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An Introduction to the Finite Element Method

Third Edition

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PREFACE

The third edition of the book, like the previous two editions, represents an effort to select and present certain aspects of the finite element method that are most useful in developing and analyzing linear problems of engineering and science. In revising the book, students taking courses that might use this book as their textbook have been kept in mind. This edition is prepared to bring more clarity to the concepts being discussed while maintaining the necessary mathematical rigor and providing physical interpretations and engineering applications at every step.

This edition of the book is a revision of the second edition, which was well received by engineers as well as researchers in the fields of engineering and science. Most of the revisions in the current edition took place in Chapters 1 through 6 and 14. Chapter 5 on error analysis from the second edition is eliminated and a section on the same subject is added to Chapter 14. Chapter 3 from the second edition is now divided into two chapters, Chapter 3 on theoretical formulation and Chapter 4 on applications. Chapter 4 on beams from the second edition now became Chapter 5 in the current edition, making the total number of chapters in both editions the same. Another change is the interchanging of Chapters 10 and 11 to facilitate the natural transition from plane elasticity to plate bending (Chapter 12). New material on three-dimensional finite element formulations and nonlinear formulations is added in Chapter 14. In all the chapters, material is added and reorganized to aid the reader in understanding the concepts.

Another change in this edition is the removal of appendices containing the program listing of **FEM1D** and **FEM2D**, which would have taken up over 80 pages. The program listings, executables, and instructor resources are available on the text's website at www.mhhe.com/reddy3e. The source files of these programs may be obtained from the author for a small fee.

Most people who have used the earlier editions of the book liked the "differential equations approach" adopted here. This is natural because everyone, engineer or scientist, is looking for a way to solve differential equations arising in the study of physical phenomena. It is hoped that the third edition of the book serves the student even better than the previous editions in understanding the finite element method as applied to linear problems of engineering and science.

The author has benefited by teaching an introductory course from this book for many years. While it is not possible to name hundreds of students and colleagues who have contributed to the author's ability to explain concepts in a clear manner, the author

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expresses his sincere appreciation to the following people for their helpful comments on this edition:

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Tejashwina vadheetamasthu
(*May what we study be well studied*)

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Chapter 1
INTRODUCTION

1.1 GENERAL COMMENTS

One of the most important things engineers and scientists do is to model physical phenomena. Virtually every phenomenon in nature, whether aerospace, biological, chemical, geological, or mechanical can be described, with the aid of the laws of physics or other fields in terms of algebraic, differential, and/or integral equations relating various quantities of interest. Determining the stress distribution in a pressure vessel with oddly shaped holes and numerous stiffeners and subjected to mechanical, thermal, and/or aerodynamic loads, finding the concentration of pollutants in lakes and seawater or in the atmosphere, and simulating weather in an attempt to understand and predict the formation of thunderstorms and tornadoes are a few examples of many important practical problems that engineers deal with.

Analytical descriptions of physical phenomena and processes are called *mathematical models*. Mathematical models of a process are developed using assumptions concerning how the process works and using appropriate axioms or laws governing the process, and they are often characterized by very complex differential and/or integral equations posed on geometrically complicated domains. Consequently, the processes to be studied, until the advent of electronic computation, were drastically simplified so that the governing equations can be solved analytically. Over the last three decades, however, computers have made it possible, with the help of suitable mathematical models and numerical methods, to solve many practical problems of engineering. The use of a numerical method and a computer to evaluate the mathematical model of a process and estimate its characteristics is called a *numerical simulation*. There now exists a new and growing body of knowledge connected with the development of mathematical models and use of numerical simulations of physical systems, and it is known as *computational mechanics* [1].

Any numerical simulation, such as the one by the finite element method, is not an end in itself but rather an aid to design and manufacturing. There are several reasons why an engineer or a scientist should study a numerical method, especially the finite element method:

1. Most practical problems involve complicated domains (both geometry and material constitution), loads, and nonlinearities that forbid the development of analytical solutions. Therefore, the only alternative is to find approximate solutions using numerical methods.

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2. A numerical method, with the advent of a computer, can be used to investigate the effects of various parameters (e.g., geometry, material parameters, or loads) of the system on its response to gain a better understanding of the process/system being analyzed. It is cost-effective and saves time and material resources compared to the multitude of physical experiments needed to gain the same level of understanding.
3. Because of the power of numerical methods and electronic computation, it is possible to include all relevant features in a mathematical model of a physical process without worrying about its solution by exact means.
4. Those who are quick to use a computer program rather than *think* about the problem to be analyzed may find it difficult to interpret or explain the computer-generated results. Even to develop proper input data to the computer program, a good understanding of the underlying theory of the problem as well as the numerical method (on which the computer program is based) is required.
5. The finite element method and its generalizations are the most powerful computer-oriented methods ever devised to analyze practical engineering problems. Today, finite element analysis is an integral and major component in many fields of engineering design and manufacturing. Major established industries such as the automobile, aerospace, chemical, pharmaceutical, petroleum, electronics, and communications, as well as emerging technologies such as nanotechnology and biotechnology rely on the finite element method to simulate complex phenomenon at different scales for design and manufacture of high-technology products.

1.2 MATHEMATICAL MODELS

A *mathematical model* can be broadly defined as a set of equations that expresses the essential features of a physical system in terms of variables that describe the system. The mathematical models of physical phenomena are often based on fundamental scientific laws of physics such as the principle of conservation of mass, conservation of linear momentum, and conservation of energy. Below we consider three simple examples drawn from dynamics, heat transfer, and solid mechanics to illustrate how mathematical models of physical problems are formulated.

Example 1.2.1 (A Dynamics Problem)

A simple pendulum (such as the one used in a clock) consists of a bob of mass m (kg) attached to one end of a rod of length ℓ (m) and the other end is pivoted to a fixed point O, as shown in Fig. 1.2.1. In order to derive the governing equation of the problem, we must make certain assumptions concerning the system (the bob and rod) consistent with the goal of the analysis. If the goal is to study the simplest linear motion of the pendulum, we assume that the bob as well as the rod are rigid (i.e., not deformable) and the rod is massless (i.e., compared to that of the bob). In addition, we assume that there is no friction at the pivot point O and the resistance offered by the surrounding medium to the pendulum is also negligible.

Under these assumptions, the equation governing the motion of the system can be formulated using the *principle of conservation of linear momentum* (or simply Newton's second law),

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Figure 1.2.1 Simple pendulum.

which states, in the present case, that the vector sum of externally applied forces on a system is equal to the time rate of change of the linear momentum (mass times velocity) of the system:

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\mathbf{a} \tag{1.2.1}$$

where **F** is the vector sum of all forces acting on the system, m is the mass of the system, **v** is the velocity vector, and **a** is the acceleration vector of the system. To write the equation governing the angular motion, we set up a coordinate system, as shown in Fig. 1.2.1. Applying Newton's second law to the x-direction (note that the dynamic equilibrium of forces in the y-direction gives the reaction force $R(N)$ in terms of the weight mg of the bob), we obtain

$$F_x = m\frac{dv_x}{dt} \tag{1.2.2}$$

where $F_x = -mg\ \sin\theta$, $v_x = \ell\frac{d\theta}{dt}$, θ is the angular displacement (radians), v_x is the component of velocity (m/s) along the x coordinate, and t denotes time (s). Thus, the equation for angular motion becomes

$$-mg\ \sin\theta = m\ell\frac{d^2\theta}{dt^2} \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0 \tag{1.2.3}$$

Equation (1.2.3) is nonlinear on account of the term $\sin\theta$. For small angular motions (consistent with the goal of the study), $\sin\theta$ is approximated as θ. Thus, the angular motion is described by the linear differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0 \tag{1.2.4}$$

Equations (1.2.3) and (1.2.4) represent mathematical models of nonlinear and linear motions, respectively, of a rigid pendulum. Their solution requires knowledge of conditions at time $t = 0$ on θ and its time derivative $\dot{\theta}$ (angular velocity). These conditions are known as the *initial conditions*. Thus, the linear problem involves solving the differential equation (1.2.4) subjected to the initial conditions

$$\theta(0) = \theta_0, \qquad \frac{d\theta}{dt}(0) = v_0 \tag{1.2.5}$$

The problem described by Eqs. (1.2.4) and (1.2.5) is called an *initial-value problem*.

The linear problem described by Eqs. (1.2.4) and (1.2.5) can be solved analytically. The general analytical solution of the linear equation (1.2.4) ($\ddot{\theta} + \lambda^2\theta = 0$) is given by

$$\theta(t) = A\ \sin\lambda t + B\ \cos\lambda t \tag{1.2.6}$$

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where $\lambda = \sqrt{\frac{g}{l}}$ and A and B are constants to be determined using the initial conditions in (1.2.5). We obtain

$$A = \frac{v_0}{\lambda}, \quad B = \theta_0 \tag{1.2.7}$$

and the solution to the linear problem is

$$\theta(t) = \frac{v_0}{\lambda} \sin \lambda t + \theta_0 \cos \lambda t \tag{1.2.8}$$

For zero initial velocity and nonzero initial position θ_0, we have

$$\theta(t) = \theta_0 \cos \lambda t \tag{1.2.9}$$

which represents a simple harmonic motion.

If we were to solve the nonlinear equation (1.2.3) subject to the conditions in (1.2.5), we may consider using a numerical method because it is not possible to solve it exactly for large values of θ. We will revisit this issue in the sequel.

Example 1.2.2 (A Heat Transfer Problem)

Here we wish to derive the governing equations (i.e., develop the mathematical model) of steady-state heat transfer through a cylindrical bar of nonuniform cross section. The bar is subject to a known temperature T_0 (°C) at the left end and exposed, both on the surface and at the right end, to a medium (such as cooling fluid or air) at temperature T_∞. We assume that temperature is uniform at any section of the bar, $T = T(x)$. Due to the difference between the temperatures of the bar and the surrounding medium, there is convective heat transfer across the surface of the body and at the right end. The principle of conservation of energy (or the *second law of thermodynamics*) can be used to derive the governing equations of the problem. The principle of conservation of energy requires that the rate of change (increase) of internal energy is equal to the sum of heat gained by conduction, convection, and internal heat generation (radiation not included). For a steady process, the time rate of internal energy is zero.

Consider a volume element of length Δx and having an area of cross section $A(x)$ (m^2) normal to the x axis (see Fig. 1.2.2). If q denotes the heat flux (heat flow per unit area, W/m^2), then $[Aq]_x$ is the net heat flow into the volume element at x, $[Aq]_{x+\Delta x}$ is the net heat flow out of the volume element at $x + \Delta x$, and $\beta P \Delta x(T_\infty - T)$ is the heat flow through the surface of the rod into the body. Here β denotes the film (that is formed between the body and the medium around) conductance [W/(m^2·°C)], T_∞ is the temperature of the surrounding medium, and P is the perimeter (m). We also assume that there is a heat source within the rod generating energy at a rate of g (W/m^3). In practice, such energy source can be due to nuclear fission or chemical reactions taking place within the rod, or due to the passage of electric current through the medium (i.e., volume heating). Then the energy balance gives

$$[Aq]_x - [Aq]_{x+\Delta x} + \beta P \, \Delta x(T_\infty - T) + gA \, \Delta x = 0 \tag{1.2.10}$$

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Figure 1.2.2 Heat transfer in a bar.

or, dividing throughout by Δx,

$$-\frac{[Aq]_{x+\Delta x}-[Aq]_x}{\Delta x}+\beta P(T_\infty-T)+Ag=0$$

and taking the limit $\Delta x \to 0$, we obtain

$$-\frac{d}{dx}(Aq)+\beta P(T_\infty-T)+Ag=0 \tag{1.2.11}$$

We can relate the heat flux q (W/m^2) to the temperature gradient. Such a relation is provided by the *Fourier law*

$$q(x)=-k\frac{dT}{dx} \tag{1.2.12}$$

where k denotes the thermal conductivity [W/(m·°C)] of the material. The minus sign on the right side of the equality in the above equation indicates that heat flows from high temperature to low temperature. Equation (1.2.12) is known as a *constitutive relation* because it contains a material property.

Now using the Fourier law (1.2.12) in (1.2.11), we arrive at the heat *conduction equation*

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right)+\beta P(T_\infty-T)+Ag=0 \tag{1.2.13}$$

which can also be written as

$$-\frac{d}{dx}\left(kA\frac{dT}{dx}\right)+\beta P(T-T_\infty)=Ag \tag{1.2.14}$$

Equation (1.2.14) is a linear, (nonhomogeneous) second-order differential equation, which can be solved with known conditions on temperature T or heat Aq (not both) at a boundary

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point (e.g., ends of the bar). The known end conditions for the present case (recall the problem description at the start of this example) can be expressed as

$$T(0) = T_0, \quad \left[kA\frac{dT}{dx} + \beta A(T - T_\infty)\right]_{x=L} = 0 \tag{1.2.15}$$

These conditions are called *boundary conditions* because they represent conditions at the boundary points of the bar. The first condition in (1.2.15) is obvious. The second condition represents the balance of heat due to conduction $[kA(dT/dx)]$ and convection $[\beta A(T - T_\infty)]$. The problem described by Eqs. (1.2.14) and (1.2.15) is called, for obvious reasons, a *boundary-value problem*. This completes the mathematical model development of the problem.

Equations (1.2.14) and (1.2.15) can be simplified for various special cases. First let

$$\theta \equiv T - T_\infty, \quad a \equiv kA \text{ (W·m/°C)}, \quad c \equiv \beta P \text{ [W/(m·°C)]}, \quad f \equiv Ag \text{ (W/m)} \tag{1.2.16}$$

so that Eqs. (1.2.14) and (1.2.15) can be written as (note that a and c are always positive in heat transfer problems)

$$-\frac{d}{dx}\left(a\frac{d\theta}{dx}\right) + c\theta = f \tag{1.2.17}$$

$$\theta(0) = T_0 - T_\infty, \quad \left[a\frac{d\theta}{dx} + \beta A\theta\right]_{x=L} = 0 \tag{1.2.18}$$

Now if a and c are constant (e.g., homogeneous bar of constant cross section) and $g = 0$ (no internal heat generation), Eqs. (1.2.17) and (1.2.18) become

$$-\frac{d^2\theta}{dx^2} + \frac{c}{a}\theta = 0, \text{ where } 0 < x < L \tag{1.2.19}$$

$$\theta(0) = T_0 - T_\infty \equiv \theta_0, \quad \left[\frac{d\theta}{dx} + \frac{\beta}{k}\theta\right]_{x=L} = 0 \tag{1.2.20}$$

The general solution of (1.2.19) ($\theta'' - m^2\theta = 0$) is

$$\theta(x) = C_1 \cosh mx + C_2 \sinh mx, \quad m \equiv \sqrt{\frac{c}{a}} = \sqrt{\frac{\beta P}{kA}} \tag{1.2.21a}$$

where C_1 and C_2 are constants that can be determined using the boundary conditions (1.2.20). Using $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$, we have

$$C_1 = \theta_0, \quad C_2 = -\theta_0\left[\frac{\sinh mL + (\beta/mk)\cosh mL}{\cosh mL + (\beta/mk)\sinh mL}\right] \tag{1.2.21b}$$

Hence, the solution of Eqs. (1.2.19) and (1.2.20) becomes

$$\theta(x) = \theta(0)\left[\frac{\cosh m(L - x) + (\beta/mk)\sinh m(L - x)}{\cosh mL + (\beta/mk)\sinh mL}\right] \tag{1.2.22}$$

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Example 1.2.3 (A Solid Mechanics Problem)

The last example is concerned with the mathematical formulation of the axial deformation of a bar of variable cross section. The term *bar* is used in solid and structural mechanics to mean a structural element that carries only axial loads (tensile as well as compressive). Practical examples of the problem are provided by the deformation of a slender body under its own weight and by a concrete pier supporting a bridge.

In the case of the concrete pier, it may be treated as a three-dimensional problem. The weight of the bridge as well as the weight of vehicles on the bridge is carried by the piers. The piers at the ends of a bridge take slightly different load than the ones in the interior. In a preliminary design of structures, we often try to obtain the orders-of-magnitude of the significant deformation and stress components in a typical pier rather than determine the detailed three-dimensional deformation and stress fields. With the former in mind, we make several simplifying assumptions to reduce the problem to a one-dimensional problem that allows us to determine the compression and associated stress in a typical pier. First, we know that the significant load taken by a pier is vertical (along the height of the pier). So the axial deformation (compression) and stress will be the significant quantities, and the lateral deformation due to the Poisson effect may be neglected. Second, we wish to consider the static case (as if all of the traffic on the bridge is standing still—like during heavy rush hour) with a magnitude of the axial load to represent the worst case scenario of heaviest traffic as well as snow fall (a quasi-static case may be included by magnifying the static loads). We may also take into account the weight (which is a volume force, called body force) of the pier itself. The governing equations of this simplified problem can be obtained using Newton's second law and a uniaxial stress-strain (constitutive) relation (note the similarity in the derivation of the equations of this solid mechanics problem with the heat transfer problem discussed earlier).

Figure 1.2.3 shows an element of length Δx with axial forces acting at both ends of the element, where σ_x denotes stress (i.e., force per unit area; N/m^2) in the x direction, which is taken positive downward, and $g(x)$ denotes the body force measured per unit volume (N/m^3). Hence, $[A\sigma_x]_x$ is the net tensile force on the volume element at x and $[A\sigma]_{x+\Delta x}$ is the net

Figure 1.2.3 Axial deformation of a bar.

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tensile force at $x + \Delta x$. Then setting the sum of the forces to zero (i.e., applying Newton's second law in the x-direction) yields

$$-[A\sigma_x]_x + [A\sigma_x]_{x+\Delta x} + gA\Delta x = 0$$

where $g(x)$ is the body force (i.e., weight of the body). Dividing throughout by Δx and taking the limit $\Delta x \to 0$, we obtain

$$\frac{d}{dx}(A\sigma_x) + Ag(x) = 0 \tag{1.2.23}$$

which represents the equilibrium of forces in the x-direction.

The stress σ_x can be related to the axial displacement using *Hooke's law*

$$\sigma_x = E\varepsilon_x, \quad \varepsilon_x = \frac{du}{dx} \tag{1.2.24}$$

where E is Young's modulus (N/m^2), u denotes the axial displacement (m), and ε_x is the axial strain (m/m). Again, we see that Eq. (1.2.24) is a constitutive equation. Note that a system can have several constitutive relations, each depending on the phenomenon being studied. The study of heat transfer in a bar (with nonuniform temperature) required us to employ Fourier's law to relate temperature gradient to heat flow, and the study of deformation of the same bar (subjected to axial forces) requires us to use Hooke's law to connect stress to displacement gradient.

Now using Eq. (1.2.24) in Eq. (1.2.23), we arrive at the equilibrium equation in terms of the displacement

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + gA = 0, \quad 0 < x < L \tag{1.2.25}$$

This second-order equation can be solved using known boundary conditions at $x = 0$ and $x = L$. The boundary conditions of a bar involve specifying either the displacement u or the force $A\sigma_x$ at a boundary point.

The known boundary conditions of the problem at hand are (see Fig. 1.2.3)

$$u(L) = 0, \quad \left[EA\frac{du}{dx}\right]_{x=0} = -P \tag{1.2.26}$$

where P is the load carried by the pier (i.e., weight of the bridge and the load it carries divided by the number of piers supporting the bridge). The second boundary condition in (1.2.26) represents the force equilibrium at $x = L$. Equations (1.2.25) and (1.2.26) may not admit an analytical solution when $a = a(x)$, requiring us to seek approximate solutions using numerical methods.

The analytical solution of Eq. (1.2.25) subject to the boundary conditions in (1.2.26) may be found for the simple case in which $a \equiv EA$ and $f \equiv Ag$ are constant (homogeneous and uniform cross-sectional, pier). The general solution of (1.2.25) in this case is given by

$$u(x) = \frac{1}{a}\left(-\frac{f}{2}x^2 + C_1 x + C_2\right) \tag{1.2.27}$$

Use of the boundary conditions (1.2.26) give $C_1 = -P$ and $C_2 = fL^2/2 - PL$, and the solution becomes

$$u(x) = \frac{1}{E}\left[\frac{g}{2}(L^2 - x^2) + \frac{P}{A}(L - x)\right] \tag{1.2.28}$$

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1.3 NUMERICAL SIMULATIONS

While the derivation of the governing equations for most problems is not unduly difficult, their solution by exact methods of analysis is often difficult due to geometric and material complexities. In such cases, numerical methods of analysis provide alternative means of finding solutions. By a *numerical simulation* of a process, we mean the solution of the governing equations (or mathematical model) of the process using a numerical method and a computer. Numerical methods typically transform differential equations governing a continuum to a set of algebraic equations of a discrete model of the continuum that are to be solved using computers.

There exist a number of numerical methods, many of which are developed to solve differential equations. In the finite difference approximation of a differential equation, the derivatives in the latter are replaced by difference quotients (or the function is expanded in a Taylor series) that involve the values of the solution at discrete mesh points of the domain. The resulting algebraic equations are solved for the values of the solution at the mesh points after imposing the boundary conditions. These ideas are illustrated with the help of two examples, one for an initial-value problem and another for a boundary-value problem.

