Available online at www.sciencedirect.com

ScienceDirect

(2017) –??

Journal Logo

Analysis of the nature of trigonometric functions and hyperbolic functions☆

Álvaro I. Plasencia[a,1,*], C. Aznarán[a,2]

[a]Facultad de Ciencias - Escuela profesional de Matemática,
Universidad Nacional de Ingeniería, Av. Túpac Amaru 210, Rímac, Lima 25, Peru

Abstract

In this paper, we will review the definitions about the hyperbolic functions, trigonometric functions and deduce their inverse functions in terms of logarithms and exponentials introducing complex numbers. Hence, we will show the mathematical interrelation between this types of functions. Having already present the established definitions for the hyperbolic functions will look for an analog representation for the trigonometric functions.

Keywords: unit imaginary, trigonometric functions, hyperbolic functions

1. Introduction and statement of the results

The objective of this paper is to find their derivatives and integrals.

Theorem 1. *F*

$$.... \tag{1}$$

$$i(\cos\theta + i\sin\theta) = (i\cos\theta - \sin\theta)$$

$$i = (\cos\theta + i\sin\theta)^{-1}(i\cos\theta - \sin\theta)$$

$$id\theta = (\cos\theta + i\sin\theta)^{-1}(id(\sin\theta) + d(\cos\theta))$$

$$id\theta = (\cos\theta + i\sin\theta)^{-1}(d(\cos\theta) + d(i\sin\theta)$$

$$d(i\theta) = (\cos\theta + i\sin\theta)^{-1}d(\cos\theta + i\sin\theta)$$

$$d(i\theta) = (\cos\theta + i\sin\theta)^{-1}\ln(\cos\theta + i\sin\theta)$$

$$\int d(i\theta) = \int d(\ln(\cos\theta + i\sin\theta))$$

$$i\theta = \ln(\cos\theta + i\sin\theta)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\boxed{(e^{i\theta}) = cis\ \theta} \tag{2}$$

De la ecuación 3.

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{3}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{4}$$

De las ecuaciones 3 y 4

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Habiéndose representado en forma análoga a las funciones hiperbólicas las dos funciones fundamentales en

☆This paper are available on GitHub.

*Corresponding author

[1]First author partially supported by the Universidad Nacional de Ingeniería project.

[2]Second author partially supported by the Undergraduate Mathematics Group P156250.

lo que respecta a las llamadas trigonométricas, pues es asequible representar las demás en términos de estas, se pone en evidencia la relación entre estas dos familias de funciones, empezando por las fundamentales y acabando con las que subyacen en forma inmediata de las ecuaciones 3 y ecuación 4, concatenando posteriormente de las funciones fundamentales:

$$\cos\theta = \cosh i\theta \tag{5}$$

$$\sin\theta = -i\sinh(i\theta) \tag{6}$$

$$\cot\theta = i\coth(i\theta) \tag{7}$$

$$\tan\theta = -i\tanh(i\theta) \tag{8}$$

$$\csc\theta = \mathrm{csch}(i\theta) \tag{9}$$

$$\sec\theta = \mathrm{sech}(i\theta) \tag{10}$$

Siguiendo la hilación de lo expuesto, es cuestión subsiguiente determinar las funciones inversas, siendo factible para este propósito aseverar las inversas hiperbólicas para posteriormente obtener las inversas trigonométricas.
Sea $\mathrm{arcCosh}\ \lambda = \theta$ tal que:

$$\cosh\theta = \lambda$$

$$\frac{1}{2}(e^{\theta} + e^{-\theta}) = \lambda$$

$$e^{2\theta} - 2\lambda e^{\theta} + 1 = 0$$

$$e^{\theta} = \lambda \pm \sqrt{\lambda^2 - 1}$$

Suponiendo $\theta = \ln(\lambda - \sqrt{\lambda^2 - 1})$
Veamos:

$$\theta = \ln(\cosh\theta - \sqrt{\cosh^2\theta - 1})$$

$$\theta = \ln(\cosh\theta - \sinh\theta$$

$$\theta = \ln(e^{-\theta})$$

$$\theta = -\theta\ldots \text{absurdo}$$

$$\therefore\ \theta = \ln(\lambda + \sqrt{\lambda^2 - 1})$$

$$\mathrm{arcCosh}\,\lambda = \ln(\lambda + \sqrt{\lambda^2 - 1}) \tag{11}$$

Para $\arccos\lambda = \theta$, de la ecuación 3 tenemos:

$$\cos(\arccos\lambda) = \lambda$$

$$\cosh(i\theta) = \lambda$$

$$\mathrm{arcCosh}(\cosh(i\theta)) = \mathrm{arcCosh}\,\lambda$$

$$\theta = -i\,\mathrm{arcCosh}\,\lambda$$

$$\arccos\lambda = -i\,\mathrm{arcCosh}\,\lambda \tag{12}$$

Sea $\mathrm{arcsinh}\,\lambda = \theta$, tal que

$$\sinh\theta = \lambda$$

$$\frac{1}{2}(e^{\theta} - e^{-\theta}) = \lambda$$

$$e^{2\theta} - 2\lambda e^{\theta} - 1 = 0$$

$$e^{\theta} - \lambda = \sqrt{\lambda^2 + 1}$$

$$\theta = \ln(\lambda \pm \sqrt{\lambda^2 - 1})$$

$$\text{pero, } \lambda - \sqrt{\lambda^2 - 1} < 0$$

$$\therefore \theta = \ln(\lambda + \sqrt{\lambda^2 + 1})$$

$$\mathrm{arcsinh}\,\lambda = \ln(\lambda + \sqrt{\lambda^2 + 1}) \tag{13}$$

Para $\mathrm{arcsinh}\,\lambda = \theta$, de la ecuación 7 tenemos:

$$\sinh(\mathrm{arcsinh}\,\lambda) = \lambda$$

$$-i\sinh(i\theta) = \lambda$$

$$\mathrm{arcsinh}(\sinh(i\theta)) = \mathrm{arcsinh}(i\lambda)$$

$$\theta = -i\,\mathrm{arcsinh}(i\lambda)$$

$$\arcsin\lambda = -i\,\mathrm{arcsinh}(i\lambda) \tag{14}$$

Sea $\mathrm{arcCoth}\,\lambda = \theta$, tal que

$$\coth\theta = \lambda$$

$$\frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}} = \lambda$$

$$\frac{2}{e^{2\theta}} = \frac{2}{\lambda - 1} + 1$$

$$\theta = \frac{1}{2}\ln(\frac{\lambda + 1}{\lambda - 1})$$

$$\mathrm{arcCoth}\,\lambda = \frac{1}{2}\ln(\frac{\lambda + 1}{\lambda - 1}) \tag{15}$$

Para $\mathrm{arcCot} = \theta$ de la ecuación 8 tenemos:

$$\cot(\mathrm{arcCot}\,\lambda) = \lambda$$

$$i\coth(i\theta) = \lambda$$

$$\mathrm{arcCoth}(\coth(i\theta)) = \mathrm{arcCoth}(-i\lambda)$$

$$\theta = -i \operatorname{arcCoth}(-i\lambda)$$

$$\operatorname{arcCot}\lambda = -i \operatorname{arcCoth}(-i\lambda) \tag{16}$$

Sea $\operatorname{arctanh}\lambda = \theta$ tal que:

$$\tanh\theta = \lambda$$

$$\frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} = \lambda$$

$$1 - \frac{2}{e^{2\theta} + 1} = \lambda$$

$$e^{2\theta} = \frac{2}{1-\lambda} - 1$$

$$\theta = \frac{1}{2}\ln(\frac{1+\lambda}{1-\lambda})$$

$$\operatorname{arctanh}\lambda = \frac{1}{2}\ln(\frac{1+\lambda}{1-\lambda}) \tag{17}$$

