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Problem Set 8 – Gascoigne Workshop
Summer Term 2013

Time-dependent problems

In this problem set, we consider time-dependent PDE's.

Problem 8.1:

Copy the problem set 8 as usual. The template is a running implementation of the heat equation. Change the problem to solve the problem

$$\partial_t u - \frac{1}{\pi^2}\Delta u = 0 \text{ in } [0,1]\times(0,1)^2,$$

with $u = 0$ on the boundary $[0,1]\times\partial\Omega$ and the initial solution

$$u(x,0) = \sin(\pi x)\sin(\pi y).$$

The exact solution of this problem is given by

$$u(x,t) = e^{-2t}\sin(\pi x)\sin(\pi y).$$

Use this exact solution and estimate the error at the final time $t = 1$. Determine the order of convergence (with regard to the time-step size k) using the implicit Euler method $\theta = 1$ and the Crank-Nicolson scheme $\theta = \frac{1}{2}$.

Note: Measuring the convergence rate is not simple, as errors in time and in space get mixed. For measuring the error in time, it will be necessary to consider a space-discretization that is fine enough (small h) while keeping the time-step k still large.

Problem 8.2:

Implement the wave equation

$$\partial_{tt}u - \frac{1}{2}\Delta u = 0, \text{ in } [0,2] \times (0,1)^2,$$

with the boundary condition $u = 0$ on $\partial\Omega$ and the initial condition

$$u(x,0) = \sin(\pi x)\sin(\pi y), \quad \partial_t u(x,0) = 0.$$

For implementing this second order equation (in time) we split the system into two first order equations:

$$\begin{aligned} \partial_t v - \frac{1}{2}\Delta u &= 0 \\ \partial_t u - v &= 0 \end{aligned}$$

For the exact solution of this problem, it holds

$$u(x,t) = \cos(\pi t)\sin(\pi x)\sin(\pi y).$$

In order to control your implementation, note that

$$u(x,0) = u(x,2).$$