In the solution of a differential equation by a classical variational method, the equation is put into an equivalent weighted-integral form and then the approximate solution over the domain is assumed to be a linear combination ($\sum_j c_j \phi_j$) of appropriately chosen approximation functions ϕ_j and undetermined coefficients c_j. The coefficients c_j are determined such that the integral statement equivalent to the original differential equation is satisfied. Various variational methods, e.g., the Ritz, Galerkin, collocation, and least-squares methods, differ from each other in the choice of the integral form, weight functions, and/or approximation functions. A more detailed discussion of variational methods will be given in Chapter 2. The classical variational methods, which are truly meshless methods, are powerful methods that provide globally continuous solutions but suffer from the disadvantage that the approximation functions for problems with arbitrary domains are difficult to construct (the modern meshless methods seem to provide a way to construct approximation functions for arbitrary domains).

Here we consider two examples of numerical simulations using the finite difference method to give the reader a taste of numerical methods and, in the process, introduce the finite difference method. The examples discussed here are based on the problems introduced in Examples 1.2.1 and 1.2.2. The bar problem of Example 1.2.3 is mathematically the same as the heat transfer problem in Example 1.2.2; therefore, its numerical solution is not discussed here.

Example 1.3.1 (Pendulum Problem)

Here we consider the numerical solution of Eq. (1.2.4) governing a simple pendulum. To introduce the finite difference method, we consider the first-order differential equation

$$\frac{du}{dt} = f(t, u) \tag{1.3.1}$$

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where f is a known function of the unknown. Equation (1.3.1) must be solved for $t > 0$ subject to the (initial) condition $u(0) = u_0$. We approximate the derivative at t_i by

$$\left(\frac{du}{dt}\right)\bigg|_{t=t_i} \approx \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \tag{1.3.2}$$

Note that we replaced the derivative at $t = t_i$ by its definition except that we did not take the limit $\Delta t \equiv t_{i+1} - t_i \to 0$; that is why it is an approximation. For increasingly small values of Δt, we expect the approximation to have decreasingly small error. Substituting (1.3.2) into (1.3.1) at $t = t_i$, we obtain

$$u_{i+1} = u_i + \Delta t\ f(u_i, t_i), \quad u_i = u(t_i), \quad \Delta t = t_{i+1} - t_i \tag{1.3.3}$$

Equation (1.3.3) can be solved, starting from the known value u_0 of $u(t)$ at $t = 0$, for $u_1 = u(t_1) = u(\Delta t)$. This process can be repeated to determine the values of u at times $t = \Delta t, 2\Delta t, \ldots, n\Delta t$. This is known as *Euler's explicit scheme* (or first-order Runga–Kutta method), also known as the *forward difference scheme*. Of course there are higher-order finite difference schemes that are more accurate than Euler's scheme, and we will not discuss them as they are outside the scope of this study. Note that we are able to convert the ordinary differential equation (1.3.1) to an algebraic equation (1.3.3) that needs to be evaluated at different times to construct the time history of $u(t)$.

We now apply Euler's explicit scheme to the second-order equation (1.2.4) subjected to the initial conditions (1.2.5). To apply the procedure described above to the equation at hand, we rewrite Eq. (1.2.4) as a pair of coupled (i.e., one cannot be solved without the other) first-order equations

$$\frac{d\theta}{dt} = v, \quad \frac{dv}{dt} = -\lambda^2\theta \tag{1.3.4}$$

Figure 1.3.1 Analytical and numerical solutions $\theta(t)$ of the simple pendulum.

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where $\lambda^2 = g/\ell$. Applying the scheme of Eq. (1.3.3) to the equations at hand, we obtain

$$\theta_{i+1} = \theta_i + \Delta t\, v_i; \quad v_{i+1} = v_i - \Delta t\, \lambda^2 \theta_i \tag{1.3.5}$$

The expressions for θ_{i+1} and v_{i+1} in Eq. (1.3.5) are repeatedly computed using the known solution (θ_i, v_i) from the previous time step. At time $t = 0$, we use the known initial values (θ_0, v_0). Thus, one needs a computer and a computer language like Fortran or C++ to implement the logic of repeatedly computing θ_{i+1} and v_{i+1} with the help of Eq. (1.3.5).

The numerical solutions of equation (1.3.5) for three different time steps, $\Delta t = 0.05, 0.025$, and 0.001, along with the exact linear solution (1.2.9) (with $\ell = 2.0$, $g = 32.2$, $\theta_0 = \pi/4$, and $v_0 = 0$) are presented in Fig. 1.3.1. The accuracy of the numerical solutions is dependent on the size of the time step; the solution is more accurate with smaller time steps. For large time steps, the solution even diverges from the true solution.

Example 1.3.2 (Heat Transfer Problem)

Here we consider the boundary-value problem of Example 1.2.2. The finite difference approximation of a boundary-value problem differs from that of an initial value problem. First we divide the domain $(0, L)$ into a finite set of N intervals of equal length Δx, as shown in Fig. 1.3.2. The ends of each interval is called a mesh point. Thus, there are $N + 1$ mesh points in the domain. Then we approximate the second-derivative directly using the centered difference scheme where the error is of order $O(\Delta x)^2$

$$\left(\frac{d^2\theta}{dx^2}\right)_{x=x_i} \approx \left(\frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{(\Delta x)^2}\right) \tag{1.3.6}$$

Using the above approximation in Eq. (1.2.19), we obtain

$$-(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + (m\Delta x)^2\theta_i = 0 \quad \text{or} \quad -\theta_{i-1} + [2 + (m\Delta x)^2]\theta_i - \theta_{i+1} = 0 \tag{1.3.7}$$

Equation (1.3.7) is valid for any mesh point $x = x_i$, $i = 1, 2, \ldots, N$, and it contains values of θ both at $x = x_{i-1}$ and $x = x_{i+1}$. Note that Eq. (1.3.7) is not used at mesh point $x = x_0 = 0$ because the temperature is known there [see Eq. (1.2.20)]. However, use of Eq. (1.3.7) at mesh point $x = x_N = L$ requires the knowledge of the fictitious value θ_{N+1} (as we see in the sequel,

Figure 1.3.2 Heat transfer in a bar and a typical finite difference mesh.

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we never have to deal with such fictitious values in the finite element method). The forward finite difference approximation of the second boundary condition in (1.2.20) at mesh point $x = L$ gives

$$\frac{\theta_{N+1} - \theta_N}{\Delta x} + \frac{\beta}{k}\theta_N = 0$$

from which we have

$$\theta_{N+1} = \left(1 - \frac{\beta \Delta x}{k}\right)\theta_N \tag{1.3.8}$$

Application of the formula (1.3.7) to mesh points at $x_1, x_2, \ldots, x_N$ yields

$$\begin{aligned} -\theta_0 + D\theta_1 - \theta_2 &= 0 \\ -\theta_1 + D\theta_2 - \theta_3 &= 0 \\ -\theta_2 + D\theta_3 - \theta_4 &= 0 \\ \cdots \quad \cdots \quad \cdots \\ -\theta_{N-1} + D\theta_N - \theta_{N+1} &= 0 \end{aligned} \tag{1.3.9}$$

where $D = [2 + (m\Delta x)^2]$; Eq. (1.3.9) consists of N equations in N unknowns, $\theta_1, \theta_2, \ldots, \theta_N$.

As a specific example, consider a steel rod of diameter $d = 0.02$ m, length $L = 0.05$ m, and thermal conductivity $k = 50$ W/(m·°C). Suppose that the temperature at the left end is $T_0 = 320$°C, ambient temperature is $T_\infty = 20$°C and film conductance (or heat transfer coefficient) $\beta = 100$ W/(m²·°C). For this data, we have

$$\frac{\beta}{k} = 2, \quad m^2 = \frac{\beta P}{kA} = \frac{\beta(\pi d)}{k(\pi d^2/4)} = 400, \quad \theta_0 = \theta(0) = 300°\text{C}$$

For a subdivision of four intervals ($N = 4$), we have $\Delta x = 0.0125$ m and $D = 2 + (20 \times 0.0125)^2 = 2.0625$. For this case, there are four equations in four unknowns:

$$\begin{array}{rrrrcr} 2.0625\theta_1 & -\theta_2 & & & = & 300 \\ -\theta_1 & +2.0625\theta_2 & -\theta_3 & & = & 0 \\ & -\theta_2 & +2.0625\theta_3 & -\theta_4 & = & 0 \\ & & -\theta_3 & +1.0875\theta_4 & = & 0 \end{array} \tag{1.3.10}$$

The above *tridiagonal* system of algebraic equations can be solved using the Gauss elimination method (this is where we need a computer!). The solution is given by

$$\{\theta\} = \{245.81,\ 206.98,\ 181.10,\ 166.52\}^{\mathrm{T}}$$

The analytical solution at the same points is

$$\{\theta\} = \{248.75,\ 213.13,\ 190.90,\ 180.66\}^{\mathrm{T}}$$

The maximum error is about 7.8 percent. When the number of mesh points is doubled, the maximum error goes down to 4.2 percent, and it is 1 percent when the number of mesh points is increased to 32.

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1.4 THE FINITE ELEMENT METHOD

1.4.1 The Basic Idea

The finite element method is a numerical method like the finite difference method but is more general and powerful in its application to real-world problems that involve complicated physics, geometry, and/or boundary conditions. In the finite element method, a given domain is viewed as a collection of subdomains, and over each subdomain the governing equation is approximated by any of the traditional variational methods. The main reason behind seeking approximate solution on a collection of subdomains is the fact that it is easier to represent a complicated function as a collection of simple polynomials, as can be seen from Fig. 1.4.1. Of course, each individual segment of the solution should fit with its neighbors in the sense that the function and possibly derivatives up to a chosen order are continuous at the connecting points. These ideas will be more clear in the sequel.

1.4.2 The Basic Features

The method is endowed with three distinct features that account for its superiority over other competing methods. First, a geometrically complex domain Ω of the problem, such as the one in Fig. 1.4.2(a), is represented as a collection of geometrically simple subdomains, called *finite elements* [see Fig. 1.4.2(b)]. Each finite element Ω_e [Fig. 1.4.2(c)] is viewed as an independent domain by itself. Here the word "domain" refers to the geometric region over which the equations are solved. Second, over each finite element, algebraic equations among the quantities of interest are developed using the governing equations of the problem. Third, the relationships from all elements are assembled (i.e., elements are put back into their original positions of the total domain) as indicated in Fig. 1.4.2(d), using certain interelement relationships. These statements may not make complete sense to the reader until some specific examples to illustrate the features are discussed in the coming pages.

Approximations enter engineering analyses at several stages. The division of the whole domain into finite elements may not be exact (i.e., the assemblage of elements, Ω_h, does not match the original domain Ω), introducing error in the domain being modeled. The second stage is when element equations are derived. Typically, the dependent unknowns (u) of the problem are approximated using the basic idea that any continuous function can be

Figure 1.4.1 Piecewise approximation of a function.

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Figure 1.4.2 Representation of a two-dimensional domain by a collection of triangles and quadrilaterals.

represented by a linear combination of known functions ϕ_i and undetermined coefficients c_i ($u \approx u_h = \sum c_i \phi_i$). Algebraic relations among the undetermined coefficients c_j are obtained by satisfying the governing equations, in a weighted-integral sense, over each element. The approximation functions ϕ_i are often taken to be polynomials, and they are derived using concepts from interpolation theory. Therefore, they are termed *interpolation functions*. Thus, approximation errors in the second stage are introduced both in representing the solution u as well as in evaluating the integrals. Finally, errors are introduced in solving the assembled system of equations. Obviously, some of the errors discussed above can be zero. When all the errors are zero, we obtain the exact solution of the problem (which is not the case in most problems).

Next, the basic ideas and some terminology of the finite element method are introduced via several simple examples. The reader should try to understand the basic features rather than to question the use of an approximation method to solve such a simple problem (which can possibly be solved exactly without using the finite element method).

Example 1.4.1 (Approximation of the Perimeter of a Circle)

This discussion is an expansion of an article written by the author in 1978 for a student magazine at the University of Oklahoma (Reddy, 1978). Consider the problem of determining the perimeter (quantity of interest) of a circle of radius R [see Fig. 1.4.3(a)]. We *pretend*

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Figure 1.4.3 Determination of the perimeter of a circle.

that we do not know the formula ($P = 2\pi R$) for the perimeter (circumference) of a circle. Ancient mathematicians estimated the value of the perimeter of a circle by approximating it by (straight) line segments, whose lengths they were able to measure. The approximate value of the circumference is obtained by summing the lengths of the line segments used to represent it.

The three basic features of the finite element method in the present case take the following form. First, the division of the perimeter of a circle into a collection of line segments. In theory, we need an infinite number of such line elements to represent the perimeter; otherwise, the value computed will have some error. Second, writing an equation for the quantity of interest (perimeter) over an element (line segment) in this case is exact, as we shall see. Third, assembly of elements amounts to simply adding the element lengths to obtain the total value.

Although this is a trivial example, it illustrates several (but not all) ideas and steps involved in the finite element analysis of a problem. We outline the steps involved in computing an approximate value of the perimeter of the circle. In doing so, we introduce certain terms that are used in the finite element analysis of any problem.

1. *Finite element discretization.* First, the domain (i.e., the perimeter of the circle) is represented as a collection of a finite number n of subdomains, namely, line segments [Fig. 1.4.3(b)]. This is called *discretization of the domain*. Each subdomain (i.e., line segment) is called an *element*. The collection of elements is called the *finite element mesh*. The elements are connected to each other at points called *nodes*. In the present case, we discretize the perimeter into a mesh of five ($n = 5$) line segments [Fig. 1.4.3(c)]. The line segments can be of different lengths. When all the elements are of the same length, the mesh is said to be *uniform*; otherwise, it is called a *nonuniform* mesh.
2. *Element equations.* A typical element (i.e., line segment, Ω_e) is isolated and its required properties, i.e., length, are computed by some appropriate means. Let h_e be the length of element

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Ω_e in the mesh. For a typical element Ω_e, h_e is given by [see Fig. 1.4.3(d)]

$$h_e = 2R \sin \frac{1}{2}\theta_e \tag{1.4.1}$$

where R is the radius of the circle and $\theta_e < \pi$ is the angle subtended by the line segment. The above equations are called *element equations*. Ancient mathematicians most likely made measurements, rather than using Eq. (1.4.1), to find h_e (they did not know π).

3. *Assembly of element equations and solution.* The approximate value of the perimeter of the circle is obtained by putting together the element properties in a meaningful way; this process is called the *assembly* of the element equations. It is based, in the present case, on the simple idea that the total perimeter of the polygon Ω_h (assembly of elements) is equal to the sum of the lengths of individual elements:

$$P_n = \sum_{e=1}^{n} h_e \tag{1.4.2}$$

Then P_n represents an approximation to the actual perimeter, P. If the mesh is uniform, or h_e is the same for each of the elements in the mesh, then $\theta_e = 2\pi/n$, and we have

$$P_n = n\left(2R \sin \frac{\pi}{n}\right) \tag{1.4.3}$$

4. *Convergence and error estimate.* For this simple problem, we know the exact solution: $P = 2\pi R$. We can estimate the error in the approximation and show that the approximate solution P_n converges to the exact value P in the limit as $n \to \infty$. Consider a typical element Ω_e. The error in the approximation is equal to the difference between the length of the sector and that of the line segment [see Fig. 1.4.3(d)]

$$E_e = |S_e - h_e| \tag{1.4.4}$$

where $S_e = R\theta_e$ is the length of the sector. Thus, the error estimate for an element in the mesh is given by

$$E_e = R\left(\frac{2\pi}{n} - 2 \sin \frac{\pi}{n}\right) \tag{1.4.5}$$

The total error (called *global* error) is given by multiplying E_e by n:

$$E = nE_e = 2R\left(\pi - n \sin \frac{\pi}{n}\right) = 2\pi R - P_n = P - P_n \tag{1.4.6}$$

We now show that E goes to zero as $n \to \infty$. Letting $x = 1/n$, we have

$$P_n = 2Rn \sin \frac{\pi}{n} = 2R \frac{\sin \pi x}{x}$$

and

$$\lim_{n\to\infty} P_n = \lim_{x\to 0}\left(2R \frac{\sin \pi x}{x}\right) = \lim_{x\to 0}\left(2\pi R \frac{\cos \pi x}{1}\right) = 2\pi R \tag{1.4.7}$$

Hence, E_n goes to zero as $n \to \infty$. This completes the proof of convergence.

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Example 1.4.2 (Geometric Centroid of an Irregular Body)

It should be recalled, from a first course on statics of rigid bodies, that the calculation of the mass center or the geometric centroid (quantity of interest) of an irregular volume makes use of the so-called method of composite bodies, in which a body is conveniently divided (mesh discretization) into several parts (elements) of simple shape for which the mass and the center of mass (element properties) can be computed readily. The center of mass of the whole body is then obtained using the moment *principle of Varignon* (a basis for the assembly of element equations):

$$(m_1 + m_2 + \cdots + m_n)\bar{X} = m_1\bar{x}_1 + m_2\bar{x}_2 + \cdots + m_n\bar{x}_n \tag{1.4.8}$$

where $\bar{X}$ is the x-coordinate of the center of mass of the whole body, m_e is the mass of the eth part, and $\bar{x}_e$ is the x-coordinate of the center of mass of the eth part. Similar expressions hold for the y and z coordinates of the center of mass of the whole body. Analogous relations hold for composite lines, areas, and volumes, wherein the masses are replaced by lengths, areas, and volumes, respectively.

When a given body is not expressible in terms of simple geometric shapes (elements) for which the mass and the center of the mass can be represented mathematically, it is necessary to use a method of approximation to represent the "properties" of an element. As an example, consider the problem of finding the centroid $(\bar{X}, \bar{Y})$ of the irregular region shown in Fig. 1.4.4(a). The region can be divided into a finite number of trapezoidal strips (elements), a typical element

Figure 1.4.4 Determination of the centroid of an irregular region.

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having width h_e and heights b_e and b_{e+1} [see Fig. 1.4.4(*b*)]. The area of the eth strip is given by

$$A_e = \frac{1}{2} h_e (b_e + b_{e+1}) \tag{1.4.9}$$

The area A_e is an approximation of the true area of the element because $(b_e + b_{e+1})/2$ is an estimated average height of the element. The coordinates of the centroid of the region are obtained by applying the moment principle:

$$\bar{X} = \frac{\sum_e A_e \bar{x}_e}{\sum_e A_e}, \quad \bar{Y} = \frac{\sum_e A_e \bar{y}_e}{\sum_e A_e} \tag{1.4.10}$$

where $\bar{x}_e$ and $\bar{y}_e$ are the coordinates of the centroid of the eth element with respect to the coordinate system used for the whole body. It should be noted that the accuracy of the approximation will be improved by increasing the number of strips (i.e., decreasing their width, h_e).

When the center of mass is required, A_e in the above equations is replaced by the mass $m_e = \rho_e A_e$, ρ_e being the mass density of the eth element. For a homogeneous body, ρ_e is the same for all elements, and therefore Eq. (1.4.10) also gives the coordinates of the center of mass of a homogeneous body.

The two examples considered above illustrates how the idea of piecewise approximation is used to approximate irregular geometry and calculate required quantities. Thus, subdividing a geometrically complex domain into parts that allow the evaluation of desired quantities is a very natural and practical approach. The idea can be extended to approximate functions representing physical quantities. For example, the temperature variation in a two-dimensional domain can be viewed as a curved surface, and it can be approximated over any part of the domain, i.e., over a subdomain, by a polynomial of desired degree. A curved surface over a triangular subregion may be approximated by a planar surface,

Figure 1.4.5 Approximation of a curved surface by a planar surface.

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as shown in Fig. 1.4.5. Such ideas form the basis of finite element approximations. The next example illustrates this idea for a one-dimensional continuous system described by a differential equation.

Example 1.4.3 (Solution of a Differential Equation)

Consider the temperature variation in a composite cylinder consisting of two coaxial layers in perfect thermal contact, as shown in Fig. 1.4.6(*a*). Heat dissipation from a wire (with two insulations) carrying an electric current and heat flow across a thick-walled composite circular cylindrical tube are typical examples. Because of the axisymmetric geometry, boundary conditions, material and loading, the temperature T is independent of the angular coordinate θ, i.e., every radial line has the same temperature T at any radial distance r, $T = T(r)$. Therefore, it is sufficient to consider an arbitrary radial line and determine temperature along it. We wish to determine an approximation $T_e(r)$ to $T(r)$ over the radius of the cylinder. The equation governing the axisymmetric heat transfer can be formulated using the principle of conservation of energy (as was done in Example 1.2.2):

$$-\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = g(r) \tag{1.4.11}$$

Figure 1.4.6 (*a*) Coaxial cylinder made of two different materials. (*b*) Typical cross section of the composite cylinder. (*c*) Finite element discretization by line segments (linear elements).

