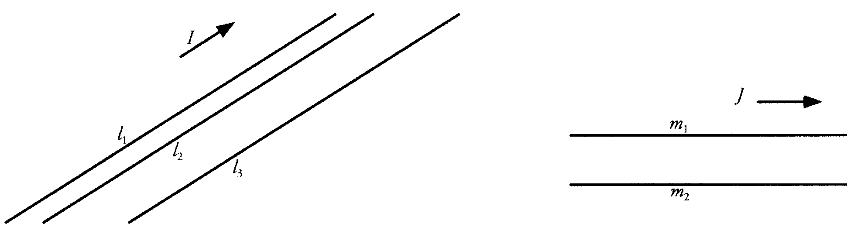
(i) Every ordinary line of the plane contains exactly one ideal point.

(ii) Parallel lines have the same ideal point.

(iii) Nonparallel lines do not have the same ideal point.

The plane together with all of the ideal points is called the ***extended Euclidean plane*** or simply the ***extended plane***. The plane without the ideal line is called the ***Euclidean plane*** to distinguish it from the extended plane.[3]

We designate an ideal point by an arrow indicating the direction in which it lies, and an arrow pointing in exactly the opposite direction designates the same ideal point. In the figure below, parallel lines l_1, l_2, and l_3 all meet at the ideal point I. The lines m_1 and m_2 meet at a different ideal point J.



In the extended plane, every two distinct points determine a unique line, and every two distinct lines determine a unique point. Technically, in the extended plane there are no parallel lines. Nevertheless, we will continue to use the word *parallel* to describe ordinary lines in the extended plane that meet only at an ideal point.

If I is the ideal point on the line AB, we cannot assign a real number to the directed distance AI. However, Ceva's Theorem deals with directed ratios, and we adopt the convention that

$$\frac{\overline{AI}}{\overline{IB}} = -1$$

for the ideal point on the line AB.

[3]The extended plane is also called the ***projective plane*** and is the setting for what is called "projective geometry" in Part III.

This makes a certain amount of sense, for two reasons:

1. We have already observed that there is no ordinary point X on the line for which

$$\frac{\overline{AX}}{\overline{XB}} = -1.$$

2. As the ordinary point X on the ordinary line AB moves along the line away from A, the ratio

$$\frac{\overline{AX}}{\overline{XB}}$$

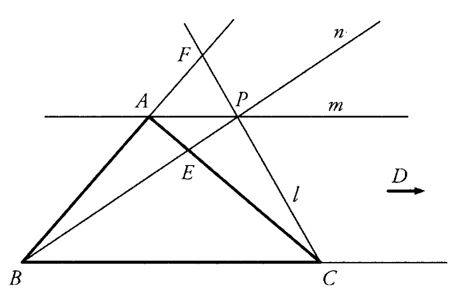
tends to -1.

4.6.1 Ceva's Theorem in the Extended Plane

In the extended Euclidean plane, a ***cevian*** is defined to be any line that contains exactly one vertex of an *ordinary* triangle. (An **ordinary triangle** is one whose edge segments do not contain ideal points.) Thus, the notion of a cevian has been modified to include lines through a vertex that in the Euclidean plane would be parallel to the opposite side.

The following two examples show that Ceva's Theorem is valid in the extended Euclidean plane with the modified notion of a cevian.

Example 4.6.2. *Given an ordinary triangle ABC and three concurrent cevians l, m, and n, with m parallel to BC, show that the cevian product is* 1.



Solution. We have

$$\begin{aligned}\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} &= \left(-\frac{AF}{FB}\right) \cdot (-1) \cdot \frac{CE}{EA} \\ &= \frac{AF}{FB} \cdot \frac{CE}{EA},\end{aligned}$$

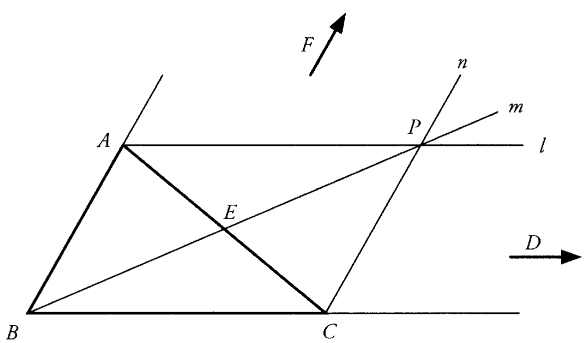
and by similar triangles,

$$\begin{aligned}\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} &= \frac{AP}{BC} \cdot \frac{BC}{AP} \\ &= 1\,,\end{aligned}$$

showing that the cevian product is 1.

□

Example 4.6.3. *Given $\triangle ABC$ and three concurrent cevians l, m, and n, with l parallel to BC and n parallel to AB, show that the cevian product is* 1.



Solution.

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = (-1) \cdot (-1) \cdot \frac{CE}{EA} = +1\,,$$

since the diagonals of parallelogram $ABCP$ bisect each other.

□

The previous two examples show that in the extended plane, Ceva's Theorem can be stated more succinctly.

Theorem 4.6.4. *(Ceva's Theorem for the Extended Plane)*

A necessary and sufficient condition that the cevians AD, BE, and CF of a triangle ABC be concurrent is that

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = +1.$$

The theorem includes the possibility that the point of concurrency is an ideal point; that is, it includes the possibility that the three cevians are parallel in the Euclidean plane. It also includes the possibility that some of the vertices are ideal points, but one must be careful about the sufficiency part of the theorem.

4.6.2 Menelaus' Theorem in the Extended Plane

In the extended Euclidean plane, the notion of a transversal is modified to include those that are parallel to a side of an ordinary triangle: any ordinary line that does not contain a vertex of the triangle is called a ***transversal*** of the triangle.

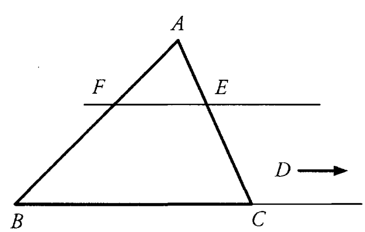
Example 4.6.5. *Show that if a transversal l is parallel to side BC of an ordinary triangle ABC, then the corresponding cevian product is -1.*

Solution. Let us suppose that l meets sides BC, CA, and AB at D, E, and F, respectively. Then D is an ideal point and, since EF is parallel to BC,

$$\frac{\overline{AF}}{\overline{FB}} = \frac{\overline{AE}}{\overline{EC}},$$

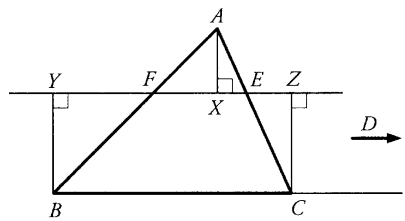
so that

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{\overline{AF}}{\overline{FB}} \cdot (-1) \cdot \frac{\overline{CE}}{\overline{EA}} = -1.$$



An alternate proof: drop perpendiculars from the vertices, as in the figure on the right. Then

$$\begin{aligned}\frac{\overline{AF}}{\overline{FB}}\cdot\frac{\overline{BD}}{\overline{DC}}\cdot\frac{\overline{CE}}{\overline{EA}} &= \frac{\overline{AF}}{\overline{FB}}\cdot(-1)\cdot\frac{\overline{CE}}{\overline{EA}}\\ &= \frac{\overline{AX}}{\overline{YB}}\cdot(-1)\cdot\frac{\overline{CZ}}{\overline{XA}}\\ &= -1,\end{aligned}$$



since $\overline{YB} = -\overline{CZ}$.

□

In the extended Euclidean plane, the statement of Menelaus' Theorem is the same as before.

Theorem 4.6.6. *(Menelaus' Theorem for the Extended Plane)*

Let ABC be an ordinary triangle, and let D, E, F be nonvertex points on the (possibly extended) sides BC, CA, and AB of the triangle. Then the points D, E, and F are collinear if and only if

$$\frac{\overline{AF}}{\overline{FB}}\cdot\frac{\overline{BD}}{\overline{DC}}\cdot\frac{\overline{CE}}{\overline{EA}} = -1\,.$$

Although the statement is the same, the meaning is a bit different, for in the extended version neither the vertices nor the menelaus points are restricted to being ordinary points.

Remark. The increased generality of the theorems does have some cost. If we wish to apply the extended versions to Euclidean problems, we must first restate the problem in the extended plane. For example, the theorem about the concurrency of the medians of a triangle would, in the extended plane, be stated as follows:

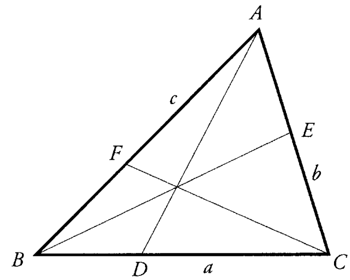
The medians of an ordinary triangle are concurrent at an ordinary point.

To prove this by the extended version of Ceva's Theorem, we would have to show that the point of concurrency is not an ideal point; that is, we still have to show that the cevians in question are not parallel.

4.7 Problems

1. Show that the lines drawn from a vertex to a point halfway around the perimeter of a triangle are concurrent.

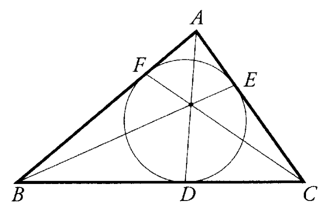
   This point of concurrency is called the ***Nagel point*** of the triangle. In the diagram, a, b, and c denote the lengths of the sides.



2. Given $\triangle ABC$ with its incircle, the lines drawn from the vertices to the opposite points of tangency are concurrent.

   This point of concurrency is known as the ***Gergonne point*** of the triangle.

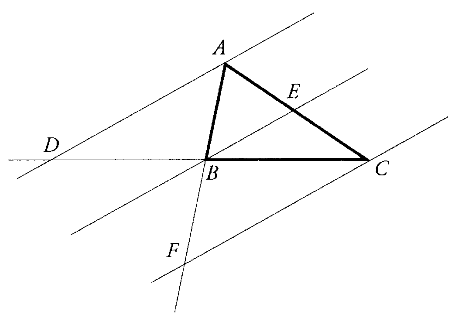
   The reader is cautioned that the Gergonne point and the incenter are usually different points, and the associated cevians are neither altitudes nor angle bisectors.

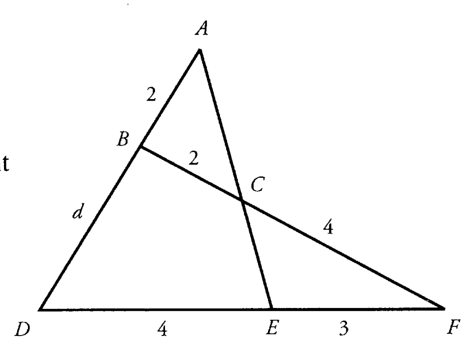


3. Prove the necessary part of Ceva's Theorem for parallel cevians; that is, prove the following:

   If the cevians AD, BE, and CF are parallel cevians for triangle ABC, then

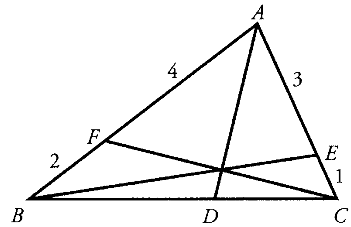
$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$



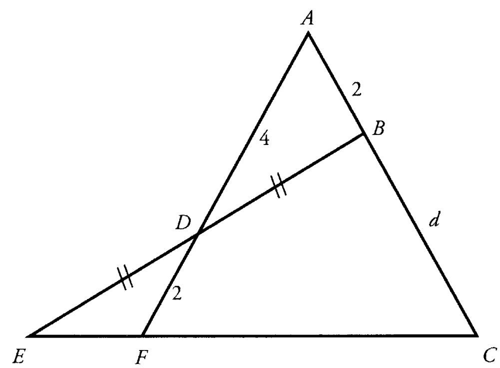


4. Find the length d of the segment BD in the following figure.

5. Prove that AD is an angle bisector in the figure below.

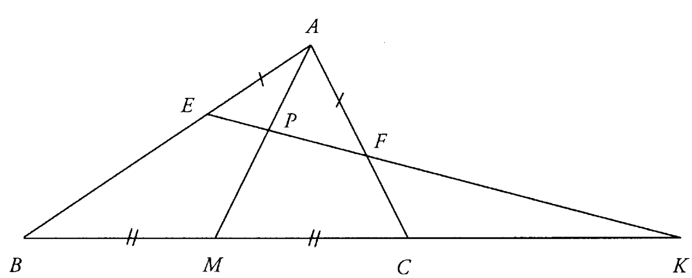


6. In the figure below, find d, the length of the segment BC.



7. (a) Prove that an external angle bisector is parallel to the opposite side if and only if the triangle is isosceles.
   (b) Show that in a nonisosceles triangle, the external angle bisectors meet the opposite sides in three collinear points.

8. In the figure, show that $EP \cdot AB = FP \cdot AC$.



CHAPTER 5

AREA

5.1 Basic Properties

In this chapter we will use the word ***polygon*** to refer to a polygon together with its interior, even though properly we should use the term ***polygonal region***. This should not cause any confusion.

Suppose one polygon is inside another. When treated as wire frames, the polygons would be considered as being disjoint; in the present context, they overlap. In general, if two figures share interior points, they will be considered as ***overlapping***; otherwise, they will be considered as ***nonoverlapping***.

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

5.1.1 Areas of Polygons

We will associate with each simple polygon a nonnegative number called its ***area***, and we will assume that area has certain reasonable properties.

Postulates for polygonal areas:

(i) To each simple polygon is associated a nonnegative number called its ***area***.

(ii) ***Invariance Property***: Congruent polygons have equal area.

(iii) ***Additivity Property***: The area of the union of a finite number of nonoverlapping polygons is the sum of the areas of the individual polygons.

(iv) ***Rectangular Area***: The area of an $a \times b$ rectangle is ab.

Square brackets will be used to denote area. So, for example, the area of a quadrilateral $ABCD$ will be denoted $[ABCD]$.

Properties (ii) and (iii) certainly conform to our preconceived notions about area. We expect figures to have the same area if they have the same shape and size, and we also expect to be able to find the area of a large shape by summing the areas of the individual pieces making up the shape.

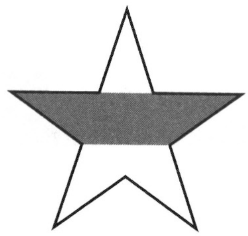
To develop a concrete theory of area, we must specify how it is to be measured, and our specification must satisfy (ii) and (iii). The easiest (and most useless) way would be to designate the area of every figure to be zero. To avoid this, a fundamental shape is chosen and defined to have a positive area, which is what statement (iv) accomplishes.

As an alternative to (iv), we could have used the ***unit square***[4] as the fundamental region, defined its area to be one square unit, and derived statement (iv) from it. This approach, although logical, presents some obstacles, especially when the rectangle has sides of irrational length.[5] Because of this, we have chosen to use the set of ***all rectangles*** as a family of fundamental regions on which to base the computation of areas.

[4]The **unit square** is a 1×1 square.

[5]A **rational** number is a real number that can be expressed as one integer divided by another nonzero integer. A number that cannot be expressed this way is called **irrational**.

Exercise 5.1.1. *The following figure is a regular five-pointed star. Which is larger, the area of the shaded part or the total area of the unshaded parts?*



Since a square is a special case of a rectangle, we immediately have the following formula.

Theorem 5.1.2. *(Area of a Square)*

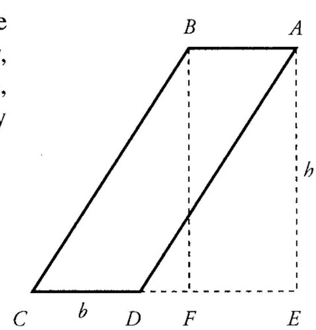
The area of an $a \times a$ square is a^2.

Now let us consider a parallelogram, of which a rectangle is a special case. We may choose any of its sides as the ***base***. We will use the same symbol to designate both the length of the base and the base itself. The context will make it clear which meaning is intended. The distance from the base to the opposite side is called the ***altitude*** on that base.

Theorem 5.1.3. *(Area of a Parallelogram)*

The area of a parallelogram with altitude h on a base b is bh.

Proof. Let $ABCD$ be a parallelogram with $AB = b$. Drop perpendiculars AE and BF to CD from A and B, respectively. Then $ABFE$ is a rectangle with $AE = h$, so that $[ABFE] = bh$. Now, triangles ADE and BCF are congruent. Hence, $[ADE] = [BCF]$ by the Invariance Property. By the Additivity Property,



$$\begin{aligned}[ABCD] &= [ABCE] - [ADE] \\ &= [ABCE] - [BCF] \\ &= [ABFE] \\ &= bh,\end{aligned}$$

which completes the proof. $\square$

Any side of a triangle may be designated as the ***base***. The perpendicular from the opposite vertex to the base is called the ***altitude*** on that base. As with parallelograms, the word *base* has a dual meaning, referring to a specific side of a triangle and also to the length of that side. For triangles, the word *altitude* has a similar double meaning.

Theorem 5.1.4. *(Area of a Triangle: Base-Altitude Formula)*

The area of a triangle with altitude h on a base b is $\frac{1}{2}bh$.

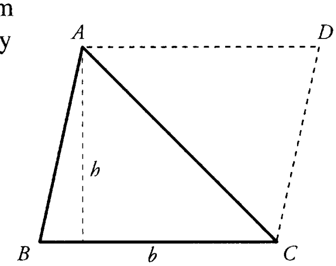
Proof. Let ABC be the triangle. Complete the parallelogram $ABCD$. Then the area of parallelogram $ABCD$ is bh. Now, since $\triangle ABC \equiv \triangle CDA$, by the Invariance Property, we have

$$[ABC] = [CDA],$$

and, by the Additivity Property,

$$[ABC] + [CDA] = [ABCD],$$

so that $[ABC] = \frac{1}{2}bh$.

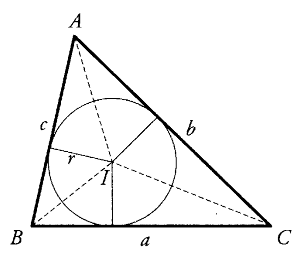


□

Theorem 5.1.5. *(Area of a Triangle: Inradius Formula)*

The area of a triangle with inradius r and semiperimeter s is rs.

(The ***semiperimeter*** is half the perimeter.)



Proof. Let I be the incenter of triangle ABC. Then

$$\begin{aligned}[ABC] &= [IBC] + [ICA] + [IAB] \\ &= \frac{r}{2}(BC + CA + AB) \\ &= rs.\end{aligned}$$

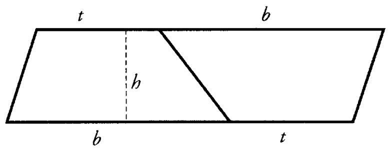
□

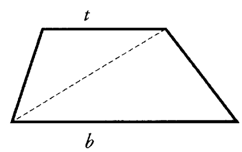
Example 5.1.6. *Find the area of a trapezoid whose parallel bases have lengths b and t and whose altitude is h using:*

(1) the formula for the area of a parallelogram;

(2) the formula for the area of a triangle.

Solution.





(1) Two copies of the trapezoid will form a parallelogram with base $t + b$ and altitude h. Its area is $(t + b)h$, so the area of the trapezoid is $\frac{1}{2}(t + b)h$.

(2) A diagonal divides the trapezoid into two triangles with altitudes h and respective bases t and b. The areas of the triangles are $\frac{1}{2}th$ and $\frac{1}{2}bh$, from which the desired result follows.

□

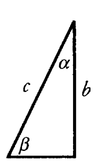
The following example shows how area can be used to give a simple proof of Pythagoras' Theorem.

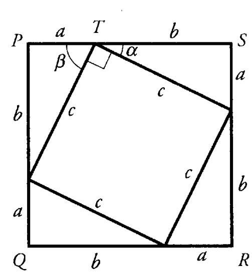
Example 5.1.7. *(Pythagoras' Theorem)*

Given a right triangle with sides of length a and b and with hypotenuse of length c,

$$a^2 + b^2 = c^2.$$

Proof. Draw a square with side c and place four copies of the triangle around it, as shown below.





Since $\alpha + \beta = 90°$, it follows that PTS is a straight line, and so $PQRS$ is a square. Using the Additivity Property of area, we have

$$(a+b)^2 = c^2 + 4 \cdot \frac{ab}{2},$$

so that

$$a^2 + 2ab + b^2 = c^2 + 2ab,$$

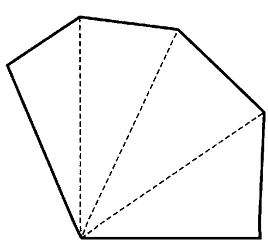
and so

$$a^2 + b^2 = c^2.$$

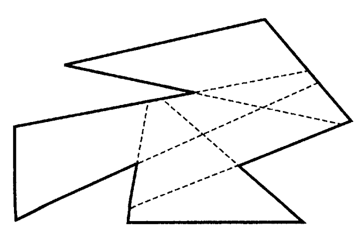
□

5.1.2 Finding the Area of Polygons

The area of an arbitrary polygon can be found by decomposing the polygon into triangular regions. If the polygon is convex, all diagonals are internal, so we can choose an arbitrary vertex and join it to all others by diagonals, as in figure (a) below, thereby dividing the polygon into triangles. Such a process is called ***triangulation***. Since we can determine the area of each triangle, the Additivity Property yields the area of the polygon.



(a)

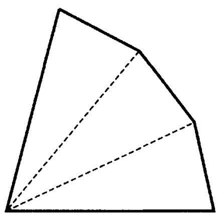


(b)

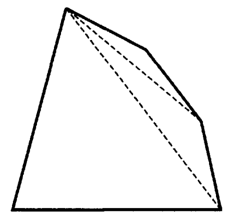
If the polygon is not convex, the following process can be used: extend the sides of all reflex angles until they meet the perimeter of the polygon. This will divide the polygon into convex polygons, each of which can be triangulated as in the preceding paragraph. Again, the Additivity Property yields the desired result.

The methods above show that the problem of finding the area of any polygon can always be reduced to the problem of finding the area of triangles.

However, from a computational point of view, triangulation methods are extremely clumsy, and we should find a better method. There are many different ways to triangulate a given polygon, and a good choice can ease the computation of the area. For example, we would likely prefer the triangulation of Figure (c) below rather than that of Figure (d), because the former decomposes the pentagon into congruent triangles.

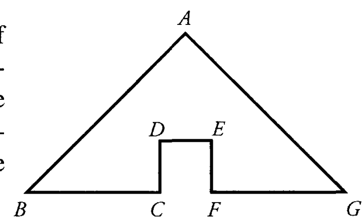


(c)



(d)

Also, we should keep in mind the possibility of using other methods that do not involve triangulation. In the figure on the right, it is surely more convenient to view the area of the 7-gon as the difference in the areas of triangle ABG and the square $CDEF$.

