Chapter 1

P_1 Finite elements on triangles - tetraedrons

Let $V = H_0^0(\Omega)$. We consider the Poisson problem $u \in H_0^1(\Omega)$

$$-\Delta u = f \text{ in } \Omega$$

and its weak formulation

1.1 Lecture 2: Accuracy of Finite Element Discretizations

Outline for today:

1. A priori error estimates in H^1

$$\|\nabla(u - u_h)\|_{L^2(h)}.$$

C dependes of the solution. One is usually intereset in the error

$$\|u - u_h\|_X = ?$$

for a certain norm $\| \cdot \|_X$.

- **A priori error estimates:**
  Information about the error in terms of mesh size asymptotics, e.g, for P_1 or Q_1 elements

$$\|\nabla(u - u_h)\|_{L^2(h)} \leq ch|u|_{H^2(\Omega)}$$
$$\|u - u_h\|_{L^2(\Omega)} \leq ch^2|u|_{H^2(\Omega)}$$

- **A posteriori error estimates:** Information about the error in terms of u_h, e.g,

$$\|\nabla(u - u_h)\|^2_{L^2(\Omega)} \leq \sum$$

1.2 Galerkin Orthogonality

- Continuous problem with $A\colon V \times V \to \mathbb{R}$ bilinear:
$$u \in V\colon A(u,\phi) = (f,\phi) \quad \forall \phi \in V.$$
Most simple example:
$$A(u,\phi) = (\nabla u, \nabla \phi) = \int_\Omega \nabla u \nabla \phi \mathrm{d}x$$
- Discrete problem:
$$u_h \in V_N\colon A(u_h,\phi) = (f,\phi) \quad \forall \phi \in V_h.$$
- Discretization error
$$e_h =$$

1.3 Cea's lemma

Suppose that the bilinear form $A\colon V \times V \to \mathbb{R}$ satisfies the conditions of Lax-Milgram's theorem (continuous and V-coercive with $\alpha_1, \alpha_2 > 0$). Further, let $V_h \subseteq V$ a subspace. Then:
$$\|u - u_h\|_V \leq \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V}$$

1.4 Continuity and coercivity

- **Continuity:** There exists $\alpha_1 \geq 0$ such that.
$$A(u,\phi) \leq \alpha_1$$

1.5 Proof of Cea's lemma

Let $v_h \in V_h$ be arbitrary,
$$\alpha_2 \|u - u_h\|_V^2 \leq A(u - u_h, u - u_h) \quad (\text{ coercivity })$$

- $V = H_0^1(\Omega)$.
- associated norm
$$\|u\|_V = \|\nabla u\|_\Omega = \left(\int_\Omega |\nabla u(x)|^2 \mathrm{d}x \right)^{1/2}$$
- $\alpha_1 = \alpha_2 = 1$
$$\|\nabla (u - u_h)\|_\Omega = \inf_{v_h \in V_h} \|\nabla (u - v_h)\|_\Omega$$

1.6 Interpolation error

- Let $I_h \colon V \to V_h$ be an arbitrary interpolation. Then

$$\begin{aligned}\|u-u_h\|_V &\le \frac{\alpha_1}{\alpha_2}\inf_{v_h\in V_h}\|u-v_h\|_V\\ &\le \frac{\alpha_1}{\alpha_2}\|u-I_h u\|_V\end{aligned}$$

- We only need do get and idea about the interpolation error

$$\|u-I_h u\|_V$$

- Most simple is the nodal interpolation of continuous functions

$$I_h u(N) = u(N).$$

  for nodes N of the mesh.

- But: Are $H^1(\Omega)$ functions continuuous?

$$d=1 \quad \text{(yes)} \qquad d\ge 2 \quad \text{(no)}$$

1.7 More regular Sobolev functions

- Higher orden Sobolev spaces of order $k \ge 1$:

$$H^k(\Omega) =$$

1.8 H^2 functions are continuous

- For $d=1$

$$H^1(\Omega) \subseteq C(\Omega)$$

- For $d=2$ and $d=3$

$$H^2(\Omega) \subseteq C(\Omega)$$

- If $\partial\Omega$ is Lipschitz ...