Para $\arctan\lambda = \theta$ de la ecuación 9 tenemos:

$$\tan(\arctan\lambda) = \lambda$$

$$-i\tanh(i\theta) = \lambda$$

$$\operatorname{arctanh}(\tanh(i\theta)) = \operatorname{arctanh}(i\lambda)$$

$$\theta = -i\operatorname{arctanh}(i\lambda)$$

$$\arctan\lambda = -i\operatorname{arctanh}(i\lambda) \tag{18}$$

Sea $\operatorname{arcCsch}\lambda = \theta$ tal que:

$$\operatorname{csch}\theta = \lambda$$

$$\frac{2}{e^{\theta} - e^{-\theta}} = \lambda$$

$$\lambda e^{2\theta} - 2e^{\theta} - \lambda = 0$$

$$e^{\theta} = \frac{1 \pm \sqrt{1+\lambda^2}}{\lambda}$$

$$\theta = \ln(\frac{1 \pm \sqrt{1+\lambda^2}}{\lambda})$$

Pero $1 - \sqrt{1+\lambda^2} < 0$

$$\therefore \theta = \ln(\frac{1 + \sqrt{1+\lambda^2}}{\lambda})$$

$$\operatorname{arcCsch}\lambda = \ln(\frac{1 + \sqrt{1+\lambda^2}}{\lambda}) \tag{19}$$

Para $\operatorname{arcCsc}\lambda = \theta$ de la ecuación 10 tenemos:

$$\csc(\operatorname{arcCsc}\lambda = \lambda$$

$$i\operatorname{csch}(i\theta) = \lambda$$

$$\operatorname{arcCsch}(\operatorname{csch}(i\theta)) = \operatorname{arcCsch}(-i\lambda)$$

$$\theta = -i\operatorname{arcCsch}(-i\lambda)$$

$$\operatorname{arcCsc}\lambda = -i\operatorname{arcCsch}(-i\lambda) \tag{20}$$

Sea $\operatorname{arcsech} = \theta$ tal que:

$$\operatorname{sech}\theta = \lambda$$

$$\frac{2}{e^{\theta} + e^{-\theta}} = \lambda$$

$$\lambda e^{2\theta} - 2e^{\theta} + \lambda = 0$$

$$e^{\theta} = \frac{1 \pm \sqrt{1-\lambda^2}}{\lambda}$$

$$\theta = \ln(\frac{1 \pm \sqrt{1-\lambda^2}}{\lambda})$$

Suponiendo $\theta = \ln(\frac{1 - \sqrt{1-\lambda^2}}{\lambda})$
Veamos

$$\theta = \ln(\frac{1 - \sqrt{1-\operatorname{sech}^2\theta}}{\operatorname{sech}\theta})$$

$$\theta = \ln(\cosh\theta - \sinh\theta)$$

$$\theta = \ln(e^{-\theta})$$

$$\theta = -\theta \ldots \text{ absurdo}$$

$$\therefore \theta = \ln(\frac{1 + \sqrt{1-\lambda^2}}{\lambda})$$

$$\operatorname{arcsech}\lambda = \ln(\frac{1 + \sqrt{1-\lambda^2}}{\lambda}) \tag{21}$$

Para $\operatorname{arcsec}\lambda = \theta$ de la ecuación 11 tenemos:

$$\sec(\operatorname{arcsec}\lambda) = \lambda$$

$$\operatorname{sech}(i\theta) = \lambda$$

$$\operatorname{arcsech}(\operatorname{sech}(i\theta)) = \operatorname{arcsech}\lambda$$

$$\theta = -i\operatorname{arcsech}\lambda$$

$$\operatorname{arcsec}\lambda = -i\operatorname{arcsech}\lambda \tag{22}$$

Theorem 2. *a*

Lemma 3. *c*

Remark 1. c

PROOF. c

Acknowledgments

The authors want to thank the Universidad Nacional de Ingeniería and the Undergraduate Mathematics Group for their hospitality during the visits while preparing this paper.

The authors would like to thank professor Johny Valverde for many valuable and constructive suggestions, that have helped to improve the paper.[1]

References

[1] A. L. Neto, Funções de uma variável complexa, Instituto de Matemática Pura e Aplicada, 1993.

[2] A. D. Fitt, G. T. Q. Hoare, The closed-form integration of arbitrary functions, The Mathematical Gazette 77 (479) (1993) 227–236.
URL http://www.jstor.org/stable/3619719

[3] C. G. Denlinger, Elements of real analysis, Jones & Bartlett Publishers, 2011.