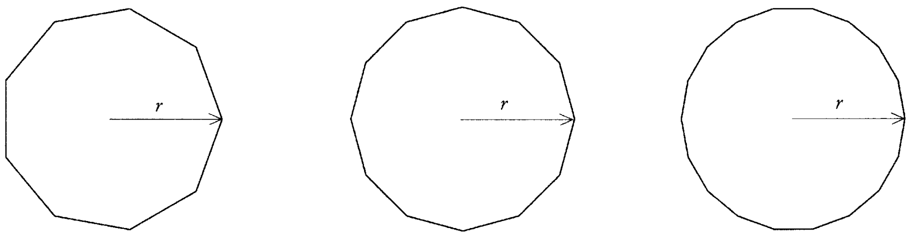


5.1.3 Areas of Other Shapes

Is it possible to extend the notion of area to *all bounded regions* of the plane in a way that satisfies both the Invariance Property and the Additivity Property?

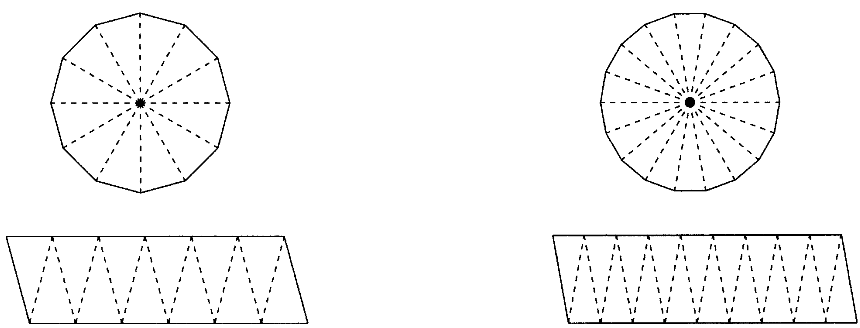
The answer to this question is yes—although the proof of this fact is well beyond the scope of this book. The positive answer means that the properties of polygonal area postulated earlier can therefore be applied to all bounded figures in the plane.

This does not mean, however, that the computation of areas of nonpolygonal regions is straightforward. In fact, one must resort to a limiting process even to find the area of such a basic region as a circular disk. The figure on the following page shows how to approximate a circle of radius r by polygons, with successive approximations coming closer and closer to the circle.



If the approximations in the figure above are cut along the dotted lines as shown below, the pie-shaped regions can be reassembled to form a parallelogram. As the approximations improve, the parallelogram comes closer and closer to becoming a rectangle whose altitude is the radius of the circle and whose base is half the circumference, which yields the following:

Theorem 5.1.8. *The area of a circle is half the product of its radius and its circumference.*



5.2 Applications of the Basic Properties

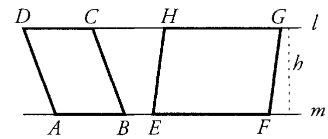
In this section we will develop several tools using the notion of area and give examples to show how these new tools can be used to solve a variety of problems. Some of the examples in this section may seem to have no connection with the notion of area . . . but they do.

Theorem 5.2.1. *Let l and m be parallel lines. Let $ABCD$ and $EFGH$ be parallelograms with AB and EF on m and CD and GH on l. Then*

$$\frac{[ABCD]}{[EFGH]} = \frac{AB}{EF}.$$

Proof. The two parallelograms have equal altitude h on the respective bases AB and EF. Hence,

$$\frac{[ABCD]}{[EFGH]} = \frac{h \cdot AB}{h \cdot EF} = \frac{AB}{EF}.$$

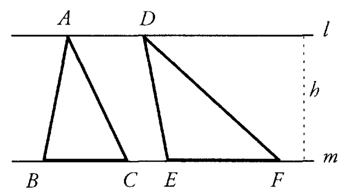


□

The same result holds for triangles.

Corollary 5.2.2. *If ABC and DEF are triangles such that A and D are on a line l and BC and EF are on a parallel line m, then*

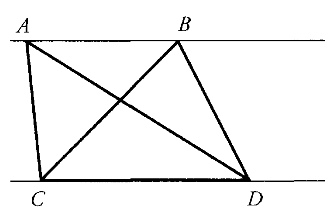
$$\frac{[ABC]}{[DEF]} = \frac{BC}{EF}.$$



Two special cases of this result that will be used frequently are:

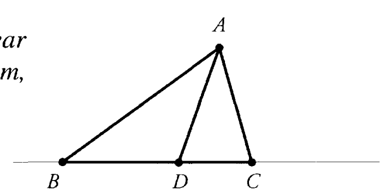
Corollary 5.2.3. *If AB and CD are parallel, then*

$$[ACD] = [BCD].$$



Corollary 5.2.4. *If B, C, and D are collinear points and if A is a point not collinear with them, then*

$$\frac{[ABD]}{[ADC]} = \frac{BD}{DC}.$$



The following is another proof that the three medians of a triangle are concurrent at the centroid.

Theorem 5.2.5. *The medians of a triangle are concurrent.*

Proof. Let D be the midpoint of the side BC of triangle ABC, and let G be a point on AD such that $AG = 2GD$. Suppose that the extension of BG meets CA at E. We will show that $AE = EC$ and from this deduce that the three medians of ABC are concurrent at G.

Let $[DEG] = x$ and $[BDG] = y$. Since

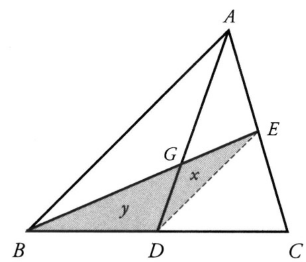
$$AG = 2GD,$$

then by Corollary 5.2.4, we have

$$[AGE] = 2x,$$
$$[BAG] = 2y,$$

and

$$[CDE] = [BDE] = x + y.$$



Hence,

$$[ABE] = 2x + 2y = [BEC],$$

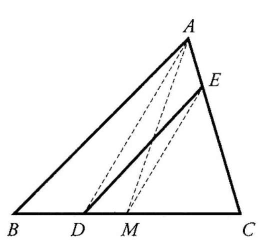
so that $AE = EC$; that is, BE is a median.

Similarly, if we let the extension of CG meet AB at F, then $AF = FB$ and AF is a median. Hence, the three medians are concurrent at G.

□

Example 5.2.6. *Let D be a point on the side BC of triangle ABC. Construct a line through D which bisects the area of ABC.*

Solution. If D is the midpoint of BC, then clearly AD is the desired line. Suppose D is between B and the midpoint M of BC. Draw a line through M parallel to AD, cutting CA at E.



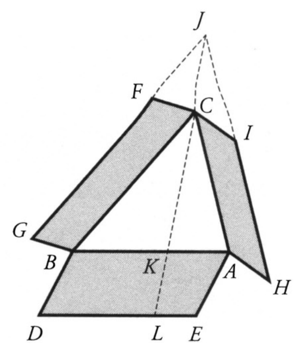
We claim that DE is the desired line. By Corollary 5.2.3, $[ADE] = [MAD]$, and so

$$\begin{aligned}[ABDE] &= [BAD] + [ADE]\\ &= [BAD] + [MAD]\\ &= [BAM]\\ &= \frac{1}{2}[ABC].\end{aligned}$$

□

Example 5.2.7. *$ABDE$, $BCFG$, and $CAHI$ are three parallelograms drawn outside $\triangle ABC$. The lines FG and HI meet at J. The extension of JC meets AB at K and the line DE at L. If $JC = KL$, prove that*

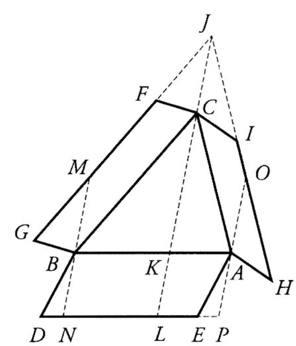
$$[ABDE] = [BCFG] + [CAHI].$$



Solution. Draw a line through B parallel to JL, cutting the line FG at M and the line DE at N. Draw a line through A parallel to JL, cutting the line HI at O and the line DE at P. By Theorem 5.2.1 and the Additivity Property of area, we get

$$\begin{aligned}[ABDE] = [ABNP] &= [KBNL] + [AKLP]\\ &= [JMBC] + [JOAC]\\ &= [BCFG] + [CAHI],\end{aligned}$$

which completes the proof.



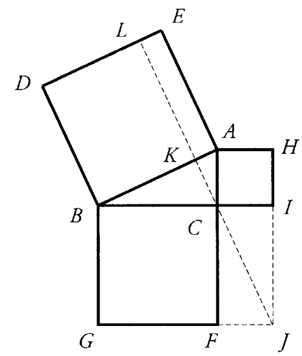
□

Example 5.2.7 is due to Pappus and has Pythagoras' Theorem as a corollary:

Corollary 5.2.8. *(Pythagoras' Theorem)*

Let ABC be a triangle. If $\angle C$ is a right angle, then

$$AB^2 = BC^2 + CA^2.$$



Proof. Suppose $\angle ACB$ is a right angle. Draw squares $ABDE$, $BCFG$, and $CAHI$ outside ABC. Let the extensions of GF and HI meet at J. Let the extension of JC meet AB at K and DE at L. Since triangles ABC and JCF are congruent, $JC = AB = AE = KL$. By Example 5.2.7,

$$\begin{aligned} AB^2 &= [ABDE] \\ &= [BCFG] + [CAHI] \\ &= BC^2 + CA^2. \end{aligned}$$

□

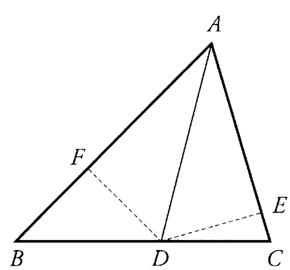
The next example uses Corollary 5.2.3 to obtain another proof of the Angle Bisector Theorem. In the proof, we make use of the fact that an angle bisector is characterized by each point on the bisector being equidistant from the arms of the angle. Before stating the theorem, we recall that the notation $\overline{XY}$ refers to the *directed distance* from X to Y. This enables us to distinguish between interior and exterior angle bisectors.

Theorem 5.2.9. *(The Angle Bisector Theorem)*

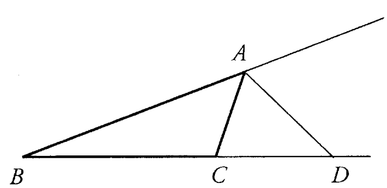
Let D be a point on the line BC and A a point not on it. Then

(1) $AB/AC = \overline{BD}/\overline{DC}$ if and only if AD bisects angle A of triangle ABC and

(2) $AB/AC = -\overline{BD}/\overline{DC}$ if and only if AD bisects the exterior angle A of triangle ABC.



(i)



(ii)

Proof.

(1) Drop perpendiculars DE and DF from D to CA and AB, respectively. Suppose AD bisects $\angle CAB$. Since D is on the angle bisector, we have $DE = DF$. By Corollary 5.2.3,

$$\overline{BD}/\overline{DC} = [BAD]/[CAD] = \left(\tfrac{1}{2}AB \cdot DF\right)/\left(\tfrac{1}{2}AC \cdot DE\right) = AB/AC.$$

Conversely, suppose $\overline{BD}/\overline{DC} = AB/AC$. Reversing the argument, we have $DE = DF$, so D is equidistant from the arms of $\angle BAC$. It follows that AD bisects $\angle CAB$.

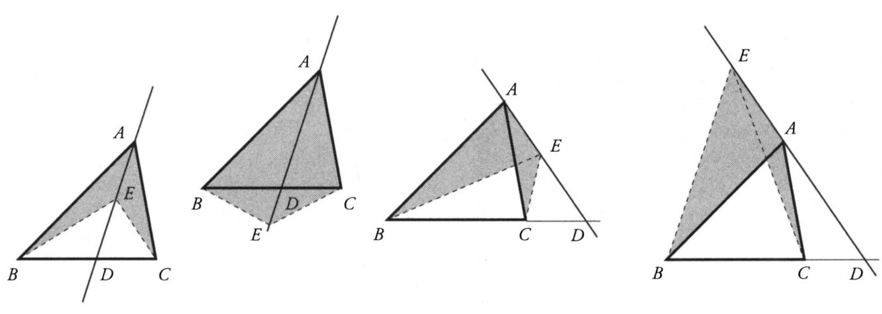
(2) Statement (2) can be proved in an analogous manner.

□

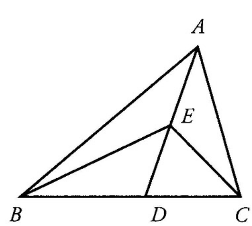
As the previous examples show, Corollary 5.2.3 is very useful. It can be generalized to the following result.

Theorem 5.2.10. *Let D be a point on the line BC and A a point not on it. If E is a point on the line AD, then*

$$\frac{[ABE]}{[ACE]} = \frac{BD}{CD}.$$



Proof. The figure above shows that there are many possibilities for the location of the points D and E. We will consider one subcase of the situation where D is between B and C, namely, the case where E is between A and D as in the figure on the right.



Let $BD/CD = t$. Corollary 5.2.3 implies that

$$\frac{[BAD]}{[CAD]} = t$$

and that

$$\frac{[BED]}{[CED]} = t.$$

Then

$$[ABE] = [BAD] - [BED] = t([CAD] - [CED]) = t[ACE],$$

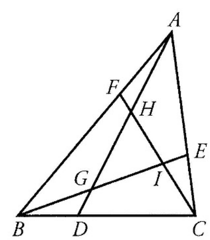
so that

$$\frac{[ABE]}{[ACE]} = t = \frac{BD}{CD}.$$

□

We now apply Theorem 5.2.10 to solve the following, a special case of a result due to Routh.[6]

Example 5.2.11. *Let D, E, and F lie respectively on the sides BC, CA, and AB of triangle ABC such that*

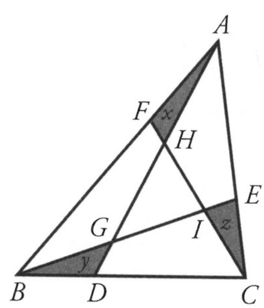


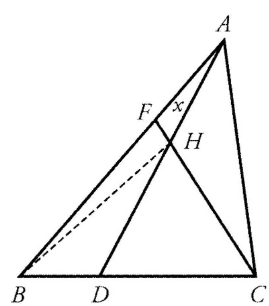
$$DC = 2BD, \quad EA = 2CE, \quad \textit{and} \quad FB = 2AF.$$

Suppose also that AD meets BE at G and CF at H and that BE meets CF at I.

Determine $\dfrac{[GHI]}{[ABC]}$.

Solution.





Let the areas of triangles HAF, GBD, and ICE be x, y, and z, respectively. We will first show that $[ABC] = 21x$ and that $x = y = z$.

[6]A generalization of this example was given by E. J. Routh in 1891 (without proof) who needed this ratio in his analysis of the stresses and tensions in mechanical frameworks.

By Corollary 5.2.3,

$$\frac{[HBF]}{[HAF]} = \frac{BF}{FA} = 2,$$

so that

$$[HBF] = 2x.$$

Similarly, since $DC = 2BD$, Theorem 5.2.10 gives us

$$[HAC] = 2[HAB] = 6x,$$

and so

$$[FAC] = [HAF] + [HAC] = 7x.$$

Since $BF = 2AF$, from Corollary 5.2.3, we get $[FBC] = 14x$ so that

$$[ABC] = 21x.$$

In the same way, we can prove that $[ABC] = 21y = 21z$. Hence $x = y = z$ and so

$$[BAG] = [CBI] = [ACH] = 6x.$$

It follows that

$$[GHI] = 21x - 18x = 3x,$$

so that

$$\frac{[GHI]}{[ABC]} = \frac{1}{7}.$$

□

5.3 Other Formulae for the Area of a Triangle

The **SAS** congruency condition asserts that a triangle is uniquely determined given two of its sides and the included angle. All other congruency conditions also describe geometric data that determine a unique triangle. Since these conditions do not explicitly specify the altitude of the triangle, the formula for the area of a triangle cannot be used without some preliminary work. It would be much more convenient to be able to compute the area of a triangle directly from the data that describes it. In this section we develop a list of formulae for the area of a triangle based on the various congruency conditions. As you might expect, some of the formulae involve trigonometric functions.

In this section we will continue to follow the practice of denoting the sides of a triangle by the lowercase letters that correspond to the labels for the opposite vertices.

We start by proving a useful trigonometric tool.

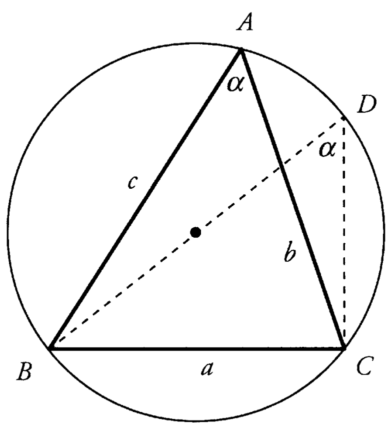
Theorem 5.3.1. *(Law of Sines)*

In triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where R is the circumradius.

Proof.



Let BD be the diameter of the circumcircle. Then $\angle BCD = 90^\circ$ and

$$\angle A = \angle BAC = \angle BDC$$

by Thales' Theorem. Hence,

$$a = BC = BD \sin A = 2R \sin A,$$

so that

$$\frac{a}{\sin A} = 2R.$$

Similarly,

$$\frac{b}{\sin B} \quad \text{and} \quad \frac{c}{\sin C}$$

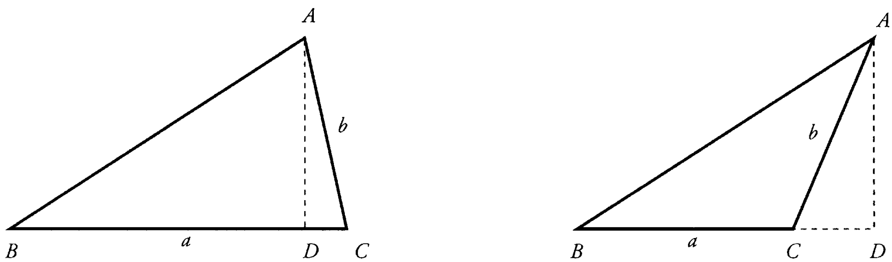
are both equal to $2R$.

Theorem 5.3.2. *(SAS Case)*

The area of a triangle with sides a, b and angle C is

$$[ABC] = \frac{1}{2}ab \sin C.$$

Proof.



Let AD be the altitude on BC. Then $AD = b \sin C = b \sin(180° - C)$. Hence,

$$[ABC] = \frac{1}{2}a\,AD = \frac{1}{2}ab \sin C\,.$$

□

Example 5.3.3. *Prove that*

$$[ABC] = \frac{abc}{4R},$$

where R is the circumradius of $\triangle ABC$.

Solution. We have

$$[ABC] = \frac{1}{2}ab \sin C = \frac{abc}{4R}$$

by the Law of Sines.

□

For the **ASA** case, we make use of all three angles.

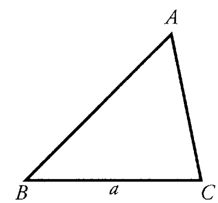
Theorem 5.3.4. *(***ASA** *Case)*

Given angles B and C and included side a of $\triangle ABC$, then

$$[ABC] = \frac{a^2 \sin B \, \sin C}{2 \sin A}.$$

Proof. By the Law of Sines, $a \sin B = b \sin A$. Hence,

$$\begin{aligned}[ABC] &= \frac{1}{2} ab \sin C \\ &= \frac{a^2 \sin B \, \sin C}{2 \sin A}.\end{aligned}$$



□

We remark that the Law of Sines is also useful when one needs to find the remaining dimensions of a triangle given a side and two angles.

The proof of the following uses the compound angle formula for the sine function, that is, $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

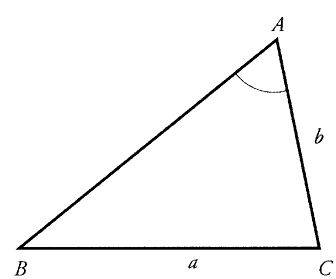
Theorem 5.3.5. *(***SSA** *Case)*

Assuming that $a > b$, the area of $\triangle ABC$ given sides a and b and angle A is

$$[ABC] = \frac{1}{2} b \sin A\big(\sqrt{a^2 - b^2 \sin^2 A} + b \cos A\big).$$

Proof. Since $a > b$, $A > B$, so that $\cos B > 0$. By the Law of Sines, $a \sin B = b \sin A$. Hence,

$$\begin{aligned}[ABC] &= \frac{1}{2} ab \sin C = \frac{1}{2} ab \sin(A + B) \\ &= \frac{1}{2} ab \sin A \cos B + \frac{1}{2} ab \cos A \sin B \\ &= \frac{1}{2} b \sin A(a \cos B + b \cos A) \\ &= \frac{1}{2} b \sin A\big(\sqrt{a^2 - a^2 \sin^2 B} + b \cos A\big) \\ &= \frac{1}{2} b \sin A\big(\sqrt{a^2 - b^2 \sin^2 A} + b \cos A\big).\end{aligned}$$



□

In the following, s denotes the semiperimeter of a triangle; that is, for a triangle with sides a, b, and c,

$$s = \frac{a+b+c}{2}.$$

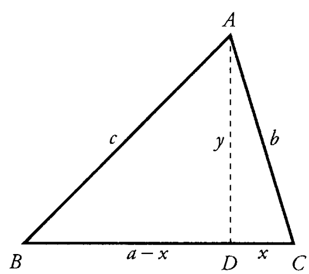
Theorem 5.3.6. *(***SSS** *Case: Heron's Formula)*

For a triangle with sides a, b, and c,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof. Let BC be the longest side. Then the foot D of the altitude AD lies between B and C. Let $CD = x$ and $AD = y$, so that $BD = a - x$. By Pythagoras' Theorem, $x^2 + y^2 = b^2$ and $(a-x)^2 + y^2 = c^2$. Subtraction yields

$$2ax - a^2 = b^2 - c^2 \quad \text{or} \quad x = (a^2 + b^2 - c^2)/2a.$$



Now,

$$\begin{aligned}
[ABC] &= \frac{1}{2}ay = \frac{1}{2}a\sqrt{b^2 - x^2} \\
&= \frac{1}{2}a\sqrt{b^2 - [\,(a^2+b^2-c^2)/2a\,]^2} \\
&= \frac{1}{4}\sqrt{4a^2b^2 - (a^2+b^2-c^2)^2} \\
&= \frac{1}{4}\sqrt{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)} \\
&= \frac{1}{4}\sqrt{((a+b)^2 - c^2)(c^2 - (a-b)^2)} \\
&= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)} \\
&= \sqrt{s(s-a)(s-b)(s-c)}.
\end{aligned}$$

□

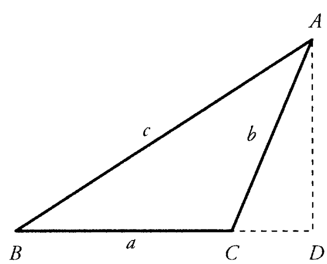
Heron's Formula was derived from Pythagoras' Theorem, but it is possible to reverse directions and derive Pythagoras' Theorem from Heron's Formula, meaning that the two theorems are equivalent. (To derive Pythagoras' Theorem, apply Heron's Formula to a right triangle. The details are left as an exercise.)

The next result, which is far more familiar than Heron's Formula, is also equivalent to Pythagoras' Theorem.

