Christian-Albrechts-Universität zu Kiel

OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

Teoría de elementos finitos y su implementación

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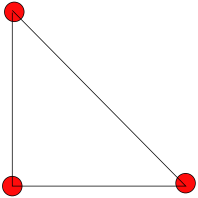
P_1 Finite elements on triangles / tetraedrons

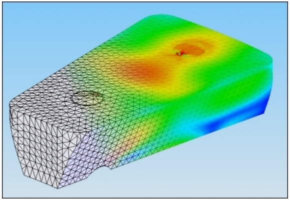
$$
\begin{aligned}
V = H_0^1(\Omega) &= \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega), u|_{\partial\Omega} = 0\} \\
P_1(\mathcal{T}_h) &:= \{\varphi \in C(\Omega_h) \,:\, \varphi|_T \in P_1 \,\forall T \in \mathcal{T}_h\} \\
V_h &:= V \cap P_1(\mathcal{T}_h)
\end{aligned}
$$

In 2D and 3D on each element:

$$
\begin{aligned}
\varphi(x, y) &= \alpha_0 + \alpha_1 x + \alpha_2 y \\
\varphi(x, y, z) &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z
\end{aligned}
$$

One dof on each vertex





Why only Poisson problem ?

- We consider the Poisson problem $u \in H_0^1(\Omega)$

$$-\Delta u \quad = \quad f \quad \text{in } \Omega$$

  and its weak formulation

$$u \in V : \quad (\nabla u, \nabla \phi) = (f, \phi) \quad \forall \phi \in V$$

- This is only the most simple example of a PDE.
- Extensions of more complicated PDEs are possible, e.g. systems of convection-diffusion-reaction equations

$$-\nabla \cdot (A_i \nabla u_i) + (b_i \cdot \nabla) u_i \quad = \quad f_i(u)$$

  with certain modifications of weak formulation

Outline for today:

1. A priori error estimates in H^1

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \quad \leq \quad Ch^r, \qquad r = ?$$

2. A priori error estimates in L^2

$$\|u - u_h\|_{L^2(\Omega)} \quad \leq \quad Ch^p, \qquad p = ?$$

A priori and a posteriori error estimates

One is usually interested in the error

$$\|u - u_h\|_X \quad = \quad ?$$

for a certain norm $\| \cdot \|_X$.

- **A priori error estimates:**
  Information about the error in terms of mesh size asymptotics, e.g. for P_1 or Q_1 elements

$$\begin{aligned} \|\nabla(u - u_h)\|_{L^2(\Omega)} &\leq ch|u|_{H^2(\Omega)} \\ \|u - u_h\|_{L^2(\Omega)} &\leq ch^2|u|_{H^2(\Omega)} \end{aligned}$$

- **A posteriori error estimates:**
  Information about the error in terms of u_h, e.g.:

$$\|\nabla(u - u_h)\|^2_{L^2(\Omega)} \leq \sum_{T \in \Omega_h} \left\{ h_T^2 \|f + \Delta u_h\|^2_{L^2(T)} + \frac{1}{2} \sum_{e \in \partial T} h_e \|[n \cdot \nabla u_h]\|^2_{L^2(e)} \right\}$$

Galerkin orthogonality

- Continuous problem with $A : V \times V \to \mathbb{R}$ bilinear:

$$u \in V : \quad A(u, \phi) = (f, \phi) \qquad \forall \phi \in V$$

Most simple example:

$$A(u, \phi) = (\nabla u, \nabla \phi) = \int_\Omega \nabla u \nabla \phi \, dx$$

- Discrete problem:

$$u_h \in V_h : \quad A(u_h, \phi) = (f, \phi) \qquad \forall \phi \in V_h$$

- Discretization error

$$e_h = u - u_h$$

- Galerkin orthogonality if $V_h \subseteq V$:

$$A(e_h, \phi) = 0 \qquad \forall \phi \in V_h$$

Cea's lemma

Lemma (Cea's Lemma)

