CHAPTER 8

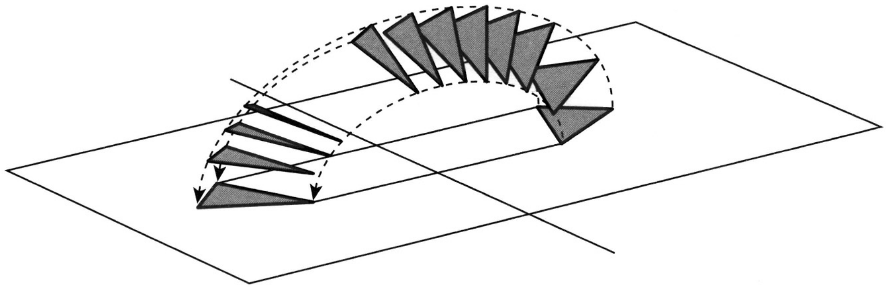
THE ALGEBRA OF ISOMETRIES

8.1 Basic Algebraic Properties

Consider a large iron grate (G) in the shape of a right triangle, as shown in the figure below. The grate has to move from its current position to cover the hole (H). It must fit exactly within the dotted lines.

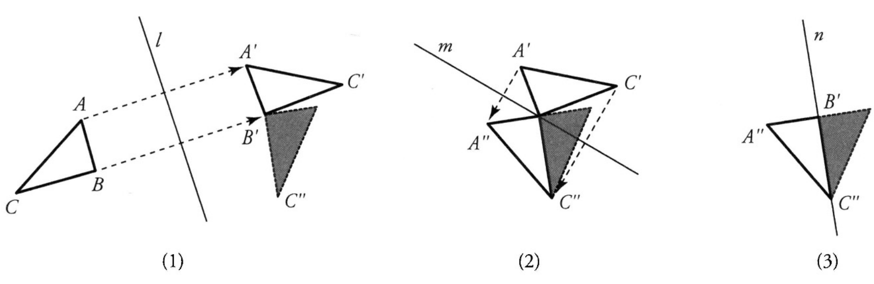
Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

The grate is far too heavy to be lifted by hand, so a machine has been rigged that can lift the grate and flip it around any desired axis, thereby placing it on the other side of the turning axis. The action of the machine is shown in the figure below. If necessary, the axis can pass through the grate.



Find a sequence of flips that will move the grate to cover the hole. What is the minimum number of flips it will take to cover the hole?

The machine is a "reflecting" device—after the machine does its work, the new position of the grate is the reflection through the axis of the original position of the grate. The solution to the problem is that the grate can be moved in a step-by-step manner to cover the hole in three flips, and the minimum number of flips necessary is three. A sequence of flips is depicted in the figure below.



The first step uses a reflection through the line l, which is the right bisector of the segment joining the right angle vertex B of the grate with the corresponding vertex of the hole. With this reflection, the grate ABC is moved into position $A'B'C'$.

We now have one vertex in the correct position. Vertices A' and C' are not in the correct position, and the next step is to do another flip to get one of them into the correct place.

In (2), vertex C'' of the hole corresponds to vertex C' of the moved grate, and by relecting $A'B'C'$ about the right bisector of $C'C''$ we can move C' to the correct position.

Note that the right bisector of $C'C''$ passes through B' because $B'C' = B'C''$. After the second step, the grate is in position $A''B'C''$. The third and final step is to reflect $A''B'C''$ about the line $B'C''$.

The Composition of Transformations

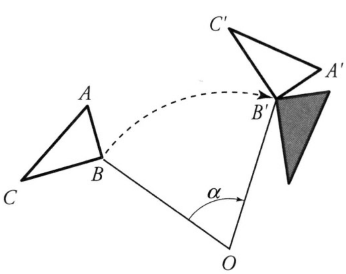
Earlier we defined the composition of two isometries. This terminology is used for any collection of transformations. If T and S are two transformations, the ***composition*** or ***product*** of T and S is denoted by either $S \circ T$ or ST. In terms of the individual points in the plane, ST acts as follows. First, the point X is mapped by T to $T(X)$, then $T(X)$ is mapped by S to $S\,(T(X))$. In other words, for each point X in the plane:

$$ST(X) = S\,(T(X))\,.$$

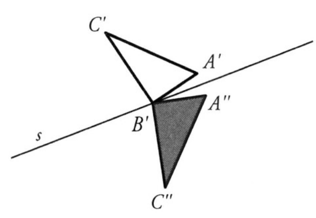
To move the grate to its desired position, we applied the product $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ of three reflections. Again, we emphasize that we follow the "right-to-left" rule. When evaluating $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l(X)$, first find $\mathbf{R}_l(X)$, then $\mathbf{R}_m\,(\mathbf{R}_l(X))$, and finally $\mathbf{R}_n\,(\mathbf{R}_m\,(\mathbf{R}_l(X)))$.

Equal Transformations

In the iron grate problem, we used the product of three reflections to shift the grate to its desired position. If we had a better machine—one that could perform rotations as well as reflections—we could accomplish the same thing by performing just two operations, as shown in the figure below.



(1)



(2)

In this case, the grate is moved by applying the product $\mathbf{R}_s\mathbf{R}_{O,\alpha}$. In the first solution, we used $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$. The individual transformations that make up the two products are quite different, but the net effect is the same. So we can write:

$$\mathbf{R}_s\mathbf{R}_{O,\alpha} = \mathbf{R}_n\mathbf{R}_m\mathbf{R}_l.$$

Two transformations are said to be ***equal*** if they have the same effect on every point in the plane. In other words, saying that $T = S$ means that $T(X) = S(X)$ for *every* point X in the plane.

As the example shows, $T = \mathbf{R}_s\mathbf{R}_{O,\alpha}$ and $S = \mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ does not mean that T and S necessarily have the same description. The situation is similar to equality of functions in trigonometry: if

$$f(x) = 1 - 2\sin^2 x \qquad \text{and} \qquad g(x) = \cos 2x,$$

then the functions f and g are equal (since we have pointwise equality), although their descriptions are quite different.

Other examples of equal transformations that are defined differently are the *halfturn* and a *reflection in a point.*

The ***halfturn*** about a point O, denoted by $\mathbf{H}_O$, is the transformation $\mathbf{R}_{O,180^\circ}$. The ***reflection in the point*** O is the transformation that takes each point P to the point P' so that O is the midpoint of PP'. The point O is called the ***center of the halfturn*** or the ***center of reflection***. In the plane, since a reflection through a point is identical to a halfturn, there is no further need to talk about reflections through a point, and there is really no need for a special symbol to denote it.

Closure

In Chapter 7, we mentioned that the composition of two isometries results in a transformation that is also an isometry. In algebra, we would describe the situation by saying that the set of isometries of the plane is *closed* under the operation of composition.

More generally, if $\mathcal{S}$ is a set of elements and if $\circ$ is a binary operation on $\mathcal{S}$, we say that $\mathcal{S}$ is ***closed*** under the operation $\circ$ if for every pair of elements a and b in $\mathcal{S}$ the product $a \circ b$ is also in $\mathcal{S}$. For example, the set of positive integers is closed under addition.

Associativity

If R, S, and T are three transformations, the product TSR means first apply R, then S, and then T. For a point X, the notation $TSR(X)$ means $T\left(S\left(R(X)\right)\right)$.

We can overrule this by using parentheses since the operations inside parentheses are carried out first. The notation $(TS)\,R$ is interpreted as follows: first, determine what (TS) is. It will be some transformation, call it H, and H is usually different than either T or S. The notation $(TS)\,R$ means HR; that is, first apply the transformation R, then apply the transformation H.

For example, let us suppose that S and T are reflections about two different parallel lines and R is a rotation. Then, as we will see later, TS is some translation H. So we interpret $(TS)\,R$ as meaning first do the rotation R, then follow it by the translation H.

In a similar way, the notation $T\,(SR)$ means that we should first determine what SR is, namely, some different isometry L, and then take the product of L and T: $T\,(SR) = TL$. Note, however, that the parentheses in this case could be omitted, because in the absence of parentheses, the expression is evaluated from right to left.

The following theorem shows that $(TS)\,R$ and $T\,(SR)$ are equal.

Theorem 8.1.1. *(Associative Law)*

The **associative law** *holds for the product of transformations; that is, given three transformations T, S, and R in the plane,*

$$T\,(SR) = (TS)\,R.$$

Proof. Let X be a point in the plane. We will evaluate $T\,(SR)\,(X)$ and $(TS)\,R(X)$.

The notation $T\,(SR)\,(X)$ means "first evaluate $SR(X)$, then evaluate $T\,(SR(X))$." By definition, $SR(X) = S\,(R(X))$, so

$$T\,(SR)\,(X) = T\,(S\,(R(X)))\,.$$

The notation $(TS)\,R(X)$ tells us to evaluate $TS\,(R(X))$. Now, $TS(Z) = T\,(S(Z))$ for all Z in the plane. In particular, when $Z = R(X)$, we get

$$TS(Z) = T\,(S(Z)) = T\,(S\,(R(X)))\,.$$

Thus,

$$(TS)\,R(X) = T\,(S\,(R(X)))\,.$$

Since $(TS)\,R$ and $T\,(SR)$ have the same effect on every point X in the plane, we conclude that they are equal transformations.

□

8.2 Groups of Isometries

In algebra, a set of elements $\mathcal{G}$ together with a binary operation $\cdot$ is called a ***group*** and is denoted by $(\mathcal{G}, \cdot)$ if it possesses the following properties:

1. The set $\mathcal{G}$ is closed under the binary operation.
2. The associative law holds: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, and c in $\mathcal{G}$.
3. $\mathcal{G}$ has an ***identity element***: there is some element e in $\mathcal{G}$ such that $e \cdot a = a \cdot e = a$ for every a in $\mathcal{G}$.
4. For every a in $\mathcal{G}$, there is an ***inverse element*** a': there is an element a' in $\mathcal{G}$ such that $a \cdot a' = a' \cdot a = e$.

If it is also true that the commutative law holds (that is, if $a \cdot b = b \cdot a$ for all a and b in $\mathcal{G}$), then $\mathcal{G}$ is called an ***Abelian group***.

The notation for the binary operation $\cdot$ is usually omitted, so that we write ab instead of $a \cdot b$.

We learned in Chapter 7 that every isometry has an inverse. This, along with the results of the previous section, show that the family of isometries in the plane, together with the composition operation, satisfy all four of the conditions listed above. We can summarize this with the following theorem:

Theorem 8.2.1. *The set of all isometries of the plane, together with the operation of composition, forms a group.*

Given an isometry $\mathbf{T}$, we denote its inverse by $\mathbf{T}^{-1}$. Theorem 7.2.1 stated that

$$(\mathbf{R}_{P,\theta})^{-1} = \mathbf{R}_{P,-\theta},$$

$$(\mathbf{R}_l)^{-1} = \mathbf{R}_l,$$

$$(\mathbf{T}_{AB})^{-1} = \mathbf{T}_{BA}.$$

Example 8.2.2. *What are the inverses of* $\mathbf{G}_{l,AB}$ *and* $\mathbf{H}_O$*?*

Solution. We have

$$(\mathbf{G}_{l,AB})^{-1} = \mathbf{G}_{l,BA} \qquad \text{and} \qquad (\mathbf{H}_O)^{-1} = \mathbf{H}_O.$$

□

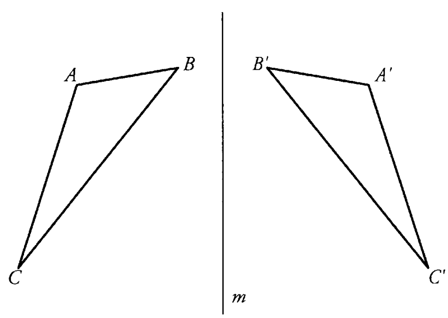
In any group, an element a other than the identity e such that $a^2 = a \cdot a = e$ is called an ***involution***. In other words, an involution is an element other than the identity that is its own inverse.

Any transformation that is an involution in a group of transformations is said to be an ***involutory*** or ***involutoric*** transformation; that is, T is involutory if and only if $TT = T \circ T = I$. Among the transformations

$$\mathbf{R}_{P,\theta}, \quad \mathbf{R}_l, \quad \mathbf{T}_{AB}, \quad \mathbf{G}_{l,AB}, \quad \text{and} \quad \mathbf{H}_P,$$

the involutions are $\mathbf{R}_l$, $\mathbf{H}_P$, and $\mathbf{R}_{P,\theta}$ when θ is a multiple of $180°$.

8.2.1 Direct and Opposite Isometries



In the figure above, triangle ABC has been carried into $A'B'C'$ by a reflection in the line m. The orientation of the triangle has changed—the path ABC proceeds in a clockwise (or negative) direction around the triangle, while the image path $A'B'C'$ goes in a counterclockwise (or positive) direction. This reversal of orientation will obviously happen to every triangle to which $\mathbf{R}_m$ is applied, and $\mathbf{R}_m$ is therefore called an *opposite* isometry. In general, any isometry T that reverses the orientation of every triangle is called an ***opposite isometry***. On the other hand, an isometry T that preserves the orientation of every triangle is said to be a ***direct isometry***.

There is a strong analogy between the products of direct and opposite isometries and the products of positive and negative numbers: direct isometries are like positive numbers and opposite isometries are like negative numbers. Some people prefer to draw an analogy to the addition of even and odd numbers, with direct isometries corresponding to the even numbers, so that the product of a direct and opposite isometry is like the sum of an even and an odd number. Some texts use the terms *even* and *odd* isometries instead of *direct* and *opposite* isometries.

Theorem 8.2.3. *The product of two direct isometries is a direct isometry. The product of two opposite isometries is a direct isometry. The product of a direct isometry and an opposite isometry is an opposite isometry.*

Theorem 8.2.4. *The direct isometries of the plane form a group $\mathcal{D}$.*

Proof. It is clear that $\mathcal{D}$ is closed (under multiplication). Since the associative law holds for all transformations, it also holds for $\mathcal{D}$. The identity map is a direct isometry; that is, I is in $\mathcal{D}$. Every isometry has an inverse, and an isometry and its inverse are either both direct or both indirect. Therefore, whenever T is in $\mathcal{D}$, so is T^{-1}. This completes the proof.

□

It is relatively easy to invent a transformation of the plane that is neither direct nor opposite. For example, let B and C be two different points in the plane, and let T be the transformation that interchanges B and C but leaves every other point where it is. The transformation T is almost the identity map, and there are many triangles PQR that remain unchanged under T. However, if A is a point that forms a triangle with B and C, the transformation maps A, B, and C to A, C, and B, respectively, thereby reversing the orientation. In other words, T preserves the orientation of some triangles while reversing the orientation of others. Thus, T is neither direct nor opposite.

The situation where T is an isometry is quite different as a consequence of two fundamental theorems about isometries in the plane, namely, that a plane isometry is completely determined by its action on three noncollinear points and that every plane isometry is the product of at most three reflections.

We begin the proofs of these facts with a theorem that characterizes the identity transformation.

Theorem 8.2.5. *If T is an isometry that fixes each of three noncollinear points, then T is the identity.*

Proof. Let A, B, and C be three collinear points that are fixed by T, so that

$$T(A) = A, \qquad T(B) = B, \qquad \text{and} \qquad T(C) = C.$$

We want to show that $T(X) = X$ for every point X in the plane. Suppose that this is not the case. Then there is a point P such that

$$P' = T(P) \neq P.$$

Since T is an isometry with $T(A) = A$ and $T(P) = P'$, we have

$$\text{dist}(A, P) = \text{dist}\,(T(A), T(P)) = \text{dist}(A, P').$$

This means that A is on the right bisector of PP'. Similarly, B and C must also be on the right bisector of PP', contradicting the fact that A, B, and C are noncollinear. This contradiction shows that our supposition that T is not the identity must be false and completes the proof.

$\square$.

When we say that a transformation T *fixes* a point X we mean that $T(X) = X$, and the point X is called a ***fixed point*** of T. In Martin's text [50], he uses the phrase "T *fixes* a set $\mathcal{S}$" to mean that under T the image of $\mathcal{S}$ is $\mathcal{S}$. He uses the phrase "T *fixes* $\mathcal{S}$ *pointwise*" to means that $T(X) = X$ for each point X of $\mathcal{S}$. Another way of saying the same thing is to say that the set $\mathcal{S}$ is *invariant* under the transformation T.

As an example of the difference in the meaning of this language, consider the mapping of an equilateral triangle ABC onto itself by a rotation of $120°$ around the centroid. The triangle ABC is fixed by the transformation, but it is *not fixed pointwise*.

Question 8.2.6. *Suppose that an isometry fixes each of two different points. Can we assume that the isometry is the identity?*

One of the virtues of group theory is that once you have proven a theorem about groups, it is valid for every group and does not have to be proven separately for each different group. Here is a very simple yet useful result:

Theorem 8.2.7. *Let a and b be elements of a group $(\mathcal{G}, \cdot)$ such that $a \cdot b = e$, where e is the identity. Then $a = b^{-1}$ and $b = a^{-1}$.*

Proof. Since $\mathcal{G}$ is a group, the element a has an inverse a^{-1}. Multiplying each side of the equation

$$a \cdot b = e$$

by a^{-1} on the left, we have

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot e,$$

so that

$$\left(a^{-1} \cdot a\right) \cdot b = a^{-1},$$

and so

$$e \cdot b = a^{-1},$$

and thus, $b = a^{-1}$.

The proof that $a = b^{-1}$ can be obtained by multiplying the equation $a \cdot b = e$ on the right by b^{-1}.

$\square$

The following theorem can be proven in much the same way as Theorem 8.2.5, but we can give a more satisfying proof by using the fact that the isometries of the plane form a group. This proof illustrates how nicely algebra fits with geometry.

Theorem 8.2.8. *An isometry of the plane is completely determined by its action on three noncollinear points.*

Proof. Let A, B, and C be three noncollinear points in the plane, and let S and T be two isometries such that

$$S(A) = T(A), \qquad S(B) = T(B), \qquad \text{and} \qquad S(C) = T(C).$$

We want to show that $S = T$. Now, since the isometries of the plane form a group, we know that T has an inverse T^{-1} so that

$$T^{-1}S(A) = A, \qquad T^{-1}S(B) = B, \qquad \text{and} \qquad T^{-1}S(C) = C.$$

Thus, the isometry $T^{-1}S$ fixes each of the points A, B, and C, and by Theorem 8.2.5, this means that $T^{-1}S$ must be the identity. By the previous theorem, S must be equal to the inverse of T^{-1}; that is,

$$S = \left(T^{-1}\right)^{-1} = T.$$

$\square$

Theorem 8.2.9. *Every isometry of the plane that is not the identity can be decomposed into the product of at most three reflections.*

Proof. Let T be a given isometry, and let A, B, and C be three noncollinear points.

From the iron grate example, we know how to map A, B, and C to $T(A)$, $T(B)$, and $T(C)$, respectively, by a sequence of at most three reflections.

Suppose that it actually took three reflections, say,

$$\mathbf{R}_l, \qquad \mathbf{R}_m, \qquad \text{and} \qquad \mathbf{R}_n,$$

in that order. Then

$$\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l(A) = A' = T(A),$$

$$\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l(B) = B' = T(B),$$

$$\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l(C) = C' = T(C).$$

Thus, $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ and T both have exactly the same effect on A, B, and C, so $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l = T$, by Theorem 8.2.8.

□

Theorem 8.2.9, together with the fact that a reflection in a line is an opposite isometry, now allows us to show that every isometry in the plane is either a direct isometry or an opposite isometry.

Theorem 8.2.10. *Every isometry in the plane is either a direct isometry or an opposite isometry.*

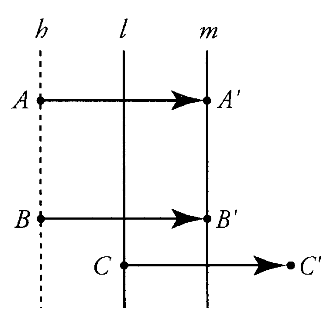
Proof. If the isometry T is the identity or the product of two reflections, it is a direct isometry. If T is a reflection or the product of three reflections, it is an opposite isometry.

□

8.3 The Product of Reflections

In this section, we use reflections to show that there are only four different types of plane isometries: reflections (in a line), rotations, translations, and glide reflections. We begin by examining the product of two reflections.

Example 8.3.1. *Let l and m be two distinct parallel lines. Show that* $\mathbf{R}_m\mathbf{R}_l = \mathbf{T}_{XY}$, *where $\overline{XY}$ is a directed segment perpendicular to l and m and twice the distance from l to m.*



Solution. Let h be a line parallel to l and m such that l is midway between h and m, as in the figure above.

Let A and B be points on h and let C be a point on l. $\mathbf{R}_l$ maps A and B to points A' and B' on m and leaves C where it is.

Note that the two directed segments $\overline{AA'}$ and $\overline{BB'}$ are parallel and twice the distance from l to m.

The reflection $\mathbf{R}_m$ leaves A' and B' where they are and maps C to C', where $\overline{CC'}$ is parallel to and equal in length and direction to $\overline{AA'}$.

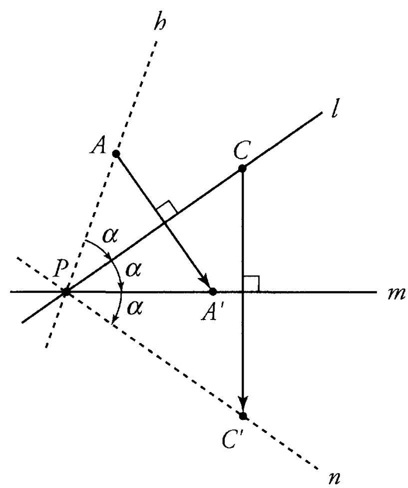
Thus, the effect of $\mathbf{R}_m\mathbf{R}_l$ is to map A, B, and C to A', B', and C', respectively.

However, the translation $\mathbf{T}_{AA'}$ also does the same thing, and by Theorem 8.2.8,

$$\mathbf{R}_m\mathbf{R}_l = \mathbf{T}_{AA'},$$

which completes the proof. □

Example 8.3.2. *Let l and m be two different lines intersecting at a point P. Show that $\mathbf{R}_m\mathbf{R}_l = \mathbf{R}_{P,\theta}$, where θ is twice the directed angle α from l to m.*



Solution. Let h be the line through P such that the directed angle from h to l is α, and let n be the line through P such that the directed angle from m to n is α, as in the figure above.

Let A be a point on h and let C be a point on l. Note that A, C, and P cannot be collinear. If they were collinear, then $h = l$, and it would follow that l, h, and m are all the same line, which is a contradiction.

Consider the effect of $\mathbf{R}_m\mathbf{R}_l$ on the points A, P, and C. It is clear that $\mathbf{R}_{P,\theta}$ has exactly the same effect, so by Theorem 8.2.8, $\mathbf{R}_m\mathbf{R}_l = \mathbf{R}_{P,\theta}$.

□

If l and m are the same line, then $\mathbf{R}_m\mathbf{R}_l$ is the identity. This fact, together with the previous two examples, proves the following theorem:

Theorem 8.3.3. *The only direct isometries of the plane are the identity, the translations, and the rotations.*

We next turn our attention to the product of three reflections.

Theorem 8.3.4. *Let l, m, and n be three lines with a common point P. Then $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ is a reflection.*

Proof. If l and m coincide, then

$$\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l = \mathbf{R}_n\left(\mathbf{R}_m\mathbf{R}_l\right) = \mathbf{R}_n\,\mathbf{I} = \mathbf{R}_n.$$

If n and m coincide, then

$$\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l = \left(\mathbf{R}_n\mathbf{R}_m\right)\mathbf{R}_l = \mathbf{I}\,\mathbf{R}_l = \mathbf{R}_l.$$

If neither of these two cases occurs, we proceed as follows: let α be the directed angle from l to m. Now, there is a unique line h through P such that the angle from h to n is α.

Thus,

$$\mathbf{R}_m\mathbf{R}_l = \mathbf{R}_n\mathbf{R}_h = \mathbf{R}_{P,2\alpha},$$

and, therefore,

$$\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l = \mathbf{R}_n\mathbf{R}_n\mathbf{R}_h = \mathbf{I}\,\mathbf{R}_h = \mathbf{R}_h.$$

□

To handle the combination $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ when the lines l, m, and n are all different and have no point in common, we break it down into separate cases. First, we consider the cases where l and m are both perpendicular to n, then the cases where m and n are both perpendicular to l.

We note first that if all three lines are parallel, then $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ is a reflection. We leave the proof of this as an exercise.

Theorem 8.3.5. *Let l, m, and n be three different lines.*

(1) *If l and m are perpendicular to n, then $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ is the glide reflection $\mathbf{G}_{n,AB}$, where AB is a line segment parallel to n and twice the directed distance from l to m.*

(2) *If m and n are perpendicular to l, then $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ is the glide reflection $G_{l,CD}$, where CD is a line segment parallel to l and twice the directed distance from m to n.*

Proof. For (1), we have $\mathbf{R}_m\mathbf{R}_l = \mathbf{T}_{AB}$ by Example 8.3.1, and so $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ is the glide reflection $\mathbf{R}_n\mathbf{T}_{AB} = \mathbf{G}_{n,AB}$, as claimed.

For (2), Let $A = l \cap m$, let $B = l \cap n$, and let C be any point on m other than A. We leave it as an exercise to show that

$$\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l(X) = \mathbf{R}_l\mathbf{R}_n\mathbf{R}_m(X)$$

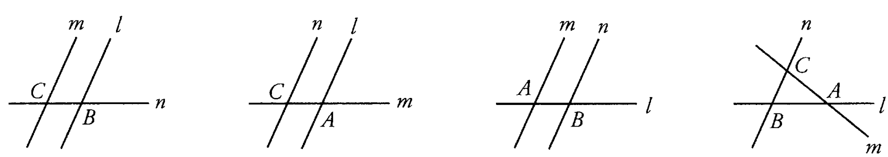
for each of the points $X = A, B, C$ so that

$$\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l = \mathbf{R}_l\mathbf{R}_n\mathbf{R}_m$$

by the fundamental Theorem 8.2.8, and $\mathbf{R}_l\mathbf{R}_n\mathbf{R}_m$ is the glide reflection $G_{l,CD}$ by (1).

□

The cases not covered by the previous theorem are depicted in the figures below, which show the possible arrangements of the reflecting lines.



(1) (2) (3) (4)

Although arrangements (1), (2), and (3) appear to be identical, they are not the same because the order of the reflections matters. It may appear that these cases are complicated, but the associative law comes to the rescue in a rather magnificent way.

Theorem 8.3.6. *Let l, m, and n be three nonconcurrent lines. Then $\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l$ is a glide reflection.*

Proof. The proof of each of the four cases is very similar. Here is how it is done for case (4).

The combination $\mathbf{R}_m\mathbf{R}_l$ is a rotation about A through the angle 2α, where α is the directed angle from l to m, as in (1) below. Let m' be a line through A perpendicular to n, intersecting n at C'. Let l' be the line through A such that the angle from l' to m' is α, as in (2) below. Then

$$\mathbf{R}_n\mathbf{R}_m\mathbf{R}_l = \mathbf{R}_n\left(\mathbf{R}_m\mathbf{R}_l\right) = \mathbf{R}_n\left(\mathbf{R}_{m'}\mathbf{R}_{l'}\right).$$

Now apply the associative law:

$$\mathbf{R}_n\left(\mathbf{R}_{m'}\mathbf{R}_{l'}\right) = \left(\mathbf{R}_n\mathbf{R}_{m'}\right)\mathbf{R}_{l'}.$$

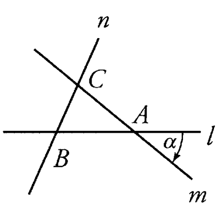
The transformation $\mathbf{R}_n\mathbf{R}_{m'}$ is the rotation $\mathbf{H}_{C'}$. Let n'' be the line through C' perpendicular to l', and let m'' be the line through C' perpendicular to n'', as in (3) below. Then

$$\mathbf{R}_{n''}\mathbf{R}_{m''} = \mathbf{H}_{C'}.$$

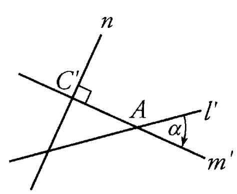
Therefore,

$$\left(\mathbf{R}_n\mathbf{R}_{m'}\right)\mathbf{R}_{l'} = \left(\mathbf{R}_{n''}\mathbf{R}_{m''}\right)\mathbf{R}_{l'} = \mathbf{R}_{n''}\left(\mathbf{R}_{m''}\mathbf{R}_{l'}\right),$$

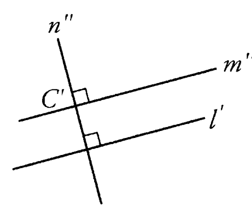
and since m'' and l' are parallel and both perpendicular to n'', we have a glide reflection.



(1)



(2)



(3)

□

8.4 Problems

1. Let S and T be two involutive transformations of the plane.
   (a) Prove that ST is involutive if and only if $ST = TS$.
   (b) Assume that S, T, and I are distinct transformations, where I is the identity, such that
   $$ST = TS = X.$$
   Let $\Gamma = \{I, S, T, X\}$. Prove that Γ is a commutative subgroup of $\mathcal{G}$, the group of all transformations on the plane, by constructing the multiplication table.

2. Let P, Q, and R be three points in the plane, and let P', Q', and R', respectively, be their images under an isometry T. Show that the points P, Q, and R are collinear, with Q between P and R, if and only if the points P', Q', and R' are collinear, with Q' between P' and R'.

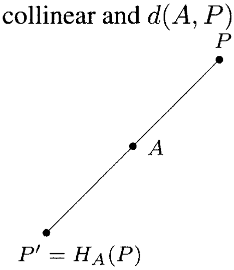
   Hint: When does equality hold in the Triangle Inequality?

3. Let T be an isometry of the plane. Show that if P and Q are fixed points of T, then every point X on the line through P and Q is a fixed point of T.

4. Let T be an isometry of the plane. Show that if T has three fixed points that are not collinear, then $T = I$, the identity.

5. Let S and T be isometries and let A, B, and C be three noncollinear points for which
   $$S(A) = T(A), \qquad S(B) = T(B), \qquad \text{and} \qquad S(C) = T(C).$$
   Show that $S = T$.

6. Let H_A be a halfturn about a point A so that
   $$H_A(P) = P',$$
   where A, P, and P' are collinear and $d(A, P) = d(A, P')$.



   (a) Show that H_A is an isometry.
   (b) Show that H_A is an involution; that is, $H_A = H_A^{-1}$.
   (c) Show that if ℓ is a line in the plane, then $H_A(\ell)$ is a line parallel to ℓ.

7. Let D, E, and F be the midpoints of the sides BC, AC, and AB, respectively, of $\triangle ABC$ and let T be the transformation of the plane that is the product of the three halfturns

$$T = H_E\, H_D\, H_F.$$

Show that the vertex A of $\triangle ABC$ is a fixed point of T; that is, that $T(A) = A$.

Hint. Draw the picture.

8. Let T be an isometry of the plane that admits an invariant line ℓ (that is, $T(\ell) = \ell$) and a fixed point P. Prove that there is a point $Q \in \ell$ such that $T(Q) = Q$ and a line ℓ' through P such that $T(\ell') = \ell'$.

9. Show that if a circle is invariant under the isometry T, then its center is a fixed point of T.

10. Let $T \neq I$ be an involutive isometry. Show that T has at least one fixed point.

11. Let T be an isometry that is an involution and has **exactly** one fixed point O in the plane. Show that T is the halfturn H_O about the point O.

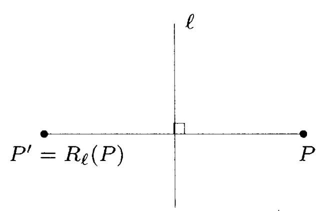
12. If the isometry T is an involution, show that for any point P in the plane the midpoint of the line segment joining P and $T(P)$ is a fixed point of T.

13. Let T be an isometry of the plane and let ℓ be the perpendicular bisector of the segment $\overline{AB}$. Prove that $T(\ell)$ is the perpendicular bisector of the segment $\overline{T(A)\,T(B)}$.

14. Let R_ℓ be a reflection in the line ℓ so that

$$R_\ell(P) = P',$$

where either ℓ is the perpendicular bisector of the segment PP' for each P or P and P' coincide on the line ℓ for each P.



(a) Show that R_ℓ is an isometry.

(b) Show that R_ℓ is an involution; that is, that $R_\ell = R_\ell^{-1}$.

(c) Show that if m is a line distinct from ℓ, then $R_\ell(m)$ is distinct from ℓ.

15. Show that if m and n are perpendicular lines that intersect at a point P in the plane, then
$$R_n\, R_m = H_P.$$

16. Given a point O and a directed segment $\overline{AB}$.

    (a) Find the point Q such that
    $$T_{AB} H_O T_{AB}^{-1} = H_Q.$$

    (b) What is the product $H_O T_{AB}$?

17. Let A and C be distinct points in the plane. Show that B is the midpoint of the segment $\overline{AC}$ if and only if
$$H_C\, H_B = H_B\, H_A.$$

18. In the triangle $\triangle ABC$, show that G is the centroid if and only if
$$H_G H_C H_G H_B H_G H_A = I,$$
where I is the identity.

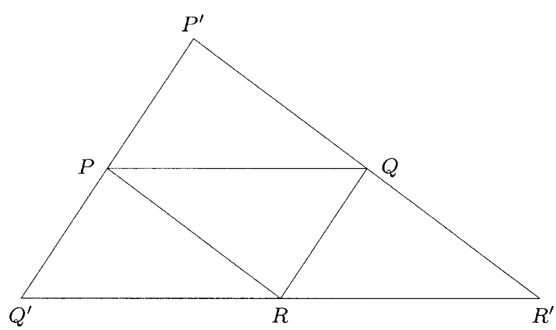
19. Using halfturns, prove that the diagonals of a parallelogram bisect each other.

    Hint. Show that if N is the midpoint of the diagonal AC of parallelogram $ABCD$ so that $H_A\, H_N = H_N\, H_C$, then N is the midpoint of BD also, that is, $H_D\, H_N = H_N\, H_B$.

20. Using halfturns, prove that if $ABCD$ and $EBFD$ are parallelograms, $EAFC$ is also a parallelogram.

21. Find all triangles such that three given noncollinear points are the midpoints of the sides of the triangle.

    Hint. Given P, Q, and R, $H_R H_Q H_P$ fixes a vertex of a unique triangle $\triangle P'Q'R'$, as in the figure below.



22. Given $\angle ABC$, construct a point P on AB and a point Q on BC such that $PQ = AB$, and the line PQ intersects the line BC at an angle of $60°$.

    Hint. Take a point D such that $AB = BD$ and BD intersects BC at an angle of $60°$.

23. Prove that if $R_n R_m$ fixes the point P and $m \neq n$, then the point P is on both lines m and n.

24. Show that if m and n are distinct lines in the plane, then

$$R_n R_m = R_m R_n$$

    if and only if m and n are perpendicular.

25. Let m be a line with equation $2x + y = 1$. Find the equations of the transformation R_m.

26. Suppose that the lines ℓ and m have equations $x + y = 0$ and $x - y = 1$, respectively. Find the equations for the transformation $R_\ell R_m$.

27. Given triangles ABC and DEF, where $\triangle ABC \equiv \triangle DEF$ and where

$$A(0,0),\ B(5,0),\ C(0,10),\ D(4,2),\ E(1,-2),\ F(12,-4),$$

    find the equations of the lines such that the product of reflections in the lines maps $\triangle ABC$ to $\triangle DEF$.

28. Let A_0 be a given point and $\ell_1,\ \ell_2,\ \ldots,\ \ell_n$ be given lines. For $1 \leq k \leq n$, let A_k be obtained from A_{k-1} by a reflection across ℓ_k, and let A_{n+k} be obtained from A_{n+k-1} by a reflection across ℓ_k.

    (a) Prove that A_{2n} will coincide with A_0 if n is odd.

    (b) Can the same conclusion be drawn if n is even?

29. Let $A_0 = B_0$ be a given point and $\ell_1,\ \ell_2,\ \ldots,\ \ell_n$ be given lines. For $1 \leq k \leq n$, let A_k be obtained from A_{k-1} by a reflection across ℓ_k, and let B_k be obtained from B_{k-1} by a reflection across ℓ_{n-k+1}. For what values of n will A_n coincide with B_n?

CHAPTER 9

THE PRODUCT OF DIRECT ISOMETRIES

Given two direct isometries, say the rotation $\mathbf{R}_{P,\phi}$ and the translation $\mathbf{T}_{AB}$, we know that their product $\mathbf{R}_{P,\phi}\mathbf{T}_{AB}$ is a direct isometry, and we may even suspect that it is a rotation $\mathbf{R}_{Q,\theta}$. The question is, what is Q and what is θ? This chapter will describe some of the ways that we can determine the values of the parameters that describe the result, in this case, Q and θ.

9.1 Angles

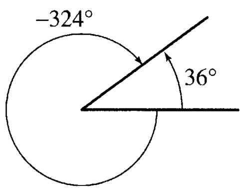
We first need to clear up some possible ambiguities about directed angles.

In Example 8.3.2, we showed that the product of two reflections in nonparallel lines is a rotation whose center is the intersection point of the two lines and whose angle of rotation is twice the angle from the first line to the second line; that is,

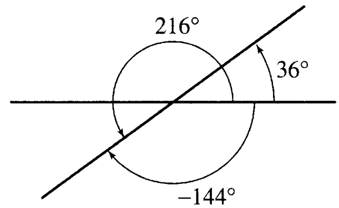
$$\mathbf{R}_m\mathbf{R}_l = \mathbf{R}_{Q,\alpha},$$

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

where $Q = l \cap m$ and where α is twice the directed angle from l to m. If l and m are rays, as in (a) below, there is no ambiguity in the meaning of the phrase "the directed angle from l to m." We can think of it as being $36°$ or $-324°$, since we identify the angles of θ and $\theta + 360n$ for all integers n.[8]



(a)



(b)

When we talk about the directed angle from one line to another, however, there are other possible interpretations. In (b) above, in addition to $36°$ and $-324°$, the directed angle from l to m can be legitimately interpreted as $216°$ or $-144°$. Does this affect the validity of the example? In fact, the answer is no, it does not matter which of the four angles we use. This is illustrated in the following table:

θ	2θ	modulo 360
36	72	72
-324	-648	72 $[= -648 + 360(2)]$
216	432	72 $[= 432 + 360(-1)]$
-144	-288	72 $[= -288 + 360(1)]$

In each case, 2θ is always 72 plus or minus some integral multiple of 360, so the entries in the 2θ column all represent exactly the same angle.

[8]This is the same as saying that α and β differ by a multiple of 360. In number theory, this is written mathematically as follows:

$$\alpha \equiv \beta \pmod{360}.$$

The expression is called a *congruence* and is expressed verbally by saying "α is congruent to β modulo 360." Although the two notions coincide in this specific case, generally there does not have to be any connection between geometric congruence and number theoretical congruence.

9.2 Fixed Points

Under the identity, every point of the plane is fixed. Under a reflection, each point in the line of reflection is fixed. Under a rotation, the center of rotation is the only fixed point. Translations and glide reflections have no fixed points.

Conversely, given an isometry T, we know from Theorem 8.2.5 that if T fixes each of three noncollinear points, then T must be the identity. The following theorem tells us what we can conclude if we know that either one or two points are fixed.

Theorem 9.2.1. *(Fixed Points)*

(1) An isometry that fixes a point P is either the identity, a rotation centered at P, or a reflection in a line that passes through P.

(2) An isometry that fixes each of two given points is either the identity or a reflection in the line determined by those points.

Proof. In each case, these are the only isometries that have (at least) the specified number of fixed points.

□

It is possible to distinquish between translations and glide reflections in terms of fixed sets.

Theorem 9.2.2. *An isometry that fixes exactly one line but does not fix any points is a glide reflection.*

Proof. By the previous theorem, the isometry is either a translation or a glide reflection, since it does not fix any points. A translation $\mathbf{T}_{AB}$ fixes all lines parallel to AB, and since this is not the case, we must conclude that the isometry is a glide reflection.

□

It should be noted that halfturns and translations each fix infinitely many lines. A halfturn fixes each line that contains the center of the turn, and a translation fixes each line parallel to the direction of translation. A halfturn also fixes the center of rotation, as the point common to all of the lines that it fixes.

9.3 The Product of Two Translations

The effect of a translation $\mathbf{T}_{UV}$ is to map each point P to a point P' such that $\overline{PP'}$ is congruent to $\overline{UV}$. This means that

$$\mathbf{T}_{UV} = \mathbf{T}_{PP'}.$$

Thus, if we know that an isometry T is a translation, we can write

$$T = \mathbf{T}_{PP'},$$

where $P' = T(P)$. This handy fact is used in the next theorem.

Theorem 9.3.1. *The product of two translations is a translation or the identity.*

Proof. Let $\mathbf{T}_{AB}$ and $\mathbf{T}_{CD}$ be the two translations. The product $\mathbf{T}_{CD}\mathbf{T}_{AB}$ must be a direct isometry, so it is either the identity, a translation, or a rotation. We will show that the only way the product can be a rotation is if $\mathbf{T}_{CD}$ and $\mathbf{T}_{AB}$ are inverses of each other, and so the rotation is actually the identity.

Suppose that $\mathbf{T}_{CD}\mathbf{T}_{AB} = \mathbf{R}_{P,\theta}$. Then

$$\mathbf{T}_{CD}\mathbf{T}_{AB}(P) = P.$$

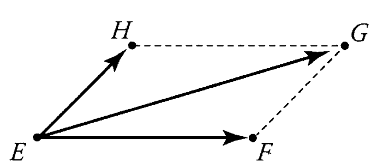
Let $P' = \mathbf{T}_{AB}(P)$. Then $\mathbf{T}_{AB} = \mathbf{T}_{PP'}$. Similarly, $\mathbf{T}_{CD} = \mathbf{T}_{P'P}$, but then

$$\mathbf{T}_{CD}\mathbf{T}_{AB} = \mathbf{T}_{P'P}\mathbf{T}_{PP'} = \mathbf{I}.$$

Thus, the rotation would actually be the identity.

□

Note that if $\overline{AB}$ is not congruent to $\overline{CD}$, then $\mathbf{T}_{CD}\mathbf{T}_{AB}$ must be a translation T. To pin down this translation geometrically, consider the effect of $\mathbf{T}_{CD}\mathbf{T}_{AB}$ on a point P. $\mathbf{T}_{AB}$ maps P to P' and $\mathbf{T}_{CD}$ maps P' to P''. Therefore, $T = \mathbf{T}_{PP''}$. We can construct a directed line segment that is congruent to PP'' by completing the parallelogram $EFGH$, where $\overline{EF}$ and $\overline{EH}$ are congruent to $\overline{AB}$ and $\overline{CD}$, respectively. Thus, $\mathbf{T}_{CD}\mathbf{T}_{AB}$ maps E to G, and the diagonal $\overline{EG}$ is congruent to PP''.



9.4 The Product of a Translation and a Rotation

We will determine the product of a rotation and a translation in two stages. First, we will show that the product is a rotation, and then we will describe how to find the center of rotation and the angle of rotation.

The product of a translation and rotation is a direct isometry, so it is either the identity, a translation, or a rotation. Supposing that $\overline{AB}$ is not of length zero and that the angle of rotation is θ, where θ is not a multiple of $360°$, a little bit of algebra shows that the product $\mathbf{T}_{AB}\mathbf{R}_{O,\theta}$ cannot be either the identity or a translation.

- It cannot be the identity. The reason is that

$$\mathbf{T}_{AB}\mathbf{R}_{O,\theta}(O) = \mathbf{T}_{BA}(O) = O',$$

  and since $\text{dist}(O, O') = \text{dist}(A, B) \neq 0$, it follows that $O \neq O'$. Since O is not fixed, the product cannot be the identity.

- It cannot be a translation. Let $S = \mathbf{T}_{AB}\mathbf{R}_{O,\theta}$. Then multiplying both sides of this equation by $\mathbf{T}_{BA}$ (the inverse of $\mathbf{T}_{AB}$), we get

$$\mathbf{T}_{BA}S = \mathbf{T}_{BA}\left(\mathbf{T}_{AB}\mathbf{R}_{O,\theta}\right) = \left(\mathbf{T}_{BA}\mathbf{T}_{AB}\right)\mathbf{R}_{O,\theta} = \mathbf{R}_{O,\theta}.$$

  If S were a translation, then Theorem 9.3.1 tells us that $\mathbf{T}_{BA}S$ (and hence $\mathbf{R}_{O,\theta}$) is the identity or a translation, which contradicts our assumptions about $\overline{AB}$ and θ.

The only possibility left is that the product is a rotation.

In a similar way, we can show that $\mathbf{R}_{O,\theta}\mathbf{T}_{AB}$ cannot be a translation unless $\mathbf{R}_{O,\theta}$ is the identity or a translation, which is again a contradiction. Thus, we have shown:

Theorem 9.4.1. *The product of a nontrivial translation and a rotation is a rotation, unless the angle of rotation is a multiple of* $360°$.

Although we know that the result has to be a rotation $\mathbf{R}_{Q,\phi}$, the theorem does not tell us how to find Q or ϕ. The next example does this.

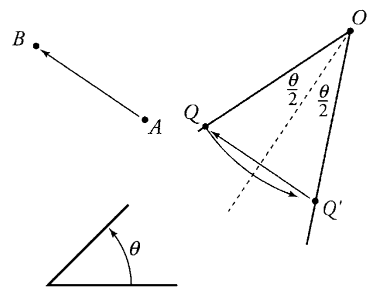
Example 9.4.2. *Given $\overline{AB}$, the point O, and the angle θ, where $\theta \neq n \cdot 180°$, find Q and ϕ such that*

$$\mathbf{T}_{AB}\mathbf{R}_{O,\theta} = \mathbf{R}_{Q,\phi}.$$

Solution. The key here is to look for the fixed point Q. In other words, we are looking for the point Q such that

$$\mathbf{T}_{AB}\mathbf{R}_{O,\theta}(Q) = Q.$$

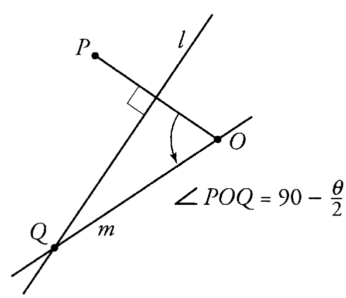
The point Q is transported to Q' by $\mathbf{R}_{O,\theta}$ and then back to Q by $\mathbf{T}_{AB}$. The solution can be derived from the figure on the right. The points Q and Q' are on a circle centered at O, with $\angle QOQ' = \theta$. The segment QQ' is a chord of a circle, and so its right bisector passes through O.



The center Q can be constructed as follows (see the figure on the right).

Through O, draw a line parallel to $\overline{AB}$, and let P be the point such that $\overline{OP} \cong \overline{AB}$. Construct the right bisector l of $\overline{OP}$, and construct the line m through O so that the directed angle from $\overline{OP}$ to m is $90° - \theta/2$. The point where m intersects l is Q.

This constructs the center Q of the rotation $\mathbf{R}_{Q,\phi}$. The angle of rotation ϕ is equal to the directed angle θ, and this can be confirmed by noting that $\angle OQP = \theta$ and that O is mapped to P by $\mathbf{T}_{AB}\mathbf{R}_{O,\theta}$.



□

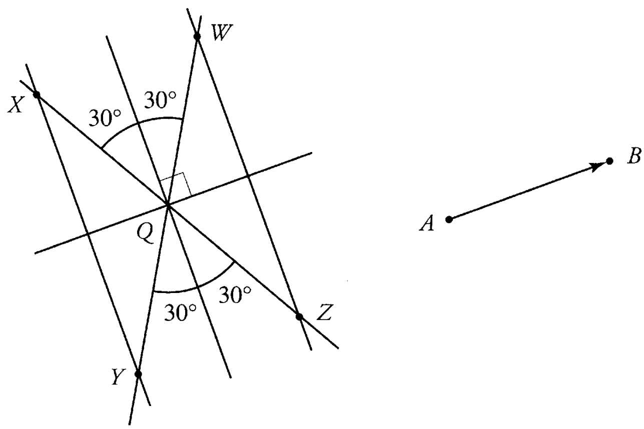
The product $\mathbf{R}_{O,\theta}\mathbf{T}_{AB}$ can be analyzed in a similar way. Here, the center would be a point X such that $\mathbf{T}_{AB}$ maps X to Z and $\mathbf{R}_{O,\theta}$ maps Z back to X. In the first figure above, this would be the point Q.

Care must also be taken when the angle of rotation is negative.

Example 9.4.3. *The figure below represents the centers of the rotations resulting from the four products*

$$\mathbf{T}_{AB}\mathbf{R}_{Q,30^\circ}, \quad \mathbf{T}_{AB}\mathbf{R}_{Q,-30^\circ}, \quad \mathbf{R}_{Q,30^\circ}\mathbf{T}_{AB}, \quad \mathbf{R}_{Q,-30^\circ}\mathbf{T}_{AB}.$$

The directed segment $\overline{AB}$ is as shown. The points W, X, Y, and Z are the centers of rotation, although not necessarily in that order. Determine the correct center for each rotation.



Solution. The centers are as follows:

W is the center of $\mathbf{T}_{AB}\mathbf{R}_{Q,30^\circ}$,

X is the center of $\mathbf{R}_{Q,30^\circ}\mathbf{T}_{AB}$,

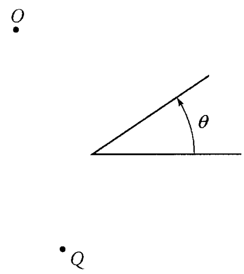
Y is the center of $\mathbf{R}_{Q,-30^\circ}\mathbf{T}_{AB}$,

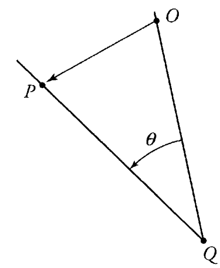
Z is the center of $\mathbf{T}_{AB}\mathbf{R}_{Q,-30^\circ}$.

□

Example 9.4.4. *Given the point Q and rotation $\mathbf{R}_{O,\theta}$, where θ is not a multiple of* 360°, *as in the figure on the left on the following page, find a translation $\mathbf{T}_{CD}$ such that*

$$\mathbf{T}_{CD}\mathbf{R}_{O,\theta} = \mathbf{R}_{Q,\theta}.$$





Solution. Again, we exploit the idea that if for some point X we can find its image X' under $\mathbf{T}_{CD}$, then $\mathbf{T}_{CD} = \mathbf{T}_{XX'}$.

We first solve for $\mathbf{T}_{CD}$. Multiply the equation

$$\mathbf{T}_{CD}\mathbf{R}_{O,\theta} = \mathbf{R}_{Q,\theta}$$

on the right by the inverse of $\mathbf{R}_{O,\theta}$, as follows:

$$(\mathbf{T}_{CD}\mathbf{R}_{O,\theta})\,\mathbf{R}_{O,-\theta} = \mathbf{R}_{Q,\theta}\mathbf{R}_{O,-\theta}.$$

Next use the associative law and the fact that $\mathbf{R}_{O,\theta}\mathbf{R}_{O,-\theta} = \mathbf{I}$ to obtain $\mathbf{T}_{CD}$:

$$\mathbf{T}_{CD} = \mathbf{R}_{Q,\theta}\mathbf{R}_{O,-\theta}.$$

Now apply $\mathbf{T}_{CD}$ to the point O (we chose O because it is fixed under $\mathbf{R}_{O,-\theta}$). Then

$$\mathbf{T}_{CD}(O) = \mathbf{R}_{Q,\theta}\mathbf{R}_{O,-\theta}(O) = \mathbf{R}_{Q,\theta}(O).$$

Letting P be the point $\mathbf{R}_{Q,\theta}(O)$, we therefore have $\mathbf{T}_{CD} = \mathbf{T}_{OP}$, as in the figure on the right at the top of the page.