Theorem 5.3.7. *(The Law of Cosines)*

In triangle ABC,

$$c^2 = a^2 + b^2 - 2ab\cos C.$$



Proof. If $\angle C = 90°$, then $\cos C = 0$ and the result is just Pythagoras' Theorem. Suppose $\angle C > 90°$. Then the foot D of the altitude AD lies on the extension of BC. By Pythagoras' Theorem,

$$\begin{aligned} c^2 &= AD^2 + BD^2 = AD^2 + (a + CD)^2 \\ &= AD^2 + CD^2 + a^2 + 2a \cdot CD \\ &= b^2 + a^2 - 2ab\cos C, \end{aligned}$$

since $\cos C < 0$.

If $\angle BCA < 90°$, the argument is similar.

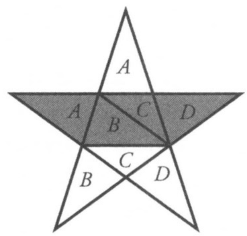
□

The Law of Cosines is not used to find the area of a triangle. It is indispensable, however, when one needs to find the remaining sides and angles in the **SAS** and **SSS** cases.

5.4 Solutions to the Exercises

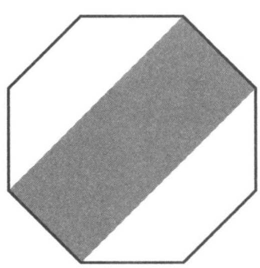
Solution to Exercise 5.1.1

In the figure below, the shaded region has been divided into four parts, each of which is congruent to a part of the unshaded region. Therefore, the shaded region is equal in area to the total of the unshaded regions.



5.5 Problems

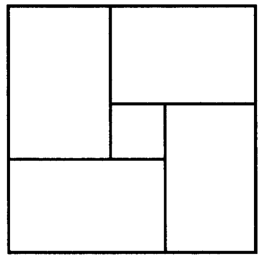
1. In the regular octagon shown below, is the area of the shaded region larger or smaller than the total area of the unshaded regions?



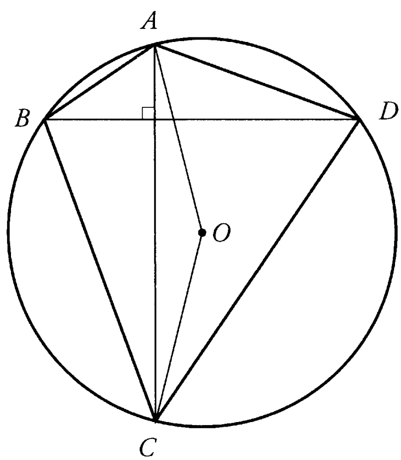
2. P is a point inside square $ABCD$. Show that

$$[APB] + [DPC] = [APD] + [BPC].$$

3. A square is divided into five nonoverlapping rectangles, with four of the rectangles completely surrounding the fifth rectangle, as shown in the diagram. The outer rectangles are the same area. Prove that the inner rectangle is a square.



4. $ABCD$ is a quadrilateral with perpendicular diagonals inscribed in a circle with center O. Prove that $[ADCO] = [ABCO]$.



5. A paper rectangle $ABCD$ of area 1 is folded along a straight line so that C coincides with A. Prove that the area of the pentagon obtained is less than $\frac{3}{4}$.

6. $ABCD$ is a parallelogram. E is a point on BC and F a point on CD. AE cuts BF at G, AF cuts DE at H, and BF cuts DE at K. Prove that

$$[AGKH] = [BEG] + [CEKF] + [DFH].$$

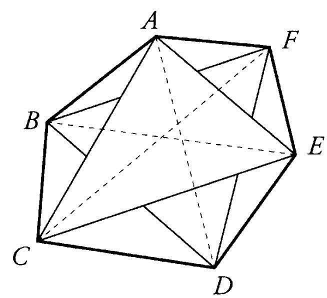
7. $ABCDEF$ is a convex hexagon in which opposite sides are parallel. Prove that $[ACE] = [BDF]$.

8. G is a point inside triangle ABC such that $[GBC] = [GCA] = [GAB]$. Show that G is the centroid of ABC.

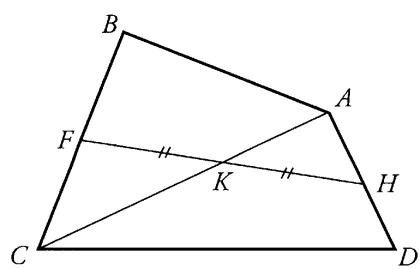
9. Seven children share a square pizza whose crust may be considered to consist only of the perimeter. Show how they make straight cuts to divide the pizza into seven pieces such that all pieces have the same amount of pizza and the same amount of crust.

10. AB and CD are four points such that $AB = CD$. Find the set of all points P such that $[PAB] = [PCD]$.

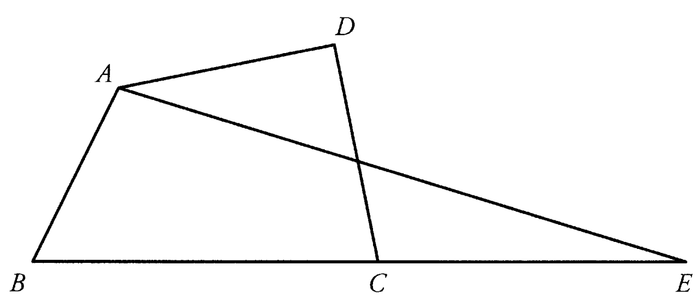
11. $ABCDEF$ is a convex hexagon with side AB parallel to CF, side CD parallel to BE, and side EF parallel to AD. Prove that $[ACE] = [BDF]$.



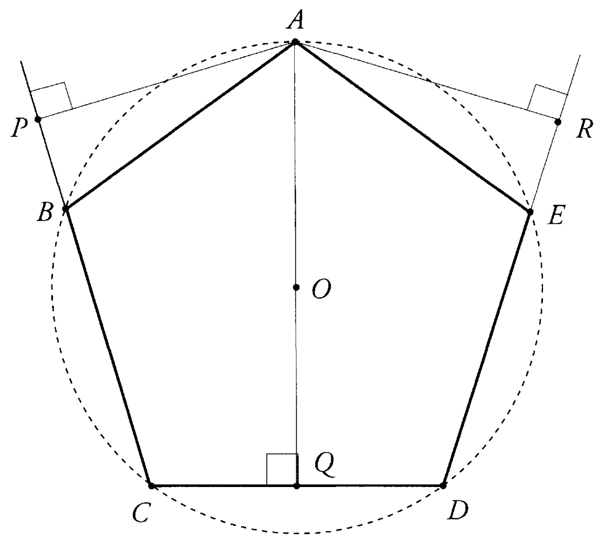
12. $ABCD$ is a convex quadrilateral, and F and H are the midpoints of BC and AD, respectively. If AC cuts FH at the midpoint K of FH, show that $[ABC] = [ADC]$.



13. Given a convex quadrilateral $ABCD$, construct a point E on the extension of BC such that the area $[ABCD] = [ABE]$.



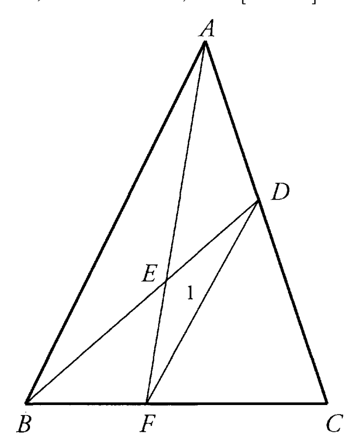
14. $ABCDE$ is a regular pentagon. The points P, Q, and R are the feet of the perpendiculars from A to BC, CD, and DE, respectively. The center O of the pentagon lies on AQ. If $OQ = 1$, compute $AP + AQ + AR$.



15. Given the figure below with $3BF = 2FC$, $AE = 2EF$, and $[DEF] = 1$:

(a) Find $[DFC]$.

(b) Find $\dfrac{AC}{AD}$.



16. M is a point interior to the rectangle $ABCD$. Prove that $AM \cdot CM + BM \cdot DM > ABCD$.

17. Given a rectangle, construct a square having the same area.

18. A triangle is inside a parallelogram. Prove that the area of the triangle is at most half that of the parallelogram.

19. P is a point inside an equilateral triangle ABC. Perpendiculars PD, PE, and PF are dropped from P onto BC, CA, and AB, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = [PAE] + [PCD] + [PBF].$$

20. Find the area of a triangle of sides 13, 18, and 31.

21. A triangle has sides 13, 14, and 15. Find its altitude on the base of length 14.

22. The sides of a triangle are 5, 7, and 8.

    (a) Calculate its area.

    (b) Calculate its inradius.

    (c) Calculate its circumradius.

23. $ABDE$, $BCGF$, and $CAHI$ are three squares drawn on the outside of triangle ABC, which has a right angle at C. Prove that

$$GD^2 - HE^2 = 3([BCFG] - [CAHI]).$$

24. M is a point interior to the rectangle $ABCD$. Prove that

$$AM \cdot CM + BM \cdot DM \geq [ABCD].$$

25. $ABCDE$ is a convex pentagon such that

$$AB = AC, \quad AD = AE, \qquad \text{and} \qquad \angle CAD = \angle ABE + \angle AEB.$$

If M is the midpoint of BE, prove that $CD = 2AM$.

CHAPTER 6

MISCELLANEOUS TOPICS

6.1 The Three Problems of Antiquity

In some of the earlier chapters we had sections on construction problems. In this chapter, we expand further and describe some useful techniques.

The object is to draw geometric figures in the plane using two simple ***tools***:

1. A ***straightedge***. This is a device for drawing a straight line through any two given points. It is of arbitrary length—that is, you can draw a line as long as you need that passes through the given points. Note that this means that given a segment AB you can extend this segment: take two points on the segment and draw the line through those two points. Note also that a straightedge is *not a ruler*. You cannot use a straightedge to measure distances. A ruler is a different tool.

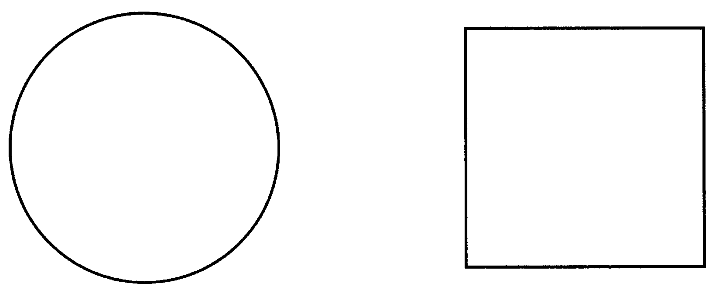
Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

2. A ***modern compass***. This is a device for drawing arcs and circles given any point as center and the length of any given segment as radius. The modern compass holds its radius when it is lifted from the page, as opposed to the ***classical compass*** which collapses to zero radius when removed from the page.

The constructions that can be accomplished using these two basic tools are called ***Euclidean constructions***.

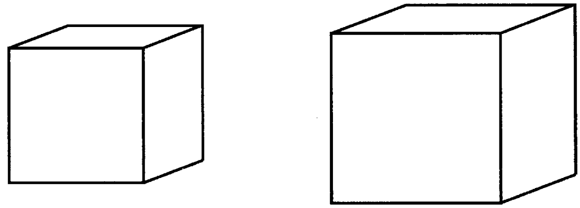
There are certain famous construction problems that have been shown to be impossible in the sense that they cannot be accomplished with a straightedge and compass. These are:

1. ***Squaring the circle***. The problem is to construct a square of the same area as a given circle.



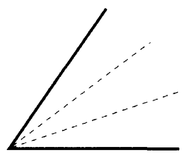
Squaring the circle amounts to the following: given a segment of length 1, construct a segment of length $\sqrt{\pi}$.

2. ***Doubling the cube***. Given a cube, construct another cube of twice its volume.



Doubling the cube amounts to the following: given a segment of length 1, construct a segment of length $2^{1/3}$.

3. ***Trisecting any general angle***. Given an arbitrary angle, construct an angle that is one-third its size.



Note that some angles *can* be trisected—for example, it is possible to trisect a right angle (because we can construct an angle of $30°$). We can construct an angle of $60°$. Trisecting this angle would amount to constructing an angle of $20°$, and once we have an angle of $20°$ and a segment of length 1 unit, we could construct a segment equal in length to the cosine of $20°$. The proof that there is no general method for trisecting an angle is accomplished by proving that, given only a segment of length 1, it is impossible to construct a segment equal in length to $\cos 20°$.

6.2 Constructing Segments of Specific Lengths

Given a segment of length 1 unit, what other lengths can we construct? The construction must use only a straightedge and a compass and must be accomplished in a finite number of steps. This section describes what lengths we can construct. A complete discussion would prove that these are the only lengths that we can construct, but that is well beyond the scope of this book.

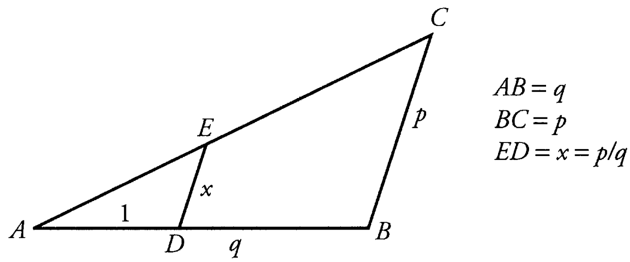
It is clear that, given a segment of length 1, we can construct a segment of length n where n is any positive integer.

Given segments of length p and q, we can construct segments of length $p + q$ and $p - q$.

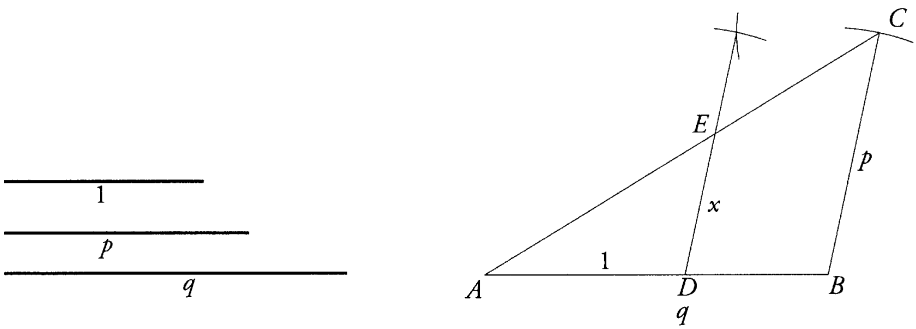
What about constructing segments of length p/q or pq?

Example 6.2.1. *Given segments of lengths* 1, *p, and q, construct a segment of length* p/q.

Solution.



The diagram above reveals how to use similar triangles to accomplish the task. The actual construction is as follows:



(1) Construct a segment AB equal in length to the given length q.

(2) Construct a segment BC equal in length to the given length p so that ABC is a triangle.

(3) On AB, cut off a segment AD of length 1.

(4) Through D, construct a line parallel to BC cutting AC at E; then DE has length p/q.

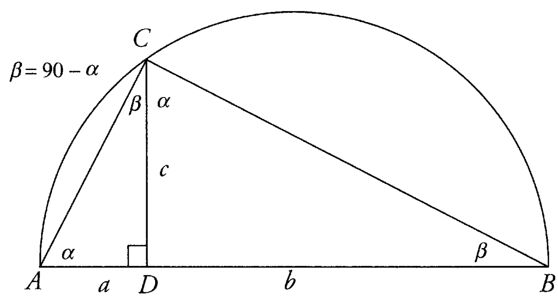
The proof that DE has length p/q follows from the fact that $\triangle ABC \sim \triangle ADE$.

□

Exercise 6.2.2. *Given segments of lengths* 1, p, *and* q, *construct a segment of length* pq.

Example 6.2.3. *Given segments of length* a *and* b, *show how to construct a segment of length* $\sqrt{ab}$.

Solution. The key to this is to somehow use Pythagoras' Theorem. It is actually used in conjunction with Thales' Theorem, which tells us that the angle inscribed in a semicircle is $90°$. Here is the analysis figure:



Since $\triangle ADC \sim \triangle CDB$, we have

$$\frac{DC}{AD} = \frac{DB}{CD},$$

so that

$$\frac{c}{a} = \frac{b}{c},$$

and it follows that $c = \sqrt{ab}$.

Thus the construction is as follows:

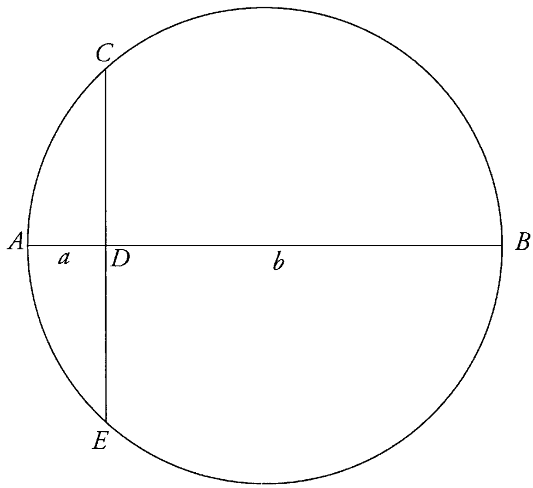
(1) Construct a segment AB of length $a + b$, with D the point on AB such that $AD = a$ and $DB = b$.

(2) Construct the semicircle with AB as diameter.

(3) Construct the perpendicular to AB at D cutting the semicircle at C; then CD has length $\sqrt{ab}$.

$\square$

Note. Using the *power of a point* also yields the same construction:



(1) Construct a segment AB of length $a + b$, with D the point on AB such that $AD = a$ and $DB = b$.

(2) Construct the circle with AB as diameter.

(3) Construct the chord CE perpendicular to AB at D; then CD has length $\sqrt{ab}$.

Since CE is perpendicular to the diameter AB, we must have $CD = DE$. By the power of the point D, we have $CD \cdot DE = AD \cdot DB$, that is, $CD = \sqrt{ab}$.

Constructible Numbers

The ancient Greeks described a number x as being ***constructible*** if, starting with a segment of length 1, you could construct a segment of length x. For example, $3/5$ is a constructible number: starting with a segment of length 1, construct segments of length 3 and 5, and then using Example 6.2.1, construct a segment of length $3/5$.

Combining Examples 6.2.1, 6.2.2, and 6.2.3 we have the following theorem:

Theorem 6.2.4. *(Constructible Numbers)*

If the nonnegative numbers a and b are constructible, then so are the following numbers:

$$a+b, \qquad a-b, \qquad \frac{a}{b} \quad (\textit{if } b \neq 0), \qquad ab, \qquad \sqrt{a}.$$

We can build many constructible numbers by taking a succession of these operations. For example:

5 and 6 are constructible, so $\sqrt{5}$ and $\sqrt{6}$ are constructible.

3 and $\sqrt{5}$ are constructible, so $3 + \sqrt{5}$ is constructible.

$3 + \sqrt{5}$ is constructible, so $\sqrt{3 + \sqrt{5}}$ is constructible.

4 and $\sqrt{6}$ are constructible, so $4\sqrt{6}$ is constructible.

$4\sqrt{6}$ and $\sqrt{3 + \sqrt{5}}$ are constructible, so $\dfrac{4\sqrt{6}}{\sqrt{3 + \sqrt{5}}}$ is constructible.

And so on.

Example 6.2.5. *Show that* $(1 + \sqrt{2})^{1/4}$ *is constructible.*

Solution. The numbers 1 and $\sqrt{2}$ are constructible, so $1 + \sqrt{2}$ is constructible, thus $(1 + \sqrt{2})^{1/2}$ is also constructible and then so is $\left((1 + \sqrt{2})^{1/2}\right)^{1/2}$. Since

$$(1 + \sqrt{2})^{1/4} = \left((1 + \sqrt{2})^{1/2}\right)^{1/2},$$

we are finished.

□

Starting with the number 1, and by taking a finite succession of additions, subtractions, ratios, products, and square roots, with repetitions allowed, we can obtain ***all*** of the constructible numbers. For example, the number

$$\sqrt{\frac{2}{3} + \frac{\sqrt{1 + \sqrt{2}}}{\sqrt{5}}}$$

can be obtained in the way described, so it is constructible.

These are the only types of numbers that are constructible. This can be rather tricky, because there are many ways of expressing the same number. For example, at first glance

$$\left(\frac{2^6}{3^6}\right)^{1/3}$$

may not appear to be constructible, but it is constructible because

$$\left(\frac{2^6}{3^6}\right)^{1/3} = \frac{4}{9}.$$

Some numbers that are known ***not*** to be constructible are

$$\pi, \quad e, \quad \sqrt[3]{2}, \quad \cos 20^\circ.$$

This explains why the Three Problems of Antiquity cannot be solved.

Remark. To reiterate, a number a is ***constructible*** if and only if, given a segment of unit length, it is possible to construct a segment of length $|a|$ using only a straightedge and compass. It can be shown using Galois theory that the only numbers that are constructible are the following:

- Integers
- Rational numbers
- Square roots of rational numbers
- Sums, differences, ratios, and products of the above
- Sums, differences, ratios, and products of the above
- Sums, differences, ratios, and products of the above

6.3 Construction of Regular Polygons

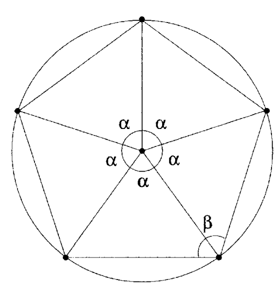
Recall that a simple polygon is called a ***regular polygon*** if all of its sides are congruent and all of its vertex angles are congruent. This section deals with the problem of how to construct some of the regular polygons.

There is little difficulty in constructing the regular n-gon for $n = 3$ (an equilateral triangle), for $n = 4$ (a square), and for $n = 6$ (a regular hexagon).

In an earlier problem, we saw that all of the regular polygons have their vertices on a circle, so construction of the regular n-gon amounts to finding n equally spaced points on a circle. The general problem then becomes:

> Given a circle of radius 1, find n equally spaced points on that circle.

The situation for the pentagon is shown on the right.



The ***central angle*** α is given by

$$\alpha = \frac{360}{n},$$

while the ***vertex angle*** β is given by

$$\beta = 180 - \alpha = 180\left(\frac{n-2}{n}\right),$$

since $\alpha + 2 \cdot (\beta/2) = 180$.

Construction Tips:

(i) We can construct a regular n-gon if and only if we can construct its central angle

$$\alpha = 360/n.$$

(ii) We can construct a regular n-gon if and only if we can construct the vertex angle

$$\beta = 180\left(\frac{n-2}{n}\right).$$

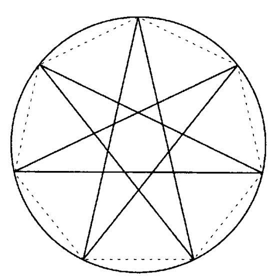
(iii) If we can construct a regular n-gon, then we can construct a regular $2n$-gon by bisecting the central angles. In general, we can construct a regular $2^k \cdot n$-gon by continually bisecting the central angles.

(iv) If we can construct a regular $2n$-gon, then we can construct a regular n-gon by joining alternate vertices.

It should be noted that nonsimple polygons can also have all of their sides congruent and all of their vertex angles congruent. The vertices for these polygons also lie on a circle. Such polygons are known as ***regular star polygons***.