Suppose that the bilinear form $A : V \times V \to \mathbb{R}$ satisfies the conditions of Lax-Milgram thm (continuous and V-coercive with $\alpha_1, \alpha_2 > 0$). Further, let $V_h \subseteq V$ a subspace. Then:

$$\|u - u_h\|_V \quad \leq \quad \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V_h} \|u - v_h\|_V \, .$$

- Approximation error

$$\inf_{v_h \in V_h} \|u - v_h\|_V$$

  Better is not possible.
- Discretization error

$$\|u - u_h\|_V$$

- Galerkin approximations are quasi-optimal, i.e.

$$\text{discretization error} \quad \leq \quad C \text{ approximation error}$$

  with $C = \alpha_1/\alpha_2$ independent of h.

Repetiton: Continuity and coercivity

- **Continuity:** There exists $\alpha_1 \geq 0$ s.t.

$$A(u, \phi) \quad \leq \quad \alpha_1 \|u\|_V \|\phi\|_V \qquad \forall u, \phi \in V$$

- **Coercivity:** There exists $\alpha_2 > 0$ s.t.

$$A(u, u) \quad \geq \quad \alpha_2 \|u\|_V^2 \qquad \forall u \in V$$

Proof of Cea's lemma

Let $v_h \in V_h$ be arbitrary.

$$
\begin{aligned}
\alpha_2 \|u - u_h\|_V^2 &\leq A(u - u_h, u - u_h) && \text{(coercivity)} \\
&= A(u - u_h, u - u_h) \\
&\quad + A(u - u_h, u_h - v_h) && \text{(Galerkin ortho.)} \\
&= A(u - u_h, u - v_h) && \text{(linearity)} \\
&\leq \alpha_1 \|u - u_h\|_V \|u - v_h\|_V && \text{(continuity)}
\end{aligned}
$$

Dividing by $\alpha_2 \|u - u_h\|_V$ leads to

$$
\|u - u_h\|_V \leq \frac{\alpha_1}{\alpha_2} \|u - v_h\|_V
$$

$v_h \in V_h$ was arbitrary, hence

$$
\|u - u_h\|_V \leq \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V_h} \|u - v_h\|_V^2
$$

Cea's lemma for the Poisson problem

- $V = H_0^1(\Omega)$
- associated norm

$$\|u\|_V \quad = \quad \|\nabla u\|_\Omega \;=\; \left(\int_\Omega |\nabla u(x)|^2\, dx\right)^{1/2}$$

- $\alpha_1 = \alpha_2 = 1$

$$\boxed{\|\nabla(u - u_h)\|_\Omega \quad = \quad \inf_{v_h \in V_h} \|\nabla(u - v_h)\|_\Omega}$$

Independently of the type of finite elements, as long as these are conforming finite elements, i.e. $V_h \subseteq H_0^1(\Omega)$.

Interpolation error

- Let $I_h : V \to V_h$ be an arbitrary interpolation. Then

$$\begin{aligned} \|u - u_h\|_V &\leq \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V_h} \|u - v_h\|_V \\ &\leq \frac{\alpha_1}{\alpha_2} \|u - I_h u\|_V \end{aligned}$$

- We only need do get an idea about the interpolation error

$$\|u - I_h u\|_V$$

- Most simple is the nodal interpolation of continuous functions

$$I_h u(N) = u(N)$$

  for nodes N of the mesh.

- But: Are $H^1(\Omega)$ functions continuous?

$$\begin{aligned} d = 1 &\quad : \quad \text{yes} \\ d \geq 2 &\quad : \quad \text{no} \end{aligned}$$

More regular Sobolev functions

- Higher order Sobolev spaces of order $k \geq 1$:

$$H^k(\Omega) \quad := \quad \{u : \Omega \to \mathbb{R} \,:\, D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leq k\}$$

- Semi-norms and norm:

$$|u|_{H^m(\Omega)} \quad := \quad \Big(\sum_{|\alpha|=m} \|D^\alpha u\|_\Omega^2 \Big)^{1/2}$$

$$\|u\|_{H^m(\Omega)} \quad := \quad \Big(\sum_{k=0}^{m} |u|_{H^k(\Omega)}^2 \Big)^{1/2}$$

- $(H^m(\Omega), \|\cdot\|_{H^m(\Omega)})$ are Banach spaces.