□

9.5 The Product of Two Rotations

The product of two rotations with the same center is a third rotation, also with the same center, through an angle that is the sum of the two angles of rotation; that is,

$$\mathbf{R}_{A,\beta}\mathbf{R}_{A,\alpha} = \mathbf{R}_{A,\alpha+\beta}.$$

To analyze $\mathbf{R}_{B,\beta}\mathbf{R}_{A,\alpha}$ when A and B are different, we must first show the following:

Theorem 9.5.1. *When A and B are different, $\mathbf{R}_{B,\beta}\mathbf{R}_{A,\alpha}$ is a rotation through an angle $\alpha + \beta$ about some point P.*

Proof. We rely on the fact that we can use a translation to "move" the center of a rotation $\mathbf{R}_{Q,\theta}$ to any point we like, as in Example 9.4.4.

Let $\mathbf{T}_{CD}$ be the translation such that

$$\mathbf{T}_{CD}\mathbf{R}_{B,\beta} = \mathbf{R}_{A,\beta}.$$

Let S be the product $\mathbf{R}_{B,\beta}\mathbf{R}_{A,\alpha}$; that is,

$$S = \mathbf{R}_{B,\beta}\mathbf{R}_{A,\alpha}.$$

Multiply this equation on the left by $\mathbf{T}_{CD}$ to get

$$\mathbf{T}_{CD}S = \mathbf{T}_{CD}\left(\mathbf{R}_{B,\beta}\mathbf{R}_{A,\alpha}\right),$$

which implies that

$$\begin{aligned}\mathbf{T}_{CD}S &= \left(\mathbf{T}_{CD}\mathbf{R}_{B,\beta}\right)\mathbf{R}_{A,\alpha}\\ &= \mathbf{R}_{A,\beta}\mathbf{R}_{A,\alpha}\\ &= \mathbf{R}_{A,\alpha+\beta}.\end{aligned}$$

Now multiply this equation on the left by $\mathbf{T}_{DC}$, the inverse of $\mathbf{T}_{CD}$, to get

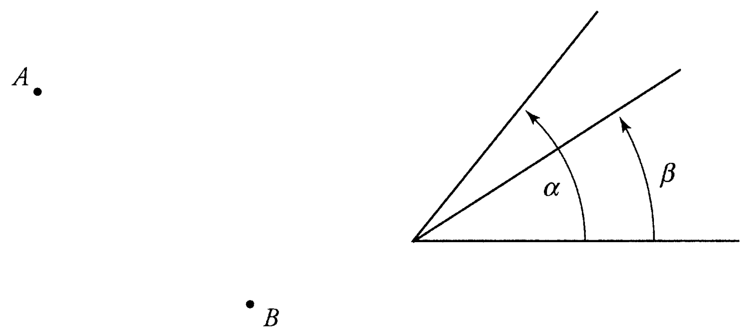
$$S = \mathbf{T}_{DC}\mathbf{R}_{A,\alpha+\beta}.$$

From the previous section, we know that $\mathbf{T}_{DC}\mathbf{R}_{A,\alpha+\beta}$ is a rotation $\mathbf{R}_{P,\alpha+\beta}$ for some point P (unless $\alpha + \beta$ is a multiple of $360°$, in which case $\mathbf{T}_{DC}\mathbf{R}_{A,\alpha+\beta} = \mathbf{T}_{DC}$).

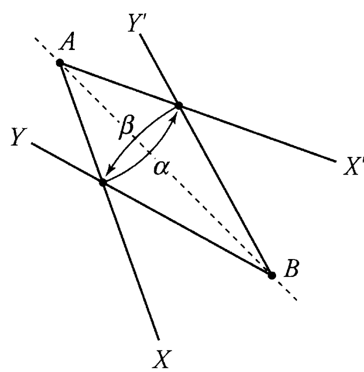
□

To find the center P when $\alpha + \beta$ is not a multiple of $360°$, we can trace our progress through the preceding equations. For $\overline{CD}$, we can take $C = A$ and $D = \mathbf{R}_{A,\beta}(B)$, as in the discussion following Example 9.4.4. Applying the inverse transformation $\mathbf{T}_{DC}$ to $\mathbf{R}_{A,\alpha+\beta}$ then allows us to geometrically construct the center P. A better option is to look for the fixed point P of the product $\mathbf{R}_{B,\beta}\mathbf{R}_{A,\alpha}$, as in the following example.

Example 9.5.2. *Given $\mathbf{R}_{B,\beta}$ and $\mathbf{R}_{A,\alpha}$, with α and β as shown in the figure below, construct the center Q of the rotation $\mathbf{R}_{B,\beta}\mathbf{R}_{A,\alpha}$.*



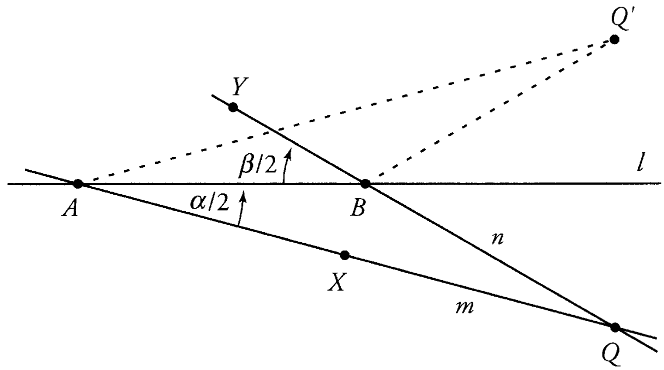
Solution. The figure below shows how to solve this problem. Construct the line AB and an angle $\angle XAX'$ of size α such that AB is the bisector of $\angle XAX'$. Construct the angle $\angle YAY'$ of size β so that AB is the bisector of $\angle YAY'$. One of the points of intersection of these two angles will be the desired point Q.



□

Remark. In general, when trying to find the center for the product $\mathbf{R}_{B,\beta}\mathbf{R}_{A,\alpha}$, the angles should be drawn so that the directed angle $\angle XAB = \alpha/2$ and the directed angle $\angle ABY = \beta/2$. The point of intersection of the line XA and the line YB will be the center Q of the product.

For example, this is what the construction looks like when $\alpha = 30°$ and $\beta = -60°$. The point Q will be mapped by $\mathbf{R}_{A,\alpha}$ to Q', and Q' will then be mapped by $\mathbf{R}_{B,\beta}$ back to Q.



9.6 Problems

1. If ℓ, m, and n are the perpendicular bisectors of the sides AB, BC, and CA, respectively, of $\triangle ABC$, then

$$T = R_n\, R_m\, R_\ell$$

is a reflection in which line?

2. If $R_n\, R_m\, R_\ell$ is a reflection, show that the lines ℓ, m, and n are concurrent or have a common perpendicular.

3. Find Cartesian equations for lines m and n such that

$$R_m R_n(x, y) = (x + 2, y - 4).$$

4. Show that

$$H_P\, R_\ell\, H_P\, R_\ell\, H_P\, R_\ell\, H_P$$

is a reflection in a line parallel to ℓ.

5. Let C be a point on the line ℓ, and show that

$$R_\ell\, R_{C,\theta}\, R_\ell = R_{C,-\theta}.$$

6. Given nonparallel lines AB and CD, show that there is a rotation T such that

$$T(AB) = CD.$$

7. Show that if S, T, $T\,S$, and $T^{-1}\,S$ are rotations, then the centers of S, $T\,S$, and $T^{-1}\,S$ are collinear.

8. In a given acute triangle, inscribe a triangle PQR having minimum perimeter. This is called *Fagnano's problem.*

9. Prove *Thomsen's Relation*: for any lines a, b, and c, we have

$$R_c\,R_a\,R_b\,R_c\,R_a\,R_b\,R_a\,R_b\,R_c\,R_a\,R_b\,R_c \\ \times R_b\,R_a\,R_c\,R_b\,R_a\,R_b\,R_a\,R_c\,R_b\,R_a = I,$$

where I is the identity transformation.

10. Show that Thomsen's Relation is still true if each reflection R_x is replaced by a halfturn H_X; that is, show that if A, B, and C are three distinct points in the plane, then

$$H_C\,H_A\,H_B\,H_C\,H_A\,H_B\,H_A\,H_B\,H_C\,H_A\,H_B\,H_C \\ \times H_B\,H_A\,H_C\,H_B\,H_A\,H_B\,H_A\,H_C\,H_B\,H_A = I,$$

where I is the identity transformation.

11. If $x' = ax + by + c$ and $y' = bx - ay + d$ with $a^2 + b^2 = 1$ are the equations for an isometry T, show that T is a reflection if and only if

$$ac + bd + c = 0 \qquad \text{and} \qquad ad - bc - d = 0.$$

12. Find the Cartesian equation of the line m if the equations for a reflection in the line are

$$x' = \frac{3}{5}x + \frac{4}{5}y \qquad \text{and} \qquad y' = \frac{4}{5}x - \frac{3}{5}y.$$

13. If the equations for the rotation $R_{P,\theta}$ are

$$2x' = -\sqrt{3}x - y + 2 \qquad \text{and} \qquad 2y' = x - \sqrt{3}y - 1,$$

find the center of rotation P and and the angle of rotation θ.

14. If a and b are lines in the plane, show that the following are equivalent:

(a) $a = b$ or a and b are perpendicular,

(b) $R_a\,R_b = R_b\,R_a$,

(c) $R_b(a) = a$,

(d) $(R_b\,R_a)^2 = I$,

(e) $R_b\,R_a$ is either the identity or a halfturn.

15. If the isometry H_P is a halfturn, show that given any two perpendicular lines m and n that intersect at the point P, we have $H_P = R_m\, R_n$.

16. Let m be a line with equation $2x + y = 1$. Find the equations of the transformation R_m.

17. Given a line b and a point A, show that the following conditions are equivalent:

    (a) $A \in b$,
    (b) $H_A\, R_b = R_b\, H_A$,
    (c) $R_b(A) = A$,
    (d) $H_A(b) = b$,
    (e) $R_b\, H_A$ (or $H_A\, R_b$) is an involution,
    (f) $R_b\, H_A$ is a reflection in the line through A perpendicular to b.

18. If $A \neq C$, show that the following conditions are equivalent:

    (a) B is the midpoint of AC,
    (b) $H_C\, H_B = H_B\, H_A$,
    (c) $H_B(A) = C$,
    (d) $H_B\, H_A\, H_B = H_C$.

19. Show that nonidentity rotations of the plane with different centers do not commute.

20. Let A and B be distinct points in the plane and let S be an isometry. Show that

$$S\, T_{AB}\, S^{-1} = T_{CD},$$

    where $C = S(A)$ and $D = S(B)$.

21. Given a point A and two lines ℓ and m, construct a square $ABCD$ such that B lies on ℓ and D lies on m.

22. Given four distinct points, find a square such that each of the lines containing a side of the square passes through one of the four given points.

    Hint. Given A, B, C, and D, we want to find the lines a, b, c, and d. Take P such that $R_{P,90}(B) = C$. Let $R_{P,90}(D) = E$. Then take a to be AE.

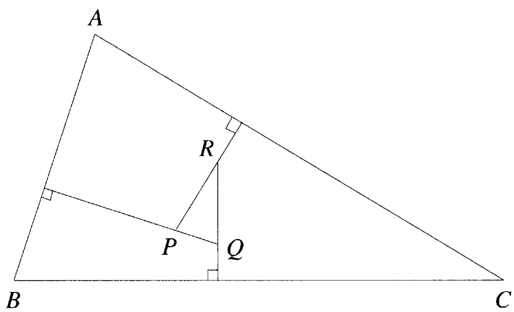
23. Consider a triangle $\triangle ABC$ (oriented counterclockwise) with positive angles α, β, γ at A, B, C. Show that

$$R_{A,2\alpha}\, R_{B,2\beta}\, R_{C,2\gamma} = I.$$

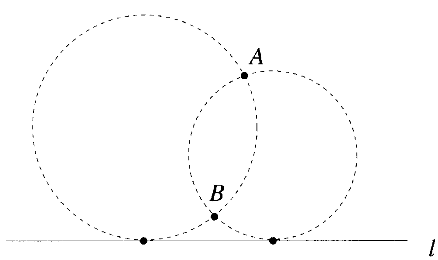
    Are there other similar formulas?

24. Construct over each side of $\triangle ABC$ an equilateral triangle. The centroids G_1, G_2, and G_3 of these triangles form a new triangle, the so-called ***Napoleon triangle***. Show that $\triangle G_1G_2G_3$ is equilateral. This theorem is attributed to Napoleon Bonaparte.

25. Let perpendiculars erected at arbitrary points on the sides of triangle $\triangle ABC$ meet in pairs at points P, Q, and R. Show that the triangle PQR is similar to the given triangle.



26. Let A and B be two points lying on one side of a line l. Explain how to construct one of the two circles that are tangent to ℓ and pass through A and B.



27. Given points A, B, and P in the plane, construct the reflection of P in the line AB using a Euclidean compass alone.

28. Let $ABCD$ be a square with the vertices in clockwise order. For each of the following translations, find a counterclockwise rotation that brings the image of the translation back to the same physical space as $ABCD$:

 (a) distance AB, direction AB,
 (b) distance $\frac{1}{2}AB$, direction AB,
 (c) distance AC, direction AC,
 (d) distance $\frac{1}{2}AC$, direction AC.

29. Let $A_0B_0C_0$ be an equilateral triangle with the vertices in clockwise order. We first rotate it 60° counterclockwise about A_0 to obtain $A_1B_1C_1$, then about B_1 to obtain $A_2B_2C_2$, and finally about C_2 to obtain $A_3B_3C_3$. We continue to rotate about A_3, B_4, C_5, and so on, until $A_nB_nC_n$ occupies the same physical space as $A_0B_0C_0$. What is the minimum positive value of n?

30. Let $A_0B_0C_0$ be a triangle with the vertices in counterclockwise order, where $\angle A = 40^\circ$, $\angle B = 60^\circ$, and $\angle C = 80^\circ$. We first rotate it 40° counterclockwise about A_0 to obtain $A_1B_1C_1$, then 60° counterclockwise about B_1 to obtain $A_2B_2C_2$, and finally 80° counterclockwise about C_2 to obtain $A_3B_3C_3$. We continue to rotate about A_3, B_4, C_5, and so on, until $A_nB_nC_n$ occupies the same physical space as $A_0B_0C_0$. What is the minimum positive value of n?

31. A ***halfturn*** about a point O is a 180° rotation about the point O. Prove that the composition of:

    (a) two halfturns is a translation or the identity;

    (b) a translation and a halfturn is a halfturn.

32. Prove that the composition of:

    (a) an even number of halfturns is a translation or the identity;

    (b) an odd number of halfturns is a halfturn.

33. Let $A_0,\ B_0,\ O_1,\ O_2,\ \ldots,\ O_n$ be given points. For $1 \le k \le n$, let A_kB_k be obtained from $A_{k-1}B_{k-1}$ by a halfturn about O_k.

    (a) Prove that $A_0A_n = B_0B_n$ if n is even.

    (b) What conclusion may be drawn if n is odd?

34. Let $A_0,\ O_1,\ O_2,\ \ldots,\ O_n$ be given points. For $1 \le k \le n$, let A_k be obtained from A_{k-1} by a halfturn about O_k, and let A_{n+k} be obtained from A_{n+k-1} by a halfturn about O_k.

    (a) Prove that A_{2n} will coincide with A_0 if n is odd.

    (b) Can the same conclusion be drawn if n is even?

35. Let $A_0 = B_0,\ O_1,\ O_2,\ \ldots,\ O_n$ be given points. For $1 \le k \le n$, let A_k be obtained from A_{k-1} by a halfturn about O_k, and let B_k be obtained from B_{k-1} by a halfturn about O_{n-k+1}. For what values of n will A_n coincide with B_n?

CHAPTER 10

SYMMETRY AND GROUPS

10.1 More About Groups

Recall that a set $\mathcal{G}$ together with a binary operation $\cdot$ is called a ***group*** if the following conditions are satisfied:

1. $\mathcal{G}$ is closed under the binary operation; that is, if x and y are elements of $\mathcal{G}$, then so is $x \cdot y$.
2. The associative law holds. If x, y, and z are elements of $\mathcal{G}$, then
$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$
3. There is an identity element e in $\mathcal{G}$. For every x in $\mathcal{G}$, $e \cdot x = x \cdot e = x$.
4. $\mathcal{G}$ is closed with respect to inversion. For every member x in $\mathcal{G}$, there is another member x' also in $\mathcal{G}$ such that $x \cdot x' = x' \cdot x = e$.

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

When dealing with groups in general, the binary operation is typically called ***group multiplication*** or simply ***multiplication***, and the symbol for the operation is omitted.

There are a few simple but useful facts about groups that can occasionally save us some work.

Theorem 10.1.1. *A group has only one identity element.*

Proof. Suppose that e and f are identity elements in a group $(\mathcal{G}, \cdot)$. Since $xe = x$ for every x in $\mathcal{G}$, we have $fe = f$. Since $fx = x$ for every x in $\mathcal{G}$, we have $fe = e$. Therefore, $f = fe = e$.

□

Theorem 10.1.2. *Each element of a group has only one inverse.*

Proof. Let x be an element of $\mathcal{G}$ and suppose that x' and x'' are both inverses for x. Then,

$$(x'x)x'' = ex'' = x''$$

and

$$x'(xx'') = x'e = x'.$$

However, by the associative law,

$$x'' = (x'x)x'' = x'(xx'') = x'.$$

□

Since there is exactly one inverse for each x in $\mathcal{G}$, there is no ambiguity in denoting it by x^{-1}.

An important consequence of the previous theorem is that the ***cancellation laws*** hold:

If $ax = ay$, then $x = y$; similarly, if $xa = ya$, then $x = y$.

If $ya = ax$, can we conclude that $y = x$? We certainly could if it were true that $ax = xa$ or that $ay = ya$. If a group $\mathcal{G}$ has the property that $xy = yx$ for *every* pair of elements x and y in $\mathcal{G}$, we say that the ***commutative law*** holds, and the group is called a ***commutative group*** or an ***abelian group***. The group of all isometries of the plane is *not* a commutative group, although it is true that $TS = ST$ for *some* pairs of isometries.

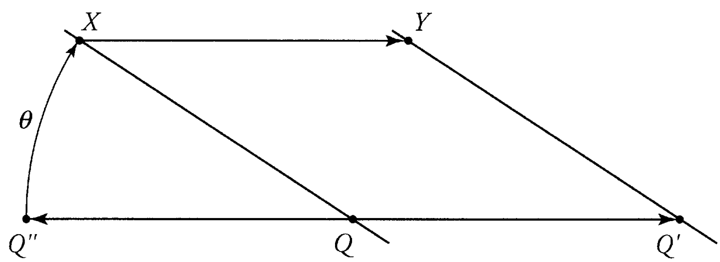
Returning to the question of whether $ya = ax$ implies that $x = y$, the answer is that this may not be the case. If $ya = ax$, then multiplying on the right by a^{-1} gives us

$$y = axa^{-1}.$$

This is a fairly important operation in group theory and is called ***conjugation***. More specifically, we say that y is the ***conjugate*** of x by a if $y = axa^{-1}$.

Example 10.1.3. *Let $a = \mathbf{T}_{AB}$ and let $x = \mathbf{R}_{Q,\theta}$. Find the conjugate of x by a.*

Solution. We know that a^{-1} is $\mathbf{T}_{BA}$, and so axa^{-1} is the product of three direct isometries and must therefore be a direct isometry. To figure out what it is, we know from previous exercises that we need only look at how it affects two convenient points.



Let Q' be the point that is mapped to Q by $\mathbf{T}_{BA}$. In other words, let $Q' = \mathbf{T}_{AB}(Q)$. Then

$$axa^{-1}(Q') = \mathbf{T}_{AB}\mathbf{R}_{Q,\theta}\mathbf{T}_{BA}(Q') = Q',$$

so axa^{-1} is either the identity or a rotation about Q'.

Now apply axa^{-1} to the point Q. Let $Q'' = \mathbf{T}_{BA}(Q)$. Then Q is the midpoint of $Q'Q''$. Let $X = \mathbf{R}_{Q,\theta}(Q'')$ and let $Y = \mathbf{T}_{AB}(X)$. The segments $\overline{XY}$ and $\overline{QQ'}$ are equal in length and in the same direction, so $Q'QXY$ is a parallelogram, and so $\angle Q''QX$ is congruent to $\angle QQ'Y$. It therefore follows that $axa^{-1} = \mathbf{R}_{Q',\theta}$.

□

The image of a straight line under any isometry A is another straight line. The following theorem says that any conjugate of a reflection is again a reflection, but possibly in a different line.

Theorem 10.1.4. *Let A be any isometry in the plane and let m be any line. Then*

$$A\mathbf{R}_m A^{-1} = \mathbf{R}_{A(m)}.$$

Proof. If A is a direct isometry, so is A^{-1}, and if A is an opposite isometry, then so is A^{-1}. It follows that $A\mathbf{R}_m A^{-1}$ is an opposite isometry.

Let X be a point on $A(m)$; that is, let $X = A(Y)$ for some point Y on m. Then

$$A\mathbf{R}_m A^{-1}(X) = A\mathbf{R}_m A^{-1}A(Y) = A\mathbf{R}_m(Y) = A(Y) = X.$$

This says that $A\mathbf{R}_m A^{-1}$ fixes every point on the line $A(m)$, and since $A\mathbf{R}_m A^{-1}$ is an opposite isometry, it must be the reflection $\mathbf{R}_{A(m)}$.

□

The image of a directed line segment under any isometry A is another directed line segment of the same length. The following theorem says that any conjugate of a translation is again a translation through the same distance, although possibly in a different direction.

Theorem 10.1.5. *Let A be any isometry in the plane, and let $\overline{CD}$ be a directed line segment. Then*

$$A\mathbf{T}_{CD}A^{-1} = \mathbf{T}_{EF},$$

where the directed segment $\overline{EF} = A\left(\overline{CD}\right)$.

Proof. We first show that $A\mathbf{T}_{CD}A^{-1}$ must be a translation. The isometry $A\mathbf{T}_{CD}A^{-1}$ must be a direct isometry, and since the only direct isometries without fixed points are translations, it suffices to show that $A\mathbf{T}_{CD}A^{-1}$ has no fixed point.

Suppose to the contrary that X is a fixed point of the isometry, that is, that

$$X = A\mathbf{T}_{CD}A^{-1}(X).$$

Now, $X = A(Y)$ for some point Y, so that

$$Y = A^{-1}(X) = A^{-1}\left(A\mathbf{T}_{CD}A^{-1}\right)(X) = \left(AA^{-1}\right)\mathbf{T}_{CD}A^{-1}(X) = \mathbf{T}_{CD}(Y).$$

This says that $Y = \mathbf{T}_{CD}(Y)$; that is, Y is a fixed point of $\mathbf{T}_{CD}$. However, this contradicts the fact that a translation has no fixed points. Thus, we must conclude that $A\mathbf{T}_{CD}A^{-1}$ has no fixed points, and therefore it must be a translation.

To pin down the translation $\mathbf{T}_{EF}$, let us consider the effect of $A\mathbf{T}_{CD}A^{-1}$ upon E:

$$A\mathbf{T}_{CD}A^{-1}(E) = A\mathbf{T}_{CD}A^{-1}A(C) = A\mathbf{T}_{CD}(C) = A(D) = F,$$

so we have a translation that maps E to F. In other words,

$$A\mathbf{T}_{CD}A^{-1} = \mathbf{T}_{EF}.$$

□

Theorem 10.1.6. *Let Q be a point on the line m. Then* $\mathbf{R}_l\mathbf{R}_{Q,\theta}\mathbf{R}_l = \mathbf{R}_{Q,-\theta}$.

We leave the proof as an exercise.

10.1.1 Cyclic and Dihedral Groups

If a is an element of a group $\mathcal{G}$ and m is a positive integer, then a^m, a^{-m}, and a_0 are defined as follows:

$$a^m = \underbrace{a \cdot a \cdots a \cdot a}_{m \text{ factors}},$$

$$a^{-m} = \left(a^{-1}\right)^m,$$

$$a^0 = e \quad (e \text{ being the identity}).$$

With these definitions, it is not difficult to verify that the following two laws of exponents hold for all integers m and n:

$$a^m a^n = a^{m+n} \qquad \text{and} \qquad (a^m)^n = a^{mn}.$$

In general, we can expect that $(ab)^m \neq a^m\, b^m$, unless the group is commutative.

The ***order*** of an element a of a group is the smallest positive integer n such that $a^n = e$. If there is no positive integer n such that $a^n = e$, then the order of a is infinite.

For example, in the group of all isometries of the plane, the order of $\mathbf{I}$ is 1, the order of $\mathbf{R}_{Q,90^\circ}$ is 4, the order of $\mathbf{R}_{Q,180^\circ}$ is 2, the order of $\mathbf{R}_{Q,270^\circ}$ is 4, and the order of $\mathbf{T}_{AB}$ is infinite.

The ***order*** of a group $\mathcal{G}$ is the number of elements in that group.

A group $\mathcal{G}$ is said to be ***generated by a subset*** S ***of*** $\mathcal{G}$ if every element of $\mathcal{G}$ can be expressed as a product of elements of S and inverses of such elements (by this we mean a *finite* product, not something that is the limit of some infinite process). In this case, we write

$$\mathcal{G} = \langle\, S \,\rangle\,.$$

A group $\mathcal{G}$ is called a ***cyclic group*** if it is generated by an element $a \in \mathcal{G}$; that is, $\mathcal{G} = \langle\, a \,\rangle$. Here, the order of $\mathcal{G}$ could be finite or infinite.

- For example, the group $\mathbb{Z}$ of integers, with addition as the binary operation, is generated by the set $\{\, 1 \,\}$; that is, $(\mathbb{Z}, +) = \langle\, 1 \,\rangle$.

  Here, the group multiplication is addition of integers, and the inverse of 1 is -1. To see why $\mathbb{Z}$ is generated by $\{\, 1 \,\}$, note that, for an integer m,

$$m = \begin{cases} \underbrace{1 + 1 + \cdots + 1}_{m \text{ times}} & \text{if } m > 0, \\ \underbrace{(-1) + (-1) + \cdots + (-1)}_{m \text{ times}} & \text{if } m < 0, \\ 1 + (-1) & \text{if } m = 0. \end{cases}$$

  Thus, $\mathbb{Z} = \langle\, 1 \,\rangle$, and the order of the group is infinite.

- On the other hand, if $a \in \mathcal{G}$ and $a^n = e$ for some integer $n \geq 1$, then the group $\mathcal{G} = \langle\, a \,\rangle$ has n elements, namely

$$a,\ a^2,\ a^3,\ \ldots\ ,\ a^n = e,$$

  and the order of the group is n. This group is denoted by $\mathcal{C}_n$ and is called the ***cyclic group of order*** n.

A group $\mathcal{G}$ is called a ***dihedral group*** if it is generated by two elements a and b for which

1. $a^n = e$,
2. $b^2 = e$, and
3. $bab^{-1} = a^{-1}$.

The elements of this group are

$$\begin{array}{cccc} a, & a^2, & \ldots & a^n = e, \\ ba, & ba^2, & \ldots & ba^n = b. \end{array}$$

Consequently, the group is called the ***dihedral group of order*** **2*n*** and is denoted by $\mathcal{D}_{2n}$.

Unlike the situation with cyclic groups, it is not immediately obvious that the $2n$ elements listed on the previous page are the only elements of $\mathcal{D}_{2n}$.

- For example, how do we know that a product like

$$a^8 b^3 \left(a^{-1}\right)^{10} a^4 b^5$$

is actually one of the 10 elements that are said to comprise $\mathcal{D}_{10}$ or that the inverses of such products belong to $\mathcal{D}_{10}$?

Here, $n = 5$, and the 10 elements are

$$\begin{array}{ccccc} a, & a^2, & a^3 & a^4, & a^5 = e, \\ ba, & ba^2, & ba^3 & ba^4, & ba^5 = b. \end{array}$$

Since $b^2 = e$, we can replace the factors b^k with b if k is odd or with e (that is, omit the factor) if k is even. The product $a^8 b^3 \left(a^{-1}\right)^{10} a^4 b^5$ reduces to

$$a^8 b \left(a^{-1}\right)^{10} a^4 b = a^8 b a^{-10} a^4 b = a^8 b a^{-6} b.$$

Then, recalling that $a^5 = e$, replace a^8 with $a^5 a^3 = a^3$ and replace a^{-6} with $a^{-6}\left(a^{10}\right) = a^4$ to get

$$a^3 b a^4 b.$$

Now replace a^4 by $a\left(b^{-1}b\right)a\left(b^{-1}b\right)a\left(b^{-1}b\right)a$ to get

$$a^3 ba \left(b^{-1}b\right) a \left(b^{-1}b\right) a \left(b^{-1}b\right) ab = a^3 \left(bab^{-1}\right)\left(bab^{-1}\right)\left(bab^{-1}\right) bab^{-1}.$$

Since $bab^{-1} = a^{-1}$, this becomes

$$a^3 a^{-4} = a^{-1} = a^4,$$

which is one of the 10 elements, as claimed.

It should be clear that by applying the same process to any finite product, we will end up with one of a^s or ba^s.

Example 10.1.7. *What is the inverse of ba^s in $\mathcal{D}_{2n}$?*

Solution. It is its own inverse. Here is the proof:

$$(ba^s)^{-1} = (a^s)^{-1} b^{-1} = \left(bab^{-1}\right)^s b.$$

Expanding $\left(bab^{-1}\right)^s$, several b and b^{-1} cancel each other out, and we get $ba^s b^{-1}$, so that

$$(ba^s)^{-1} = \left(bab^{-1}\right)^s b = ba^s b^{-1} b = ba^s,$$

as claimed.

□

Example 10.1.8. *Assuming that $0 < k < n$, what is $a^k b$ in $\mathcal{D}_{2n}$?*

Solution. We have

$$a^k b = \left(ba^{-1}b^{-1}\right)^k b = b\left(a^{-1}\right)^k = ba^{n-k}.$$

□

Here is the prototypical concrete example of a dihedral group.

Example 10.1.9. *Show that the group of symmetries of a square is $\mathcal{D}_8$.*

Solution. Label the vertices of the square A, B, C, and D counterclockwise, and let Q be the center point of the square. Any isometry T that carries the square onto itself must map the vertex A to $T(A)$, where $T(A)$ is one of the four vertices. The vertex B is mapped to $T(B)$, which must be one of the two vertices that are adjacent to $T(A)$. Since T must also map Q to Q, the action of T on A and B completely determines the isometry.

Two particular symmetries of the square are $\mathbf{R}_{Q,90^\circ}$ and $\mathbf{R}_m$, where m is a diagonal of the square. Letting $a = \mathbf{R}_{Q,90^\circ}$ and $b = \mathbf{R}_m$, we see that

$$a^4 = e,$$

$$b^2 = e,$$

$$bab^{-1} = a^{-1}.$$

The last equality follows by Theorem 10.1.6.

Consequently, the isometries $\mathbf{R}_{Q,90^\circ}$ and $\mathbf{R}_m$ generate the dihedral group $\mathcal{D}_8$, and since there are exactly eight different symmetries of the square, we are finished.

□

The preceding example has an obvious generalization whose proof is virtually identical to the proof for the square:

Theorem 10.1.10. *The group of symmetries of the regular n-gon is the dihedral group $\mathcal{D}_{2n}$.*

Remark. This result is often used as the definition of $\mathcal{D}_{2n}$.

10.2 Leonardo's Theorem

Leonardo da Vinci apparently worked out all possible symmetries for the floor plans of many chapels. Because of this, the following theorem is known as Leonardo's Theorem.

Theorem 10.2.1. *(Leonardo's Theorem)*

Every finite group of isometries of the plane is either a cyclic group or a dihedral group.

The proof of this theorem is long but not difficult and amounts to checking what can happen. In order to prove it, we need some facts about the product

$$\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta}$$

that were developed earlier. We marshall them here for convenience.

(1) When $P = Q$: $\mathbf{R}_{P,\phi}\mathbf{R}_{P,\theta} = \mathbf{R}_{P,\phi+\theta}$.

(2) When $P \neq Q$ and $\phi + \theta \equiv 0 \pmod{360}$: $\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta} = \mathbf{T}_{AB}$, where $A = Q$ and $B = \mathbf{R}_{P,\phi}(Q)$.

(3) When $P \neq Q$ and $\phi + \theta \not\equiv 0 \pmod{360}$: $\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta} = \mathbf{R}_{S,\phi+\theta}$ for some point S.

We also need the following facts about the interaction between rotations and reflections:

(4) When P is on m: $\mathbf{R}_m\mathbf{R}_{P,\phi} = \mathbf{R}_l$, where l is the line through P such that the angle from l to m is $\phi/2$.

(5) When P is not on m: $\mathbf{R}_m\mathbf{R}_{P,\phi}$ is a glide reflection.

Note that points (2) and (5) imply the following:

Lemma 10.2.2. *Suppose that $\mathcal{G}$ is a finite group of isometries.*

(1) Any two rotations that are in $\mathcal{G}$ must have the same center.

(2) For any rotation and any reflection that are in $\mathcal{G}$, the center of rotation must be on the line of reflection.

Proof. The proof uses the fact that if a group contains a translation, then it must be infinite. To see why this is true, note that if $\mathbf{T}_{AB}$ is in the group $\mathcal{G}$, then so are $(\mathbf{T}_{AB})^2$, $(\mathbf{T}_{AB})^3$, and so on. However, each of $(\mathbf{T}_{AB})^2$, $(\mathbf{T}_{AB})^3$, ..., is a translation, and all of them are different.

(1) We want to show that if $\mathbf{R}_{P,\phi}$ and $\mathbf{R}_{Q,\theta}$ are two different members of $\mathcal{G}$, then $P = Q$. To establish this, we proceed by contradiction and assume that $P \neq Q$. We consider two cases.

(a) $\phi + \theta \equiv 0 \pmod{360}$:
If $P \neq Q$, then the product $\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta}$ would be a translation, contradicting the fact that $\mathcal{G}$ is finite.

(b) $\phi + \theta \not\equiv 0 \pmod{360}$:
The fact that $\mathcal{G}$ is a group means that the product

$$(\mathbf{R}_{P,\phi})^{-1}(\mathbf{R}_{Q,\theta})^{-1}\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta}$$

is a member of $\mathcal{G}$. However, we would then have

$$\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta} = \mathbf{R}_{X,\phi+\theta}$$

for some X and

$$(\mathbf{R}_{P,\phi})^{-1}(\mathbf{R}_{Q,\theta})^{-1} = \mathbf{R}_{P,-\phi}\mathbf{R}_{Q,-\theta} = \mathbf{R}_{Y,-(\phi+\theta)}$$

for some point Y. Consequently,

$$(\mathbf{R}_{P,\phi})^{-1}(\mathbf{R}_{Q,\theta})^{-1}\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta} = \mathbf{R}_{Y,-(\phi+\theta)}\mathbf{R}_{X,\phi+\theta}.$$

This is either the identity, if $X = Y$, or a translation, if $X \neq Y$. However, it cannot be the case that $X = Y$, for this would mean that $\mathbf{R}_{Y,-(\phi+\theta)}$ is the inverse of $\mathbf{R}_{X,\phi+\theta}$, or in other words, that

$$\mathbf{R}_{P,-\phi}\mathbf{R}_{Q,-\theta} = (\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta})^{-1}.$$

However,

$$(\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta})^{-1} = (\mathbf{R}_{Q,\theta})^{-1}(\mathbf{R}_{P,\phi})^{-1} = \mathbf{R}_{Q,-\theta}\mathbf{R}_{P,-\phi},$$

and we would have

$$\mathbf{R}_{P,-\phi}\mathbf{R}_{Q,-\theta} = \mathbf{R}_{Q,-\theta}\mathbf{R}_{P,-\phi},$$

which contradicts the fact that rotations with different centers do not commute. Since the centers X and Y are different, it follows that the product

$$(\mathbf{R}_{P,\phi})^{-1}(\mathbf{R}_{Q,\theta})^{-1}\mathbf{R}_{P,\phi}\mathbf{R}_{Q,\theta}$$

is a translation, again contradicting the fact that $\mathcal{G}$ is finite.

(2) Suppose that $\mathbf{R}_{P,\phi}$ and $\mathbf{R}_{Q,\theta}$ both belong to $\mathcal{G}$, and suppose for a contradiction that P is not on m. Then $(\mathbf{R}_m\mathbf{R}_{P,\phi})^2$ is a translation, meaning that $\mathcal{G}$ would have to be infinite.

This completes the proof.

□

We next prove a theorem that is part of Leonardo's Theorem but which is useful in its own right.

Theorem 10.2.3. *If $\mathcal{G}$ is a finite group of isometries that consists of exactly n rotations (counting the identity as a rotation through $360°$), then $\mathcal{G}$ is the cyclic group $\mathcal{C}_n$.*

Proof. From Lemma 10.2.2, we know that $\mathcal{G}$ consists of rotations through various angles with all rotations centered at a common point Q. We may also assume that all rotations belonging to $\mathcal{G}$ are through a positive angle no greater than $360°$. Since there are only a finite number of rotations, one of them has the smallest positive angle of rotation, say θ. We claim that every other rotation, including $\mathbf{R}_{Q,360°}$, must be a multiple of θ.

We again use a proof by contradiction. Supposing that this were not the case, there would be a rotation $\mathbf{R}_{Q,\phi}$ in $\mathcal{G}$ where $\phi > 0$ and where ϕ is not a multiple of θ. Let α be the remainder when ϕ is divided by θ; that is,

$$\phi = m\theta + \alpha,$$

where m is a positive integer and $0 < \alpha < \theta$. Since $\mathcal{G}$ is a group, then

$$(\mathbf{R}_{Q,\theta})^{-m} = \mathbf{R}_{Q,-m\theta}$$

is in $\mathcal{G}$, and therefore so is the product

$$\mathbf{R}_{Q,\phi}\mathbf{R}_{Q,-m\theta} = \mathbf{R}_{Q,\phi-m\theta} = \mathbf{R}_{Q,\alpha}.$$

However, this is impossible, since $\mathbf{R}_{Q,\theta}$ is the rotation with the smallest positive angle of rotation.

This shows that all of the rotations in $\mathcal{G}$ are multiples of $\mathbf{R}_{Q,\theta}$ and, conversely, since $\mathcal{G}$ is a group, all multiples of $\mathbf{R}_{Q,\theta}$ must be in $\mathcal{G}$. Letting a denote $\mathbf{R}_{Q,\theta}$, this means that $\mathcal{G}$ consists of

$$a, \quad a^2, \quad a^3, \quad \ldots, \quad a^n$$

for some integer n, where $a^n = \mathbf{R}_{Q,360^\circ} = \mathbf{I}$. This completes the proof of Theorem 10.2.3.

□

Theorem 10.2.4. *Suppose that $\mathcal{G}$ is a finite group that contains a rotation (other than the identity) and a reflection. Then there is a rotation $\mathbf{R}_{Q,\alpha}$ in the group that generates all of the rotations in the group. That is, the set of all rotations in $\mathcal{G}$ is*

$$\mathbf{R}_{Q,\alpha}, \quad \mathbf{R}_{Q,2\alpha}, \quad \ldots, \quad \mathbf{R}_{Q,n\alpha},$$

where $n\alpha = 360^\circ$.

Furthermore, if R_m is any reflection in $\mathcal{G}$, then every reflection in $\mathcal{G}$ must be one of the following:

$$\mathbf{R}_m\mathbf{R}_{Q,\alpha}, \quad \mathbf{R}_m\mathbf{R}_{Q,2\alpha}, \quad \ldots, \quad \mathbf{R}_m\mathbf{R}_{Q,n\alpha}.$$

In particular, this means that $\mathcal{G}$ is the dihedral group $\mathcal{D}_{2n}$.

Proof. Let $\mathcal{S}$ be the set of all rotations that are in $\mathcal{G}$, including the identity. Then by the previous theorems, these rotations form a group, and all of the rotations have the same center Q. Hence, by Theorem 10.2.3, $\mathcal{S}$ is a cyclic group and so the members of $\mathcal{S}$ are

$$\mathbf{R}_{Q,\alpha}, \quad \mathbf{R}_{Q,2\alpha}, \quad \ldots, \quad \mathbf{R}_{Q,n\alpha},$$

where $n\alpha = 360^\circ$.

To complete the remainder of the proof, we have to show that if $\mathbf{R}_l$ is any reflection that is in $\mathcal{G}$, then there is some integer k such that

$$\mathbf{R}_l = \mathbf{R}_m\mathbf{R}_{Q,k\alpha}.$$

Now, consider the product $\mathbf{R}_m\mathbf{R}_l$. Since both reflections belong to $\mathcal{G}$, it follows that the lines l and m both contain Q, and so $\mathbf{R}_m\mathbf{R}_l$ is a rotation with center Q.

Since $\mathbf{R}_m\mathbf{R}_l$ is a member of $\mathcal{G}$, and since $\mathcal{S}$ contains all of the rotations in $\mathcal{G}$, it follows that

$$\mathbf{R}_m\mathbf{R}_l = \mathbf{R}_{Q,k\alpha}$$

for some integer k. Therefore, we have

$$\mathbf{R}_m\mathbf{R}_{Q,k\alpha} = \mathbf{R}_m\left(\mathbf{R}_m\mathbf{R}_l\right) = \left(\mathbf{R}_m\mathbf{R}_m\right)\mathbf{R}_l m = \mathbf{R}_l.$$

□

Leonardo's Theorem now follows from Theorem 10.2.3 and Theorem 10.2.4.

One of the consequences of Leonardo's Theorem is:

Theorem 10.2.5. *The group of symmetries of a polygon in the plane is either a cyclic group or a dihedral group.*

Proof. Given a vertex A of a polygon, together with an adjacent vertex B, any symmetry of the polygon must map A onto one of the vertices of the polygon, in which case there are at most two possible adjacent vertices that can be the image of B. In other words, there are only a finite number of symmetries. Leonardo's Theorem now tells us that the group of symmetries is either a cyclic group or a dihedral group.

□

10.3 Problems

1. Prove that a finite group of isometries cannot contain two halfturns about distinct points.
2. Prove that the set of all halfturns and all translations forms a group.
3. Prove that if a triangle is invariant under a reflection, then the triangle must be isosceles.
4. Which of the following sets of transformations form a group, and which do not form a group?
   (a) All translations.
   (b) All reflections.
   (c) All glide reflections.
   (d) All rotations.
   (e) All direct isometries.
   (f) All opposite isometries.

5. If $H_{O_1} H_{O_2} = H_{O_2} H_{O_1} = T$, prove that $T = I$, the identity transformation.

6. Find a plane figure P such that its group of symmetries equal

   (a) the cyclic group C_2 of order 2,

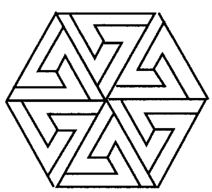
   (b) the cyclic group C_1 of order 1.

7. Find a plane figure P such that its group of symmetries equal

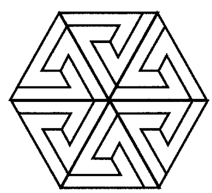
   (a) the dihedral group D_2 of order 4,

   (b) the dihedral group D_1 of order 2.

8. Find the group of symmetries of each of the following figures.



(a)



(b)

9. Let $\mathcal{G}$ be a group of isometries whose subgroup of translations is generated by T_{AB}, where $AB \neq 0$. Prove that if $R_\ell \in \mathcal{G}$, then either $\overline{AB}$ is parallel to ℓ or $\overline{AB}$ is perpendicular to ℓ.

10. Find the group of isometries of an ellipse.

11. If a and b are elements of a group $\mathcal{G}$ and

$$(ab)^2 = a^2b^2,$$

show that $ab = ba$.

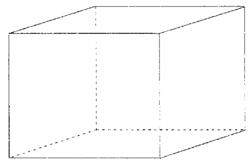
12. Let a and b be elements of a group $\mathcal{G}$ such that b has order 2 and $ab = ba^{-1}$.

   (a) Show that $a^n b = ba^{-n}$ for all integers n.
   Hint: Evaluate the product $(bab)(bab)$ in two different ways to show that $ba^2b = a^{-2}$, and then extend this method.

   (b) Show that the set $S = \{a^n, ba^n \mid n \in \mathbb{Z}\}$ is closed under multiplication and in fact forms a group.

   (c) Show that $S = \langle a, b \rangle$, the dihedral group with generators a and b.

13. A *cuboid* is a rectangular parallelepiped; that is, a parallelepiped where each plane face is orthogonal to four other faces and parallel to the fifth, as in the figure.

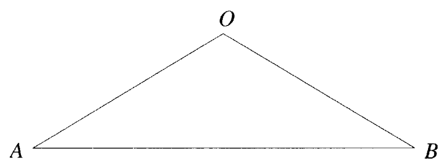


Find the group of symmetries of a cuboid with three unequal sides.

14. What is the symmetry group of a rhombus that is not a square? Find all the symmetries of the rhombus and construct the Cayley table or multiplication table for the group of symmetries.

Hint: The diagonals of a parallelogram bisect each other, and a parallelogram is a rhombus if and only if its diagonals are perpendicular.

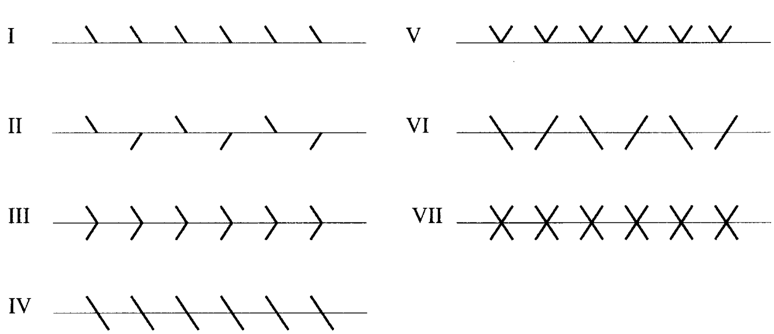
15. Let $\mathcal{T}$ be a (nonequilateral) isosceles triangle.



Find the group of symmetries of $\mathcal{T}$ and construct the Cayley table for the group.

- In the plane, the discrete groups fixing a line are the groups of symmetries of ribbons or friezes. To study the frieze groups, we first find an isometry fixing a line and then compose it with an isometry that has a fixed point. The group $\mathcal{F}$ generated by this isometry is called a ***frieze group***.

16. For each of the seven patterns given in the figure below, assuming each extends to infinity both to the left and to the right, name the types of isometries in the symmetry group of each pattern.

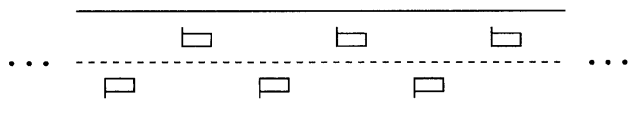


17. Prove that if $H_{O_0} \in \mathcal{F}$, a frieze group with translation subgroup

$$\mathcal{T} = \{T_{nAB}\} = \langle T_{AB} \rangle ,$$

then $H_{O_{n/2}} \in \mathcal{F}$ for every integer n, where $\overline{O_0 O_{n/2}} = \frac{n}{2}\overline{AB}$.

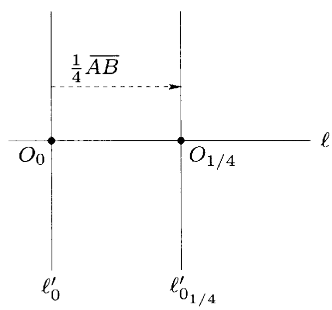
18. Find the group of symmetries of the following repeated pattern on an infinite horizontal strip, as shown below.



19. Find a different set of two generators for the frieze group

$$\mathcal{F} = \left\langle H_{O_{1/4}}, G_{\ell, \frac{1}{2}AB} \right\rangle ,$$

where $\ell \parallel \overline{AB}$, and O_0 and $O_{\frac{1}{4}}$ are two fixed points on ℓ with $\overline{OO_{\frac{1}{4}}} = \frac{1}{4}\overline{AB}$, as in the figure below.



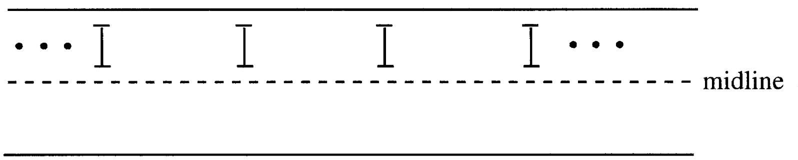
20. With repeated use of only the symbol

construct a repeated pattern on a horizontal strip whose frieze group is

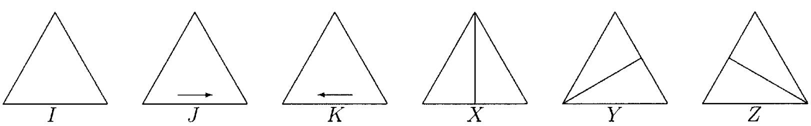
$$\mathcal{F} = \left\langle H_{O_{1/4}}, G_{\ell, \frac{1}{2}AB} \right\rangle ,$$

where $\ell \parallel \overline{AB}$, and O_0 and $O_{\frac{1}{4}}$ are points on ℓ such that $\overline{OO_{\frac{1}{4}}} = \frac{1}{4}\overline{AB}$, as in the figure in Problem 10.19.

21. Find the frieze group of an infinite horizontal strip consisting of repeated I's if the I's lie above the midline of the strip as shown below.

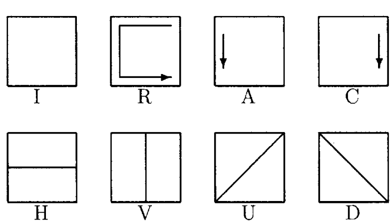


22. Consider a particular vertex of an equilateral triangle. Under a symmetry, it can land on any of the three vertices. The remaining vertices must follow in order, either counterclockwise or clockwise. Thus, there are only $3 \times 2 = 6$ symmetries for the equilateral triangle. They are: the identity **I**, a 120° counterclockwise rotation J, a 120° clockwise rotation K, and three reflections X, Y, and Z, as shown in the figure on the following page.



These six symmetries form a group under the operation of composition, the dihedral group of the equilateral triangle. Construct the operation table of this group.

23. Consider a particular vertex of a square. Under a symmetry of the square, this vertex can land on any one of the four vertices. The remaining vertices must follow in order, either counterclockwise or clockwise. Thus, there are only $4 \times 2 = 8$ symmetries for the square. They are: the identity I, a 180° rotation or halfturn R, a 90° counterclockwise rotation A, a 90° clockwise rotation C, and four reflections H, V, D, and U, as shown in the figure below.



These eight symmetries of the square form a group under the operation of composition, the dihedral group of the square. Construct the operation table of this group.