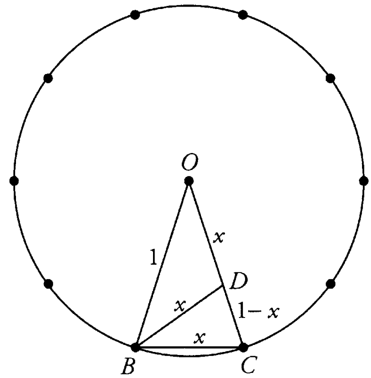
The regular $\left\{ \begin{matrix} n \\ d \end{matrix} \right\}$ star polygon can be obtained from the regular n-gon by joining every dth point.

The regular $\left\{ \begin{matrix} 7 \\ 3 \end{matrix} \right\}$ star polygon is shown on the right.



6.3.1 Construction of the Regular Pentagon

To construct the regular pentagon, we actually first construct the regular decagon and then join alternate vertices. The analysis figure for the regular decagon is as follows:



The circle is of radius 1 and center O, and there are 10 vertices, each at distance x from its immediate neighbours. If B and C are two consecutive vertices, then $\angle BOC = 360/10$; that is, in the isosceles triangle OBC, the angles are 36°, 72°, and 72°. Thus, if BD bisects the base angle OBC, we have $\angle OBD = 36$, so triangles OBD and CBD are isoceles. Thus, $OD = BD = BC = x$.

Since $\triangle OBC \sim \triangle BCD$ we have

$$\frac{BC}{OB} = \frac{CD}{BC},$$

so that

$$\frac{x}{1} = \frac{1-x}{x};$$

that is,

$$x^2 + x - 1 = 0.$$

Solving for x, we get the roots

$$x = \frac{\sqrt{5}-1}{2} \quad \text{and} \quad x = \frac{-\sqrt{5}-1}{2}.$$

The positive number

$$\frac{\sqrt{5}-1}{2}$$

is constructible, so given the radius 1 of the circle, we can construct the segment of length x and therefore we can construct the regular decagon and the regular pentagon.

Exercise 6.3.1. *Given a circle with radius $OP = 1$, construct the segment of length*

$$\frac{\sqrt{5}-1}{2}$$

and complete the construction of the regular pentagon.

Remark. The number

$$\frac{\sqrt{5}+1}{2}$$

(note the plus sign) is called the ***golden ratio*** or ***golden section*** or ***golden mean*** and is usually denoted by the Greek letter ϕ.

Note that

$$\frac{\sqrt{5}+1}{2} \cdot \frac{\sqrt{5}-1}{2} = 1,$$

so that

$$\frac{\sqrt{5}-1}{2}$$

is the reciprocal of ϕ. Note also that

$$\frac{\sqrt{5}+1}{2} \quad \text{and} \quad \frac{\sqrt{5}-1}{2}$$

differ by 1.

6.3.2 Construction of Other Regular Polygons

We can construct a regular n-gon for $n = 3, 4, 5$, and it is not very difficult to construct a $2n$-gon if we can construct an n-gon. For $n \leq 10$, this leaves $n = 7$ and $n = 9$. Unfortunately, we cannot construct these regular n-gons with straightedge and compass alone. In fact, there are very few n-gons that are constructible, and this section describes all of them. First, we need the following definitions.

A ***Fermat number*** is a number of the form $2^{2^n} + 1$, where $n \geq 0$.

A ***prime number*** is a positive integer $p > 1$ such that p has exactly two positive divisors, namely 1 and p.

A ***Fermat prime*** is a Fermat number that is also a prime number.

Here are the first few Fermat numbers:

n	F_n	$2^{2^n}+1$	Prime?
0	F_0	3	yes
1	F_1	5	yes
2	F_2	17	yes
3	F_3	257	yes
4	F_4	65,537	yes
5	F_5	4,294,967,297	no

Remark. As can be seen from the table, the first five Fermat numbers

$$F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65,537$$

are all primes, and this led Fermat to conjecture that every Fermat number F_n is a prime.

In 1732, almost 100 years later, Euler showed that this conjecture was false, and he gave the following counterexample:

$$F_5 = 4,294,967,297 = 641 \cdot 6,700,417.$$

Even today we do not know if there are an infinite number of Fermat primes. In fact, the only known Fermat primes are the ones in the table above.

In 1796, Gauss found what is probably the most important aspect of the Fermat numbers, the connection between the Fermat primes and the straightedge and compass construction of regular polygons. His result is as follows:

Theorem 6.3.2. *(Gauss' Theorem)*

A regular n-gon is constructible if and only if

$$n = 2^{k+2} \qquad \text{or} \qquad n = 2^k \cdot p_1 \cdot p_2 \cdots p_s,$$

where $k \geq 0$ and $p_1, p_2, \ldots, p_s$ are distinct Fermat primes.

The early Greeks knew how to construct regular polygons with 2^k, $3 \cdot 2^k$, $5 \cdot 2^k$, and $15 \cdot 2^k = 2^k \cdot 3 \cdot 5$ sides. They also knew how to construct regular polygons with $3, 4, 5, 6, 8, 10, 12, 15,$ and 16 sides, but not one with 17 sides. Gauss, however, did this at age 19 and so reportedly decided to devote the rest of his life to mathematics. He also requested that a 17-sided regular polygon be engraved on his tombstone (it wasn't).

Corollary 6.3.3. *The regular 7-gon and the regular 9-gon are not constructible.*

Proof. When $n = 7$, n is a prime, but not a Fermat prime. When $n = 9$, n is the product of Fermat primes since $n = 3 \cdot 3$, but it is not the product of *distinct* Fermat primes.

□

Example 6.3.4. *Is an angle of* $3°$ *constructible?*

Solution. This is the central angle formed by the edges of a 120-gon, since

$$n = \frac{360}{3} = 120.$$

The question amounts to asking whether we can construct a 120-gon.

Since

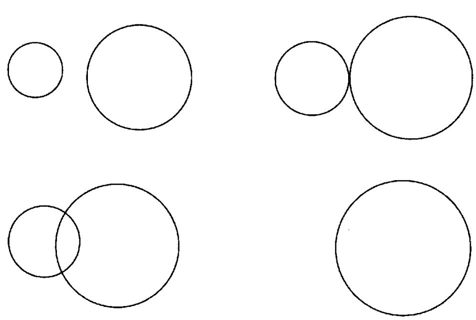
$$120 = 2^3 \cdot 3 \cdot 5,$$

and since 3 and 5 are distinct Fermat primes, the construction is possible.

□

6.4 Miquel's Theorem

Now we return to the ideas of concurrency and collinearity. First we note that two circles intersect in zero, one, or two points, or they coincide, as in the figure, and therefore:



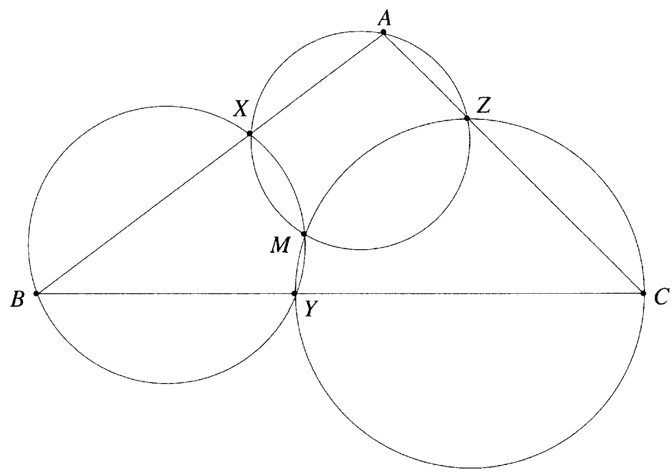
(i) If two circles have three distinct points of intersection, then they must coincide.

(ii) The circumcircle of any triangle is unique.

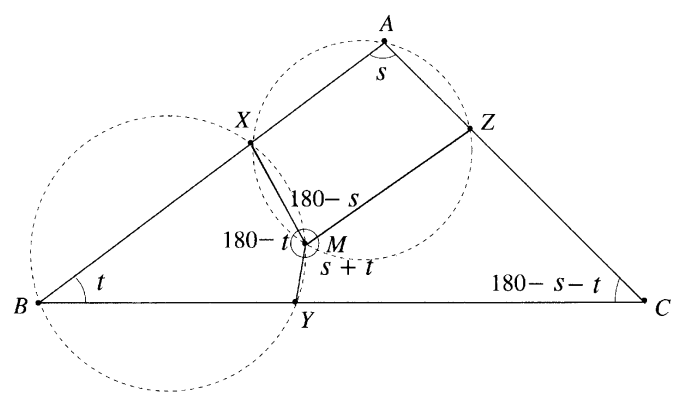
Theorem 6.4.1. *(Miquel's Theorem)*

Given $\triangle ABC$ and three menelaus points X, Y, and Z, one on each side (possibly extended) of the triangle, then the circles formed using a vertex and its two adjacent menelaus points are concurrent at a point M.

The point of concurrency is called the ***Miquel point***.



Proof. Let two of the circles have a second point of intersection M. We want to show that the third circle also goes through the point M.



We know that the quadrilateral $AXMZ$ is cyclic and that $BYMX$ is also cyclic, and this implies that the angles are as shown in the figure.

Since

$$\angle YMZ = s + t \qquad \text{and} \qquad \angle YCZ = 180 - s - t,$$

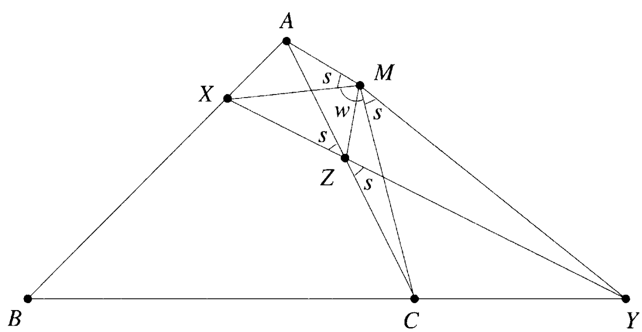
then

$$\angle YMZ + \angle YCZ = s + t + 180 - s - t = 180,$$

and the quadrilateral $YCZM$ is also cyclic. Therefore, the circumcircle of $\triangle YZC$ also passes through M, and the three circles are concurrent at the Miquel point M.

□

Corollary 6.4.2. *Given $\triangle ABC$ and three menelaus points X, Y, and Z, one on each side (possibly extended) of the triangle, if X, Y, and Z are collinear, then the circumcircle of $\triangle ABC$ passes through the Miquel point M.*



Proof. We recall that the Miquel point M lies on each of the circumcircles of $\triangle AXZ$, $\triangle BXY$, and $\triangle CYZ$, and therefore:

(1) Since $AXZM$ is cyclic, from Thales' Theorem we have

$$\angle AMX = \angle AZX = s.$$

(2) Since $BXMY$ is cylic, then

$$\angle XBY + \angle XMY = 180.$$

(3) Since $CZMY$ is cyclic, from Thales' Theorem we have

$$\angle CMY = \angle CZY = s.$$

Therefore, if we let $\angle XMC = w$, then

$$\angle B + \angle XMY = 180;$$

that is,

$$\angle B + w + s = 180,$$

so that

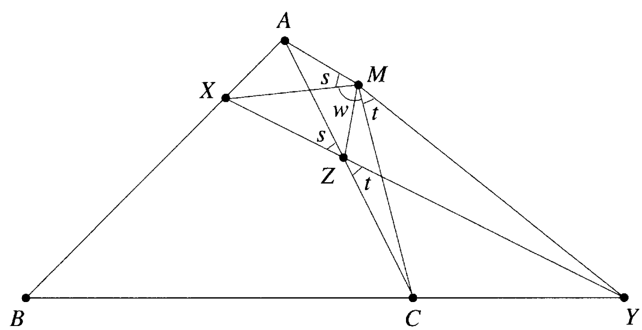
$$\angle B + \angle AMC = 180,$$

and thus $ABCM$ is cyclic.

□

The converse is also true.

Corollary 6.4.3. *Given $\triangle ABC$ and three menelaus points X, Y, and Z, one on each side (possibly extended) of the triangle, if the circumcircle of $\triangle ABC$ also goes through the Miquel point M, then the three menelaus points X, Y, and Z must be collinear.*



Proof. In the figure we let $\angle XMC = w$. Now:

(1) Since M is on the circumcircle of $\triangle ABC$, then $ABCM$ is cyclic.

(2) Since M is on the circumcircle of $\triangle AXY$, then $AXZM$ is cyclic.

(3) Since M is on the circumcircle of $\triangle CZY$, then $CZMY$ is cyclic.

(4) Since M is on the circumcircle of $\triangle BXY$, then $BXMY$ is cyclic.

From (2), using Thales' Theorem, we have

$$\angle AZX = \angle AMX = s,$$

and from (3), using Thales' Theorem, we have

$$\angle CZY = \angle CMY = t,$$

and to show that X, Y, and Z are collinear, it is enough to show that $s = t$.

However, from (1), since $ABCM$ is cyclic, the opposite angles are supplementary, so that

$$\angle B + w + s = 180,$$

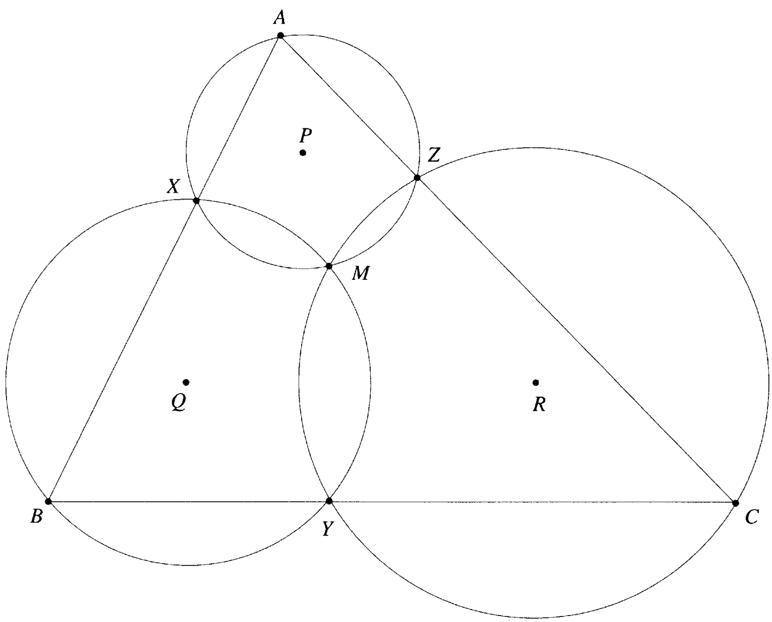
while from (4), since $BXMY$ is cyclic, the opposite angles are supplementary, so that

$$\angle B + w + t = 180.$$

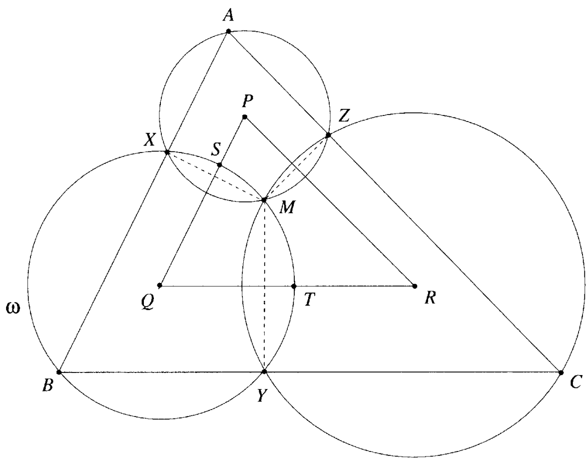
Therefore $s = t$, and we are done.

□

Example 6.4.4. *Given $\triangle ABC$ and three menelaus points X, Y, and Z, one on each side, where M is the Miquel point, as in the figure, show that if P, Q, and R are the centers of the circumcircles of $\triangle AXZ$, $\triangle BXY$, and $\triangle CYZ$, respectively, then $\triangle PQR$ is similar to $\triangle ABC$.*



Solution. First, observe that quadrilateral $XPMQ$ is a kite. Let ω be the circumcircle of triangle BXY and draw the common chords $\overline{XM}$, $\overline{YM}$, and $\overline{ZM}$. Let the side $\overline{PQ}$ meet ω at S and the side $\overline{QR}$ meet ω at T, as in the figure on the following page.



Since the line joining the centers of two circles is the perpendicular bisector of their common chord, $\overline{PQ}$ is the perpendicular bisector of $\overline{XM}$ and therefore $\overline{PQ}$ also bisects $\angle XQM$, so that

$$\angle XQS = \angle MQS.$$

Similarly, $\overline{QR}$ bisects $\angle MQY$, so that

$$\angle MQT = \angle TQY.$$

Since

$$\angle SQT = \angle SQM + \angle MQT = \frac{1}{2}\angle XQY,$$

and from Thales' Theorem we have

$$\angle XBY = \frac{1}{2}\angle XQY,$$

then

$$\angle B = \angle XBY = \frac{1}{2}\angle XQY = \angle SQT = \angle Q.$$

Similarly,

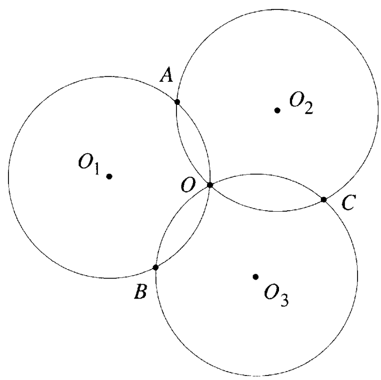
$$\angle A = \angle P \qquad \text{and} \qquad \angle C = \angle R,$$

and therefore $\triangle PQR \sim \triangle ABC$.

□

Example 6.4.5. *Prove Johnson's Theorem (1916).*

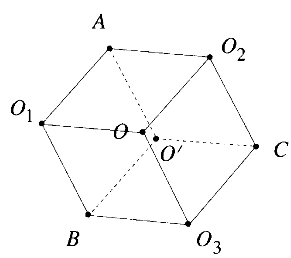
Given three circles concurrent at O, all with the same radius r, as in the figure below, then the circumcircle of the other three intersection points A, B, and C has radius r also.



Solution. Let O_1, O_2, and O_3 be the centers of the three circles. Then the quadrilaterals

$$AO_1OO_2, \qquad O_1OO_3B, \qquad CO_2OO_3$$

are rhombii, since all the sides have length r, as in the figure below.



Let AO_1BO' be the completed parallelogram of triangle AO_1B, which is in fact a rhombus. Since

$$\overline{BO'} \parallel \overline{O_1A} \parallel \overline{OO_2} \parallel \overline{O_3C} \qquad \text{and} \qquad BO' = O_1A = OO_2 = O_3C,$$

then $BO'CO_3$ is also a rhombus, and therefore the circumcircle of triangle ABC is $\mathcal{C}(O', r)$.

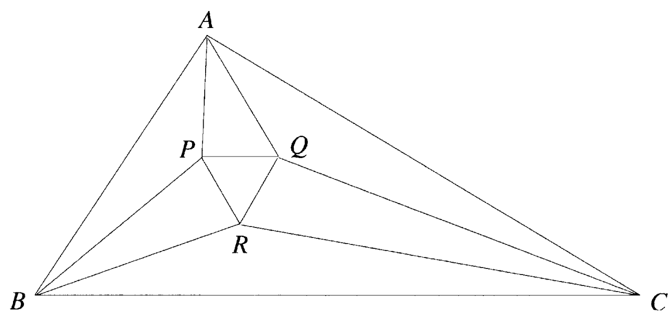
Note. In general, the center O' of the circumcircle of ABC is different from O.

□

6.5 Morley's Theorem

The following result was discovered by Frank Morley in about 1900. He mentioned it to friends in Cambridge and published it about 20 years later in Japan.

Morley's Theorem states that the points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle, as in the figure below.



Before proving this theorem we need a lemma, which is yet another characterization of the incenter of a triangle.

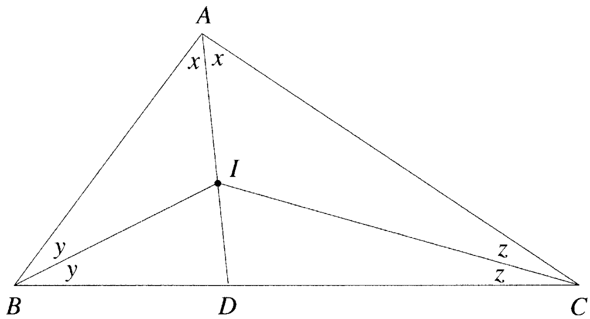
Lemma 6.5.1. *(Another Characterization of the Incenter)*

The incenter of a triangle $\triangle ABC$ is the unique point I interior to the triangle which satisfies the following two properties:

(1) it lies on an angle bisector of one of the angles (say at A) and

(2) it subtends an angle $90 + \frac{1}{2}\angle A$ with the side BC.

Proof. The incenter has property (1) by definition, since it is the intersection of the internal angle bisectors of the triangle.

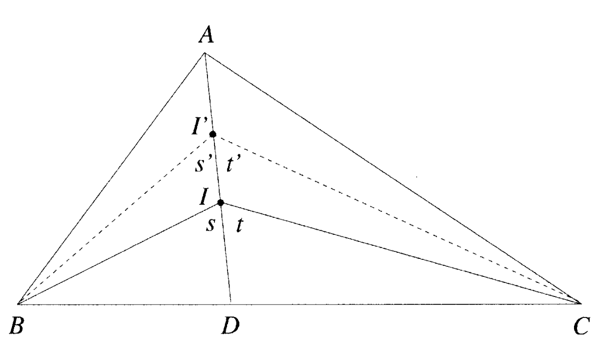


Also, from the External Angle Theorem, we have

$$\begin{aligned}
\angle BIC &= \angle BID + \angle CID \\
&= x + y + x + z \\
&= x + \frac{1}{2}(2x + 2y + 2z) \\
&= x + \frac{1}{2}\,180 \\
&= 90 + x \\
&= 90 + \frac{1}{2}\angle A,
\end{aligned}$$

and property (2) holds also.

To prove uniqueness, suppose that the point $I' \neq I$ lies on the angle bisector of $\angle A$, and suppose that I' also subtends an angle $90 + \frac{1}{2}\angle A$ with the side BC.



Note that in the figure on the previous page, where I' is between A and I, the External Angle Inequality implies that

$$s > s' \qquad \text{and} \qquad t > t',$$

so that

$$90 + \frac{1}{2}\angle A = \angle BIC = s + t > s' + t' = \angle BI'C = 90 + \frac{1}{2}\angle A,$$

which is a contradiction. Similarly, if I is between A and I', we again get a contradiction. Therefore, the point I satisfying (1) and (2) is unique.

$\square$

Theorem 6.5.2. *(Morley's Theorem)*

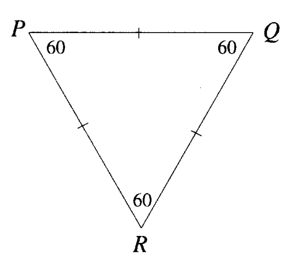
The points of intersection of adjacent trisectors of the angle of any triangle form an equilateral triangle.

Proof. Let $\triangle ABC$ be a fixed triangle, so that the angles at the vertices A, B, and C are fixed angles. It is enough to prove Morley's Theorem for a triangle $\triangle A'B'C'$ that is similar to the given triangle, since scaling the sides by a proportionality factor k does not change the angles, and so an equilateral triangle remains equilateral.