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Equation (1.4.11) is subject to appropriate boundary conditions, for example, insulated at $r = R_i$ and subjected to a temperature T_0 at $r = R_0$:

$$kr\frac{dT}{dr} = 0 \quad \text{at} \quad r = R_i; \quad T(r) = T_0 \quad \text{at} \quad r = R_0 \tag{1.4.12}$$

Here k denotes the thermal conductivity, which varies from layer to layer, R_i and R_0 are the inner and outer radii of the cylinder [Fig. 1.4.6(b)], and g is the rate of energy generation in the medium. When it is difficult to obtain an exact solution of Eqs. (1.4.11) and (1.4.12), either because of complex geometry and material properties or because $g(r)$ is a complicated function that does not allow exact evaluation of its integral, we seek an approximate one. In the finite element method, the domain (R_i, R_0) is divided into N subintervals [Fig. 1.4.6(c)], and the approximate solution is sought in the form

$$\begin{aligned}
T_1(r) &= \sum_{j=1}^{n} T_j^{(1)}\psi_j^{(1)}(r), \quad R_i \le r \le R_i + h_1 \text{ (first interval)} \\
T_2(r) &= \sum_{j=1}^{n} T_j^{(2)}\psi_j^{(2)}(r), \quad R_i + h_1 \le r \le R_i + h_1 + h_2 \text{ (second interval)} \\
&\vdots \\
T_N(r) &= \sum_{j=1}^{n} T_j^{(N)}\psi_j^{(N)}(r), \quad R_i + h_1 + \cdots + h_{N-1} \le r \le R_0 \text{ (}N\text{th interval)}
\end{aligned} \tag{1.4.13}$$

where h_e denotes the length of the eth interval, T_j (the element label "e" is omitted in the interest of brevity) is the value of the temperature $T_e(r)$ at the jth geometric point (called the node) of the eth interval, ψ_j are the interpolation polynomials on the eth interval (finite element), and T_j are termed the nodes values (which are unknown). Thus, the continuous function $T(r)$ is approximated in each finite element by a desired degree of interpolation polynomials, and the polynomial is expressed in terms of the values of the function at a selected number of points in the interval. The number of points is equal to the number of parameters in the polynomial. For example, a linear polynomial approximation of the temperature over the interval requires two values, and hence two points, are identified in the interval. The endpoints of the interval are selected for this purpose because the two points, called nodes, also define the length of the interval (see Fig. 1.4.7). For higher-order polynomial approximations, additional nodes are

Figure 1.4.7 (a) Linear approximation $T_e(r)$ of a function $T(r)$ over an element. (b) Quadratic approximation $T_e(r)$ of $T(r)$.

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identified inside the element domain. The importance of these geometric points (i.e., nodes) is that they are the base points required to define the interpolation polynomials. The nodal values T_j are determined such that $T_e(r)$ satisfies the differential equation (1.4.11) and boundary conditions (1.4.12) in some sense, often, in integral sense over each element:

$$0 = \int_{r_a}^{r_b} w(r) \left[-\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr} \right) - g(r) \right] r\,dr \tag{1.4.14}$$

for all independent choices of the weight function, w. Often we select $w = \psi_i$ $(i = 1, 2, \cdots)$ to obtain n relations among the n unknowns. These ideas will be discussed in more detail in Chapter 3.

The piecewise (i.e., elementwise) approximation of the solution allows us to include any discontinuous data, such as the material properties, and to use meshes of many lower-order elements or a mesh of few higher-order elements to represent large gradients of the solution. Polynomial approximations of the form (1.4.13) can be derived systematically for any assumed degree of variation. The satisfaction of the differential equation in a weighted-integral sense leads, for steady-state problems, to algebraic relations among nodal temperatures T_j and heats Q_j of the element. The algebraic equations of all elements are assembled (i.e., related to each other) such that the temperature is continuous and the heats are balanced at nodes common to elements. Then the assembled equations are solved for the nodal values of temperatures and heats after imposing the boundary conditions of the problem. More details of this example will be presented in Chapter 3.

1.4.3 Some Remarks

In summary, in the finite element method a given domain is divided into subdomains, called finite elements, and an approximate solution to the problem is developed over each element. The subdivision of a whole domain into parts has two advantages:

1. Allows accurate representation of complex geometry and inclusion of dissimilar material properties.
2. Enables easy representation of the total solution by functions defined within each element that capture local effects (e.g., large gradients of the solution).

The three fundamental steps of the finite element method that are illustrated via the examples are:

1. Divide the whole domain into parts (both to represent the geometry and solution of the problem).
2. Over each part, seek an approximation to the solution as a linear combination of nodal values and approximation functions, and derive the algebraic relations among the nodal values of the solution over each part.
3. Assemble the parts and obtain the solution to the whole.

Although the above examples illustrate the basic idea of the finite element method, there are several other features that are either not present or not apparent from the examples. Some remarks are in order.

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1. We can discretize the geometry of the domain, depending on its shape, into a mesh of more than one type of element (by shape or order). For example, in the approximation of an irregular domain, we can use a combination of rectangles and triangles. However, the element interfaces must be compatible in the sense that the solution is continuous.
2. If more than one type of element is used in the representation of the domain, one of each kind should be isolated and its equations developed.
3. The governing equations of problems considered in this book are differential equations. In most cases, the equations cannot be solved over an element for two reasons. First, they do not permit the exact solution. It is here that the variational methods come into play. Second, the discrete equations obtained in the variational methods cannot be solved independent of the remaining elements because the assemblage of the elements is subjected to certain continuity, boundary, and/or initial conditions.
4. There are two main differences in the form of the approximate solution used in the finite element method and in the classical variational methods (i.e., variational methods applied to the whole domain). First, instead of representing the solution u as a linear combination ($u_h = \sum_j c_j \phi_j$) in terms of arbitrary parameters c_j as in the variational methods, in the finite element method the solution is often represented as a linear combination ($u_h = \sum_j c_j \phi_j$) in terms of the values u_j of u_h (and possibly its derivatives as well) at the nodal points. Second, the approximate functions in the finite element method are often polynomials that are derived using interpolation theory. However, the finite element method is not restricted to the use of approximations that are linear combinations of nodal values u_j and interpolation functions ψ_j that are algebraic polynomials. One may use nodeless variables and nonpolynomial functions to approximate a function (like in meshless or element-free methods).
5. The number and the location of the nodes in an element depend on (a) the geometry of the element, (b) the degree of the polynomial approximation, and (c) the weighted-integral form of the equations. By representing the required solution in terms of its values at the nodes, we directly obtain the approximate solution at the nodes.
6. The assembly of elements, in a general case, is based on the idea that the solution (and possibly its derivatives for higher-order equations) is continuous at the interelement boundaries.
7. In general, the assemblage of finite elements is subjected to boundary and/or initial conditions. The discrete equations associated with the finite element mesh are solved only after the boundary and/or initial conditions have been imposed.
8. There are three sources of error in a finite element solution: (a) those due to the approximation of the domain (this was the only error present in the first two examples); (b) those due to the approximation of the solution; and (c) those due to numerical computation (e.g., numerical integration and round-off errors in a computer). The estimation of these errors, in general, is not simple. However, under certain conditions, they can be estimated for an element and problem.
9. The accuracy and convergence of the finite element solution depends on the differential equation, its integral form, and the element used. "Accuracy" refers to the difference between the exact solution and the finite element solution, while "convergence" refers to the accuracy as the number of elements in the mesh is increased.

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Figure 1.4.8 Finite element discretization of a connecting rod and piston.

10. For time-dependent problems, a two-stage formulation is usually followed. In the first stage, the differential equations are approximated by the finite element method to obtain a set of ordinary differential equations in time. In the second, the differential equations in time are solved exactly or further approximated by either variational methods or finite difference methods to obtain algebraic equations, which are then solved for the nodal values (see Chapter 6).
11. The desktop computers of today are more powerful than the supercomputers that existed when the finite element method was first implemented. Consequently, the analysis time is considerably reduced, provided the mesh used to model the problem is adequate (a finite element mesh of a connecting rod and piston is shown in Fig. 1.4.8). Even the automatic mesh generation programs cannot guarantee meshes that are free of irregularly shaped elements or have sufficient number of elements in a region containing high gradients in the solution, both of which result in loss of accuracy or, in the case of nonlinear problems, nonconvergence of solutions.
12. Element-free methods that require no assembly (because there are no elements) are being developed. Such methods have applications in fracture mechanics and wave propagation problems where re-meshing is necessary.

1.4.4 A Brief Review of History and Recent Developments

The idea of representing a given domain as a collection of discrete parts is not unique to the finite element method. It was recorded that ancient mathematicians estimated the value of π by noting that the perimeter of a polygon inscribed in a circle approximates the circumference of the latter. They predicted the value of π to accuracy of almost 40 significant digits by representing the circle as a polygon of a finitely large number of sides (Reddy, 1978, 1993). In modern times, the idea found a home in aircraft structural analysis, where, for example, wings and fuselages are treated as assemblages of stringers, skins, and shear panels. In 1941, Hrenikoff [1941] introduced the so-called framework method in which a plane elastic medium was represented as a collection of bars and beams. The use of piecewise-continuous functions defined over a subdomain to approximate an unknown function can be found in the work of Courant (1943), who used an assemblage of triangular elements and the principle of minimum total potential energy to study the St. Venant torsion problem. Although certain key features of the finite element method can be found in

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the works of Hrenikoff (1941) and Courant (1943), its formal presentation is attributed to Turner, Clough, Martin, and Topp (1956) and Argyris and Kelsey (1960). The term "finite element" was first used by Clough (1960). Since its inception, the literature on finite element applications has grown exponentially, and today there are numerous journals that are primarily devoted to the theory and application of the method. Additional information on the history of the finite element method can be found in the review article by Babuska (1994) and in the early books on the finite element method [e.g., Zienkiewicz and Cheung (1967), Oden (1972), Strang and Fix (1973), and Oden and Reddy (1976)].

In recent years, extensions and modifications of the finite element method have been proposed. These include the *partition of unity method* (PUM) of Melenk and Babuska (1996), the *h-p Cloud method* of Duarte and Oden (1996) and *meshless methods* advanced by Belytschko and his colleagues (1996). All of these methods and numerous other methods not named here are very closely related.

1.5 THE PRESENT STUDY

This is a book on the finite element method and its application to linear problems in engineering and applied sciences. Most introductory finite element textbooks written for use in engineering schools are intended for students of solid and structural mechanics, and these introduce the method as an offspring of matrix methods of structural analysis. A few texts that treat the method as a variationally based technique leave the variational formulations and the associated methods of approximation to an appendix. The approach taken in this book is one in which the finite element method is introduced as a numerical technique of solving classes of problems, each class having a common mathematical structure in the form of governing differential equations. This approach makes the student understand the generality of the finite element method, irrespective of the student's subject background. It also enables the student to see the mathematical structure common to various physical problems, and thereby to gain additional insight into various engineering problems.

1.6 SUMMARY

Scientists and engineers develop conceptual and mathematical models of phenomena and systems that they wish to understand. The understanding may be used to develop and improve systems that contribute to human convenience and comfort. Mathematical models are developed using axioms or laws of nature that govern the phenomena. Mathematical models consist of algebraic, differential, and/or integral equations, which are often difficult to solve for the desired variables of the system for a variety of input parameters (called data), necessitating the use of numerical simulations of the phenomena.

In a numerical simulation of physical processes, we employ a numerical method and computer to evaluate the mathematical model of the process. The finite element method is a powerful numerical method of solving algebraic, differential, and integral equations, and it is devised to study complex physical processes. The method is characterized by three features:

1. The domain of the problem is represented by a collection of simple subdomains, called finite elements. The collection of finite elements is called the finite element mesh.

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2. Over each finite element, the physical process is approximated by functions of the desired type (polynomials or otherwise), and algebraic equations relating physical quantities at selective points, called nodes, of the element are developed.
3. The element equations are assembled using continuity and/or "balance" of physical quantities.

In the finite element method, we seek an approximation u_h of u in the form

$$u \approx u_h = \sum_{j=1}^{n} u_j \psi_j + \sum_{j=1}^{m} c_j \phi_j$$

where u_j are the values of u_h at the element nodes, ψ_j are the interpolation functions, c_j are coefficients that are *not* associated with nodes, and ϕ_j are the associated approximation functions. Direct substitution of such approximations into the governing differential equations does not always result, for an arbitrary choice of the data of the problem, in a necessary and sufficient number of equations for the undetermined coefficients u_j and c_j. Therefore, a procedure whereby a necessary and sufficient number of equations can be obtained is needed. One such procedure is provided by a weighted-integral form of the governing differential equation. Chapter 2 is devoted to the study of weighted-integral formulations of differential equations and their solution by variational methods of approximation.

There is only one method of finite elements that is characterized by the three features stated above. Of course, there can be more than one *finite element model* of the same problem (i.e., governing equations). The type of model depends on the differential equations, methods used to derive the algebraic equations (i.e., the weighted-integral form used) for the undetermined coefficients over an element, and nature of the approximation functions used. Although the Ritz-Galerkin methods and polynomial approximations are used frequently to generate the finite element equations, any appropriate method and approximations can be used, in principle, to generate the algebraic equations. In this spirit, the collocation method, subdomain method, boundary integral methods, etc., can be used to generate the algebraic equations among discrete values of the primary and secondary variables (see Chapter 14).

The basic theory of the finite element method can be found in more than three dozen textbooks. For the beginner, it is not necessary to consult any of the other books on the finite element method as the present book provides complete details of the method as applied to linear field problems, with examples drawn from fluid mechanics, heat transfer, and solid and structural mechanics.

PROBLEMS

1.1 Newton's second law can be expressed as

$$\mathbf{F} = m\mathbf{a} \tag{1}$$

where $\mathbf{F}$ is the net force acting on the body, m mass of the body, and $\mathbf{a}$ is the acceleration of the body in the direction of the net force. Use Eq. (1) to determine the mathematical model, i.e., governing equation, of a free-falling body. Consider only the forces due to gravity and the air resistance. Assume that the air resistance is linearly proportional to the velocity of the falling body.

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1.2 A cylindrical storage tank of diameter D contains a liquid at depth (or head) $h(x, t)$. Liquid is supplied to the tank at a rate of q_i (m^3/day) and drained at a rate of q_0 (m^3/day). Use the principle of conservation of mass to arrive at the governing equation of the flow problem.

1.3 Consider the simple pendulum of Example 1.3.1. Write a computer program to numerically solve the nonlinear equation (1.2.3) using the Euler method. Tabulate the numerical results for two different time steps $\Delta t = 0.05$ and $\Delta t = 0.025$ along with the exact linear solution.

1.4 An improvement of Euler's method is provided by Heun's method, which uses the average of the derivatives at the two ends of the interval to estimate the slope. Applied to the equation

$$\frac{du}{dt} = f(t, u) \tag{1}$$

Heun's scheme has the form

$$u_{i+1} = u_i + \frac{\Delta t}{2}\left[f(t_i, u_i) + f(t_{i+1}, u_{i+1}^0)\right], \; u_{i+1}^0 = u_i + \Delta t \; f(t_i, u_i) \tag{2}$$

Equation (2) is called the *predictor* equation, and Eq. (1) is called the *corrector* equation. Apply Heun's method to Eqs. (1.3.4) and obtain the numerical solution for $\Delta t = 0.05$.

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Chapter 2

MATHEMATICAL PRELIMINARIES, INTEGRAL FORMULATIONS, AND VARIATIONAL METHODS

2.1 GENERAL INTRODUCTION

2.1.1 Variational Principles and Methods

This chapter is devoted to a review of some mathematical preliminaries that prove to be useful in the sequel and to a study of integral formulations and more commonly used variational methods such as the Ritz, Galerkin, collocation, and least-squares methods. Since the finite element method can be viewed as an elementwise application of a variational method (see Section 1.4), it is useful to learn how variational methods work. We begin with a discussion of the general meaning of the phrases "variational methods" and "variational formulations" used in the context of finite element formulations.

The phrase "direct variational methods" refers to methods that make use of variational principles, such as the principles of virtual work and the principle of minimum total potential energy in solid and structural mechanics, to determine approximate solutions of problems [see Oden and Reddy (1983) and Reddy (2002)]. In the classical sense, a *variational principle* has to do with finding the extremum (i.e., minimum or maximum) or stationary values of a functional with respect to the variables of the problem. The functional includes all the intrinsic features of the problem, such as the governing equations, boundary and/or initial conditions, and constraint conditions, if any. In solid and structural mechanics problems, the functional represents the total energy of the system, and in other problems, it is simply an integral representation of the governing equations.

Variational principles have always played an important role in mechanics (see the references at the end of the chapter). First, many problems of mechanics are posed in terms of finding the extremum (i.e., minima or maxima) and thus, by their nature, can be formulated in terms of variational statements. Second, there are problems that can be formulated by other means, such as the conservation laws (as illustrated in Chapter 1), but these can also be formulated by means of variational principles. Third, variational formulations form a powerful basis for obtaining approximate solutions to practical problems, many of which

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are intractable otherwise. The principle of minimum total potential energy, for example, can be regarded as a substitute to the equations of equilibrium of an elastic body as well as a basis for the development of displacement finite element models that can be used to determine approximate displacement and stress fields in the body [Reddy (2002)]. Variational formulations can also serve to unify diverse fields, suggest new theories, and provide a powerful means to study the existence and uniqueness of solutions to problems. Similarly, Hamilton's principle [see Reddy (2002)] can be used in lieu of the equations governing dynamical systems, and the variational forms presented by Biot (1972) replace certain equations in linear continuum thermodynamics.

2.1.2 Variational Formulations

The classical use of the phrase "variational formulations" refers to the construction of a functional (whose meaning will be made clear shortly) or a variational principle that is equivalent to the governing equations of the problem. The modern use of the phrase refers to the formulation in which the governing equations are translated into equivalent weighted-integral statements that are not necessarily equivalent to a variational principle. Even those problems that do not admit variational principles in the classical sense (e.g., the Navier–Stokes equations governing the flow of viscous or inviscid fluids) can now be formulated using weighted-integral statements.

The importance of variational formulations of physical laws, in the modern or general sense of the phrase, goes far beyond its use as simply an alternate to other formulations [Oden and Reddy (1983)]. In fact, variational forms of the laws of continuum physics may be the only natural and rigorously correct way to think of them. While all sufficiently smooth fields lead to meaningful variational forms, the converse is not true: There exist physical phenomena which can be adequately modeled mathematically only in a variational setting; they are nonsensical when viewed locally.

The starting point for the discussion of the finite element method is differential equations governing the physical phenomena under study. As such, we shall first discuss why integral statements of the differential equations are needed.

2.1.3 Need for Weighted-Integral Statements

In almost all approximate methods used to determine the solution of differential and/or integral equations, we seek a solution in the form

$$u(\mathbf{x}) \approx U_N(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}) \tag{2.1.1}$$

where u represents the solution of a particular differential equation and associated boundary conditions, and U_N is its approximation that is represented as a linear combination of unknown parameters c_j and known functions ϕ_j of position $\mathbf{x}$ in the domain Ω on which the problem is posed. We shall shortly discuss the conditions on ϕ_j. The approximate solution U_N is completely known only when c_j are known. Thus, we must find a means to determine c_j such that U_N satisfies the equations governing u. If somehow we can find U_N that satisfies the differential equation at every point $\mathbf{x}$ of the domain Ω

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and conditions on the boundary Γ of Ω, then $U_N(\mathbf{x}) = u(\mathbf{x})$, which is the exact solution of the problem. Of course, approximate methods are not about problems for which exact solutions can be determined by some methods of mathematical analysis; the role of approximate methods is to find an approximate solution of problems that do not admit analytical solutions. When the exact solution cannot be determined, the alternative is to find a solution U_N that satisfies the governing equations in an approximate way. In the process of satisfying the governing equations approximately, we obtain (not accidentally but by planning) N algebraic relations among the N parameters $c_1, c_2, \cdots, c_N$. A detailed discussion of these ideas is given in the next few paragraphs in connection with a specific problem.

Consider the problem of solving the differential equation

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u = f(x) \quad \text{for } 0 < x < L \tag{2.1.2a}$$

subjected to the boundary conditions

$$u(0) = u_0, \qquad \left[a(x)\frac{du}{dx}\right]_{x=1} = Q_0 \tag{2.1.2b}$$

where $a(x)$, $c(x)$, and $f(x)$ are known functions, u_0 and Q_0 are known parameters, and $u(x)$ is the function to be determined. The set $a(x)$, $c(x)$, $f(x)$, u_0, and Q_0 is called the problem *data*. An example of the above problem is given by the heat transfer in an uninsulated rod (see Example 1.2.2): here u denotes the temperature (θ), $f(x)$ is the internal heat generation per unit length (Ag), $a(x)$ is the thermal resistance (kA), $c = \beta P$, u_0 is the specified temperature (θ_0), and Q_0 is the specified heat.