24. Consider the set of symmetries of a non-square rectangle. It has only four elements: I, R, H, and V, analogous to the corresponding symmetries for the square. These four symmetries form a group with respect to composition. Construct the multiplication table of this group.

25. The complex numbers 1, -1, i, and $-i$ form a group under multiplication. Construct the operation table of this group.

26. The ***Quaternion group*** consists of the eight elements

$$1, \quad -1, \quad i, \quad -i, \quad j, \quad -j, \quad k, \quad -k,$$

with the operation of multiplication defined by

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = k, \quad jk = i, \quad ki = j, \\ &ji = -k, \quad kj = -i, \quad ik = -j. \end{aligned}$$

Construct the operation table.

27. Let

$$\begin{aligned} a(x) &= \frac{1}{1-x}, \\ b(x) &= \frac{x-1}{x}, \\ c(x) &= 1-x, \\ d(x) &= \frac{x}{x-1}, \\ e(x) &= x, \\ f(x) &= \frac{1}{x}. \end{aligned}$$

These functions form a group with respect to the operation of composition. Construct the operation table.

CHAPTER 11

HOMOTHETIES

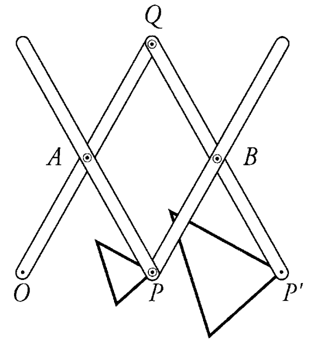
11.1 The Pantograph

Isometries provide a dynamic way of dealing with congruency. In this chapter, we study a transformation which serves the same purpose for the notion of similarity.

Without employing photography, photocopying, or computer graphics, it is not a simple matter to produce an enlarged or reduced copy of a figure. There is, however, a physical instrument called a ***pantograph*** that allows us to accomplish this.

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

A pantograph is formed from four thin flat rods of equal length that are joined together by four hinge pins P, Q, A, and B so that $APBQ$ is a parallelogram and $OA = AP$. The instrument lies flat on the drawing board and is fixed to the board at the ***pivot point*** O. Pencils are attached to the instrument at points P and P'.



If an enlargement is desired, the pencil at P is used to trace the original feature. As this is being done, the pencil at P' draws a copy magnified by a factor equal to OQ/OA. If a reduction is desired, the pencil at P' is used to trace the figure so that the pencil at P draws the reduced copy.

To see why this works, note that OAP and OQP' are similar triangles by **sAs**. It follows that O, P, and P' are collinear. Moreover,

$$\frac{OP'}{OP} = \frac{OQ}{OA},$$

and thus the figure that is traced by P may be considered a "contraction" towards O of the figure that is traced by P'.

It should be noted that the magnification factor OQ/OA for the pantograph illustrated in the figure above is fixed. In an actual pantograph, the positions of the hinge pins at A and B may be adjusted so that OQ/OA can be set as desired. The pins A and B must be adjusted in such a manner that $APBQ$ remains a parallelogram and so that $OA = AP$.

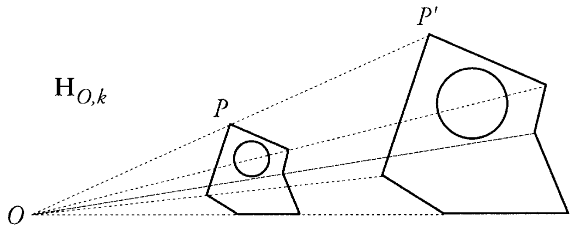
11.2 Some Basic Properties

A similarity in which a figure contracts towards a point or expands away from a point in this manner is called a *homothety*.

More formally, let O be a point and let k be a nonzero real number. The ***homothety*** centered at O with ratio k, denoted by $\mathbf{H}_{O,k}$, maps O to O and each point $P \neq O$ onto another point P' on the line OP such that

$$\overline{OP'} = k\,\overline{OP}.$$

Since $k \neq 0$, it follows that $\mathbf{H}_{O,k}$ is a transformation, and its inverse is easily seen to be $\mathbf{H}_{O,1/k}$.



Remark. In this chapter, the word *parallel* includes the case where two lines coincide. In other words, two lines are considered to be parallel if there is a translation that maps one onto the other, including a translation through a distance of magnitude zero. Also, the notation $\overline{AB}$ is used to denote the *directed segment* from A to B.

A ***similarity*** is the composition of an isometry and a homothety. Thus, any similarity that is not an isometry is

(a) either a rotation followed by a homothety with the same center (the isometry is direct)

(b) or a reflection followed by a homothety whose center lies on the line of the reflection (the isometry is opposite).

Theorem 11.2.1. *Let A and B be two points whose images under the homothety $\mathbf{H}_{O,k}$ are A' and B', respectively. Then $A'B'$ is parallel to AB and $\overline{A'B'} = k\,\overline{AB}$.*

Proof.
Case 1. O, A, and B are collinear.
In this case, $A'B'$ and AB are both contained in the line OA, so they are parallel. Moreover,

$$\overline{A'B'} = \overline{OB'} - \overline{OA'} = k\,\overline{OB} - k\,\overline{OA} = k\,\overline{AB}.$$

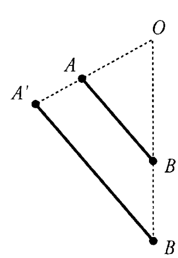
Case 2. O, A, and B are noncollinear (as in the figure).
If the points O, A, and B do not lie on a line, then angle O is common to both $\triangle OA'B'$ and $\triangle OAB$. Also,

$$\overline{OA'} = k\,\overline{OA} \qquad \text{and} \qquad \overline{OB'} = k\,\overline{OB},$$

so the triangles are similar by **sAs**. It follows that

$$\overline{A'B'} = k\,\overline{AB}$$

and that the segments $A'B'$ and AB are parallel.



□

Corollary 11.2.2. *Let A', B', and C' be the images of A, B, and C, respectively, under a homothety.*

(1) If B is between A and C, then B' is between A' and C'.

(2) If A, B, and C are the vertices of a triangle, then A', B', and C' are the vertices of a similar triangle.

Proof. Suppose the homothety is $\mathbf{H}_{O,k}$.

(1) By the Triangle Inequality, $AC = AB + BC$. Using Theorem 11.2.1, we get

$$A'C' = |k|\, AC = |k|\,(AB + BC) = |k|\, AB + |k|\, BC = A'B' + B'C',$$

and so by the Triangle Inequality, B' must be between A' and C'.

(2) By Theorem 11.2.1,

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC} = |k|,$$

and the triangles are similar by SSS.

□

A homothety preserves any geometric relationship that can be completely characterized by ratios of distances. For example, midpoints, centroids, and angle bisectors can all be characterized by distance ratios. This means that the midpoint of BC is mapped to the midpoint of $B'C'$, the centroid of $\triangle ABC$ is mapped to the centroid of $\triangle A'B'C'$, and the bisector of $\angle ABC$ is mapped to the bisector of $\angle A'B'C'$. We will use these facts freely throughout this chapter.

All of these facts can be proved using Theorem 11.2.1 and Corollary 11.2.2. For example, here is a proof that the midpoint M of BC is mapped to the midpoint M' of $B'C'$ by the homothety $\mathbf{H}_{O,k}$.

Corollary 11.2.2 tells us that B', C', and M' are collinear. Also, by Theorem 11.2.1,

$$\overline{B'M'} = k\,\overline{BM} \qquad \text{and} \qquad \overline{B'C'} = k\,\overline{BC},$$

so that

$$\frac{\overline{B'M'}}{\overline{B'C'}} = \frac{\overline{BM}}{\overline{BC}} = \frac{1}{2}.$$

Thus, the image M' of M is indeed the midpoint of $B'C'$.

□

11.2.1 Circles

Theorem 11.2.3. *The image of a circle $\mathcal{C}$ with center C and radius r under the homothety $\mathbf{H}_{O,k}$ is a circle with center $D = \mathbf{H}_{O,k}(C)$ and radius $|k|\,r$.*

Proof. The homothety maps each point X of $\mathcal{C}$ to a point Y such that

$$DY = |k|\,CX = |k|\,r,$$

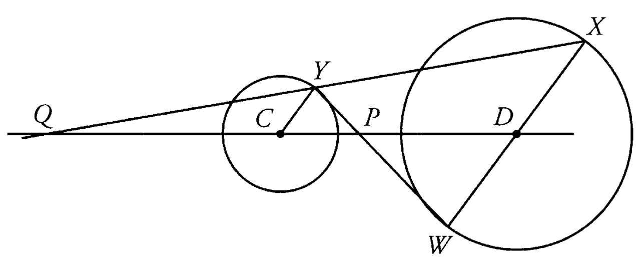
so the points Y form a circle centered at D with radius $|k|\,r$.

□

If $\mathcal{C}$ and $\mathcal{D}$ are two circles, the center of any homothety that transforms one circle into the other is called a ***center of similitude***.

Example 11.2.4. *Given two circles with two different centers and two different radii, explain how to construct all centers of similitude for the circles.*

Solution. The process is illustrated in the figure below.



Construct any diameter of one of the circles, say WX. Then construct a radius CY of the other circle that is parallel to this diameter. The points P and Q, where the lines WY and XY intersect the line CD through the centers of the circles, will be the centers of similitude. There are no more centers of similitude for these two circles, for if O is a center of similitude, then O must be on the line CD, and the homothety $\mathbf{H}_{O,k}$ that carries C to D must transform the radius CY into a parallel radius, either DW or DX. This leaves only two possible locations for O.

□

Theorem 11.2.5. *Let P be a point on the line joining the centers A_1 and A_2 of two circles with radii r_1 and r_2, respectively. If*

$$\frac{PA_1}{PA_2} = \frac{r_1}{r_2},$$

then P is a center of similitude of the circles.

Proof. Note that there are at most only two points P on A_1A_2 such that

$$\frac{PA_1}{PA_2} = \frac{r_1}{r_2}.$$

For one of them the ratio

$$\frac{\overline{PA_1}}{\overline{PA_2}}$$

is positive, for the other it is negative. Letting

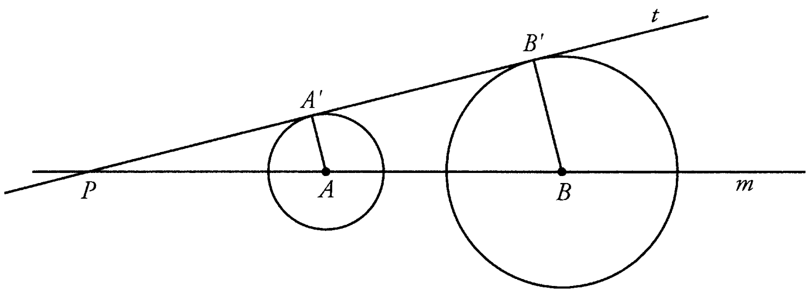
$$k = \frac{\overline{PA_1}}{\overline{PA_2}},$$

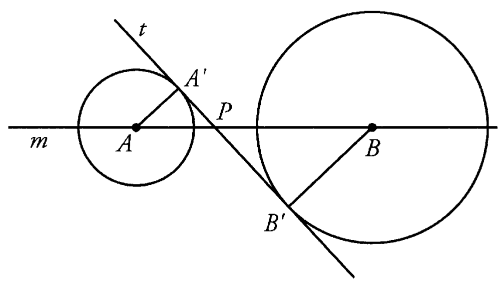
we see that the homothety $\mathbf{H}_{P,k}$ maps the circle centered at A_1 to the circle centered at A_2.

□

Example 11.2.6. *Show that any common tangent to two circles of unequal radii passes through a center of similitude.*

Solution. The tangent t cannot be parallel to the the line m joining the centers A and B. Suppose that t meets m at P. Let A' and B' be the points of tangency, as in the figure on the following page.





Then $\triangle PA'A \sim \triangle PB'B$ by **AAA** similarity. Consequently,

$$\frac{PA}{PB} = \frac{AA'}{BB'},$$

and it follows from Theorem 11.2.5 that P is a center of similitude.

□

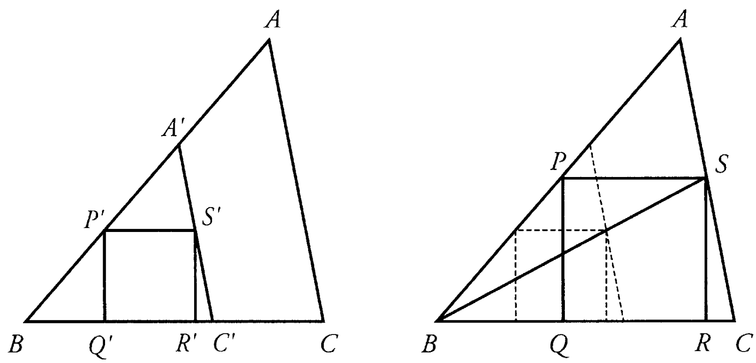
11.3 Construction Problems

If one figure is the image of another under a homothety, the figures are said to be ***homothetic*** to each other. Homothetic figures are similar, but similar figures need not be homothetic, since similar figures need not be oriented the same way.

Some construction problems can be solved by first constructing a homothetic image of the figure and then enlarging or shrinking the image to get the desired solution.

Example 11.3.1. *Given an acute triangle ABC, construct a square $PQRS$ with P on AB, S on AC, and edge QR on BC.*

Solution. Let P' be any point AB. Drop the perpendicular $P'Q'$ to BC and complete the square $P'Q'R'S'$, as in figure (a) below.



(a) (b)

If we draw the line $A'C'$ through S' parallel to AC, we note that the square $P'Q'R'S'$, together with the triangle $A'B'C'$, is homothetic to the desired solution, with B being the center of the homothety.

Thus, we need a homothet of $P'Q'R'S'$ so that the image of S' is the point S on AC. To accomplish this, draw the line BS'. Then S is the point where BS' meets AC, as in figure (b) above. Now drop the perpendicular SR to BC and complete the desired square.

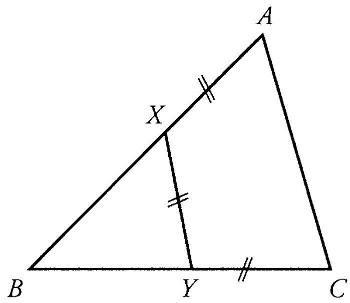
□

Here is another example that uses the same idea.

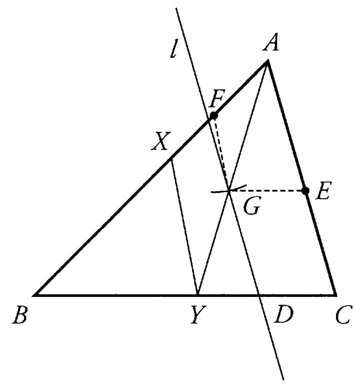
Example 11.3.2. *Construct points X and Y on the sides AB and BC, respectively, of a given triangle ABC such that*

$$AX = XY = YC.$$

Solution. The desired result is shown in figure (a) on the following page. Take a point D on BC and a point F on AB such that $AF = CD$, as in figure (b) on the following page.



(a)



(b)

Draw a line l through D parallel to AC. Draw a circle with center F and radius FA, cutting l at a point G inside $\triangle ABC$. Complete the parallelogram $CDGE$, with E on AC. We have

$$AF = FG = GE.$$

Join A and G and extend the segment AG to cut BC at Y. Draw a line through Y parallel to FG, cutting AB at X. Then $AFGE$ and $AXYC$ are homothetic to each other with A being the center of homothety. Hence,

$$AX = XY = YC.$$

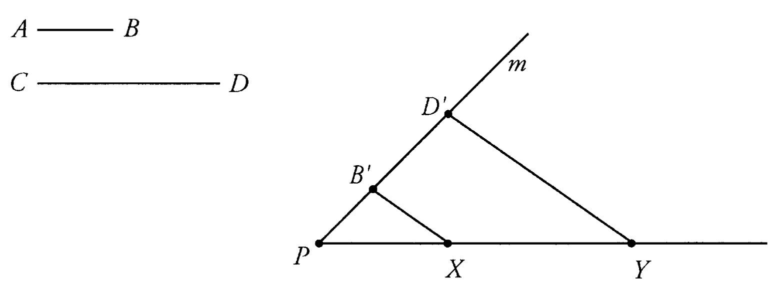
□

Example 11.3.3. *Given points P and X and line segments AB and CD and letting $k = CD/AB$, construct the point $\mathbf{H}_{P,k}(X)$.*

Solution. Construct a line m through P at an angle to PX and construct points D' and B' on m so that $PB' = AB$ and $PD' = CD$. Join B' to X and construct the line through D' parallel to $B'X$ meeting PX at Y. Therefore,

$$\frac{PY}{PX} = \frac{CD}{AB},$$

since triangles $PB'X$ and $PD'Y$ are similar. Thus, $Y = \mathbf{H}_{P,k}(X)$.



This shows that given any point X, we can construct the image of X under the homothety $\mathbf{H}_{P,k}$ where $k = CD/AB$.

$\square$

Example 11.3.3 can be considered a basic construction. It shows us how to construct the homothetic images of all standard geometric figures.

For example, to construct the image of $\triangle ABC$ under $\mathbf{H}_{P,k}$, we can construct the images A', B', and C' of the vertices A, B, and C, respectively, under the homothety.

Also, to construct the image under $\mathbf{H}_{P,k}$ of a circle with center C and radius r, let X be a point on the circle and construct $\mathbf{H}_{P,k}(C)$ and $\mathbf{H}_{P,k}(X)$. These are, respectively, the center of the image circle and a point on the image circle, and we can now construct the image circle.

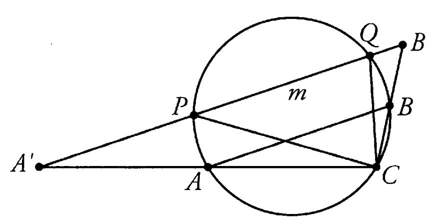
Example 11.3.4. *Given three points A, B, and C on a circle $\mathcal{C}$, explain how to find all chords through the point C that are bisected by AB.*

Solution. There are two ways to solve this problem.

The first way is to construct the image m of the line AB under $\mathbf{H}_{C,2}$, as in the figure on the right. Then for any point X' on the line m, the point

$$X = X'C \cap AB$$

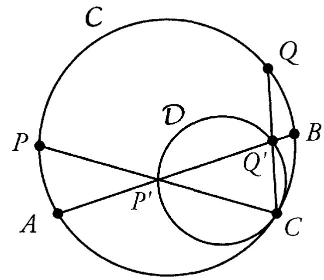
is the midpoint of $X'C$. Thus, the points P and Q, where m meets the circle, will provide the chords PC and QC.



The second way is to apply $\mathbf{H}_{C,1/2}$ to the circle, obtaining another circle $\mathcal{D}$, as in the figure on the right. Let P' and Q' be the points where $\mathcal{D}$ intersects AB. Let P and Q be the points where CP' and CQ' meet $\mathcal{C}$; that is,

$$P = CP' \cap \mathcal{C} \quad \text{and} \quad Q = CQ' \cap \mathcal{C},$$

so that, again, CP and CQ are the desired chords.



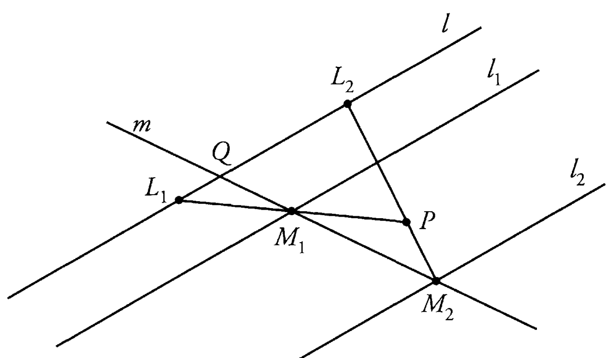
□

Example 11.3.5. *Given two lines l and m intersecting at Q and given a point P not on either line, construct a line through P cutting l at L and m at M such that $PL = 2PM$.*

Solution. There are two pairs of points L and M. To find the first pair L_1 and M_1, construct the line

$$l_1 = \mathbf{H}_{P,1/2}(l)$$

and let $M_1 = l_1 \cap m$, and then let $L_2 = PM_1 \cap l$.



To find the second pair, construct the line

$$l_2 = \mathbf{H}_{P,-1/2}(l)$$

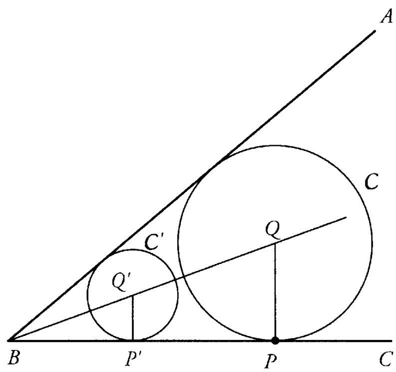
and let $M_2 = l_2 \cap m$, and then let $L_1 = PM_2 \cap l$.

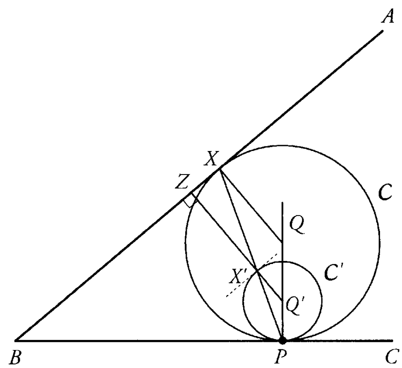
□

Example 11.3.6. *Given $\angle ABC$ and a point P on the arm BC, construct a circle passing through P that is tangent to both arms of the angle.*

Solution. There are two possible solutions to this problem.

(1) Draw the angle bisector of ABC and let Q' be any point on it, as in the figure on the left. Drop the perpendicular $Q'P'$ to BC and construct the circle $\mathcal{C}'$ with center Q' and radius $Q'P'$. The circle $\mathcal{C}'$ is tangent to both arms of the angle. Thus, the desired circle $\mathcal{C}$ is obtained by constructing the image of $\mathcal{C}'$ under the homothety $\mathbf{H}_{B,k}$ where $k = PB/P'B$.





(2) Construct the line through P perpendicular to BC and let Q' be any point on it, as in the figure on the right above. Construct the circle $\mathcal{C}'$ with center Q' and radius $Q'P$, which is therefore tangent to BC. Drop the perpendicular from Q' to AB and let X' be the point where it intersects $\mathcal{C}'$. Note that the tangent to $\mathcal{C}'$ at X' is parallel to AB. Let X be the point $PX' \cap AB$, and the circle $\mathcal{C}$ is now obtained by constructing the image of $\mathcal{C}'$ under the homothety $\mathbf{H}_{P,k}$ where $k = PX/PX'$.

□

11.4 Using Homotheties in Proofs

Recall that the four major concurrency points of a triangle are:

1. the ***incenter***, the point where the angle bisectors meet,
2. the ***circumcenter***, the point where the right bisectors of the sides meet,
3. the ***centroid***, the point where the medians meet, and
4. the ***orthocenter***, the point where the altitudes meet.

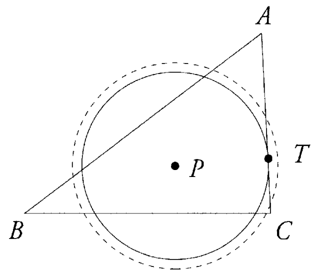
The ***incircle*** is the circle centered at the incenter and is internally tangent to all three sides of the triangle.

The ***circumcircle*** is the circle centered at the circumcenter that passes through the three vertices of the triangle.

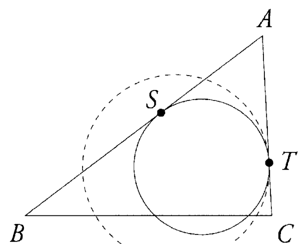
Despite their suggestive names, neither the centroid nor the orthocenter is the center of any significant circle associated with the triangle.

Theorem 11.4.1. *The incircle is the smallest circle that meets all three sides of a given triangle.*

Proof. Let $\mathcal{C}$ be a circle with center P that intersects all three sides of triangle ABC. If none of the sides are tangent to the circle $\mathcal{C}$, contract it using a homothety centered at P, so that the image circle intersects all sides of ABC but is tangent to at least one side. Let T be the point of tangency.

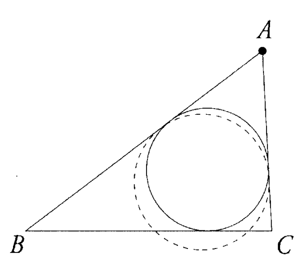


If the image circle is not tangent to two sides of triangle ABC, then contract it using a homothety centered at T, so that the new image circle still intersects all three sides of ABC and is tangent to at least two sides. Let the second point of tangency be denoted by S.



If this image circle is not tangent to all three sides of triangle ABC, then contract it again using a homothety centered at the vertex that is common to the two sides tangent to the circle.

This image circle is the incircle of triangle ABC. Since it was obtained using only *contractions*, it is smaller than the original circle $\mathcal{C}$, unless $\mathcal{C}$ was already the incircle.



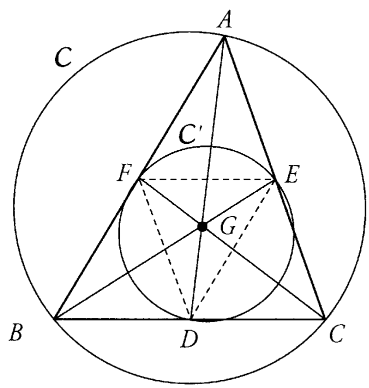
□

Example 11.4.2. *(Euler's Inequality)*
Prove that $R \geq 2r$, where R is the circumradius and r the inradius of a triangle. Equality holds if and only if the triangle is equilateral.

Solution. As in the figure below, let ABC be the triangle. Let D, E, and F be the midpoints of BC, CA, and AB, respectively. Let G be the centroid of $\triangle ABC$, and let $\mathcal{C}$ be the circumcircle.

Recalling that G is a trisection point of each median, we see that $\mathbf{H}_{G,-1/2}$ maps A, B, and C to D, E, and F, respectively. Hence, $\mathcal{C}'$, the image of $\mathcal{C}$ under $\mathbf{H}_{G,-1/2}$, is the circumcircle of triangle DEF, and the radius of $\mathcal{C}'$ is $R/2$.

Thus, $\mathcal{C}'$ is a circle that has a point in common with all three sides of ABC, and the smallest such circle is its incircle, so we can conclude that $R/2 \geq r$.



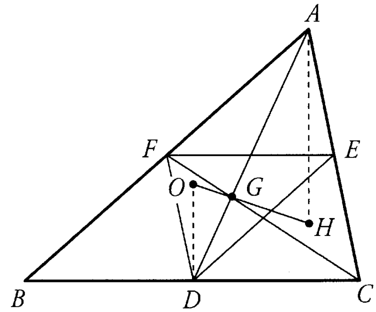
For equality to hold, $\mathcal{C}'$ must be the incircle of ABC, touching the sides at D, E, and F. Hence, the circumcenter coincides with the incenter of ABC, which must therefore be equilateral.

□

Example 11.4.3. *(The Euler Line)*
The centroid G, the circumcenter O, and the orthocenter H of a triangle are collinear. Moreover, G is between O and H and $\overline{GH} = 2\overline{GO}$.

Solution. This is also proved by applying the homothety $H_{G,-1/2}$. Given triangle ABC, let D, E, and F be the midpoints of BC, CA, and AB, respectively. As in the previous example, $\mathbf{H}_{G,-1/2}$ maps A, B, and C to D, E, and F, respectively.

The homothety maps the altitude AH of $\triangle ABC$ into a line DH' that is parallel to AH. In other words, the homothety maps the altitude from A into the right bisector of the opposite side. A similar thing happens to the other two altitudes, so the homothety maps the intersection of the altitudes to the intersection of the right bisectors. In other words, $\mathbf{H}_{G,-1/2}$ maps the orthocenter H of ABC into the circumcenter O of ABC.



The definition of $\mathbf{H}_{G,-1/2}$ tells us that G, H, and O are collinear and also that

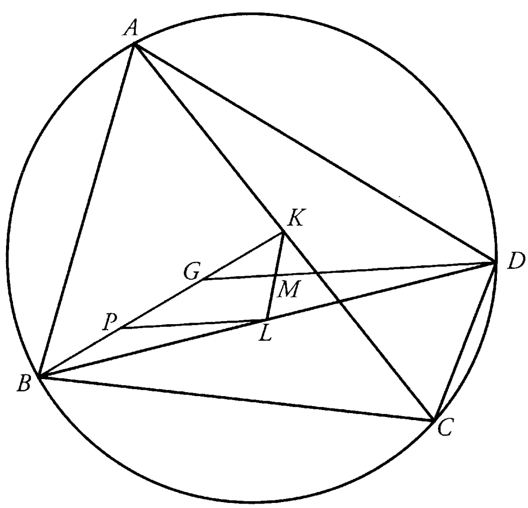
$$\overline{GH} = -2\overline{GO}.$$

□

Example 11.4.4. *$ABCD$ is a cyclic quadrilateral. Prove that the centroids of the triangles ABC, BCD, CDA, and DAB are cyclic, that is, that they lie on a circle.*

Solution. Let K and L be the midpoints of the diagonals AC and BD, and let M be the midpoint of KL. We will prove the result by showing that the quadrilateral formed by the centroids is the image $A'B'C'D'$ of $ABCD$ under $\mathbf{H}_{M,-1/3}$. Since $ABCD$ is a cyclic quadrilateral, it follows that the image is also a cyclic quadrilateral, since the circumcircle $\mathcal{C}$ of $ABCD$ will be mapped into the circumcircle of $A'B'C'D'$.

Let G be the centroid of triangle ABC. We will show that G is actually D', the image of D under $\mathbf{H}_{m,-1/3}$.



Let P be the midpoint of BG. Then P and G trisect BK. In triangle BGD, the points P and L are the midpoints of BG and BD, and the segments PL and GD are parallel, with $\overline{GD} = 2\overline{PL}$.

In triangle KPL, the points G and M are the midpoints of KP and KL, and the segments PL and GM are parallel, with $\overline{PL} = 2\overline{GM}$.

Since GD and GM are both parallel to PL, it follows that G, M, and D are collinear. Since $\overline{GD} = 2\overline{PL}$ and $\overline{PL} = 2\overline{GM}$, it follows that $\overline{GD} = 4\overline{GM}$ or, equivalently, that $\overline{MD} = -3\overline{MG}$.

This shows that $\mathbf{H}_{M,-1/3}(D) = G$, as claimed. Similarly, A', B', and C' are the centroids of BCD, CDA, and DAB, respectively. Hence, the four points A', B', C', and D' lie on the circle $\mathcal{C}'$, which is the image of $\mathcal{C}$ under $\mathbf{H}_{M,-1/3}$.

□

11.5 Dilatation

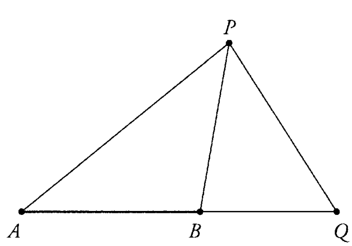
Any transformation of the plane that transforms a line segment into a parallel line segment is called a ***dilatation*** or a ***dilation***. The following theorem follows directly from the definition, and its proof is left to the reader.

Theorem 11.5.1. *The collection of all dilatations of the plane is a group.*

Examples of dilatations are translations, halfturns, homotheties, and the identity. It turns out that these are the only ones.

Theorem 11.5.2. *A dilatation that has two fixed points must be the identity.*

Proof. Suppose the dilatation T has A and B as fixed points. Let P be any point not on AB and let $T(P) = P'$.



Then $P'A \parallel PA$ and $P'B \parallel PB$, so it follows that P' is on both PA and PB. Since PA and PB have only one point of intersection, it follows that $P' = P$.

Suppose now that Q is a point on AB other than A or B. Then Q is not on the line AP, and since T fixes A and P, the same proof shows that T maps Q onto itself.

This shows that T maps every point of the plane onto itself; that is, T is the identity.

□

Theorem 11.5.3. *A dilatation is completely determined by its action on any two given points.*

Proof. Suppose that the dilatations T and S map A and B to A' and B', respectively. Then T^{-1} maps A' to A and B' to B, and so

$$T^{-1}S(A) = A \qquad \text{and} \qquad T^{-1}S(B) = B.$$

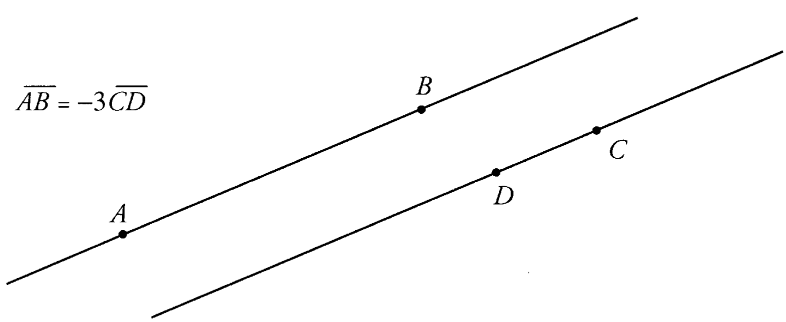
By the previous theorem, $T^{-1}S = \mathbf{I}$, and multiplying by T shows that $S = T$.

□

Given parallel lines l and m, it is possible to compare directed segments on them so that if A and B are on l and C and D are on m, the meaning of an equation such as

$$\overline{AB} = -3\,\overline{CD}$$

is unambiguous (see the figure below).



Corollary 11.5.4. *Suppose that the dilatation T maps AB to $A'B'$. Then:*

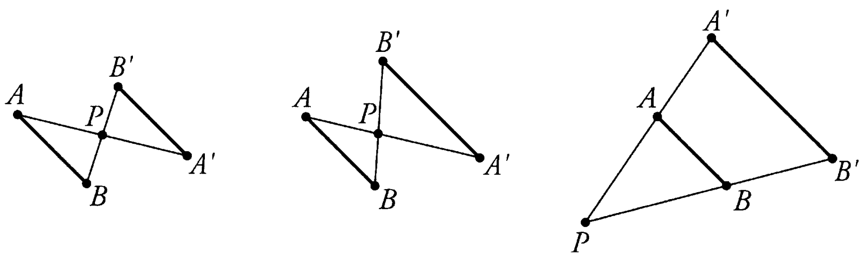
(1) $\overline{AB} = \overline{A'B'}$ if and only if T is a translation (or the identity).

(2) $\overline{AB} = -\,\overline{A'B'}$ if and only if T is a halfturn.

Theorem 11.5.5. *Any dilatation T that is not a translation or the identity is a homothety.*

Proof. Since T is not the identity, there is at least one point A that is not fixed. Let its image be A'. Now, let B be a point that is not on the line AA'. The segment $A'B'$ is parallel to AB, which is not parallel to AA', so B' cannot be on AA'.

The lines AA' and BB' cannot be parallel, for if they were, $AA'B'B$ would be a parallelogram; that is, $\overline{AB} = \overline{A'B'}$, which would contradict the fact that T is not a translation.



Thus, we may assume that the lines AA' and BB' intersect at a unique point P, as in one of the three situations depicted in the figure above. It follows that

$$\triangle PAB \sim \triangle PA'B',$$

and so

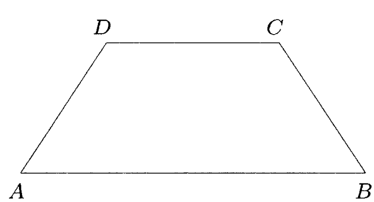
$$\frac{\overline{PA'}}{\overline{PB'}} = \frac{\overline{PA}}{\overline{PB}}.$$

Thus, the homothety $\mathbf{H}_{P,k}$, where $k = \overline{PA}/\overline{PB}$, maps A to A' and B to B', and therefore $T = \mathbf{H}_{P,k}$ by Theorem 11.5.3.

□

11.6 Problems

1. Find the two centers of homothety for the top and bottom of an isosceles trapezoid.



2. Show that two homotheties commute if and only if they have the same center or at least one of the ratios is $k = 1$.

3. Show that the product of two homotheties with the same center is a homothety and find its center and ratio.

4. Show that the product of two homotheties whose ratios are k and $1/k$ is a translation.

5. Show that if $\triangle ABC$ and $\triangle A'B'C'$ are similar, with AB parallel to $A'B'$, AC parallel to $A'C'$, and BC parallel to $B'C'$, then the lines joining corresponding vertices are concurrent, and there is a homothety $\mathbf{H}(O, k)$ such that $\triangle A'B'C'$ is the image of $\triangle ABC$ under $\mathbf{H}(O, k)$. Find the center O and the ratio k.

6. Show that a homothety preserves angles between lines.

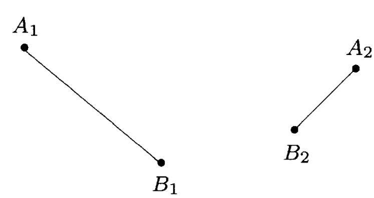
7. Show that the inverse of the homothety $\mathbf{H}(O, k)$ is the homothety $\mathbf{H}(O, 1/k)$.

8. Show that the center of a homothety of ratio $k \neq 1$ is the only fixed point of the homothety and lines through the center are the only fixed lines.

9. Show that a product of three homotheties is a homothety or a translation.

10. Show that a similarity preserves angles.

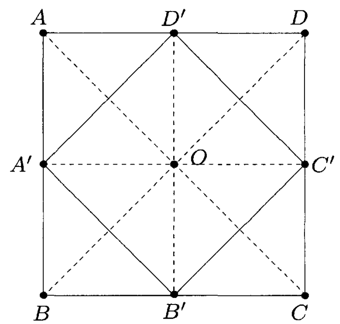
11. $\overline{A_1B_1}$ and $\overline{A_2B_2}$ are two nonparallel segments such that $A_1B_1 = 2A_2B_2$, as in the figure below.



    (a) Find a point O such that $\overline{A_2B_2}$ may be obtained from $\overline{A_1B_1}$ by means of a homothety centered at O with ratio $\frac{1}{2}$ followed by a rotation about O.

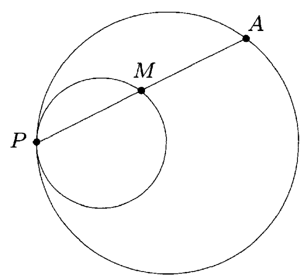
    (b) Find a line ℓ and a point O on ℓ such that $\overline{A_2B_2}$ may be obtained from $\overline{A_1B_1}$ by a homothety centered at O with ratio $\frac{1}{2}$ followed by a reflection across ℓ.

12. Using homotheties, show that the figure formed by joining the midpoints of the sides of a square is a square having half the area of the original square.



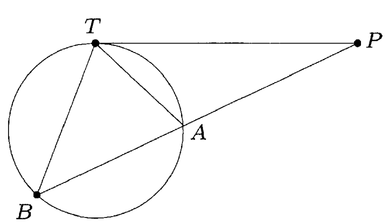
Hint: Show that the similarity composed of a $45°$ rotation and a homothety of ratio OA/OA', both with center O, maps the square $ABCD$ onto the quadrilateral $A'B'C'D'$.

13. Let P be a fixed point on a circle. Using homotheties, find the locus of midpoints of all chords PA.



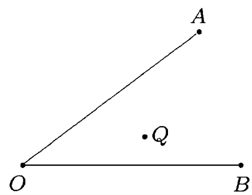
Hint: Consider the image of the given circle under the homothety $\mathbf{H}(P, 1/2)$.

14. If PT is a tangent and PAB a secant from an external point P to a circle $\mathcal{C}$, show that $\overline{PA}\,\overline{PB} = PT^2$.



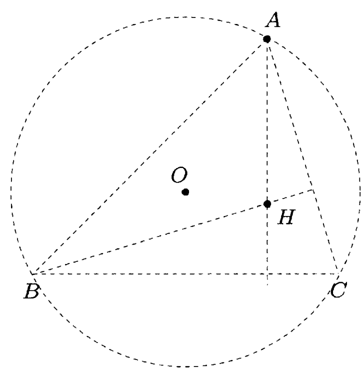
Hint: Reflect $\triangle PAT$ in the internal bisector of $\angle P$ and apply the homothety $H(P, PB/PT)$.

15. The point Q is a point inside $\angle AOB$, as in the figure.

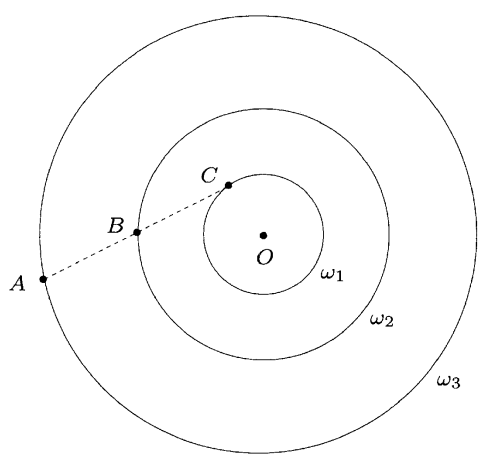


Construct a line through Q intersecting OA and OB at P and Q, respectively, such that $PQ = 2QR$.

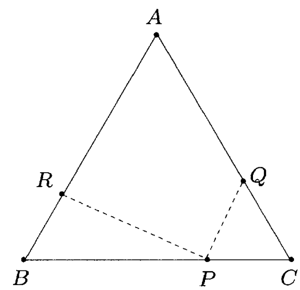
16. Construct $\triangle ABC$ given its circumcenter O, its orthocenter H, and the vertex A, as in the figure.



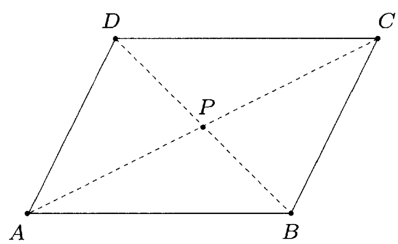
17. Construct a line intersecting three given concentric circles ω_1, ω_2, and ω_3 at A, B, and C, respectively, so that $AB = BC$.



18. Let P be a point on side BC of equilateral triangle $\triangle ABC$ that is closer to C than to B. Construct a point Q on CA and a point R on AB so that $\angle RPQ = 90°$ and $PR = 2PQ$.

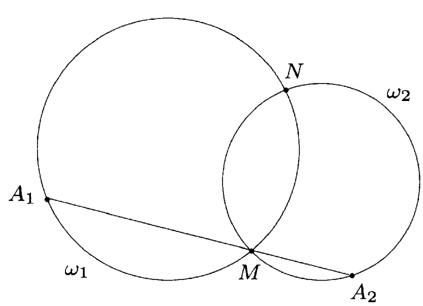


19. Two rods AD and BC are hinged at fixed points A and B on the ground. They are also connected to each other by means of a third rod CD, as in the figure, so that $ABCD$ is a parallelogram.

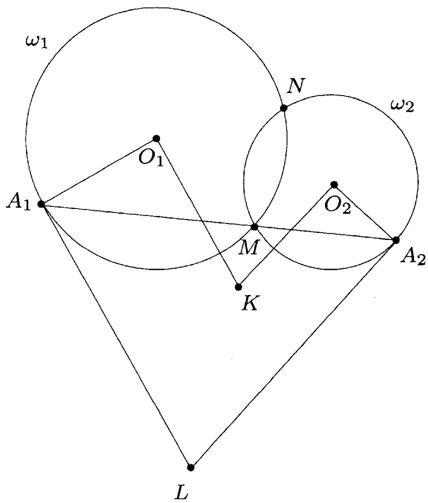


As the hinged rods move in a vertical plane, what is the locus of the point P of intersection of AC and BD?

20. The circles ω_1 and ω_2 intersect at M and N, and A_1 is a variable point on ω_1. A_2 is the point of intersection of the line $\overline{A_1M}$ with ω_2. B is the third vertex of an equilateral triangle A_1A_2B, with the vertices in counterclockwise order. Prove that the locus of B is a circle.



21. The circles ω_1 and ω_2 intersect at M and N, and A_1 is a variable point on ω_1. A_2 is the point of intersection of the line $\overline{A_1M}$ with ω_2, and L is the point of intersection of the tangent to ω_1 at A_1 and the tangent to ω_2 at A_2. Let O_1 and O_2 be the respective centers of ω_1 and ω_2. The line through O_1 parallel to LA_1 intersects the line through O_2 parallel to LA_2 at K, as in the figure below.



Prove the following:

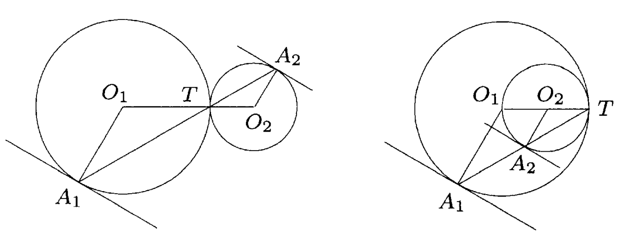
(a) $\triangle A_1NA_2$ and $\triangle O_1NO_2$ are similar.

(b) A_1LA_2N is a cyclic quadrilateral.

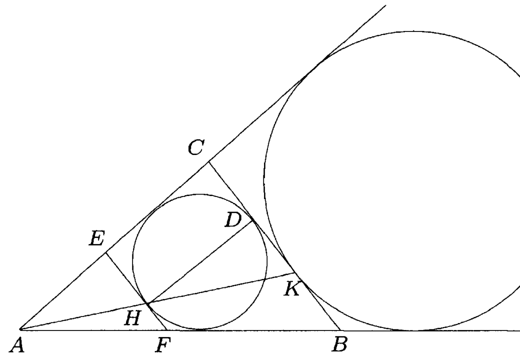
(c) O_1KO_2N is a cyclic quadrilateral.

(d) K, L, and N are collinear.

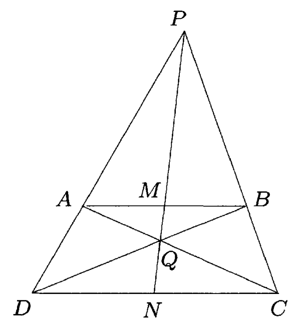
22. Two circles ω_1 and ω_2 are tangent to each other at the point T. A line through T intersects ω_1 at A_1 and ω_2 at A_2. Prove that the tangent to ω_1 at A_1 is parallel to the tangent to ω_2 at A_2.



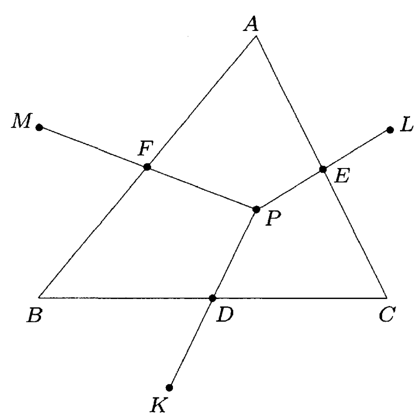
23. The incircle of triangle ABC touches BC at D. The excircle of triangle ABC opposite A touches BC at K. The line AK intersects the incircle at two points, and we let H be the one closer to A. Prove that DH is perpendicular to BC.



24. $ABCD$ is a quadrilateral with AB parallel to DC. The extensions of DA and CB intersect at P, and the diagonals AC and BD intersect at Q. Prove that PQ passes through the midpoints of AB and CD.



25. D, E, and F are the respective midpoints of the sides BC, CA, and AB of triangle ABC. P is a point inside ABC. K, L, and M are points such that D, E, and F are also the respective midpoints of PK, PL, and PM. Prove that AK, BL, and CM bisect one another at a common point.



CHAPTER 12

TESSELLATIONS

12.1 Tilings

A ***tiling*** or ***tessellation*** of the plane is a division of the plane into regions

$$T_1,\ T_2,\ \ldots\ ,$$

called ***tiles***, in such a manner that:

1. No region contains an interior point of another region.
2. Every point in the plane belongs to one of the regions.

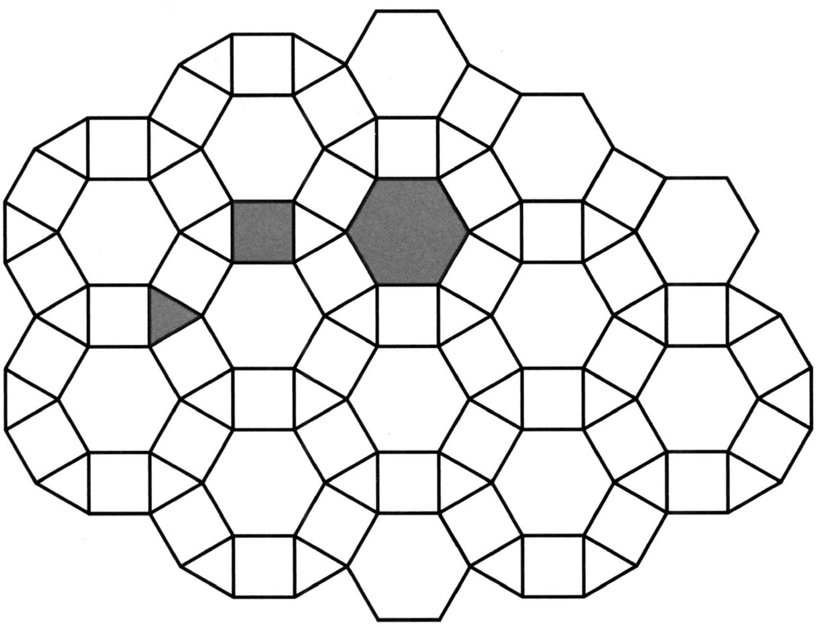
In other words, the plane is completely covered by nonoverlapping tiles.

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

There is no requirement that the tiles be related in any way, but our interest is primarily in tilings where there are only a finite number of differently shaped tiles. A tessellation is of ***order-k***, or ***k-hedral***, if there is a finite set of k incongruent tiles S such that:

1. Every tile in the tessellation is congruent to some member of S.
2. Every member of S occurs at least once in the tessellation.

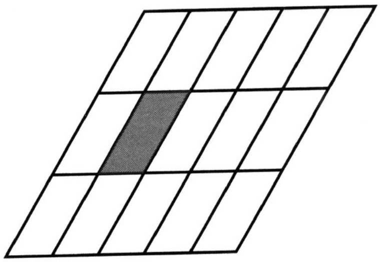
The members of S are called ***prototiles***, and we say that S ***tiles the plane***. The tiling in the figure below has a set of three prototiles, so it is an order-3 tiling.



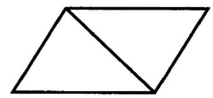
12.2 Monohedral Tilings

The most natural question is "Which polygons are monohedral prototiles?" The words ***monohedral***, ***dihedral***, and ***trihedral*** are commonly used as synonyms for ***1-hedral***, ***2-hedral***, and ***3-hedral***, respectively.

It is not difficult to see that every parallelogram tiles the plane, as in (a) below.



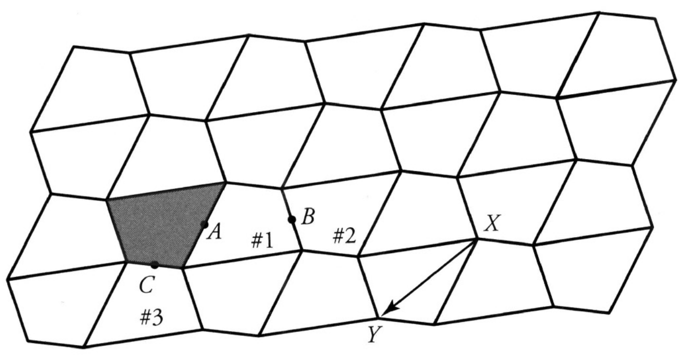
(a)



(b)

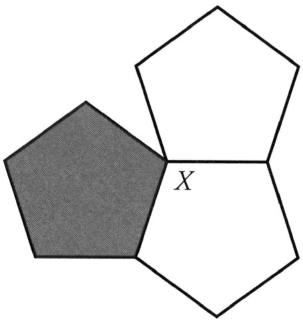
As a consequence, every triangle tiles the plane because two copies of the triangle can be combined to form a parallelogram, as in (b) above.