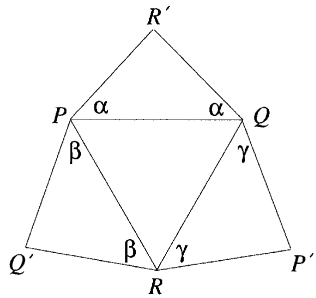
In the proof we start with an equilateral triangle $\triangle PQR$ and then construct a triangle $\triangle A'B'C'$ that is similar to $\triangle ABC$ and has the property that the adjacent trisectors form the given equilateral triangle $\triangle PQR$.

Construction Phase

Step 1. Construct an equilateral triangle $\triangle PQR$.



Step 2. Construct three isosceles triangles on the sides of $\triangle PQR$, with angles as shown,



where

$$\alpha = 60 - \frac{1}{3}\angle A,$$
$$\beta = 60 - \frac{1}{3}\angle B,$$
$$\gamma = 60 - \frac{1}{3}\angle C.$$

Observations:

(1) $0 < \alpha, \beta, \gamma < 60$.

Since, for example, $\angle A > 0$ implies that $60 > 60 - \frac{1}{3}\angle A = \alpha$, while $\angle A < 180$ implies that $\alpha = 60 - \frac{1}{3}\angle A > 60 - \frac{1}{3}180 = 60 - 60 = 0$.

(2) $\alpha + \beta + \gamma = 120$.

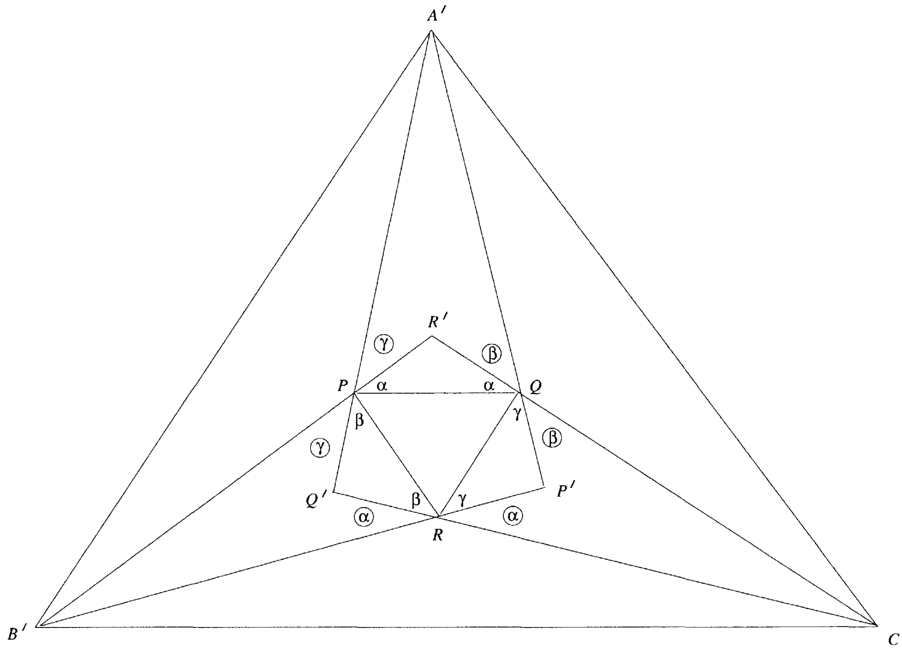
Since $\alpha+\beta+\gamma = 180-\frac{1}{3}(\angle A+\angle B+\angle C) = 180-\frac{180}{3} = 180-60 = 120$.

(3) The sum of any two of the angles α, β, γ is greater than 60.

Since, for example,

$$\alpha + \beta = 120 - \tfrac{1}{3}(\angle A + \angle B) > 120 - \tfrac{180}{3} = 120 - 60 = 60.$$

Step 3. Extend the sides of the isosceles triangles until they meet as shown to produce a larger triangle $\triangle A'B'C'$.



We claim that $\triangle A'B'C'$ is similar to $\triangle ABC$ and that the lines

$$A'P, \quad A'Q, \quad B'P, \quad B'R, \quad C'Q, \quad C'R$$

are the angle trisectors of the angles at A', B', and C'.

Argument Phase

Step 1. $\alpha + \beta + \gamma + 60 = 180$, so that the angles are as shown on the figure.

For example, the angle $\angle B'PR'$ is a straight angle and $\angle QPR = 60$ so that $\alpha + 60 + \beta + \angle B'PQ' = 180$, which implies that $\gamma = \angle B'PQ'$ and that the vertically opposite angle $\angle A'PR' = \gamma$ also.

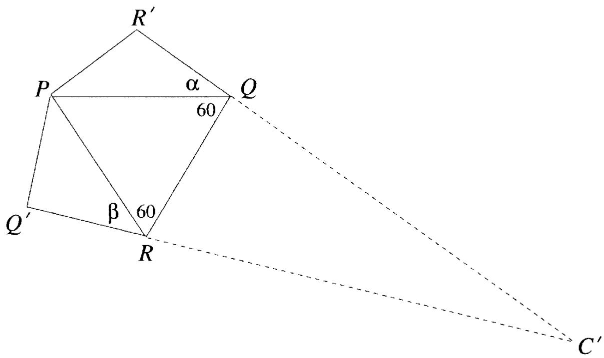
We should also show that the extensions of the sides of the isosceles triangles actually do intersect at the points A', B', and C'.

For example, if we isolate part of the figure, we can show that QR' and $Q'R$ intersect at a point C', as shown in the figure on the following page.

If we consider the sum $\angle RQR' + \angle Q'RQ$, then

$$\angle RQR' + \angle Q'RQ = \alpha + 60 + \beta + 60 = \alpha + \beta + 120 > 60 + 120 = 180,$$

and therefore the parallel postulate says that RQ' and $R'Q$ intersect on the side of the transversal QR where the sum of the interior angles is less than 180; that is, RQ' and $R'Q$ intersect on the side opposite P at some point C', as shown.



Similarly, PQ' and $P'Q$ intersect at some point A', while PR' and $P'R$ intersect at some point B', as shown.

Step 2. $\alpha + \beta + \gamma = 120$, and therefore

$$\angle PA'Q = 60 - \alpha = \frac{1}{3}\angle A,$$
$$\angle PB'R = 60 - \beta = \frac{1}{3}\angle B,$$
$$\angle QC'R = 60 - \gamma = \frac{1}{3}\angle C.$$

For example, in $\triangle PB'R$, the sum of the interior angles is

$$\angle PB'R + \gamma + \beta + \beta + \alpha = 180,$$

so that

$$\angle PB'R = 180 - \beta - (\alpha + \beta + \gamma) = 180 - \beta - 120 = 60 - \beta,$$

and

$$\angle PB'R = 60 - \beta = \frac{1}{3}\angle B.$$

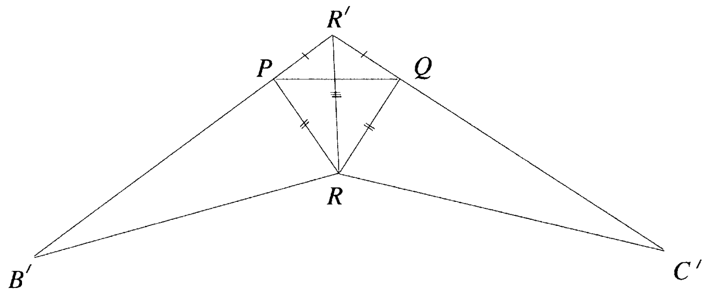
Similarly,

$$\angle PA'Q = 60 - \alpha = \frac{1}{3}\angle A \quad \text{and} \quad \angle QC'R = 60 - \gamma = \frac{1}{3}\angle C.$$

Step 3. R is the incenter of $\triangle B'R'C'$. Similarly, P is the incenter of $\triangle A'B'P'$, while Q is the incenter of $\triangle A'C'Q'$.

We will show that R is the incenter of $\triangle B'R'C'$. The other two results follow in the same way.

(a) Note that R lies on the angle bisector of $\angle B'R'C'$ since $\triangle PR'R$ is congruent to $\triangle QR'R$ by the **SSS** congruency theorem.



(b) Note that

$$\angle B'RC' = 180 - \alpha = 90 + (90 - \alpha) = 90 + \frac{1}{2}(180 - 2\alpha) = 90 + \frac{1}{2}\angle B'R'C'.$$

By the characterization theorem for the incenter proven in the lemma, (1) and (2) imply that R is the incenter of $\triangle B'R'C'$.

Therefore,

$$\angle PB'R = \angle RB'C' = \frac{1}{3}\angle B$$

so that PB' and RB' are angle trisectors of $\angle B'$. Similarly, PA' and QA' are angle trisectors of $\angle A'$, and QC' and RC' are angle trisectors of $\angle C'$.

Therefore, $\angle A' = \angle A$, $\angle B' = \angle B$, and $\angle C' = \angle C$, and by the **AAA** similarity theorem, $\triangle A'B'C'$ is similar to $\triangle ABC$, and the corresponding segments in $\triangle ABC$ are the angle trisectors. Therefore, in $\triangle ABC$ the points of intersection of adjacent trisectors of the angles form an equilateral triangle.

$\square$

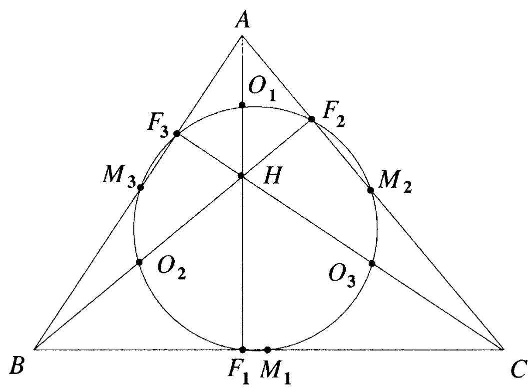
6.6 The Nine-Point Circle

In any triangle $\triangle ABC$, the following nine points all lie on a circle, called the ***9-point circle***, and they occur naturally in three groups.

(a) The three feet of the altitudes: F_1, F_2, F_3.

(b) The three midpoints of the sides: M_1, M_2, M_3.

(c) The three midpoints of the segments from the vertices to the orthocenter: O_1, O_2, O_3.



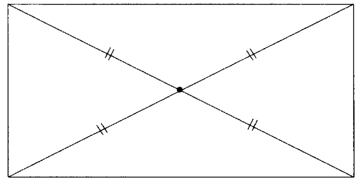
Note. In general, the orthocenter H is *not* the center of the 9-point circle.

Recall the following facts:

Fact 1. In a rectangle,

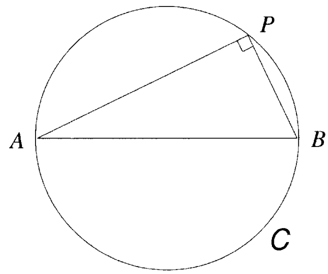
(a) the diagonals have the same length and

(b) the diagonals bisect each other.



Conclusion. The center of the circumcircle of a rectangle is the intersection of the diagonals and is also the midpoint of either diagonal.

Fact 2. The Thales' Locus of points subtending an angle of 90° with a segment $\overline{AB}$ is exactly the circle with $\overline{AB}$ as a diameter.



Conclusion. If a point P subtends an angle of 90° with any diameter of a circle $\mathcal{C}$, then P must be on the circle.

The 9-point circle theorem was first proved by Feuerbach (1800 – 1834).

Theorem 6.6.1. *(Feuerbach's Theorem)*

In any triangle ABC, the following nine points all lie on a circle, called the **9-point circle***:*

(i) The three feet of the altitudes: F_1, F_2, F_3.

(ii) The three midpoints of the sides: M_1, M_2, M_3.

(iii) The three midpoints of the segments from the vertices to the orthocenter: O_1, O_2, O_3.

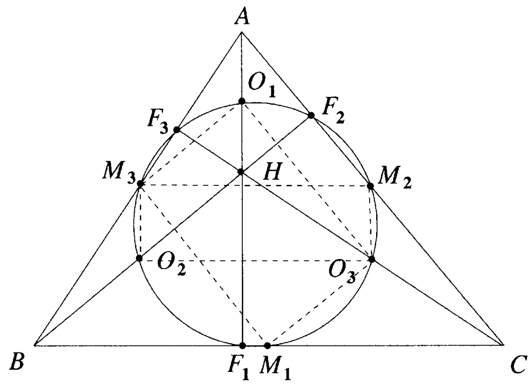
The center of the 9-point circle is denoted by N.

Proof. In the figure on the following page, the following constructions have been performed.

Step 1.

Join M_3 to M_2, so that M_2M_3 is parallel to BC, and $M_2M_3 = \frac{1}{2}BC$ by the Midline Theorem.

Join O_2 to O_3, so that O_2O_3 is parallel to BC, and $O_2O_3 = \frac{1}{2}BC$ by the Midline Theorem.



Step 2.

Join M_3 to O_2, so that M_3O_2 is parallel to AH, and $M_3O_2 = \frac{1}{2}AH$ by the Midline Theorem.

Join M_2 to O_3, so that M_2O_3 is parallel to AH, and $M_2O_3 = \frac{1}{2}AH$ by the Midline Theorem.

Since AH is an altitude and is perpendicular to the side BC, then $M_3M_2O_3O_2$ is a rectangle. Similarly, $M_3M_1O_3O_1$ is a rectangle.

Now note the following:

(1) M_3O_3 is a common diagonal of both rectangles so that the circumcircles of both rectangles coincide. This means that M_1, M_2, M_3 and O_1, O_2, O_3 all lie on the same circle $\mathcal{C}$.

(2) M_3O_3, M_2O_2, and M_1O_1 are all diameters of $\mathcal{C}$.

$$(3) \begin{cases} F_1 \quad \text{is on} \quad \mathcal{C} \quad \text{since} \quad \angle M_1F_1O_1 = 90°, \\ F_2 \quad \text{is on} \quad \mathcal{C} \quad \text{since} \quad \angle M_2F_2O_2 = 90°, \\ F_3 \quad \text{is on} \quad \mathcal{C} \quad \text{since} \quad \angle M_3F_3O_3 = 90°. \end{cases}$$

□

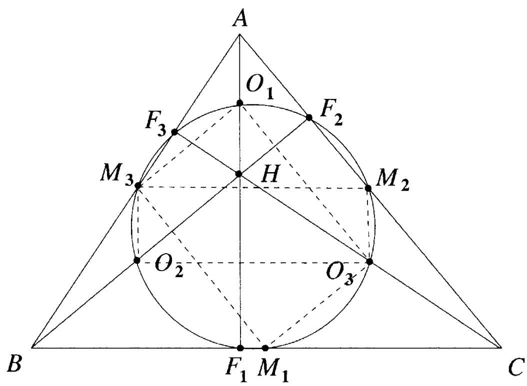
Note. In the proof, we joined

$$O_2 M_3, \quad O_3 M_2, \quad O_2 O_3, \quad M_2 M_3$$

and

$$O_1 O_3, \quad O_1 M_3, \quad M_3 M_1, \quad M_1 O_3$$

to get rectangles with common diagonals.

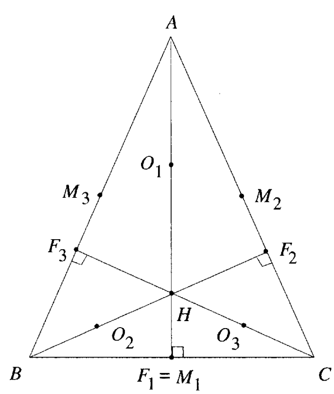


The segments O_1M_1, O_2M_2, and O_3M_3 are diameters of the 9-point circle and are also the diagonals of these rectangles. Thus, the point N, the center of the 9-point circle, is the midpoint of each of these segments.

6.6.1 Special Cases

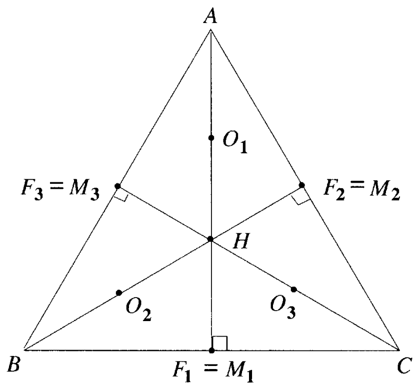
Isosceles Triangle

For an isosceles triangle which is not an equilateral triangle, only eight of the nine points on the 9-point circle are distinct.



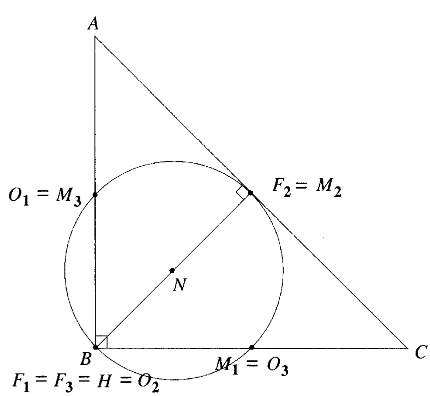
Equilateral Triangle

For an equilateral triangle, only six of the nine points on the 9-point circle are distinct. The center N of the 9-point circle is H, and the circle is just the *incircle*.



Right-Angled Isosceles Triangle

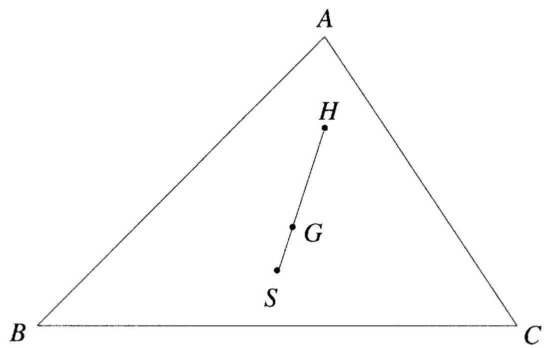
For a right-angled isosceles triangle, as below, where $AB = BC$ and $\angle B$ is a right angle, only four of the nine points on the 9-point circle are distinct.



The Euler Line

Theorem 6.6.2. *(Euler Line)*

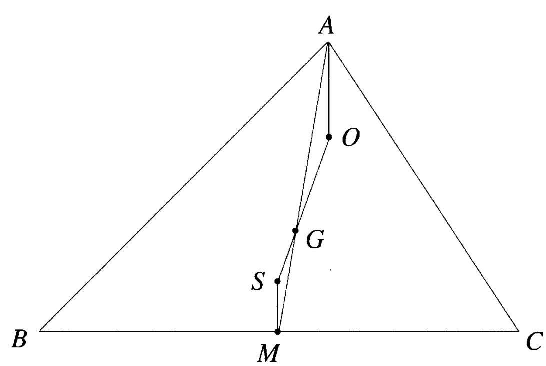
Given a nonequilateral triangle ABC, the circumcenter S, the centroid G, and the orthocenter H are collinear and form the ***Euler line****.*



In fact, G is a trisection point of HS; that is, G is between H and S and $GH = 2GS$.

Note. If $\triangle ABC$ is equilateral, then $S = G = H$, and conversely, if $S = G = H$, then $\triangle ABC$ is equilateral.

Proof. Since $\triangle ABC$ is nonequilateral, then $G \neq S$. Extend SG to SO, with $S - G - O$ and $GO = 2GS$. If we can show that $O = H$, then we are done.



Let M be the midpoint of BC. Since S is the circumcenter of $\triangle ABC$, then SM is perpendicular to BC.

Now join A and M. Then AM is a median and so passes through G. Next, join A to O, and by construction $GO = 2GS$.

Also, since G is the centroid of $\triangle ABC$ and AM is a median, $GA = 2GM$, and since vertically opposite angles are equal, $\angle AGO = \angle MGS$.

By the **sAs** similarity theorem, $\triangle AGO \sim \triangle MGS$, with proportionality constant $k = 2$, so that $SG = 2GO$ and $AO = 2SM$.

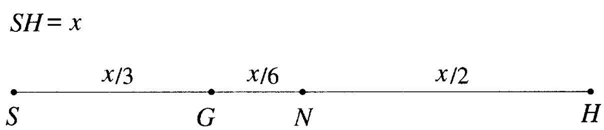
Now $\angle SMG = \angle OAG$, and the alternate interior angles formed by the transversal AM of SM and AO are equal so that SM is parallel to AO, and if AO is extended it hits BC at $90°$. Thus, the altitude from A goes through O. Similarly, the altitudes from B and C also pass through O, and $O = H$.

We note for future reference that $AH = 2SM$.

□

Theorem 6.6.3. *The center N of the 9-point circle of $\triangle ABC$ lies midway between the circumcenter S and the orthocenter H.*

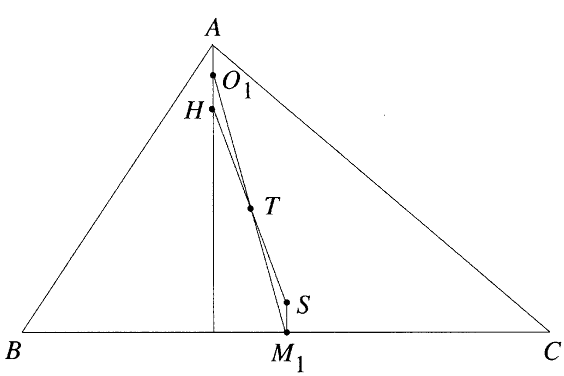
Note. This gives us four special points on the Euler line



and we can compute all of these distances.

Proof. Note that if the triangle is equilateral, then $S = N = G = H$ and there is nothing to prove.

Suppose that $\triangle ABC$ is nonequilateral, so that $S \neq H$. Introduce the midpoint M_1 of BC so that SM_1 is perpendicular to BC.



Join AH and let O_1 be the midpoint of AH, and then join O_1M_1 intersecting SH at T. Since H is the orthocenter and AH (extended) is perpendicular to BC, then AH is parallel to SM_1.

Since SH and O_1M_1 are criss-crossing transversals of the parallel lines AH and SM_1, then

$$\triangle STM_1 \sim \triangle HTO_1.$$

However,

$$AH = 2SM_1 \qquad \text{and} \qquad O_1H = \tfrac{1}{2}AH = SM_1,$$

so the proportionality constant is 1, and

$$\triangle STM_1 \equiv \triangle HTO_1.$$

Therefore, $ST = TH$, and T is the midpoint of SH.

Also, $TO_1 = TM_1$ and T is also the midpoint of O_1M_1, but O_1M_1 is a diameter of the 9-point circle, and therefore $T = N$, the center of the 9-point circle.