We seek an approximate solution over the entire domain $\Omega = (0, L)$ in the form

$$U_N \equiv \sum_{j=1}^{N} c_j\phi_j(x) + \phi_0(x) \tag{2.1.3}$$

where the c_j are coefficients to be determined and $\phi_j(x)$ and $\phi_0(x)$ are functions chosen such that the specified boundary conditions of the problem are satisfied by the N-parameter approximate solution U_N. Note that the particular form in (2.1.3) has two parts: one containing the unknowns ($\sum c_j\phi_j$) that is termed the homogeneous part and the other is the nonhomogeneous part (ϕ_0) that has the sole purpose of satisfying the specified boundary conditions of the problem. Since ϕ_0 satisfies the boundary conditions, the sum $\sum c_j\phi_j$ must satisfy, for arbitrary c_j, the homogeneous form of the boundary conditions ($Bu = \hat{u}$ is said to be a nonhomogeneous boundary condition when $\hat{u} \neq 0$, and it is termed a homogeneous boundary condition when $\hat{u} = 0$; here B denotes some operator). Thus, in the present case, the actual boundary conditions are both nonhomogeneous ($B = 1$ and $\hat{u} = u_0$ at $x = 0$, and $B = a(x)(d/dx)$ and $\hat{u} = Q_0$ at $x = L$). The particular form (2.1.3) is convenient in selecting ϕ_0 and ϕ_j. Thus, ϕ_0 and ϕ_j satisfy the conditions

$$B\phi_0 = \hat{u}, \qquad B\phi_j = 0 \ \text{ for all } j = 1, 2, \cdots, n \tag{2.1.4}$$

To be more specific, let $L = 1$, $u_0 = 1$, $Q_0 = 0$, $a(x) = x$, $c(x) = 1$, $f(x) = 0$, and $N = 2$. Then we choose the approximate solution in the form

$$U_2 = c_1\phi_1 + c_2\phi_2 + \phi_0 \ \text{ with } \ \phi_0 = 1, \ \phi_1(x) = x^2 - 2x, \ \phi_2 = x^3 - 3x$$

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that satisfies the boundary conditions (2.1.2*b*) of the problem for any values of c_1 and c_2 because

$$\phi_0(0)=1, \quad \left(x\frac{d\phi_0}{dx}\right)_{x=1}=0; \quad \phi_j(0)=0, \quad \left(\frac{d\phi_j}{dx}\right)_{x=1}=0 \text{ for } j=1,2 \tag{2.1.5}$$

To make U_2 satisfy the differential equation (2.1.2*a*), we must have

$$\begin{aligned}-\frac{dU_2}{dx}-x\frac{d^2U_2}{dx^2}+U_2 &= -2c_1(x-1)-3c_2(x^2-1)-2c_1x-6c_2x^2\\ &\quad +c_1(x^2-2x)+c_2(x^3-3x)+1=0\end{aligned} \tag{2.1.6}$$

Since this expression must be zero for any value of x, the coefficients of the various powers of x must be zero:

$$\begin{aligned}1+2c_1+3c_2&=0\\ -(6c_1+3c_2)&=0\\ c_1-9c_2&=0\\ c_2&=0\end{aligned}$$

The above relations are inconsistent; hence, there is *no solution* to the equations. On the other hand, we can require the approximate solution U_N to satisfy the differential equation (2.1.2*a*) in the weighted-integral sense,

$$\int_0^1 w(x)R\,dx=0 \tag{2.1.7}$$

where R denotes the left side of the equality in (2.1.6) and is called the *residual*,

$$R\equiv-\frac{dU_N}{dx}-x\frac{d^2U_N}{dx^2}+U_N$$

and $w(x)$ is called a *weight function*. From (2.1.7), we obtain as many linearly independent equations as there are linearly independent functions for $w(x)$. The number of linearly independent choices of w must be restricted to $N=2$ so that we have exactly the same number of equations as the number of unknown coefficients, c_j. For example, in the present example, if we take $w=1$ and $w=x$, we obtain

$$0=\int_0^1 1\cdot R\,dx=(1+2c_1+3c_2)+\frac{1}{2}(-6c_1-3c_2)+\frac{1}{3}(c_1-9c_2)+\frac{1}{4}c_2$$

$$0=\int_0^1 x\cdot R\,dx=\frac{1}{2}(1+2c_1+3c_2)+\frac{1}{3}(-6c_1-3c_2)+\frac{1}{4}(c_1-9c_2)+\frac{1}{5}c_2$$

or

$$\frac{2}{3}c_1+\frac{5}{4}c_2=1, \quad \frac{3}{4}c_1+\frac{31}{20}c_2=\frac{1}{2} \tag{2.1.8}$$

which provides two linearly independent equations for c_1 and c_2 (whose solution is $c_1=\frac{222}{23}$ and $c_2=-\frac{100}{23}$).

The above discussion clearly demonstrates the need for weighted-integral statements of the type in (2.1.7); they provide the means for obtaining as many algebraic equations

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as there are unknown coefficients in the approximate solution. This chapter deals with the construction of different types of integral statements used in different variational methods. The variational methods differ from each other in the choice of the weight function w and/or the integral statement used, which in turn dictates the choice of the approximation functions ϕ_j. In the finite element method, a given domain is viewed as an assemblage of subdomains (i.e., finite elements), and an approximate solution is sought over each subdomain in the same way as in variational methods. Therefore, it is informative to study variational methods before we study the finite element method.

Our goal in this chapter is to illustrate the basic steps in the weighted-integral formulations and associated approximations of boundary value, eigenvalue, and initial value problems. Toward this goal, we first introduce the necessary terminology and notation.

2.2 SOME MATHEMATICAL CONCEPTS AND FORMULAE

2.2.1 Coordinate Systems and the Del Operator

In the analytical description of physical phenomena, a coordinate system in the chosen frame of reference is introduced, and various physical quantities involved in the description are expressed in terms of measurements made in that system. The vector and tensor quantities are expressed in terms of their components in that coordinate system. For example, a vector $\mathbf{A}$ in a three-dimensional space may be expressed in terms of its components (a_1, a_2, a_3) and *basis vectors* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ ($\mathbf{e}_i$ are not necessarily unit vectors) as*

$$\mathbf{A} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{2.2.1}$$

When the basis vectors of a coordinate system are constants, i.e., with fixed lengths and directions, the coordinate system is called a *Cartesian coordinate system*. The general Cartesian system is oblique. When the Cartesian system is orthogonal, it is called *rectangular Cartesian*. The rectangular Cartesian coordinates are denoted by

$$(x_1, x_2, x_3) \text{ or } (x, y, z) \tag{2.2.2}$$

The familiar rectangular Cartesian coordinate system is shown in Fig. 2.2.1. We shall always use a right-hand coordinate system. When the basis vectors are of unit lengths and mutually orthogonal, they are called *orthonormal*. In many situations an *orthonormal basis* simplifies calculations. We denote an orthonormal Cartesian basis by

$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \text{ or } (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z) \tag{2.2.3}$$

For an orthonormal basis, the vector $\mathbf{A}$ can be written as

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

where $\hat{\mathbf{e}}_i$ $(i = 1, 2, 3)$ is the orthonormal basis and A_i are the corresponding *physical components* (i.e., the components have the same physical dimensions as the vector). Although the analytical description depends upon the chosen coordinate system and may appear different

*Vectors and matrices in this book are written with boldface letters.

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Figure 2.2.1 A rectangular Cartesian coordinate system, $(x_1, x_2, x_3) = (x, y, z)$; $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) = (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$ are the unit basis vectors.

in another type of coordinate system, one must keep in mind that *the laws of nature are independent of the choice of coordinate system.*

It is useful to abbreviate a summation of terms by understanding that a repeated index means summation over all values of that index. Thus, the summation

$$\mathbf{A} = \sum_{i=1}^{3} A_i \mathbf{e}_i$$

can be shortened to

$$\mathbf{A} = A_i \mathbf{e}_i \tag{2.2.4}$$

The repeated index is a *dummy index* and thus can be replaced by *any other symbol that has not already been used.* Thus, we can also write

$$\mathbf{A} = A_i \mathbf{e}_i = A_m \mathbf{e}_m$$

and so on.

The "dot product" $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ and "cross product" $\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j$ of base vectors in a right-handed system are defined by

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \equiv \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \tag{2.2.5}$$

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \equiv \varepsilon_{ijk} \hat{\mathbf{e}}_k \tag{2.2.6}$$

where δ_{ij} is the *Kronecker delta* and ε_{ijk} is the *alternating symbol* or *permutation symbol*

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ are in cyclic order} \\ & \text{and not repeated } (i \neq j \neq k) \\ -1, & \text{if } i, j, k \text{ are not in cyclic order} \\ & \text{and not repeated } (i \neq j \neq k) \\ 0, & \text{if any of } i, j, k \text{ are repeated} \end{cases} \tag{2.2.7}$$

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Figure 2.2.2 Cylindrical coordinate system.

Differentiation of vector functions with respect to the coordinates is common in science and engineering. Most of the operations involve the "del operator," denoted by ∇. In a rectangular Cartesian system, it has the form

$$\nabla \equiv \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \tag{2.2.8}$$

It is important to note that the del operator has some of the properties of a vector, but it does not have them all because it is an operator. The operation $\nabla\phi(\mathbf{x})$ is called the *gradient* of a scalar function ϕ whereas $\nabla \times \mathbf{A}(\mathbf{x})$ is called the *curl* of a vector function $\mathbf{A}$. The operator $\nabla^2 \equiv \nabla \cdot \nabla$ is called the Laplace operator. In a 3-D rectangular Cartesian coordinate system, it has the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.2.9}$$

We have the following relations between the rectangular Cartesian coordinates (x, y, z) and cylindrical coordinates (r, θ, z) (see Fig. 2.2.2):

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{2.2.10}$$

$$\hat{\mathbf{e}}_r = \cos\theta\, \hat{\mathbf{e}}_x + \sin\theta\, \hat{\mathbf{e}}_y, \quad \hat{\mathbf{e}}_\theta = -\sin\theta\, \hat{\mathbf{e}}_x + \cos\theta\, \hat{\mathbf{e}}_y, \quad \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \tag{2.2.11}$$

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = -\sin\theta\, \hat{\mathbf{e}}_x + \cos\theta\, \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\cos\theta\, \hat{\mathbf{e}}_x - \sin\theta \hat{\mathbf{e}}_y = -\hat{\mathbf{e}}_r \tag{2.2.12}$$

and all other derivatives of the base vectors are zero. For more on vector calculus, see Reddy and Rasmussen (1982) and Reddy (2002), among other references.

2.2.2 Boundary Value, Initial Value, and Eigenvalue Problems

The objective of most analyses is to determine unknown functions, called *dependent variables,* that are governed by a set of differential equations posed in a given domain Ω and some conditions on the boundary Γ of the domain. Often, a domain not including its boundary is called an open domain. A domain Ω with its boundary Γ is called a closed domain and is denoted by $\bar{\Omega} = \Omega \cup \Gamma$.

A function u of several variables $(x, y, \cdots)$ is said to be of class $C^m(\Omega)$ in a domain Ω if all its partial derivatives with respect to $(x, y, \cdots)$ of order up to and including m exist

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and are *continuous* in Ω. Thus, if u is of class C^0 in a two-dimensional domain Ω, then u is continuous in Ω (i.e., $\partial u/\partial x$ and $\partial u/\partial y$ exist but may not be continuous). Similarly, if u is of class C^1, then u, $\partial u/\partial x$, and $\partial u/\partial y$ exist and are continuous (hence, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$, and $\partial^2 u/\partial y \partial x$ exist but may not be continuous).

When the dependent variables are functions of one independent variable (say, x), the domain is a line segment (i.e., one-dimensional) and the end points of the domain are called boundary points. When the dependent variables are functions of two independent variables (say, x and y), the domain is two-dimensional and the boundary is the closed curve enclosing it. In a three-dimensional domain, dependent variables are functions of three coordinates (say x, y, and z) and the boundary is a two-dimensional surface.

As discussed in Section 1.2, a differential equation is said to describe a *boundary value problem* over the domain Ω if the dependent variable and possibly its derivatives are required to take specified values on the boundary Γ of Ω. An *initial value problem* is one in which the dependent variable and possibly its derivatives are specified initially (i.e., at time $t = 0$). Initial value problems are generally time-dependent problems. Examples of boundary and initial value problems were discussed in Section 1.2. A problem can be both a boundary value and initial value problem if the dependent variable is subject to both boundary and initial conditions. Another type of problem we encounter is one in which a differential equation governing the dependent unknown also contains an unknown parameter and we are required to find both the dependent variable and the parameter such that the differential equation and associated boundary conditions are satisfied. Such problems are called *eigenvalue problems*. Examples of various types of problems we encounter in science and engineering are given below (the mathematical classification of differential equations into elliptic, parabolic, and hyperbolic is of no interest at the moment).

Boundary Value Problems. *Steady State Heat Transfer in a Fin and Axial Deformation of a Bar* [Fig. 2.2.3(a)]: Find $u(x)$ that satisfies the second-order differential equation and *boundary conditions*:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu = f \quad \text{for} \quad 0 < x < L \tag{2.2.13a}$$

$$u(0) = u_0, \quad \left(a\frac{du}{dx}\right)_{x=L} = q_0 \tag{2.2.13b}$$

Bending of Elastic Beams under Transverse Load: Find $u(x)$ that satisfies the fourth-order differential equation and *boundary conditions*:

$$\frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right) + cu = f \quad \text{for} \quad 0 < x < L \tag{2.2.14a}$$

$$u(0) = u_0, \quad \left(\frac{du}{dx}\right)_{x=0} = d_0$$

$$\left[\frac{d}{dx}\left(b\frac{d^2u}{dx^2}\right)\right]_{x=L} = m_0, \quad \left(b\frac{d^2u}{dx^2}\right)_{x=L} = v_0 \tag{2.2.14b}$$

Steady Heat Conduction in a Two-Dimensional Region and *Transverse Deflections of a Membrane* [Fig. 2.2.3(b)]: Find $u(x, y)$ that satisfies the second-order partial differential

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(*a*) A one-dimensional domain

(*b*) A two-dimensional domain

Figure 2.2.3 (*a*) One-dimensional domain. (*b*) Two-dimensional domain.

equation and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+cu=f \quad \text{in } \Omega \tag{2.2.15a}$$

$$u=u_0 \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial x}n_y\right)=q_0 \text{ on } \Gamma_q \tag{2.2.15b}$$

where (n_x, n_y) are the direction cosines of the unit normal vector $\hat{\mathbf{n}}$ to the boundary Γ_q.

Initial Value Problems. *A General First-Order Equation:* Find $u(t)$ that satisfies the first-order differential equation and *initial conditions*:

$$a\frac{du}{dt}+cu=f \quad \text{for} \quad 0<t\le T \tag{2.2.16a}$$

$$u(0)=u_0 \tag{2.2.16b}$$

A General Second-Order Equation: Find $u(t)$ that satisfies the second-order differential equation and *initial conditions*:

$$a\frac{du}{dt}+b\frac{d^2u}{dt^2}+cu=f \quad \text{for} \quad 0<t\le T \tag{2.2.17a}$$

$$u(0)=u_0, \left(b\frac{du}{dt}\right)_{t=0}=v_0 \tag{2.2.17b}$$

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Boundary and Initial Value Problems. *Unsteady Heat Transfer in a Rod:* Find $u(x, t)$ that satisfies the partial differential equation and *initial* and *boundary conditions*:

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\rho\frac{\partial u}{\partial t}=f(x,t) \quad \text{for} \quad 0<x<L,\ 0<t\le T \tag{2.2.18a}$$

$$u(0,t)=d_0(t), \quad \left(a\frac{du}{dx}\right)_{x=L}=q_0(t), \quad u(x,0)=u_0(x) \tag{2.2.18b}$$

Unsteady Motion of a Membrane: Find $u(x, y, t)$ that satisfies the partial differential equation and *initial* and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]+\rho\frac{\partial^2 u}{\partial t^2}=f(x,y,t) \quad \text{in } \Omega,\ 0<t\le T \tag{2.2.19a}$$

$$u=u_0(t) \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)=q_0(t) \text{ on } \Gamma_q \tag{2.2.19b}$$

$$u(x,y,0)=d_0, \quad \dot{u}(x,y,0)=v_0 \tag{2.2.19c}$$

where the superposed dot indicates a derivative with respect to time t.

Eigenvalue Problems. *Axial Vibrations of a Bar:* Find $u(x)$ and λ that satisfy the differential equation and *boundary conditions*:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)-\lambda u=0 \quad \text{for} \quad 0<x<L \tag{2.2.20a}$$

$$u(0)=0, \quad \left(a\frac{du}{dx}\right)_{x=L}=0 \tag{2.2.20b}$$

Transverse Vibrations of a Membrane: Find $u(x, y)$ and λ that satisfy the partial differential equation and *boundary conditions*:

$$-\left[\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right)\right]-\lambda u=0 \quad \text{in } \Omega \tag{2.2.21a}$$

$$u=0 \text{ on } \Gamma_u, \quad \left(a_1\frac{\partial u}{\partial x}n_x+a_2\frac{\partial u}{\partial y}n_y\right)=0 \text{ on } \Gamma_q \tag{2.2.21b}$$

The values of λ are called *eigenvalues*, and the associated functions u are called *eigenfunctions*.

The set of specified functions and parameters (e.g., a, b, c, ρ, f, u_0, d_0, q_0, v_0, and so on) are called the *data* of the problem. Differential equations in which the right-hand side f is zero are called *homogeneous differential equations*, and boundary (initial) conditions in which the specified data is zero are called homogeneous boundary (initial) conditions. The *exact solution* of a differential equation is the function that identically satisfies the differential equation at every point of the domain and for all times $t > 0$, and satisfies the specified boundary and/or initial conditions.

2.2.3 Integral Identities

Integration by parts is frequently used in the integral formulation of differential equations. In two- and three-dimensional cases, integration by parts is carried out with the help of

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the gradient and divergence theorems. In this section, we derive some useful identities for future use.

Integration-by-Parts Formulae. Let u, v, and w be sufficiently differentiable functions of the coordinate x. Then the following integration-by-parts formula holds:

$$\int_a^b w\,\frac{dv}{dx}\,dx = \int_a^b w\,dv = -\int_a^b v\,dw + [wv]_a^b$$
$$= -\int_a^b v\,\frac{dw}{dx}\,dx + w(b)\,v(b) - w(a)\,v(a) \tag{2.2.22}$$

This identity can easily be established. First, note the following identity from the product rule of differentiation:

$$\frac{d}{dx}(wv) = \frac{dw}{dx}\,v + w\,\frac{dv}{dx}$$

Therefore,

$$w\frac{dv}{dx} = \frac{d}{dx}(wv) - \frac{dw}{dx}\,v$$

Integrating both sides over the interval (a, b), we obtain the identity in (2.2.22)

$$\int_a^b w\frac{dv}{dx}\,dx = \int_a^b \left[\frac{d}{dx}(wv) - \frac{dw}{dx}\,v\right]dx$$
$$= \int_a^b \frac{d}{dx}(wv)\,dx - \int_a^b \frac{dw}{dx}\,v\,dx$$
$$= [wv]_a^b - \int_a^b \frac{dw}{dx}\,v\,dx$$

Next, consider the expression

$$\int_a^b w\,\frac{d^2u}{dx^2}\,dx = \int_a^b w\,\frac{d}{dx}\left(\frac{du}{dx}\right)dx = \int_a^b w\,\frac{dv}{dx}\,dx$$

where $v \equiv \frac{du}{dx}$. Using (2.2.22), we obtain

$$\int_a^b w\,\frac{d^2u}{dx^2}\,dx = -\int_a^b v\,\frac{dw}{dx}\,dx + w(b)\,v(b) - w(a)\,v(a)$$
$$= -\int_a^b \frac{du}{dx}\,\frac{dw}{dx}dx + w(b)\frac{du}{dx}\bigg|_b - w(a)\frac{du}{dx}\bigg|_a \tag{2.2.23a}$$

or

$$-\int_a^b \frac{du}{dx}\,\frac{dw}{dx}\,dx = \int_a^b w\,\frac{d^2u}{dx^2}\,dx + w(a)\frac{du}{dx}\bigg|_a - w(b)\frac{du}{dx}\bigg|_b \tag{2.2.23b}$$

Similarly,

$$\int_a^b v\,\frac{d^4w}{dx^4}\,dx = \int_a^b v\,\frac{d^2}{dx^2}\,\frac{d^2w}{dx^2}\,dx$$
$$= \int_a^b v\,\frac{d^2u}{dx^2}\,dx$$

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where $u \equiv \frac{d^2w}{dx^2}$. Using (2.2.23$a$) with $w = v$, we can write the right-hand side as

$$-\int_a^b \frac{du}{dx}\frac{dv}{dx}\,dx + v(b)\frac{du}{dx}\bigg|_b - v(a)\frac{du}{dx}\bigg|_a \tag{2.2.24a}$$

We use (2.2.23b) with $w = u$ and $u = v$ to write (2.2.24a) as

$$\int_a^b u\frac{d^2v}{dx^2}\,dx + u(a)\frac{dv}{dx}\bigg|_a - u(b)\frac{dv}{dx}\bigg|_b + v(b)\frac{du}{dx}\bigg|_b - v(a)\frac{du}{dx}\bigg|_a \tag{2.2.24b}$$

and, finally, replacing u by its actual value $u = d^2w/dx^2$, we arrive at

$$\int_a^b v\frac{d^4w}{dx^4}\,dx = \int_a^b \frac{d^2w}{dx^2}\frac{d^2v}{dx^2}\,dx + \frac{d^2w}{dx^2}\bigg|_a\frac{dv}{dx}\bigg|_a - \frac{d^2w}{dx^2}\bigg|_b\frac{dv}{dx}\bigg|_b + v(b)\frac{d^3w}{dx^3}\bigg|_b - v(a)\frac{d^3w}{dx^3}\bigg|_a \tag{2.2.25}$$

Gradient and Divergence Theorems. Let ∇ and ∇^2 denote, respectively, the gradient operator and the Laplace operator in the two-dimensional Cartesian rectangular coordinate system (x, y):

$$\nabla = \hat{\mathbf{e}}_x\frac{\partial}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial}{\partial y}, \quad \nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.2.26}$$

where $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ denote the unit basis vectors along the x and y coordinates, respectively. If $F(x, y)$ and $G(x, y)$ are scalar functions of class $C^0(\Omega)$ in the two-dimensional domain Ω shown in Fig. 2.2.3(b), the following gradient and divergence theorems hold.