H^2 functions are continuous

- For $d = 1$

$$H^1(\Omega) \quad \subseteq \quad C(\Omega)$$

- For $d = 2$ and $d = 3$

$$H^2(\Omega) \quad \subseteq \quad C(\Omega)$$

- If $\partial\Omega$ is Lipschitz (or piecewise polynomial and Ω convex), then

$$H^2(\Omega) \quad \subseteq \quad C(\overline{\Omega})$$

with continuous embedding, i.e.

$$\sup_{x\in\Omega} |u(x)| \quad \leq \quad C\|u\|_{H^2(\Omega)} \qquad \forall u \in H^2(\Omega)$$

Then, the nodal interpolant is well-defined

$$I_h : H^2(\Omega) \to C(\overline{\Omega})$$

Nodal interpolation of H^2-functions

Hence, if $u \in H^2(\Omega)$, then it holds for the Poisson pb

$$\|\nabla(u - u_h)\|_\Omega \quad \leq \quad \|\nabla(u - I_h u)\|_\Omega$$

Questions:

1. What is the order of approximation with respect to h ?

   $$\|\nabla(u - I_h u)\|_\Omega \quad = \quad O(h^r)$$

   $r =?$

2. In which cases holds $u \in H^2(\Omega)$?

Structure to address the interpolation error

1. Localization:

$$\|\nabla(u - I_h u)\|_\Omega^2 = \sum_{T \in \mathcal{T}_h} \|\nabla(u - I_h u)\|_T^2$$

2. Transformation to the reference cell:

$$\|\nabla(u - I_h u)\|_T \rightarrow \|\nabla(\widehat{u} - \widehat{I}\widehat{u})\|_{\widehat{T}}$$

3. Interpolation error on the reference cell, for P_1/Q_1 elements:

$$\|\nabla(\widehat{u} - \widehat{I}\widehat{u})\|_{\widehat{T}} \leq c|\widehat{u}|_{H^2(\widehat{T})}$$

4. Backward transformation

$$|\widehat{u}|_{H^2(\widehat{T})} \rightarrow h_T^2 |u|_{H^2(T)}$$

5. Assembling together

$$\sum_{T \in \mathcal{T}_h} h_T^2 |u|_{H^2(T)}^2 \leq h^2 |u|_{H^2(\Omega)}^2$$

with $h = \max\{h_T : T \in \mathcal{T}_h\}$

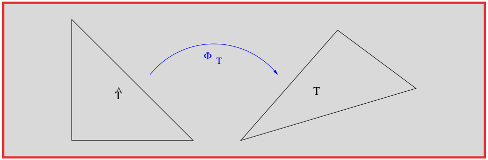
Step 2: Transformation to the reference cell

How to transform an expression as ($w = u - I_h u$)

$$\|\nabla(u - I_h u)\|_T^2 \quad = \quad \int_T |\nabla w(x)|^2 dx.$$

onto the reference triangle $\widehat{T}$ by an affine linear transformation

$$\Phi_T(\widehat{x}) = x_0 + B_T \widehat{x}$$



Partial derivative:

$$\begin{aligned}
\frac{\partial w(x)}{\partial x_i} &= \sum_{j=1}^{d} \frac{\partial \widehat{w}(\widehat{x})}{\partial \widehat{x}_j} \frac{\partial \widehat{x}_j}{\partial x_i} \\
&= \sum_{j=1}^{d} \frac{\partial \widehat{w}(\widehat{x})}{\partial \widehat{x}_j} (B_T^{-1})_{ji} \\
&= (B_T^{-t} \widehat{\nabla} \widehat{w}(\widehat{x}))_i
\end{aligned}$$