Every quadrilateral tiles the plane. The tiling can be obtained by successively rotating the quadrilateral around the midpoints of the sides, as in the figure below.

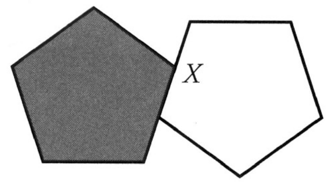


Tile #1 is obtained by applying $\mathbf{H}_A$ to the shaded tile, and tile #2 is obtained by applying $\mathbf{H}_B$ to tile #1. Continuing in this way, we can tile the entire horizontal strip containing tiles #1 and #2. The strip below this one can be obtained by similar rotations—for example, tile #3 may be obtained by applying $\mathbf{H}_C$ to the shaded tile. Alternatively, we can apply $\mathbf{T}_{XY}$ to the entire strip containing #1 and #2.

The regular pentagon will not tile the plane. One way to see this is to examine what happens when you try to tile around a vertex.



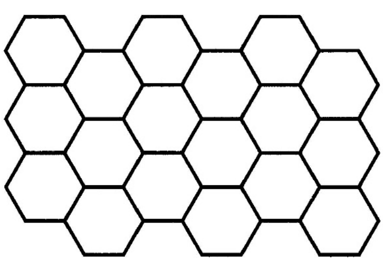
(a)



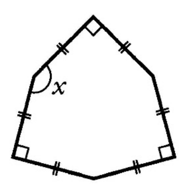
(b)

The figures above show what happens when you try to tile around vertex X of the shaded pentagon. You can either try a ***vertex-to-vertex tiling***, as in (a), or you can try an ***edge-to-vertex tiling***, as in (b). In (a) there is an angular gap of $36°$, and in (b) there is a $72°$ gap, neither of which can be covered without overlapping tiles.

Some hexagons tile the plane, but not all do. Everyone has seen the tiling in (a) below.



(a)



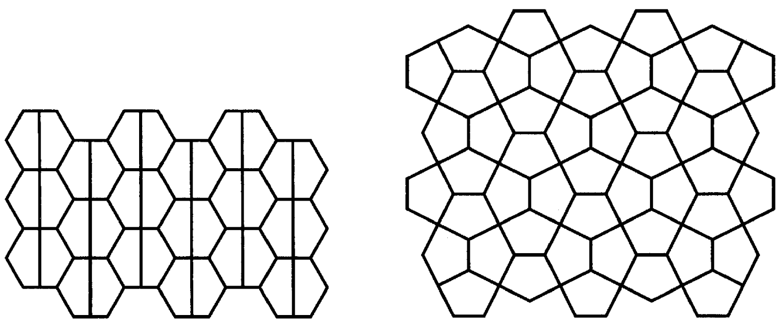
(b)

The hexagon in (b) above will not tile the plane, and, as before, this can be verified by trying to tile around the vertex of $x°$.

Although the regular pentagon does not tile the plane, there are some pentagons that do, and the figure below shows two of them.

The tiling in (a) is obtained by splitting the regular hexagonal tiling.

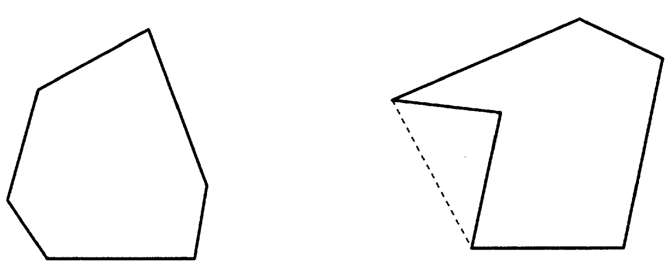
The tiling in (b), which also uses hexagons, is the beautiful "Cairo" tiling.



(a) (b)

Convex Hexagons that Tile

A polygon is ***convex*** if all of its diagonals are interior to the polygon.

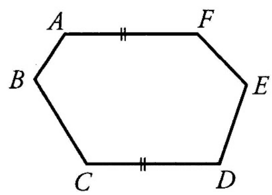


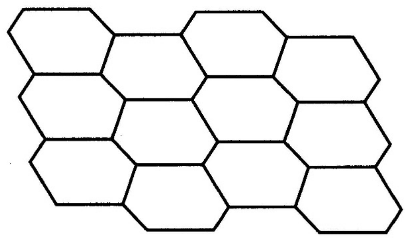
convex nonconvex

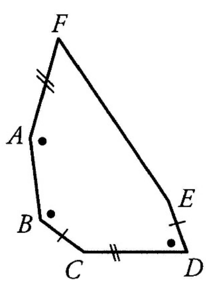
As shown earlier, some convex pentagons and some convex hexagons tile the plane. The situation for convex hexagons is completely understood—a convex polygon $ABCDEF$ will tile the plane if and only if it satisfies one of the following three criteria:

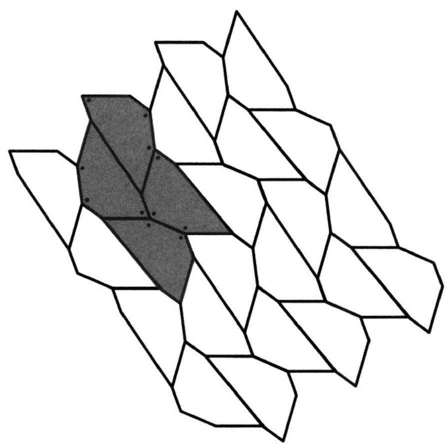
1. $\angle A + \angle B + \angle C = 360°$ and $AF = CD$.
2. $\angle A + \angle B + \angle D = 360°$, $AF = CD$, and $BC = DE$.
3. $\angle A = \angle C = \angle E = 120°$, $AF = AB$, $BC = CD$, and $DE = EF$.

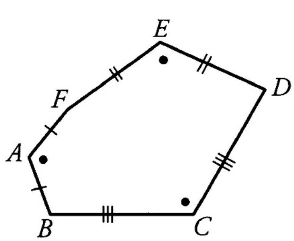
The figures below give an example of each type.

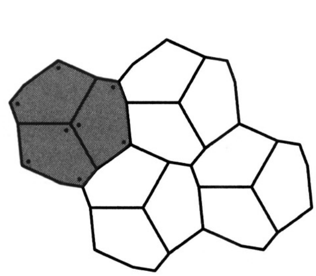












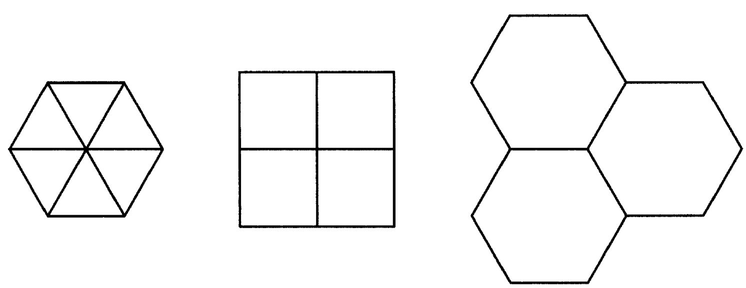
However, it is still not known which convex pentagons tile the plane. At the time of writing this text, we know that 15 different types of convex pentagons tile the plane. This problem has a fascinating history, and you can find much more information about the state of affairs of this problem on the web or in the following articles:

"Tiling with Convex Polygons," Chapter 13 of *Time Travels and Other Mathematical Bewilderments* (Martin Gardner; W. H. Freeman and Company, New York, 1988).

"In Praise of Amateurs," Doris Schattschneider, in *The Mathematical Gardner* (David A. Klarner, Editor; Prindle, Weber, and Schmidt, Boston, 1981).

12.3 Tiling with Regular Polygons

In this section, we will investigate tessellations or tilings with regular polygons whose sides are of unit length and such that two neighboring tiles share a complete edge. Such tessellations are said to be ***edge-to-edge tilings***. The diagram below shows three familiar examples of monohedral tessellations of this type.



Recall that in order to form a tessellation, the tiles must be able to completely surround a vertex without overlapping.

The sequence of regular polygonal tiles that surround a point generates a ***vertex sequence of the point*** which gives the number of sides of each tile.

For example, in the first tessellation above, the vertex sequence of each vertex is (3,3,3,3,3,3). In the second and third tessellations, the vertex sequences are (4,4,4,4) and (6,6,6), respectively.

We will show that there are other combinations of regular polygonal tiles that can surround a point. In order to do this, we need to find a set of angles whose total measure is 360° and such that the measure of each angle is the measure of a vertex angle of an n-sided regular polygon, that is,

$$\frac{n-2}{n} \cdot 180^\circ,$$

where n is an integer greater than or equal to 3.

Combinations of Three Polygons

We first consider combinations with three polygons. Let the vertex sequence be (a, b, c) with $a \leq b \leq c$. Then

$$\frac{a-2}{a} \cdot 180^\circ + \frac{b-2}{b} \cdot 180^\circ + \frac{c-2}{c} \cdot 180^\circ = 360^\circ,$$

which simplifies to

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}.$$

Case (i). $a = 3$.

In this case, the equation above becomes

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{6}.$$

Solving for c in terms of b, we have

$$\frac{1}{c} = \frac{b-6}{6b}$$

so that

$$c = \frac{6b}{b-6},$$

and this implies that $b > 6$.

For $b = 7$, we have $c = 42$; for $b = 8$, we have $c = 24$; for $b = 9$, $c = 18$; for $b = 10$, $c = 15$; for $b = 11$, c is not an integer; and for $b = 12$, $c = 12$.

Since $b \leq c$, the only solutions in this case are $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, and $(3, 12, 12)$.

Note. Vertex sequences that are not monohedral can have equivalent forms. For instance, the vertex sequence (3,7,42) has five other equivalent forms, namely,

$$(3,42,7), \quad (7,3,42), \quad (7,42,3), \quad (42,3,7), \quad \text{and} \quad (42,7,3).$$

They are listed in lexicographical order, that is, a vertex sequence with a smaller first term comes before one with a larger first term. Among those with equal first terms, a vertex sequence with a smaller second term comes before one with a larger second term, and so on.

Case (ii). $a = 4$.

In this case, the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$$

becomes

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{4}.$$

Solving for c in terms of b, we have

$$\frac{1}{c} = \frac{b-4}{4b}$$

so that

$$c = \frac{4b}{b-4},$$

and this implies that $b > 4$.

For $b = 5$, $c = 20$; for $b = 6$, $c = 12$; for $b = 7$, c is not an integer; and for $b = 8$, $c = 8$.

Since $b \le c$, the only solutions in this case are

$$(4,5,20), \quad (4,6,12), \quad \text{and} \quad (4,8,8).$$

Case (iii). $a = 5$.

In this case, the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$$

becomes

$$\frac{1}{b} + \frac{1}{c} = \frac{3}{10}.$$

Solving for c in terms of b, we have

$$\frac{1}{c} = \frac{3b-10}{10b}$$

so that

$$c = \frac{10b}{3b - 10},$$

and we have $b \geq a = 5$.

For $b = 5$, $c = 10$; for $b = 6$, 7, c is not an integer; and for $b = 8$, we have $b > c$. Thus, the only solution in this case is $(5, 5, 10)$.

Case (iv). $a = 6$.

In this case, if $c \geq 7$, then

$$\frac{1}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{2}{6} + \frac{1}{7} = \frac{20}{42},$$

which is a contradiction. Hence,

$$a = b = c = 6,$$

and the only solution in this case is (6,6,6).

Case (v). $a \geq 7$.

In this case,

$$\frac{1}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{3}{7},$$

which is a contradiction. Hence, there are no further solutions.

Note. In the case (3,7,42), neither the regular 7-gon nor the regular 42-gon has angles with integral measures. Thus, this combination is not easy to discover just by inspection.

Combinations of Four Polygons

Next we consider combinations with four polygons. Let the vertex sequence be a permutation of (a, b, c, d) with $a \leq b \leq c \leq d$. As before, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1.$$

Case (i). $a = 3$.

In this case, the equation above becomes

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{2}{3}.$$

If $b = 3$, we have

$$\frac{1}{c} + \frac{1}{d} = \frac{1}{3},$$

and solving for d in terms of c, we have

$$\frac{1}{d} = \frac{c-3}{3c}$$

so that

$$d = \frac{3c}{c-3},$$

and therefore $c > 3$.

For $c = 4$, $d = 12$; for $c = 5$, d is not an integer; and for $c = 6$, $d = 6$.

Since $c \leq d$, the only solutions for $b = 3$ are (3,3,4,12) and (3,3,6,6).

If $b = 4$, we have

$$\frac{1}{c} + \frac{1}{d} = \frac{5}{12},$$

and solving for d in terms of c, we have

$$\frac{1}{d} = \frac{5c-12}{12c}$$

so that

$$d = \frac{12c}{5c-12},$$

and therefore $c \geq b = 4$.

For $c = 4$, $d = 6$; for $c = 5$, we already have $c > d$. Thus, the only solution when $b = 4$ is $(3, 4, 4, 6)$.

For $b \geq 5$,

$$\frac{2}{3} = \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{3}{5},$$

which is a contradiction. Hence, there are no further solutions in this case.

Case (ii). $a = 4$.

In this case, for $d \geq 5$,

$$1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{3}{4} + \frac{1}{5} = \frac{19}{20},$$

which is a contradiction. Hence $a = b = c = d = 4$, and the only solution in this case is $(4, 4, 4, 4)$.

Case (iii). $a \geq 5$.

In this case,

$$1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{4}{5},$$

which is a contradiction. Hence there are no further solutions.

Combinations of Five Polygons

Now we consider combinations with five polygons. Let the vertex sequence be a permutation of (a, b, c, d, e) where $a \leq b \leq c \leq d \leq e$. As before, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = \frac{3}{2}.$$

For $c \geq 4$,

$$\frac{3}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \leq \frac{2}{3} + \frac{3}{4} = \frac{17}{12},$$

which is a contradiction.

Hence, $a = b = c = 3$, so that

$$\frac{1}{d} + \frac{1}{e} = \frac{1}{2}.$$

As before, we have

$$(d, e) = (3, 6) \qquad \text{or} \qquad (4, 4),$$

so that the only solutions in this case are $(3, 3, 3, 3, 6)$ and $(3, 3, 3, 4, 4)$.

Combinations of Six Polygons

Finally, we consider combinations with six polygons. Let the vertex sequence be a permutation of (a, b, c, d, e, f) where $a \leq b \leq c \leq d \leq e \leq f$. As before, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} = 2.$$

For $f \geq 4$,

$$2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} = \frac{5}{3} + \frac{1}{4} = \frac{23}{12},$$

which is a contradiction. Hence, $a = b = c = d = e = f = 3$, and the only solution in this case is (3,3,3,3,3,3).

Note. We cannot surround a point with seven or more regular polygons since the smallest of the angles at this point is at least $60°$, and the sum of these angles will exceed $6 \times 60° = 360°$.

Several of our solutions give rise to more than one vertex sequence. The combination (3,3,4,12) may be permuted as (3,4,3,12). The combination (3,3,6,6) may be permuted as (3,6,3,6). The combination (3,4,4,6) may be permuted as (3,4,6,4). Finally, the combination (3,3,3,4,4) may be permuted as (3,3,4,3,4). There are left-handed and right-handed versions of (3,3,3,3,6), but they are not considered to be different. This brings the total number of possible vertex sequences to 21.

12.4 Platonic and Archimedean Tilings

Now we consider tilings that are named after the Greek philosophers Plato and Archimedes.

For each vertex sequence, we wish to know if we can tile the entire plane with regular polygons such that every vertex has this vertex sequence. In this case, such a tessellation is said to be ***semiregular***.

Moreover, if all the terms in the vertex sequences are identical, the tessellation is said to be ***regular***. The regular tessellations of the plane are called ***Platonic*** tilings.

Semiregular tessellations that are not regular are called ***Archimedean*** tilings.

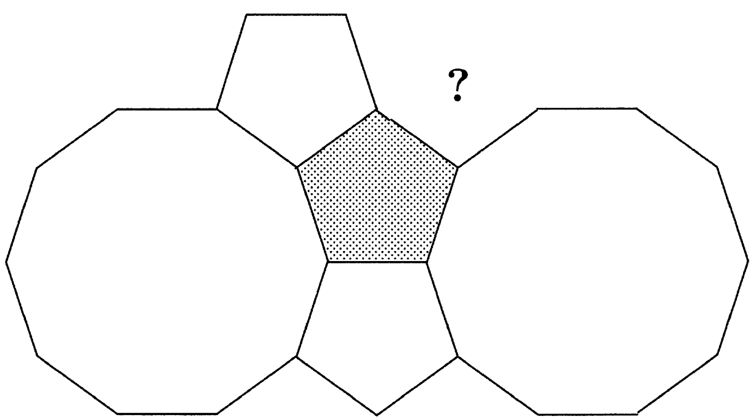
We divide the 21 possible vertex sequences into three groups.

• Group I.

$$(3,7,42), \quad (3,8,24), \quad (3,9,18), \quad (3,10,15), \quad (4,5,20), \quad (5,5,10).$$

These vertex sequences are all of the form (a, b, c) where a is odd and $b \neq c$ if we read (5,4,20) for (4,5,20).

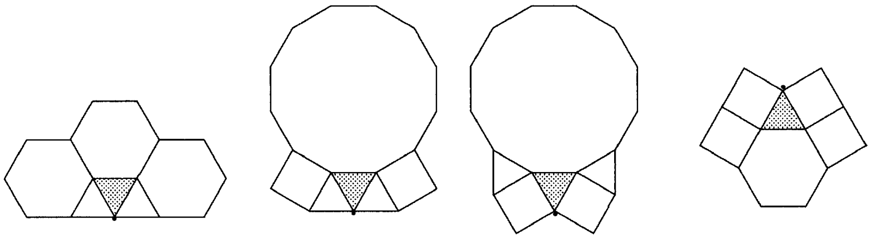
If there is a tessellation in which every vertex has this vertex sequence, we must be able to surround the a-sided polygon. Thus, its neighbours must be the b-sided polygon and the c-sided polygon, alternately. However, this is impossible since a is odd. The figure below illustrates the case (5,5,10). Moreover, no combination of regular polygons can fill the void.



• Group II.

$$(3,3,4,12), \quad (3,3,6,6), \quad (3,4,3,12), \quad (3,4,4,6).$$

Each of these vertex sequences contains a triangular tile. We can surround two of its vertices properly, but the third vertex, indicated with a black dot in the diagram on the following page, requires a different vertex sequence.



• Group III.

$$\begin{aligned}&(3,12,12),\quad (4,6,12),\quad (4,8,8),\quad (6,6,6),\\&(3,4,6,4),\quad (3,6,3,6),\quad (4,4,4,4),\\&(3,3,3,3,6),\quad (3,3,3,4,4),\quad (3,3,4,3,4),\\&(3,3,3,3,3,3).\end{aligned}$$

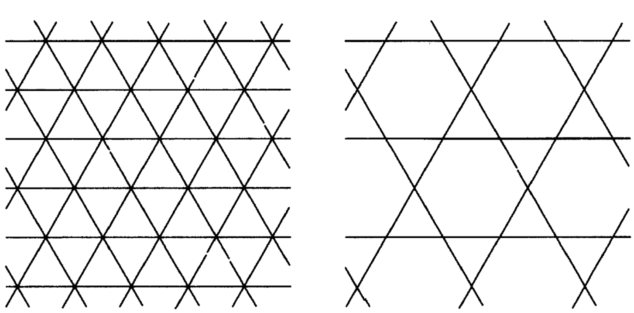
Here we have no local problem. It turns out that each of these vertex sequences leads to a semiregular tessellation. There are exactly three Platonic tessellations, namely,

$$(3,3,3,3,3,3),\quad (4,4,4,4),\quad \text{and}\quad (6,6,6),$$

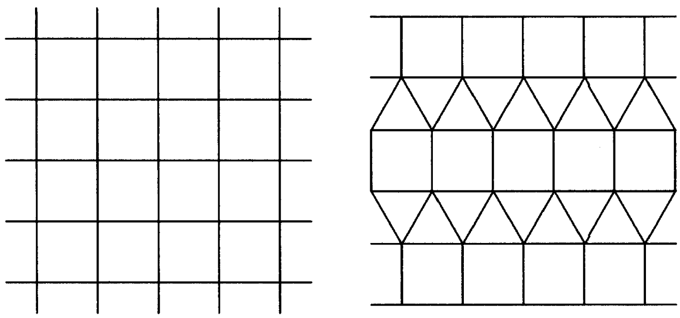
which we saw at the beginning of Section 12.3.

The other eight are the only Archimedean tilings, and they will be discussed a later. However, we still have to prove that we have no global problem with any of these 11 vertex sequences. We shall use a direct approach and construct each of the 11 tessellations.

(a) The (3,3,3,3,3,3) and the (3,6,3,6) tessellations may be constructed with three infinite families of evenly spaced parallel lines forming 60° angles across the families, as shown in the diagram below.



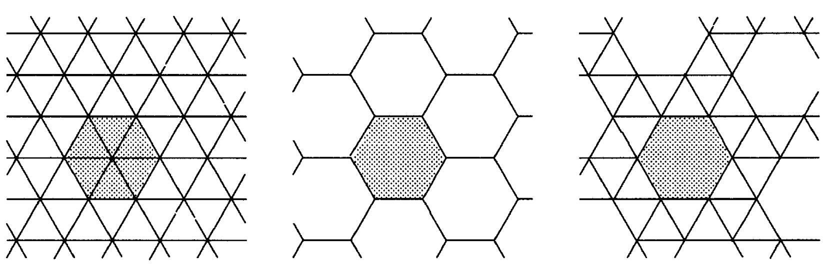
(b) The (4,4,4,4) tessellation may be constructed with two infinite familes of evenly spaced parallel lines forming 90° angles across families, as shown in the figure below on the left. This tessellation and the (3,3,3,3,3,3) and (3,6,3,6) tessellations are the three *basic* tessellations.



(c) The (3,3,3,4,4) tessellation is obtained by taking alternate strips from the basic (3,3,3,3,3,3) and (4,4,4,4,4) tessellations, as in the figure above on the right.

The remaining tessellations are obtained from others by the *cut-and-merge* method.

(d) From the basic (3,3,3,3,3,3) tessellation, we can merge a set of six equilateral triangles into a regular hexagon, as in the figure below on the left.



(e) The (6,6,6) and the (3,3,3,3,6) tessellations may be obtained from the basic (3,3,3,3,3,3) tessellations by merging various sets of six equilateral triangles, as shown in the figure above in the middle and on the right. No cutting is required in either case.

(f) The (3,4,6,4) tessellation can also be obtained from the basic (3,3,3,3,3,3) tessellation. As shown in the figure below on the left, we cut each triangular tile into seven pieces consisting of an equilateral triangle, three congruent half-squares, and three congruent kites with angles $120°$, $90°$, $60°$, and $90°$.

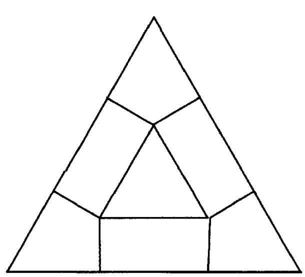
Let the edge length of the triangular tile be 1 and the length of the side of the equilateral triangle be x. Then the short sides of the kite have length $x/2$ and the long sides $\sqrt{3}x/2$. Since

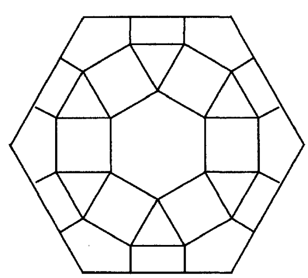
$$\frac{\sqrt{3}x}{2} + x + \frac{\sqrt{3}x}{2} = 1,$$

we have

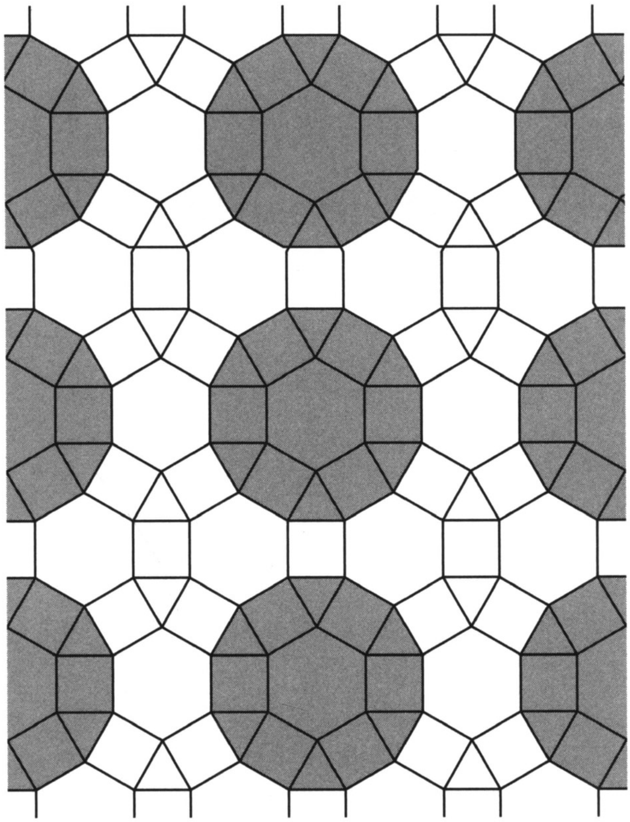
$$x = \frac{\sqrt{3} - 1}{2} \approx 0.366.$$

When we merge the kites and half-squares across six triangular tiles, we obtain the $(3, 4, 6, 4)$ tessellations, as shown in the figure below on the right.





(g) The (4,6,12) tessellation can now be obtained from the (3,4,6,4) tessellation without cutting. Each dodecagon in the new tessellation is obtained by merging one regular hexagon, six squares and six equilateral triangles in the old tessellation, as shown in the figure on the following page.



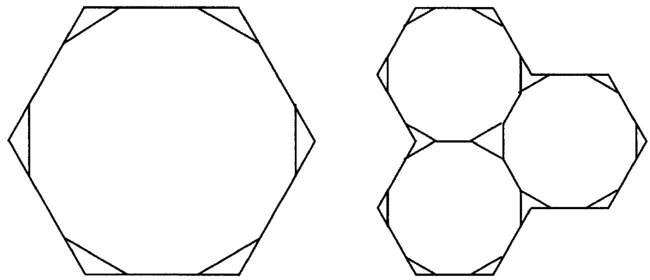
(h) The (3,12,12) tessellation may be constructed from the (6,6,6) tessellation. We cut each hexagonal tile into seven pieces consisting of a regular dodecagon and six congruent isosceles triangles with vertical angles $120°$, as in the figure on the following page on the left. Let the edge length of the hexagonal tile be 1 and the length of the base of the triangles be x. The equal sides of the triangles have length $x/\sqrt{3}$, and since

$$\frac{x}{\sqrt{3}} + x + \frac{x}{\sqrt{3}} = 1,$$

we have

$$x = 2\sqrt{3} - 3 \approx 0.464.$$

When we merge the triangles across three hexagonal tiles, we obtain the (3,12,12) tessellation, as in the figure on the following page on the right.



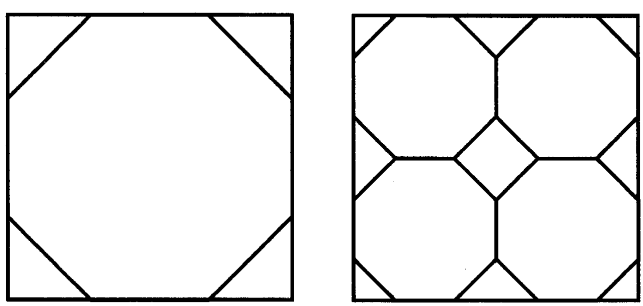
(i) Next, we construct the (4,8,8) tessellation from the basic (4,4,4,4) tessellation. We cut each square tile into five pieces consisting of a regular octagon and four congruent right isosceles triangles, as in the figure below on the left. Let the edge length of the square tile be 1 and the length of the hypotenuse of the triangles be x. Then the legs of the triangles have length $x/\sqrt{2}$. Since

$$\frac{x}{\sqrt{2}} + x + \frac{x}{\sqrt{2}} = 1,$$

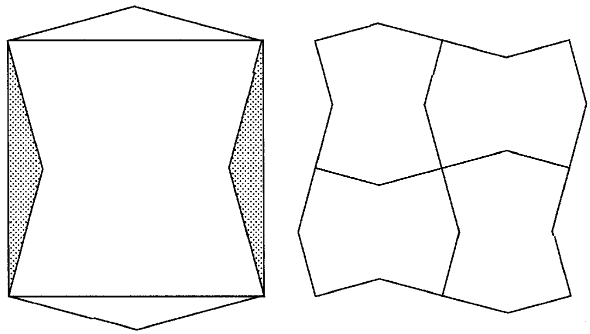
we have

$$x = \sqrt{2} - 1 \approx 0.412.$$

When we merge the triangles across four square tiles, we obtain the (4,8,8) tessellation, as in the figure below on the right.

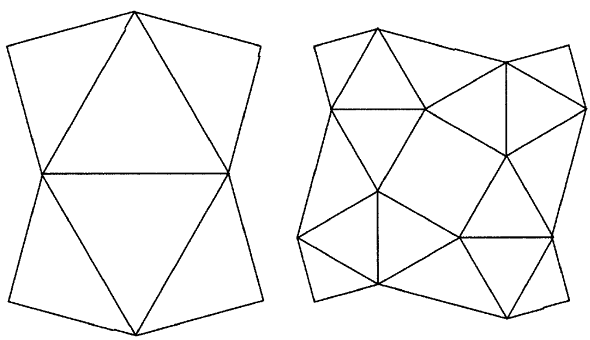


(j) The last tessellation, (3,3,4,3,4), is the most difficult to construct. It is obtained from the basic (4,4,4,4) tessellation with an intermediate step. We first modify the square tile as in the figure on the following page on the left. We cut out two isosceles triangles with vertical angles 150°, based on two opposite sides of the square, and attach them to the other two sides. This modified tile can also tile the plane, as in the figure on the following page on the right.



We now cut each modified tile into six pieces consisting of two congruent equilateral triangles and four congruent right isosceles triangles, as in the figure below on the left,

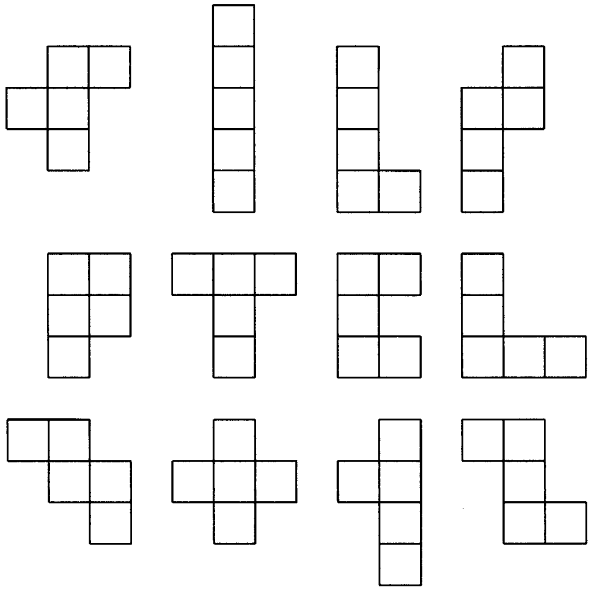
When we merge the right isosceles triangles across four modified tiles, we obtain the (3,3,4,3,4) tessellations, as in the figure below on the right.



The eight Archimedean tilings are graphically illustrated on page 222 in the book *Sphere Packing, Lewis Carroll and Reversi* by Martin Gardner, published in 2009 by the Mathematical Association of America, Washington, DC.

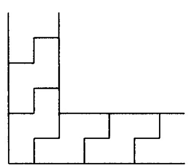
12.5 Problems

1. There are 12 ways in which five unit squares can be joined edge to edge. The resulting figures are called ***pentominoes***, as shown in the figure on the following page.



They are given letter names, F, I, L, N, P, T, U, V, W, X, Y, and Z, respectively. Identify those that can tile a rectangle.

2. The figure below shows the tiling of a bent strip by a figure consisting of four unit squares joined edge to edge.



Obviously, a pentomino that can tile a rectangle can also tile some bent strip. Identify those that can tile a bent strip but not a rectangle.

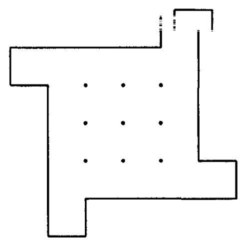
3. Prove that a pentomino that can tile a bent strip can also tile some infinite strip.

4. Identify those pentominoes that can tile an infinite strip but cannot tile a bent strip.

5. Obviously, a pentomino that can tile an infinite strip can also tile the plane. Identify those that can tile the plane but cannot tile an infinite strip.

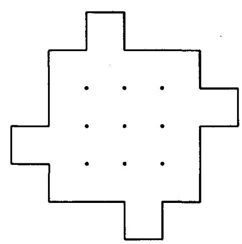
6. For each of the I-pentomino, L-pentomino, N-pentomino, U-pentomino, V-pentomino, and W-pentomino, dissect it into four pieces and reassemble them into a square.

7. (a) Show that the figure below can tile the plane.



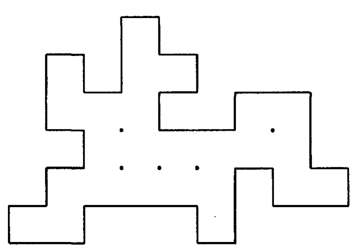
(b) For each of the P-pentomino, Y-pentomino, and Z-pentomino, dissect it into three pieces and reassemble them into a square.

8. (a) Show that the figure below can tile the plane.

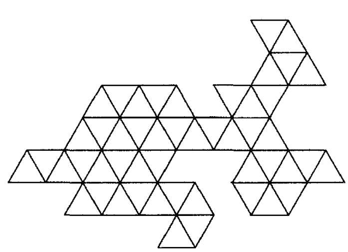


(b) For each of the F-pentomino, T-pentomino, and X-pentomino, dissect it into four pieces and reassemble them into a square.

9. Dissect the figure below into three pieces and reassemble them into a square.



10. Dissect the figure below into three pieces and reassemble them into an equilateral triangle.



11. (a) Prove that the sum of measures of the exterior angles of a regular n-gon is $360°$ for any $n \geq 3$.

    (b) Use part (a) to prove that the sum of the measures of the central angles of a regular n-gon is given by $(n - 2)180°$.

12. For which values of n is the measure of the central angle of an n-sided regular polygon a positive integer?

13. Find three ways of obtaining the basic (3,6,3,6) tessellation from other tessellations.

14. Find another way of obtaining the (6,6,6) tessellation from the basic (3,3,3,3,3,3) tessellation.

15. Find a way of obtaining the (3,4,6,4) tessellation from the (6,6,6) tessellation.

16. Find a tessellation that has exactly two kinds of vertex sequences, one of which is (3,3,4,12).

17. Find a tessellation that has exactly two kinds of vertex sequences, one of which is (3,4,3,12).

18. Find three tessellations that have exactly two kinds of vertex sequences, one of which is (3,3,6,6).

19. Find three tessellations that have exactly two kinds of vertex sequences, one of which is (3,4,4,6).

PART III

INVERSIVE AND PROJECTIVE GEOMETRIES

CHAPTER 13

INTRODUCTION TO INVERSIVE GEOMETRY

13.1 Inversion in the Euclidean Plane

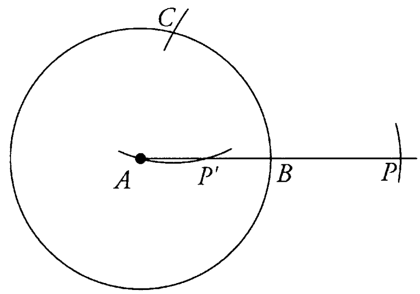
We introduce the concept of inversion with a simple example, that of constructing the midpoint of a line segment using only a compass.

Example 13.1.1. *Given a line through A and B, find the midpoint of the segment AB using only a compass.*

Solution. With center A and radius $r = AB$, draw the circle $\mathcal{C}(A, r)$ and locate the point P on the line AB so that B is the midpoint of AP.

With center P, draw the circle $\mathcal{C}(P, AP)$ intersecting the first circle at C, as in the figure on the following page.

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky



Finally, draw $\mathcal{C}(C, r)$ intersecting the line AB at P'. Then P' is the midpoint of AB.

To see that this is the case, note that the triangles $AP'C$ and ACP are similar isosceles triangles since they share a vertex angle at A, so that

$$\frac{AP'}{AC} = \frac{AC}{AP},$$

which implies that

$$\frac{AP'}{r} = \frac{r}{2r},$$

and this implies that

$$AP' = \frac{r}{2} = \frac{AB}{2}.$$

□

Note that with

$$AP = 2r \qquad \text{and} \qquad AP' = \tfrac{1}{2}r,$$

we have $AP \cdot AP' = r^2$. This relationship between P and P' is called an *inversion*. More generally, we have the following definition:

Definition. (*Inverse of a Point*)

Given a circle $\mathcal{C}(O, r)$ and a point P other than O, the point P' on the ray $\overrightarrow{OP}$ is the ***inverse*** of P if and only if $OP \cdot OP' = r^2$.

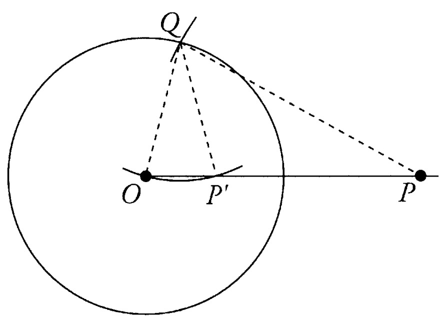
The circle $\mathcal{C}(O, r)$ is called the ***circle of inversion***, the point O is called the ***center of the inversion***, r is called the ***radius of inversion***, and r^2 is called the ***power of the inversion***.

Remark. Suppose P is a point other than the center of inversion. If P is outside the circle of inversion, then its inverse P' is interior to the circle of inversion. If P is on the circle, then it is its own inverse. If P is inside the circle, then its inverse P' is exterior to the circle of inversion.

Compass Method of Finding the Inverse

Note that given the ray $\overrightarrow{OP}$ and the circle $\mathcal{C}(O, r)$, the compass-only construction described above also works to find the inverse of P when P is outside the circle of inversion.

With center P and radius OP, draw an arc intersecting $\mathcal{C}(O, r)$ at Q. With center Q and radius OQ, draw an arc intersecting OP at P'. Then P' is the inverse of P.



To see this, note that the isosceles triangles OQP and $OP'Q$ are similar by the **AA** similarity condition so that

$$\frac{OP}{OQ} = \frac{OQ}{OP'},$$

and therefore

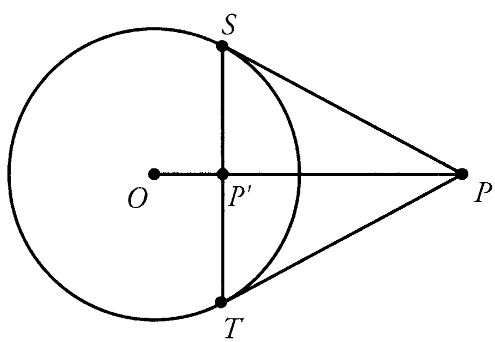
$$OP \cdot OP' = OQ^2 = r^2.$$

The Tangent Method of Finding the Inverse

Another construction for finding the inverse using a compass and straightedge is as follows (several construction lines have been omitted).

Here we are given the circle $C(O, r)$ and the point P outside the circle.

Draw the segment OP, and construct the tangents PS and PT to the circle with S and T being the points of tangency, as in the figure below.



Let $P' = ST \cap OP$. Then P' is the inverse of P.

We leave the proof as an exercise.

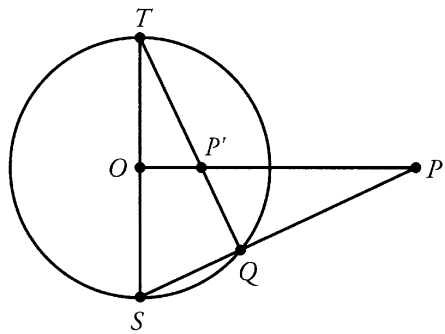
Note that an easy modification works to find P' when P is inside the circle: draw the line through P perpendicular to OP intersecting the circle at S and T. Then draw the tangents at S and T meeting at P'.

The Perpendicular Diameter Method of Finding the Inverse

Another method that works when P is inside or outside the circle, as in the figure on the following page, is as follows:

1. Draw ST, the diameter perpendicular to OP.
2. Let Q be the point where the line SP meets the circle.
3. Let P' be the point where the line TQ meets OP.

Then P' is the inverse of P.



The proof is left as an exercise.

The Inversive Plane

Given a circle C, every point P in the plane has an inverse with respect to C *except* the center of the circle O. The point O has no inverse and is not itself the inverse of any point. As far as inversion is concerned, the point O may as well not exist.

In order to overcome this omission, we append a single ***point at infinity*** I to the plane so that the inversion maps O to I and vice-versa. The point I is also called the ***ideal point***, and it is considered to be on every line in the plane. The Euclidean plane together with this single ideal point is called the ***inversive plane***.

Note. There is only *one* ideal point in the inversive plane, in contrast to the extended Euclidean plane discussed in the earlier chapters, which has infinitely many ideal points that make up the ideal line.

When we want to exclude the ideal point from the discussion of the inversive plane, we refer to the nonideal points as ***ordinary points***.

In the inversive plane, all lines pass through the ideal point. Two lines that meet at a single point in the Euclidean plane meet at two points in the inversive plane. Two lines that are parallel in the Euclidean plane meet only at the ideal point in the inversive plane.

Technically speaking, there are no parallel lines in the inversive plane, although we continue to use the term "parallel lines" to mean that the lines meet only at the ideal point. As in the Euclidean plane, lines that coincide are also said to be parallel.

The following facts are immediate consequences of the definition of inversion in $\mathcal{C}(O, r)$:

Theorem 13.1.2.

(1) The point P' is the inverse of the point P if and only if P is the inverse of P'.

(2) If $OP = kr$, then $OP' = \frac{1}{k}r$.

(3) The inversion maps every point outside the circle to some point inside the circle and vice-versa.

(4) Each point on the circle of inversion is mapped onto itself.

Example 13.1.3. *Suppose that P and Q are points on the ray $\overrightarrow{OP}$. Let P' and Q' be the respective inverses. Show that if $OQ = k \cdot OP$, then $OP' = k \cdot OQ'$.*

Solution. Let r be the radius of inversion. Then

$$OP \cdot OP' = r^2 \qquad \text{and} \qquad OQ \cdot OQ' = r^2.$$

Multiplying both sides of the equation

$$OQ = k \cdot OP$$

by

$$OP' \cdot OQ',$$

we get

$$OP' \cdot OQ' \cdot OQ = OP' \cdot OQ' \cdot k \cdot OP,$$

which implies that

$$OP' \cdot r^2 = OQ' \cdot k \cdot r^2.$$

Thus,

$$OP' = k \cdot OQ'.$$

□

13.2 The Effect of Inversion on Euclidean Properties

A ***Euclidean property*** is one that is preserved by the *Euclidean transformations* (translations, rotations, reflections, and combinations thereof). Euclidean properties include distance, shape, and size. Inversion does *not* preserve Euclidean properties, but it does affect them in useful ways.

In this section, we denote the inversion in the circle $\mathcal{C}(O, r)$ by $I(O, r^2)$. Note that given a figure $\mathcal{G}$, its inverse $\mathcal{G}'$ is obtained by taking the inverse of each point of $\mathcal{G}$. Thus, it follows that $\mathcal{G}'$ is the inverse of $\mathcal{G}$ if and only if $\mathcal{G}$ is the inverse of $\mathcal{G}'$.

Lines and Circles

Before proving the result about how inversion affects lines and circles, we first prove a useful lemma.

Lemma 13.2.1. *Under the inversion $I(O, r^2)$, suppose that P and Q have inverses P' and Q', respectively. Then $\triangle OPQ \sim \triangle OQ'P'$.*

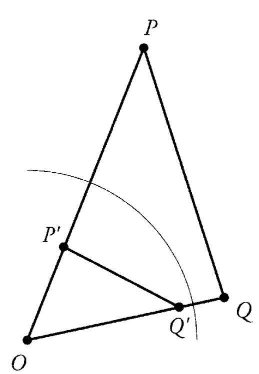
Proof. We have

$$OP \cdot OP' = r^2 = OQ \cdot OQ',$$

which implies that

$$\frac{OP}{OQ} = \frac{OQ'}{OP'}.$$

Since $\angle POQ \equiv \angle Q'OP'$, then $\triangle OPQ \sim \triangle OQ'P'$ by the **sAs** similarity criterion.



□

Remark. Note that in the similar triangles above we have

$$\angle OP'Q' = \angle OQP \qquad \text{and} \qquad \angle OQ'P' = \angle OPQ.$$

Note also that if we were discussing *signed* or *directed* angles, the direction of the angles would be reversed; that is,

$$m(\angle OP'Q') = -m(\angle OQP) \qquad \text{and} \qquad m(\angle OQ'P') = -m(\angle OPQ),$$

where $m(\angle ABC)$ denotes the measure of the angle $\angle ABC$, as defined in Part I, Section 1.1.

And now the theorem describing the effect of inversion on lines and circles.

Theorem 13.2.2. *The inversion operator $I(O, r^2)$ affects lines and circles as follows:*

(1) The inverse of a line through O is the same line through O.

(2) The inverse of a line not through O is a circle through O.

(3) The inverse of a circle through O is a line not through O.

(4) The inverse of a circle not through O is a circle not through O.

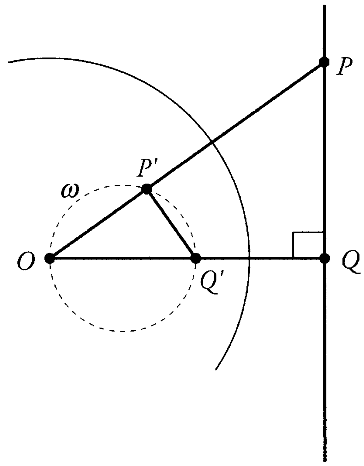
Proof.

(1) The first assertion is straightforward. However, note that a point of the line inverts to a different point of the line, except for the point where the line intersects the circle of inversion. The points I and O of the line are inverses of each other.

(2) Let Q be the foot of the perpendicular from O to the line, and let Q' be the inverse of Q. Let P be any other point on the line other than I, and let P' be the inverse of P, as in the figure on the following page. It follows from Lemma 13.2.1 that $\triangle OP'Q' \sim \triangle OQP$ and, therefore,

$$\angle OP'Q' = \angle OQP = 90^\circ.$$

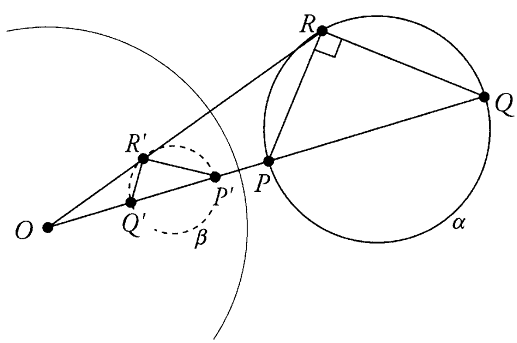
Thus, P' is on the circle ω with diameter OQ' by the converse to Thales' Theorem. In a similar way, every point X on ω is the inverse of some point X' on the line.



(3) This follows from the fact that since a circle through O is the inverse of a line not through O, then the line not through O is the inverse of a circle through O.

(4) Referring to the figure below, where α is the circle with diameter PQ to be inverted, Lemma 13.2.1 tells us that

$$\triangle OR'P' \sim \triangle OPR \qquad \text{and} \qquad \triangle OR'Q' \sim \triangle OQR.$$



Thus,

$$\angle OR'P' = \angle OPR \qquad \text{and} \qquad \angle OR'Q' = \angle OQR.$$

Now,

$$\angle Q'R'P' = \angle OR'P' - \angle OR'Q',$$

so that

$$\angle Q'R'P' = \angle OPR - \angle OQR.$$

From the External Angle Theorem, we have

$$\angle OPR - \angle OQR = \angle PRQ,$$

while from Thales' Theorem, we have

$$\angle PRQ = 90^\circ.$$

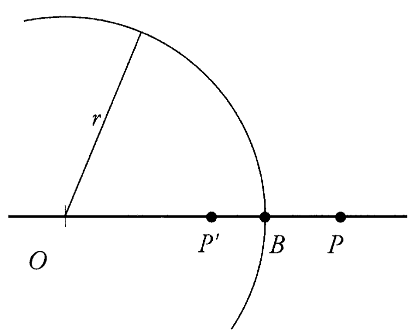
Thus, $\angle QR'P' = 90^\circ$, and from the converse to Thales' Theorem we can conclude that R' is on β, the circle with diameter $Q'P'$. Similarly, any point on the circle β is the inverse of some point on the circle α.

□

Inversion and Distances

Theorem 13.2.3. *Let P and P' be inverse points with P outside the circle of inversion, and let B be the point where PP' meets the circle of inversion, as in the figure below. Then,*

$$BP' = \frac{BP}{1 + BP/r} \quad \textit{and} \quad BP = \frac{BP'}{1 - BP'/r}.$$



Proof. We have

$$\begin{aligned} BP' &= r - OP' \\ &= r - \frac{OP' \cdot OP}{OP} \\ &= r - \frac{r^2}{r + BP} \\ &= \frac{r \cdot BP}{r + BP} \\ &= \frac{BP}{1 + BP/r}. \end{aligned}$$

That is,

$$BP' = \frac{BP}{1 + BP/r}.$$

The proof that $BP = \dfrac{BP'}{1 - BP'/r}$ is similar.

□

Theorem 13.2.4. *If P' and Q' are inverse points for P and Q, respectively, under the inversion $I(O, r^2)$, then*

$$P'Q' = \frac{PQ}{OP \cdot OQ} r^2 .$$

Proof. We consider three distinct cases.

Case (i). P, Q, and O are not collinear.

By Lemma 13.2.1, $\triangle OP'Q' \sim \triangle OQP$, so that

$$\frac{P'Q'}{PQ} = \frac{OQ'}{OP} ,$$

which implies that

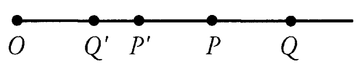
$$\begin{aligned} P'Q' &= \frac{PQ \cdot OQ'}{OP} \\ &= \frac{PQ \cdot OQ' \cdot OQ}{OP \cdot OQ} \\ &= \frac{PQ}{OP \cdot OQ} r^2. \end{aligned}$$

That is,

$$P'Q' = \frac{PQ}{OP \cdot OQ} r^2.$$

Case (ii). P, Q, and O are collinear, with O not between P and Q.

We may assume that $OP < OQ$, as in the figure below.