Gradient Theorem

$$\int_\Omega \text{grad}F\,dxdy \equiv \int_\Omega \nabla F\,dxdy = \oint_\Gamma \hat{\mathbf{n}}\,F\,ds$$

$$\int_\Omega \left(\hat{\mathbf{e}}_x\frac{\partial F}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial F}{\partial y}\right)dxdy = \oint_\Gamma \left(n_x\,\hat{\mathbf{e}}_x + n_y\,\hat{\mathbf{e}}_y\right)F\,ds \tag{2.2.27a}$$

The second equation implies, because two vectors are equal if and only if their components are equal, that the following relations hold:

$$\int_\Omega \frac{\partial F}{\partial x}\,dxdy = \oint_\Gamma n_x\,F\,ds, \quad \int_\Omega \frac{\partial F}{\partial y}\,dxdy = \oint_\Gamma n_y\,F\,ds \tag{2.2.27b}$$

Divergence Theorem

$$\int_\Omega \text{div}\mathbf{G}\,dxdy \equiv \int_\Omega \nabla\cdot\mathbf{G}\,dxdy = \oint_\Gamma \hat{\mathbf{n}}\cdot\mathbf{G}\,ds \tag{2.2.28a}$$

$$\int_\Omega \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}\right)dxdy = \oint_\Gamma (n_x\,G_x + n_y\,G_y)ds \tag{2.2.28b}$$

Here the dot denotes the scalar product of vectors, $\hat{\mathbf{n}}$ denotes the unit vector normal to the surface Γ of the domain Ω; n_x and n_y (G_x and G_y) are the rectangular components of $\hat{\mathbf{n}}(\mathbf{G})$; and the circle on the boundary integral indicates that the integration is taken over the entire

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boundary [see Fig. 2.2.3(*b*)]. The direction cosines n_x and n_y of the unit vector $\hat{\mathbf{n}}$ can be written as

$$n_x = \cos(x, \hat{\mathbf{n}}) = \hat{\mathbf{e}}_x \cdot \hat{\mathbf{n}}, \quad n_y = \cos(y, \hat{\mathbf{n}}) = \hat{\mathbf{e}}_y \cdot \hat{\mathbf{n}} \tag{2.2.29}$$

where $\cos(x, \hat{\mathbf{n}})$, for example, is the cosine of the angle between the positive x direction and the unit vector $\hat{\mathbf{n}}$.

The following identities, which can be derived using the gradient and divergence theorems, will be useful in the sequel. Let w and G be scalar functions defined in a two-dimensional domain Ω. Then

$$\int_\Omega (\nabla G)\, w\, dxdy = -\int_\Omega (\nabla w)\, G\, dxdy + \int_\Gamma \hat{\mathbf{n}}\, w\, G\, ds \tag{2.2.30a}$$

and

$$-\int_\Omega \left(\nabla^2 G\right) w\, dxdy = \int_\Omega \nabla w \cdot \nabla G\, dxdy - \oint_\Gamma \frac{\partial G}{\partial n}\, w\, ds \tag{2.2.30b}$$

where $\partial/\partial n$ denotes the normal derivative operator,

$$\frac{\partial}{\partial n} = \hat{\mathbf{n}} \cdot \nabla = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y} \tag{2.2.31}$$

The following component form of (2.2.30a), with an appropriate change of variables, is useful in the sequel:

$$\int_\Omega w \frac{\partial G}{\partial x}\, dxdy = -\int_\Omega \frac{\partial w}{\partial x}\, G\, dxdy + \oint_\Gamma n_x\, w\, G\, ds \tag{2.2.32a}$$

$$\int_\Omega w \frac{\partial G}{\partial y}\, dxdy = -\int_\Omega \frac{\partial w}{\partial y}\, G\, dxdy + \oint_\Gamma n_y\, w\, G\, ds \tag{2.2.32b}$$

Equations (2.2.32*a*) and (2.2.32*b*) can easily be established by means of (2.2.27*b*).

2.2.4 Linear and Bilinear Functionals

Consider the integral expression of the form

$$I(u) = \int_a^b F(x, u, u')\, dx, \quad u = u(x), \quad u' = \frac{du}{dx} \tag{2.2.33}$$

where the integrand $F(x, u, u')$ is a given function of the coordinate x (independent variable), dependent variable u, and its derivative du/dx. For a given real function $u = u(x)$, $I(u)$ is a real number. Therefore, I can be viewed as an operator that transforms functions $u(x)$ into real numbers, and such operators are called *functionals*. We shall use the term "functional" to describe functions defined by integrals whose arguments themselves are functions. Thus, loosely speaking, a functional is a "function of functions." A formal definition from functional analysis is that a functional is an operator I mapping functions u from a linear vector space into real number field. The following integrals qualify as functionals:

$$I(u) = \int_a^b \left(p(x)\frac{du}{dx} + q(x)u^2\right) dx + Pu(a)$$

$$I(u, v) = \int_\Omega \left(p(x, y)\frac{du}{dx}\frac{dv}{dx} + q(x, y)v\right) dxdy + \int_\Gamma Q\, u\, ds \tag{2.2.34}$$

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where u and v are the dependent variables and all other parameters are either constants or functions of position.

A functional $l(u)$ is said to be *linear* in u if and only if it satisfies the relation

$$l(\alpha u + \beta v) = \alpha l(u) + \beta l(v) \tag{2.2.35}$$

for any real numbers α and β and dependent variables u and v. Examples of linear functionals are provided by

$$I(u) = \int_a^b f(x)\, u\, dx + q\, u(b), \quad I(u, v) = \int_\Omega (f(x, y)\, u + q(x, y) v)\, dx dy$$

Note that the functionals

$$I_1(u) = \int_a^b u \frac{du}{dx}\, dx, \quad I_2(u) = \int_a^b f(x)\, u\, dx + c, \quad c \text{ is a constant}$$

do not qualify as linear functionals (why?).

A functional $B(u, v)$ is said to be *bilinear* if it is linear in each of its arguments u and v:

$$B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v) \quad \text{linear in the first argument} \tag{2.2.36a}$$

$$B(u, \alpha v_1 + \beta v_2) = \alpha B(u, v_1) + \beta B(u, v_2) \quad \text{linear in the second argument} \tag{2.2.36b}$$

where u, u_1, u_2, v, v_1, and v_2 are dependent variables and α and β are real numbers. Note that a bilinear functional necessarily contains two arguments (dependent variables), and it must be linear with respect to each argument. Examples of bilinear forms are

$$\begin{aligned} B_1(u, v) &= \int_a^b \left(p(x) u v + q(x) \frac{du}{dx} \frac{dv}{dx} \right) dx + k u(a) v(a) \\ B_2(\mathbf{u}, \mathbf{v}) &= \int_a^b \left(p(x) \mathbf{u} \cdot \mathbf{v} + q(x) \frac{d\mathbf{u}}{dx} \cdot \mathbf{v} \right) dx \\ B_3(\mathbf{u}, \mathbf{v}) &= \int_\Omega (p(\mathbf{x}) \mathbf{u} \cdot \mathbf{v} + q(\mathbf{x})\, \nabla \mathbf{u} \cdot \nabla \mathbf{v})\, d\mathbf{x} \end{aligned} \tag{2.2.37}$$

The functionals

$$\begin{aligned} I_1(u, v) &= \int_a^b \left[p(x) u^2 + q(x) \left(\frac{dv}{dx} \right)^2 \right] dx \\ I_2(\mathbf{u}, \mathbf{v}) &= \int_a^b \left(p(x) \mathbf{u} \cdot \mathbf{u} + q(x) \frac{d\mathbf{v}}{dx} \cdot \frac{d\mathbf{v}}{dx} \right) dx \\ I_3(u, v) &= \int_\Omega \left[p(x, y) \left(\frac{\partial u}{\partial x} \right)^2 v + q(x, y) u \right] dx dy \end{aligned} \tag{2.2.38}$$

are not linear in their arguments.

A bilinear form $B(u, v)$ is said to be *symmetric* in its arguments u and v if

$$B(u, v) = B(v, u) \tag{2.2.39}$$

for all u and v. Examples of symmetric bilinear forms are provided by $B_1(u, v)$ and $B_3(\mathbf{u}, \mathbf{v})$ listed above. Note that $B_2(\mathbf{u}, \mathbf{v})$ is not symmetric.

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A *quadratic* functional $Q(u)$ is one that satisfies the relation

$$Q(\alpha u) = \alpha^2 Q(u) \tag{2.2.40}$$

for all real numbers α.

2.3 ELEMENTS OF CALCULUS OF VARIATIONS

2.3.1 Introduction

Calculus of variations is a branch of mathematics that deals with extrema and stationary behavior of functionals. As discussed in the introduction to this chapter, there are many problems that can be posed only as finding the extremum of functionals [see, for example, the *Brachistochrone problem, geodesic problem,* and *isoperimetric problem* discussed on pages 129 and 130 of Reddy (2002)]. The field of solid and structural mechanics heavily depends on the use of energy principles that are stated in terms of finding the extrema or stationary values of functionals to construct finite element models (e.g., the principle of minimum total potential energy, the principle of maximum total complementary energy, and the Hu–Washizu and Hellinger–Reissner mixed variational principles). The direct variational methods like the Ritz method use variational principles to obtain approximate solutions directly, bypassing the derivation of the governing equations.

In this section we study the concept of variational operator and its properties, first variation of functionals, and Euler equations resulting from the condition of vanishing of the first variation of a functional. In the interest of keeping the scope of the study within reasonable limits, only necessary concepts are covered.

Although the material covered in this section is not absolutely necessary for an understanding of the finite element method, it is deemed useful, at least for those readers who have solid and structural mechanics background, to understand the concepts from calculus of variations. Other readers may skip the section and go directly to Section 2.4.

2.3.2 Variational Operator and First Variation

Consider the function $F(x, u, u')$. For an arbitrarily fixed value of the independent variable x, F depends on u and u'. The change ϵv in u, where ϵ is a constant and v is a function, is called the *variation* of u and is denoted by δu (see Fig. 2.3.1):

$$\delta u \equiv \epsilon v \tag{2.3.1}$$

The operator δ is called the *variational operator*. The variation δu of a function u represents an admissible change in the function $u(x)$ at a *fixed* value of the independent variable x. If u is specified at a point (usually on the boundary), the variation of u is zero there because the specified value cannot be varied. Thus, the variation of a function u should satisfy the homogeneous form of the boundary conditions for u. The variation δu represents a *virtual* but admissible change in u. Associated with this change in u (i.e., when u is changed to $u + \epsilon v$), there is a change in F,

$$\Delta F = F(x, u + \epsilon v, u' + \epsilon v') - F(x, u, u')$$

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Figure 2.3.1 The variations of $u(x)$.

Expanding in powers of ϵ gives (treating $u+\epsilon v$ and $u'+\epsilon v'$ as dependent functions)

$$\begin{aligned}\Delta F &= F(x,u,u') + \epsilon v\frac{\partial F}{\partial u} + \epsilon v'\frac{\partial F}{\partial u'} + \frac{(\epsilon v)^2}{2!}\frac{\partial^2 F}{\partial u^2} \\ &\quad + \frac{(\epsilon v)(\epsilon v')}{2!}\frac{\partial^2 F}{\partial u\partial u'} + \frac{(\epsilon v')^2}{2!}\frac{\partial^2 F}{\partial u'^2} + \cdots - F(x,u,u') \\ &= \epsilon v\frac{\partial F}{\partial u} + \epsilon v'\frac{\partial F}{\partial u'} + \epsilon R_1(\epsilon) \qquad (2.3.2)\end{aligned}$$

where $lim_{\epsilon\to 0}R_1(\epsilon)=0$. The *first variation* of F is defined by

$$\begin{aligned}\delta F &= \epsilon\left[\lim_{\epsilon\to 0}\frac{F(x,u+\epsilon v,u'+\epsilon v') - F(x,u,u')}{\epsilon}\right] = \epsilon\left[\lim_{\epsilon\to 0}\frac{\Delta F}{\epsilon}\right] \\ &\equiv \epsilon\left[\frac{d}{d\epsilon}(F(u+\epsilon v))\right]_{\epsilon=0} \\ &= \epsilon\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right) = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \qquad (2.3.3)\end{aligned}$$

Thus, the first variation of F can be written in terms of the variations of the dependent variable u and its derivatives. Note that in the special case when $F=u$, Eq. (2.3.3) gives the result in Eq. (2.3.1). Further, note the analogy between the first variation, (2.3.3), and the total differential of F,

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du' \qquad (2.3.4)$$

Since x is *not varied* during the variation of u to $u+\delta u$, $dx=0$ and the analogy between δF and dF becomes apparent, i.e., *δ acts as a differential operator with respect to dependent variables.*

The above discussion can be extended to two dimensions and to functions F that depend on more than one dependent variable in two or more dimensions. Let $F = F(x,y,u,v,u_x,v_x,u_y,v_y)$, where $u=u(x,y)$ and $v=v(x,y)$ are dependent variables,

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and $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, and so on. The first variation of F is given by

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial v_y}\delta v_y$$

It can easily be verified that the laws of variation of sums, products, ratios, powers, and so forth are completely analogous to the corresponding laws of differentiation. For example, if $F_1 = F_1(u)$ and $F_2 = F_2(u)$, then

$$\begin{aligned} \delta(F_1 \pm F_2) &= \delta F_1 \pm \delta F_2 \\ \delta(F_1\, F_2) &= \delta F_1\, F_2 + F_1 \delta F_2 \\ \delta\left(\frac{F_1}{F_2}\right) &= \frac{\delta F_1\, F_2 - F_1\, \delta F_2}{F_2^2} \\ \delta(F_1)^n &= n(F_1)^{n-1}\delta F_1 \end{aligned} \tag{2.3.5}$$

If $G = G(u, v, w)$ is function of several dependent variables u, v, and w (and possibly their derivatives), the total variation is the sum of the partial variations:

$$\delta G = \delta_u\, G + \delta_v\, G + \delta_w\, G \tag{2.3.6}$$

Furthermore, the variational operator can commute with differential and integral operators (as long as the coordinates x and y are the fixed coordinates):

$$\frac{d}{dx}(\delta u) = \frac{d}{dx}(\epsilon v) = \epsilon\frac{dv}{dx} = \epsilon v' = \delta u' = \delta\left(\frac{du}{dx}\right) \tag{2.3.7a}$$

$$\delta\int_a^b u(x)\, dx = \int_a^b \delta u(x)\, dx \tag{2.3.7b}$$

The first variation of a functional can now be computed readily. Consider the functional in Eq. (2.2.33). The first variation of $I(u)$ is

$$\begin{aligned} \delta I(u) &= \delta\int_a^b F(x, u, u')\, dx = \int_a^b \delta F(x, u, u')\, dx \\ &= \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right) dx \end{aligned} \tag{2.3.8}$$

More specific functionals are considered in the next example.

Example 2.3.1

Consider the functionals in (2.2.34). We wish to find their first variations. We have

$$\begin{aligned} \delta I(u) &= \int_a^b \left(p(x)\frac{d\delta u}{dx} + 2q(x)u\delta u\right) dx + P\delta u(a) \\ \delta I(u, v) &= \int_\Omega \left[p(x, y)\left(\frac{d\delta u}{dx}\frac{dv}{dx} + \frac{du}{dx}\frac{d\delta v}{dx}\right) + q(x, y)\delta v\right] dxdy \\ &\quad + \int_\Gamma Q\, \delta u\, ds \end{aligned} \tag{2.3.9}$$

Note that functions of position, p and q, do not undergo variation because they are not functions of the dependent variables.

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2.3.3 Fundamental Lemma of Variational Calculus

The *fundamental lemma of calculus of variations* can be stated as follows: *for any integrable function $G(x)$, if the statement*

$$\int_a^b G(x)\,\eta(x)\,dx = 0 \tag{2.3.10}$$

holds for any arbitrary continuous function $\eta(x)$, for all x in (a, b), then it follows that $G(x) = 0$ in (a, b). A simple proof of the lemma follows from setting $\eta(x)$, which is arbitrary, equal to G. We have

$$\int_a^b [G(x)]^2\,dx = 0$$

Since an integral of a positive function, G^2, is positive, the above statement implies that $G(x) = 0$ in the domain $\Omega = (a, b)$.

A more general statement of the fundamental lemma is as follows: If $\eta(x)$ is arbitrary in $a < x < b$ and $\eta(a)$ is arbitrary, then the statement

$$\int_a^b G\eta\,dx + B(a)\eta(a) = 0 \tag{2.3.11a}$$

implies that

$$G = 0 \text{ in } a < x < b \text{ and } B(a) = 0 \tag{2.3.11b}$$

because $\eta(x)$ is independent of $\eta(a)$.

2.3.4 The Euler Equations

As stated in the introduction, certain problems are formulated as one of seeking the extremum of functionals (i.e., functions of dependent unknowns of the problem). For example, problems of solid mechanics can be formulated as one of minimizing the total potential energy of the system (Reddy, 2002). Typically, the total potential energy (functional) is written in terms of the displacement field and applied loads. Then it is useful to derive the differential equations that govern the displacement field from this minimum principle. Here we outline the steps in obtaining such equations.

Consider, for example, the problem of finding a function $u = u(x)$ such that

$$u(a) = u_a, \quad u(b) = u_b \tag{2.3.12a}$$

and

$$I(u) = \int_a^b F(x, u(x), u'(x))\,dx \tag{2.3.12b}$$

is a minimum. In analyzing the problem, we are not interested in all functions u but only in those functions that satisfy the stated boundary (or end) conditions. The set of all such functions is called, for obvious reasons, the *set of competing functions* (or set of admissible functions). We shall denote the set by $\mathcal{C}$. The problem is to seek an element u from $\mathcal{C}$ that renders I a minimum. If $u \in \mathcal{C}$ (the symbol $\in$ means 'an element of'), then $(u + \epsilon v) \in \mathcal{C}$ for every v satisfying the conditions $v(a) = v(b) = 0$. The space of all such elements is called the space of admissible variations, as already mentioned. Figure 2.3.1 shows a typical competing function $\bar{u}(x) = u(x) + \epsilon v(x)$ and a typical admissible variation $v(x)$.

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Let $I(u)$ be a differentiable functional in the sense that

$$\frac{dI(u+\epsilon v, u'+\epsilon v')}{d\epsilon}$$

exists, and let $\mathcal{C}$ denote the space of competing functions. Then, an element u in $\mathcal{C}$ is said to yield a *relative minimum (maximum)* for $I(\bar{u})$ in $\mathcal{C}$ if

$$I(\bar{u}) - I(u) \geq 0 \qquad (\leq 0) \tag{2.3.13}$$

If $I(\bar{u})$ assumes a relative minimum (maximum) at $u \in \mathcal{C}$ relative to elements $\bar{u} \in \mathcal{C}$, then it follows from the definition of the space of admissible variations $\mathcal{H}$ and Eq. (2.3.13) that

$$I(u+\epsilon v) - I(u) \geq 0 \qquad (\leq 0) \tag{2.3.14}$$

for all $v \in \mathcal{H}$, $||v|| < \varepsilon$, and ϵ a real number. Since u is the minimizer, any other function $u \in \mathcal{C}$ is of the form $\bar{u} = u + \epsilon v$, and the actual minimizer is determined by setting $\epsilon = 0$. Once $u(x)$ and $v(x)$ are assigned, $I(\bar{u})$ is a function of ϵ alone, say $\bar{I}(\epsilon)$. Now a necessary condition for $I(\bar{u}) = \bar{I}(\epsilon)$ to attain a minimum is that

$$\frac{d\bar{I}(\epsilon)}{d\epsilon} = \frac{d}{d\epsilon}[I(u+\epsilon v)] = 0 \tag{2.3.15}$$

On the other hand, $I(\bar{u})$ attains its minimum at u, i.e., $\epsilon = 0$. These two conditions together imply $(d\bar{I}(\epsilon)/d\epsilon)|_{\epsilon=0} = 0$, which is nothing but

$$\delta I(u) = 0 \tag{2.3.16}$$

Analogous to the sufficient condition for ordinary functions, the sufficient condition for a functional to assume a relative minimum (maximum) is that the second variation $\delta^2 I(u)$ is greater (less) than zero. The second variation $\delta^2 I(u)$ of a functional $I(u)$ is given by

$$\delta^2 I(u) \equiv \frac{\epsilon^2}{2}\left[\frac{d^2}{d\epsilon^2} I(u+\epsilon v)\right]_{\epsilon=0} \tag{2.3.17}$$

for all $v \in \mathcal{C}$ and real number ϵ.

It is clear that any candidate for the minimizing functional should satisfy the end conditions in Eq. (2.3.12*a*) and be sufficiently differentiable (twice in the present case, as we shall see shortly). The set of all such functions is the set of admissible functions or competing functions for the present case. Functions from the admissible set can be viewed as smooth (i.e., differentiable twice) functions passing through points (a, u_a) and (b, u_b), as shown in Fig. 2.3.1. Clearly, any element $\bar{u}$ in $\mathcal{C}$ (the set of competing functions) has the form

$$\bar{u} = u + \epsilon v \tag{2.3.18}$$

where ϵ is a small number and v is a sufficiently differentiable function that satisfies the homogeneous form of the end conditions (because $\bar{u}$ must satisfy the specified end conditions) in Eq. (2.3.12*a*)

$$v(a) = v(b) = 0 \tag{2.3.19}$$

and u is the function that minimizes the functional in Eq. (2.3.12*b*). The set of all functions v is the set of admissible variations, $\mathcal{H}$. Now assuming that, for each admissible function $\bar{u}$, $F(x, \bar{u}, \bar{u}')$ exists and is continuously differentiable with respect to its arguments, and $I(\bar{u})$ takes one and only one real value, we seek the particular function $u(x)$ that makes the integral a minimum.

