1 Laplace Equation

If does not have boundary conditions, ill-possed problem. The equation needs two conditions. It is very easy to partition the interval $[a, b]$.

Definition 1 (Domain)**.** We call $\Omega \subset \mathbb{R}^d$ for $d = 1, 2, 3$ a **domain** iff

1. Ω is open.
2. Ω is connected. It has no holes. It must be smooth.

Definition 2 (Boundary)**.** We call $\Gamma = \partial\Omega$ the **boundary** of the domain Ω.

Definition 3 (Unit normal vector)**.** By $\vec{n}$ we denote the **unit normal vector** (facing outwards) on the boundary.

Definition 4. We define **function space of differentiable functions**

$$C^m(\Omega) = \{f \colon \Omega \to \mathbb{R} \mid f(x_1.x_2, \ldots, x_d)\}.$$

Definition 5. We define the **Laplace operator**

- Let $f \in C^0(\Omega)$ be the **right hand side function**.
- Let $g \in C^0(\Gamma)$ be the **boundary value function**.
- **Dirichtlet Problem** we are looking for $u \in C^2(\Omega)$ such that

$$-\Delta u = f \text{ in }.$$

- Let Ω be the unit sphere

$$\Omega = \left\{\boldsymbol{x} = (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\right\}.$$

- Let $f = 1$ and $g = 0$.
- There is **no solution** to the Dirichlet Problem

$$-\Delta u = 1 \text{ in } u = 0 \text{ on } \Gamma$$

  which is 2 times differentiable.

1.1 The variational formulation

- Assume that $u \in C^2(\Omega)$ is a solution to the Laplace problem

$$-\Delta u\,(x,y) = f\,(x,y) \ \text{ in } \Omega \text{ with } u = 0 \text{ on } \Gamma.$$

- Then, we can multiply this equation with a **test function** ϕ

$$-\Delta u\,(x,y) \cdot \phi\,(x,y) = f\,(x,y) \cdot \phi\,(x,y) \ \text{ in } \Omega.$$

- Then, we can **integrate by parts over the domain**

$$-\int_D \Delta u\,(x,y) \cdot \phi\,(x,y)\,\mathrm{d}x\mathrm{d}y = \int_\Omega f\,(x,y) \cdot \phi\,(x,y)\,\mathrm{d}x\mathrm{d}y.$$

- We assume that the test function is differentiable $\phi \in C^1(\Omega)$. Then, we can **integrate by parts**

$$\int_\Omega \nabla u\,(x,y) \cdot \phi\,(x,y)\,\mathrm{d}x\mathrm{d}y - \int_\Gamma (\vec{n} \cdot \nabla)\,u \cdot \phi \mathrm{d}S = \int_\Omega f\,(x,y) \cdot \phi\,(x,y)\,\mathrm{d}x\mathrm{d}y.$$

- We assume that the test function is zero on the boundary. Then

$$\int_\Omega \nabla u\,(x,y) \cdot \nabla\phi\,(x,y)\,\mathrm{d}x\mathrm{d}y = \int_\Omega f\,(x,y) \cdot \phi\,(x,y)\,\mathrm{d}x\mathrm{d}y.$$

If the boundary is given by he graph of a function in C^2, then there exists a classical solution $u \in C^2\,(\Omega)$.

- We introduce L^2 scalar product.

$$(u,\phi) = \int_\Omega .$$

Theorem 1. Let $\Omega \subset \mathbb{R}^d$ for $d = 1,2,3$ be a domain and $f \in L^2\,(\Omega)$. Then, there exists a solution

$$u \in \mathcal{V} = H_0^1\,(\Omega)$$

to the *Laplace problem* in variational formulation.

2 Finite Element Method

Steps for a finit element discretization

1. We discretize the domain Ω by a mesh Ω_h.
2. On Ω_h we discretize the function space $\mathcal{V} = H_0^1(\Omega)$ by a finite element space V_h.
3. We restrict the variational formulation to V_h

$$u_h \in V_h \left(\nabla u_h, \nabla \phi_h\right) = \left(f, \phi_h\right) \quad \forall \phi_h.$$

4. We solve a linear system of equations.

2.1 Construction

- We discretize the domain Ω by splitting it into simple **open elements**, e.g, triangles, quadrilaterals (in $2D$) or tetrahedras, prisms, hexaedras, pyramids (in $3D$)
- The **finite element mesh** Ω_h.

2.2 Some examples

2.3 Shape assumption

2.3.1 Local Finite Element space

- On every element $T \in \Omega_h$ define the basis functions of a simple polynomial space.
- **bi-linear finite elements**.
- Let T be and quadrilateral with the points $x^{(1)} = (0,0)$, $x^{(2)} = (h,0)$, $x^{(3)} = (o,h)$, $x^{(4)} = (h,h)$.
- $\phi^{(1)}(x,y) = \left(1 - \frac{x}{h}\right)\left(1 - \frac{y}{h}\right), \phi^{(2)}(x,y) = \frac{x}{h}\left(1 - \frac{y}{h}\right), \phi^{(3)}(x,y) = \left(1 - \frac{x}{h}\right)\frac{y}{h}, \phi^{(1)}(x,y) = \frac{xy}{h^2}$.
- The Lagrange basis of the finite element space is given as

$$V_h = \left\{\phi_h \in C(\Omega) \mid \phi\big|_T \in Q^1 = \operatorname{span}\left(\phi_h^{(1)}, \phi_h^{(2)}, \phi_h^{(3)}, \phi_h^{(4)}\right)\right\}.$$

- The **Lagrange basis** of **nodal basis** is given by

$$V.$$

- Starting point: weak formulation of Laplace equation

$$u \in \mathcal{V}.$$

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$$u_h \in V_h \, (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h.$$

- The finite element is given by a local basis

$$V_h = \operatorname{span}\left\{\phi_h^{(1)}, \dots, \phi_h^{(N)}\right\} \quad \forall i = 1, \dots, N.$$

- We write the unknown solution $u_h \in V_h$.

2.4 Assembling the matrix

- We must compute the matrix entries

$$A_{ij}\left(\nabla \phi_h^{(j)}, \nabla \phi_h^{(i)}\right) = \int_\Omega \nabla \phi_h^{(j)} \cdot \nabla \phi_h^{(i)} \mathrm{d}x = \sum_{T \subset \Omega_h} \int_T \nabla \phi_h^{(j)} \cdot \nabla \phi_h^{(i)} \mathrm{d}x.$$

- For every **nodal**....
- We combine the result in a **stencil**

$$S = \begin{bmatrix} s_{31} & s_{32} & s_{33} \\ s_{21} & s_{22} & s_{23} \\ s_{11} & s_{12} & s_{13} \end{bmatrix}.$$

- The finite element matrix on a small mesh with $16 = 4 \cdot 4$ nodes like

$$A = \frac{1}{3}\left[1\right].$$

- The main difference between $1D$ and $2D$ (or $3D$).