This implies that $OQ' < OP'$, since $OP \cdot OP' = OQ \cdot OQ'$, so that

$$\begin{aligned} P'Q' &= OP' - OQ' \\ &= \frac{r^2}{OP} - \frac{r^2}{OQ} \\ &= \frac{OQ - OP}{OP \cdot OQ} r^2 \\ &= \frac{PQ}{OP \cdot OQ} r^2. \end{aligned}$$

That is,

$$P'Q' = \frac{PQ}{OP \cdot OQ} r^2 .$$

Case (iii). P, Q, and O are collinear, with O between P and Q.

The proof is almost the same as Case (ii) and is left as an exercise.

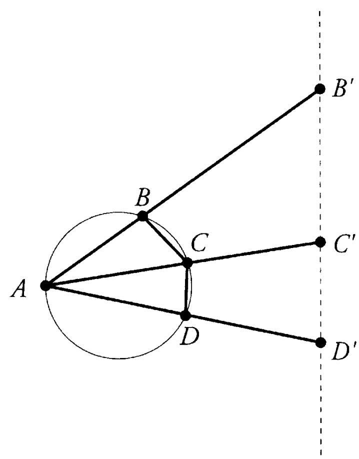
□

Theorem 13.2.5. *(Ptolemy's Theorem)*

If $ABCD$ is a convex cyclic quadrilateral, then

$$AC \cdot BD = BC \cdot AD + CD \cdot AB.$$

Proof. Consider the effect of $I(A, r^2)$ on the circumcircle of the cyclic quadrilateral. The circle inverts into a straight line, and the inverse points B', C', and D' are on this line, as shown in the figure below.



The convexity of $ABCD$ guarantees that C' is between B' and D', so that

$$B'D' = B'C' + C'D',$$

and thus,

$$\frac{BD}{AB \cdot AD} r^2 = \frac{BC}{AB \cdot AC} r^2 + \frac{CD}{AC \cdot AD} r^2,$$

from which the theorem follows.

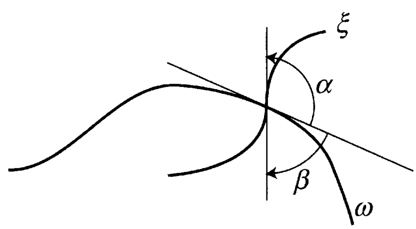
□

Inversion and Angles

The angle between two curves at a point of intersection P is defined as the angle between the tangents to these curves at P.

If one or both of the curves fail to have a tangent at P, then the angle is not defined. We will be dealing only with curves that do have tangents.

Given two curves ω and ξ, there are two different magnitudes that are associated with the angle from ω to ξ: one measured counterclockwise from ω to ξ, shown as angle α in the figure below, and the other measured clockwise, as shown by β. In general, when we refer to the angle ***from*** ω ***to*** ξ without mentioning the direction, we mean the counterclockwise one.



The main result concerning inversion and angles is:

Theorem 13.2.6. *Inversion preserves the magnitude of the angle between intersecting curves but reverses the direction.*

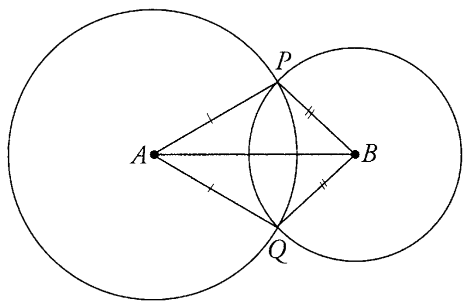
We omit the proof.

For us, the main consequence of Theorem 13.2.6 is that inversion preserves tangencies and orthogonality. For example, if two circles are tangent to each other, then their inverses are also tangent to each other. If two lines meet at $90°$, then their inverses, which may be circles, also meet at $90°$.

13.3 Orthogonal Circles

Theorem 13.3.1. *If two circles meet at P and Q, then the magnitude of the angles between the circles is the same at P and Q.*

Proof. Referring to the figure below, we have $\triangle APB \equiv \triangle AQB$ by the **SSS** congruency condition, so $\angle APB \equiv \angle AQB$. Since the tangents to the circles at P are perpendicular to the radii AP and BP, it follows that the angle between the tangents at P is equal in measure to $m(\angle APB)$. Likewise, the angle between the tangents at Q is equal in measure to $m(\angle AQB)$.



□

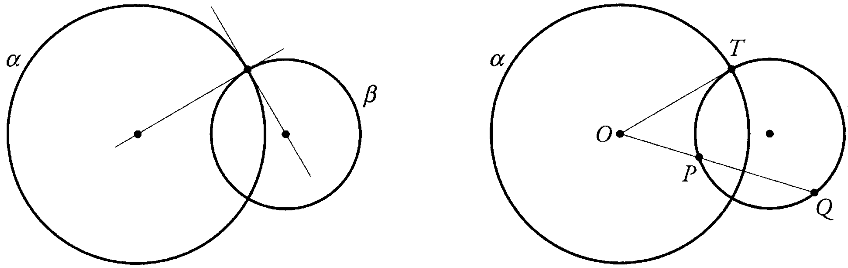
Remark. The previous theorem means that to determine the magnitude of the angle between two circles intersecting at P and Q, we only need to check one of the angles. Note, however, that the directions of the angles at P and Q are opposite.

Definition. Two intersecting circles α and β are said to be ***orthogonal*** if the angle between them is $90°$. We sometimes write $\alpha \perp \beta$ to indicate orthogonality.

Theorem 13.3.2. *If two circles α and β are orthogonal, then:*

(1) The tangents at each point of intersection pass through the centers of the other circle (figure (a) on the following page).

(2) Each circle is its own inverse with respect to the other.



(a) (b)

Proof.

(1) This follows because a line through the point of tangency perpendicular to the tangent must pass through the center of the circle.

(2) Let P be a point on the circle β. Join P to O, the center of α, and let r be the radius of α. Let Q be the other point where the ray $\overrightarrow{OP}$ meets β. Let T be the point of intersection of the two circles so that OT is the tangent to β by (1) above. By the power of the point O with respect to α, we have

$$OP \cdot OQ = OT^2 = r^2 ,$$

showing that the inverse of any point on β is another point on β.

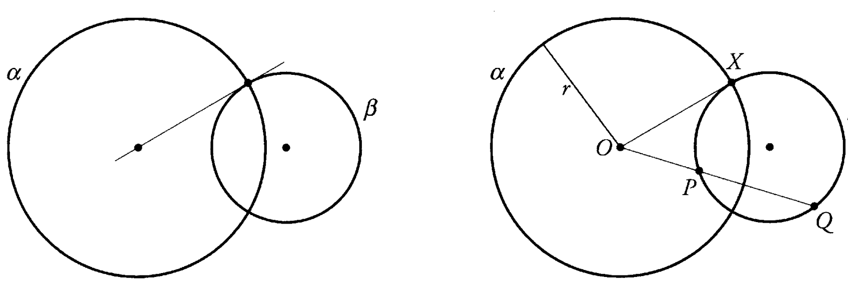
□

The preceding theorem has the following converse:

Theorem 13.3.3. *Two intersecting circles α and β are orthogonal if any one of the following statements is true.*

(1) The tangent to one circle at one point of intersection passes through the center of the other circle (figure (a) below).

(2) One of the circles passes through two distinct points that are inverses with respect to the other circle.



(a) (b)

Proof.

(1) This implies that the two tangents at the point of intersection must be perpendicular.

(2) Suppose that the circle β passes through P and Q, which are inverses with respect to α. Let O be the center of α and let OX be tangent to β at X, as in figure (b) on the previous page. Then we have

$$OP \cdot OQ = OX^2$$

by the power of O with respect to β, and

$$OP \cdot OQ = r^2,$$

since P and Q are inverses.

This implies that $OX = r$, so X must be on α as well as on β; that is, X is a point of intersection of α and β, and the tangent to β at this point passes through the center of α. By (1), the circles must be orthogonal.

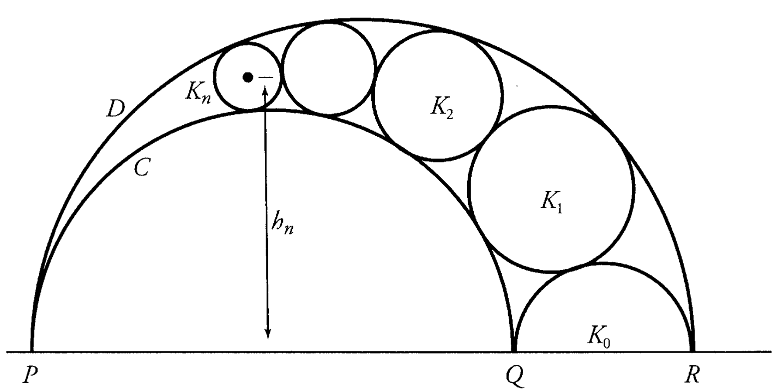
□

The Arbelos Theorem

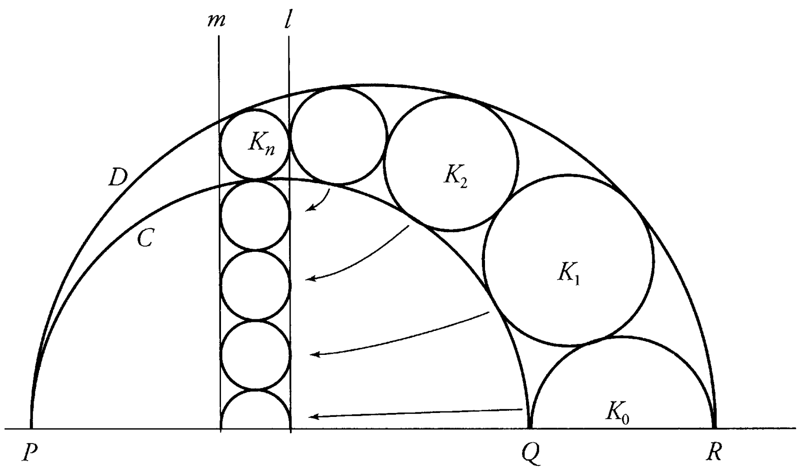
Theorem 13.3.4. *(The Arbelos Theorem, a.k.a. Pappus' Ancient Theorem)*

Suppose that P, Q, and R are three collinear points with C, D, and K_0 being semicircles on PQ, PR, and QR, respectively. Let K_1, K_2, ..., be circles touching C and D, with K_1 touching K_0, K_2 touching K_1, and so on.

Let h_n be the distance of the center of K_n from PR and let r_n be the radius of K_n. Then $h_n = 2nr_n$.



Proof. In the figure below, let t be the tangential distance from P to the circle K_n and apply $I(P, t^2)$.



K_n is orthogonal to the circle of inversion, so it is its own inverse.

C inverts into a line l.

D inverts into a line m parallel to l.

K_0 inverts into a semicircle K_0' tangent to l and m, since inversion preserves tangencies.

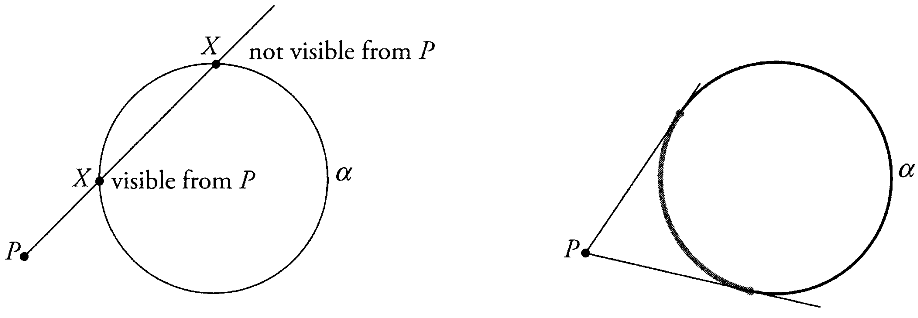
K_i inverts into a circle K_i' tangent to l and m.

Thus, all of the K_i's have the same radius, namely r_n, and the theorem follows.

□

Steiner's Porism

Given a point P outside a circle α, a point X of α is said to be ***visible*** from P if the segment PX meets α only at X.



(a) (b)

Figure (b) above shows the set of points that are visible from P, namely, the two tangent points and the points on the arc between the tangent points. In other words, a point X of α is visible from P if and only if

- either PX is tangent to α
- or the tangent to α at X has α and P on opposite sides.

Note also that if a line m is tangent to α at X, and if P is on the same side of m as α but not on the line m, then X is not visible from P.

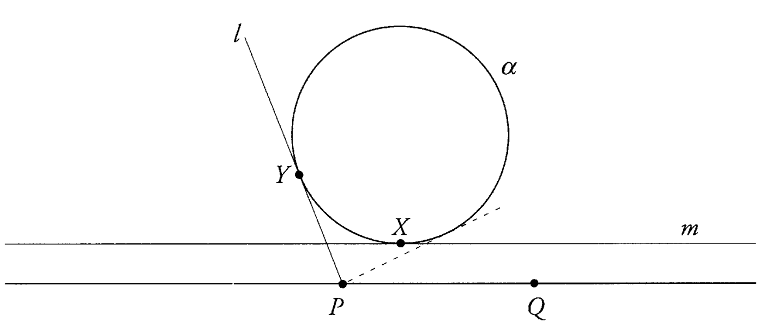
Lemma 13.3.5. *Suppose the line PQ misses the circle α. Then*

(1) there is a point X visible from both P and Q,

(2) there is a point Y visible from P but not from Q,

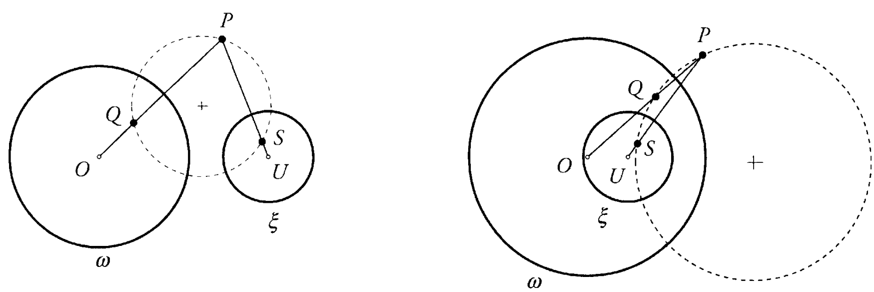
(3) there is a point Z visible from Q but not from P.

The figure below illustrates how to find points X and Y. Let m be a line parallel to PQ and tangent to α. X is the point of tangency of m with α. There are two lines from P tangent to α. Let l be the tangent line such that α and Q are both on the *same* side of l. Then Y is the point where l is tangent to α.



Lemma 13.3.6. *Given two circles ω and ξ and given a point P not on either circle, there is a circle through P orthogonal to both ω and ξ.*

Proof. Let Q be the inverse of P with respect to ω, and let S be the inverse of P with respect to ξ. Then there is a unique circle through P, Q, and S, and this circle must be orthogonal to both ω and ξ by Theorem 13.3.3.

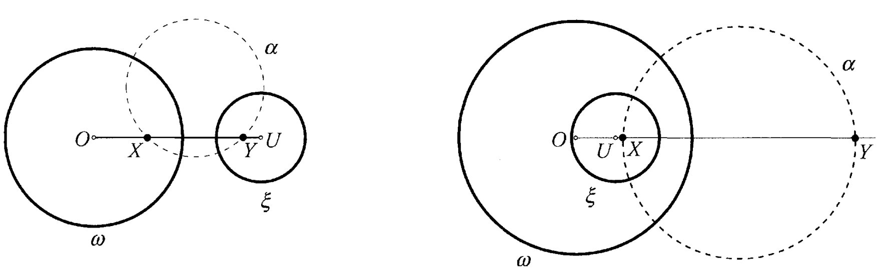


□

Note. If O, U, and P are collinear, then the orthogonal "circle" is a line. If O, U, and P are not collinear, then the orthogonal circle is a true circle.

Lemma 13.3.7. *Let ω and ξ be two nonintersecting circles with centers O and U, $O \neq U$. Then we can find points X and Y that are inverses to each other with respect to both ω and ξ.*

Proof. Let α be any circle other than a line that is orthogonal to both ω and ξ. We claim that the line OU intersects α in two points. In this case, the two points are X and Y and they are inverses to each other with respect to both ω and ξ.



case (i) case (ii)

There are two cases to consider: (i) when the circles are exterior to each other and (ii) when one circle is inside the other.

(i) Suppose for a contradiction that OU misses α. Then there is a point Z on α that is visible from both O and U. Since Z is visible from O, it is inside or on ω. Since Z is visible from U, it is inside or on ξ. Then Z is inside or on both ω and ξ, which contradicts the fact that ω and ξ are exterior to each other. This proves case (i).

(ii) The proof of this case is left as an exercise.

□

Definition. Let ω and ξ be two nonintersecting circles.

Let α_1 be a circle tangent to both ω and ξ.

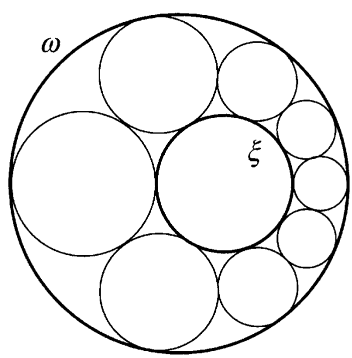
Let α_2 be a circle tangent to α_1, ω, and ξ.

Let α_3 be a circle tangent to α_2, ω, and ξ.

Continuing in this fashion, if at some point α_k is tangent to α_1, then we say that

$$\alpha_1, \quad \alpha_2, \quad \ldots, \quad \alpha_k$$

is a ***Steiner chain*** of k circles.

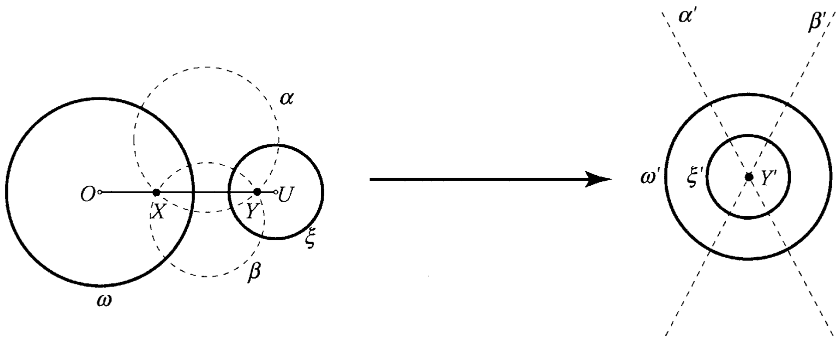


Remark. Given two circles ω and ξ, there is no guarantee that a Steiner chain exists for ω and ξ.

In order to prove the next theorem, we need the following lemma:

Lemma 13.3.8. *Given two nonintersecting circles ω and ξ that are not concentric, there is an inversion that transforms them into concentric circles.*

Proof. Using Lemma 13.3.6, we can find two circles α and β simultaneously orthogonal to both ω and ξ. These two circles intersect at the points X and Y referred to in Lemma 13.3.7.



Perform the inversion $I(X, r^2)$ for some radius r. Then:

- α transforms to α', a straight line through Y' and not through X.
- β transforms to β', a straight line through Y' and not through X.
- ω transforms to a circle ω' and

$$\omega' \perp \alpha' \qquad \text{and} \qquad \omega' \perp \beta'$$

  since orthogonality is preserved.
- ξ transforms to a circle ξ' and

$$\xi' \perp \alpha' \qquad \text{and} \qquad \xi' \perp \beta'.$$

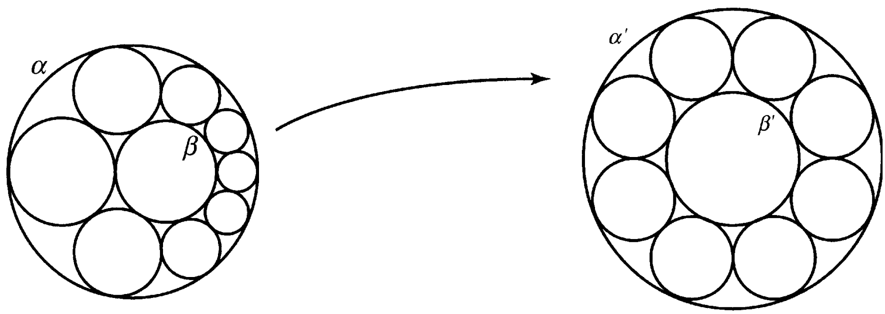
Since the circle ω' is orthogonal to the line α', then ω' must be centered at some point of α'. Similarly, ω' must be centered at some point of β'. Thus, ω' is centered at Y'. By the same argument, ξ' must also be centered at Y'.

$\square$

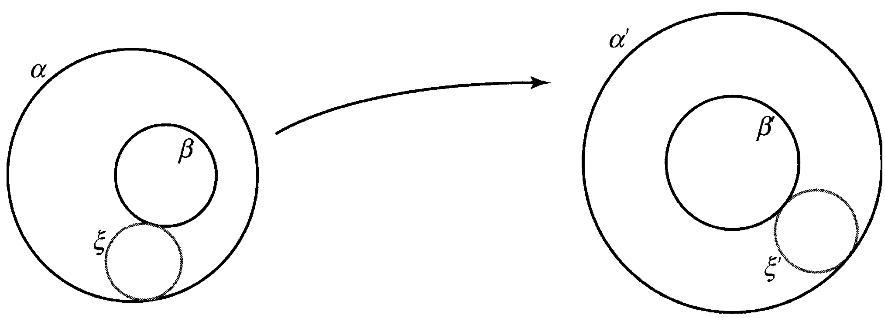
Now we are able to prove the following:

Theorem 13.3.9. *(Steiner's Porism)*
Suppose that two nonintersecting circles ω and ξ have a Steiner chain of k circles. Then any circle tangent to ω and ξ is a member of some Steiner chain of k circles.

Proof. In the figure below, we invert α and β into concentric circles. The inversion preserves the Steiner chain of k circles.



Using the same inversion, we transform ξ into a circle ξ', as in the figure below.



The circle ξ' is obviously part of a Steiner chain of k circles, so by the reverse inversion, ξ must also be part of a Steiner chain of k circles.

□

13.4 Compass-Only Constructions

We will use some special notation for this section only:

- $A(P)$ the circle with center A passing through the point P.
- $A(r)$ the circle with center A and radius r.
- $A(BC)$ the circle with center A and radius BC.

Note that when we say "draw $A(B)$," we will often draw only an arc of the circle rather than the entire circle. Also, *in this section only*, "construct" means "construct using only a compass."

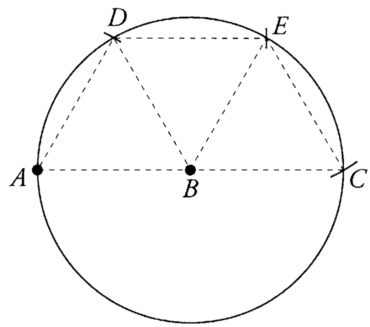
The objective of this section is to show that, insofar as constructing points is concerned, a compass alone is just as powerful as the combination of compass and straightedge. To put it another way, even if a construction involves drawing straight lines, we can carry out the construction in such a manner that we postpone drawing the straight lines until the very end. Furthermore, at the end we only use the straightedge to draw lines between existing points—we never need to use the straightedge to perform any new construction. This does not mean, however, that using a compass alone will be as efficient as a compass and straightedge together.

We begin with some examples that have been discussed previously.

Example 13.4.1. *Given points A and B, construct the point C such that B is the midpoint of AC.*

Solution. We perform the following constructions, as shown in the figure on the following page:

1. construct $B(A)$,
2. construct $A(B)$, yielding point D,
3. construct $D(AB)$, yielding point E,
4. construct $E(AB)$, yielding point C.



The justification is as follows. As in the figure, $\triangle ABD$, $\triangle DBE$, and $\triangle EBC$ are all equilateral triangles, so that $\angle ABD = \angle DBE = \angle EBC = 60°$. This means that ABC is a straight line, and so AC is a diameter of $B(A)$. Therefore, B is the midpoint of AC.

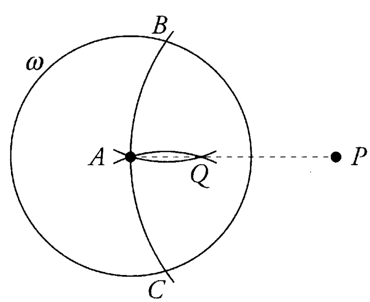
□

Example 13.4.2. *Given a point P outside a circle ω with center A, construct the inverse of P with respect to ω.*

Solution. We perform the following constructions, as in the figure below:

1. draw $P(A)$ meeting ω at B and C,
2. draw $B(A)$ and $C(A)$ meeting at Q.

Then Q is the inverse of P.

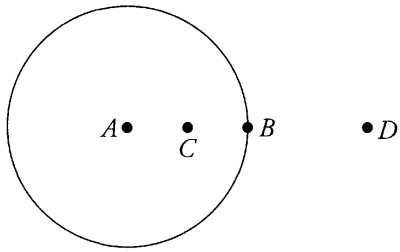


The justification is as follows. As in the figure, P' is on $B(A)$ and the line AP. Also, P' is on $C(A)$ and the line AP, so P' is the point $B(A) \cap C(A)$ other than A; that is, $P' = Q$.

□

Example 13.4.3. *Given points A and B, find the midpoint C of AB.*

Solution. Use Example 13.4.1 to construct the point D such that B is the midpoint of AD.



Now draw $A(B)$, and use Example 13.4.2 to find the inverse C of D with respect to $A(B)$. Then C is the midpoint of AB.

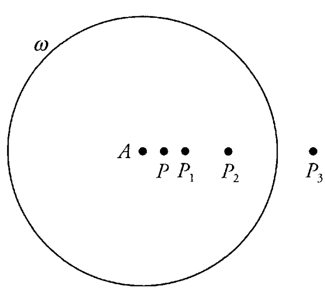
□

Example 13.4.4. *Given a circle ω with center A and radius r, and given a point P inside ω, construct the inverse of P with respect to ω.*

Solution. Repeatedly use Example 13.4.1 to construct points $P_1, P_2, \ldots, P_k$ so that P_k is outside ω, with

$$\begin{aligned} AP_1 &= 2AP, \\ AP_2 &= 2AP_1 = 4AP \\ AP_3 &= 2AP_2 = 8AP \\ &\vdots \\ AP_k &= 2AP_{k-1} = 2^k AP \end{aligned}$$

For example, with $k = 3$, we would have the following figure.



Use Example 13.4.2 to find the inverse S of P_k. Then $AS \cdot AP_k = r^2$.

Now use Example 13.4.1 to find points $S_1, S_2, \ldots, S_k$ so that

$$\begin{aligned} AS_1 &= 2AS, \\ AS_2 &= 2AS_1 = 4AS, \\ &\vdots \\ AS_k &= 2AS_{k-1} = 2^k AS. \end{aligned}$$

Then S_k is the inverse of P, since

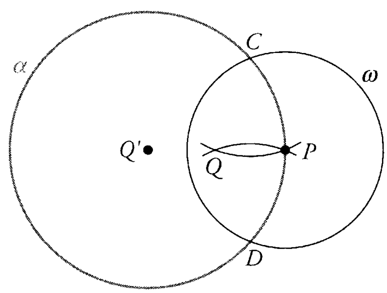
$$AS_k \cdot AP = 2^k AS \cdot AP = AS \cdot 2^k AP = AS \cdot AP_k = r^2.$$

□

The following example is a famous problem due to Mohr.

Example 13.4.5. *Given a circle α with unknown center A, construct its center.*

Solution. We perform the following constructions, as shown in the figure below.



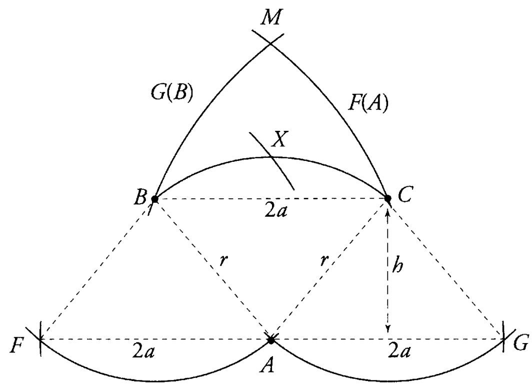
1. With any point P on α, construct a circle ω meeting α at C and D, with the radius of ω less than the radius of α.
2. Draw $C(P)$ and $D(P)$ meeting at Q.
3. Using Example 13.4.4, construct Q', the inverse of Q with respect to the circle ω. Then Q' is the center of α.

The justification is as follows. Let A be the center of α, and referring to the figure above, note that if we knew where A was, then Q would be the inverse of A by Example 13.4.2, so A must be Q'.

□

Example 13.4.6. *Given an arc BC with center A, construct the midpoint of the arc.*

Solution. It suffices to construct a point X of arc BC that is on the right bisector of chord BC, as in the analysis figure below.



Construction:

1. Construct $A(BC)$, $B(A)$, yielding F, the fourth point of parallelogram $ACBF$.
2. Construct $A(BC)$, $C(A)$, yielding G, the fourth point of parallelogram $ABCG$. Then A is collinear with F and G.
3. Construct $F(C)$ and $G(B)$, yielding M. Note that AM is the right bisector of both BC and FG.
4. Construct $F(AM)$ intersecting the arc BC at X. Then X is the desired point.

Justification:

(a) Let BC have length $2a$. Then $FA = GA = BC = 2a$.

(b) Let h be the perpendicular distance from BC to FG. Then from Pythagoras' Theorem, we have

$$\begin{aligned} FX^2 = AM^2 &= FM^2 - 4a^2 \\ &= FC^2 - 4a^2 \\ &= \left[(3a)^2 + h^2\right] - 4a^2 \\ &= 9a^2 + r^2 - a^2 - 4a^2 \\ &= 4a^2 + r^2 \\ &= FA^2 + AX^2. \end{aligned}$$

Since $FX^2 = FA^2 + AX^2$, by the converse to Pythagoras' Theorem, we conclude that $\triangle FCX$ is a right triangle; that is, X is on the right bisector of FG and BC.

$\square$

The Mohr-Mascheroni Theorem

A straightedge and compass construction allows only the following:

- Drawing a straight line through two different points.
- Drawing a circle centered at one point passing through another point.
- Drawing a circle centered at one point with a particular radius (typically specified by two other points).

All constructions are just a sequence of these basic operations. By using these operations we can construct new points and then use the new points to carry out more of the basic operations.

In the Euclidean plane, there are only three ways to construct new points:

(1) Construct a point as the intersection of two circles.

(2) Construct a point as the intersection of a line and a circle.

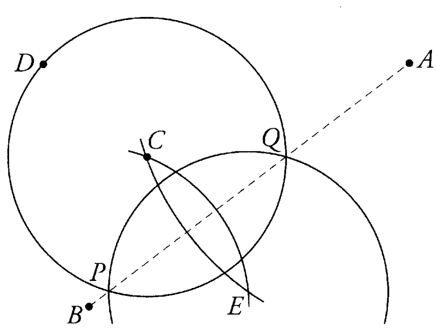
(3) Construct a point as the intersection of two lines.

The purpose of this section is to show that we can carry out all of these constructions with a compass alone. In a sense, this means that any construction we can perform with a straightedge and compass we can also perform with a compass alone. Thus, the objective here is to accomplish (2) and (3) using only a compass. We restate (2) as an example:

Example 13.4.7. *Given points A, B, C, and D, construct the intersection of $C(D)$ with the line AB.*

Solution. There are two cases to consider:

Case (i). A, B, C are not collinear. The analysis figure is shown below.



Construction:

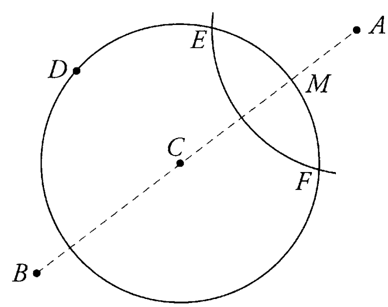
1. Draw $C(D)$.
2. Draw $A(C)$, $B(C)$ intersecting at E. Then $ACBE$ is a kite, and hence AB is the right bisector of CE.
3. Draw $E(CD)$ intersecting $C(D)$ at P and Q. Then P and Q are the desired points.

Justification:

(a) $EPCQ$ is a rhombus, so PQ is the right bisector of CE.

(b) P and Q are on AB; that is, P and Q are the points where the line AB meets the circle $C(D)$.

Case (ii). A, B, C are collinear. The analysis figure is shown below.



Construction:

1. Draw $C(D)$.
2. Draw $A(r)$ for some convenient r, intersecting $C(D)$ at E and F.
3. Use Example 13.4.6 to find the midpoint M of the arc EF. Then M is one of the desired points, and the other can be found in a similar way.

The justification is left as an exercise.

□

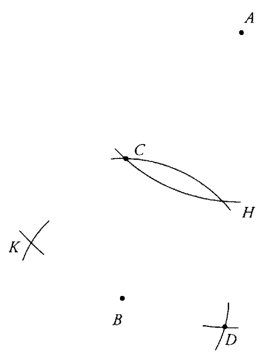
Accomplishing (3), that is, constructing the intersection of two lines, requires the construction of 12 different circles. We restate (3) as an example:

Example 13.4.8. *Given points A, B, C, and D, construct the intersection of the lines AB and CD.*

Solution. As in the figure below, we make the following constructions.

1. Draw $A(C)$, $B(C)$, yielding H. Then AB is the right bisector of CH because $ACBH$ is a kite.

2. Draw $B(D)$ and AD, yielding K. Then AB is the right bisector of KD because $AKBD$ is a kite. Note that this means that $HK = CD$ because of trapezoid $KCHD$.



Next we perform the following constructions, as shown in the figure on the following page.

3. Draw $C(DK)$ and $K(CD)$, yielding point G. The point G is collinear with C and H since $KDCG$ is a parallelogram and $CH \parallel DK \parallel CG$.

4. Draw $H(G)$, $G(K)$, giving point E (one of two possible points). Note that

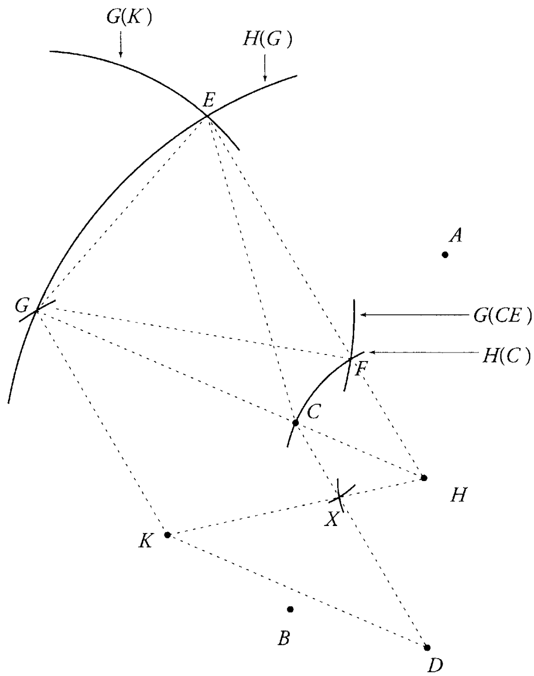
$$GE = GK = CD.$$

5. Draw $H(C)$, $G(CE)$ giving point F. Then F is collinear with H and E. This follows since $\triangle GHF \equiv \triangle EHC$ by the **SSS** congruency condition, and hence,

$$\angle GHF = \angle EHC = \angle EHG,$$

and the lines HE and HF coincide.

6. Draw $C(F)$ and $H(CF)$, yielding point X, which is the desired point.



Now the justification. We need to show that $X = AB \cap CD$.

Since $ACXH$ is a kite with $XC = XH$, then X is on the right bisector of CH, but AB is the right bisector of CH by step 1.

To show that X is also on CD, we will show that $\angle HCX = \angle HCD$.

Consider the triangles HCX and HGK. We have

$$HX = CX = CF,$$

while from steps 2 and 4 we have

$$HK = KG = GE.$$

Thus,

$$\frac{HX}{HK} = \frac{CX}{KG} = \frac{CF}{GE}.$$

However, since $\triangle HCF$ and $\triangle HGE$ are isosceles with a common vertex angle at H, they are similar, and

$$\frac{CF}{GE} = \frac{HC}{HG}.$$

Therefore,

$$\frac{HX}{HK} = \frac{CX}{KG} = \frac{HC}{HG},$$

which shows that triangles HCX and HGK are similar by the **sss** similarity condition, so that $\angle HCX = \angle HGK$.

Hence $\angle HGK = \angle HCD$, and since $CD \parallel GK$, then $\angle HCX = \angle HCD$; that is, the points C, X, and D are collinear, which completes the proof.

$\square$

The solutions to Example 13.4.7 and Example 13.4.8 prove:

Theorem 13.4.9. *(The Mohr-Mascheroni Construction Theorem)*

Any Euclidean construction, insofar as the given and required elements are points, may be completed with the compass alone.

13.5 Problems

1. Suppose that P and Q are inverse points with respect to a circle with center S, that $SP = m$, and that the radius of the circle of inversion is n. Find SQ.

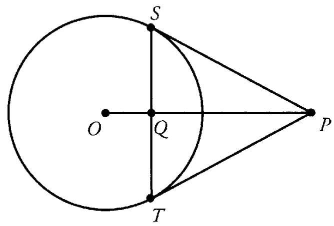
2. Given that A and A' are two inverse points ($A \neq A'$) with respect to some circle ω, find:

   (a) the radius of ω and

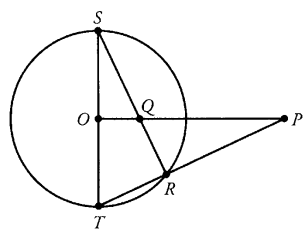
   (b) the center O of ω.

3. Given points P and Q with $PQ = 8$, draw all circles ω of radius 3 such that P and Q are inverses with respect to ω.

4. In the figure below, the tangents from P to the circle $\mathcal{C}(O, r)$ meet the circle at S and T. The point Q is the intersection of OP and ST. Prove that P and Q are inverses of each other with respect to the circle.



5. In the figure below, O is the center of the circle. The diameter ST is perpendicular to OP. PT intersects the circle at R, and SR intersects OP at Q. Prove that P and Q are inverses of each other with respect to the circle.



6. If the circle ξ passes through the center O of the circle ω, and if a diameter of ω meets the common chord of ω and ξ at P and meets the circle ξ at Q, show that P and Q are inverse points with respect to ω.

7. Draw the figure obtained by inverting a square with respect to its circumcircle.

8. What is the image under inversion $I(O, r^2)$ of the set of lines passing through P, where P is different from O and I? Include a sketch.

9. What is the inverse of a set of parallel lines?

10. Let P and P' be inverses under $I(O, r^2)$ with P outside the circle of inversion. Let B be the point where PP' meets the circle of inversion. Show that

$$BP = \frac{BP'}{1 - BP'/r}.$$

11. Let P and Q have inverses P' and Q', respectively, under $I(O, r^2)$, with O between P and Q. Show that

$$P'Q' = \frac{PQ}{OP \cdot OQ} r^2.$$

This is called the ***distortion teorem***.

12. If A and B are two distinct points inside some circle α, use inversion to show that there are exactly two circles through both A and B that are tangent to α.

13. A circle and an intersecting line (nontangential) can be inverses to each other in two different ways. Illustrate this by showing how to find two circles of inversion α and β such that the line and the given circle are inverses of each other.

14. If $ABCD$ is a convex noncyclic quadrilateral, show that

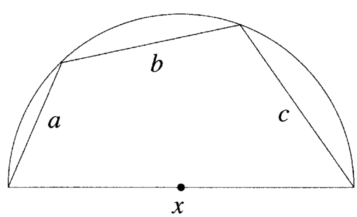
$$AC \cdot BD < AB \cdot CD + AD \cdot BC.$$

15. Given a triangle ABC with circumcenter O, let A', B', and C' be the images of the points A, B, and C under the inversion $I(O, r^2)$. Prove that

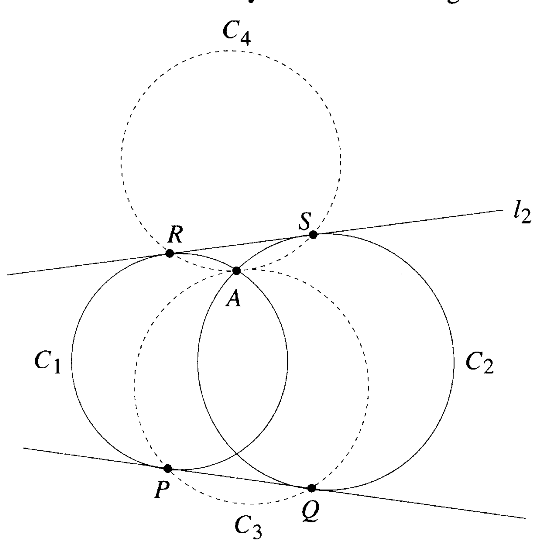
$$\triangle A'B'C' \sim \triangle ABC.$$

16. If a quadrilateral with sides of length a, b, c, and x is inscribed in a semicircle of diameter x, as shown, prove that

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0.$$



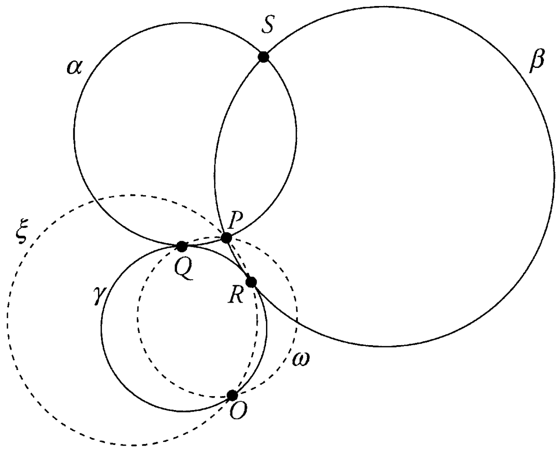
17. If PQ and RS are common tangents to two circles PAR and QAS, respectively, prove that the circles PAQ and RAS are tangent to each other.



18. Given a circle ω with center O and a point A outside ω, construct the circle with center A orthogonal to ω.

19. Given a circle ω and two noninverse points P and Q inside ω, construct the circle through P and Q orthogonal to ω.

20. Two circles α and β intersect orthogonally at P. O is any point on a circle γ tangent to both former circles at Q and R. Prove that the circles ω and ξ through OPQ and OPR, respectively, intersect at an angle of $45°$.



21. Construct (using a straightedge and compass) a circle orthogonal to a given circle having within it one-third of the circumference of the given circle.

22. Construct (using a straightedge and compass) a circle orthogonal to a given circle so that one-third of the circumference of the constructed circle lies within the given circle.

23. Let AC be a diameter of a given circle and chords AB and CD intersect (produced if necessary) in a point O. Prove that the circle OBD is orthogonal to the given circle.

24. Let ω and ξ be orthogonal circles intersecting at P and Q. Let AB be a straight line tangent to both circles at A and B. Show that one of $\angle APB$ and $\angle AQB$ is $45°$ and the other is $135°$.

25. In the Arbelos Theorem, show that the points of contact of K_i and K_{i+1}, $i = 0, 1, \ldots$, lie on a circle.

CHAPTER 14

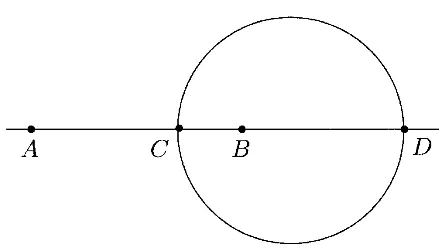
RECIPROCATION AND THE EXTENDED PLANE

14.1 Harmonic Conjugates

If A and B are two points on a line, any pair of points C and D on the line for which

$$\frac{AC}{CB} = \frac{AD}{DB}$$

is said to ***divide*** AB ***harmonically***. The points C and D are then said to be ***harmonic conjugates*** with respect to A and B.



Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

Lemma 14.1.1. *Given ordinary points A and B, and given a positive integer k where $k \neq 1$, there are two ordinary points C and D such that*

$$\frac{AC}{CB} = \frac{AD}{DB} = k.$$

One of the points C and D is between A and B, while the other is exterior to the segment AB.

Proof. Choose a point C on the line AB such that

$$CB = \frac{AB}{1+k}.$$

Since $k > 0$, then $CB < AB$, and we may assume that C lies between A and B.

Now, we have

$$CB = \frac{AB}{1+k} = \frac{AC + CB}{1+k},$$

so that

$$CB + kCB = AC + CB;$$

that is,

$$\frac{AC}{CB} = k.$$

Now we find the point D, which will be exterior to the segment AB—beyond B if $k > 1$ and beyond A if $0 < k < 1$.

Assuming that $k > 1$, we set

$$k = \frac{AD}{DB} = \frac{AD}{AD - AB}$$

and solve for AD to get

$$AD = \frac{k}{k-1} \cdot AB.$$

Therefore, if k is a positive number such that $k > 1$ and C satisfies

$$\frac{AC}{CB} = k,$$

then there always exists a point $D \neq C$ such that

$$\frac{AD}{DB} = k.$$

We simply choose D such that

$$AD = \frac{k}{k-1} \cdot AB.$$

A similar argument will find the point D when $0 < k < 1$.

□

Note. The midpoint C of AB satisfies

$$\frac{AC}{CB} = 1,$$

and we will adopt the convention that

$$\frac{AI}{IB} = 1,$$

where I is the ideal point in the inversive plane.

Using this convention, given two ordinary points A and B, for every positive number k there are harmonic conjugates C and D with respect to A and B for which

$$\frac{AC}{CB} = \frac{AD}{DB} = k.$$

Recall that three positive numbers a, b, and c form a ***harmonic progression*** if and only if

$$\frac{1}{a}, \quad \frac{1}{b}, \quad \text{and} \quad \frac{1}{c}$$

form an arithmetic progression. A similar definition holds for an infinite sequence of positive numbers.

For example, the sequence

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \ldots$$

forms a harmonic progression, since the sequence

$$1, \quad 2, \quad 3, \quad 4, \quad \ldots$$

forms an arithmetic progression.

Theorem 14.1.2. *Given four ordinary points A, B, C, and D, if AB is divided harmonically by C and D, then CD is divided harmonically by A and B.*

This terminology is explained by the following:

Theorem 14.1.3. *Suppose that P, Q, R, and S are consecutive ordinary points on a line and that Q and S divide PR harmonically. Then the sequence of distances PQ, PR, and PS forms a harmonic progression.*

Proof. The hypothesis says that

$$\frac{RQ}{QP} = \frac{RS}{SP}.$$

We want to show that

$$\frac{1}{PQ}, \quad \frac{1}{PR}, \quad \frac{1}{PS}$$

are in an arithmetic progression; that is, that

$$\frac{1}{PQ} - \frac{1}{PR} = \frac{1}{PR} - \frac{1}{PS}.$$

From the first equation, we have

$$\frac{RQ}{QP \cdot PR} = \frac{RS}{SP \cdot PR},$$

which implies that

$$\frac{PR - PQ}{PQ \cdot PR} = \frac{PS - PR}{PR \cdot PS},$$

which in turn implies that

$$\frac{1}{PQ} - \frac{1}{PR} = \frac{1}{PR} - \frac{1}{PS},$$

which is what we wanted to show.

□

The Circle of Apollonius

If we are given points A and B and a positive number $k \neq 1$, we can find precisely two ordinary points X on the line AB such that

$$\frac{AX}{XB} = k.$$

However, there are also points X not on the line AB for which

$$\frac{AX}{XB} = k,$$

and in fact, as we show in the following theorem, the set of all such points X lie on a circle.

Theorem 14.1.4. *(Circle of Apollonius)*

Given two ordinary points A and B, and a positive number $k \neq 1$, the set of all points X in the plane for which

$$\frac{AX}{XB} = k$$

forms a circle called the ***Circle of Apollonius for*** *A, B,* ***and*** *k.*

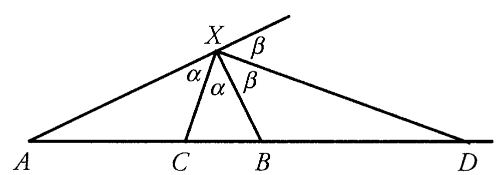
Proof. Let C and D be the two points on AB for which

$$\frac{AC}{CB} = \frac{AD}{DB} = k,$$

and let ξ be the circle with diameter CD.

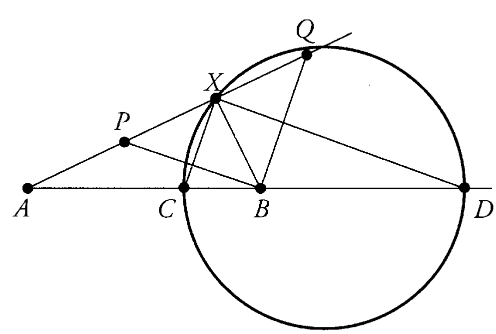
We show first that every point X for which $AX/XB = k$ is on ξ.

Let X be a point such that $AX/XB = k$, and since $AC/CB = k$ and $AD/DB = k$, we know from the Angle Bisector Theorem that XC and XD are, respectively, internal and external bisectors of angle AXB. Referring to the figure below, we see that $\alpha + \beta = 90°$; that is, $\angle CXD$ is a right angle. Therefore, by the converse to Thales' Theorem, this means that X is on the circle ξ.



We show next that every point X on the circle ξ satisfies $AX/XB = k$.

Let X be a point on the circle ξ and draw $BP \parallel DX$ and $BQ \parallel CX$, as shown below.



Since X is on the circle, then $\angle CXD = 90°$, and it follows that $\angle PBQ = 90°$.

Also, since

$$\triangle APB \sim \triangle AXD \quad \text{and} \quad \triangle AQB \sim \triangle AXC,$$

we have the following:

$$\frac{AX}{XP} = \frac{AD}{DB} \quad \text{and} \quad \frac{AX}{XQ} = \frac{AC}{CB}.$$

Since

$$\frac{AD}{DB} = \frac{AC}{CB} = k,$$

it follows that

$$\frac{AX}{XP} = \frac{AX}{XQ},$$

from which we get $XP = XQ$.

Now, $\angle PBQ$ is a right angle, and so the point B is on the circle centered at X with radius XP, by Thales' Theorem. Thus, $XB = XP$, so that

$$\frac{AX}{XB} = \frac{AX}{XP} = \frac{AD}{DB} = k.$$

□

Now we give some facts concerning the Circle of Apollonius.

Theorem 14.1.5. *Let O be the center and r the radius of the Circle of Apollonius for A, B, and k. Then:*

(1) O is on the line AB.

(2) The points A and B are to the same side of O.

(3) A and B are inverses with respect to the circle.

(4) If the circle meets AB at C and D, then C and D divide AB harmonically in the ratio k.

Proof. Statements (1) and (4) follow directly from Theorem 14.1.4.

(2) We may assume that the line AB is horizontal, that A is to the left of B, and that C is between A and B, but D is not.

Thus, D is located either to the left of A as in figure (a) below or to the right of B as in figure (b) below. We will show that statement (2) is true for case (a). The proof for case (b) is similar (see Problem 14.5 in this chapter).