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The necessary condition (2.3.15) for I to attain a minimum gives

$$0 = \frac{dI(u+\epsilon v)}{d\epsilon}\bigg|_{\epsilon=0} = \left[\frac{d}{d\epsilon}\int_a^b F(x,\bar{u},\bar{u}')dx\right]_{\epsilon=0}$$

$$= \int_a^b \left(\frac{\partial F}{\partial \bar{u}}\frac{\partial \bar{u}}{\partial \epsilon} + \frac{\partial F}{\partial \bar{u}'}\frac{\partial \bar{u}'}{\partial \epsilon}\right)\bigg|_{\epsilon=0} dx = \int_a^b \left(\frac{\partial F}{\partial u}v + \frac{\partial F}{\partial u'}v'\right)dx \qquad (2.3.20)$$

where $\bar{u} = u + \epsilon v$. Integrating the second term in the last equation by parts to transfer differentiation from v to u, we obtain

$$0 = \int_a^b v\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx + \left(\frac{\partial F}{\partial u'}v\right)\bigg|_a^b \qquad (2.3.21)$$

The boundary term vanishes because v is zero at $x = a$ and $x = b$ [see Eq. (2.3.19)]. The fact that v is arbitrary inside the interval (a, b) and yet the equation should hold implies, by the fundamental lemma of the calculus of variations, that the expression in the square brackets is zero identically:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \text{ in } a < x < b \qquad (2.3.22)$$

Equation (2.3.22) is called the *Euler equation* of the functional in Eq. (2.3.12*b*). Of all the admissible functions, the one that satisfies (i.e., the solution of) Eq. (2.3.22) is the true minimizer of the functional I.

Next consider the problem of finding (u, v), defined on a two-dimensional region Ω, such that the following functional is to be minimized:

$$I(u,v) = \int_\Omega F(x, y, u, v, u_x, v_x, u_y, v_y)dxdy \qquad (2.3.23)$$

where $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, and so on. For the moment we assume that u and v are specified on the boundary Γ of Ω. The vanishing of the first variation of $I(u, v)$ is written as

$$\delta I(u,v) = \delta_u I(u,v) + \delta_v I(u,v) = 0$$

Here δ_u and δ_v denote (partial) variations with respect to u and v, respectively. We have

$$\delta I = \int_\Omega \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v_x}\delta v_x + \frac{\partial F}{\partial v_y}\delta v_y\right)dxdy \qquad (2.3.24)$$

The next step in the development involves the use of integration by parts, or the gradient theorem on the second, third, fifth, and sixth terms in Eq. (2.3.24). Consider the second term. We have

$$\int_\Omega \frac{\partial F}{\partial u_x}\frac{\partial \delta u}{\partial x}dxdy = \int_\Omega \left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u\right]dxdy$$

$$= \oint_\Gamma \frac{\partial F}{\partial u_x}\delta u\, n_x\, ds - \int_\Omega \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u\, dxdy \qquad (2.3.25)$$

Using a similar procedure on the other terms and collecting the coefficients of δu and δv separately, we obtain

$$0 = \int_\Omega \left\{\left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\delta u\right.$$

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$$+\left[\frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)\right]\delta v\Bigg\}\,dxdy$$

$$+\oint_\Gamma\left[\left(\frac{\partial F}{\partial u_x}n_x+\frac{\partial F}{\partial u_y}n_y\right)\delta u+\left(\frac{\partial F}{\partial v_x}n_x+\frac{\partial F}{\partial v_y}n_y\right)\delta v\right]ds \qquad (2.3.26)$$

Since (u, v) are specified on Γ, $\delta u=\delta v=0$ and the boundary expressions vanish. Then, since δu and δv are arbitrary and independent of each other in Ω, the fundamental lemma yields the Euler equations

$$\delta u:\quad \frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)=0 \qquad (2.3.27a)$$

$$\delta v:\quad \frac{\partial F}{\partial v}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right)=0 \qquad (2.3.27b)$$

2.3.5 Natural and Essential Boundary Conditions

First, consider the problem of minimizing the functional in Eq. (2.3.12*b*) subject to *no* end conditions [hence, an element v of the set of admissible variation is arbitrary even at the end points, i.e., $v(a)\neq 0$ and $v(b)\neq 0$]. Then the functional $I(u)$ has the form

$$I(u)=\int_a^b F(x, u(x), u'(x))\,dx-Q_a\,u(a)-Q_b\,u(b) \qquad (2.3.28)$$

where Q_a and Q_b are known values. The necessary condition for I to attain a minimum yields [cf. Eq. (2.3.21)]

$$0=\int_a^b v\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx+\left(\frac{\partial F}{\partial u'}v\right)\Bigg|_a^b-Q_a\,v(a)-Q_b\,v(b) \qquad (2.3.29)$$

Now suppose that $\frac{\partial F}{\partial u'}$ and v are selected such that

$$\left(-\frac{\partial F}{\partial u'}-Q_a\right)v=0 \text{ for } x=a,\qquad \left(\frac{\partial F}{\partial u'}-Q_b\right)v=0 \text{ for } x=b \qquad (2.3.30)$$

Then using the fundamental lemma of the calculus of variations, we obtain the same Euler equation as in Eq. (2.3.22).

Equations in (2.3.30) are satisfied identically for any of the following combinations:

$$\begin{aligned}
&(1)\quad v(a)=0,\quad v(b)=0\\
&(2)\quad v(a)=0,\quad \frac{\partial F}{\partial u'}\bigg|_b-Q_b=0\\
&(3)\quad -\frac{\partial F}{\partial u'}\bigg|_a-Q_a=0,\quad v(b)=0\\
&(4)\quad -\frac{\partial F}{\partial u'}\bigg|_a-Q_a=0,\quad \frac{\partial F}{\partial u'}\bigg|_b-Q_b=0
\end{aligned} \qquad (2.3.31)$$

The requirement that $v=0$ at an end point is equivalent to the requirement that u is specified (to be some value) at that point. The end conditions in (2.3.31) are classified into two types:

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essential boundary conditions, which require v (and possibly its derivatives) to vanish at the boundary, and *natural boundary conditions*, which require the specification of *the coefficient of* v (and possibly its derivatives). Thus, we have

Essential Boundary Conditions:

$$\text{specify } v = 0 \text{ or } u = \hat{u} \text{ on the boundary} \tag{2.3.32a}$$

Natural Boundary Conditions:

$$\text{specify } \frac{\partial F}{\partial u'} = Q \text{ on the boundary} \tag{2.3.32b}$$

where $Q = -Q_a$ at $x = a$ and $Q = Q_b$ at $x = b$. In a given problem, only one of the four combinations given in Eq. (2.3.31) can be specified. Problems in which all of the boundary conditions are of the essential type are called *Dirichlet boundary-value problems*, and those in which all of the boundary conditions are of the natural type are called *Neumann boundary-value problems*. *Mixed boundary-value problems* are those in which both essential and natural boundary conditions are specified. Essential boundary conditions are also known as *Dirichlet* or *geometric* boundary conditions, and natural boundary conditions are known as *Neumann* or *dynamic* boundary conditions.

As a general rule, the vanishing of the variation v (or δu) at a point implies that u is specified there (in general, to be nonzero). The specification of u on the boundary constitutes the essential boundary condition. Vanishing of the coefficient of the variation v in the boundary expression constitutes the natural boundary condition. This rule applies to any functional in one, two, and three dimensions, and to integrands that are functions of one or more dependent variables and their derivatives of any order.

Next consider the functional in Eq. (2.3.23), and suppose that (u, v) are arbitrary on Γ for the moment. It is easy to identify the natural and essential boundary conditions of the problem from Eq. (2.3.26): In each of the pairings on boundary Γ, specifying the first element (which contains no variations of the dependent variables) constitutes the natural boundary condition, and vanishing of the second element (or, equivalently, specifying the quantity in front of the variational operator) constitutes the essential boundary condition. Thus, we have either

$$u = \hat{u} \text{ (specified) so that } \delta u = 0 \quad \text{on } \Gamma \tag{2.3.33a}$$

$$v = \hat{v} \text{ (specified) so that } \delta v = 0 \quad \text{on } \Gamma \tag{2.3.33b}$$

or

$$\frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y = 0 \quad \text{on } \Gamma \tag{2.3.34a}$$

$$\frac{\partial F}{\partial v_x} n_x + \frac{\partial F}{\partial v_y} n_y = 0 \quad \text{on } \Gamma \tag{2.3.34b}$$

Equations (2.3.33*a*) and (2.3.33*b*) represent the essential boundary conditions and Eqs. (2.3.34*a*) and (2.3.34*b*) the natural boundary conditions. The pair of elements (u, v) are called the *primary variables* and

$$Q_x \equiv \frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y \quad \text{and} \quad Q_y \equiv \frac{\partial F}{\partial v_x} n_x + \frac{\partial F}{\partial v_y} n_y$$

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are called the *secondary variables*. Thus, specification of the primary variables constitute essential boundary conditions and specification of the secondary variables constitute natural boundary conditions. In general, one element of each pair (u, Q_x) and (v, Q_y) (but not both elements of the same pair) may be specified at any point of the boundary. Thus, there are four possible combinations of natural and essential boundary conditions for the problem under discussion.

Example 2.3.2

Consider an elastic bar of length L, modulus of elasticity E, and area of cross section A. Assume that it is fixed at the left end, spring supported at the right end, and subjected to distributed axial load $f(x)$ (see Fig. 2.3.2). The total potential energy functional $\Pi = U + V$ of the bar is (here we assumed that one knows how to write the total potential energy of the bar)

$$\Pi(u) = \int_0^L \left[\frac{EA}{2}\left(\frac{du}{dx}\right)^2 - fu \right] dx + \frac{k}{2}[u(L)]^2 \tag{2.3.35}$$

where u denotes the axial displacement of the bar. The first term in $\Pi(u)$ represents the strain energy U stored in the bar, the second term denotes the work done V on the bar by the distributed load f, and the last term denotes the strain energy stored in the linear elastic spring. We wish to determine the Euler equation of the bar by requiring that $\Pi(u)$ be a minimum subject to the geometric boundary condition $u(0) = 0$. The statement $\delta\Pi = 0$ is known as *the principle of minimum total potential energy*.

The first variation of Π is given by

$$\delta\Pi(u) = \int_0^L \left(EA\frac{du}{dx}\frac{d\delta u}{dx} - f\delta u \right) dx + ku(L)\,\delta u(L)$$

where δu is arbitrary in $0 < x < L$ and at $x = L$ but satisfies the condition $\delta u(0) = 0$. To use the fundamental lemma of variational calculus, we must relieve δu of any differentiation.

Figure 2.3.2 Elastic bar problem discussed in Example 2.3.2.

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Integrating the first term by parts, we get

$$\delta\Pi(u) = \int_0^L \left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f\right]\delta u\, dx + \left[EA\frac{du}{dx}\delta u\right]_0^L + ku(L)\,\delta u(L)$$

$$= \int_0^L \delta u\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f\right]dx + \delta u(L)\left[EA\frac{du}{dx} + ku(L)\right]_{x=L}$$

$$- \delta u(0)\left[EA\frac{du}{dx}\right]_{x=0}$$

The last term is zero because $\delta u(0) = 0$. Setting the coefficients of δu in $(0, L)$ and δu at $x = L$ to zero separately, we obtain the Euler equation and the natural (or force) boundary condition of the problem:

Euler Equation:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f = 0, \quad 0 < x < L \tag{2.3.36a}$$

Natural Boundary Condition:

$$EA\frac{du}{dx} + ku(L) = 0 \text{ at } x = L \tag{2.3.36b}$$

Thus, the solution u of Eqs. (2.3.36*a*) and (2.3.36*b*) that satisfies $u(0) = 0$ is the minimizer of the energy functional $\Pi(u)$ in Eq. (2.3.35).

Equations (2.3.36*a*) and (2.3.36*b*) can be obtained directly from Eq. (2.3.22) and (2) of Eq. (2.3.31) (with $Q_b = -ku(L)$ as shown in Fig. 2.3.2) by substituting

$$F(x, u, u') = \frac{EA}{2}\left(\frac{du}{dx}\right)^2 - f(x)u(x), \quad \frac{\partial F}{\partial u} = -f, \quad \frac{\partial F}{\partial u'} = EA\frac{du}{dx} \tag{2.3.37}$$

Example 2.3.3

The total potential energy of a linear elastic body in three dimensions, subjected to body force $\mathbf{f}$ (measured per unit volume) and surface traction $\hat{\mathbf{t}}$ (measured per unit area) on portion S_2 of the surface [see Fig. 2.3.3(*a*)] is given by (summation on repeated indices is implied)

$$\Pi(\mathbf{u}) = \int_V \left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij} - f_i u_i\right)dV - \int_{S_2} \hat{t}_i u_i\, dS \tag{2.3.38}$$

where u_i denote the displacement components, σ_{ij} are the stress components [see Fig. 2.3.3(*b*)], and ε_{ij} are strain components. It is assumed that the body is subjected to specified displacements on the remaining portion S_1 of the surface, and therefore the virtual displacements vanish there:

$$u_i = \hat{u}_i \quad \text{and} \quad \delta u_i = 0 \quad \text{on} \quad S_1 \tag{2.3.39}$$

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(*a*) A 3-D domain (*b*) Stress components

Figure 2.3.3 (*a*) An elastic body with specified displacements and tractions. (*b*) The stress components at a typical point inside the body and on the boundary.

In Eq. (2.3.38), and in the following discussion, summation on repeated indices is assumed. Use of the index notation and the summation convention in the 3-D case saves considerable space (i.e., the explicit form of Eq. (2.3.38) and subsequent manipulations would take considerably more space). The first term under the volume integral represents the strain energy density of the elastic body and the second term represents the work done by the body force $\mathbf{f}$; and the surface integral denotes the work done by the specified traction $\hat{\mathbf{t}}$.

For an isotropic body the stress-strain relations are given by (the generalized Hooke's law)

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \tag{2.3.40}$$

where μ and λ are the Lamé (material) constants. Hence,

$$\sigma_{ij}\varepsilon_{ij} = 2\mu\varepsilon_{ij}\varepsilon_{ij} + \lambda\varepsilon_{ii}\varepsilon_{kk} \tag{2.3.41}$$

The linear strain-displacement relations of the linear theory are given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.3.42}$$

where $u_{i,j} = (\partial u_i/\partial x_j)$.

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Substituting Eqs. (2.3.41) and (2.3.42) into Eq. (2.3.38), we obtain

$$\Pi(\mathbf{u}) = \int_V \left[\frac{\mu}{4}(u_{i,j}+u_{j,i})(u_{i,j}+u_{j,i}) + \frac{\lambda}{2}u_{i,i}u_{k,k} - f_i u_i\right] dV - \int_{S_2} \hat{t}_i u_i \, dS \tag{2.3.43}$$

Now we wish to derive the Euler equations associated with the functional in Eq. (2.3.43) using the principle of minimum total potential energy, $\delta\Pi = 0$ (i.e., minimize Π) and $\delta u_i = 0$ on S_1.

Setting the first variation of Π to zero, we obtain

$$0 = \int_V \left[\frac{\mu}{2}(\delta u_{i,j}+\delta u_{j,i})(u_{i,j}+u_{j,i}) + \lambda \delta u_{i,i}u_{k,k} - f_i \delta u_i\right] dV - \int_{S_2} \hat{t}_i \delta u_i \, dS \tag{2.3.44}$$

wherein the product rule of variation is used and similar terms are combined. Using integration by parts

$$\int_V \delta u_{i,j}(u_{i,j}+u_{j,i})dV = -\int_V \delta u_i (u_{i,j}+u_{j,i})_{,j} \, dV + \oint_S \delta u_i (u_{i,j}+u_{j,i}) n_j \, dS$$

where n_j denotes the jth direction cosine of the unit normal to the surface, we obtain

$$\begin{aligned} 0 &= \int_V \left[-\frac{\mu}{2}(u_{i,j}+u_{j,i})_{,j}\delta u_i - \frac{\mu}{2}(u_{i,j}+u_{j,i})_{,i}\delta u_j - \lambda u_{k,ki}\delta u_i - f_i \delta u_i\right] dV \\ &\quad + \oint_S \left[\frac{\mu}{2}(u_{i,j}+u_{j,i})(n_j\delta u_i + n_i \delta u_j) + \lambda u_{k,k} n_i \delta u_i\right] dS - \int_{S_2} \delta u_i \hat{t}_i \, dS \\ &= \int_V \left[-\mu(u_{i,j}+u_{j,i})_{,j} - \lambda u_{k,ki} - f_i\right] \delta u_i \, dV \\ &\quad + \oint_S \left[\mu(u_{i,j}+u_{j,i}) + \lambda u_{k,k}\delta_{ij}\right] n_j \delta u_i \, dS - \int_{S_2} \delta u_i \hat{t}_i \, dS \end{aligned} \tag{2.3.45}$$

In arriving at the last step, a change of dummy indices is made to combine terms. Recognizing that the expression inside the square brackets of the closed surface integral is nothing but σ_{ij}, and $\sigma_{ij}n_j = t_i$ by Cauchy's formula [which is nothing but a statement of the equilibrium of forces on the tetrahedral element; see Fig. 2.3.3(b)], we can write

$$\oint_S t_i \delta u_i \, dS = \int_{S_1} t_i \delta u_i \, dS + \int_{S_2} t_i \delta u_i \, dS = \int_{S_2} t_i \delta u_i \, dS$$

The integral over S_1 is zero by virtue of Eq. (2.3.39). Hence, we have

$$0 = \int_V \left[-\mu\left(u_{i,j}+u_{j,i}\right)_{,j} - \lambda u_{k,ki} - f_i\right] \delta u_i dV + \int_{S_2} \delta u_i \left(t_i - \hat{t}_i\right) dS$$

Using the fundamental lemma of calculus of variations, we set the coefficients of δu_i in V and δu_i on S_2 to zero separately and obtain

$$\mu(u_{i,jj}+u_{j,ij}) + \lambda u_{k,ki} + f_i = 0 \text{ in } V \tag{2.3.46}$$

$$\sigma_{ij}n_j - \hat{t}_i = 0 \text{ on } S_2 \tag{2.3.47}$$

for $i = 1, 2, 3$.

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Equations (2.3.46) are the well-known Navier's equations of equilibrium of three-dimensional elasticity, and (2.3.47) are the traction boundary conditions. The explicit forms of Eqs. (2.3.40), (2.3.42), (2.3.46), and (2.3.47) in a rectangular Cartesian coordinate system (x, y, z) are given below ($x_1 = x$, $x_2 = y$, $x_3 = z$, $f_1 = f_x$, $f_2 = f_y$, $f_3 = f_z$, $\sigma_{11} = \sigma_{xx}$, $\sigma_{13} = \sigma_{xz} = \sigma_{zx}$, and so on).

Stress-Strain Relations (Hooke's Law for an Orthotropic Material):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{2.3.48a}$$

where $\gamma_{xy} = 2\varepsilon_{xy}$, $\gamma_{xz} = 2\varepsilon_{xz}$ and $\gamma_{yz} = 2\varepsilon_{yz}$ are the engineering shear strains. The elastic material stiffness coefficients C_{ij} are known in terms of the engineering material constants by

$$\begin{aligned} C_{11} &= \frac{1 - \nu_{23}\nu_{32}}{E_2 E_3 \Delta}, \quad C_{12} = \frac{\nu_{21} + \nu_{31}\nu_{23}}{E_2 E_3 \Delta} = \frac{\nu_{12} + \nu_{32}\nu_{13}}{E_1 E_3 \Delta} \\ C_{13} &= \frac{\nu_{31} + \nu_{21}\nu_{32}}{E_2 E_3 \Delta} = \frac{\nu_{13} + \nu_{12}\nu_{23}}{E_1 E_2 \Delta} \\ C_{22} &= \frac{1 - \nu_{13}\nu_{31}}{E_1 E_3 \Delta}, \quad C_{23} = \frac{\nu_{32} + \nu_{12}\nu_{31}}{E_1 E_3 \Delta} = \frac{\nu_{23} + \nu_{21}\nu_{13}}{E_1 E_3 \Delta} \\ C_{33} &= \frac{1 - \nu_{12}\nu_{21}}{E_1 E_2 \Delta}, \quad C_{44} = G_{23} \quad C_{55} = G_{31} \quad C_{66} = G_{12} \\ \Delta &= \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1 E_2 E_3} \end{aligned} \tag{2.3.48b}$$

Here E_1, E_2, and E_3 are Young's moduli in 1, 2, and 3 material directions, respectively; ν_{ij} are Poisson's ratios, defined as the ratio of transverse strain in the jth direction to the axial strain in the ith direction when stressed in the ith direction; and G_{23}, G_{13}, and G_{12} are shear moduli in the 2-3, 1-3, and 1-2 planes, respectively. In addition, the following reciprocity relations among the engineering constants hold:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2} \tag{2.3.49}$$

For the isotropic case, we have $E_1 = E_2 = E_3 = E$, $G_{12} = G_{13} = G_{23} = G$, and $\nu_{12} = \nu_{13} = \nu_{23} = \nu$ with $G = 0.5E/(1 + \nu)$. The Lamé constants are related to the engineering constants by

$$\lambda = K - \frac{2}{3}G, \quad \mu = G \tag{2.3.50}$$

where K is the bulk modulus.