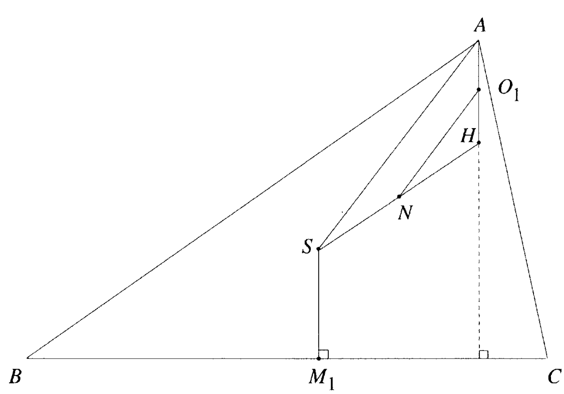
□

Theorem 6.6.4. *The radius of the* 9*-point circle is one half the radius of the circumcircle.*

Proof. In the figure below, NO_1 is the radius of the 9-point circle, AS is the radius of the circumcircle, and

$$NO_1 = \tfrac{1}{2}AS$$

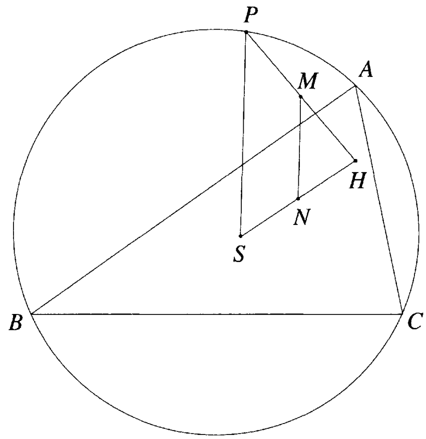
by the Midline Theorem.



□

Theorem 6.6.5. *The* 9*-point circle of* $\triangle ABC$ *bisects every segment connecting the orthocenter* H *to a point* P *on the circumcircle.*

Proof. In the figure below, let M be the midpoint of PH. We will show that the 9-point circle passes through M.



By the Midline Theorem, $MN = \frac{1}{2}PS = \frac{1}{2}R$, where R is the radius of the circumcircle, but MN is the radius of the 9-point circle by the previous theorem, so M is on the 9-point circle.

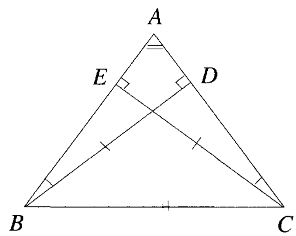
$\square$

6.7 The Steiner-Lehmus Theorem

We note the following two theorems.

Theorem 6.7.1. *If the altitudes of a triangle are congruent, then the triangle is isosceles.*

Proof. In the figure on the following page, if $BD = EC$, then $\triangle BDC \equiv \triangle CEB$ by the **HSR** congruence theorem.



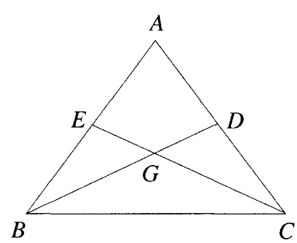
Therefore, $\triangle ADB \equiv \triangle AEC$ by the **ASA** congruence theorem, so that $AB = AC$.

Therefore, $\angle B = \angle C$ and $\triangle ABC$ is isosceles.

□

Theorem 6.7.2. *If two medians of a triangle are congruent, then the triangle is isosceles.*

Proof. In the figure below, G is the centroid of $\triangle ABC$ and medians BD and CE are equal.



Since G is the centroid, we have

$$BG = \tfrac{2}{3}BD = \tfrac{2}{3}CE = CG$$

and

$$DG = \tfrac{1}{3}BD = \tfrac{1}{3}CE = EG,$$

and since $\angle EGB = \angle DGC$, then by the **SAS** congruency theorem we have

$$\triangle EGB \equiv \triangle DGC.$$

Therefore, $BE = DC$ and $2BE = 2DC$; that is, $AB = AC$, so $\triangle ABC$ is isosceles.

□

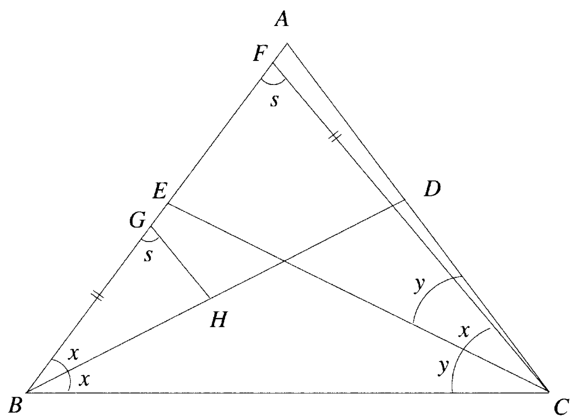
Thus, we see that if two altitudes of a triangle are congruent, or if two medians of a triangle are congruent, then the triangle must be isosceles.

In 1880, Lehmus conjectured that the result was also true for the angle bisectors, and in about 1884 Steiner proved that Lehmus' conjecture was true. Today almost 60 proofs have been given. We give below one of the simplest proofs.

Theorem 6.7.3. *(Steiner-Lehmus Theorem)*

If two internal angle bisectors of a triangle are congruent, then the triangle is isosceles.

Proof. In the figure below, let BD and EC be the internal angle bisectors at B and C, respectively, and suppose that $BD = CE$ but that $\triangle ABC$ is not isosceles, so that $x < y$.



Transfer x to C, as shown. Then by the Angle-Side Inequality applied to $\triangle BFC$, since $2x < x + y$, we have $FC < FB$.

Now transfer CF to BG as shown, and draw GH, making an angle $s = \angle EFC$ at G. Then by the **ASA** congruency theorem, we have

$$\triangle BGH \equiv \triangle CFE,$$

so that $BH = CE$. However, $BH < BD$, which implies that $CE = BH < BD$, a contradiction.

Similarly, $x > y$ is impossible. Hence $x = y$, and $\angle B = \angle C$, so that $\triangle ABC$ is isosceles.

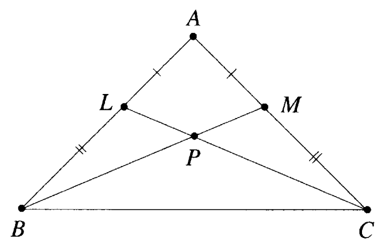
$\square$

Example 6.7.4. *Let ABC be a triangle, and let L and M be points on $\overline{AB}$ and $\overline{AC}$, respectively, such that $AL = AM$. Let P be the intersection of $\overline{BM}$ and $\overline{CL}$. Prove that $PB = PC$ if and only if $AB = AC$.*

Solution. Suppose that $AB = AC$, so that $\triangle ABC$ is isosceles. Then

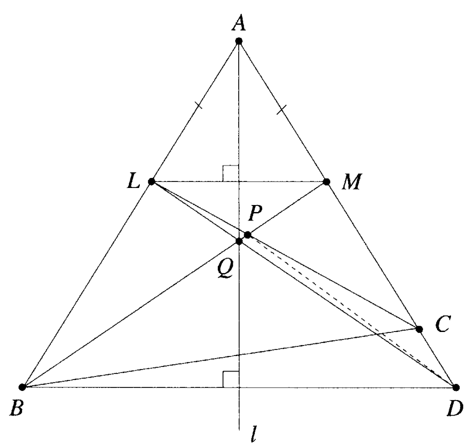
$$BL = AB - AL = AC - AM = CM,$$

and by the **SAS** congruency theorem, $\triangle LBC \equiv \triangle MCB$.



Therefore, $\angle PCB = \angle PBC$ and $\triangle BPC$ is isosceles, and thus, $BP = PC$.

Conversely, suppose that $AL = AM$ but $AB > AC$. Extend the side AC to D so that $AB = AD$, as in the figure below. We will show that this implies $PB > PC$.



Let l be the common perpendicular bisector of LM and BD. By symmetry, BM and DL intersect at a point Q on l.

Since C is between M and D, the point P *is on the same side of* l *as* D *so that* $PB > PD$.

Now,

$$\angle DCP > \angle ALC > \angle ALM = \angle ADB > \angle CDP.$$

Hence, $PD > PC$ and the conclusion follows.

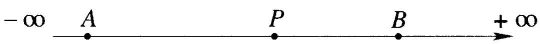
We have used the fact that an exterior angle of a triangle is greater than either of the interior angles and the fact that the larger angle is opposite the longer side. □

6.8 The Circle of Apollonius

In this section we will prove the Apollonian circle theorem, but first we give a characterization of points P on a line ℓ determined by distinct points A and B in terms of the ratio

$$\gamma = \gamma(P) = \frac{PA}{PB}.$$

We give the line an orientation such that the positive direction corresponds to going from A towards B, and we use the notation $A - P - B$ to mean that P is between A and B.



We have the following result.

Lemma 6.8.1. *If A, B, and P are distinct points on the line ℓ, and*

$$\gamma = \gamma(P) = \frac{PA}{PB},$$

then:

(1) For $A - P - B$, we have $0 < \gamma(P) < \infty$ and $\gamma(P) \uparrow +\infty$ as P goes from A to B.

(2) For $A - B - P$, we have $1 < \gamma(P) < \infty$ and $\gamma(P) \downarrow 1$ as P goes from B to $+\infty$.

(3) For $P - A - B$, we have $0 < \gamma(P) < 1$ and $\gamma \uparrow 1$ as P goes from A to $-\infty$.

Proof.

(1) For $A - P - B$, we have $PB = AB - PA$, so that

$$\gamma(P) = \frac{PA}{PB} = \frac{PA}{AB - PA} = \frac{\dfrac{PA}{AB}}{1 - \dfrac{PA}{AB}}.$$

As P approaches B, PA/AB approaches 1 through positive values so that the denominator goes to 0 through positive values. Therefore, $\gamma(P) \uparrow +\infty$ as P goes from A to P.

(2) For $A - B - P$, we have $PA = AB + PB$, so that

$$\gamma(P) = \frac{PA}{PB} = \frac{PB + AB}{PB} = 1 + \frac{AB}{PB} > 1.$$

As P approaches $+\infty$, AB/PB approaches 0 through positive values so that $\gamma(P) \downarrow 1$ as P goes from B to $+\infty$.

(3) For $P - A - B$, we have $PB = PA + AB$, so that

$$\gamma(P) = \frac{PA}{PB} = \frac{PB - AB}{PB} = 1 - \frac{AB}{PB} < 1.$$

As P approaches $-\infty$, AB/PB approaches 0 through positive values so that $\gamma(P) \uparrow 1$ as P goes from A to $-\infty$.

□

Theorem 6.8.2. *(Circle of Apollonius)*

Given two fixed points A and B, with $A \neq B$, together with a fixed positive constant $\gamma \neq 1$, then the locus of points P that satisfy

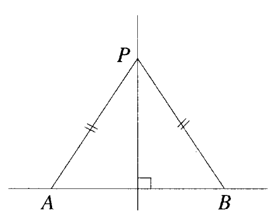
$$\frac{PA}{PB} = \gamma$$

is a circle with center on the line joining A and B, called the **circle of Apollonius**.

Note. We have avoided $\gamma = 1$, since

$$\{P \in \mathbb{R}^2 \mid \gamma = PA/PB = 1\}$$

is the perpendicular bisector of the segment AB.



Proof. Let C be an internal point of the segment AB such that

$$\frac{CA}{CB} = \gamma.$$

The previous lemma implies that there is only one such point C.

Let D be an external point to the segment AB such that

$$\frac{DA}{DB} = \gamma.$$

Again, the lemma implies that there is only one such point D.

Let $\mathcal{C}$ be the circle with CD as diameter. Then we claim that $\mathcal{C}$ is the circle of Apollonius, that is,

$$\mathcal{C} = \{P \in \mathbb{R}^2 \mid \gamma = PA/PB\}.$$

(1) Suppose first that $P \in \mathbb{R}^2$ and $PA/PB = \gamma$. Then

$$\frac{PA}{PB} = \gamma = \frac{CA}{CB},$$

and C is internal to the segment AB.

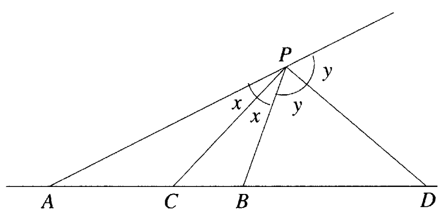
Thus, in $\triangle APB$, PC is the internal bisector of $\angle APB$ by the converse of the internal bisector theorem.

Also,

$$\frac{PA}{PB} = \gamma = \frac{DA}{DB},$$

and D is external to the segment AB.

Thus, in $\triangle APB$, PD is the external bisector of the external angle at P.



Now, $2x + 2y = 180$, so that $x + y = 90$, and $\angle CPD = 90$. Therefore, P is on the circle $\mathcal{C}$ with diameter CD.

(2) Suppose now that p is on $\mathcal{C}$, and assume for a contradiction that

$$\frac{PA}{PB} < \gamma = \frac{CA}{CB}.$$

From the lemma, the internal bisector of $\angle P$ divides the side opposite P in the ratio

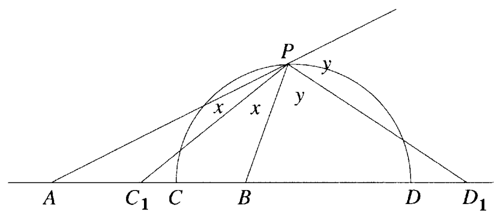
$$\frac{PA}{PB} = \frac{C_1A}{C_1B} < \gamma,$$

at a point C_1 to the *left* of C.

Again, from the lemma, the extenal bisector of $\angle P$ divides the side opposite P in the ratio

$$\frac{PA}{PB} = \frac{D_1A}{D_1B} < \gamma$$

at a point D_1 to the *right* of D.



However, $\angle C_1PD_1 = 90$ since the internal and external angle bisectors are perpendicular, and $\angle CPD = 90$ since P is on the circle $\mathcal{C}$ with diameter CD. This is a contradiction since $\angle CPD < \angle C_1PD_1$. Thus, it is not true that $PA/PB < \gamma$. Similarly, if we assume that $PA/PB > \gamma$, we get a contradiction. Therefore, we must have $PA/PB = \gamma$.

□

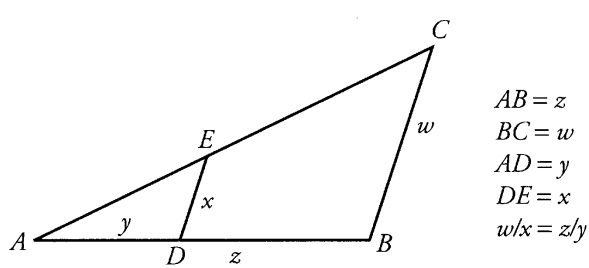
6.9 Solutions to the Exercises

Analysis Figure for Exercise 6.2.2

In the following figure, triangles ABC and ADE are similar, so

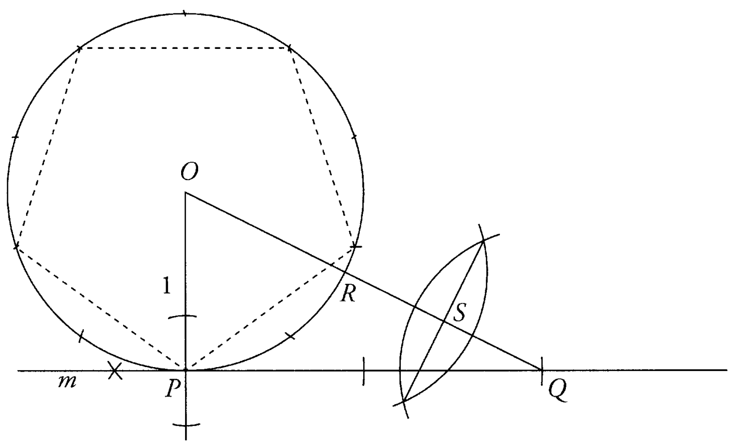
$$\frac{w}{x} = \frac{z}{y}.$$

If we choose $y = 1$ we get $w = xz$. Let one of x or z be p, the other be q, and then $w = pq$.



Solution for Exercise 6.3.1

Here is a fairly efficient construction.



(1) Construct a line m perpendicular to OP at P.

(2) Strike off a point Q such that $PQ = 2$. By Pythagoras' Theorem, $OQ = \sqrt{5}$.

(3) Let R be the point where the circle intersects OQ so that $RQ = \sqrt{5} - 1$.

(4) Bisect RQ at S. Then $RS = x = \frac{1}{2}(\sqrt{5} - 1)$.

(5) Beginning at P, strike off the 10 successive points around the circle at distances x from each other. Connect every second point to construct the regular pentagon.

6.10 Problems

1. Construct a triangle given the three midpoints of its sides.

2. Construct a triangle given the length of one side, the size of an adjacent angle, and the length of the median from that angle.

3. Construct a triangle given the length of one side, the distance from an adjacent vertex to the incenter, and the radius of the incircle.

4. Construct $\triangle ABC$ given the length of side BC and the lengths of the altitudes from B and C.

5. Construct a triangle given the measure of one angle, the length of the internal bisector of that angle, and the radius of the incircle.

6. Given segments of length 1 and a, construct a segment of length $1/a$.

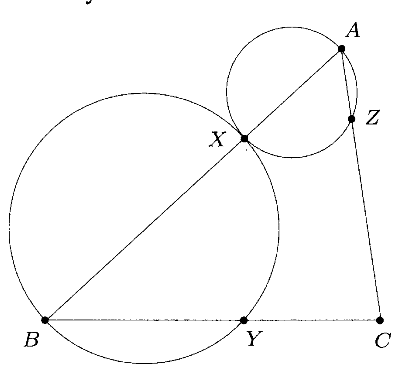
7. Given segments of length 1, a, and b, explain how to solve the following geometrically:
$$x^2 = a + b^2.$$
Explain how to construct the segment of length x.

8. Which of the following are constructible numbers?
   (a) 1
   (b) 3.1416
   (c) π
   (d) $\sqrt{203}$
   (e) $\sqrt{3+\sqrt{2}}$
   (f) $4^{1/3}$

9. Can we construct an angle of $2°$?

10. Given a unit line segment $\overline{AB}$ as shown:
   (a) Construct a segment of length $\dfrac{1+\sqrt{5}}{2}$.
   (b) Construct a segment of length $\sqrt{\dfrac{2}{3}}$.

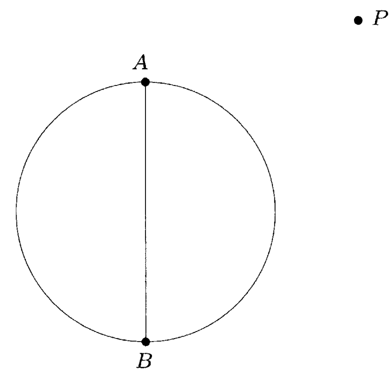
11. Prove Miquel's Theorem for the case where two of the circles are tangent. That is, given $\triangle ABC$ with menelaus points X, Y, and Z, as shown, where the circumcircles of $\triangle AXZ$ and $\triangle BXY$ are tangent at X, show that the quadrilateral $XYCZ$ is cyclic.



Hint. Tangent circles at X have the same tangent line to both circles at X.

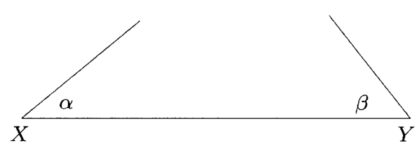
12. Construct a triangle given the foot F of an altitude, the orthocenter H, and the center N of the 9-point circle.

13. Given a diameter $\overline{AB}$ of a circle and a point P as shown, construct a perpendicular from P to $\overleftrightarrow{AB}$, *with a straightedge alone.*



14. (a) List all the regular n-gons with $n \leq 100$ sides that are constructible with a straightedge and compass.
    (b) Use Gauss' Theorem to prove that an angle of $20°$ is not constructible.
    (c) Use Gauss' Theorem to decide whether or not an angle of $6°$ is constructible.

15. Construct $\triangle ABC$ given a line segment $\overline{XY}$ and two adjacent angles, as in the figure, where the length of the perimeter is XY, $\alpha = \angle B$, and $\beta = \angle C$.



16. Construct triangle ABC given the location of its circumcenter S, the location of its orthocenter H, and the foot of an altitude F.

17. Construct a triangle given the foot F of an altitude, the circumcenter S, and the center N of the 9-point circle.

18. Construct triangle ABC given the location of vertex A, the location of its circumcenter S, and the center N of the 9-point circle.

19. Show that the incenter of a triangle is the Nagel point of its medial triangle.

PART II

TRANSFORMATIONAL GEOMETRY

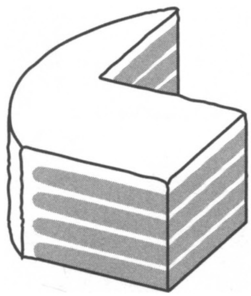
CHAPTER 7

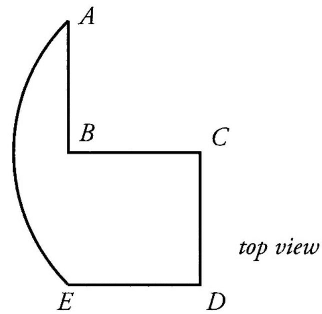
THE EUCLIDEAN TRANSFORMATIONS OR ISOMETRIES

7.1 Rotations, Reflections, and Translations

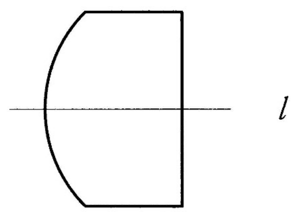
Two children want to share a piece of cake of an unusual shape, as shown in the figure on the following page, where $BCDE$ is a square, the curve AE is a circular arc with center C, and the points A, B, and E are collinear. Being children, they are not enthusiastic about receiving pieces that differ in shape. In other words, they want to cut the cake into two congruent pieces. How can this be done?

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky



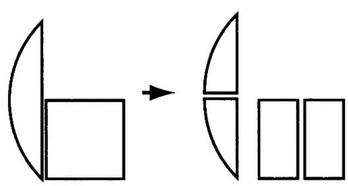


If the cake included the square space above $BCDE$, so that it was of the shape shown in the figure on the right, the task would be very simple. The cake could be cut down the middle along the line l.



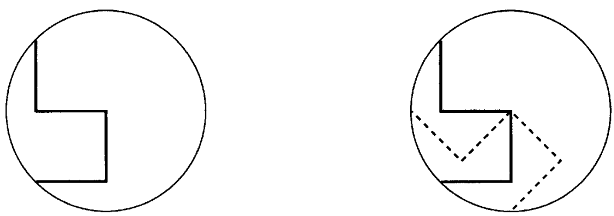
The cake in this figure has ***reflectional symmetry*** about the line l: if you hold a mirror vertical to the page with one edge of the mirror along l, the reflection of one half of the figure will coincide with the other half (this type of symmetry will be defined more precisely later).

When we wish to divide cakes and pies equally, we tend to look for reflectional symmetry and often overlook other possibilities. This might be why many people try to divide the cake by cutting it into two pieces and then dividing each of the pieces along an axis of reflectional symmetry, as in the figure on the right.



Although it can be cut into two pieces, each of which has reflectional symmetry, the cake itself does not have reflectional symmetry. But this does not mean that there is no solution to the original problem.

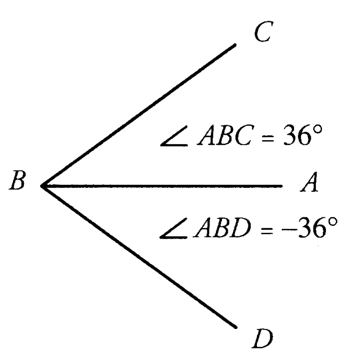
The cake was probably one quarter of a round cake, as shown in the figure below. A spin around the center C through a 45° angle shows how to cut the cake into two congruent pieces.