(a)

(b)

For case (a), we have $CB < DB$, and since C and D are on the Circle of Apollonius, we also have

$$\frac{AC}{CB} = \frac{AD}{DB},$$

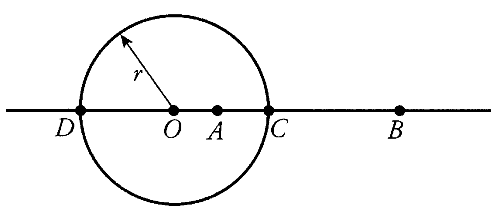
so that

$$\frac{AC}{CB} \cdot CB < \frac{AD}{DB} \cdot DB,$$

which implies that $AC < AD$. Thus, the midpoint O of CD is to the left of A and hence to the left of both A and B.

(3) Assuming that O is to the left of A, we have the following relationships, as in the figure below:

$$AC = r - OA, \qquad AD = r + OA, \qquad CB = OB - r, \qquad DB = OB + r.$$



Since C and D are on the circle,

$$\frac{AC}{CB} = \frac{AD}{DB},$$

which implies that

$$\frac{r - OA}{OB - r} = \frac{r + OA}{OB + r},$$

and solving for r^2, we have $OA \cdot OB = r^2$.

□

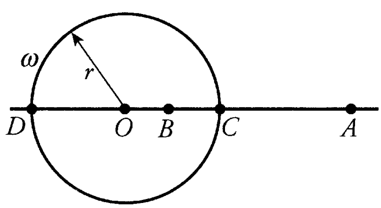
Harmonic Conjugates and Inverses

Theorem 14.1.6. *A and B are harmonic conjugates with respect to C and D if and only if A and B are inverses with respect to the circle with diameter CD.*

Proof. Suppose that A and B are harmonic conjugates for CD. Then C and D are harmonic conjugates for AB; that is,

$$\frac{AC}{CB} = \frac{AD}{DB}.$$

Letting r be the radius of the circle ω with diameter CD, as in the figure below.



We want to show that $OA \cdot OB = r^2$, and the proof proceeds as in the proof of statement (3) of Theorem 14.1.5. We have

$$AC \cdot BD = AD \cdot BC,$$

that is,

$$(OA - r)(OB + r) = (OA + r)(r - OB),$$

which simplifies to $OA \cdot OB = r^2$.

Conversely, suppose that A and B are inverses with respect to the circle ω with diameter CD. Assuming that A is outside ω, as shown above, then to prove that A and B are harmonic conjugates for CD, it suffices to show that

$$\frac{CA/AD}{CB/BD} = 1.$$

Referring to the figure above, we have

$$\begin{aligned}\frac{CA/AD}{CB/BD} &= \frac{CA \cdot BD}{AD \cdot CB} = \frac{(OA - r) \cdot (OB + r)}{(OA + r) \cdot (r - OB)} \\ &= \frac{OA \cdot OB - r \cdot OB + r \cdot OA - r^2}{r \cdot OA + r^2 - OA \cdot OB - r \cdot OB},\end{aligned}$$

and since $OA \cdot OB = r^2$, we get

$$\frac{CA/AD}{CB/BD} = \frac{r \cdot OA - r \cdot OB}{r \cdot OA - r \cdot OB} = 1,$$

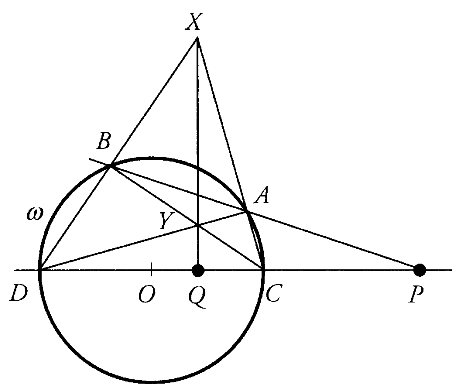
which completes the proof.

□

The relationship between harmonic conjugates and inverses allows us to show how a straightedge alone can be used to find the inverse of a point P that is outside the circle of inversion.

Example 14.1.7. *Given a point P outside a circle ω with center O, construct the inverse of P using only a straightedge.*

Solution. The analysis figure is shown below.



Construction:

(1) Draw the line OP intersecting ω at C and D.

(2) Draw a second line through P intersecting ω at A and B, as shown.

(3) Draw AC and BD intersecting at X.

(4) Draw AD and BC intersecting at Y.

(5) Draw the line through X and Y intersecting OP at Q.

Then Q is the inverse of P.

Justification:

(a) Apply Ceva's Theorem to $\triangle XCD$ and cevians XQ, CB, and DA. The cevians are concurrent at Y so that

$$\frac{XA}{AC} \cdot \frac{CQ}{QD} \cdot \frac{DB}{BX} = 1.$$

(b) Apply Menelaus' Theorem to $\triangle XCD$ with menelaus points P, A, and B. The points P, A, and B are collinear so that

$$\frac{XA}{AC} \cdot \frac{CP}{PD} \cdot \frac{DB}{BX} = 1.$$

(c) From (a) and (b), we get

$$\frac{CQ}{QD} = \frac{CP}{PD}.$$

(d) Thus, from (c), we see that P and Q are harmonic conjugates with respect to CD.

By Theorem 14.1.6, this means that P and Q are inverses with respect to ω.

□

Inversion and the Circle of Apollonius

We state here several theorems that are easy consequences of the results in the preceding sections.

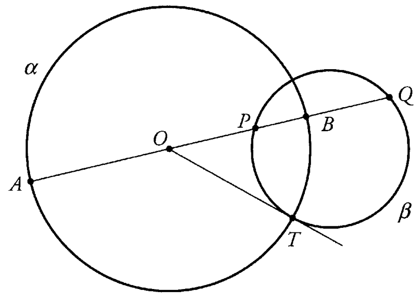
Theorem 14.1.8. *If ω is the Circle of Apollonius for A, B, and k, then A and B are inverses with respect to ω.*

Theorem 14.1.9. *The Apollonian circle for A, B, and k is the same as the Apollonian circle for B, A, and $\frac{1}{k}$.*

Remark. Note the change in the order of the points A and B in the previous theorem.

Theorem 14.1.10. *If A and B are inverse points for a circle ω, then ω is the Circle of Apollonius for A, B, and some positive number k.*

Theorem 14.1.11. *If α and β are orthogonal circles, then whenever either circle intersects a diameter of the other, it divides that diameter harmonically.*



Proof. Referring to the figure, we know that if β cuts the diameter AB of α at P and Q, then P and Q are inverse points for α, since β is its own inverse by Theorem 13.3.2.

Thus, P and Q are harmonic conjugates with respect to A and B by Theorem 14.1.6.

□

The following is the converse of the previous theorem.

Theorem 14.1.12. *If α and β are two circles and β divides a diameter of α harmonically, then the two circles are orthogonal.*

14.2 The Projective Plane and Reciprocation

We augment the Euclidean plane with infinitely many ***ideal points***, or ***points at infinity***, in such a way that:

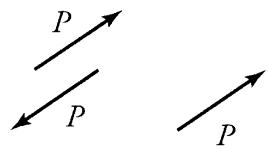
1. All lines parallel to a given line (including the given line) pass through the same ideal point.
2. Lines that are not parallel do not pass through the same ideal point.

Collectively, the ideal points form the ***ideal line*** or ***line at infinity***.

The resulting structure is called the ***extended plane*** or the ***projective plane***.

Nonideal points and nonideal lines are called ***ordinary points*** and ***ordinary lines***, respectively. The words "point" and "line" by themselves may refer to either an ordinary or ideal point and line.

Unlike the situation in inversive geometry, we can illustrate ideal points by using arrows or vectors to indicate the ideal point. Parallel vectors indicate the same ideal point. In the figure below, all three arrows indicate the same ideal point, and any line parallel to these vectors passes through that ideal point.



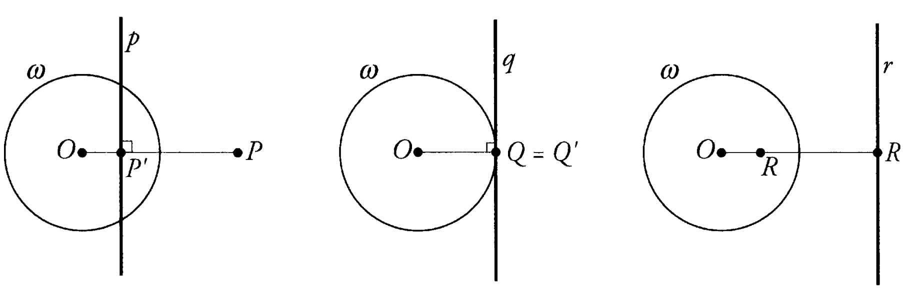
Immediate consequences of the definitions in the projective plane are as follows:

1. Every ordinary line contains exactly one ideal point.
2. Every two lines meet at exactly one point.
3. Every two points determine a unique line.

Reciprocation

Definition. Given a circle ω centered at an ordinary point O and given an ordinary point $P \neq O$, the ***polar*** of P is the line p that is perpendicular to OP passing through the inverse P' of P.

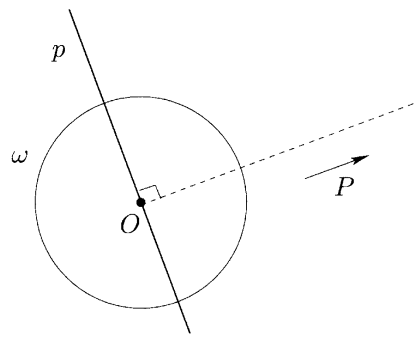
The circle ω is called the ***circle of reciprocation***. The center O is called the ***center of reciprocation***.



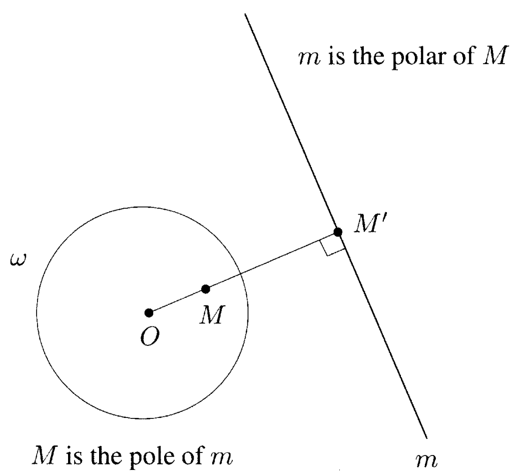
Note. We use the convention that uppercase letters denote points and corresponding lowercase letters denote the polars of those points.

- The polar of O is defined to be the ideal line.

- If P is an ideal point, then its polar is a line through O perpendicular to OP, that is, perpendicular to the arrow that points to P, as in the figure below.

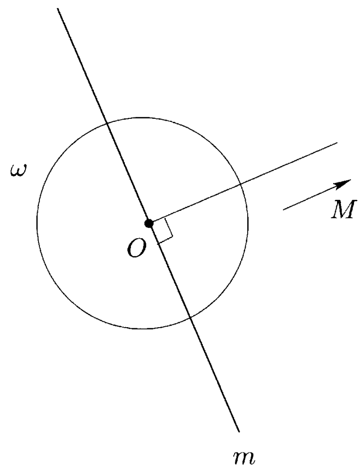


Definition. If m is a line, then the ***pole*** of m is the point M such that m is the polar of M, as in the figure below.



- The pole of the ideal line is the center O of the circle ω.

- If m is a line through the center O of the circle ω, then the pole of m is the ideal point M on any line perpendicular to m, as in the figure on the following page.



The most useful theorem about poles and polars follows:

Theorem 14.2.1. *(Reciprocation Theorem)*

P is on q if and only if Q is on p.

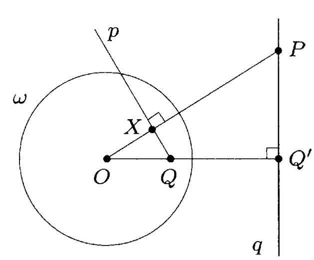
Proof. We will show that if P is on q, then Q is on p, and the converse can then be obtained by interchanging P and Q. We consider three cases.

Case (i). P is an ordinary point and q is an ordinary line.

Suppose that P is on q and let Q' be the inverse of Q.

On OP, let X be the foot of the perpendicular from Q, as in the figure below. Then from the **AA** similarity condition, we have

$$\triangle OXQ \sim \triangle OQ'P.$$



Therefore,

$$\frac{OX}{OQ} = \frac{OQ'}{OP},$$

which implies that

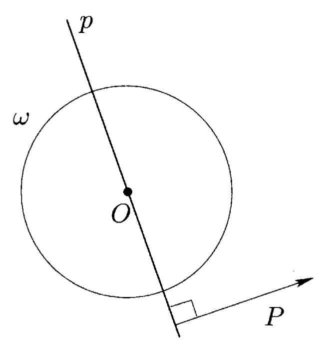
$$OX \cdot OP = OQ \cdot OQ' = r^2.$$

Thus, X is the inverse of P.

The definition of p now shows that p is the line QX, which means that if P is on q, then Q is on p.

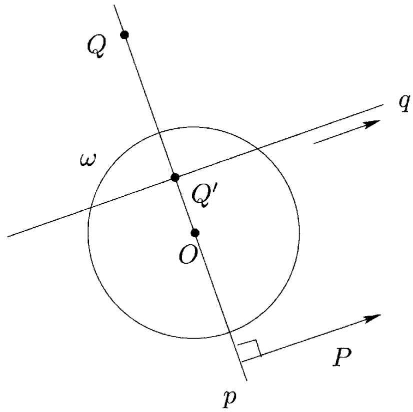
Case (ii). P is an ideal point and q is the ideal line.

Since q is the ideal line, Q is the center of the reciprocating circle; that is, $Q = O$. Therefore, p is the line through O perpendicular to any line pointing to P, as in the figure below. Thus, Q is on p.



Case (iii). P is an ideal point and q is an ordinary line passing through P.

Let Q be the pole of q. Since q is an ordinary line, we can draw the line OQ so that OQ is perpendicular to q, as in the figure on the following page.



Thus, the line OQ is the polar of P; that is, $p = \overleftrightarrow{OQ}$ and Q is on p.

We leave as an exercise the situation where $P = O$, the center of the reciprocating circle.

$\square$

Theorem 14.2.1 can be stated in different ways:

- P is on the polar of Q if and only if Q is on the polar of P.
- p is on the pole of q if and only if q is on the pole of p.

Definition. A ***range*** of points is a set of collinear points, and a ***pencil*** of lines is a set of concurrent lines.

An immediate consequence of Theorem 14.2.1 is the following:

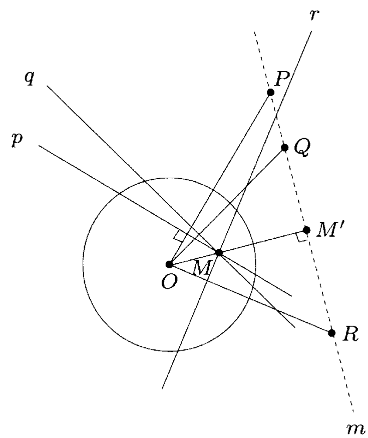
Corollary 14.2.2. *The polars of a range of points on a line m are a pencil of lines concurrent at M, and vice versa; that is, the poles of a pencil of lines are a range of points.*

Example 14.2.3. *Given a circle ω with center O, suppose that P, Q, and R are a range of points that lie on the line m, and let M be the pole of m. Show that the polars p, q, and r are concurrent at M.*

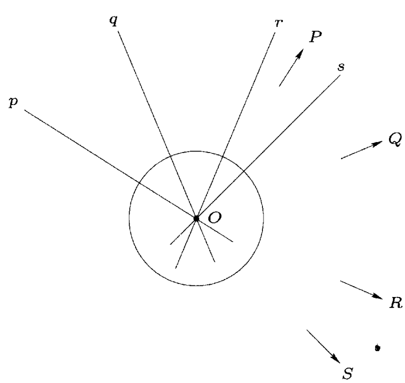
Solution. We have the following:

1. P is on m, so that M is on p.
2. Q is on m, so that M is on q.
3. R is on m, so that M is on r.

Therefore, the polars p, q, and r of P, Q, and R, respectively, are concurrent at the point M; that is, polars p, q, and r form a pencil of lines through M, as in the figure below.



In the extreme situation where all the points are ideal points, we have a range of points on the ideal line, and their polars form a pencil of lines through the center of the circle of reciprocation, as in the figure below, where we are given ideal points P, Q, R, and S.



□

In the following, $\overleftrightarrow{AB}$ denotes the line through A and B.

Theorem 14.2.4. *Let ω be the circle of reciprocation. Then:*

(1) A is outside ω if and only if a cuts ω.

(2) A is on ω if and only if a is tangent to ω.

(3) A is inside ω if and only if a misses ω.

(4) The pole of $\overleftrightarrow{AB}$ is $a \cap b$.

(5) The polar of $a \cap b$ is $\overleftrightarrow{AB}$.

Proof. We will prove statements (4) and (5) and leave the others as exercises.

(4) Let $m = \overleftrightarrow{AB}$ and let the pole of m be M.

Since A is on m, then M is on a, and since B is on m, then M is on b. Therefore, M is on $a \cap b$; that is, $M = a \cap b$.

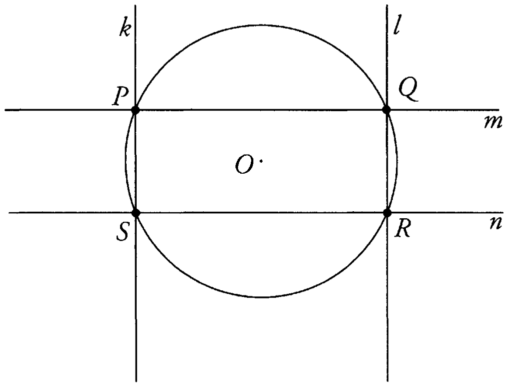
(5) We have $P = a \cap b$ if and only if P is on a and P is on b, and this is true if and only if A is on p and B is on p; that is, if and only if $p = \overleftrightarrow{AB}$.

□

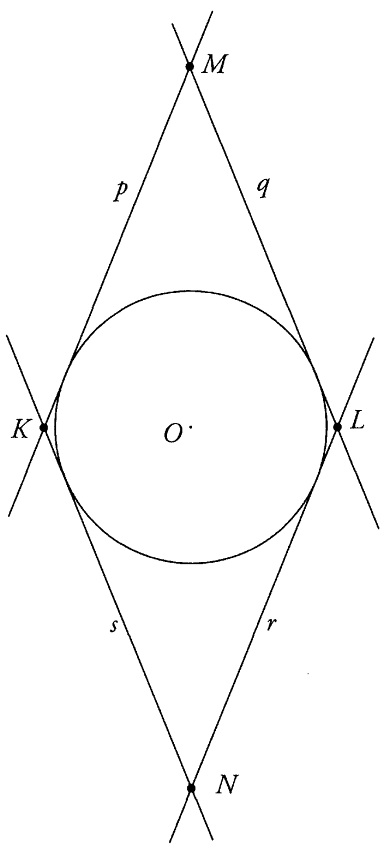
Duality

Given a figure $\mathcal{F}$ that consists of lines (entire lines, not just segments) and points (which may or may not be on the lines), then the ***polar*** or ***dual*** of $\mathcal{F}$ is the figure that is obtained by taking the poles of the lines of $\mathcal{F}$ and the polars of the points of $\mathcal{F}$.

Example 14.2.5. *Draw the dual of the figure on the following page.*



Solution. The dual is shown below.



□

The following is a translation table for obtaining the dual of a figure or the dual of a statement.

- To obtain the dual, any word or phrase that appears in one column must be replaced by the corresponding word or phrase in the other column.
- The symbol ω is the circle of reciprocation.

point	line
lie on	pass through
ω	ω
concurrent	collinear
pole	polar
locus	envelope
point on a curve	line tangent to a curve
inscribed in ω	circumscribed about ω
hexagon	hexagon

The translation table is a direct consequence of Theorem 14.2.1.

Theorem 14.2.6. *(Principle of Duality)*

If a statement that involves only points, lines, and their incidence properties is true, then the dual statement is automatically true.

We will illustrate the use of the translation table by using it to obtain the dual of Pascal's Mystic Hexagon Theorem. The dual theorem is called Brianchon's Theorem, and it was discovered by Brianchon by taking the dual, as illustrated on the following page.

In the left column, we state Pascal's Theorem using only the incidence properties of points and lines. The right column is the translation obtained by using the table above.

Pascal's Theorem	**Brianchon's Theorem**
If A, B, C, D, E, and F	If a, b, c, d, c, and f
are the vertices of a hexagon	are the edges of a hexagon
inscribed in ω	circumscribed about ω
then the points $\overleftrightarrow{AB} \cap \overleftrightarrow{DE}$,	then the lines through $a \cap b$ and $d \cap e$,
$\overleftrightarrow{BC} \cap \overleftrightarrow{EF}$,	through $b \cap c$ and $e \cap f$,
and $\overleftrightarrow{CD} \cap \overleftrightarrow{FA}$	and through $c \cap d$ and $f \cap a$
are collinear.	are concurrent.

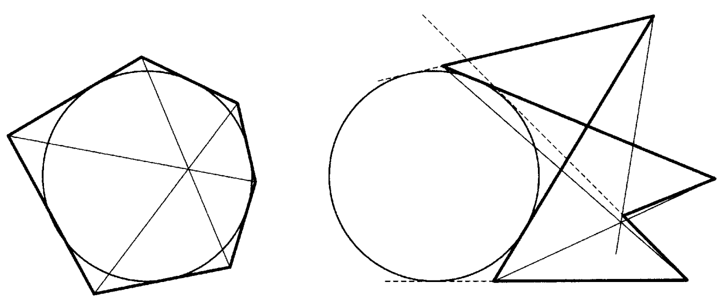
The points

$$a \cap b, \quad b \cap c, \quad c \cap d, \quad d \cap e, \quad e \cap f, \quad f \cap a$$

are the vertices of a hexagon, so in more familiar language, Brianchon's Theorem says:

Theorem 14.2.7. *(Brianchon's Theorem)*

If a hexagon is circumscribed about a circle, then the lines joining the opposite vertices are concurrent.



It is worth mentioning that in Brianchon's Theorem the hexagon is considered to circumscribe the circle if each edge, possibly extended, is tangent to the circle. In the figure above, the "hexagons" are shown as bold and the lines joining the opposite vertices are lighter.

If you try the same exercise to dualize Desargues' Theorem, you will find that you get nothing new—Desargues' Theorem is self-dual.

14.3 Conjugate Points and Lines

Let ω be a fixed circle with center O.

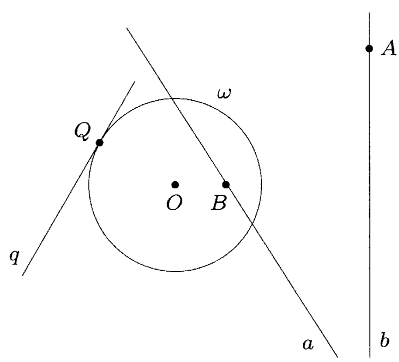
Two points A and B are said to be ***conjugate points*** with respect to ω if each lies on the polar of the other (that is, if A lies on b and B lies on a).

Two lines a and b are ***conjugate lines*** with respect to ω if each passes through the pole of the other.

A point or line which is conjugate to itself is said to be ***self-conjugate***.

Examples

The following properties concerning conjugate points and conjugate lines with respect to a circle ω are illustrated in the figure below and are easily proven.



A and B are conjugate points

a and b are conjugate lines

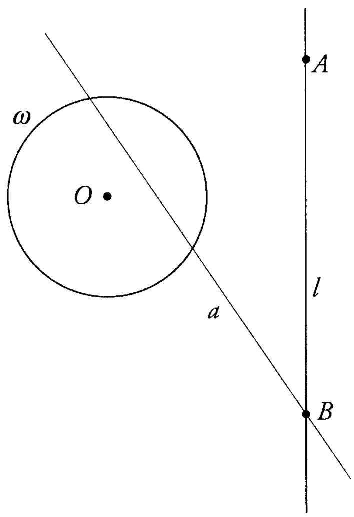
Q and q are self-conjugate

1. A and B are conjugate points if and only if a and b are conjugate lines.
2. B is conjugate to A if and only if B lies on a.
3. b is conjugate to a if and only if b passes through A.
4. The set of lines conjugate to a is the pencil of lines through A.
5. The set of points conjugate to B is the range of points on b.
6. The following are equivalent:
   (a) A is self-conjugate.
   (b) A is on ω.
   (c) a is tangent to ω.

Example 14.3.1. *Each point on a line has a conjugate point on that line.*

Solution. Let A be on l and let $B = a \cap l$. Note that if l and a are parallel, then B is an ideal point.

If l and a are not parallel, then B is on a and, by the basic reciprocation theorem, A is on b.



□

Example 14.3.2. *Each line through a point A has a conjugate line through A.*

Solution. This is the dual of the previous example.

A direct proof is as follows:

Let b be a line through A. Then $\overleftrightarrow{BA}$ is a line conjugate to b.

Recall that the pencil of lines at B is the set of all lines conjugate to b.

□

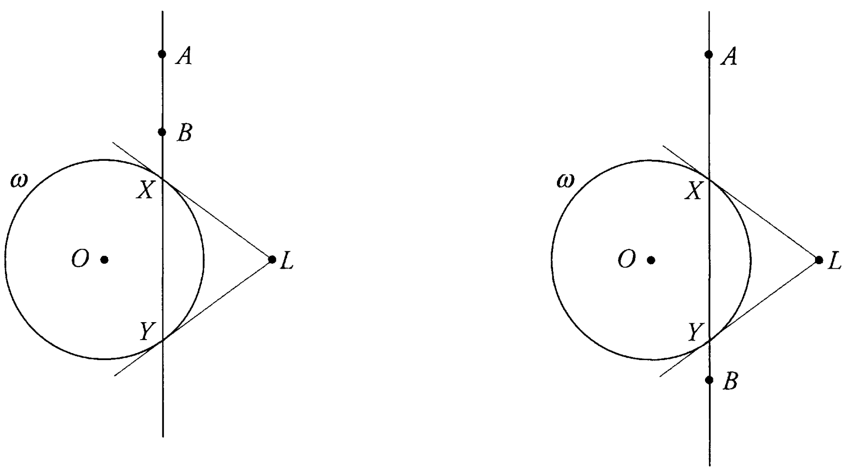
Example 14.3.3. *Of two distinct conjugate points on a line that cuts the circle of reciprocation, one point is inside or on the circle and the other point is outside the circle.*

Solution. Let ω be the circle of reciprocation and suppose that A and B are conjugate points on the line l that cuts ω.

If A is inside ω, then a misses ω, and since B is on a, B must be outside ω.

If A is on ω, then a is tangent to ω at A, and since B is on a and is different from A, then B must be outside ω.

If A is outside ω, then a cuts ω. Suppose for a contradiction that B is also outside ω, and let L be the pole of l. Since we are given that l cuts ω, L is outside ω. The situation is as shown in either of the two diagrams below.



Now, since A is on l, L is on a, and since B is conjugate to a, B is on a. Together these imply that $a = \overleftrightarrow{LB}$.

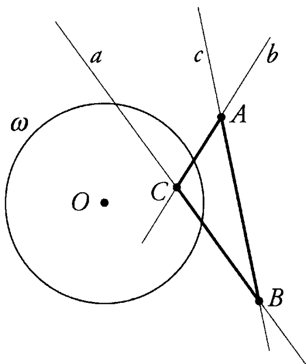
Let X and Y be the points of tangency from L to ω. Then B is outside the segment XY and LB misses ω. However, this contradicts the fact that a cuts ω. Thus, we must conclude that B is on or inside ω. $\square$

The following is the dual of the previous example and we leave the direct proof as an exercise.

Example 14.3.4. *Of two distinct conjugate lines that intersect outside a circle ω, one cuts the circle or is tangent to it, and the other misses the circle.*

Self-Polar Triangles

A triangle is ***self-polar*** if each vertex is the pole of the opposite side. Here, the sides are considered as lines.



a, b, c are polars of *A, B, C,* respectively

Remark. For a self-polar triangle:

- Any two vertices are conjugate points.
- Any two sides are conjugate lines.
- Given any two conjugate points, $A \neq B$, they are the vertices of some self-polar triangle, and the third vertex is $C = a \cap b$; that is, $c = \overleftrightarrow{AB}$.

Theorem 14.3.5. *Every nondegenerate self-polar triangle is obtuse, with the obtuse angle inside the circle of reciprocation* ω.

Proof. Let ABC be self-polar. Then exactly one vertex must be inside ω.

Here are the reasons:

(1) Suppose one vertex (say, A) is inside ω. Then a misses ω, but both other vertices are on a, so B and C are outside ω. This shows that there is at most one vertex inside ω.

(2) It is impossible for A to be on ω, because B and C would have to be on a, in which case all three of A, B, and C would be on a; that is, ABC would be degenerate.

(3) Suppose A, B, and C are all outside ω. Then a cuts ω, and B and C are on a. Therefore, by Example 14.3.3, one of B or C is inside ω and the other is outside.

This proves that exactly one vertex is inside ω. Supposing that A is inside ω, it remains to show that $\angle BAC$ is obtuse.

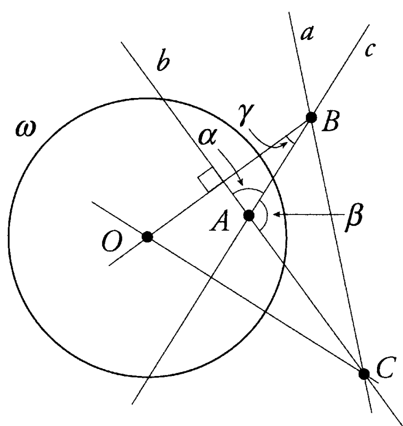
In the figure below, A is on b. Join OB. Then b is the line through A perpendicular to OB. Note that C is on b and B is on c, since C and B are conjugates.

A is on c. Join OC. Then c is the line through A perpendicular to OC.

Referring to the figure, for angles α, β, and γ, we have

$$\beta = 180 - \alpha = 180 - (90 - \gamma) = 90 + \gamma,$$

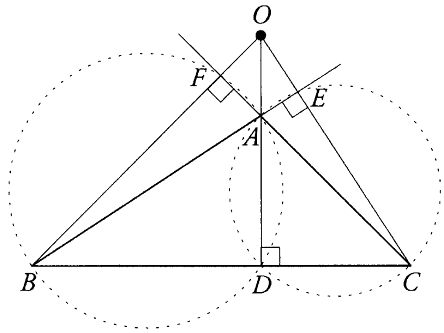
showing that β is obtuse.



□

Theorem 14.3.6. *Every obtuse triangle ABC is self-polar with respect to a unique circle ω, which is called the* **polar circle** *for the triangle.*

Proof. In the figure on the following page, let O be the orthocenter of $\triangle ABC$.



Referring to the figure above,

$$\angle AEC = 90^\circ = \angle ADC.$$

Therefore, $\square AECD$ is cyclic, and by the power of the point O with respect to the circumcircle of $\square AECD$, we have

$$OA \cdot OD = OC \cdot OE\,.$$

Similarly, $\square ADBF$ is cyclic, and by the power of the point O with respect to the circumcircle of $\square ADBF$, we have

$$OA \cdot OD = OF \cdot OB\,.$$

Let $OA \cdot OD = k^2$. Then

$$OC \cdot OE = k^2 \qquad \text{and} \qquad OF \cdot OB = k^2.$$

Now, let ω be the circle with center O and radius k. Then ω is the polar circle for triangle ABC.

Note that this works because:

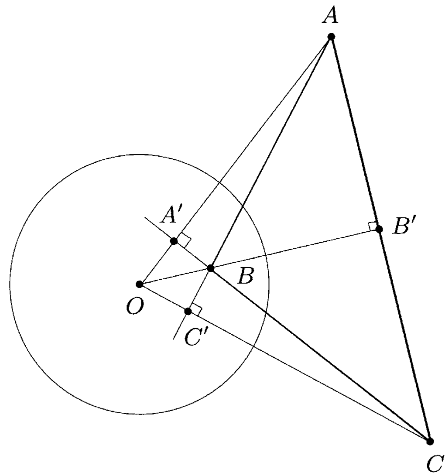
- $OA \cdot OD = k^2$, which means that $D = A'$, so $BC = a$ (with respect to ω),
- $OC \cdot OE = k^2$, which means that $E = C'$, so $AB = c$, and
- $OF \cdot OB = k^2$, which means that $F = B'$, so $AC = b$.

Thus, each vertex is the pole of the opposite side, and the obtuse triangle ABC is self-polar with respect to ω.

□

Example 14.3.7. *Show that given an obtuse triangle, the circumcircle and the 9-point circle invert into each other with respect to the polar circle.*

Solution. Let ABC be an obtuse triangle with the obtuse angle at vertex B. From the previous theorem, $\triangle ABC$ is self-polar with respect to the polar circle, and the vertices A, B, and C invert into the points A', B', and C', respectively, as in the figure below.



The circle through A, B, and C is the circumcircle of $\triangle ABC$. Since A', B', and C' are the feet of the altitudes, the inverse of $\triangle ABC$ with respect to the polar circle is just the 9-point circle of $\triangle ABC$.

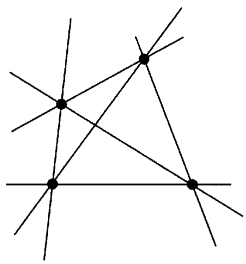
□

14.4 Conics

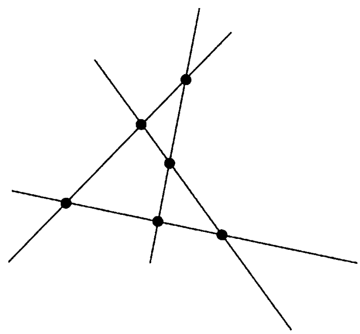
Duality Revisited

A ***complete quadrangle*** consists of four points, A, B, C, and D, no three of which are collinear, and the six joins or lines determined by these points.

A ***complete quadrilateral*** consists of four lines, a, b, c, and d, no three of which are concurrent, and the six points determined by these lines.



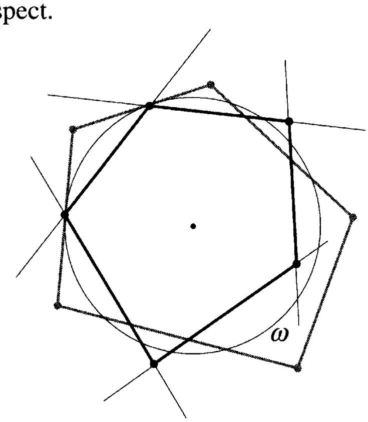
a complete quadrangle



a complete quadrilateral

Exercise 14.4.1. *Show that the reciprocal of a complete quadrangle is a complete quadrilateral, and vice-versa.*

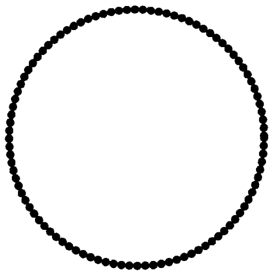
We can think of an n-gon as being composed of n points and the successive lines between these points, or, in the *opposite aspect*, as n lines and the successive points of intersections of these lines. In general, the dual or reciprocal of an n-gon is another n-gon of the opposite aspect.

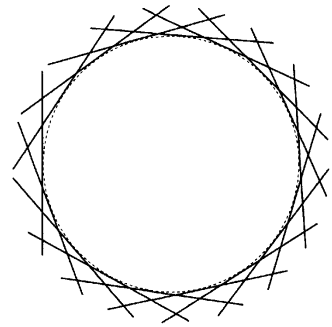


Reciprocals of Circles

The problem we are concerned with here is this: what is the image of a circle under reciprocation?

We can view a circle as a locus of points or as an envelope of lines, as in the figures on the following page, and in fact, every smooth curve can be viewed in these two aspects.



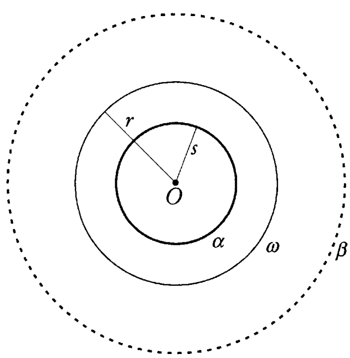


To find the reciprocal of the circle, view the circle in one aspect and see what curve is generated by the reciprocal aspect. We will use ω throughout to denote the reciprocating circle.

Special Cases

The reciprocal of the reciprocating circle ω is ω itself.

If the circle α is concentric with ω, and if the radius of α is s and the radius of ω is r, the reciprocal of α is another circle β concentric with ω with radius r^2/s.

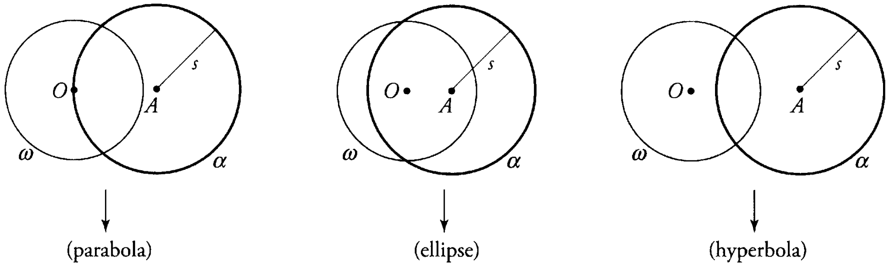


Theorem 14.4.2. *Let α and ω be two nonconcentric circles with centers A and O, respectively. The reciprocal of α with respect to ω is*

(1) an ellipse, if O is inside α;
(2) a parabola, if O is on α;
(3) a hyperbola, if O is outside α.

In each case, the focus of the conic section is O and the directrix is the polar of A. If the radius of α is s, the eccentricity ϵ of the conic is given by

$$\epsilon = OA/s\,.$$



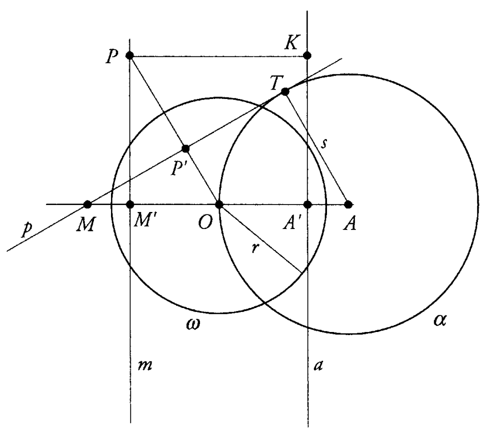
Proof. We will prove case (2). The proofs of (1) and (3) are similar.

Recall that a parabola with focus O and directrix a is the set of all points P such that $\text{dist}(P, a) = PO$. (See the next two sections for the connection between the focus-directrix definition and the more common Cartesian definition.)

As in the figure below, let p be tangent to α, let P be its pole, and let P' be the inverse of P.

Let $M = p \cap \overleftrightarrow{OA}$, let M' be the inverse of M, and let m be the polar of M. Note that P is on m since M is on p.

We will show that $PK/PO = 1$.



We have

$$\frac{PK}{PO} = \frac{MO' + OA'}{PO} = \frac{r}{PO}\left(\frac{M'O}{r} + \frac{OA'}{r}\right)$$

$$= \frac{P'O}{r}\left(\frac{r}{MO} + \frac{r}{OA}\right) = P'O\left(\frac{1}{MO} + \frac{1}{OA}\right).$$

That is,

$$\frac{PK}{PO} = P'O\left(\frac{MO + OA}{MO \cdot OA}\right) = \frac{P'O}{MO} \cdot \frac{MA}{AO},$$

and since $\triangle MP'O \sim \triangle MTA$, we have

$$\begin{aligned} \frac{PK}{PO} &= \frac{TA}{MA} \cdot \frac{MA}{AO} \\ &= \frac{TA}{AO} = \frac{s}{s} = 1. \end{aligned}$$

□

Focus-Directrix Definition of a Conic

Let d be a fixed line, let F be a fixed point not on the line, and let $\text{dist}(X, d)$ denote the perpendicular distance from the point X to the line d. If ϵ is a fixed positive constant, then the set of all points X for which

$$\frac{FX}{\text{dist}(X, d)} = \epsilon$$

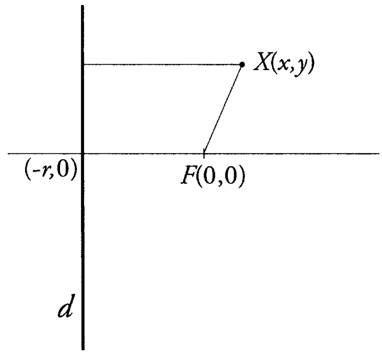
is a ***conic section***. The point F is the ***focus*** of the conic and the line d is the ***directrix*** of the conic. The positive constant ϵ is called the eccentricity of the conic, and:

- If $\epsilon = 1$, the conic is a parabola.
- If $0 < \epsilon < 1$, the conic is an ellipse.
- If $\epsilon > 1$, the conic is a hyperbola.

In the next section, we will show how to recover the Cartesian definitions of the conic sections from the focus-directrix definitions.

Cartesian Definitions from Focus-Directrix Definitions

Let d be the vertical line through $(-r, 0)$ and let F be the point $(0, 0)$, as in the figure on the following page.



We have

$$\frac{FX}{\text{dist}(X,d)} = \epsilon,$$

which implies that

$$\frac{FX^2}{(\text{dist}(X,d)\ )^2} = \epsilon^2,$$

which in turn implies that

$$\frac{x^2 + y^2}{(x+r)^2} = \epsilon^2.$$

After rearranging, we get

$$(1-\epsilon^2)x^2 - 2r\epsilon^2 x + y^2 = \epsilon^2 r^2.$$

If $\epsilon = 1$, the x^2 term disappears and the equation takes the form

$$y^2 - 2rx = r^2,$$

which we recognize as the Cartesian equation for a parabola.

If $\epsilon \neq 1$, then after some additional rearrangement we get

$$\left(x - \frac{r\epsilon^2}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \left(\frac{r\epsilon}{1-\epsilon^2}\right)^2.$$

This is of the form

$$(x-h)^2 + \frac{y^2}{1-\epsilon^2} = a^2,$$

which is the Cartesian equation for an ellipse if $\epsilon < 1$ or a hyperbola if $\epsilon > 1$.

Pascal's Mystic Hexagon Theorem

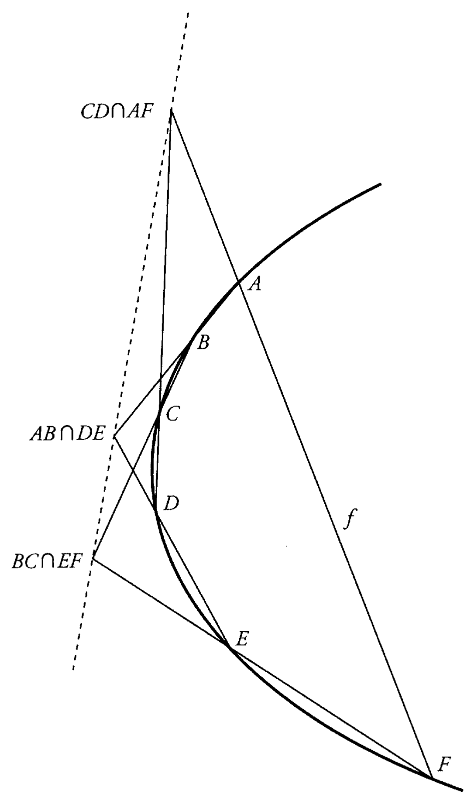
Two facts that we will not prove in this text:

1. Given any conic in the plane, there is a reciprocating circle ω and a circle α such that the conic is the reciprocal of α with respect to ω.
2. The reciprocal of a conic is a conic.

Theorem 14.4.3. *Pascal's Mystic Hexagon Theorem holds for conics.*

Proof. Let the vertices of the hexagon inscribed in the conic be A, B, C, D, E, and F, and let α and ω be circles such that the conic is the reciprocal of α with respect to ω. Then α is the reciprocal of the conic.

The points A, B, C, D, E, and F on the conic have polars a, b, c, d, e, and f that are tangent to the circle α. Brianchon's Theorem, which holds for the circle α, tells us that the joins of $a \cap b$ and $d \cap e$, $b \cap c$ and $e \cap f$, $c \cap d$ and $a \cap f$ are concurrent. Thus, taking reciprocals, the points $AB \cap DE$, $BC \cap EF$, and $CD \cap AF$ are collinear.



□

Using the same idea, the following can be seen to be true.

Theorem 14.4.4. *Brianchon's Theorem holds for conic sections.*

Another approach to showing that Pascal's Mystic Hexagon Theorem and Brianchon's Theorem are true for conics is to use the fact that all proper conics can be obtained via a central perspectivity of a circle. (See Theorem 16.6.1.)

14.5 Problems

1. Prove case (b) of statement (2) of Theorem 14.1.5.

2. Prove Theorem 14.1.9: The Apollonian circle for A, B, and k is the same as the Apollonian circle for B, A, and $1/k$.

3. Prove Theorem 14.1.10: If A and B are inverse points for a circle ω, then ω is the Circle of Apollonius for A, B, and some positive number k.

4. If PR is a diameter of circle α orthogonal to a circle β with center O, and if OP meets α in Q, prove that the line QR is the polar of P for β.

5. Show that one of the angles between the polars of A and B is equal to $\angle AOB$, where O is the center of the reciprocating circle.

6. Prove or disprove:

   (a) The reciprocal of a simple convex quadrilateral is a simple convex quadrilateral.

   (b) The reciprocal of a simple n-gon is a simple n-gon.

7. Use reciprocation to prove that given a triangle inscribed in a circle, then the points of intersection of the tangent lines at the vertices with the opposite sides are collinear.

8. If P and Q are conjugate points for a circle α, prove that the circle on PQ as diameter is orthogonal to α.

9. If two circles are orthogonal, prove that the extremities of any diameter of one are conjugate points for the other.

10. Sketch a circle α (with center A) and its polar with respect to a circle ω if the center O of ω is on α.

11. Given r and ϵ, $0 < \epsilon < 1$, then the equation

$$\frac{FX}{\text{dist}(X, d)} = \epsilon$$

in Cartesian coordinates becomes

$$\left(x - \frac{r\epsilon^2}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \left(\frac{r\epsilon}{1-\epsilon^2}\right)^2.$$

Show that given positive numbers a and b, there are suitable values for r and ϵ (in terms of a and b) so that the Cartesian equation above becomes

$$\frac{(x-f)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

12. Find the foci, directrices, and eccentricities of the following:

(a) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1.$

(b) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1.$

13. Find the Cartesian equations of the following conic sections:

(a) foci: $(\pm 8, 0)$, $e = 0.2$.

(b) foci: $(\pm 4, 0)$, directrix: $x = \dfrac{16}{3}$.

CHAPTER 15

CROSS RATIOS

15.1 Cross Ratios

Directed Distances

Recall that we denote directed distances, or signed distances, with a bar over the distance, as in $\overline{AB}$, and that

$$\overline{AB} = -\overline{BA}.$$

Also recall the following facts:

1. Given points A, B, and C on a line, if $\overline{AB} = \overline{AC}$, then $B = C$.
2. Given points A, B, C, and X on a line, if $\overline{AB}/\overline{BC} = \overline{AX}/\overline{XC}$, then $B = X$.

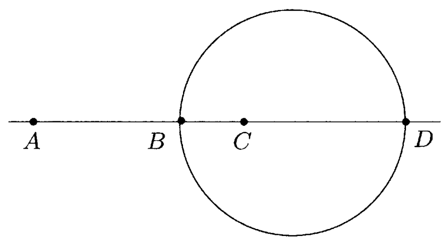
Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

Directed Distances and Harmonic Conjugates

Recall that given points A, B, C, and D, then B and D are harmonic conjugates with respect to A and C if and only if

$$\frac{AB}{BC} = \frac{AD}{DC},$$

as in the figure below.



Here, we are using unsigned distances; for *signed distances*, B and D are harmonic conjugates with respect to A and C if and only if

$$\frac{\overline{AB}}{\overline{BC}} = -\frac{\overline{AD}}{\overline{DC}}.$$

Properties of Cross Ratios

Given a range of four points, A, B, C, and D, we define the quantity (AB, CD) by

$$(AB, CD) = \frac{\overline{AC}/\overline{CB}}{\overline{AD}/\overline{DB}}$$

and call it the ***cross ratio*** of the points A, B, C, and D, taken in that order.

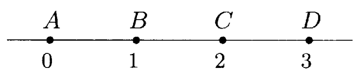
Note.

1. The order refers to the order of the points in the notation, *not* the order of the points on the line.
2. If $(AB, CD) = -1$, then A and B are harmonic conjugates with respect to C and D.
3. The value of (AB, CD) is independent of the direction of the line on which the range of points lie.

Example 15.1.1. *Find (AB, CD), (AC, BD), and (BA, DC) where A, B, C, and D are collinear with coordinates along the line given by 0, 1, 2, and 3, respectively.*

Solution. Working directly from the definition of cross ratios, we get

$$(AB, CD) = \frac{\overline{AC}/\overline{CB}}{\overline{AD}/\overline{DB}} = \frac{2/(-1)}{3/(-2)} = \frac{4}{3}.$$



Similarly,

$$(AC, BD) = -\frac{1}{3},$$

and

$$(BA, DC) = \frac{4}{3}.$$

□

Theorem 15.1.2. *If $(AB, CD) = k$, then:*

(1) If we interchange any pair of points and also interchange the other pair of points, then the resulting cross ratio has the same value k. Thus, (AB, CD), (BA, DC), (CD, AB), and (DC, BA) all have the value k.

(2) Interchanging only the first pair or only the last pair of points results in a cross ratio with the value $1/k$. Thus, $(BA, CD) = (AB, DC) = 1/k$.

(3) Interchanging only the middle pair or only the outer pair of points results in a cross ratio with the value $1 - k$. Thus, $(AC, BD) = (DB, CA) = 1 - k$.

Proof. (1) and (2) follow directly from the definition of the cross ratio. To prove (3), we will use the fact that for three collinear points X, Y, and Z, the directed distances are related by $\overline{XZ} = \overline{XY} + \overline{YZ}$, whether Y is between X and Z or not.

Interchanging the middle pair, we have

$$\begin{aligned}(AC,BD) &= \frac{\overline{AB}/\overline{BC}}{\overline{AD}/\overline{DC}} = \frac{\overline{AB}\cdot\overline{CD}}{\overline{AD}\cdot\overline{CB}}\\ &= \frac{(\overline{AC}+\overline{CB})(\overline{CB}+\overline{BD})}{\overline{AD}\cdot\overline{CB}}\\ &= \frac{\overline{AC}\cdot\overline{BD}}{\overline{AD}\cdot\overline{CB}} + \frac{\overline{AC}\cdot\overline{CB}+\overline{CB}(\overline{CB}+\overline{BD})}{\overline{AD}\cdot\overline{CB}}\\ &= -\frac{\overline{AC}\cdot\overline{DB}}{\overline{AD}\cdot\overline{CB}} + \frac{\overline{CB}(\overline{AC}+\overline{CB}+\overline{BD})}{\overline{AD}\cdot\overline{CB}}\\ &= -k+1.\end{aligned}$$

Interchanging the outer pair, from (1) we have

$$(DB,CA) = (CA,DB),$$

and interchanging the middle pair on the right-hand side, we have

$$(DB,CA) = 1-(CD,AB),$$

and again by (1), we have

$$\begin{aligned}(DB,CA) &= 1-(AB,CD)\\ &= 1-k.\end{aligned}$$

□

Remark. Any permutation of the letters A, B, C, and D can be obtained by successively interchanging pairs using (1), (2), or (3) of Theorem 15.1.2.