Gradient oin T:

$$|\nabla w(x)|^2 \quad \leq \quad \|B_T^{-t}\|_F^2 |\widehat{\nabla}\widehat{w}(\widehat{x})|^2$$

with Frobenius norm $\|B_T^{-t}\|_F = \sqrt{\sum_{i,j} |(B_T^{-t})_{i,j}|^2}$.

$$\begin{aligned}
\int_T |\nabla w(x)|^2 \, dx \quad &\leq \quad \|B_T^{-t}\|_F^2 \int_T |\widehat{\nabla}\widehat{w}(\widehat{x})|^2 dx \\
&= \quad \|B_T^{-t}\|_F^2 |\det B_T| \int_{\widehat{T}} |\widehat{\nabla}\widehat{w}(\widehat{x})|^2 d\widehat{x} \\
&= \quad \|B_T^{-t}\|_F^2 |\det B_T| \|\widehat{\nabla}\widehat{w}\|_{\widehat{T}}^2
\end{aligned}$$

What's about

$$\|\widehat{\nabla}(\widehat{u} - \widehat{I}\widehat{u})\|_{\widehat{T}}^2 \qquad ?$$

Step 3: Interpolation error on the reference cell

Theorem (Bramble-Hilbert lemma)

Let $T \subset \mathbb{R}^d$ a Lipschitz domain, F a normed space, $\Phi : H^m(T) \to F$ linear and continuous, $m > d/2$, such that

$$P_{m-1}(T) \quad \subseteq \quad Ker(\Phi).$$

Then there exists a constant $c = c(T, \Phi)$ s.t.

$$\|\Phi u\|_F \quad \leq \quad c|u|_{H^m(T)} \quad \forall u \in H^m(T).$$

- Interpolation $\widehat{I} : H^2(\widehat{T}) \to P_1$ by linear functions.
- Application of Bramble-Hilbert lemma to $\Phi = Id - \widehat{I}$, $F = H^1(\widehat{T})$, $m = 2$:

$$\|\widehat{\nabla}(\widehat{u} - \widehat{I}\widehat{u})\|_{\widehat{T}} \quad \leq \quad c|\widehat{u}|_{H^2(\widehat{T})}$$

because $\widehat{u} - \widehat{I}\widehat{u} = 0$, if u is a polynomial of maximal degree 1.

Step 4: Backward transformation

$$|\widehat{u}|_{H^2(\widehat{T})} \quad = \quad |\det B_T|^{-1/2}\|B_T^t\|_F^2|u|_{H^2(T)}$$

$$\|\nabla(u - I_h u)\|_\Omega^2 \quad \leq \quad c \sum_{T\in\mathcal{T}_h} \|B_T^{-1}\|_F^2\|B_T\|_F^4|u|_{H^2(T)}^2$$

What's about

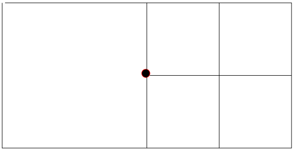
$$\|B_T^{-1}\|_F \quad \text{and} \quad \|B_T\|_F \qquad ?$$

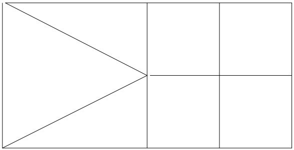
Shape regular meshes

Definition

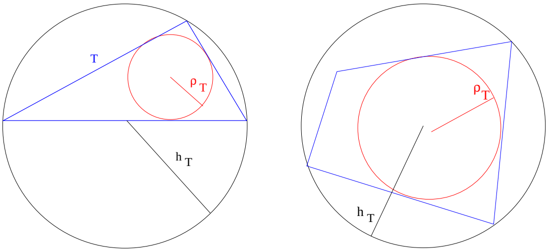
A triangulation $\mathcal{T}_h = \{T_1, \ldots, T_C\}$ of a domain $\Omega \subset \mathbb{R}^2$ consisting of triangles (or quadrilaterals) is called compatible, if $T_i \cap T_j$ for every $1 \leq i < j \leq C$ is

- is empty
- consists of a single node, or
- consists of an (entire) edge.