Directed Angles

The cake problem was solved by using a counterclockwise rotation through an angle of 45°. A rotation in the clockwise direction would also have led to a solution. The usual way to distinquish between clockwise and counterclockwise rotations is to use ***directed*** or ***signed*** angles.

Angles that are measured in a counterclockwise direction are considered positive, while those measured in a clockwise direction are negative, as shown in the figure below. For a directed angle, the symbol $\angle ABC$ is interpreted as the angle *from* the ray $\overrightarrow{BA}$ *to* the ray $\overrightarrow{BC}$.

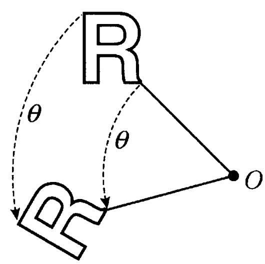


Rotations

Let O be a point and θ be a directed angle. The ***rotation*** about O through the angle θ, denoted by $\mathbf{R}_{O,\theta}$, maps each point P of the plane, where $P \neq O$, into another point P', where

$$|OP'| = |OP| \qquad \text{and} \qquad \angle POP' = \theta.$$

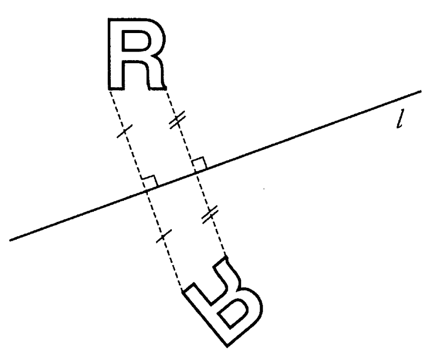
The point O, which is called the ***center of rotation***, is mapped onto itself. Since it does not move, it is called a ***fixed point*** or an ***invariant point*** under $\mathbf{R}_{O,\theta}$.



Reflections

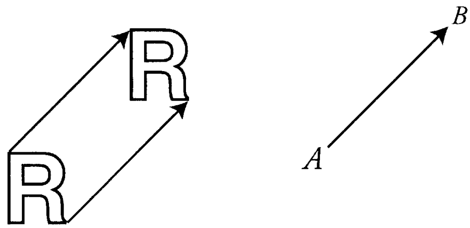
Let l be a line in the plane. The ***reflection*** about l, denoted by $\mathbf{R}_l$, maps a point P not on l to the point P' such that l is the perpendicular bisector of PP'.

Under the reflection $\mathbf{R}_l$, every point on l is mapped to itself, so every point on l is a fixed point.



Translations

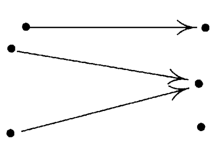
Let $\overline{AB}$ be a directed line segment. A ***translation*** by $\overline{AB}$, denoted by $\mathbf{T}_{AB}$, maps a point P to the point P' such that the directed segment $\overline{PP'}$ is congruent to $\overline{AB}$, parallel to $\overline{AB}$, and in the same direction as $\overline{AB}$.



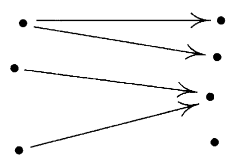
7.2 Mappings and Transformations

Mappings

We use the word ***mapping*** or ***function*** to describe an association between two sets $\mathcal{X}$ and $\mathcal{Y}$ which has the property that each point of $\mathcal{X}$ is associated with one and only one point of $\mathcal{Y}$.



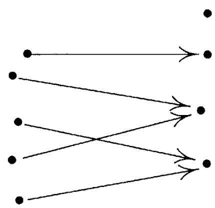
a mapping



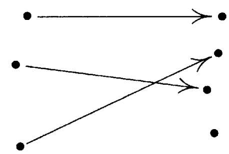
not a mapping

If a point X of $\mathcal{X}$ is associated with the point Y of $\mathcal{Y}$, we say that Y is the ***image*** of X under the mapping and that X is a ***preimage*** of Y.

There are two things that should be mentioned in connection with images and preimages. The first is that the definition of a mapping forbids that a point X of $\mathcal{X}$ have more than one image in $\mathcal{Y}$. However, it is quite permissible that a point Y of $\mathcal{Y}$ have more than one preimage in $\mathcal{X}$—many different points of $\mathcal{X}$ can have the same image in $\mathcal{Y}$. In other words, we can say that a mapping may be ***many-to-one***. Such terminology, although harmless, is a bit redundant because the definition allows such behavior. A mapping of a set $\mathcal{X}$ to a set $\mathcal{Y}$ is called ***one-to-one*** or an ***injection*** if no point of $\mathcal{Y}$ has more than one preimage in $\mathcal{X}$ or, equivalently, if distinct points of $\mathcal{X}$ have distinct images in $\mathcal{Y}$.



a *many-to-one* mapping



a *one-to-one* mapping

The second thing to note is that although the definition of a mapping from $\mathcal{X}$ to $\mathcal{Y}$ says that *every* point of $\mathcal{X}$ must have an image in $\mathcal{Y}$, it does *not* say that every point of $\mathcal{Y}$ must have a preimage in $\mathcal{X}$. When every point of $\mathcal{Y}$ has a preimage in $\mathcal{X}$, we say that the mapping is ***onto*** $\mathcal{Y}$ or that it is a ***surjection***.

When $\mathcal{X}$ and $\mathcal{Y}$ are the same set, it sometimes occurs that a point is its own image. Such a point is called a ***fixed point*** or an ***invariant point*** of the mapping. If all of the points in $\mathcal{X}$ are fixed points, the mapping is called the ***identity mapping***, or more simply the ***identity***, and it is denoted by **I**.

Transformations

A mapping that is both one-to-one and onto is called a ***bijection***, and if $\mathcal{X}$ and $\mathcal{Y}$ are the same set, then the bijection is called a ***transformation***. In other words, when we use the word *transformation* we mean a mapping with the following properties:

The mapping is from one set into the same set.

The mapping is one-to-one.

The mapping is onto.

It is easy to see that rotations, reflections, and translations have all three properties, and therefore that all three are transformations.

The Inverse of a Transformation

The ***inverse*** of a mapping T from $\mathcal{X}$ to $\mathcal{Y}$ is another mapping S from $\mathcal{Y}$ to $\mathcal{X}$ such that for every point x in $\mathcal{X}$ the point $T(x)$ is mapped back onto x by S. In other words, if $T(x) = y$, then $S(y) = x$. A mapping that is not one-to-one cannot have an inverse. However, in geometry, a *transformation* must be one-to-one and onto, and so every transformation automatically has an inverse, and the inverse mapping is itself a transformation.

The three fundamental mappings—rotations, reflections, and translations—are easily seen to be transformations whose inverses are the same type of transformation:

Theorem 7.2.1. *(Inverses of Transformations)*

(1) The inverse of the rotation $\mathbf{R}_{O,\alpha}$ is the rotation $\mathbf{R}_{O,-\alpha}$.

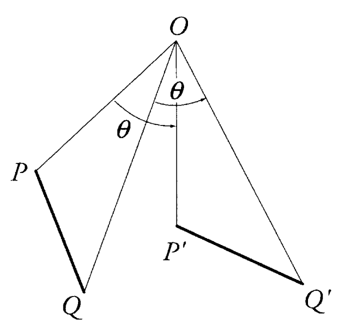
(2) The inverse of the reflection $\mathbf{R}_l$ is the same reflection.

(3) The inverse of the translation $\mathbf{T}_{AB}$ is the translation $\mathbf{T}_{BA}$.

7.2.1 Isometries

A transformation that preserves distances is called an ***isometry***.

Theorem 7.2.2. *Rotations, reflections, and translations are isometries.*



Proof. We will show that a rotation $\mathbf{R}_{O,\theta}$ is actually an isometry (the proofs that reflections and translations are also isometries are similar).

Consider the figure above, which shows a typical case. Since

$$\angle POP' = \theta = \angle QOQ',$$

we must have

$$\angle POQ = \theta - \angle QOP' = \angle P'OQ'.$$

Since

$$OP = OP' \qquad \text{and} \qquad OQ = OQ',$$

then by the **SAS** congruency theorem we have

$$\triangle OPQ \cong \triangle OP'Q',$$

and therefore $PQ = P'Q'$.

□

Composition of Isometries

What happens if an isometry T is applied to the plane and then followed by another isometry S? When a transformation T is followed by another one S, the combined result is called the ***composition*** of the two transformations and is written $S \circ T$. Notice that the first transformation is on the right, while the second is on the left.[7]

Suppose that we start with points P and Q at distance d from each other. When T is applied, these points are mapped into P' and Q', and

$$\text{dist}(P', Q') = \text{dist}(P, Q) = d.$$

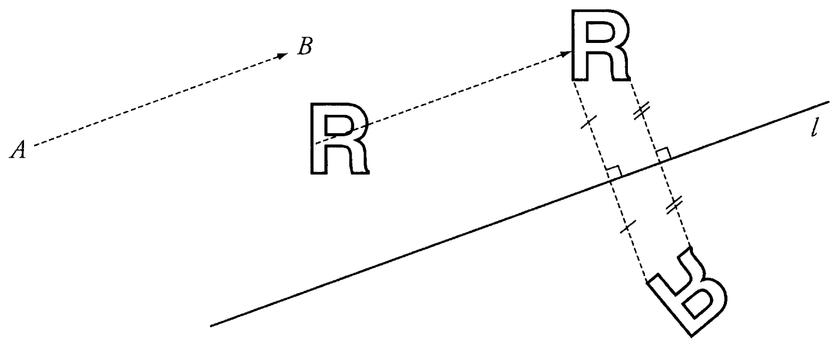
When S is applied to P' and Q', these points are mapped to P'' and Q'', respectively, and

$$\text{dist}(P'', Q'') = \text{dist}(P', Q') = d.$$

The combined effect is that P and Q are mapped to P'' and Q'', and the distance is preserved; that is, $S \circ T$ is itself an isometry.

Since we can create new isometries by composing known isometries, it seems like there is an unlimited supply of different types of isometries. For example, we could create a new isometry by first doing a rotation, then a reflection about some line, then a reflection about another line, then a translation. We will see later that we cannot really get too much that is new, and, in fact, there are only four different types of isometries in the plane. In addition to rotations, reflections, and translations, the only other type is a *glide reflection.*

[7]This is the conventional notation in geometry. It is a common convention in algebra texts to write the first transformation on the left.



A ***glide reflection*** $\mathbf{G}_{l,AB}$ is simply a translation $\mathbf{T}_{AB}$ followed by a reflection $\mathbf{R}_l$ about a line l that is parallel to $\overline{AB}$. We will prove that this is the only other additional isometry later.

It is obvious that all isometries have inverses and that the inverses themselves must also be isometries. Not quite so obvious is the fact that an isometry also preserves straight lines.

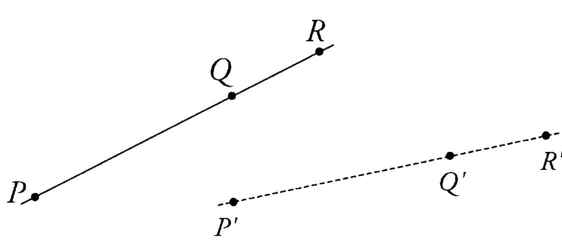
Theorem 7.2.3. *(Isometries Preserve Straight Lines)*

(1) Let P, Q, and R be three points, and let P', Q', and R' be their images under an isometry. The points P, Q, and R are collinear, with Q between P and R, if and only if the points P', Q', and R' are collinear, with Q' between P' and R'.

(2) Let l be a straight line, and let l' be the image of l under an isometry. Then l' is a straight line.

Proof. Here we write $|AB|$ for $\text{dist}(A, B)$.

(1) We will show that if Q is between P and R, then Q' must be between P' and R' (the proof of the converse may be obtained by interchanging P, Q, and R with P', Q', and R').



If Q is between P and R, then

$$|PQ| + |QR| = |PR|.$$

Since an isometry preserves distances, we must have

$$|P'Q'| = |PQ|, \qquad |Q'R'| = |QR|, \qquad \text{and} \qquad |P'R'| = |PR|.$$

Hence,

$$|P'Q'| + |Q'R'| = |P'R'|,$$

and the Triangle Inequality shows that P', Q', and R' are collinear with Q' between P' and R'.

(2) Let P and Q be two points on l, and let P' and Q' be their images under an isometry. Let m be the line passing through P' and Q'. We will show that m is the image of l under the isometry. We must check two things:

(a) that every point R on l has its image R' on m and

(b) that every point S' on m has its preimage S on l.

It follows from statement (1) above that if R is a point on l other than P or Q, then P', Q', and R' must be collinear, so R' is a point on m. Conversely, if S' is on m, then P', Q', and S' are collinear, and it follows again from statement (1) that P, Q, and S are on l.

□

The next theorem tells us that an isometry preserves the shape and size of all of the geometric figures.

Theorem 7.2.4. *Under an isometry,*

(1) the image of a triangle is a congruent triangle;

(2) the image of an angle is a congruent angle;

(3) the image of a polygon is a congruent polygon;

(4) the image of a circle is a congruent circle.

Proof. We will prove statement (1) and leave the rest as exercises. Let P, Q, and R be the vertices of a triangle. It follows from Theorem 7.2.3 that their images P', Q', and R' are the vertices of a triangle and that the edges

$$P'Q', \qquad Q'R', \qquad \text{and} \qquad P'R'$$

are the images of the edges

$$PQ, \qquad QR, \qquad \text{and} \qquad PR.$$

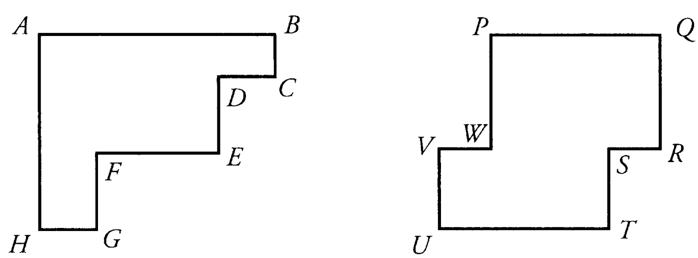
Since the isometry preserves distances, congruency now follows from the **SSS** congruence property.

□

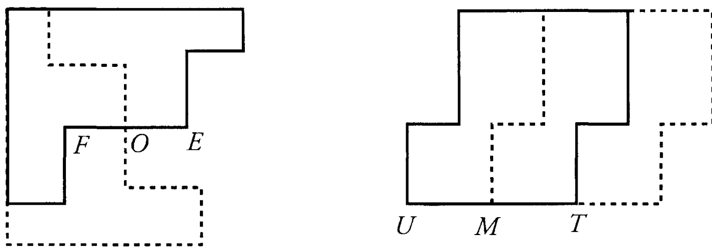
Recall that the notion of congruency is defined in different ways for different figures. For example, two triangles are congruent if the three angles and the three corresponding sides of one triangle are the same size as the three angles and the three corresponding sides of the other, while two circles are said to be congruent if they have the same radius. Theorem 7.2.4 shows that the notion of isometry encompasses and generalizes the notion of congruency.

7.3 Using Rotations, Reflections, and Translations

Example 7.3.1. *Cut each of the following figures into two congruent pieces using a single cut.*

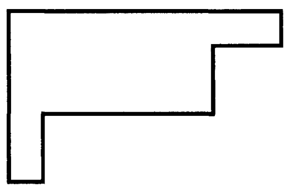


Solution. After the cut we must end up with two pieces that are congruent to each other. This means that one of the pieces must be obtainable from the other by an isometry—either a rotation, a reflection, a translation, or some combination. One way to attack the problem is to use a "trace and fit" method: trace the figure on tracing paper and place the traced figure on the original one in various positions until the two overlapping figures create the outline of the two congruent shapes (much as was done at the beginning of the chapter to cut the cake). When this is done, you can see that the solution for the polygon $ABCDEFGH$ is obtained by applying the rotation $\mathbf{R}_{O,90^\circ}$, where O is the midpoint of the edge EF. The solution for $PQRSTUVW$ may be obtained via the translation $\mathbf{T}_{UM}$, where M is the midpoint of TU.



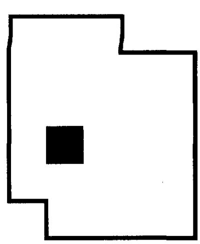
□

It is interesting to note that a slight change in the problem can result in a different solution. For example, consider the problem of cutting the polygon in the figure below into two congruent halves. It resembles the polygon $ABCDEFGH$ from the previous problem, but it is solved by a glide reflection, not a rotation.

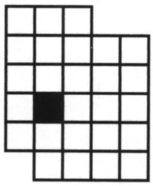


Puzzles of this type can get quite complicated, and the trace and fit method does not always lead to a solution. Here is a more complicated example.

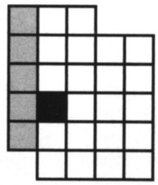
Example 7.3.2. *Cut the polygon in the figure below into two congruent pieces using a single cut. The shaded region is a hole, and cutting through the hole still counts as a single cut.*



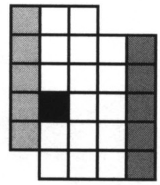
Solution. One way to approach this problems is to divide the shape into congruent squares suggested by the shape of the hole as in diagram (1) in the figure on the following page, then try to gradually build up the congruent pieces by coloring the squares. One possible sequence of events is illustrated by diagrams (2) through (7) in the figure.



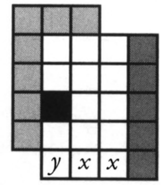
(1)



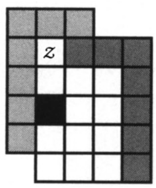
(2)



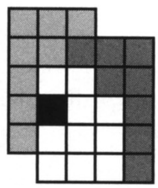
(3)



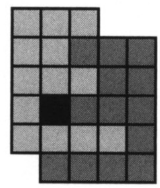
(4)



(5)



(6)



(7)

Here is an explanation of each step.

(2) We might imagine that squares along the left side all belong to the same piece, so we color them light gray.

(3) There must be a matching set of squares in the other piece, probably the five squares along the right side. Color them dark gray.

(4) A little exploration will convince us that the squares along the top cannot be dark gray, since we would not be able to find matching light gray squares, so they must be light gray.

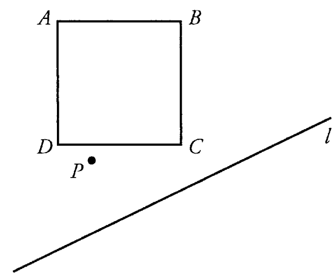
(5) We might think that the light gray squares in (4) should be matched by dark gray ones at x and x. However, this leaves y as a light gray square and then we cannot find a match for the light gray square at y. Thus, the dark gray squares must be added as in (5).

(6) This forces the square at z to be light gray, which in turn forces the corresponding dark gray one.

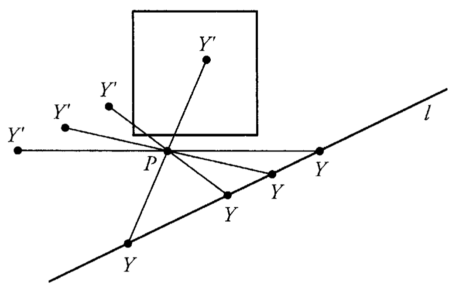
(7) Continuing in this manner, we eventually reach a solution.

□

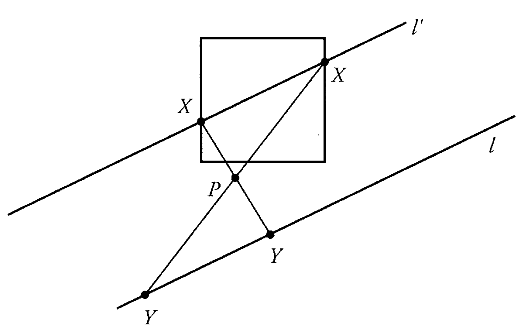
Example 7.3.3. *Given a square $ABCD$, a line l, and a point P, find all points X and Y with X in an edge of $ABCD$, Y in l, and with P being the midpoint of the segment XY.*



Solution. We will take a trial-and-error approach. Let the point Y be located on l and find the corresponding point Y' so that YY' has P as its midpoint. If Y' happens to land on an edge of the square, we have found a solution. It is more likely that Y' is not on an edge of the square, so we will try several positions for Y and see what happens to Y', as in the figure below.

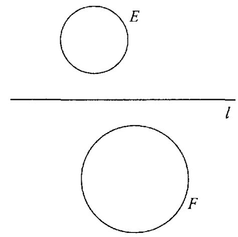


Note that each Y' is obtained from the corresponding Y by a rotation of $180°$ around P. Thus, we can get the solution by applying $\mathbf{R}_{P,180°}$ to the line l. The points X, if there are any, are the points where the image l' meets the square $ABCD$, as in the figure below.

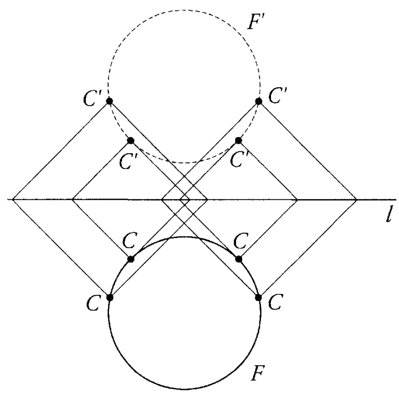


□

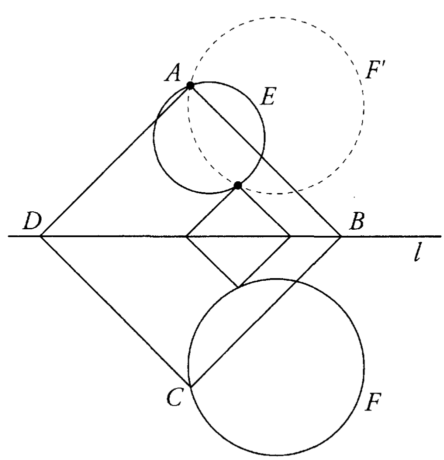
Example 7.3.4. *Given two circles E and F separated by a line l, find all squares $ABCD$ with vertex A in E, the opposite vertex C in F, and the remaining vertices on l.*



Solution. Using a trial-and-error approach again, let a point C be taken on the circle F, and find the corresponding point C' so that C and C' are opposite vertices of a square whose other two vertices are on l. Observe that l is the right bisector of the diagonal CC' of the square. As C takes up different positions in F, C' must therefore be a point on the circle F' that is obtained by reflecting F through the line l, as in the figure below.

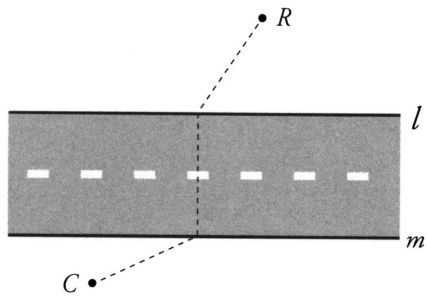


Thus, we obtain the solution by applying $\mathbf{R}_l$ to the circle F, and the points where the image F' intersects E give us the point or points A of the desired square. Having found A, we can now construct the square to complete the solution, as in the following figure.