Example 15.1.3. *Given* $(AB,CD)=k$, *find* (DA,CB).

Solution. From (3), we have

$$(DA,CB) = 1-(BA,CD),$$

while from (2), we have

$$\begin{aligned}(DA,CB) &= 1-\frac{1}{(AB,CD)}\\ &= 1-\frac{1}{k}.\end{aligned}$$

Alternatively, we have

$$(AB, CD) = k,$$

which implies that

$$(BA, CD) = 1/k,$$

which in turn implies that

$$(DA, CB) = 1 - 1/k.$$

□

Ideal Points

Suppose that *one* of the four points A, B, C, or D is an ideal point I. We use the convention that

$$\frac{\overline{XI}}{\overline{IY}} = -1 \qquad \text{and} \qquad \frac{\overline{XI}}{\overline{YI}} = +1.$$

For example, if $B = I$, then

$$(AI, CD) = \frac{\overline{AC}/\overline{CI}}{\overline{AD}/\overline{DI}} = \frac{\overline{AC}}{\overline{CI}} \cdot \frac{\overline{DI}}{\overline{AD}} = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{DI}}{\overline{CI}} = \frac{\overline{AC}}{\overline{AD}}.$$

Now suppose we interchange the two middle symbols in (AI, CD). Then by direct computation we get

$$(AC, ID) = \frac{\overline{AI}/\overline{IC}}{\overline{AD}/\overline{DC}} = \frac{-1}{\overline{AD}/\overline{DC}} = -\frac{\overline{DC}}{\overline{AD}}.$$

However,

$$1 - \left(-\frac{\overline{DC}}{\overline{AD}}\right) = \frac{\overline{AD} + \overline{DC}}{\overline{AD}} = \frac{\overline{AC}}{\overline{AD}},$$

which shows that the theorem on permutation of symbols remains true if one of the points is an ideal point.

There are 24 different arrangements of the symbols A, B, C, and D, giving rise to 24 different cross ratios: (AB, CD), (BA, CD), (BC, AD), and so on. However, there are only six different values for the 24 cross ratios, namely,

$$k, \quad \frac{1}{k}, \quad 1 - k, \quad \frac{1}{1-k}, \quad \frac{k-1}{k}, \quad \text{and} \quad \frac{k}{k-1}.$$

To see why this is so, note that all permutations of the symbols A, B, C, and D can be obtained via a sequence of interchanges of the types described in Theorem 15.1.2 and that only the operations of the second and third types produce different values. Thus, if the cross ratio (WX, YZ) has the value k, then the only new values we can obtain by operation of the second or third type are $1/k$ and $1 - k$. The value $1/k$ in turn can produce values of k or $1 - 1/k = (k - 1)/k$. If we were to continue in this manner, we would see that only the six different values listed above can be obtained.

Theorem 15.1.4. *If* $(AB, CD) = (AB, CX)$, *then* D *and* X *are the same point.*

Proof. We have

$$(AB, CD) = (AB, CX),$$

so that

$$\frac{\overline{AC}/\overline{CB}}{\overline{AD}/\overline{DB}} = \frac{\overline{AC}/\overline{CB}}{\overline{AX}/\overline{XB}},$$

which implies that

$$\overline{AD}/\overline{DB} = \overline{AX}/\overline{XB}.$$

From the remarks at the beginning of the chapter on the ratios of directed distances, this implies that $D = X$.

□

Corollary 15.1.5. *If* $(AB, CD) = (AB, XD)$, *then* $C = X$.

Proof. We have

$$(AB, CD) = (AB, XD),$$

so that

$$(AB, DC) = (AB, DX),$$

and from the previous theorem, $C = X$.

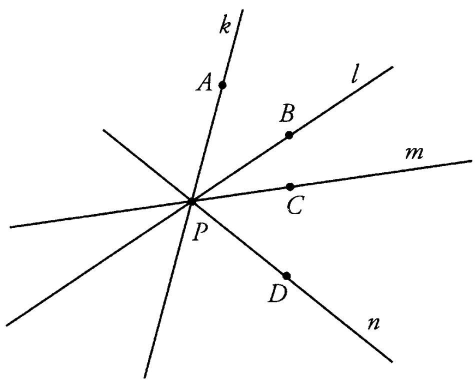
□

Cross Ratio of a Pencil of Lines

Let k, l, m, and n be a pencil of lines concurrent at a point P. This section will provide a definition of the cross ratio, denoted (kl, mn), of the pencil of lines.

Point of Concurrency is an Ordinary Point

Suppose k, l, m, and n are concurrent at an ordinary point P, and let A, B, C, and D be points other than P on k, l, m, and n, respectively, as in the figure below.



We define $P(AB, CD)$ as follows:

$$P(AB,CD) = \frac{\sin \overline{APC} / \sin \overline{CPB}}{\sin \overline{APD} / \sin \overline{DPB}},$$

where the directed angle $\overline{XPY}$ is the angle from the ray $\overrightarrow{PX}$ to the ray $\overrightarrow{PY}$ and whose magnitude is between $0°$ and $180°$.

Note that if A and A' are points on k on the opposite sides of P, then the directed angles $\overline{APC}$ and $\overline{A'PC}$ are different, as are the directed angles $\overline{APD}$ and $\overline{A'PD}$. In fact,

$$\overline{APC} = \overline{A'PC} - 180 \qquad \text{and} \qquad \overline{APD} = \overline{A'PD} - 180.$$

Since $\sin(x - 180) = -\sin x$, it follows that if A and A' are on opposite sides of P on the line k, then

$$\frac{\sin \overline{APC} / \sin \overline{CPB}}{\sin \overline{APD} / \sin \overline{DPB}} = \frac{-\sin \overline{A'PC} / \sin \overline{CPB}}{-\sin \overline{A'PD} / \sin \overline{DPB}} = \frac{\sin \overline{A'PC} / \sin \overline{CPB}}{\sin \overline{A'PD} / \sin \overline{DPB}},$$

or, in other words,

$$P(AB,CD) = P(A'B,CD).$$

Thus, we have the following result:

Theorem 15.1.6. *For a pencil of lines k, l, m, and n that are concurrent at the ordinary point P, the definition of $P(AB, CD)$ is independent of the choice of the points A, B, C, and D, as long as none of them are the point P.*

This allows us to make the following definition:

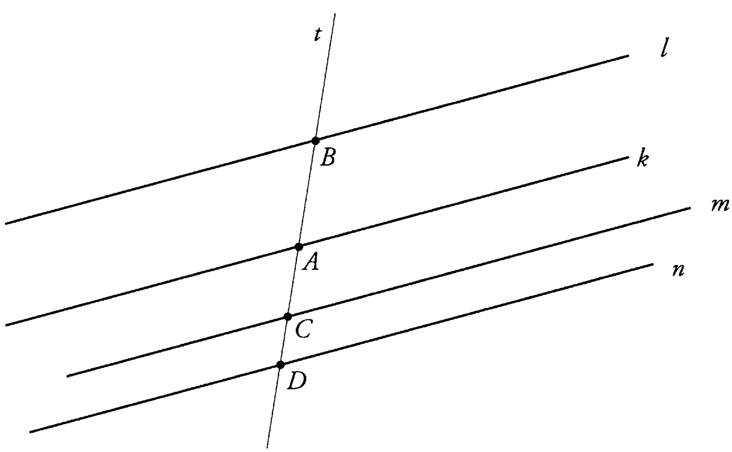
Definition. We define the ***cross ratio of a pencil of lines concurrent at an ordinary point*** P as

$$(kl, mn) = P(AB, CD),$$

where A, B, C, and D are points on k, l, m, and n other than P.

Point of Concurrency Is an Ideal Point

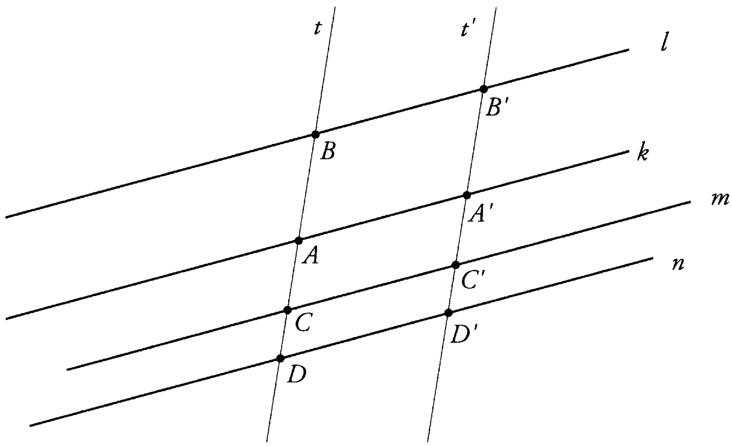
When the point of concurrency is an ideal point, the lines k, l, m, and n are parallel.



In this case, let t be any line intersecting k, l, m, and n at A, B, C, and D, respectively, and define (kl, mn) to be (AB, CD).

To check that this definition is independent of the choice of the line t, let t' be another line intersecting k, l, m, and n at A', B', C', and D'. There are two cases to consider:

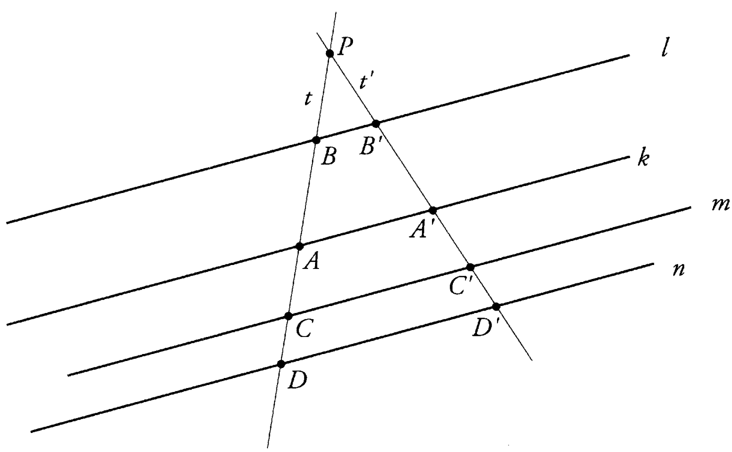
Case (i). t and t' are parallel.



In this case, obviously $AC = A'C'$, $CB = C'B'$, etc., since they are opposite sides of a parallelogram, so that

$$(AB, CD) = (A'B', C'D').$$

Case (ii). t and t' meet at an ordinary point P.



In this case, by similar triangles,

$$\frac{\overline{A'C'}}{\overline{C'B'}} = \frac{\overline{AC}}{\overline{CB}},$$

with similar results for the other ratios. From this it follows that

$$(AB, CD) = (A'B', C'D').$$

Theorem 15.1.7. *Suppose that k, l, m, and n form a pencil of lines concurrent at the ordinary point P. If a transversal cuts the lines k, l, m, and n at the points A, B, C, and D, respectively, then*

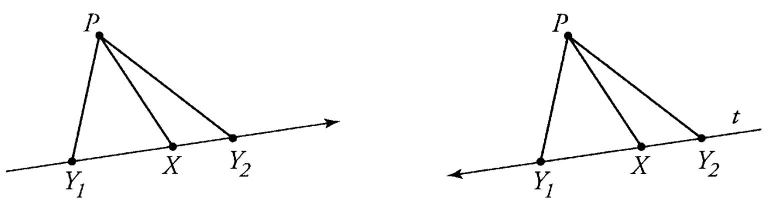
$$P(AB, CD) = (AB, CD).$$

Proof. In order to prove the theorem, we will show two things:

(1) The signs (signum) of $P(AB, CD)$ and (AB, CD) are the same.

(2) The magnitudes of $P(AB, CD)$ and (AB, CD) are the same.

Proof of (1).



To check that (1) is true, note that given a directed line t and a point P not on t, then either

$$\operatorname{sgn}(\sin \overline{XPY}) = \operatorname{sgn}(\overline{XY})$$

for all pairs of points X and Y on t or else

$$\operatorname{sgn}(\sin \overline{XPY}) = -\operatorname{sgn}(\overline{XY})$$

for all pairs of points X and Y on t.

For points A, B, C, and D on t, the value of (AB, CD) is independent of the direction of t (see statement (3) in the note following the definition of (AB, CD) at the beginning of this chapter). In particular, we can choose the direction of t so that for all pairs of points X and Y,

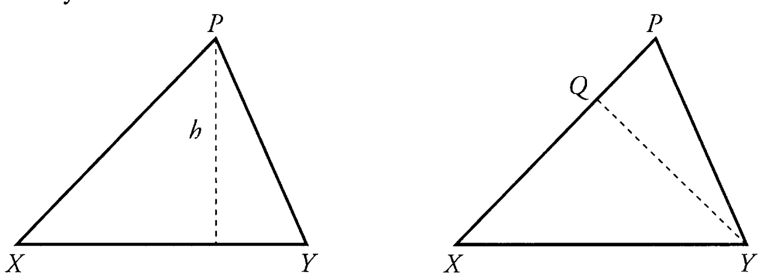
$$\operatorname{sgn}(\sin \overline{XPY}) = \operatorname{sgn}(\overline{XY}).$$

Thus,

$$\begin{aligned}
\operatorname{sgn}\left(P(AB,CD)\right) &= \operatorname{sgn}\left(\frac{\sin\overline{APC}/\sin\overline{CPB}}{\sin\overline{APD}/\sin\overline{DPB}}\right)\\
&= \frac{\operatorname{sgn}(\sin\overline{APC})/\operatorname{sgn}(\sin\overline{CPB})}{\operatorname{sgn}(\sin\overline{APD})/\operatorname{sgn}(\sin\overline{DPB})}\\
&= \frac{\operatorname{sgn}(\overline{AC}/\operatorname{sgn}(\overline{CB})}{\operatorname{sgn}(\overline{AD})/\operatorname{sgn}(\overline{DB})}\\
&= \operatorname{sgn}(AB,CD).
\end{aligned}$$

Proof of (2).

The proof uses the fact that it is possible to compute the area of a triangle in two different ways:



(a) (b)

In the figure above,

(a) area$(\triangle XPY) = XY \cdot \frac{h}{2}$, and

(b) area$(\triangle XPY) = \frac{1}{2}XP \cdot PY \cdot \sin(\angle XPY)$,

where (b) follows from the fact that $QY = PY\sin(\angle XPY)$.

Expanding $|(AB,CD)|$, we get

$$\begin{aligned}
|(AB,CD)| &= \frac{|\overline{AC}| \,/\, |\overline{CB}|}{|\overline{AD}| \,/\, |\overline{DB}|}\\
&= \frac{(|\overline{AC}|\cdot\frac{h}{2}) \;/\; (|\overline{CB}|\cdot\frac{h}{2})}{(|\overline{AD}|\cdot\frac{h}{2}) \;/\; (|\overline{DB}|\cdot\frac{h}{2})}\\
&= \frac{\text{area}(\triangle APC) \,/\, \text{area}(\triangle CPB)}{\text{area}(\triangle APD) \,/\, \text{area}(\triangle DPB)}\,.
\end{aligned}$$

Calculating the areas using (b) we get

$$\text{area}(\triangle APC) = \frac{1}{2}|AP| \cdot |PC| \cdot |\sin \overline{APC}|,$$

$$\text{area}(\triangle APD) = \frac{1}{2}|AP| \cdot |PD| \cdot |\sin \overline{APD}|,$$

$$\text{area}(\triangle CPB) = \frac{1}{2}|CP| \cdot |PB| \cdot |\sin \overline{CPB}|,$$

$$\text{area}(\triangle DPB) = \frac{1}{2}|DP| \cdot |PB| \cdot |\sin \overline{DPB}|.$$

Substituting these values into the expression for $|(AB, CD)|$ above, we get

$$|(AB, CD)| = \frac{|\sin \overline{APC}| \,/\, |\sin \overline{CPB}|}{|\sin \overline{APD}| \,/\, |\sin \overline{DPB}|} = |P(AB, CD)|\,.$$

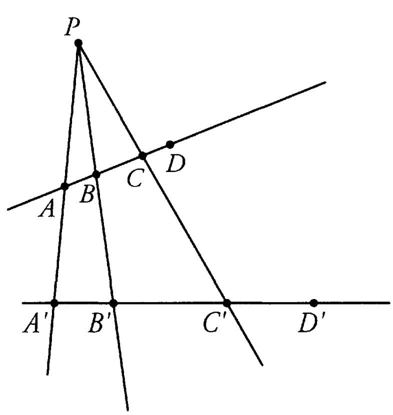
This completes the proof of the theorem. □

15.2 Applications of Cross Ratios

Four Useful Lemmas

The following lemmas connect cross ratios with concurrency and collinearity.

Lemma 15.2.1. *In the figure below, P is an ordinary point and P, A, and A' are collinear; P, B, and B' are collinear; and P, C, and C' are collinear. Transversals cut the lines PA, PB, and PC at A, B, and C, respectively. The point D is on AC, and the point D' is on $A'C'$.*



With this configuration, if $(AB, CD) = (A'B', C'D')$, then P is on DD'.

Proof. Suppose the line PD intersects $A'D'$ at some point E' (E' is not shown in the diagram). Then from Theorem 15.1.6, we have

$$P(AB,CD) = (A'B',C'E').$$

However,

$$P(AB,CD) = (AB,CD) = (A'B',C'D'),$$

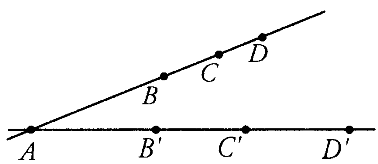
so that

$$(A'B',C'E') = (A'B',C'D'),$$

which implies that $E' = D'$, and P is on DD'.

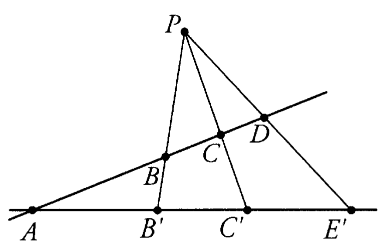
□

Lemma 15.2.2. *In the figure below, two lines intersect at the point A. The points B, C, and D are on one of the lines, while the points B', C', and D' are on the other line.*



With this configuration, if $(AB,CD) = (AB',C'D')$, then BB', CC', and DD' are concurrent.

Proof. Let $P = BB' \cap CC'$ and let $E' = PD \cap AC'$, as shown below.



It suffices to show that $E' = D'$.

We have

$$P(AB,CD) = P(AB',C'E'),$$

which implies that

$$(AB,CD) = (AB',C'E'),$$

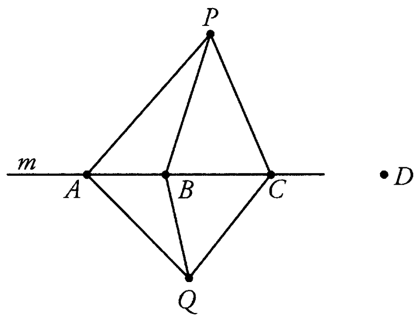
and since $(AB, CD) = (AB', C'D')$, it follows that

$$(AB', C'D') = (AB', C'E').$$

Therefore, $E' = D'$.

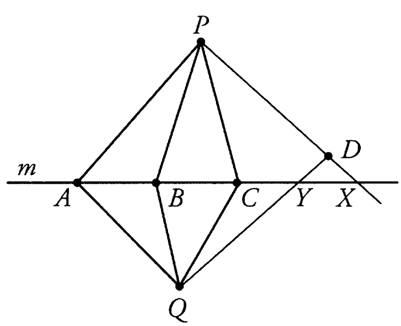
□

Lemma 15.2.3. *In the figure below, A, B, and C are points on a line m. P and Q are points not on m. D is a point other than A, B, C, P, or Q.*



With this configuration, if $P(AB, CD) = Q(AB, CD)$, then D is on m.

Proof. If D is not on m, let PD intersect m at X and let QD intersect m at Y, as shown below.



We have

$$P(AB, CD) = P(AB, CX) = (AB, CX)$$

and

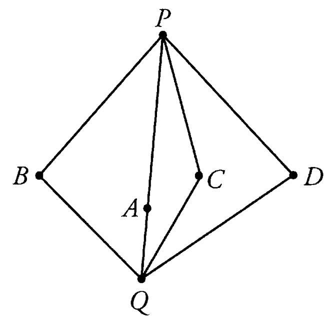
$$Q(AB, CD) = Q(AB, CY) = (AB, CY),$$

and since $P(AB, CD) = Q(AB, CD)$, we must have $(AB, CX) = (AB, CY)$.

However, this can only happen if $PD \cap QD = X = Y = D$; that is, D is on m.

□

Lemma 15.2.4. *In the figure below, A, B, C, and D are points other than P and Q, and the point A is on PQ.*



With this configuration, if

$$P(AB,CD) = Q(AB,CD),$$

then B, C, and D are collinear.

Proof. Let A' be the point $PQ \cap BC$ and let m be the line BC. Then

$$P(AB,CD) = P(A'B,CD),$$
$$Q(AB,CD) = Q(A'B,CD),$$

which implies that

$$P(A'B,CD) = Q(A'B,CD),$$

which implies that D is on m, by the previous lemma.

□

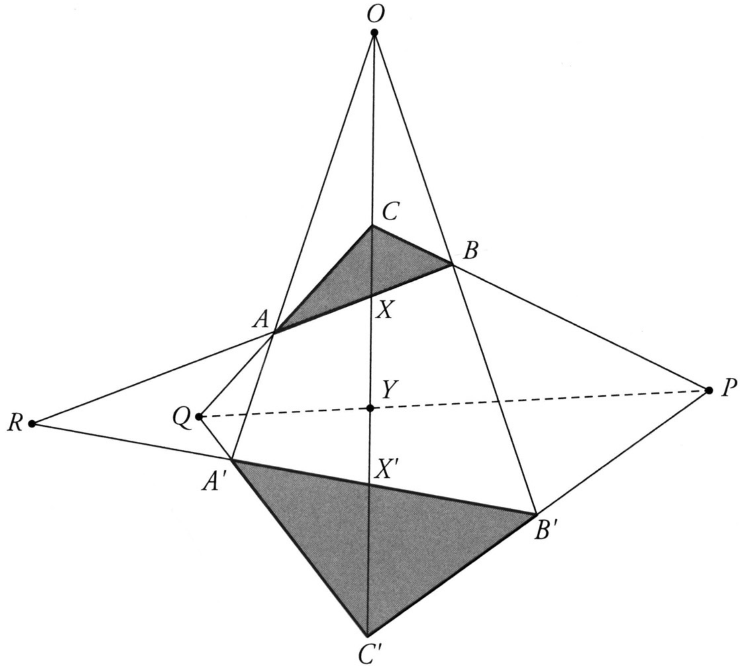
Theorems of Desargues, Pascal, and Pappus

Desargues' Two Triangle Theorem (Theorem 4.4.3) was proven in Part I using Ceva's Theorem and Menelaus' Theorem. We restate it here and prove it using cross ratios.

Theorem 15.2.5. *Copolar triangles are coaxial and conversely.*

Proof. Let the copolar triangles be ABC and $A'B'C'$. Then, as in the figure on the following page,

$$AA' \cap BB' \cap CC' = O.$$



Let P, Q, and R be the points of intersection of the corresponding sides of the triangle. In order to show that P, Q, and R are collinear, we will show that R is on the line PQ.

Let X, X', and Y be as shown; that is, let $X = OC' \cap AB$, $X' = OC' \cap A'B'$, and $Y = OC' \cap PQ$.

There are three pencils of lines: one concurrent at C, one concurrent at O, and one concurrent at C'.

Since BA is a transversal for the pencil at C, we have

$$C(PY, QR) = (BX, AR),$$

and since $B'A'$ is a transversal for the pencil at O, we have

$$(BX, AR) = O(BX, AR) = (B'X', A'R) = C'(B'X', A'R),$$

and replacing B' by P, X' by Y, and A' by Q in the last expression, we have

$$C'(B'X', A'R) = C'(PY, QR).$$

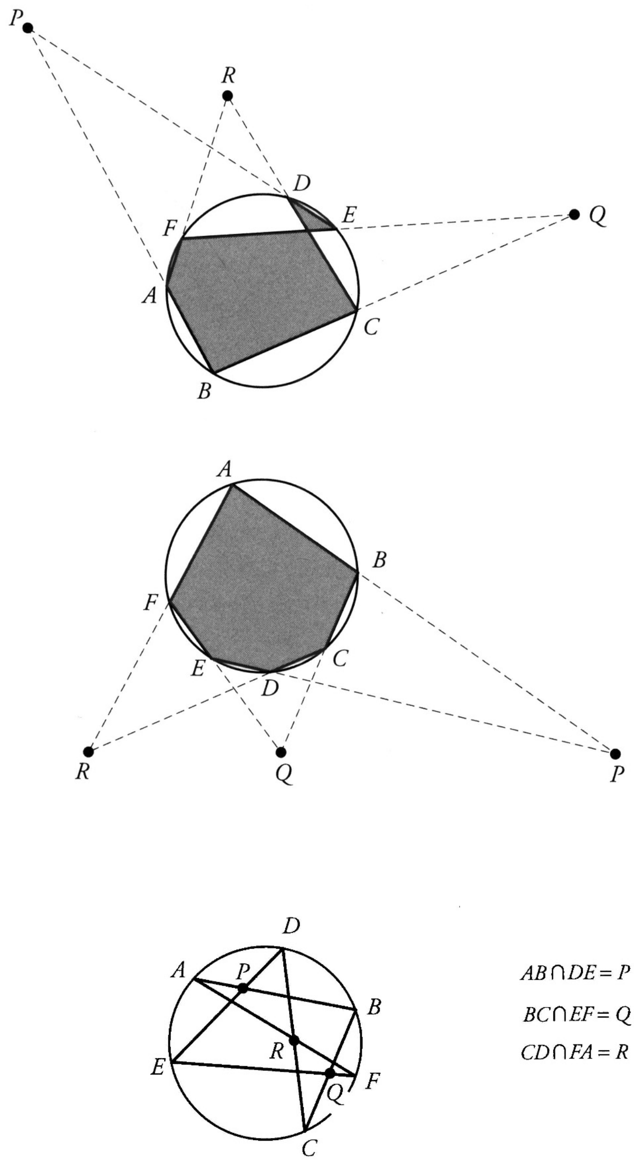
Therefore, $C(PY, QR) = C'(PY, QR)$, and by Lemma 15.2.4, P, Q, and R are collinear.

□

Pascal's Mystic Hexagon Theorem from Part I (Theorem 4.4.4) says the following:

Theorem 15.2.6. *If a hexagon is inscribed in a circle, the points of intersection of the opposite sides are collinear.*

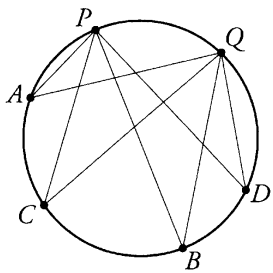
There are many possible configurations, three of which are shown in the following figure.



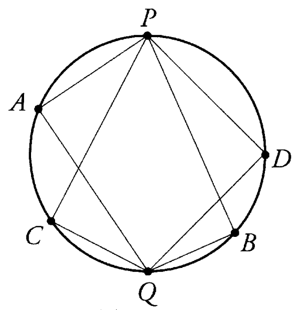
In order to prove the theorem using cross ratios we first need the following lemma:

Lemma 15.2.7. *If A, B, C, D, and P, Q are distinct points on a circle, then* $P(AB,CD) = Q(AB,CD)$.

Proof. There are two cases to consider, as illustrated in the following figures.



(1)



(2)

Case (1). P and Q are not separated by any of the points A, B, C, or D.

In this case, Thales' Theorem implies that

$$\begin{aligned}\angle\overline{APC} &= \angle\overline{AQC},\\ \angle\overline{CPB} &= \angle\overline{CQB},\\ \angle\overline{APD} &= \angle\overline{AQD},\\ \angle\overline{DPB} &= \angle\overline{DQB}.\end{aligned}$$

Thus,

$$P(AB,CD) = \frac{\sin\overline{APC}/\sin\overline{CPB}}{\sin\overline{APD}/\sin\overline{DPB}} = \frac{\sin\overline{AQC}/\sin\overline{CQB}}{\sin\overline{AQD}/\sin\overline{DQB}} = Q(AB,CD).$$

Case (2). P and Q are separated by some of the points A, B, C, or D.

The proof here is similar to that for Case (1), but now we have

$$\begin{aligned}\angle\overline{APC} &= \angle\overline{AQC},\\ \angle\overline{CPB} &= 180 + \angle\overline{CQB},\\ \angle\overline{APD} &= 180 + \angle\overline{AQD},\\ \angle\overline{DPB} &= \angle\overline{DQB}.\end{aligned}$$

The positive signs in the second and third equations arise since the signed angles are in opposite directions. From the second equation, we get

$$\sin\overline{CPB} = \sin(180 + \overline{CQB}) = -\sin\overline{CQB}.$$

Similarly, from the third equation, we get

$$\sin \overline{APD} = \sin(180 + \overline{AQD}) = -\sin \overline{AQD}\,.$$

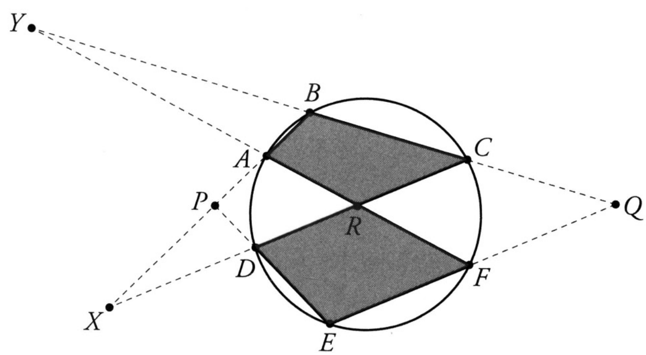
The lemma now follows from the definitions of $P(AB, CD)$ and $Q(AB, CD)$ as in Case (1).

□

For convenience, we restate Pascal's Theorem.

Theorem 15.2.8. *If $ABCDEF$ is a hexagon inscribed in a circle, then the points of intersection of the opposite sides are collinear.*

Proof. The proof works for any configuration, and for clarity we use the one in the figure below.



Let $X = AB \cap CD$ and $Y = BC \cap AF$.

Consider the pencils at D and F. Since P is on DE and X is on DC, then

$$D(AE, CB) = (AP, XB),$$

and from Lemma 15.2.7, we have

$$D(AE, CB) = F(AE, CB).$$

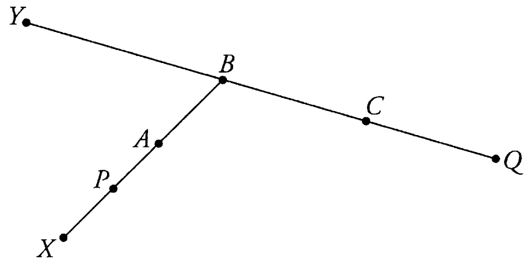
Since Y is on FA and Q is on FE, then

$$F(AE, CB) = (YQ, CB),$$

and, therefore,

$$(AP, XB) = (YQ, CB)\,.$$

Note that we have the following configuration:



Since $(AP, XB) = (YQ, CB)$, then from Lemma 15.2.2 it follows that AY, PQ, and XC are concurrent. However, $AY \cap XC = R$, so PQ passes through R.

□

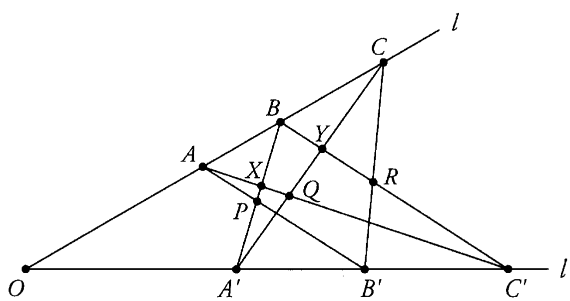
As a final example, we prove Pappus' Theorem (Theorem 4.4.5) using cross ratios:

Theorem 15.2.9. *Given points A, B, and C on a line l and points A', B', and C' on a line l', then the points of intersection*

$$P = AB' \cap A'B, \quad Q = AC' \cap A'C, \quad R = BC' \cap B'C$$

are collinear.

Proof. Introduce points $X = AC' \cap A'B$ and $Y = CA' \cap C'B$, as shown in the figure below.



Using the pencil through A, replace B by O, X by C', and P by B'. Then

$$A(BX, PA') = A(OC', B'A'),$$

and since these are transversals for the pencil through A, we have

$$(BX, PA') = (OC', B'A').$$

Using the pencil through C, replace O by B, B' by R, and A' by Y. Then

$$C(OC', B'A') = C(BC', RY),$$

and since these are transversals for the pencil through C, we have

$$(OC', B'A') = (BC', RY).$$

Thus,

$$(BX, PA') = (OC', B'A') = (BC', RY).$$

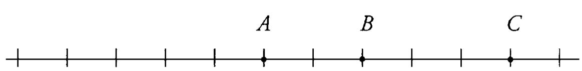
It follows from Lemma 15.2.2 that XC', PR, and $A'Y$ are concurrent, and since $XC' \cap A'Y = Q$, then Q is on PR; that is, P, Q, and R are collinear.

□

15.3 Problems

1. Given $(AB, CD) = k$, find (BC, AD) and (BD, CA).

2. Given three points A, B, and C, as shown below,



   find points D_i, $i = 1, 2, 3, 4$ such that

   (a) $(AB, CD_1) = 5/6$,

   (b) $(AB, CD_2) = -5/3$,

   (c) $(AB, CD_3) = 10/3$,

   (d) $(AB, CD_4) = 5/3$.

3. Using the definition of the cross ratio, show that $(AB, CD) = (CD, AB)$.

4. Show that for collinear points A, B, C, D, and E we have

   (a) $(AB, CE) \cdot (AB, ED) = (AB, CD)$,

   (b) $(AE, CD) \cdot (EB, CD) = (AB, CD)$.

5. Find x if $(AB, CD) = (BA, CD) = x$.

6. Let L, M, and N be the respective midpoints of the sides BC, CA, and AB of $\triangle ABC$. Prove that

$$L(MN, AB) = -1.$$

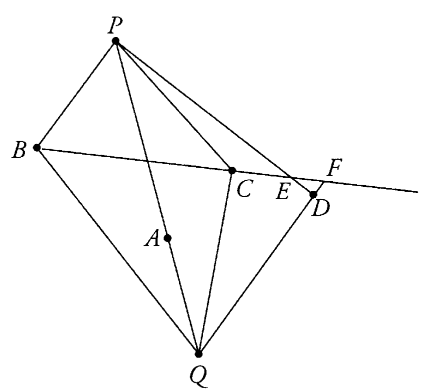
7. If P, Q, and R are the respective feet of the altitudes on the sides of BC, CA, and AB of $\triangle ABC$, show that

$$P(QR, AB) = -1.$$

8. Given $C(O, r)$ and ordinary points A, B, C, and D on a ray through O, inverting into A', B', C', and D', respectively, show that $(AB, CD) = (A'B', C'D')$; that is, that the cross ratio is invariant under inversion.

   Hint: Use the distance formula $A'B' = \dfrac{r^2}{OA \cdot OB} AB.$

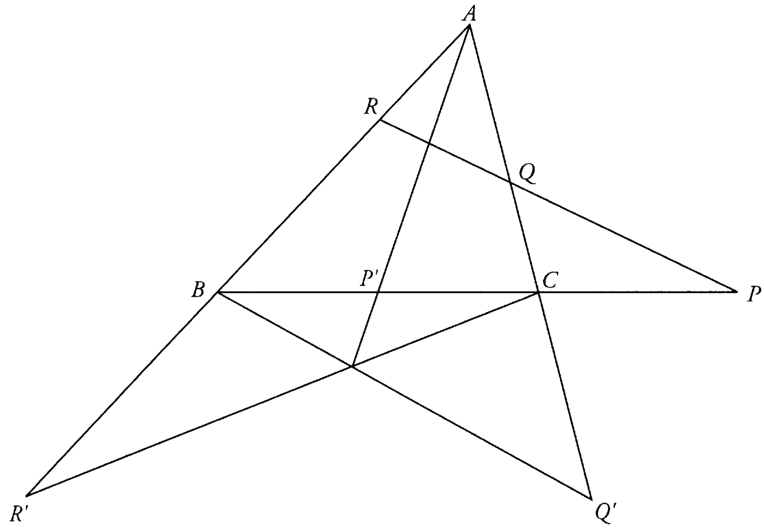
9. Prove the following: If PA, PB, PC, PD and QA, QB, QC, QD are two pencils of lines, and if $P(AB, CD) = Q(AB, CD)$ and A is on PQ, then B, C, and D are collinear.



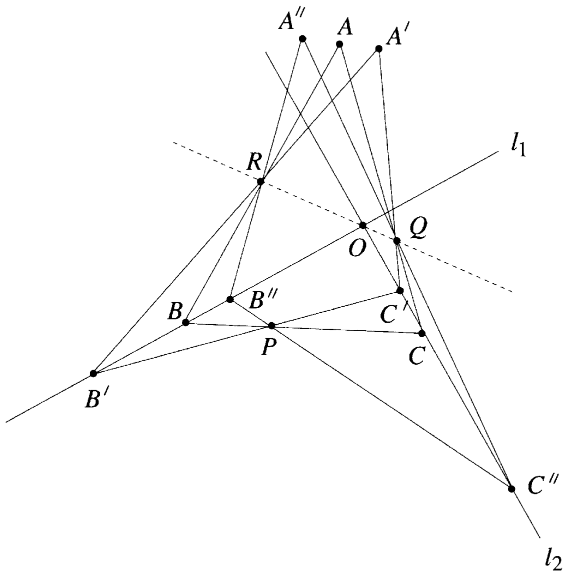
10. In $\triangle ABC$ on the following page we have

$$(BC, PP') = (CA, QQ') = (AB, RR') = -1.$$

   Show that AP', BQ', and CR' are concurrent if and only if P, Q, and R are collinear.



11. Given a variable triangle $\triangle ABC$ whose sides BC, CA, and AB pass through fixed points P, Q, and R, respectively, then if the vertices B and C move along given lines through a point O collinear with Q and R, find the locus of the vertex A.



12. If $V(AB, CD) = -1$ and if VC is perpendicular to VD, show that VC and VD are the internal bisector and external bisector of $\angle AVB$.

13. The bisector of angle A of $\triangle ABC$ intersects the opposite side at the point T. The points U and V are the feet of the perpendiculars from B and C, respectively, to the line AT. Show that U and V divide AT harmonically; that is, that $(AT, UV) = -1$.

14. A line through the midpoint A' of side BC of $\triangle ABC$ meets the side AB at the point F, side AC at the point G, and the parallel through A to side CB at the point E. Show that the points A' and E divide FG harmonically; that is, that $(FG, A'E) = -1$.

15. Prove the second part of Desargues' Theorem using cross ratios; that is, show that coaxial triangles are copolar.

CHAPTER 16

INTRODUCTION TO PROJECTIVE GEOMETRY

16.1 Straightedge Constructions

We saw earlier that a compass alone is as "powerful" as a compass combined with a straightedge. We begin this section by indicating why a straightedge alone is not as powerful as a straightedge and compass or a compass alone. There are only a few admissible operations that can be done with a straightedge by itself.

Admissible Straightedge Operations

1. Draw an arbitrary line.
2. Draw a line through a given or previously constructed point.
3. Draw a line through two given or previously constructed points.
4. Construct a point as the intersection of two different lines.

Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

A ***straightedge construction*** is a finite sequence of the above operations.

We will give informal proofs that certain well-known constructions with straightedge and compass are not possible with a straightedge alone.

One of the standard straightedge and compass constructions is bisecting a given line segment.

Theorem 16.1.1. *Using only a straightedge, we cannot construct the midpoint of a given segment.*

Proof. The idea behind the proof is that a straightedge construction is projectively invariant. Here we give an intuitive justification of the theorem.

Suppose that there is a finite sequence of the possible straightedge operations that yield the midpoint of a segment AB. In other words, there is a sequence of instructions that, when followed, produces the midpoint of AB. For example, the first few instructions might be:

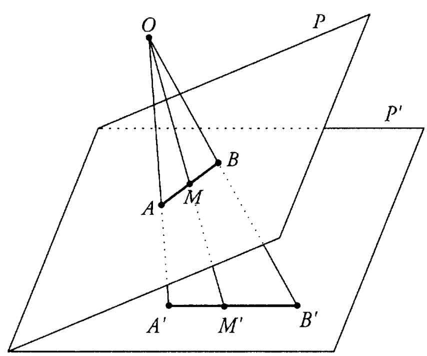
(1) Draw a line l through endpoint A.

(2) Draw a line m through endpoint B.

(3) Let C be the point of intersection of l and m.

$\vdots$

In the plane $\mathcal{P}$, carry out the instructions that yield the midpoint M of the segment AB. Now let $\mathcal{P}'$ be a plane that is not parallel to $\mathcal{P}$, as shown in the figure below.



Let O be a point not on $\mathcal{P}$ or $\mathcal{P}'$ and "project" AB onto $\mathcal{P}'$ from O.

Points in $\mathcal{P}$ are projected to points in $\mathcal{P}'$. Straight lines in $\mathcal{P}$ are projected to straight lines in $\mathcal{P}'$. The segment AB in P is projected into a segment $A'B'$ in $\mathcal{P}'$, and each point M of AB projects to a point M' in $A'B'$.

Each of the four permissible operations in $\mathcal{P}$ projects into exactly the same operation in $\mathcal{P}'$. Thus, if a finite sequence of these operations yields the point M of AB, the same finite sequence of instructions carried out in $\mathcal{P}'$ would yield the projected point M' of $A'B'$. However, projection from the point O does not preserve midpoints, so it seems that on the one hand the sequence of instructions does yield a midpoint while on the other hand it does not yield a midpoint.

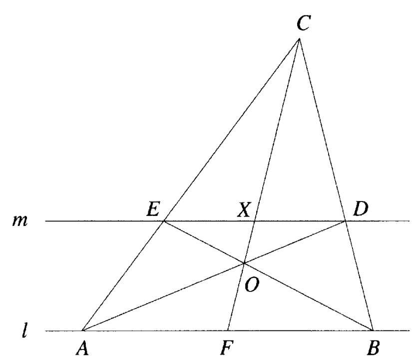
□

Thus, we see that this construction is possible with a straightedge and compass, but not with a straightedge alone. However, if we have a line parallel to the segment, then we can construct the midpoint of the segment using a straightedge alone, as in the following example.

Example 16.1.2. *Given two parallel lines and points A and B on one of them, construct the midpoint of the segment $\overline{AB}$ using a straightedge alone.*

Solution. Using the straightedge, construct a point C such that the sides AC and BC of $\triangle ABC$ are cut by the other parallel at E and D, respectively, as in the figure. Let O be the point of intersection of AD and BE, and draw the line through C and O hitting AB at F. Let CF meet m at X. Since l and m are parallel, then $\triangle CFA \sim \triangle CXE$, which implies that

$$\frac{CA}{CE} = \frac{CF}{CX},$$



so that

$$\frac{CE + EA}{CE} = \frac{CX + XF}{CX},$$

that is, that

$$\frac{EA}{CE} = \frac{XF}{CX}.$$

Also, since $\triangle CFB \sim \triangle CXD$, a similar argument shows that

$$\frac{BD}{DC} = \frac{XF}{CX},$$

so that

$$\frac{BD}{DC} = \frac{EA}{CE}.$$

Now, since the three cevians AD, BE, and CF are concurrent at O, and all divisions are internal, from Ceva's Theorem we have

$$1 = \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{AF}{FB} \cdot \frac{EA}{CE} \cdot \frac{CE}{EA} = \frac{AF}{FB},$$

so that $AF = FB$ and F is the midpoint of AB.

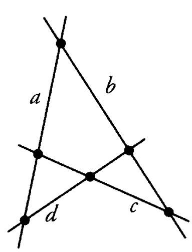
□

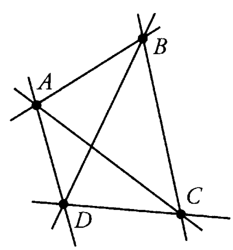
Harmonic Conjugates and Complete Quadrilaterals

What can be constructed using only a straightedge? Problem 6.10 in Section 6.10 illustrates that there are some constructions that can be done.

We will show how to construct the harmonic conjugate of a point using only a straightedge. It is convenient at this point to introduce some terminology.

The figure on the left below consists of four lines a, b, c, and d, no three of which are concurrent, along with the six points formed by the intersections of each pair of lines. Such a figure is called a ***complete quadrilateral***. In the projective plane, some lines may be parallel, in which case some points of intersection may be ideal points. Also, one of the lines may be the ideal line.





The figure on the right illustrates the dual notion, which is a configuration consisting of four points, no three of which are collinear, together with the six lines connecting each pair of points. This configuration is called a ***complete quadrangle***. In the projective plane, one or two of the points may be ideal points, and one of the lines may be the ideal line.

Before we give the construction, we prove a lemma that we alluded to at the beginning of Chapter 15 (Note 2 in Section 15.1).

Lemma 16.1.3. *Given four collinear points A, B, C, and D, the points C and D divide AB harmonically if and only if $(AB, CD) = -1$ or, equivalently, if and only if $(CD, AB) = -1$.*

Proof. Suppose C and D divide AB harmonically. Then

$$\frac{AC}{CB} = \frac{AD}{DB},$$

which implies that

$$\frac{AC/CB}{AD/DB} = 1.$$

However, one of C or D is between A and B while the other is not.

If C is between A and B, we have $\overline{AC}/\overline{CB} = 1$; if not, we have $\overline{AC}/\overline{CB} = -1$. Thus, one of $\overline{AC}/\overline{CB}$ and $\overline{AD}/\overline{DB}$ is positive and the other is negative, which implies that

$$\frac{\overline{AC}/\overline{CB}}{\overline{AD}/\overline{DB}} = -1.$$

Conversely, suppose that $(AB, CD) = -1$. Then

$$\frac{\overline{AC}/\overline{CB}}{\overline{AD}/\overline{DB}} = -1,$$

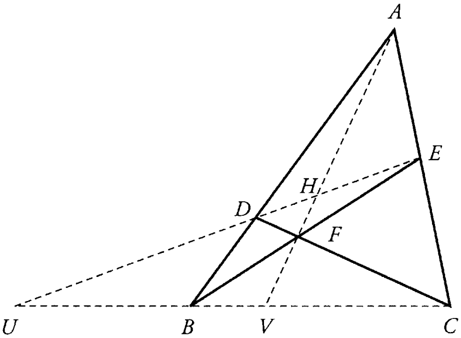
and taking absolute values, this implies that

$$\frac{AC}{CB} = \frac{AD}{DB}.$$

□

Theorem 16.1.4. *Given a complete quadrilateral, the two points of intersections of one given diagonal with the other two diagonals divide the vertices of the given diagonal harmonically.*

Proof. The following figure illustrates the situation, where the six points of intersection of the sides are A, B, C, D, E, and F, and the three possible diagonals are shown as dashed lines.



Here we wish to show that the points where diagonals ED and AF intersect BC divide BC harmonically; that is, we want to show that $(UV, BC) = -1$.

Consider the pencils at A and F. We have

$$(UV, BC) = A(UV, BC) = (UH, DE) = F(UV, CB) = (UV, CB).$$

Now recall that if $(UV, BC) = k$, then $(UV, CB) = 1/k$. Therefore,

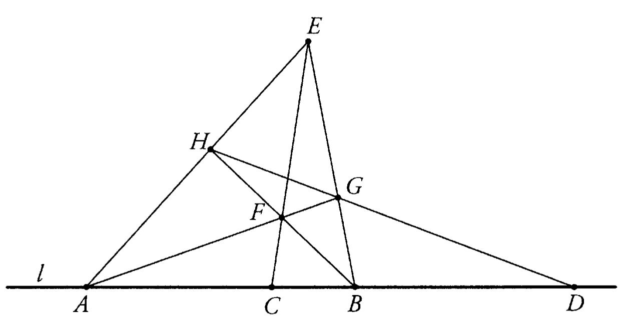
$$(UV, BC) = 1/(UV, BC)$$

so that $(UV, BC)^2 = 1$; that is, (UV, BC) is either 1 or -1. Since U and V separate B and C, we must have $(UV, BC) = -1$.

□

Example 16.1.5. *Given points A, B, and C on a line l, construct the harmonic conjugate D of C using only a straightedge.*

Solution. The analysis figure is shown below.



Construction:

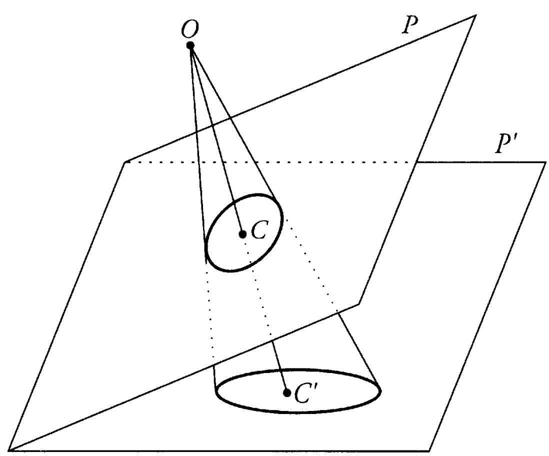
1. Choose point E not on l.
2. Construct the lines EA, EC, and EB.
3. Choose a point F on EC.
4. Construct the lines FA and FB meeting EA and EB at G and H.
5. Construct the line HG and let D be the intersection of HG and l.

Justification:

The fact that D is the harmonic conjugate of C follows from Theorem 16.1.4.

□

Theorem 16.1.6. *Given a circle with unknown center, we cannot construct its center using a straightedge alone.*



Proof. The proof is essentially the same as the proof of Theorem 16.1.1.

Using O as the center of projection, project a circle in plane $\mathcal{P}$ into an ellipse in plane $\mathcal{P}'$. The center C of the circle in plane $\mathcal{P}$ projects to a point C' in plane $\mathcal{P}'$, but C' is not the center of the ellipse in $\mathcal{P}'$.

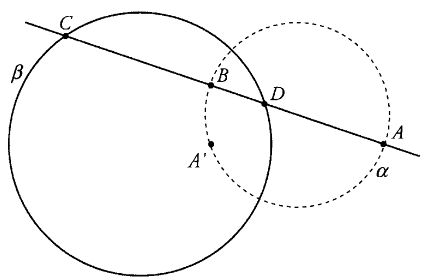
Let M be the center of the ellipse. Then there is a point C'' in $\mathcal{P}'$ (not shown in the figure) such that M is the midpoint of $C'C''$. However, a sequence of instructions for the straightedge construction of C in $\mathcal{P}$ would "project" into the same sequence of instructions for the straightedge construction of C' in $\mathcal{P}'$.

However, with regard to the ellipse in $\mathcal{P}'$, why would the sequence of instructions always yield C' and never C''? As far as the operations in $\mathcal{P}'$ are concerned, there is no distinction between C' and C''.

□

Remark. When it was realized that a straightedge alone could not be used to solve all the construction problems that could be done with a straightedge and compass, the question arose as to what sort of minimal "equipment" was needed in addition to a straightedge. In 1822, Victor Poncelet asserted that all that was needed was a single circle with its center, and in 1833, Jacob Steiner gave a systematic proof of this fact. Such constructions are called ***Poncelet-Steiner constructions***.

Theorem 16.1.7. *If A and B are conjugate points on a line that cuts the circle β at C and D, then A and B are divided harmonically by C and D.*



Proof. If B is the inverse of A, then this is part of Theorem 14.1.6.

Supposing that B is not the inverse of A, assume that A is outside β. Then B is inside since AB cuts β and B is on the polar of A by assumption. Let A' be the inverse of A and let α be the circle through A, A', and B. Recalling that two intersecting circles are orthogonal if one of the circles passes through two distinct points that are inverses with respect to the other circle (see Theorem 13.3.3), this means that α is orthogonal to β.

Since B is on the polar of A, the line $A'B$ is the polar of A, so $\angle AA'B$ is a right angle. From the converse of Thales' Theorem, AB is a diameter of α, so that β intersects the diameter of α at C and D. By Theorem 14.1.11, β divides the diameter AB of α harmonically; that is, C and D divide A and B harmonically.

□

The converse to the previous theorem is also true.

Theorem 16.1.8. *If a line AB cuts a circle β at C and D, and if C and D divide A and B harmonically, then A and B are conjugate points.*

Proof. Assume that A is outside β, and assume that E is a point on the line AB such that A and E are conjugate points. Then C and D divide A and E harmonically by the previous theorem; that is, $(AE, CD) = -1$.

However, by hypothesis, $(AB, CD) = -1$, so that

$$(AB, CD) = (AE, CD),$$

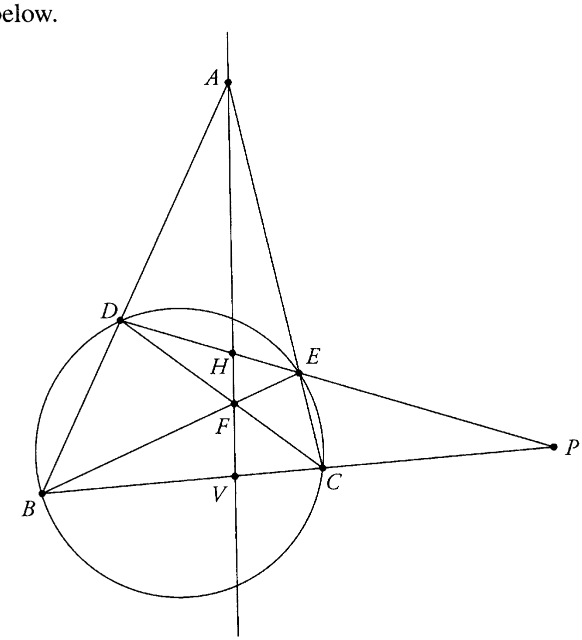
and it follows that $B = E$; that is, A and B are conjugate points.

□

The preceding theorems and examples lead to the straightedge construction of the polar of a point with respect to a given circle without its center.

Theorem 16.1.9. *Given a circle ω without its center, and given a point P outside ω, it is possible to construct the polar of P using only a straightedge.*

Proof. Through P, draw two lines intersecting the circle at B, C and D, E as shown in the figure below.



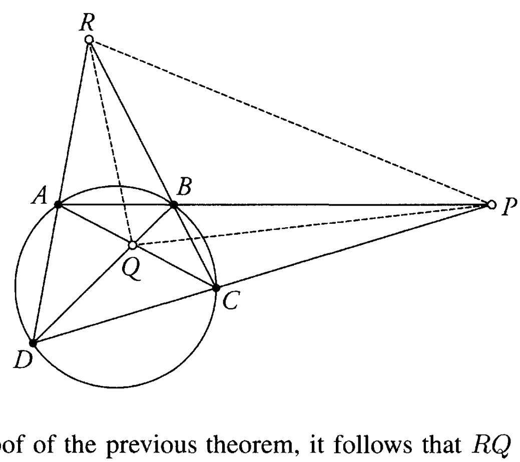
Let $A = BD \cap CE$ and $F = CD \cap BE$. Then the lines AB, AE, CD, and BE are the sides of a complete quadrilateral with diagonals AF, DE, and BC. From Theorem 16.1.4, AF and DE divide BC harmonically; that is, V is the harmonic conjugate of P, while from Theorem 16.1.8, P and V are conjugate points; that is, V is on the polar of P.

Similarly, H, the intersection of the diagonals AF and DE, is on the polar of P. Thus, VH is the polar of P, and the construction is complete.

□

Theorem 16.1.10. *Let $ABCD$ be a complete quadrangle inscribed in a circle. Let P, Q, and R be the points of intersection of the opposite sides. Then PQR is a self-polar triangle; that is, P is the pole of QR, Q is the pole of PR, and R is the pole of PQ.*

Proof. Referring to the figure below, note that the lines RD, RC, AC, and BD form a complete *quadrilateral*.



Thus, by the proof of the previous theorem, it follows that $RQ = p$, the polar of P, and similarly, $PQ = r$. Since Q is on both p and r, then by the reciprocation theorem, P and R are both on q; that is, $PR = q$.

□

Another approach to this is given at the end of Section 16.6.

16.2 Perspectivities and Projectivities

Given two planes $\mathcal{P}$ and $\mathcal{P}'$ and a point O not on either plane, a ***perspectivity*** or ***perspective transformation*** from $\mathcal{P}$ onto $\mathcal{P}'$ is a one-one correspondence between points X of $\mathcal{P}$ and points X' of $\mathcal{P}'$ such that if X is transformed into X', then the line XX' passes through O. The point O is called the ***center of perspectivity***.

The three-dimensional analogue of the projective plane is ***projective 3-space***, which is obtained by appending a "plane at infinity" to Euclidean 3-space. Note that the appended plane is a projective plane.

The same definitions of "perspectivity" and "center of perspectivity" apply to projective 3-space.

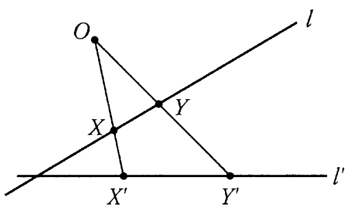
Note.

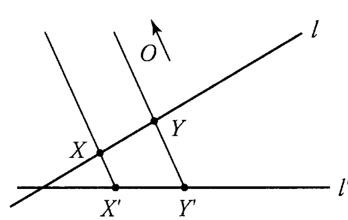
1. The transformation is one-one and onto, so every point in $\mathcal{P}$ is sent to a point in $\mathcal{P}'$, and every point in $\mathcal{P}'$ is the image of some point in $\mathcal{P}$.
2. If the center of perspectivity is an ideal point, the transformation is called a ***parallel perspectivity***.

Questions.

1. What gets mapped to the ideal points on $\mathcal{P}'$?
2. Let $\mathcal{P}''$ be a plane parallel to $\mathcal{P}$. What gets mapped to the points X' on $\mathcal{P}' \cap \mathcal{P}''$?
3. Where are the ideal points on $\mathcal{P}$ mapped to?

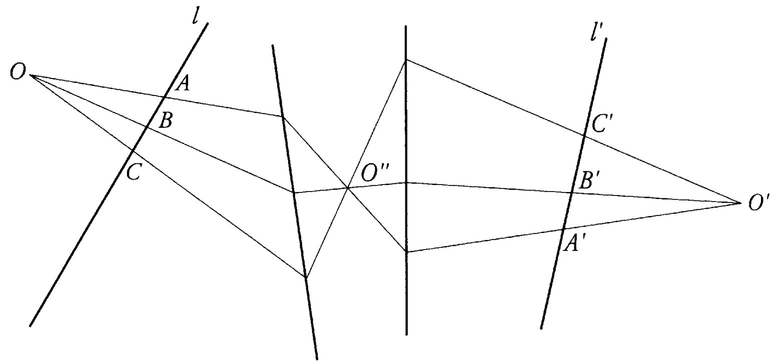
Given two lines l and l' in a plane and a point O not on either line, a ***perspectivity*** or ***perspective transformation*** from l onto l' is a one-one correspondence between points X of l and points X' of l' such that if X is transformed into X', then the line XX' passes through O. The point O is called the ***center of perspectivity***.





If O is an ideal point, as illustrated in the figure on the right above, the result is called a ***parallel perspectivity***. As before, the transformation is one-one and onto.

A finite sequence of perspectivities in a plane is called a ***projectivity***. Note that a perspectivity is always a projectivity, but a projectivity is not necessarily a perspectivity.



Perspectivities and projectivities from one plane to another in 3-space are defined similarly.

We will use lowercase Greek letters to denote perspectivities and projectivities. Two projectivities π_1 and π_2 from l to l' are ***equal*** if $\pi_1(X) = \pi_2(X)$ for each X in l. Note that for four perspectivities $\sigma_1 \neq \sigma_1'$ and $\sigma_2 \neq \sigma_2'$, it is possible that the projectivities $\sigma_2 \circ \sigma_1$ and $\sigma_2' \circ \sigma_1'$ are equal.

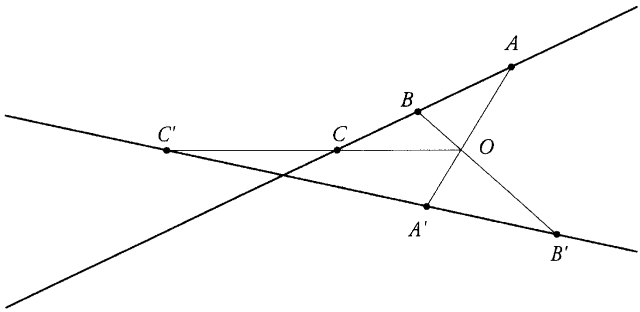
Effect on Euclidean Properties

In the following table, the properties on the left are preserved by projectivities, and the properties on the right are not preserved by projectivities.

Preserved	Not Preserved
• points	• midpoints
• straight lines	• distance
• collinearity	• angle size
• concurrency	• circularity
• incidence	• order
• triangles	
• cross ratios	

Note that the table on the previous page indicates that ***cross ratios are preserved by perspectivities***. This fact is an immediate consequence of Theorem 15.1.7, and it plays a crucial role in projective geometry.

The following figure illustrates that order is not preserved.



Projectivities in 2-Space

You may recall from linear algebra that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is completely determined by its action on n linearly independent points; that is, if the points

$$a_1, \quad a_2, \quad \ldots, \quad a_n$$

are linearly independent and are mapped respectively to

$$a_1', \quad a_2', \quad \ldots, \quad a_n',$$

then this enables us to determine what every other point is mapped to.

There is an analogous situation regarding perspectivities and projectivities that we explore in this section. In particular, the following questions arise:

1. How many points are required to completely determine a perspectivity?
2. How many points are required to completely determine a projectivity?

Before addressing these questions, we remind ourselves that two projectivities are equal if and only if they have the same effect on every point. The two projectivities could be the composition of different perspectivities, or even different numbers of perspectivities, but that is immaterial to the definition of equality.

Theorem 16.2.1. *Given two distinct points A and B on a line l and two distinct points A' and B' on a line l', with $l \neq l'$ and none of the four points being $l \cap l'$, there is a unique perspectivity that takes A to A' and B to B'.*

Proof. Let $O = AA' \cap BB'$. Then the perspectivity centered at O takes A to A' and B to B'.

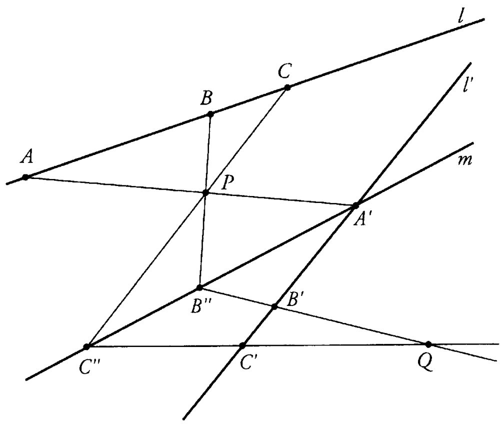
Call this perspectivity π_1, and suppose that π_2 is another perspectivity that takes A to A' and B to B'. The center of π_2 is $AA' \cap BB' = O$, and since a perspectivity is completely determined by its center, π_1 and π_2 are identical.

□

The proof of the corresponding theorem for projectivities is a bit more delicate.

Theorem 16.2.2. *Suppose l and l' are two distinct lines, that A, B, and C are three distinct points on l, and that A', B', and C' are three distinct points on l'. Then there is a unique projectivity from l to l' that takes A to A', B to B', and C to C'.*

Proof. First we prove existence. We may assume that $A \neq A'$ because at least two of A, B, and C differ from $l \cap l'$.



Draw a line m through A' that does not coincide with l' and that misses A. Pick a point P on AA' other than A or A'. Using P as the center of perspectivity, map l onto m and denote this perspectivity by σ_1. This takes B to B'' and C to C''.

Let $Q = B'B'' \cap C'C''$. Using Q as the center of perspectivity, map m onto l' and denote this perspectivity by σ_2. Let $\pi = \sigma_2 \circ \sigma_1$. The projectivity π maps A to A', B to B', and C to C'.

Next we prove uniqueness. We have to show that if π_1 and π_2 are projectivities that both take A, B, and C to A', B', and C', respectively, then

$$\pi_1(X) = \pi_2(X)$$

for every X in l.

Suppose that for some X we have

$$\pi_1(X) = X_1,$$
$$\pi_2(X) = X_2.$$

Since projectivities preserve cross ratios, we have

$$(AB, CX) = (A'B', C'X_1),$$
$$(AB, CX) = (A'B', C'X_2),$$

so that

$$(A'B', C'X_1) = (A'B', C'X_2),$$

which implies that

$$X_1 = X_2.$$

□

Theorem 16.2.3. *(The Fundamental Theorem of Projective Geometry)*

A projectivity in the plane from a line l to a line l' is completely determined by its action on three distinct points.

Proof. There are two cases to consider: where l and l' are different and where $l = l'$. The first case is the previous theorem.

For the case where $l = l'$, we will show that given three distinct points A, B, and C on l and three more distinct points A', B', and C' also on l, there is a unique projection that takes A to A', B to B', and C to C'. To establish the existence of such a projection, draw a line m different than l, and let σ be a perspectivity that maps l onto m. Let A'', B'', and C'' be the images of A, B, and C under σ. By the previous theorem, there is a projectivity π that maps m to l and which takes A'', B'', and C'' to A', B', and C', respectively. The composition $\pi \circ \sigma$ is a projectivity from l to l that takes A, B, and C to A', B', and C', respectively. This establishes the existence of a projectivity, and its uniqueness follows as before via cross ratios.

□

Corollary 16.2.4. *A projectivity from a line l to a different line l' can always be expressed as a sequence of two or fewer perspectivities or a sequence of three or fewer perspectivities if $l = l'$.*

16.3 Line Perspectivities and Line Projectivities

The perspectivities and projectivities described so far are sometimes referred to as being ***central perspectivities*** and ***central projectivities***. These two notions can be dualized. Two pencils L and L' of lines are said to be ***perspective from a line*** p if there is a mapping from L to L' that takes each line x of L to a line x' of L' in such a way that p, x, and x' are concurrent. The line p is called the ***axis of perspectivity***. Such a perspectivity is called a ***line perspectivity***, and a finite sequence of line perspectivities is called a ***line projectivity***.

It is important to realize that any theorem about central projectivities has a dual theorem about line projectivities that we can obtain free of charge. For example, here is the dual of the Fundamental Theorem:

Theorem 16.3.1. *A line projectivity from a pencil L to a pencil L' is completely determined by its action on three distinct lines a, b, and c of L.*

When the words *projectivity* and *perspectivity* are used without modifiers, they are understood to mean *central projectivity* and *central perspectivity* unless the context makes it perfectly obvious that the dual meaning should be used.

16.4 Projective Geometry and Fixed Points

When Is a Projectivity a Perspectivity?

Given a transformation π, any point A for which

$$\pi(A) = A$$

is called a ***fixed point*** of π.

Note that if π is a transformation from a line l to a line m, then for $l \neq m$ the only possible fixed point of π is $l \cap m$.

Theorem 16.4.1. *Suppose l and m are distinct lines. A projectivity π from l to m is a perspectivity if and only if π has a fixed point.*

Proof. If π is a perspectivity, then $l \cap m$ is a fixed point.

On the other hand, suppose that $A = l \cap m$ is a fixed point of π (the only point that could possibly be a fixed point). Let B and C be two other points on l, and let $B' = \pi(B)$ and $C' = \pi(C)$.

Now, by Theorem 16.2.1, there is a unique perspectivity σ from l to m that takes B and C on l to B' and C' on m, respectively.

However, σ also maps A to $A' = A$. Consequently, both π and σ map A, B, and C to A', B', and C', respectively, and so $\pi = \sigma$ by the Fundamental Theorem of Projective Geometry.

□

Theorem 16.4.2. *A projectivity π from a line to itself is a perspectivity if and only if it has at least three fixed points.*

We leave the proof as an exercise.

Alternate Characterizations

Theorem 16.4.3. *Suppose that*

$$\pi : k \to l \qquad \textit{and} \qquad \sigma : l \to m$$

are two perspectivities with different centers. Then $\sigma \circ \pi$ is a perspectivity if and only if k, l, and m are concurrent.

Proof. Let A and A' be the centers of perspectivities for π and σ.

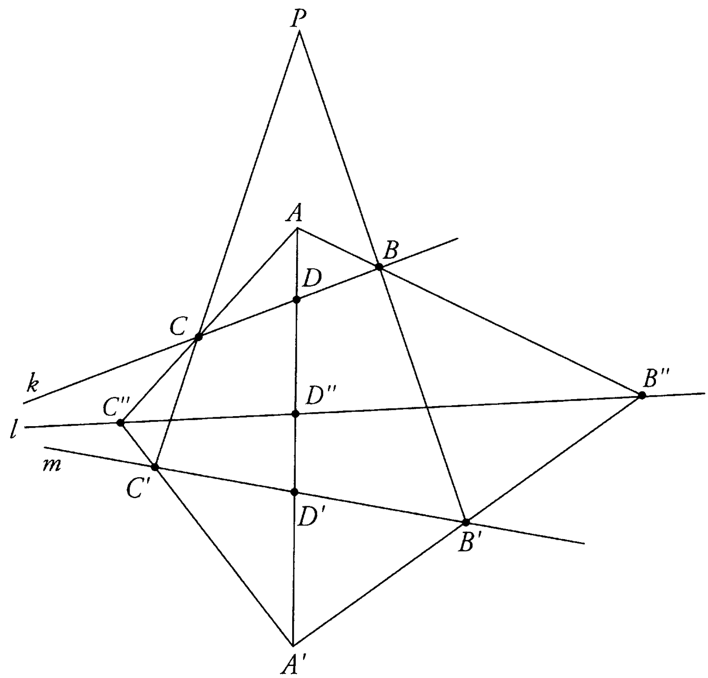
There are two things to prove:

(1) If $\sigma \circ \pi$ is a perspectivity, then k, l, and m are concurrent.

(2) If k, l, and m are concurrent, then $\sigma \circ \pi$ is a perspectivity.

We will prove (1) and leave (2) as an exercise.

Assuming that $\sigma \circ \pi$ is a perspectivity from k to m, let P be the center of the perspectivity.



Let B and C be distinct points on k, and let

$$B'' = \pi(B) \qquad \text{and} \qquad B' = \sigma(B'')$$

and

$$C'' = \pi(C) \qquad \text{and} \qquad C' = \sigma(C'').$$

Then $P = BB' \cap CC'$.

Now draw the line AA' and let D, D'', and D' be defined as follows:

$$D = AA' \cap k,$$
$$D'' = AA' \cap l,$$
$$D' = AA' \cap m.$$

Then π maps D to D'' and σ maps D'' to D'.

Consequently, $\sigma \circ \pi$ maps D to D', and since $\sigma \cap \pi$ is really a perspectivity with center P, the line DD' must pass through P. Also, since $\overleftrightarrow{DD'} = \overleftrightarrow{AA'}$, this means that $\overleftrightarrow{AA'}$ passes through P.

Therefore, $\triangle ABC$ and $\triangle A'B'C'$ are perspective from P; that is, they are *copolar*. Hence, the triangles are perspective from a line; that is, they are *coaxial* by Desargues'

Two Triangle Theorem. However,

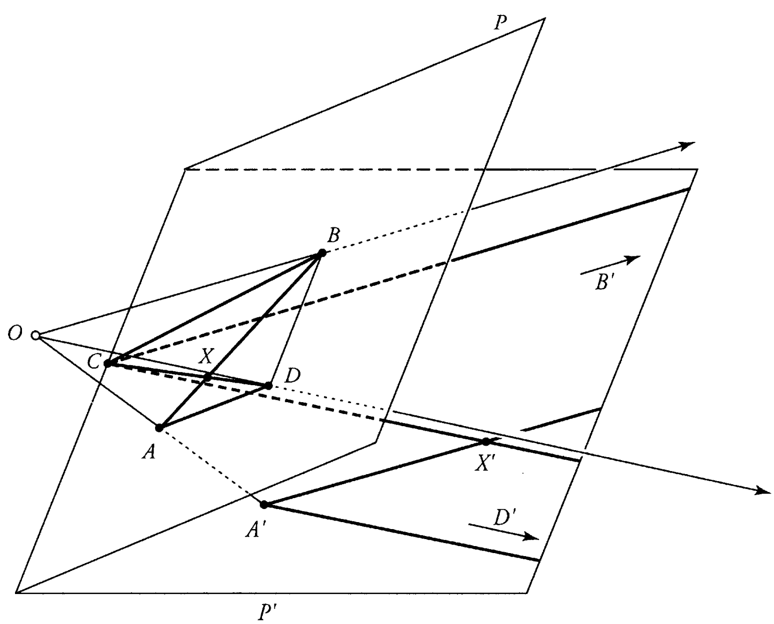
$$AB \cap A'B' = B'' \qquad \text{and} \qquad AC \cap A'C' = C'',$$

and so the axial line is $B''C''$, namely, l. Since $BC = k$ and $B'C' = m$, the point $k \cap m$ is on l, showing that k, l, and m are concurrent.

□

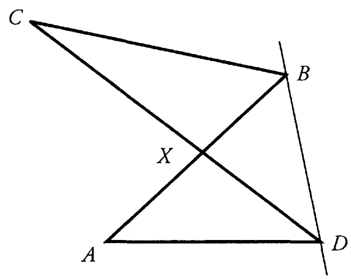
16.5 Projecting a Line to Infinity

Straight lines and incidence properties are preserved by projections, and this is sometimes advantageous. In the figure below, the quadrilateral $ABCD$ lies in a plane $\mathcal{P}$. Another plane $\mathcal{P}'$ intersects $\mathcal{P}$ in a line parallel to the diagonal BD of the quadrilateral. The point O is the center of a perspectivity from $\mathcal{P}$ to $\mathcal{P}'$, and O is positioned so that the plane defined by the points O, B, and D is parallel to $\mathcal{P}'$. With this perspectivity, points B and D are projected to ideal points B' and D' since the lines OB and OD are parallel to $\mathcal{P}'$. Points A and C are projected to ordinary points A' and C'. In this particular case, $C' = C$, since the point C happens to be on the line $\mathcal{P} \cap \mathcal{P}'$.

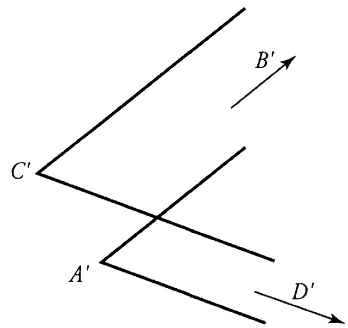


In the plane $\mathcal{P}'$, the line $B'D'$ is the ideal line, that is, the line at infinity. Consequently, the process above is called *sending the line BD to infinity*.

The projected lines $A'B'$ and $C'B'$ are necessarily parallel, as are the lines $A'D'$ and $C'D'$. This is all that we need to know in order to depict the figure with the line sent to infinity. In figure (a) below we have quadrilateral $ABCD$, with its projected image $A'B'C'D'$ shown in figure (b), and there is really no need to draw the planes $\mathcal{P}$ and $\mathcal{P}'$.



(a)



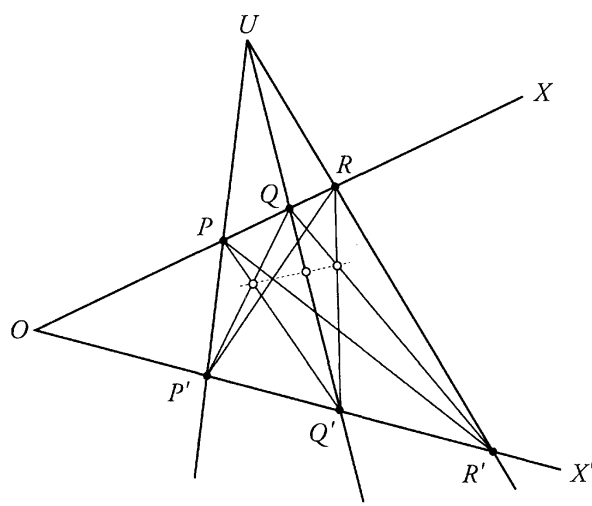
(b)

We illustrate the use of this technique to prove the following version of Pappus' Theorem.

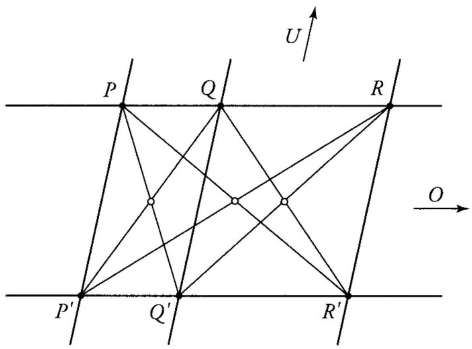
Theorem 16.5.1. *If the three lines APP', UQQ', and URR' meet two lines OX and OX' at P, Q, R and P', Q', R', respectively, then the points*

$$PQ' \cap P'Q, \qquad PR' \cap P'R, \qquad \textit{and} \qquad QR' \cap Q'R$$

are collinear.



Proof. Send the line OU to infinity. In the diagram below, we use the same letter before and after the projection.



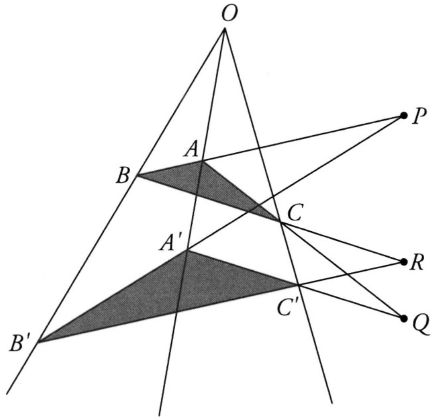
The lines PP', QQ', and RR' are parallel, as are PR and $P'R'$. Then $PQ' \cap P'Q$ is the intersection of the diagonals of the parallelogram $PQQ'P'$. Similarly, $PR' \cap P'R$ and $QR' \cap Q'R$ are the intersections of the diagonals of parallelograms, so all three points lie on a line midway between PR and $P'R'$.

□

As another example, we use the same technique to prove part of Desargues' Theorem.

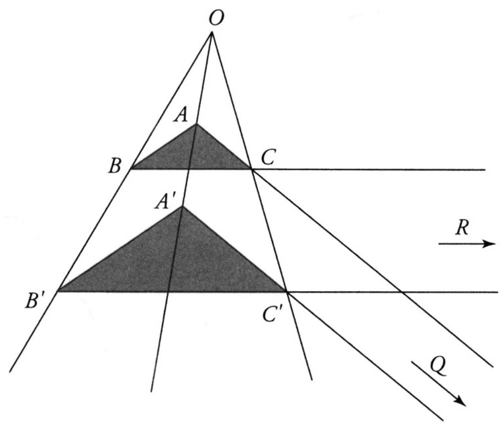
Example 16.5.2. *Show that copolar triangles are coaxial.*

Solution. In the figure below, triangles ABC and $A'B'C'$ are perspective from the point O. We want to show that the intersections of the corresponding sides, namely, P, Q, and R, are collinear.



A solution may be obtained by sending the line QR to infinity and then showing that the projection also sends point P to the ideal line. The reverse projection then maps the ideal line back to the original line QR, and since it preserves incidence, the point P must lie on the line QR.

The projection of the line QR to infinity is shown below. As before, we use the same letters before and after the projection.



For triangles OBC and $OB'C'$, since BC and $B'C'$ are parallel, we have

$$\frac{OB}{BB'} = \frac{OC}{CC'}.$$

For triangles OAC and $OA'C'$, since AC and $A'C'$ are parallel, we have

$$\frac{OA}{AA'} = \frac{OC}{CC'}.$$

Consequently,

$$\frac{OB}{BB'} = \frac{OA}{AA'},$$

which implies that AB is parallel to $A'B'$ and, thus, the projected point

$$P = AB \cap A'B'$$

is on the ideal line.

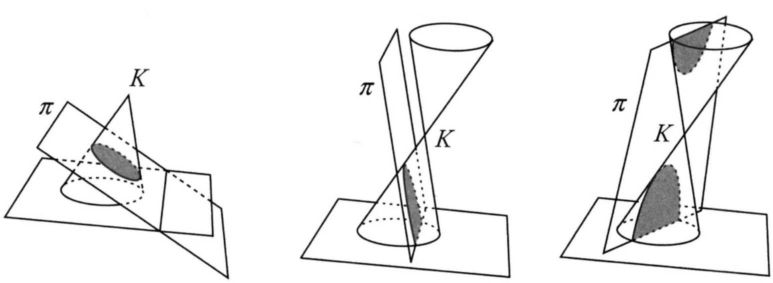
□

16.6 The Apollonian Definition of a Conic

Originally, the ancient Greeks defined conic sections as cross sections of particular types of right circular cones. Apollonius recognized that all of the conic sections can be obtained from any given circular cone. He defined a circular cone as follows:

> Let K be a point and ω a circle whose plane does not contain K. If a straight line, passing always through K, be made to move around the circumference of ω, the moving straight line will trace out the surface of a ***circular cone***.

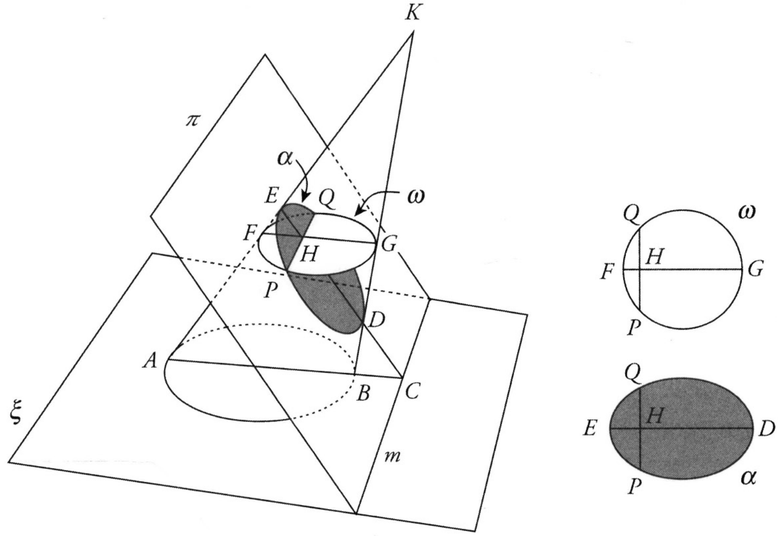
The line through K that passes through the center of ω is called the ***axis*** of the cone. If the axis of the cone is perpendicular to the plane of the circle, the cone is called a ***right circular cone***, otherwise the cone is called an ***oblique circular cone***. Note that the cone has two components, so they are often called ***two-napped cones***.



Ellipse Parabola Hyperbola

Apollonius defined a (proper) ***conic section*** as a curve that is formed by intersecting a plane with a circular cone in such a way that the plane does not contain the vertex K of the cone. The figure above shows what happens as the tilt of the plane is increased. When the plane is tilted so that it intersects only one nappe of the cone and is not parallel to any of the generating lines of the cone, the result is an ellipse. If the plane is parallel to one of the lines generating the cone, but it still only intersects one nappe, the result is a parabola. When it intersects both nappes of the cone, the result is a hyperbola.

It is not too difficult to derive the Cartesian equation of the conics from Apollonius' description. Here is how it is done for the ellipse. In the figure on the following page, the plane ξ, and any plane parallel to it, cuts the oblique circular cone in a circle. The plane π is another plane that cuts the cone, forming the curve α. The planes ξ and π meet in a straight line m.



(a) (b)

In the plane ξ, the line AB is a diameter of the circle and is perpendicular to m and $C = AB \cap m$. The plane of the triangle ABK cuts the plane π in the line EDC. To obtain a Cartesian equation for α, let P be an arbitrary point on α, and through P pass a plane ϕ (not shown in the diagram) parallel to ξ cutting the cone in a circle ω, as shown in figure (a) above.

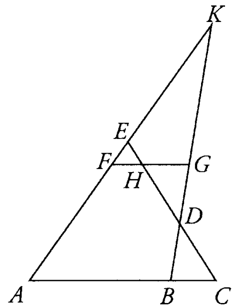
The intersection of ϕ and π is a line parallel to m cutting ω at P and Q. The plane ϕ cuts AK at F and BK at G, so that FG is a diameter of the circle ω. The lines ED, FG, and PQ are concurrent at H.

Let E be the point $(0, 0)$ in π, let x be the distance EH, and let y be the distance PH so that P is the point (x, y). Since $PQ \perp FG$, by the power of the point H with respect to ω (see figure (b) above), we have $PH^2 = FH \cdot HG$. This gives us

$$y^2 = FH \cdot HG. \tag{1}$$

From similar triangles EFH and EAC, we have $FH/AC = EH/EC$, so that

$$FH = \frac{AC}{EC}x. \tag{2}$$



From similar triangles HGD and CBD, we have $HG/CB = HD/CD$. Since $HD = ED - x$, we get

$$HG = \frac{CB}{CD}(ED - x). \tag{3}$$

Letting $ED = 2a$, and substituting (2) and (3) into (1), we obtain

$$y^2 = \frac{AC \cdot CB}{EC \cdot CD} x(2a - x).$$

Denoting the positive quantity $(AC \cdot CB)/(EC \cdot CD)$ by k^2, we have

$$y^2 = k^2 x(2a - x),$$

and so

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{a^2k^2} = 1.$$

This shows how the Cartesian equation for the ellipse arises from the Apollonian definition of the conic.

Poles and Polars of Conics

The Apollonian definition of conics can be described by saying that every conic can be interpreted as being the image of a circle under a central perspectivity. Thus, we can state without proof:

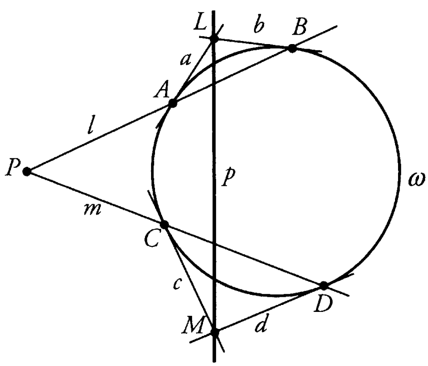
Theorem 16.6.1. *Every theorem about the incidence properties of straight lines and circles remains true when the word "circle" is replaced by the word "conic."*

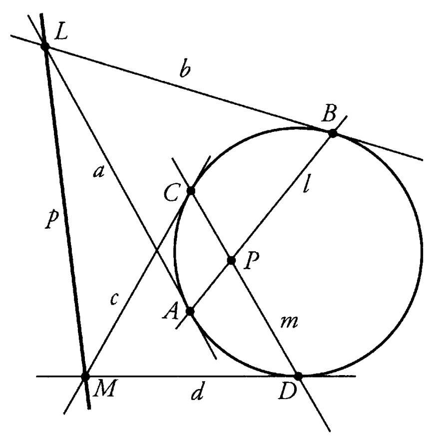
For example, both Pascal's Mystic Hexagon Theorem and Brianchon's Theorem are true for parabolas, ellipses, and hyperbolas.

One of the important consequences of Theorem 16.6.1 is that in order to prove something about the incidence properties of straight lines and conics, we can reduce it to proving the same assertion about circles. There is a caveat, however, in that the properties must be described purely in terms of incidence properties.

For example, the notions of pole and polar for a circle are useful concepts that were not initially described in terms of incidence properties. If P is a point outside a circle ω, the polar of P can be described using incidence properties: let A and B be the points where the tangents from P meet the circle. Then $\overleftrightarrow{AB}$ is the polar of P. If P is a point inside the circle, a description of the polar in terms of the incidence is not immediately evident.

Theorem 16.6.2. *Let P be a point not on the circle ω, and let l and m be lines through P, with $l \cap \omega = \{A, B\}$ and $m \cap \omega = \{C, D\}$. Let a, b, c, and d be the tangent lines through A, B, C, and D. Then the line through $a \cap b$ and $c \cap d$ is the polar of P.*





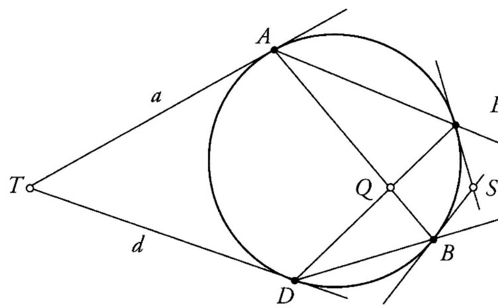
Proof. Let $L = a \cap b$ and $M = c \cap d$. Then L is the pole of l, and M is the pole of m. Since P is on l and m, the Reciprocation Theorem implies that L and M are on p. $\square$

We indicated earlier that the ***polar*** of a point P on a conic is the line p that is tangent to the conic at P. The lemma on the following page allows us to describe the polar of a point P not on the conic entirely in terms of incidence properties.

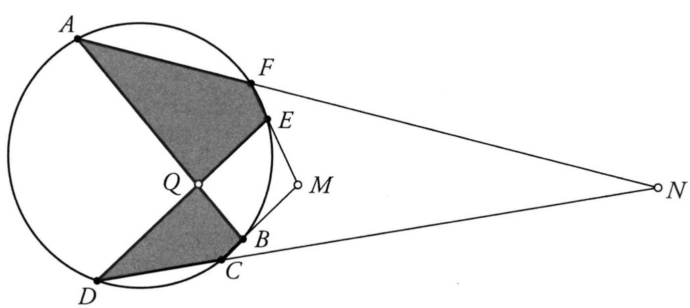
Definition. Suppose ω is a conic. If P is not on the conic, then the ***polar*** of P is the line p described in the preceding theorem with the word "circle" replaced by the word "conic".

Theorems 16.1.8 and 16.1.9 illustrate how to construct the polar using only a straightedge and without necessarily drawing tangents. The next few lemmas provide an alternate approach that illustrates the connection with Pascal's Mystic Hexagon Theorem. In the lemmas, we follow the convention that lowercase letters refer to the polars of points designated by the corresponding uppercase letters.

Lemma 16.6.3. *Suppose that A, E, B, and D are four distinct points on a circle ω. Let $P = AE \cap DB$ and $Q = AB \cap DE$. Let $S = e \cap b$ and $T = a \cap d$. Then P, Q, S, and T are collinear.*

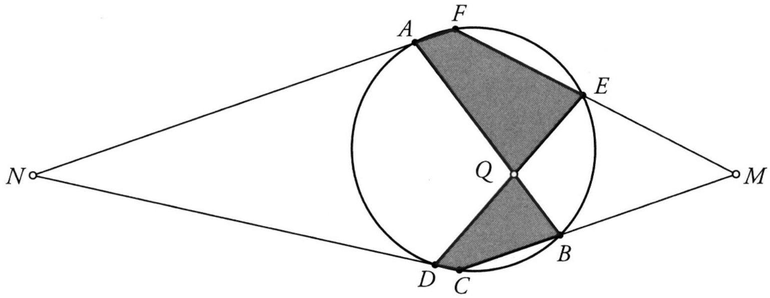


Proof. Introduce points F and C on the circle, as shown in the figure below, so that $ABCDEF$ is an inscribed hexagon. By Pascal's Mystic Hexagon Theorem, the opposite edges meet in three collinear points Q, M, and N, as depicted in the figure.



Keep the points A, E, B and D fixed, and let $F \to E$ and $C \to B$ along the circle. As this happens, M and N move but Q, M, and N remain collinear. The limiting points of M and N are S and P, respectively, so Q, S, and P are collinear.

Now apply a similar procedure to A and D, as shown on the following page. Here, the points M, Q, and N are collinear.



Letting $F \to A$ and $C \to D$ along the circle, the points M and N converge to P and T, respectively, and so P, Q, and T are collinear.

Consequently P, Q, S, and T are collinear.

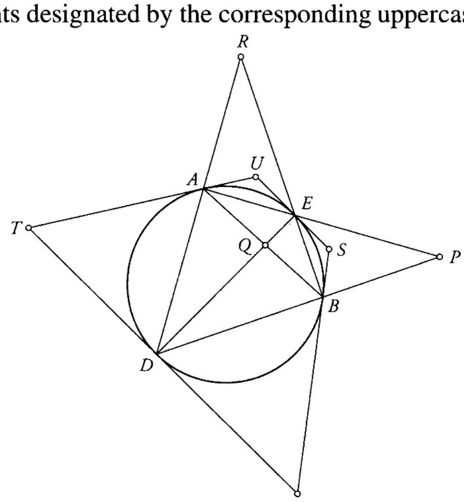
□

The following is proven in a similar way:

Lemma 16.6.4. *Suppose that A, E, B, and D are distinct points on a circle, as in the previous lemma. Let $R = AD \cap EB$ and let $Q = AB \cap DE$, as in the previous lemma, and let $U = a \cap e$ and $V = d \cap b$. Then R, Q, U, and V are collinear.*

Theorem 16.6.5. *Suppose that A, E, B, and D are distinct points on a circle. Let $P = AE \cap BD$, $Q = AB \cap DE$, and $R = AD \cap BE$. Then RQ is the polar of P, PQ is the polar of R, and PR is the polar of Q.*

Proof. In the proof we continue to employ the convention that lowercase letters refer to the polars of points designated by the corresponding uppercase letters.



Introduce the polars a, b, c, and d, and let $U = a \cap e$, $S = e \cap b$, $V = b \cap d$, and $T = a \cap d$.

By the previous lemmas, the points P, Q, S, and T are collinear, as are R, U, Q, and V. By Theorem 16.6.2, UV is the polar of P, which implies that RQ is the polar of P, as claimed. Similarly, ST is the polar of R, which implies that PQ is the polar of R.

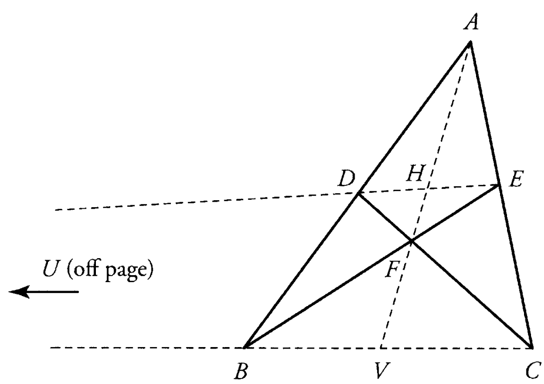
Since Q is on r, R must be on q by the Reciprocation Theorem. Similarly, since Q is on p, P must be on q. That is, R and P are on q, so $RP = q$.

□

16.7 Problems

1. Prove that the construction of the midpoint in Example 16.1.2 works by using the fact that the lines l, m, and CO are the diagonals of a complete quadrilateral.

2. (a) In the complete quadrilateral the sides intersect at points A, B, C, D, E, and F, as shown below, and the dashed lines are the three diagonals with intersection points H, U, and V. Prove that if V is the midpoint of the segment BC, then ED and BC are parallel.



   (b) Given a line segment AB with midpoint M and a point P not on the line AB, explain how to construct a line through P parallel to AB using only a straightedge.

3. Using a straightedge alone, is it possible to construct a right angle? Explain.

4. Using only a straightedge, inscribe a square in a given circle whose center is also given.

5. Given a circle ω without its center, and given a point P outside ω, construct the tangents to ω from P.

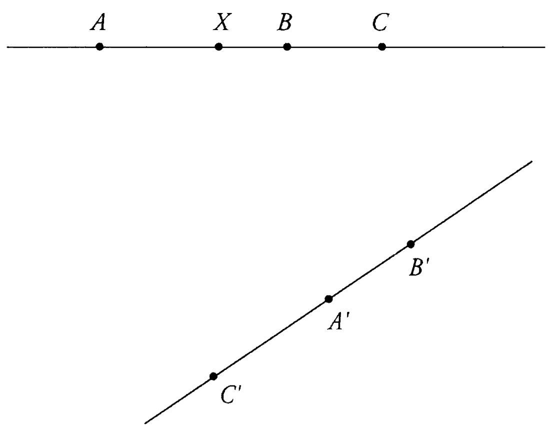
6. Given a circle ω without its center, and given a point P inside ω, construct the polar of P with respect to ω.

7. Given a circle ω without its center, and given a point P on ω, construct the tangent to ω at P.

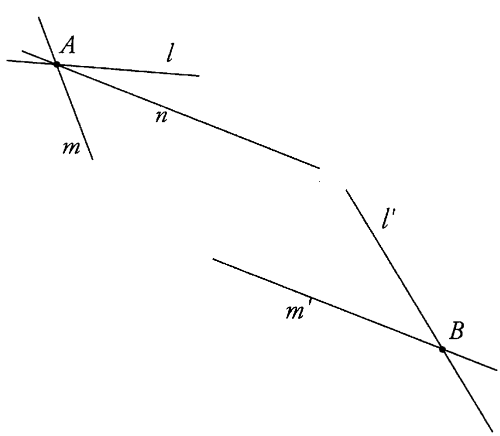
8. Prove that if $\{C, D\}$ divides AB harmonically on the line l and $\{C', D'\}$ divides $A'B'$ harmonically on the line l', then there exists a unique projectivity that maps A, B, C, and D to A', B', C', and D', respectively.

9. Given that π is a central perspectivity from l to l', show that the information that π maps A, B to A, B', respectively, is ***not*** sufficient to determine π uniquely.

10. In the figure below, there is a unique projectivity that takes A, B, and C to A', B', and C', respectively. Using only a straightedge construct the image of X under the projectivity.



11. In the figure below, there is a line perspectivity from the pencil at A to the pencil at B that takes l to l' and m to m'. Using only a straightedge, construct the image of n under this perspectivity.



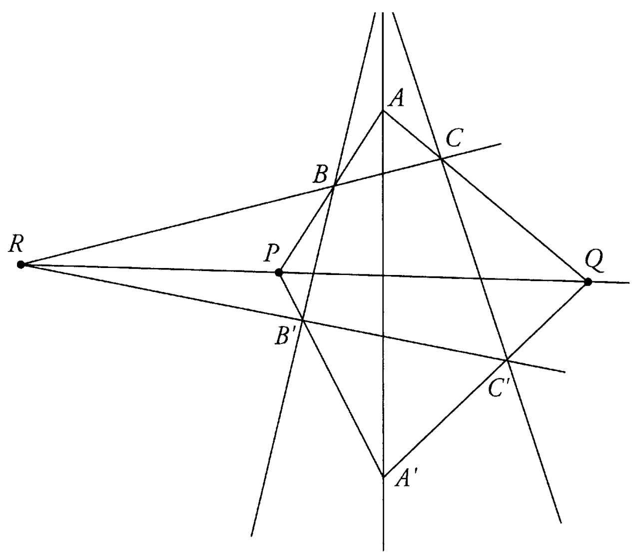
12. Theorem 16.4.3 states: suppose that $\pi : k \longrightarrow l$ and $\sigma : l \longrightarrow m$ are two perspectivities with different centers. Then $\sigma \circ \pi$ is a perspectivity if and only if k, l, and m are concurrent. State the dual of this theorem.

13. Give an example of a projectivity from m onto m, the same line, with two distinct fixed points, but which is not a perspectivity.

14. Show that coaxial triangles are copolar by projecting the polar axis to infinity; that is, given triangles ABC and $A'B'C'$ such that the points

$$P = AB \cap A'B', \qquad Q = AC \cap A'C', \qquad R = BC \cap B'C'$$

are collinear, show that AA', BB', and CC' are concurrent by projecting the line PQR to infinity.



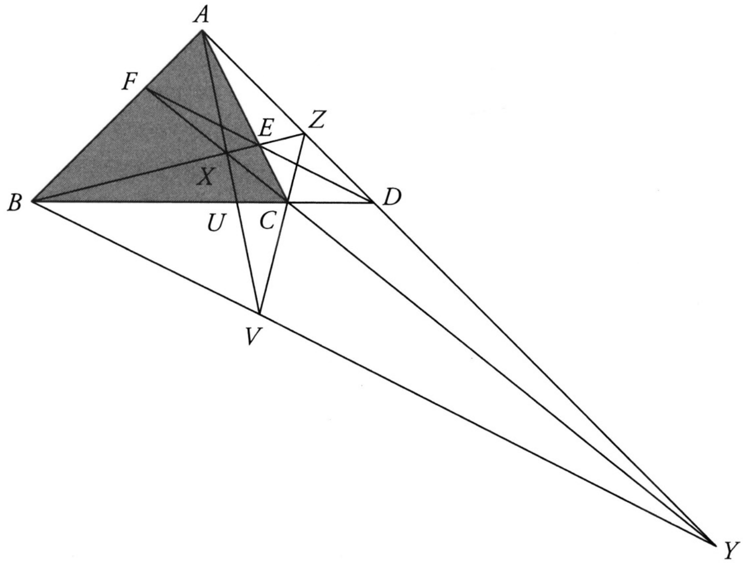
15. Let $PQRS$ be a complete quadrangle, and let

$$A = PQ \cap RS, \quad B = PR \cap SQ, \quad C = PS \cap QR, \quad D = AB \cap PS.$$

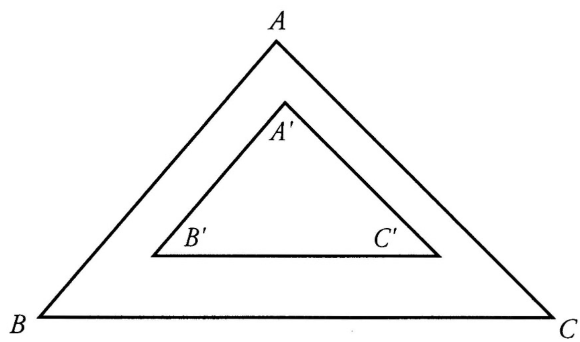
Show that $(PS, DC) = -1$ by projecting the line AC to infinity.

16. Is it possible to plant 10 trees in 10 straight rows, with 3 trees in each row?
Hint: Desargues says it is!

17. The points D, E, and F are collinear and lie on the sides BC, CA, and AB, respectively, of triangle ABC. The line BE cuts CF at X, the line CF cuts AD at Y, and the line AD cuts BE at Z. Prove that AX, BY, and CZ are concurrent by projecting the line BY to infinity.



18. Find two coaxial triangles at nonideal points P, Q, and R whose projected image is as shown in the figure below and where P', Q', and R' are ideal points.



Hint: $AB \parallel A'B'$, $AC \parallel A'C'$, and $BC \parallel B'C'$.

19. Let π be a plane that cuts an oblique circular cone in such a way that one of the generating lines of the cone is parallel to π. Prove that the intersection of π with the cone can be described by the Cartesian equation $y = kx^2$.

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