Geometrical parameters



$$\begin{aligned} h_T &= \text{outer radius} \\ \rho_T &= \text{inner radius} \\ \kappa_T &= h_T/\rho_T = \text{aspect ratio} \end{aligned}$$

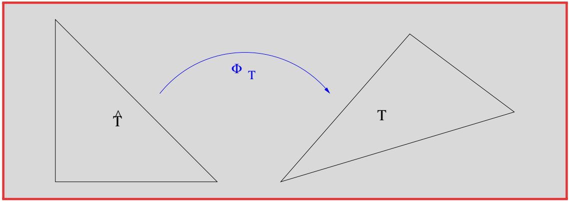
A family of meshes $\mathcal{T}_1, \mathcal{T}_2, \ldots$ is called shape regular if

$$\max_i \max_{T \in \mathcal{T}_i} \kappa_T \leq \kappa$$

A family of meshes $\mathcal{T}_1, \mathcal{T}_2, \ldots$ is called quasi uniform if

$$\frac{\max_{T \in \mathcal{T}_i} |T|}{\min_{T \in \mathcal{T}_i} |T|} \leq \kappa$$

Spectral norm of the transformation



For the spectral norm of the affin linear transformation

$$\Phi_T(\widehat{x}) = x_0 + B_T \widehat{x}$$

it holds:

$$\begin{aligned} \|B_T^{-1}\|_F &\leq h_{\widehat{T}}/\rho_T \\ \|B_T\|_F &\leq h_T/\rho_{\widehat{T}} \end{aligned}$$

with usually $h_{\widehat{T}}, \rho_{\widehat{T}} \sim 1$.
Hence, on shape regular meshes with maximal anisotropy κ, we obtain

$$\begin{aligned} \|B_T^{-1}\|_F &\leq \kappa h_T^{-1} \\ \|B_T\|_F &\leq h_T \end{aligned}$$

Step 5: Assembling together

$$\|\nabla(u - I_h u)\|_\Omega^2 \quad \leq \quad c \sum_{T \in \mathcal{T}_h} \|B_T^{-1}\|_F^2 \|B_T\|_F^4 |u|_{H^2(T)}^2$$

On shape regular meshes:

$$\|\nabla(u - I_h u)\|_\Omega^2 \quad \leq \quad c_\kappa \sum_{T \in \mathcal{T}_h} h_T^2 |u|_{H^2(T)}^2$$

Theorem

We consider the Poisson problem, discretized with P_1 finite elements on a family of shape regular meshes. If the solution u has the regularity H^2, then

$$\|\nabla(u - I_h u)\|_{L^2(\Omega)} \quad \leq \quad c_\kappa h |u|_{H^2(\Omega)}$$

where $h = \max_T h_T$ is the maximal cell size.

Remark: If Ω is convex or $\partial\Omega$ is C^2-regular, than u is a H^2 function with

$$|u|_{H^2(\Omega)} \quad \leq \quad c\|f\|_{L^2(\Omega)}$$

Error estimates in the L^2-norm

L^2 error estimates

Are you interested in a "weaker" norm

$$\|u - u_h\|_{L^2(\Omega)}$$

instead of $\|\nabla(u - u_h)\|_{L^2(\Omega)}$?

- Interpolation error

$$\|u - I_h u\|_{L^2(\Omega)} \leq c_\kappa \mathbf{h}^2 |u|_{H^2(\Omega)}$$

- Comparison with Poincare inequality:

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)} &\leq c_\Omega \|\nabla(u - u_h)\|_{L^2(\Omega)} \\ &\leq c\mathbf{h}|u|_{H^2(\Omega)} \end{aligned}$$

  is sub-optimal.

- In fact, one may do better by a duality argument

$$\|u - u_h\|_{L^2(\Omega)} \leq c\mathbf{h}^2 |u|_{H^2(\Omega)}$$

  see: Aubin-Nitsche trick.

Duality argument

- Aim: Derive error bound on

$$\|u - u_h\|_W$$

in a weaker norm, i.e. let W be an Hilbert space with continuous embedding $V \subseteq W$, i.e.

$$\|u\|_W \quad \leq \quad c\|u\|_V \qquad \forall u \in V$$

- Then

$$\|u - u_h\|_W \quad = \quad \sup_{g \in S} \langle g, u - u_h \rangle$$

with

$$S \quad := \quad \{g \in W' \; : \; \|g\|_{W'} = 1\}$$

- Our particular case:

$$V = H_0^1(\Omega)\,, \quad W = L^2(\Omega)$$

continuous embedding due to Poincaré inequality.

- Due to the continuous embedding $V \subseteq W$ it holds

$$W' \subseteq V'$$

- Hence, $g \in S \subset V'$ is a possible rhs in the dual problem:

$$z_g \in V: \quad A(\phi, z_g) \;=\; \langle g, \phi \rangle \qquad \forall \phi \in V$$

- Primal problem:

$$u \in V: \quad A(u, \phi) \;=\; \langle f, \phi \rangle \qquad \forall \phi \in V$$

- We obtain

$$\begin{aligned} \langle g, u - u_h \rangle &= A(u - u_h, z_g) \\ &= A(u - u_h, z_g - z_h) \\ &\leq \alpha_1 \|u - u_h\|_V \|z_g - z_h\|_V \end{aligned}$$

  for arbitrary $z_h \in V_h$.
- Hence

$$\|u - u_h\|_W \;\leq\; \alpha_1 \|u - u_h\|_V \sup_{g \in S} \|z_g - z_h\|_V$$

Aubin-Nitsche trick

We have shown:

Theorem (Aubin-Nitsche)

Let V, W Hilbert spaces with continuos embedding $V \subseteq W$. The bilinear form $A : V \times V \to \mathbb{R}$ is assumed to be continuos (in V with constant α_1). Then holds for the corresponding finite element solution $u_h \in V_h \subset V$:

$$\|u - u_h\|_W \quad \leq \quad \alpha_1 \sup_{g \in W', \|g\|_{W'} = 1} \left\{ \inf_{z_h \in V_h} \|z_g - z_h\|_V \right\} \|u - u_h\|_V \,,$$

where z_g is the solution of the associated dual problems .

We arrive at:

Theorem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a convex domain or a domain with C^2-boundary, $\{\mathcal{T}_h\}$ be a family of shape regular triangulations of Ω. Then it holds for the P_1 (or Q_1) finite element solution of Poisson pb:

$$\|u - u_h\|_{L^2(\Omega)} \quad \leq \quad ch^2 |u|_{H^2(\Omega)} \,.$$

Proof.

$$\|u - u_h\|_\Omega \quad \leq \quad \sup_{g \in L^2(\Omega), \|g\|_\Omega = 1} \left\{ \inf_{z_h \in V_h} \|\nabla(z_g - z_h)\|_\Omega \right\} \|\nabla(u - u_h)\|_\Omega$$

Due to the property of Ω, we know that $u, z_g \in H^2(\Omega)$. Hence:

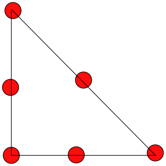
$$\begin{aligned} \|\nabla(u - u_h)\|_\Omega \quad &\leq \quad ch|u|_{H^2(\Omega)} \\ \|\nabla(z_g - z_h)\|_\Omega \quad &\leq \quad ch\|g\|_{L^2(\Omega)} \; = \; ch \end{aligned}$$

Higher order finite elements

- FEM of order $r \geq 1$:

$$P_r(\mathcal{T}_h) \quad := \quad \{\varphi : \Omega_h \to \mathbb{R} \ : \ \varphi|_T \in P_r \, \forall T \in \mathcal{T}_h\}$$