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Strain-Displacement Relations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}; \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \; \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right); \; \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{2.3.51}$$

where (u, v, w) are the displacements of a point along the (x, y, z) coordinates, respectively.

Stress Equilibrium Equations:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y &= 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z &= 0 \end{aligned} \tag{2.3.52}$$

Stress-Traction Relations (Cauchy's formula; see Fig. 2.3.3):

$$\begin{aligned} \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z &= t_x \\ \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z &= t_y \\ \sigma_{xz} n_x + \sigma_{yz} n_y + \sigma_{zz} n_z &= t_z \end{aligned} \tag{2.3.53}$$

2.3.6 Hamilton's Principle

The principle of minimum total potential energy is limited to static equilibrium of solids. Hamilton's principle is a generalization of the principle of virtual displacements (see Reddy, 2002) to dynamics (i.e., time-dependent response) of solids. The principle assumes that the system under consideration is characterized by two energy functions: *kinetic energy* K and *potential energy* Π. For *continuous* systems (i.e., systems that cannot be described by a finite number of generalized coordinates), the energies can be expressed in terms of the dependent variables (which are functions of position) of the problem.

Newton's second law of motion for a continuous body can be written in general terms as

$$\mathbf{F} - m\mathbf{a} = \mathbf{0} \tag{2.3.54}$$

where m is the mass, $\mathbf{a}$ the acceleration vector, and $\mathbf{F}$ is the resultant of *all* forces acting on the body. The actual path $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ followed by a material particle in position $\mathbf{x}$ in the body is varied, consistent with kinematic (essential) boundary conditions, to $\mathbf{u} + \delta\mathbf{u}$, where $\delta\mathbf{u}$ is the admissible variation (or virtual displacement) of the path. We suppose that the varied path differs from the actual path except at initial and final times, t_1 and t_2, respectively. Thus, an admissible variation $\delta\mathbf{u}$ satisfies the conditions,

$$\delta\mathbf{u} = \mathbf{0} \text{ on } \Gamma_u \text{ for all } t \tag{2.3.55a}$$

$$\delta\mathbf{u}(\mathbf{x}, t_1) = \delta\mathbf{u}(\mathbf{x}, t_2) = \mathbf{0} \text{ for all } \mathbf{x} \tag{2.3.55b}$$

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where Γ_u denotes the portion of the boundary of the body where the displacement vector $\mathbf{u}$ is specified. Note that the scalar product of Eq. (2.3.54) with $\delta\mathbf{u}$ gives work done at point $\mathbf{x}$, because $\mathbf{F}$, $\mathbf{a}$, and $\mathbf{u}$ are vector functions of position (whereas the work is a scalar). Integration of the product over the volume (and surface) of the body gives the total work done by all points in moving through their respective displacements.

The *work done on the body* at time t by the resultant force in moving through the virtual displacement $\delta\mathbf{u}$ is given by

$$\int_V \mathbf{f}\cdot\delta\mathbf{u}\,dV + \int_{\Gamma_\sigma} \hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS - \int_V \overleftrightarrow{\sigma}:\delta\overleftrightarrow{\varepsilon}\,dV \tag{2.3.56}$$

where $\mathbf{f}$ is the body force vector, $\hat{\mathbf{t}}$ the specified surface traction vector, and $\overleftrightarrow{\sigma}$ and $\overleftrightarrow{\varepsilon}$ are the stress and strain tensors. The "double-dot product" has the meaning $\overleftrightarrow{\sigma}:\delta\overleftrightarrow{\varepsilon} = \sigma_{ij}\varepsilon_{ji}$. The last term in Eq. (2.3.56) represents the *virtual work* of internal forces *stored in the body*. The strains $\delta\overleftrightarrow{\varepsilon}$ are assumed to be compatible in the sense that the strain-displacement relations (2.3.42) are satisfied. The work done by the inertia force $m\mathbf{a}$ in moving through the virtual displacement $\delta\mathbf{u}$ is given by

$$\int_V \rho\frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}\,dV \tag{2.3.57}$$

where ρ is the mass density (can be a function of position) of the medium. We have the result

$$\int_{t_1}^{t_2}\left\{\int_V \rho\frac{\partial^2\mathbf{u}}{\partial t^2}\cdot\delta\mathbf{u}\,dV - \left[\int_V\left(\mathbf{f}\cdot\delta\mathbf{u} - \overleftrightarrow{\sigma}:\delta\overleftrightarrow{\varepsilon}\right)dV + \int_{\Gamma_\sigma}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS\right]\right\}dt = 0$$

or

$$\int_{t_1}^{t_2}\left[\int_V \rho\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\delta\mathbf{u}}{\partial t}\,dV + \int_V\left(\mathbf{f}\cdot\delta\mathbf{u} - \overleftrightarrow{\sigma}:\delta\overleftrightarrow{\varepsilon}\right)dV + \int_{\Gamma_\sigma}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS\right]dt = 0 \tag{2.3.58}$$

In arriving at the expression in Eq. (2.3.58), integration by parts is used on the first term; the integrated terms vanish because of the initial and final conditions in Eq. (2.3.55*b*). Equation (2.3.58) is known as the general form of Hamilton's principle for a continuous medium (conservative or not, and elastic or not).

For an ideal elastic body, we recall from the previous discussions that the forces $\mathbf{f}$ and $\mathbf{t}$ are conservative,

$$\delta V = -\left(\int_V \mathbf{f}\cdot\delta\mathbf{u}\,dV + \int_{\Gamma_\sigma}\hat{\mathbf{t}}\cdot\delta\mathbf{u}\,dS\right) \tag{2.3.59a}$$

and that there exists a strain energy density function $U_0 = U_0(\varepsilon_{ij})$ such that

$$\delta U_0(\varepsilon_{ij}) = \frac{\partial U_0}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} = \sigma_{ij}\delta\varepsilon_{ij} \tag{2.3.59b}$$

Substituting Eqs. (2.3.59*a*) and (2.3.59*b*) into Eq. (2.3.58), we obtain

$$\delta\int_{t_1}^{t_2}[K - (V+U)]dt = 0 \tag{2.3.60}$$

where K and U are the kinetic and strain energies:

$$K = \int_V \frac{\rho}{2}\frac{\partial\mathbf{u}}{\partial t}\cdot\frac{\partial\mathbf{u}}{\partial t}\,dV,\quad U = \int_V U_0\,dV \tag{2.3.61}$$

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Equation (2.3.60) represents Hamilton's principle for an elastic body. Recall that the sum of the strain energy and potential energy of external forces, $U + V$, is called the total potential energy, Π, of the body. For bodies involving no motion (i.e., forces are applied sufficiently slowly such that the motion is independent of time and the inertia forces are negligible), Hamilton's principle (2.3.60) reduces to the principle of minimum total potential energy:

$$\delta(U + V) \equiv \delta\Pi = 0 \tag{2.3.62}$$

The Euler equations, known as the *Euler–Lagrange equations*, associated with the Lagrangian $L = K - \Pi$, can be obtained from Eq. (2.3.60):

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} L(\mathbf{u}, \nabla\mathbf{u}, \dot{\mathbf{u}})\, dt \\ &= \int_{t_1}^{t_2} \left[\int_V \left(\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div}\overleftrightarrow{\sigma} - \mathbf{f} \right) \cdot \delta\mathbf{u}\, dV + \int_{\Gamma_\sigma} (\mathbf{t} - \hat{\mathbf{t}}) \cdot \delta\mathbf{u}\, dS \right] dt \end{aligned} \tag{2.3.63}$$

where integration by parts, gradient theorems, and Eqs. (2.3.55*a*) and (2.3.55*b*) were used in arriving at Eq. (2.3.63) from Eq. (2.3.60). Because $\delta\mathbf{u}$ is arbitrary for t, $t_1 < t < t_2$, for $\mathbf{x}$ in V, and also on Γ_σ, it follows that

$$\begin{aligned} \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div}\overleftrightarrow{\sigma} - \mathbf{f} &= \mathbf{0} \quad \text{in } V \\ \mathbf{t} - \hat{\mathbf{t}} &= \mathbf{0} \quad \text{on } \Gamma_\sigma \end{aligned} \tag{2.3.64}$$

Equations (2.3.64) are the Euler-Lagrange equations for an elastic body [cf. Eq. (2.3.46)]. Next, we consider a specific example of application of Hamilton's principle [see Reddy (2002)].

Example 2.3.4

Consider the axial motion of an elastic bar of length L, area of cross section A, modulus of elasticity E, and mass density ρ, and subjected to distributed force f per unit length and an end load P. We wish to determine the equations of motion for the bar. The kinetic and total potential energies of the system are

$$K = \int_V \frac{\rho}{2} \left(\frac{\partial u}{\partial t} \right)^2 dV = \int_0^L \frac{\rho A}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx \tag{2.3.65a}$$

$$\begin{aligned} \Pi &= \int_V \frac{1}{2} \sigma_{ij} \varepsilon_{ij} dV - \int_0^L f u\, dx - P u(L) \\ &= \int_0^L \frac{A}{2} \sigma_{xx} \varepsilon_{xx}\, dx - \int_0^L f u\, dx - P u(L) \end{aligned} \tag{2.3.65b}$$

wherein u, σ_{xx}, and ε_{xx} are assumed to be functions of x only, and

$$\begin{aligned} u(0, t) &= 0 \quad \text{(bar is fixed at } x = 0) \\ \varepsilon_{xx} &= \frac{\partial u}{\partial x} \quad \text{(strain-displacement relation)} \end{aligned} \tag{2.3.65c}$$

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Substituting for K and Π from Eqs. (2.3.65a) and (2.3.65b) into Eq. (2.3.60), we obtain

$$
\begin{aligned}
0 &= \int_{t_1}^{t_2} \left\{ \int_0^L \left[A\rho \frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} - A\sigma_{xx}\delta\left(\frac{\partial u}{\partial x}\right) + f\delta u \right] dx + P\delta u(L) \right\} dt \\
&= \int_0^L \left[\int_{t_1}^{t_2} -\frac{\partial}{\partial t}\left(\rho A \frac{\partial u}{\partial t}\right)\delta u\, dt + \rho A \frac{\partial u}{\partial t}\delta u\Big|_{t_1}^{t_2} \right] dx \\
&\quad + \int_{t_1}^{t_2} \left\{ \int_0^L \left[\frac{\partial}{\partial x}(A\sigma_{xx}) + f \right]\delta u\, dx - (A\sigma_{xx}\delta u)\Big|_0^L + P\delta u(L) \right\} dt \\
&= -\int_{t_1}^{t_2} \left\{ \int_0^L \left[\frac{\partial}{\partial t}\left(\rho A \frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}(A\sigma_{xx}) - f \right]\delta u\, dx \right. \\
&\quad \left. - (A\sigma_{xx} - P)\big|_{x=L}\delta u(L) \right\} dt
\end{aligned}
\tag{2.3.66}
$$

where $\delta u(0,t) = 0$ and $\delta u(x,t_1) = \delta u(x,t_2) = 0$ are used to simplify the expression. The Euler-Lagrange equations are obtained by setting the coefficients of δu in $(0, L)$ and at $x = L$ to zero separately:

$$\frac{\partial}{\partial t}\left(\rho A \frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}(A\sigma_{xx}) - f = 0, \quad 0 < x < L \tag{2.3.67a}$$

$$(A\sigma_{xx})\big|_{x=L} - P = 0 \tag{2.3.67b}$$

for all $t, t_1 < t < t_2$. For linear elastic materials, we have $\sigma_{xx} = E\varepsilon_{xx} = E(\partial u/\partial x)$, and Eqs. (2.3.67$a$) and (2.3.67$b$) become

$$\frac{\partial}{\partial t}\left(\rho A \frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) - f = 0, \quad 0 < x < L \tag{2.3.68a}$$

$$\left(AE \frac{\partial u}{\partial x}\right)\Big|_{x=L} - P = 0 \tag{2.3.68b}$$

Now suppose that the bar also experiences a nonconservative (viscous damping) force proportional to the velocity,

$$F^* = -\mu \frac{\partial u}{\partial t} \tag{2.3.69}$$

where μ is the damping coefficient (a constant). Then the Euler-Lagrange equations from Eq. (2.3.58) are given by

$$\frac{\partial}{\partial t}\left(\rho A \frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(EA \frac{\partial}{\partial x}\right) - f + \mu \frac{\partial u}{\partial t} = 0, \quad 0 < x < L \tag{2.3.70a}$$

$$\left(AE \frac{\partial u}{\partial x}\right)_{x=L} - P = 0 \tag{2.3.70b}$$

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2.4 INTEGRAL FORMULATIONS

2.4.1 Introduction

Recall from Section 2.1.3 that the motivation for the use of weighted-integral statements of differential equations comes from the fact that we wish to have a means to determine the unknown parameters c_j in the approximate solution $U_N = \sum_j c_j \phi_j$. The variational methods of approximation, e.g., the Ritz, Galerkin, least-squares, collocation, or, in general, weighted-residual methods to be discussed in Section 2.5 are based on weighted-integral statements of the governing equations. Since the finite element method is a technique for constructing approximation functions required in an elementwise application of a variational method, it is necessary to study the weighted-integral formulation and the so-called weak formulation of differential equations. The weak formulations also facilitate, in a natural way, the classification of boundary conditions into natural and essential types. As we shall see shortly, this classification plays a crucial role in the derivation of the approximation functions and the selection of the nodal degrees of freedom of the finite element model.

In this section, our primary objective will be to construct the weak form of a given differential equation and to classify the boundary conditions associated with the equation. A *weak form* is defined to be a weighted-integral statement of a differential equation in which the differentiation is transferred from the dependent variable to the weight function such that all natural boundary conditions of the problem are also included in the integral statement. These ideas will be more clear in the sequel.

2.4.2 Weighted-Integral and Weak Formulations

Consider the problem of solving the differential equation

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x) \quad \text{for} \quad 0 < x < L \tag{2.4.1a}$$

for $u(x)$, subject to the boundary conditions

$$u(0) = u_0, \quad \left(a\frac{du}{dx}\right)\bigg|_{x=L} = Q_L \tag{2.4.1b}$$

Here $a(x)$ and $f(x)$ are known functions of the coordinate x; u_0 and Q_L are known values; and L is the size of the one-dimensional domain. When the specified values are nonzero ($u_0 \neq 0$ or $Q_L \neq 0$), the boundary conditions are said to be nonhomogeneous; when the specified values are zero, the boundary conditions are said to be homogeneous. The homogeneous form of the boundary condition $u(0) = u_0$ is $u(0) = 0$, and the homogeneous form of the boundary condition $(adu/dx)|_{x=L} = Q_L$ is $(adu/dx)|_{x=L} = 0$.

Equations of the type (2.4.1a) arise, for example, in the study of axial heat conduction in a rod (e.g., heat exchanger fin) or radial heat conduction in a long axisymmetric cylinder. In the former case, $a = kA$, with k being the thermal conductivity and the A the cross-sectional area, the L being the length of the rod. For the axisymmetric case, $a = 2\pi Lkx$, x being the radial coordinate r and L the length of the cylinder. In both cases, f denotes the heat generation term, u_0 is the specified temperature, and Q_L is the specified heat. Other physical problems are also described by the same equation but with different meanings for the variables. Typical examples of field problems with a description of the variables are presented in Table 2.4.1.

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Table 2.4.1 Some examples of engineering problems in which the second-order equation (2.4.1*a*) and its boundary conditions (2.4.1*b*) arise.

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)=f \quad \text{for } 0<x<L; \quad u(0)=u_0; \quad \left(a\frac{du}{dx}\right)_{x=L}=Q_L$$

Field	Primary variable u	Coefficient* a	Source term f	Secondary variable Q_0
1. Cables	Transverse deflection	T	Distributed vertical force	Axial force
2. Bars	Longitudinal displacement	EA	Distributed axial force	Axial load
3. Heat transfer	Temperature	k	Internal heat generation	Heat flux
4. Pipe flow	Hydrostatic pressure	$\frac{\pi D^4}{128\mu}$	Flow source	Flow rate
5. Viscous flows	Velocity	μ	Pressure gradient	Stress
6. Seepage	Fluid head	ε	Fluid flux	Flow
7. Electrostatics	Electrical potential	ϵ	Charge density	Electric flux

*E = Young's modulus; A = area of cross section; D = diameter of the pipe; k = thermal conductivity; μ = viscosity; T = tension; ε = permeability; and ϵ = dielectric constant.

Residual Function. Suppose that we seek an approximation of $u(x)$ in the form

$$u(x)\approx U_N(x)=\sum_{j=1}^{N}c_j\phi_j(x)+\phi_0(x) \tag{2.4.2}$$

and determine c_j such that $U_N(x)$ satisfies the differential equation (2.4.1*a*). Substitution of U_N into Eq. (2.4.1*a*) yields

$$-\frac{d}{dx}\left[a(x)\frac{dU_N}{dx}\right]=f(x) \quad \text{for} \quad 0<x<L \tag{2.4.3a}$$

Since the left side of the equality is now an approximate value, we cannot expect it to be equal, in general, to the right side of the equality. The difference

$$R(x,c_j)\equiv-\frac{d}{dx}\left[a(x)\frac{dU_N}{dx}\right]-f(x)\neq 0 \quad \text{for} \quad 0<x<L \tag{2.4.3b}$$

is called the *residual* of approximation in the differential equation. It is a function of x and c_j. Any approximate method (especially a variational method) seeks a set of N equations among c_j by making R equal to zero. Since it cannot be made zero identically at every point of the domain (as explained in Section 2.1.3), we must find an alternate way to find the necessary relations among c_j such that R is zero. If we require R to be zero at N selected points of the domain, we have

$$R(x,c_j)=0 \quad \text{for} \quad x=x_i, \; i=1,2,\cdots,N \tag{2.4.4}$$

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which is known as the *collocation method.* Another way to make R zero is to minimize the integral of the square of the residual (R is squared to make it positive) with respect to c_j:

$$\delta I \equiv \delta \int_0^L R^2\, dx = 0 \quad \text{or} \quad \frac{\partial}{\partial c_j} \int_0^L R^2\, dx = 0 \tag{2.4.5}$$

The method based on (2.4.5) is called the *least-squares method.* Equations (2.4.4) and (2.4.5), each, give N equations that can be solved for parameters c_j.

Weighted-Residual Method. Yet another way to determine the c_j is to require R to vanish in a "weighted-residual" sense:

$$\int_0^L w_i(x)\, R(x, c_j)\, dx = 0 \quad (i = 1, 2, \cdots, N) \tag{2.4.6}$$

where $w_i(x)$ are a set of linearly independent functions, called *weight functions,* which in general can be different from the approximation functions $\phi_i(x)$. This method is known as the *weighted-residual method.* Indeed, the statement in (2.4.6) includes (2.4.4) as well as (2.4.5) as special cases. When $w_i = \phi_i$, Eq. (2.4.6) is known as the *Galerkin method.* Thus, we have the following special cases of (2.4.6):

$$\begin{aligned}
&\textit{Petrov–Galerkin method:} && w_i = \psi_i \neq \phi_i \\
&\textit{Galerkin's method:} && w_i = \phi_i \\
&\textit{Least squares method:} && w_i = \frac{d}{dx}\left(a(x)\frac{d\phi_i}{dx}\right) \\
&\textit{Collocation method:} && w_i = \delta(x - x_i)
\end{aligned} \tag{2.4.7}$$

Here x_i is the ith collocation point of the domain of the problem and $\delta(\cdot)$ is the Dirac delta function defined such that its value is zero for all nonzero values of its arguments:

$$\delta(x - x_0) = 0 \text{ when } x \neq x_0, \qquad \int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = f(x_0) \tag{2.4.8}$$

Due to the different choices of w_i—even when the ϕ_i used in (2.4.4), (2.4.5) and (2.4.6) are the same—the system of algebraic equations will have different characteristics in different method. For linear differential equations of any order, only the least-squares method yields a system of matrix equations whose coefficient matrix is symmetric. One other method that has the symmetry property is the *Ritz method,* which uses the weak form of even-order (second, fourth, and so on; called self-adjoint) differential equations with $w_i = \phi_i$; the Ritz method is *not* a special case of the weighted-residual method. As we shall see shortly, the weak form of a self-adjoint differential equation always contains the same order derivatives of both the weight function w and the dependent unknown u, and the order is equal to half that of the original differential equation. In the following paragraphs, we discuss the weak form development.

Development of Weak Forms. There are three steps in the development of the weak form of any differential equation. These steps are illustrated by means of the model differential equation (2.4.1a) and boundary conditions (2.4.1b).

Step 1. (Weighted-integral statement.) This step is the same as in a weighted-residual method. Move all terms of the differential equation to one side (so that it reads $\ldots = 0$), multiply the entire equation with a function $w(x)$, and integrate over the domain $\Omega = (0, L)$

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of the problem:

$$0 = \int_0^L w \left[-\frac{d}{dx}(a\frac{du}{dx}) - f \right] dx \tag{2.4.9}$$

Recall that the expression in the square brackets is not identically zero since u is replaced by its approximation, U_N. Mathematically, in (2.4.9) the error in the differential equation (due to the approximation of the solution) is made zero in the weighted-integral sense. The integral statement (2.4.9) allows us to choose N linearly independent functions for w and obtain N equations for $c_1, c_2, \cdots, c_N$ of (2.4.2).