1.9 Nodal interpolation of H^2-functions

Hence, if $u \in H^2(\Omega)$, then it holds for the Poisson pb

$$\|\nabla(u-I_h)\|$$

1.10 Structure to addres the interpolation error

1. Location:
$$\|\nabla(u - I_h u)\|_\Omega^2 = \sum_{T \in T_h} \|\nabla(u - I_h u)\|_T^2$$

2. Transformation to the reference cell:
$$\|\nabla(u - I_h)\|$$

1.11 Step 2: Transformation to the reference cell

How to transform an expression as ($w = u - I_h u$)

$$\|\nabla(u - I_h u)\|_T^2 = \int_T |\nabla w(x)|^2 \mathrm{d}x$$

onto the reference triangle $\hat{T}$ by an affine linear transformation

$$\phi_T(\hat{x}) = x_0 + B_T \hat{x}$$

Partial derivative

$$\frac{\partial w(x)}{\partial x_i} = \sum_{j=1}^{d}$$

Gradient oin T:
$$|\nabla w(x)|^2 \leq \|B_T^{-t}\|_F^2 |\hat{\nabla} \hat{w}(\hat{x})|^2$$

with Frobenius norm $\|B_T^{-t}\|$

1.12 Step 3: Interpolation error on the reference cell

Theorem 1. Bramble-Hilbert lemma

Let $T \subset \mathbb{R}^d$ a Lipschitz domain, F a normed space, $\phi : H^m(T) \to F$

1.13 Step 4: Backward transformation

$$|\hat{u}|_{H^2(\hat{T})} = |\mathrm{det} B_T|^{-1/2} \|B_T^t\|$$

1.14 Geometrical parameters

h_T =outer radius, ρ_T =inner radius, $\kappa_T = \frac{h_T}{\rho_T}$ =aspect ratio

A family of meshes $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3, \ldots$ is called **shape regular** if $\max_i \max_{T\in\mathcal{T}_i} \kappa_T \leq \kappa$

A family of meshes $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3, \ldots$ is called **quasi uniform** if $\frac{\max}{\min} \leq \kappa$

1.15 Spectral norm of transformation

For the spectral norm of the affin linear transformation ϕ_T

1.16 Step 5: Assembling together

$$\|\nabla(u - I_h u)\|_\Omega^2 \leq c \sum_{T\in\mathcal{T}_h} \|B_T^{-1}\|_F^2 \|B_T\|$$

Error estimates in the L^2 norm

1.17 L^2 error estimates

Are you interested in a "weaker" norm

$$\|u - u_h\|_{L^2(\Omega)}$$

instead of

$$\|u - I_h u\|_{L^2(\Omega)} \leq c_\kappa \boldsymbol{h^2} |u| H^2(\Omega)$$

?

1.18 Duality argument

- **Aim**: Derive error bound on

  $$\|u - u_h\|_W$$

  in a weaker norm, i.e. let W be an Hilbert space with continuous embedding $V \subseteq W$, i.e.

  $$\|u\|_W$$

- Due to the continuous embedding $V \subseteq W$ it holds

  $$W' \subseteq V'$$

- Hence, $g \in S \subset V'$ is a possible rhs in the dual problem:

  $$z_g \in V : \quad A(\phi, z_g) = \langle g, \phi\rangle \quad \forall\phi \in V$$

- Primal problem:

  $$u \in V : \quad A(u, \phi) = \langle f, \phi\rangle.$$

1.19 Aubin-Nitsche trick

We arribe at:

Theorem 2. Let $\Omega \subset \mathbb{R}^d, d \in \{2, 3\}$, be a convex domain or a domain with C^2-boundary, $\{T_h\}$.

1.20 Higher orden finite elements

- FEM or order $r \geq 1$:

$$P_r(\mathcal{T}_h) = \{\varphi\colon \Omega_h \to \mathbb{R}\colon \varphi\big|_T \in P_r \forall T \in \mathcal{T}_h\}$$

1.21 Error estimate for higher order finite elements

Theorem 3. We consider the Poisson problem, discretized with P_r finite elments ($r \geq 1$) on a family of shape regular meshes. If the solution u has regularity H^{r+1}, then

$$\|\nabla(u - I_h u)\|_\Omega \leq c_\kappa h^r |u| H^{r+1}$$

1.22 Pro's and contras of higher order finite elements

Pro's:

- A better approximation property is expected due to better asymptotic behavoiur
- Less degrees of freedom for a given accuarancy.
- More local couplings in the stiffness matrix (can be advantageous for CPU reasons)

Contra:

- More regularity of the solution is necessary. Otherwise: reduction of accuracy \order of convergence.
- Stiffness matrix become more dense due to many couplings inside each elements
- Robust linear solvers are usually more difficult.

1.23 Accuracy of Q_r elements

$$\varphi(x, y) = \sum_{i,j=0}^{r} \alpha_{ij} x^i y^j$$

- The nodal interpolation $\hat{I}$ in the reference quadirlataral \hexaedral is exacts for polynomials of degree $\leq r$

1.24 Summary of Lecture 2

- FE for continuous, coercive bilinear forms are quasi-optimal with respect to discretization error:

Abcedario