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Exercise Nr. 2, *Summer School on Finite Elements*
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1. **Study the program:** Study the matlab program provided to you. Some details on the different files:

   dirichlet.m This function sets the dirichlet data at the boundary in the point (x, y).

   exactsolution.m This function sets the exact solution $u(x, y)$ and its derivatives $\partial_x u(x, y)$ and $\partial_y u(x, y)$ in the point (x, y)

   l2error.m This function evaluates the L^2-error

   $$\|u - u_h\|_\Omega,$$

   where u is the exact solution from `exactsolution.m` and u_h the finite element solution.

   h1error.m This function evaluates the H^1-error

   $$\|\nabla(u - u_h)\|_\Omega,$$

   where u is the exact solution from `exactsolution.m` and u_h the finite element solution.

2. **Modify the Dirichlet data:** Solve the Laplace problem

$$-\Delta u = 1 \text{ in } \Omega = (0,1)^2 \text{ and } u(x,y) = \sin(4 * \pi x) * \sin(2\pi y) \text{ on } \partial\Omega.$$

Modify the right hand side in `righthandside.m` and the dirichlet data in `dirichlet.m`.

3. **Discontinuous Dirichlet data:** Solve the Laplace problem

$$-\Delta u = 1 \text{ in } \Omega = (0,1)^2 \text{ and } u(x,y) = \begin{cases} 1 & x < y \\ -1 & x \geqslant y \end{cases} \text{ on the boundary } \partial\Omega$$

Modify the right hand side in `righthandside.m` and the dirichlet data in `dirichlet.m`.

4. We want to solve the Laplace problem

$$-\Delta u = f \text{ in } \Omega \text{ and } u = g \text{ on } \partial\Omega$$

such that the exact solution is given as

$$u(x,y) = x^2 + y^2.$$

a) Compute the right hand side $f = -\Delta u$ and implement it in `righthandside.m`.

b) Implement the Dirichlet data

$$g(x,y) = x^2 + y^2$$

in `dirichlet.m`

c) Compute the derivatives $\partial_x u(x,y)$ and $\partial_y u(x,y)$ and implement the exact solution in `exactsolution.m`.

Run the program with different values of $M = 10, M = 20, M = 40, \ldots$ and compute the convergence rate h^α.

5. **Study of convergence:** Repeat the last exercise but with the exact solution

$$u(x,y) = \left(x - \frac{1}{2})^2 + (x - \frac{1}{2})^2\right)^{\frac{1}{4}}$$

Hint: the right hand side is given by

$$f = -\Delta u = \frac{1}{4}\left(x - \frac{1}{2})^2 + (x - \frac{1}{2})^2\right)^{-\frac{3}{4}},$$

the first derivative is given by

$$\partial_x u = \frac{1}{4}(1-2x)\left(x - \frac{1}{2})^2 + (x - \frac{1}{2})^2\right)^{-\frac{3}{4}},$$
$$\partial_y u = \frac{1}{4}(1-2y)\left(x - \frac{1}{2})^2 + (x - \frac{1}{2})^2\right)^{-\frac{3}{4}}$$