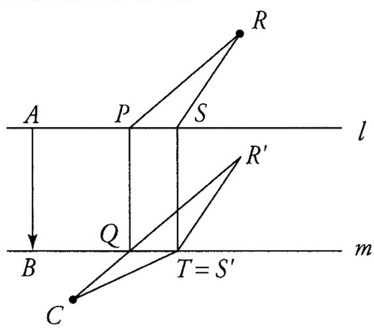
□

Example 7.3.5. *A highway is bounded by two parallel lines l and m. A horse and rider at point R wish to return to camp at a point C on the other side of the highway, and the rider wishes to cross the highway in a perpendicular direction. What is the shortest route that will fulfill both wishes?*



Solution. Let $\overline{AB}$ be the directed segment whose magnitude is equal to the distance between l and m and whose direction is perpendicular to them, going from l toward m. Let R' be the image of R under $\mathbf{T}_{AB}$, and let CR' meet m at Q. Let P be the foot of the perpendicular from Q to l, and note that Q is the image of P under $\mathbf{T}_{AB}$, as in the figure below.

The horse and rider should go from R to P, cross the highway to Q, and proceed to C. This route is the shortest. To see why, let S be any other point on l, and let T be the foot of the perpendicular from S to m.



Then T is the image of S under $\mathbf{T}_{AB}$, and

$$\begin{aligned} RS + ST + TC &= R'T + PQ + SC \\ &> PQ + R'C \qquad \text{(by the Triangle Inequality)} \\ &= PQ + R'Q + QC \\ &= PQ + RP + QC, \end{aligned}$$

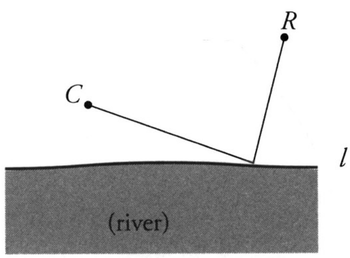
since $R'Q = RP$.

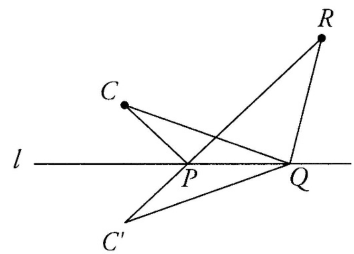
□

The previous problem is what is known as an ***extremal problem***; that is, it is a problem where the objective is to find either the largest or smallest value of some quantity.

The following is another extremal problem whose solution is obtained by a reflection rather then a translation.

Example 7.3.6. *A horse and rider at a point R wish to return to camp at a point C, but the horse wishes to take a drink from a straight river l before doing so. If C and R are on the same side of l, what is the shortest route that will fulfill both wishes?*





Solution. Let C' be the image of C under $\mathbf{R}_l$. Draw the line $C'R$, and let P be the points where it cuts l. We claim that if the horse and rider go from R to P and then from P to C, this route is the shortest. If Q is any other point on l, then by the Triangle Inequality we have

$$RQ + QC = RQ + QC' > RC' = RP + PC' = RP + PC,$$

which proves our claim.

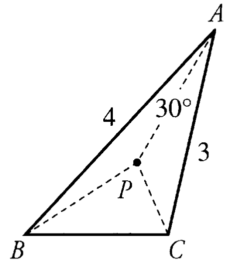
□

Example 7.3.7. *(The Fermat Point)*

Let ABC be a triangle with $AB = 4$, $AC = 3$, and $\angle BAC = 30°$. Determine the minimum value of

$$PA + PB + PC$$

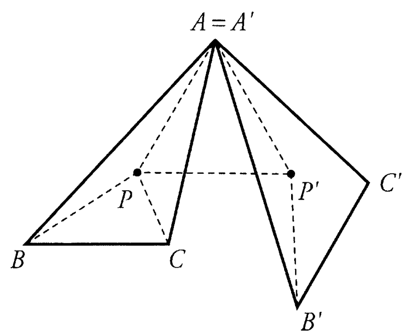
where P is any point inside the triangle ABC, and find the location of the point P that yields this minimum value.



Solution. Consider the effect of $\mathbf{R}_{A,60^\circ}$ on $\triangle ABC$, as in the figure below. This maps $\triangle ABC$ onto $\triangle A'B'C'$ and takes the point P to the point P'. Since $\angle PAP' = 60^\circ$, triangle APP' must be equilateral. Hence

$$\begin{aligned} PB + PA + PC &= BP + PP' + P'C' \\ &\geq BP' + P'C' \\ &\geq BC' \end{aligned}$$

by the Triangle Inequality.



Now, $\angle BAC' = \angle BAC + \angle CAC' = 90^\circ$, so $BC' = 5$ by Pythagoras' Theorem. Therefore, the value of $PA + PB + PC$ is at least 5, and the value will be exactly 5 if B, P, P', and C' are collinear. Thus, to attain the minimum value, P and P' would have to lie on BC'.

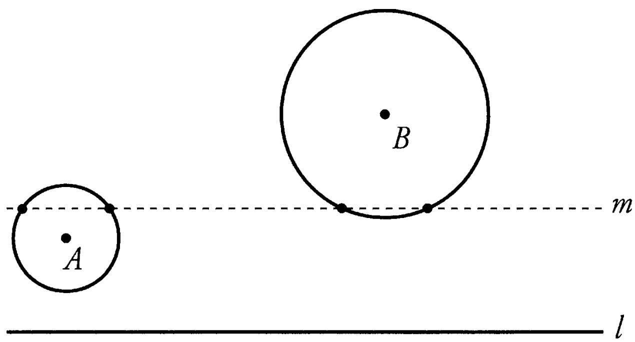
If this happens, then

$$\angle APB = 180^\circ - \angle P'PA = 120^\circ \qquad \text{and} \qquad \angle CPA = \angle C'P'A' = 120^\circ.$$

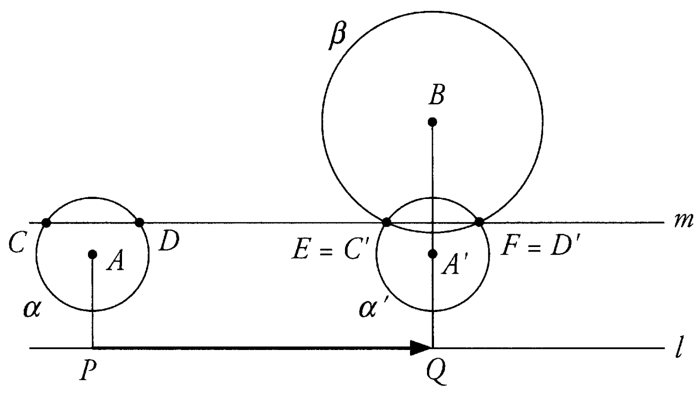
Now, this defines the point P uniquely, and since all three angles of triangle ABC are less than 120°, P is indeed inside $\triangle ABC$.

□

Example 7.3.8. *A and B are the centers of two circles on the same side of a line l. Construct a line m, parallel to l, such that the circles cut off equal segments from m.*



Solution. Let α and β denote circles centered at A and B, respectively. Let P and Q be the feet of the perpendiculars from A and B upon l, respectively. The solution here is obtained by using the translation $\mathbf{T}_{PQ}$. Under this translation, the circle α is mapped to the circle α' centered at A', as in the figure below.



Since the image of P is Q, we have $AA' = PQ$, and so $A'APQ$ is a rectangle. Thus, A' must be on BQ. Let E and F be the points at which the translated circle intersects β, and let m be the line EF, cutting α at C and D. Then $E = C'$ and $F = D'$, so that

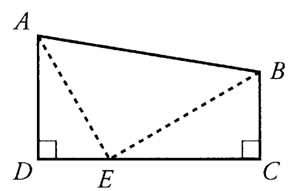
$$EF = C'D' = CD.$$

□

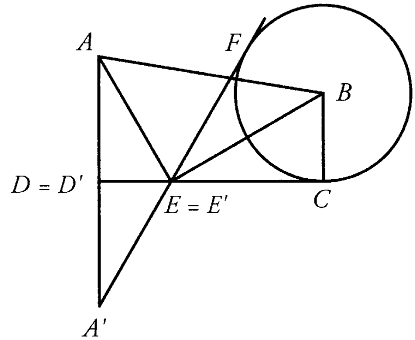
In the next example, we use a reflection to give a solution to a difficult construction problem.

Example 7.3.9. *Let $ABCD$ be a quadrilateral with right angles at C and D and with $BC < CD$. Construct the point E on CD such that*

$$\angle AED = 2\angle BEC.$$



Solution. Let A' be the image of A under $\mathbf{R}_{CD}$, as in the figure below.



Draw the circle with center B and radius BC so that DC is tangent to the circle at C. There are two tangents from A' to the circle. Draw the tangent $A'F$ that meets CD at the point E between C and D.

Note that $D' = D$ and $E' = E$, since D and E are fixed points of the reflection $\mathbf{R}_{CD}$. Therefore, the triangles ADE and $A'D'E'$ are congruent, and hence

$$\angle AED = \angle A'E'D' = \angle FEC = 2\angle BEC.$$

$\square$

7.4 Problems

1. A swimming pool is posting notices to inform visitors about their Monday closure. All the notices are identical except for the one by the diving board. It reads: "NOW NO SWIMS ON MON." What could be the reason for this?

2. The capital letters have been classified into the following subsets:

   (a) A, M, T, U, V, W, Y

   (b) B, C, D, E, K

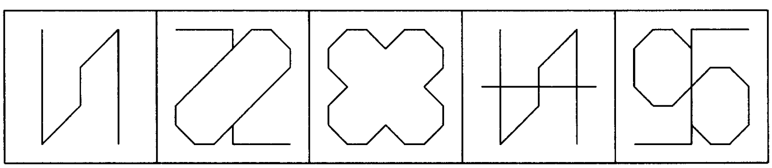
   (c) F, G, J, L, P, Q, R

   (d) H, I, O, X

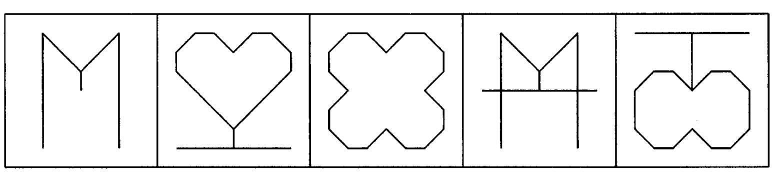
   (e) N, S, Z

   On what basis could this classification be made?

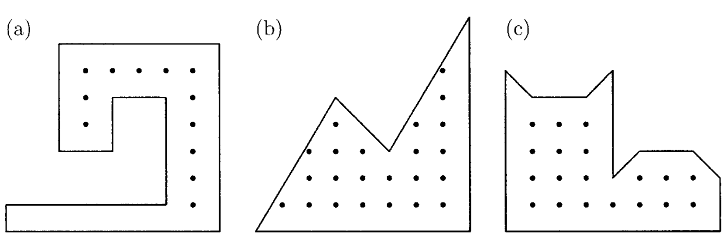
3. The diagram below shows five symbols in a sequence. What could the sixth symbol be?



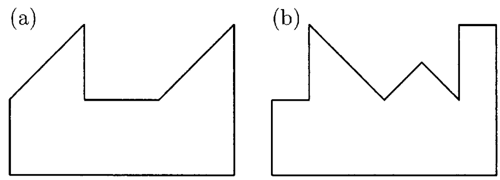
4. The diagram below shows five symbols in a sequence. What could the sixth symbol be?



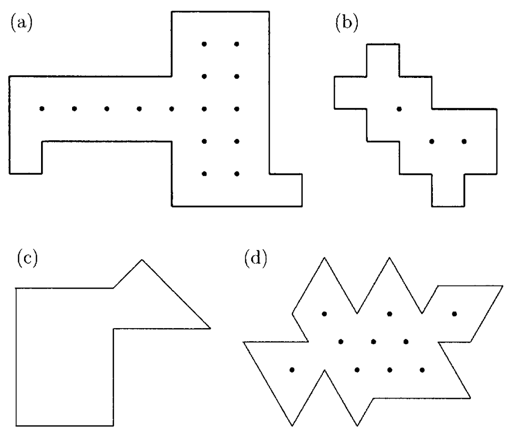
5. Dissect each of the figures below into two congruent pieces.



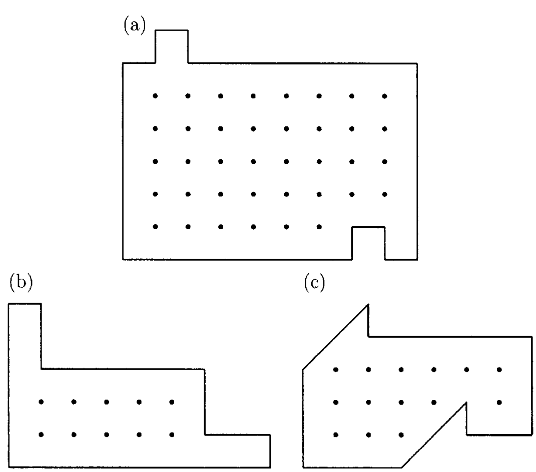
6. Dissect each of the figures in the diagram below into two congruent pieces.



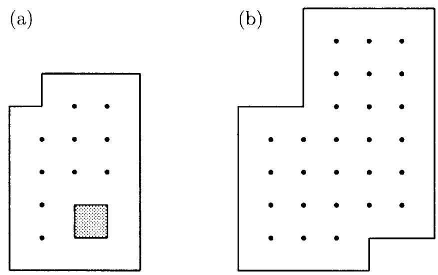
7. Dissect each of the figures below into two congruent pieces.



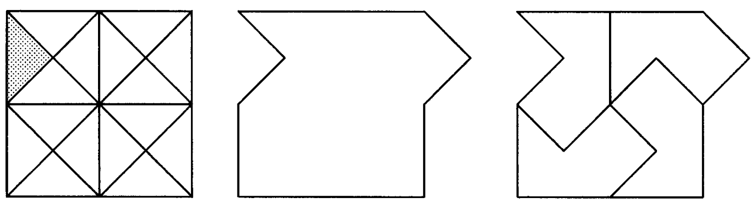
8. Dissect each of the figures in the diagram below into two congruent pieces.



9. Dissect each of the figures in the diagram below into two congruent pieces.



10. A square is divided into four smaller squares, and each small square is divided into four right isosceles triangles, as shown in the diagram on the left. One of the triangles is slid over, resulting in the figure shown in the diagram in the middle. This figure is to be dissected into four congruent pieces, and one solution is shown in the diagram on the right. Find an alternative solution.



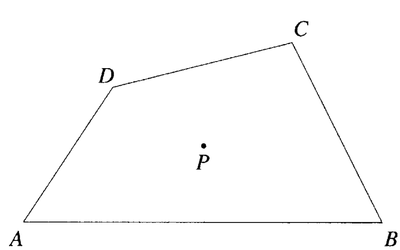
11. Dissect a circle into a number of congruent pieces such that at least one piece does not include the center of the circle, either in its interior or on its boundary. Find two solutions.

12. Find an integer root of each of the following equations:

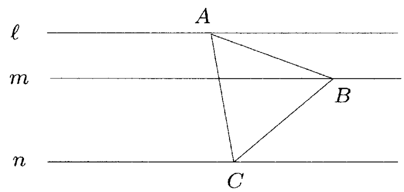
(a) $\dfrac{9(8-x)}{(9-8)x} + \dfrac{8-x}{9-8} + \dfrac{11-x}{x-1} = x.$

(b) $x = \dfrac{1-x}{x-11} + \dfrac{8-6}{x-8} + \dfrac{x(8-6)}{(x-8)6}.$

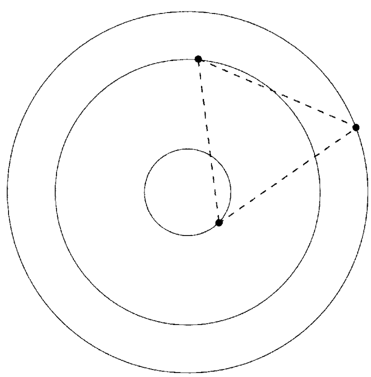
13. Describe how to inscribe a parallelogram with center P in the quadrilateral $ABCD$ shown below.



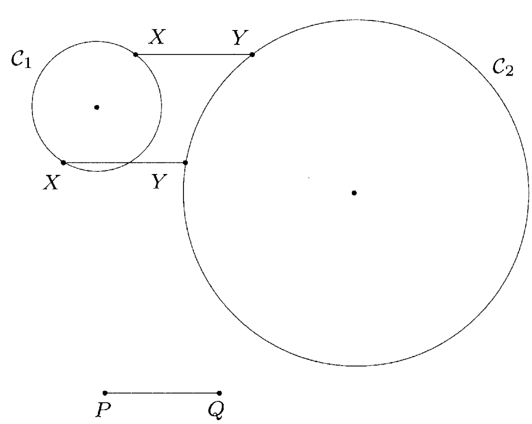
14. Given three parallel lines ℓ, m, and n construct an equilateral triangle with A on ℓ, B on m, and C on n.



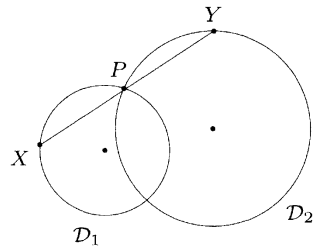
15. Given three concentric circles construct an equilateral triangle with one vertex on each circle.



16. Describe how to find all points X on circle $\mathcal{C}_1$ and Y on circle $\mathcal{C}_2$ so that the segment $\overline{XY}$ is parallel and congruent to $\overline{PQ}$.

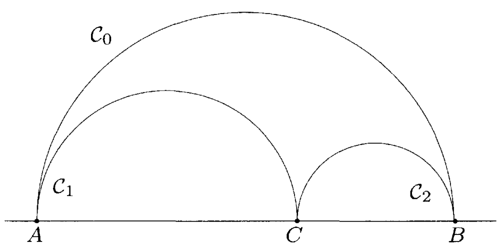


17. Describe how to find all points X on circle $\mathcal{D}_1$ and Y on circle $\mathcal{D}_2$ so that X, P, and Y are collinear and $XP = YP$.



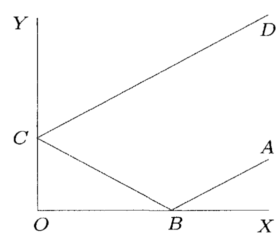
18. In the figure below, the points A, B, and C are collinear, AB is a diameter of $\mathcal{C}_0$, AC is a diameter of $\mathcal{C}_1$, and CB is a diameter of $\mathcal{C}_2$.

    (a) Construct a perpendicular to $\overline{AB}$ from C hitting $\mathcal{C}_0$ at D. Join $\overline{AD}$ and $\overline{DB}$ hitting $\mathcal{C}_1$ and $\mathcal{C}_2$ at E and F, respectively.

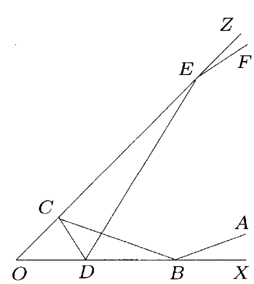


    (b) Explain why the quadrilateral $DECF$ is a rectangle and why the line through E and F is a common tangent to the circles $\mathcal{C}_1$ and $\mathcal{C}_2$.

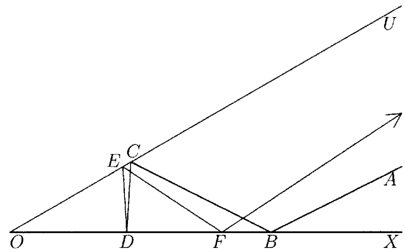
19. Two facing mirrors OX and OY form an angle at O. A light ray $ABCD$ reflects off each mirror once, as shown in the figure on the right. If the ray's final direction is opposite to its initial direction, what is the measure of the angle between the mirrors?



20. Two facing mirrors OX and OZ form an angle at O. A light ray $ABCDEF$ reflects off each mirror twice, as shown in the figure on the right. If the ray's final direction is opposite to its initial direction, what is the measure of the angle between the mirrors?



21. Two facing mirrors OX and OU form an angle at O. A light ray $ABCDEFGH$ reflects off each mirror thrice, as shown in the diagram below (G and H not shown). If the ray's final direction is opposite to its initial direction, what is the measure of the angle between the mirrors?

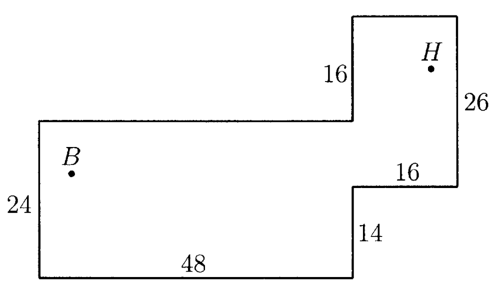


22. Prove that the perimeter of a quadrilateral with one vertex on each side of a rectangle has a perimeter no shorter than the sum of the diagonals of the rectangle.

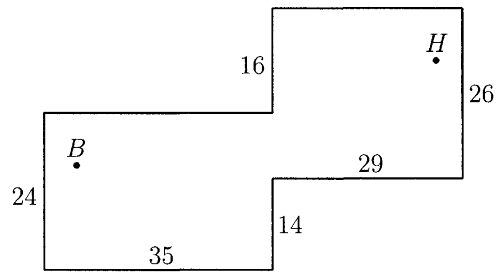
23. A woodsman's hut is in the interior of a peninsula which has the form of an acute angle. The woodsman must leave his hut, walk to one shore of the peninsula, then to the other shore, then return home. How should he choose the shortest such path?

24. A woodsman's hut is in the interior of a peninsula which has the form of an obtuse angle. The woodsman must leave his hut, walk to one shore of the peninsula, then to the other shore, then return home. How should he choose the shortest such path?

25. The figure below represents one hole on a mini golf course. The ball B is 5 units from the west wall and 16 units from the south wall. The hole H is 4 units from the east wall and 8 units from the north wall. What is the length of the shortest path for the ball to go into the hole in one stroke, bouncing off a wall only once?



26. The diagram below represents a one hole on a mini golf course. The ball B is 5 units from the west wall and 16 units from the south wall. The hole H is 4 units from the east wall and 8 units from the north wall. What is the length of the shortest path for the ball to go into the hole in one stroke, bouncing off a wall only once?



27. Two circles intersect at two points. Through one of these points P, construct a straight line intersecting the circles again at A and B such that $PA = PB$.

28. A point P lies outside a circle ω and on the same side of a straight line ℓ as ω. Construct a straight line through P intersecting ℓ at A and ω at B such that $PA = PB$. Find all solutions.

29. Two disjoint circles are on the same side of a straight line ℓ. Construct a tangent to each circle so that they intersect on ℓ and make equal angles with ℓ. Find all solutions.

30. Initially, only three points on the plane are painted: one red, one yellow and one blue. In each step, we choose two points of different colours. A point is painted in the third colour so that an equilateral triangle with vertices painted red, yellow, and blue in clockwise order is formed with the two chosen points. Note that a painted point may be painted again, and it retains all of its colours. Prove that after any number of moves, all points of the same colour lie on a straight line.

CHAPTER 8

THE ALGEBRA OF ISOMETRIES

8.1 Basic Algebraic Properties

Consider a large iron grate (G) in the shape of a right triangle, as shown in the figure below. The grate has to move from its current position to cover the hole (H). It must fit exactly within the dotted lines.

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky