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Wilker-type inequalities for hyperbolic Fibonacci functions

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Abstract
This article introduces the analog of the Wilker inequality and the parameterized Wilker inequality for the hyperbolic Fibonacci functions.

Keywords: Wilker's inequality; Wilker-Anglesio inequality; hyperbolic Fibonacci functions

1 Introduction

The inequalities

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{1}$$

and

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{2}$$

were proposed first by Wilker [1] as open problems, where $0 < x < \frac{\pi}{2}$ and c is constant. The inequality (1) was proved by some authors [2–5]. In [2], Anglesio showed that the sharp inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \frac{16}{\pi^4}x^3 \tan x \tag{3}$$

is valid and the best possible constant is $\frac{16}{\pi^4}$ for the constant c in inequality (2). Also, the following similar inequality to inequality (1) was proved by Huygens [6]:

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3. \tag{4}$$

During the past few years, some inequalities have appeared as the generalizations and improvements of the Wilker inequality (1) and the Wilker-Anglesio inequality (3) [7–11]. In [11], Wu and Srivastava gave a generalization of Wilker's inequality involving parameters of exponent and weight, as follows:

$$\frac{\lambda}{\lambda+\mu}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{\tan x}{x}\right)^q > 1, \tag{5}$$

where $0 < x < \frac{\pi}{2}$, $\lambda > 0$, $\mu > 0$, $p \leq \frac{2q\mu}{\lambda}$, $q > 0$ or $q \leq \min\{-\frac{\lambda}{\mu}, -1\}$. Moreover, Wilker and Anglesio type inequalities have been obtained for the hyperbolic functions [12, 13]. Wu and Debnath [12] established the hyperbolic analog of the Wilker-Anglesio inequality and the parameterized Wilker inequality as follows:

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 + \frac{8}{45}x^3 \tanh x, \tag{6}$$

$$\frac{\lambda}{\lambda+\mu}\left(\frac{\sinh x}{x}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{\tanh x}{x}\right)^q > 1, \tag{7}$$

where $x \neq 0$, $\lambda > 0$, $\mu > 0$, $p \geq \frac{2q\mu}{\lambda}$, and $q > 0$ or $q \leq \min\{-\frac{\lambda}{\mu}, -1\}$.

The aim of this paper is to present the analog of the Wilker-Anglesio inequality (6) and the parameterized Wilker inequality (7) together with some applications for the hyperbolic Fibonacci functions.

2 Preliminaries and some lemmas

The Fibonacci numbers are defined by the second order linear recurrence relation: $F_{n+1} = F_n + F_{n-1}$ $(n \geq 1)$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Similarly, the Lucas numbers are defined by $L_{n+1} = L_n + L_{n-1}$ $(n \geq 1)$ with the initial conditions $L_0 = 2$ and $L_1 = 1$. The characteristic equation of F_n is

$$t^2 - t - 1 = 0. \tag{8}$$

The roots of equation (8) are $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, and the Binet formula for F_n is

$$F_n = \begin{cases} \frac{\alpha^n + \alpha^{-n}}{\sqrt{5}}, & n \text{ odd}, \\ \frac{\alpha^n - \alpha^{-n}}{\sqrt{5}}, & n \text{ even}. \end{cases} \tag{9}$$

The positive root of equation (8), $\alpha = \frac{1+\sqrt{5}}{2}$, is called the golden ratio, which has been very attractive for researchers because it occurs ubiquitous such as in nature, art, architecture, and anatomy.

The Fibonacci numbers have many properties, continuous versions, and generalizations [14–20]. Stakhov and Tkachenko [14] introduced a new class of hyperbolic functions called hyperbolic Fibonacci functions replacing the discrete variable n in equation (9) with the continuous variable x that takes its values from the set of real numbers. Based on an analogy between the Binet formula, (9), and the classical hyperbolic functions,

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \tag{10}$$

Stakhov and Rozin [15] defined the so-called symmetrical hyperbolic Fibonacci functions as follows:

$$sFs(x) = \frac{\alpha^x - \alpha^{-x}}{\sqrt{5}} \quad \text{and} \quad cFs(x) = \frac{\alpha^x + \alpha^{-x}}{\sqrt{5}}, \tag{11}$$

where $sFs(x)$ and $cFs(x)$ denote symmetric hyperbolic Fibonacci sine and cosine functions, respectively. Similarly, a symmetric hyperbolic Fibonacci tangent function can be defined as

$$tFs(x) = \frac{sFs(x)}{cFs(x)} = \frac{\alpha^x - \alpha^{-x}}{\alpha^x + \alpha^{-x}}. \tag{12}$$

The graphs of the symmetrical hyperbolic Fibonacci functions have a symmetric form and are similar to the graphs of the classical hyperbolic functions. Also, the symmetrical hyperbolic Fibonacci functions $sFs(x)$ and $cFs(x)$ are increasing on $(0, +\infty)$. The graphs of the symmetrical hyperbolic Fibonacci functions are given in [15]. The symmetric hyperbolic Fibonacci functions have properties that are similar to the classical hyperbolic functions. Some of them are [15]:

$$cFs(x) = cFs(-x), \qquad sFs(x) = -sFs(-x) \quad \text{and} \quad \left[cFs(x)\right]^2 - \left[sFs(x)\right]^2 = \frac{4}{5}.$$

Also, the derivative hyperbolic Fibonacci functions are [15]

$$\left[cFs(x)\right]^{(n)} = \begin{cases} (\ln\alpha)^n sFs(x), & \text{for } n \text{ odd}, \\ (\ln\alpha)^n cFs(x), & \text{for } n \text{ even}, \end{cases}$$

$$\left[sFs(x)\right]^{(n)} = \begin{cases} (\ln\alpha)^n cFs(x), & \text{for } n \text{ odd}, \\ (\ln\alpha)^n sFs(x), & \text{for } n \text{ even}. \end{cases}$$

For more information and the generalizations as regards hyperbolic Fibonacci functions, see [15–20] the references cited therein.

Throughout this paper $sFs(x)$, $cFs(x)$, and $tFs(x)$ denote the hyperbolic Fibonacci functions given in (11) and (12) and α denotes the golden ratio, $\alpha = \frac{1+\sqrt{5}}{2}$.

Lemma 1 [21] *If* $x_i > 0$, $\lambda_i > 0$ $(i = 1, 2, \dots, n)$, *and* $\sum_{i=1}^{n} \lambda_i = 1$, *then*

$$\sum_{i=1}^{n} \lambda_i x_i \geq \prod_{i=1}^{n} x_i^{\lambda_i}.$$

Lemma 2 *For all nonzero real numbers* x, *the following inequality holds*:

$$\frac{2}{\sqrt{5}} \leq cFs(x) \leq \frac{5}{4(\ln(\alpha))^3} \left(\frac{sFs(x)}{x}\right)^3. \tag{13}$$

Proof From $cFs(0) = \frac{2}{\sqrt{5}}$, $cFs(x) = cFs(-x)$, and $cFs(x)$ is increasing on $(0, +\infty)$, the left hand side inequality of (13) is true. Let us prove the right hand side inequality of (13).

Case (I): For $x > 0$, define a function $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(x) = \frac{sFs^3(x)}{x^3 cFs(x)}.$$

By differentiating with respect to x, we get

$$f'(x) = \frac{sFs^2(x)}{x^4 cFs^2(x)} \left[2x \ln(\alpha) sFs^2(x) + \frac{12}{5} x \ln(\alpha) - 3 sFs(x) cFs(x)\right] = \frac{sFs^2(x)}{x^4 cFs^2(x)} f_1(x),$$

$$f_1'(x) = 4\ln(\alpha)sFs(x)cFs(x)\left[x\ln(\alpha) - \frac{sFs(x)}{cFs(x)}\right] = 4\ln(\alpha)sFs(x)cFs(x)f_2(x),$$

$$f_2'(x) = \ln(\alpha)\left(\frac{sFs(x)}{cFs(x)}\right)^2 > 0.$$

This means that $f_2(x)$ is increasing on $(0,+\infty)$. We thus conclude from $f_2(0) = f_1(0) = 0$ that $f_2(x) > 0$ and $f_1(x)$ is increasing and positive on $(0,+\infty)$. Hence, we see that $f(x)$ is increasing on $(0,+\infty)$. By using

$$\lim_{x\to 0^+} f(x) = \frac{4(\ln(\alpha))^3}{5},$$

we conclude that

$$cFs(x) < \frac{5}{4(\ln(\alpha))^3}\left(\frac{sFs(x)}{x}\right)^3.$$

Case (II): Let $x < 0$ $(-x > 0)$. Since $cFs(-x) = cFs(x)$ and $sFs(-x) = -sFs(x)$ and considering the result proved in Case (I), the proof is completed. □

3 Wilker-Anglesio's inequality for hyperbolic Fibonacci functions

Theorem 1 *Let x be nonzero real numbers; then the following inequality holds:*

$$\left(\frac{sFs(x)}{x}\right)^2 + \frac{tFs(x)}{x} > \frac{8(\ln(\alpha))^2}{5} + \frac{32(\ln(\alpha))^5x^3}{225}tFs(x).$$

Proof From $sFs(-x) = -sFs(x)$ and $tFs(-x) = -tFs(x)$, we have

$$\left(\frac{sFs(-x)}{(-x)}\right)^2 + \frac{tFs(-x)}{(-x)} = \left(\frac{sFs(x)}{x}\right)^2 + \frac{tFs(x)}{x}.$$

Thus, it is enough to prove that Theorem 1 is true for $x > 0$. For this purpose, define a function $g : \mathbb{R}^+ \to \mathbb{R}$ by

$$g(x) = \frac{\frac{5}{4(\ln(\alpha))^2}(\frac{sFs(x)}{x})^2 + \frac{1}{\ln(\alpha)}\frac{tFs(x)}{x} - 2}{x^3 tFs(x)}.$$

Then, upon differentiating with respect to x, we get

$$\begin{aligned} g'(x) &= \frac{1}{4\sqrt{5}\ln(\alpha)x^6 sFs^2(x)}\left[2xcFs(4x) + 24x^2\ln(\alpha)sFs(2x) - \frac{5}{\ln(\alpha)}sFs(4x)\right. \\ &\quad \left. + \frac{10}{\ln(\alpha)}sFs(2x) - 20xcFs(2x) + \left(40 - \frac{4\sqrt{5}}{5}\right)x + \frac{2\sqrt{5}}{5}\left(\ln(\alpha)\right)^2 16x^3\right] \\ &= \frac{g_1(x)}{4\sqrt{5}\ln(\alpha)x^6 sFs^2(x)}, \\ g_1'(x) &= 8\sqrt{5}cFs^2(x)\left[6x^2\left(\ln(\alpha)\right)^2 + x\ln(\alpha)sFs(x)\left[5cFs(x) - \frac{1}{cFs(x)}\right] - \frac{90}{8}sFs^2(x)\right] \\ &= 8\sqrt{5}cFs^2(x)g_2(x), \end{aligned}$$

$$\begin{aligned}
g_2'(x) &= \frac{1}{cFs^2(x)}\biggl[-\frac{35}{2}\ln(\alpha)sFs(x)cFs^3(x) - \ln(\alpha)sFs(x)cFs(x) \\
&\quad + 10x\bigl(\ln(\alpha)\bigr)^2 cFs^4(x) + 8x\bigl(\ln(\alpha)\bigr)^2 cFs^2(x) - \frac{4}{5}x\bigl(\ln(\alpha)\bigr)^2\biggr] = \frac{g_3(x)}{cFs^2(x)}, \\
g_3'(x) &= 4\bigl(\ln(\alpha)\bigr)^2 sFs(2x)\biggl[\frac{8}{\sqrt{5}}x\ln(\alpha) - 3sFs(2x) + 2x\ln(\alpha)cFs(2x)\biggr] \\
&= 4\bigl(\ln(\alpha)\bigr)^2 sFs(2x)g_4(x), \\
g_4'(x) &= 4\ln(\alpha)sFs(2x)\biggl[x\ln(\alpha) - \frac{sFs(x)}{cFs(x)}\biggr] \\
&= 4\ln(\alpha)sFs(2x)g_5(x), \\
g_5'(x) &= \ln(\alpha)\biggl(\frac{sFs(x)}{cFs(x)}\biggr)^2 > 0.
\end{aligned}$$