- P_2 elements: one dof on each vertex + one dof on each face



$$\begin{aligned} \varphi(x,y) \quad &= \quad \alpha_0 + \alpha_1 x + \alpha_2 y \\ & \qquad + \alpha_3 x^2 + \alpha_4 y^2 + \alpha_5 xy \end{aligned}$$

- A two-dimensional polynome of order $r \geq 1$ on a triangle, restricted to one edge is a one-dimensional polynome of order r.
  Hence: We need $r+1$ degrees of freedom on each edge.

Error estimate for higher order finite elements

Theorem

We consider the Poisson problem, discretized with P_r finite elements ($r \geq 1$) on a family of shape regular meshes. If the solution u has the regularity H^{r+1}, then

$$\|\nabla(u - I_h u)\|_\Omega \quad \leq \quad c_\kappa h^r |u|_{H^{r+1}(\Omega)}$$

where $h = \max_T h_T$ is the maximal cell size.

Proof: As before by Bramble-Hilbert with corresponding interpolation onto the reference cell:

$$\|\widehat{\nabla}(\widehat{u} - \widehat{I}\widehat{u})(\widehat{x})\|_{\widehat{T}} \quad \leq \quad c|\widehat{u}|_{H^{r+1}(\widehat{T})}$$

for the nodal interpolation $\widehat{I} : H^{r+1}(\widehat{T}) \to P_r$.
The powers of h result from the transformation of $|\widehat{u}|_{H^{r+1}(\widehat{T})}$ to the cell T:

$$|\widehat{u}|_{H^{r+1}(\widehat{T})} \quad = \quad |\det B_T^{-1}|^{1/2} \|B_T^t\|_F^{r+1} |u|_{H^{r+1}(T)}$$

Pro's and contras of higher order finite elements

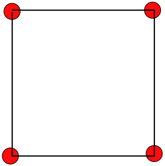
Pro's:

- A better approximation property is expected due to better asymptotic behaviour
- Less degrees of freedom for a given accuracy
- More local couplings in the stiffness matrix (can be advantageous for CPU reasons)

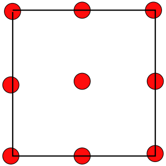
Contra:

- More regularity of the solution is necessary. Otherwise: reduction of accuracy / order of convergence.
- Stiffness matrix become more dense due to many couplings inside each element
- Robust linear solvers are usually more difficult

Accuracy of Q_r elements



$$\varphi(x, y) = \sum_{i,j=0}^{r} \alpha_{ij} x^i y^j$$



- The nodal interpolation $\widehat{I}$ on the reference quadrilateral / hexahedral is exact for polynomials of degree $\leq r$.
- Hence, Bramble-Hilbert lemma gives the same results as for P_r elements:

$$\|\nabla(u - u_h)\| \leq ch^r |u|_{H^{r+1}(\Omega)}$$

Summary of Lecture 2:

- FE for continuous, coercive bilinear forms are quasi-optimal with respect to discretization error:

$$\boxed{\|\nabla(u-u_h)\| \quad \leq \quad \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V_h} \|\nabla(u-v_h)\|}$$

- Optimal for the Poisson problem, $\alpha_1/\alpha_2 = 1$.
- P_1 finite elements are of order 1 in the energy norm

$$\|\nabla(u-u_h)\| \quad \leq \quad ch|u|_{H^2(\Omega)}$$

- The error in L^2 is one order better (if Ω is regular enough):

$$\|u-u_h\|_{L^2(\Omega)} \quad \leq \quad ch^2|u|_{H^2(\Omega)}$$

- P_r or Q_r, $r \geq 1$. finite elements are of order r in the energy norm

$$\|\nabla(u-u_h)\|_{L^2(\Omega)} \quad \leq \quad ch^r|u|_{H^{r+1}(\Omega)}$$