This means that $g_5(x)$ is increasing on the open interval $(0,+\infty)$. By $g_5(0) = g_4(0) = g_3(0) = g_2(0) = g_1(0) = 0$ this immediately shows that $g_5(x), g_4(x), g_3(x), g_2(x)$, and $g_1(x)$ are increasing and positive on $(0,+\infty)$. Thus, we see that $g(x)$ is increasing on $(0,+\infty)$. Also, we use

$$\lim_{x\to 0^+} g(x) = \frac{8(\ln(\alpha))^3}{45}.$$

Hence, we conclude from

$$\frac{\frac{5}{4(\ln(\alpha))^2}[(\frac{sFs(x)}{x})^2 + \frac{tFs(x)}{x}] - 2}{x^3 tFs(x)} > \frac{\frac{5}{4(\ln(\alpha))^2}(\frac{sFs(x)}{x})^2 + \frac{1}{\ln(\alpha)}\frac{tFs(x)}{x} - 2}{x^3 tFs(x)}$$

that

$$\left(\frac{sFs(x)}{x}\right)^2 + \frac{tFs(x)}{x} > \frac{8(\ln(\alpha))^2}{5} + \frac{32(\ln(\alpha))^5 x^3}{225} tFs(x).$$

This completes the proof. □

Next we give parameterized Wilker's inequality for hyperbolic Fibonacci functions.

Theorem 2 *For the hyperbolic Fibonacci functions, the following inequality holds*:

$$\frac{\lambda}{\lambda+\mu}\left(\frac{sFs(x)}{x}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{tFs(x)}{x}\right)^q > \left(\frac{2\ln(\alpha)}{\sqrt{5}}\right)^{\frac{p\lambda+q\mu}{\lambda+\mu}},$$

where $x \neq 0$, $\lambda > 0$, $\mu > 0$, $p \geq \frac{2q\mu}{\lambda}$, *and* $q > 0$.

Proof From Lemmas 1 and 2, we have

$$\begin{aligned}
&\frac{\lambda}{\lambda+\mu}\left(\frac{sFs(x)}{x}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{tFs(x)}{x}\right)^q \\
&\quad \geq \left(\frac{sFs(x)}{x}\right)^{\frac{p\lambda}{\lambda+\mu}}\left(\frac{tFs(x)}{x}\right)^{\frac{q\mu}{\lambda+\mu}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{sFs(x)}{x}\right)^{\frac{p\lambda}{\lambda+\mu}} \left(\frac{sFs(x)}{x}\right)^{\frac{q\mu}{\lambda+\mu}} \left(\frac{1}{cFs(x)}\right)^{\frac{q\mu}{\lambda+\mu}} \\
&> \left(\frac{sFs(x)}{x}\right)^{\frac{p\lambda+q\mu}{\lambda+\mu}} \left(\frac{sFs(x)}{x}\right)^{\frac{-3q\mu}{\lambda+\mu}} \left(\frac{5}{4(\ln(\alpha))^3}\right)^{\frac{-q\mu}{\lambda+\mu}} \\
&= \left(\frac{sFs(x)}{x}\right)^{\frac{p\lambda-2q\mu}{\lambda+\mu}} \left(\frac{5}{4(\ln(\alpha))^3}\right)^{\frac{-q\mu}{\lambda+\mu}} \\
&> \left(\frac{2\ln(\alpha)}{\sqrt{5}}\right)^{\frac{p\lambda-2q\mu}{\lambda+\mu}} \left(\frac{4(\ln(\alpha))^3}{5}\right)^{\frac{q\mu}{\lambda+\mu}} \\
&> \left(\frac{2\ln(\alpha)}{\sqrt{5}}\right)^{\frac{p\lambda+q\mu}{\lambda+\mu}}. \qquad \square
\end{aligned}
$$

Now we give some weighted and exponential Wilker-type inequalities for hyperbolic Fibonacci functions as applications of Theorem 2.

Corollary 1 *Let $x \neq 0$, $\lambda \geq \mu > 0$ and $(p,q) = (2,1)$. Then*

$$\frac{\lambda}{\lambda+\mu}\left(\frac{sFs(x)}{x}\right)^2 + \frac{\mu}{\lambda+\mu}\left(\frac{tFs(x)}{x}\right) > \frac{4(\ln(\alpha))^2}{5}.$$

Corollary 2 *Let $x \neq 0$, $p \geq q > 0$, and $(\lambda,\mu) = (2,1)$. Then*

$$2\left(\frac{sFs(x)}{x}\right)^p + \left(\frac{tFs(x)}{x}\right)^q > 3\left(\frac{2\ln(\alpha)}{\sqrt{5}}\right)^p. \tag{14}$$

It is obvious that the inequality (14) reduces to Huygens type inequality for $p = q = 1$.

Competing interests
The author declares that he has no competing interests.

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