Note that the weighted-integral statement of any differential equation can be readily written. The weighted-integral statement is equivalent only to the differential equation and does not include any boundary conditions. The weight function w in (2.4.9) can be any nonzero integrable function and has no differentiability requirements.

Step 2. While the weighted-integral statement (2.4.9) allows us to obtain the necessary number (N) of algebraic relations among c_j for N different choices of the weight function w, it requires that the approximation functions ϕ_j be such that U_N [see (2.4.2)] is differentiable as many times as called for in the original differential equation (2.4.1a) and satisfies the specified boundary conditions. If this is not a concern, one can proceed with the integral statement (2.4.9) and obtain the necessary algebraic equations for c_j (using any one of the choices listed in Eq. (2.4.7) for $w \sim w_i$).

If we plan to use the approximation functions ϕ_i for $w \sim w_i$, it makes sense to shift half of the derivatives from u to w so that both are differentiated equally, and we have fewer (or weaker) continuity requirements on ϕ_j. The resulting integral form is known as the weak form. Of course, weakening the differentiability of u (and hence ϕ_i) is purely a mathematical (and perhaps computational) consideration. As will be seen shortly, the weak formulation has two desirable characteristics. First, it requires weaker, as already indicated, continuity of the dependent variable, and for self-adjoint equations (as is the case with problems studied in this book) it always results in a symmetric coefficient matrix. Second, the natural boundary conditions of the problem are included in the weak form, and therefore the approximate solution U_N is required to satisfy only the essential boundary conditions of the problem. These two features of a weak form play an important role in the development of finite element models of a problem.

A word of caution is in order. Differentiating the weight function instead of the dependent variable (in addition to the weakening the continuity requirements on ϕ_i) is also dictated by the need to include physically meaningful boundary terms into the weak form, regardless of the effect on the continuity requirements. Therefore, this trade-off should not be performed if it results in boundary terms that are not physically meaningful.

Returning to the integral statement (2.4.9), we integrate the first term of the expression by parts to obtain

$$\begin{aligned} 0 &= \int_0^L \left\{ w \left[-\frac{d}{dx}(a\frac{du}{dx}) \right] - w\, f \right\} dx \\ &= \int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} - w\, f \right) dx - \left[w\, a\frac{du}{dx} \right]_0^L \end{aligned} \tag{2.4.10}$$

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where the integration-by-parts formula [see Eq. (2.2.22)]

$$\int_0^L w\, dv = -\int_0^L v\, dw + [wv]_0^L$$

is used with $v = -a\,du/dx$ on the first term to arrive at the second line of (2.4.10). Note that now the weight function w is required to be differentiable at least once.

An important part of Step 2 is to identify the two types of boundary conditions associated with *any* differential equation: *natural* and *essential*. The classification is important for both the variational methods of approximation considered in this chapter and the finite element formulations presented in the subsequent chapters. The following rule is used to identify the natural boundary conditions and their form. After trading between differentiating the weight function w and the variable u of the problem, examine all boundary terms of the integral statement. The boundary terms will involve both the weight function and the dependent variable. Coefficients of the weight function (and possibly its derivatives for higher-order equations) in the boundary expression(s) are termed the *secondary variables* (SV). For example, for the problem at hand the coefficient of w in the boundary term is $a(du/dx)$, which is the secondary variable. Specification of a secondary variable on the boundary constitutes the *natural boundary condition* (NBC).

The dependent variable of the problem (u), expressed in the *same form* as the weight function (w) appearing in the boundary term, is called the *primary variable* (PV), and its specification on the boundary constitutes the *essential boundary condition* (EBC). For the case under consideration, the weight function appears in the boundary expression [see (2.4.10)] as w (in higher-order equations, it may appear as w in one boundary term and as dw/dx in other). Therefore, the primary variable is u (for higher-order equations, the primary variables may include u as well as du/dx), and the essential boundary condition involves specifying u at the boundary points.

The secondary variables always have physical meaning and are often quantities of interest. In the case of heat transfer problems, the secondary variable represents heat, Q. We shall denote the secondary variable by

$$Q \equiv \left(a\frac{du}{dx}\right) n_x \tag{2.4.11}$$

where n_x denotes the direction cosine, i.e., n_x = cosine of the angle between the positive x axis and the normal to the boundary. For one-dimensional problems, the normal at the boundary points is always along the length of the domain. Thus, we have $n_x = -1$ at the left end and $n_x = 1$ at the right end of the domain.

It should be noted that the number and form of the primary and secondary variables depend on the order of the differential equation. The number of primary and secondary variables is always the same, and with each primary variable there is an associated secondary variable, i.e., they always appear in pairs (e.g., displacement and force, temperature and heat, and so on). Only one of the pair, either the primary or the secondary variable, may be specified at a point of the boundary. Thus, a given problem can have its specified boundary conditions in one of three categories: (1) all specified boundary conditions are EBC; (2) some of the specified boundary conditions are EBC and the remaining are NBC; or (3) all specified boundary conditions are NBC. For a single second-order equation, as in the present case, there is one primary variable u and one secondary variable Q. At a boundary

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point, only one of the pair (u, Q) can be specified. For a fourth-order equation, such as that for the classical (i.e., Euler-Bernoulli) theory of beams, there are two of each kind (i.e., two PVs and two SVs), as will be illustrated later (see Example 2.4.2). In general, a $2m$th-order differential equation require m integration by parts to transfer m derivatives from u to w and therefore there will be m boundary terms involving m primary variables and m secondary variables, i.e., m pairs of primary and secondary variables.

Returning to Eq. (2.4.10), we rewrite it using the notation of (2.4.11):

$$\begin{aligned} 0 &= \int_0^L \left(a \frac{dw}{dx}\frac{du}{dx} - w f \right) dx - \left[w\, a \frac{du}{dx} \right]_0^L \\ &= \int_0^L \left(a \frac{dw}{dx}\frac{du}{dx} - w f \right) dx - \left(w\, a \frac{du}{dx} n_x \right)\bigg|_{x=0} - \left(w\, a \frac{du}{dx} n_x \right)\bigg|_{x=L} \\ &= \int_0^L \left(a \frac{dw}{dx}\frac{du}{dx} - w f \right) dx - (w\,Q)_0 - (w\,Q)_L \end{aligned} \tag{2.4.12}$$

Equation (2.4.12) is called the *weak form* of the differential equation (2.4.1a). The word "weak" refers to the reduced (i.e., weakened) continuity of u, which is required to be twice-differentiable in the weighted-integral statement (2.4.9) but only once-differentiable in (2.4.12).

Step 3. The third and last step of the weak formulation is to impose the actual boundary conditions of the problem under consideration. It is here that we require the weight function w to vanish at boundary points where the essential boundary conditions are specified, i.e., w is required to satisfy the *homogeneous form* of the specified essential boundary conditions of the problem (recall Section 2.3). In weak formulations, the weight function has the meaning of a *virtual* change (or variation) of the primary variable $w \sim \delta u$. If a primary variable is specified at a point, the virtual change there must be zero. For the problem at hand, the boundary conditions are given in (2.4.1b). By the rules of classification of the boundary conditions, $u = u_0$ is the essential boundary condition and $(a du/dx)|_{x=L} = Q_L$ is the natural boundary condition. Thus, the weight function w is required to satisfy $w(0) = 0$ because $u(0) = u_0$. Since $w(0) = 0$ and

$$Q(L) = \left(a \frac{du}{dx} n_x \right)\bigg|_{x=L} = \left(a \frac{du}{dx} \right)\bigg|_{x=L} = Q_L$$

Eq. (2.4.12) reduces to the expression

$$0 = \int_0^L \left(a \frac{dw}{dx}\frac{du}{dx} - w f \right) dx - w(L)\, Q_L \tag{2.4.13}$$

which is the weak form equivalent to the original differential equation (2.4.1a) and the natural boundary condition (2.4.1b). This completes the steps involved in the development of the weak form of a differential equation.

The terms "variational form" and "weak form" will be used interchangeably. The weak form of a differential equation is a weighted-integral statement equivalent to the differential equation *and* the specified natural boundary condition of the problem. Note that the weak form exists for all problems—linear or nonlinear—that are described by second- and higher-order differential equations. When the differential equation is linear and of even order, the

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resulting weak form will have a *symmetric* bilinear form in the dependent variable u and weight function w, as we shall see shortly.

In summary, there are three steps in the development of a weak form. In the first step, we put all expressions of the differential equation on one side (so that the other side is equal to zero), then multiply the entire equation by a weight function and integrate over the domain of the problem. The resulting expression is called the weighted-integral form of the equation. In the second step, we use integration by parts to distribute differentiation evenly between the dependent variable and the weight function, and use the boundary terms to identify the form of the primary and secondary variables. In the third step, we modify the boundary terms by restricting the weight function to satisfy the homogeneous form of the specified essential boundary conditions and replacing the secondary variables by their specified values.

It should be recalled that a weighted-integral statement or the weak form of a differential equation is needed to obtain as many algebraic equations as there are unknown coefficients in the approximation of the dependent variables of the equation. For different choices of the weight function, different algebraic equations can be obtained. Because of the restrictions placed on the weight function in Step 3 ($w \sim \delta u$) of the variational formulation, it must belong to the same space of functions as the approximation functions (i.e., $w \sim \phi_i$).

2.4.3 Linear and Bilinear Forms and Quadratic Functionals

It is informative, although not necessary for the use of variational methods or the finite element method, to see the relation between the weak form and the minimum of a quadratic functional associated with the differential equation. The weak form (2.4.13) contains two types of expressions: those involving both the dependent variable u and the weight function w, and those involving only the latter. We shall denote these two types of expressions by $B(w, u)$ and $l(w)$, respectively:

$$B(w, u) = \int_0^L a \frac{dw}{dx}\frac{du}{dx}\, dx, \quad l(w) = \int_0^L w\, f\, dx + w(L)\, Q_L \tag{2.4.14}$$

Hence, the weak form (2.4.13) can be expressed in the form

$$0 = B(w, u) - l(w) \quad \text{or} \quad B(w, u) = l(w) \tag{2.4.15}$$

which is termed the *variational problem* associated with Eqs. (2.4.1*a*) and (2.4.1*b*). Using the definitions of linear and bilinear forms from Section 2.2.4, it can be verified that $B(w, u)$ is bilinear and symmetric in w and u and that $l(w)$ is linear. The variational problem associated with (2.4.1*a*) and (2.4.1*b*) can be stated as one of finding the solution u (from a suitable vector space, H) such that

$$B(w, u) = l(w) \tag{2.4.16}$$

holds for any w (in H) that satisfies the homogeneous form of the specified essential boundary conditions and continuity conditions implied by the weak form. The function w can be viewed as a variation of the actual solution $w = \delta u$, and Eq. (2.4.15) can be written as

$$0 = B(\delta u, u) - l(\delta u) \tag{2.4.17}$$

If $B(\cdot, \cdot)$ is bilinear and symmetric and $l(\cdot)$ is linear, we have

$$B(\delta u, u) = \frac{1}{2}\delta[B(u, u)], \quad l(\delta u) = \delta[l(u)] \tag{2.4.18*a*}$$

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so that (2.4.17) can be expressed as

$$0 = B(\delta u, u) - l(\delta u) = \frac{1}{2}\delta[B(u, u)] - \delta[l(u)] \equiv \delta I(u) \tag{2.4.18b}$$

where

$$I(u) = \frac{1}{2}B(u, u) - l(u) \tag{2.4.19}$$

The result in (2.4.18a) can be verified for the problem at hand:

$$\begin{aligned} B(\delta u, u) &= \int_0^L a\frac{d\delta u}{dx}\frac{du}{dx}\,dx = \delta\int_0^L \frac{a}{2}\left(\frac{du}{dx}\right)^2 dx \\ &= \frac{1}{2}\delta\int_0^L a\frac{du}{dx}\frac{du}{dx}\,dx = \frac{1}{2}\delta[B(u, u)] \\ l(\delta u) &= \int_0^L \delta u\, f\, dx + \delta u(L)Q_L \\ &= \delta\left[\int_0^L u\, f\, dx + u(L)Q_L\right] = \delta[l(u)] \end{aligned}$$

Now we can restate the variational problem (2.4.16) as one of minimizing the functional $I(u)$:

$$\delta I = 0 = B(\delta u, u) - l(\delta u), \quad \delta^2 I = B(\delta u, \delta u) > 0 \text{ for } \delta u \neq 0 \tag{2.4.20}$$

Thus, the function u that minimizes $I(u)$ is the solution of (2.4.16); conversely, the solution of (2.4.16) minimizes the functional. Mathematical proof of these assertions can be found in Reddy (1986, 2002).

Note that the key step in the derivation of the functional $I(u)$ from the weak form is the bilinearity and symmetry of the bilinear form $B(w, u)$. Thus, whenever $B(w, u)$ is bilinear and symmetric, and $l(w)$ is linear, the functional associated with the variational problem (2.4.16) is given by (2.4.19). When $B(w, u)$ is not linear in w and u, but is symmetric, the functional $I(u)$ can be derived, but not from (2.4.19). The interested reader may consult the books by Oden and Reddy (1983) and Reddy (1986).

For solid mechanics problems, $I(u)$ represents the total potential energy functional, and $\delta I = 0$ is the statement of the principle of the minimum total potential energy: Of all admissible functions u, the one that makes the total potential energy $I(u)$ a minimum also satisfies the differential equation(s) and natural boundary condition(s). In other words, the weak form of a differential equation is the same as the statement of the principle of minimum total potential energy. For problems outside solid mechanics, the functional $I(u)$, if it exists, may not have any physical meaning, but it is still useful for mathematical analysis (e.g., in proving the existence and uniqueness of solutions).

As noted earlier, every differential equation admits a weighted-integral statement, and a weak form exists provided the equation is of order two or higher. When the bilinear form is symmetric, we will also have a functional whose first variation set equal to zero is equivalent to the governing equations. Recall that we can always construct the least-squares functional associated with any set of governing equations. However, the traditional variational methods and the finite element method use only an integral statement or a weak form of the equation(s) to be solved.

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2.4.4 Examples

Here, we consider some representative examples of differential equations in one and two dimensions, and develop their weak forms. These examples are of primary interest in the study of the finite element method in the coming chapters. All problems considered here correspond to one or more physical problems of science and engineering.

Example 2.4.1 (Model Second-Order Equation in One Dimension)

Consider the differential equation

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu=f \quad \text{for} \quad 0<x<L \tag{2.4.21a}$$

subject to the boundary conditions

$$u(0)=u_0, \quad \left(a\frac{du}{dx}\right)+\beta(u-u_\infty)=Q_0 \ \text{ at } x=L \tag{2.4.21b}$$

The data of the problem consists of specifying $a(x)$, $c(x)$, $f(x)$, u_0, β, u_∞, and Q_0.

Following the three steps outlined above for the construction of variational statements, we obtain

$$\textit{Step 1:} \qquad 0=\int_0^L w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f\right]dx$$

$$\textit{Step 2:} \qquad 0=\int_0^L\left(a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right)-\left[w\,a\frac{du}{dx}\right]_0^L \tag{2.4.22}$$

From the boundary term, it is clear that the specification of u is an essential boundary condition, and the specification of $a\,du/dx$ is a natural boundary condition. Since $w=0$ at $x=0$ (because u is specified there) and

$$a\frac{du}{dx}=Q_0-\beta(u-u_\infty) \ \text{ at } x=L$$

we obtain

$$\textit{Step 3:} \qquad 0=\int_0^L\left(a\frac{dw}{dx}\frac{du}{dx}+cwu\right)dx-\int_0^L wf\,dx+w(L)\beta(u(L)-u_\infty)-w(L)Q_0 \tag{2.4.23a}$$

Variational Problem and Quadratic Functional: Equation (2.4.23*a*) can be expressed in the form

$$B(w,u)=l(w) \tag{2.4.23b}$$

where

$$\begin{aligned} B(w,u)&=\int_0^L\left(a\frac{dw}{dx}\frac{du}{dx}+cwu\right)dx+\beta w(L)u(L)\\ l(w)&=\int_0^L wf\,dx+\beta w(L)u_\infty+Q_0u(L)\end{aligned} \tag{2.4.23c}$$

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It is clear that since $B(w, u)$ is linear and symmetric in w and u, and $l(w)$ is linear in w, we can compute the quadratic functional from (2.4.19):

$$I(u) = \frac{1}{2}\int_0^L \left[a\left(\frac{du}{dx}\right)^2 + cu^2\right]dx + \frac{1}{2}\beta[u(L)]^2 - \int_0^L fu\,dx - \beta u(L)u_\infty - Q_0 u(L) \tag{2.4.24}$$

Equations of the type of (2.4.21a) arise, for example, in the study of the axial deformation of a bar or flow of heat in a bar. In the former case, u denotes the axial displacement, a the product of modulus of elasticity (E) and area of cross section (A), and c the resistance offered by an external medium to the axial deformation (often, $c = 0$). In the case of heat flow, u denotes the temperature (T), a the product of thermal conductivity (k) times the area of cross section (A) of the bar, and c is the product of film conductance (β) times the perimeter (P).

The next example illustrates the variational formulation of a fourth-order differential equation governing bending of elastic beams according to the Euler-Bernoulli beam theory [see Reddy (2002)].

Example 2.4.2 (Euler-Bernoulli Beam)

Consider the problem of finding the function $w(x)$ that satisfies the differential equation

$$\frac{d^2}{dx^2}\left[b(x)\frac{d^2w}{dx^2}\right] + c_f w - q(x) = 0 \quad \text{for} \quad 0 < x < L \tag{2.4.25}$$

This equation arises in the study of the elastic bending of beams (under the Euler-Bernoulli hypothesis that plane sections perpendicular to the axis of the beam before deformation remain plane after deformation), where w denotes the transverse deflection of the beam, L is the length of the beam, $b(x) > 0$ is the flexural rigidity of the beam (i.e., the product of modulus of elasticity E and second moment of inertia I), c_f is the foundation modulus, and $q(x)$ is the transverse distributed load, as shown in Fig. 2.4.1. At the moment, we do not have to consider any specific boundary conditions of the problem.

Figure 2.4.1 Cantilever beam with a set of loads.

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Since the equation contains a fourth-order derivative, we should integrate it twice by parts to distribute the derivatives equally between the dependent variable w and the weight function v. In this case, v must be twice differentiable and satisfy the homogeneous form of an EBC. Multiplying (2.4.25) by v, and integrating the first term by parts twice with respect to x, we obtain

$$\textit{Step 1:} \quad 0 = \int_0^L v\left[\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) + c_f w - q\right] dx \tag{2.4.26}$$

$$\begin{aligned}\textit{Step 2:} \quad 0 &= \int_0^L \left[\left(-\frac{dv}{dx}\right)\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right) + c_f vw - vq\right] dx + \left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_0^L \\ &= \int_0^L \left(\frac{d^2v}{dx^2}b\frac{d^2w}{dx^2} + c_f vw - vq\right) dx + \left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right) - \frac{dv}{dx}b\frac{d^2w}{dx^2}\right]_0^L\end{aligned} \tag{2.4.27}$$

From the last line, it follows that the specification of w and dw/dx constitutes the essential (geometric or static) boundary conditions, and the specification of

$$\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right) \text{ (shear force)} \quad \text{and} \quad b\left(\frac{d^2w}{dx^2}\right) \text{ (bending moment)} \tag{2.4.28a}$$

constitutes the natural boundary conditions for the Euler-Bernoulli beam theory.

Now consider a specific beam problem. Let us consider a beam fixed at the left end and subjected to transverse force and bending moment at $x = L$, as shown in Fig. 2.4.1:

$$w(0) = 0, \quad \left(\frac{dw}{dx}\right)\bigg|_{x=0} = 0, \quad \left(b\frac{d^2w}{dx^2}\right)\bigg|_{x=L} = -M_0, \quad \left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x=L} = -F_0 \tag{2.4.28b}$$

where M_0 is the bending moment and F_0 is the transverse load. Since w and dw/dx (both are primary variables) are specified at $x = 0$, we require the weight function v and its derivative dv/dx to be zero there

$$v(0) = \left(\frac{dv}{dx}\right)\bigg|_{x=0} = 0$$

The remaining two boundary conditions in (2.4.28b) are natural boundary conditions, which place no restrictions on v and its derivatives. Thus, Eq. (2.4.27) becomes

$$\textit{Step 3:} \quad 0 = \int_0^L \left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f vw - vq\right) dx - v(L)F_0 + \left(\frac{dv}{dx}\right)\bigg|_{x=L} M_0 \tag{2.4.29}$$

Variational Problem and Quadratic Functional: Equation (2.4.30a) can be written in the form

$$B(v, w) = l(v) \tag{2.4.30a}$$

where

$$\begin{aligned} B(v, w) &= \int_0^L \left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f vw\right) dx \\ l(v) &= \int_0^L vq\, dx + v(L)F_0 - \left(\frac{dv}{dx}\right)\bigg|_{x=L} M_0 \end{aligned} \tag{2.4.30b}$$

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The functional, known as the total potential energy of the beam, is obtained using (2.4.19):

$$I(w) = \int_0^L \left[\frac{b}{2}\left(\frac{d^2w}{dx^2}\right)^2 + \frac{c_f}{2}w^2 - wq \right] dx - w(L)F_0 + \left(\frac{dw}{dx}\right)\bigg|_{x=L} M_0 \tag{2.4.31}$$

Note that for the fourth-order equation, the essential boundary conditions involve not only the dependent variable but also its first derivative. As pointed out earlier, at any boundary point, only one of the two boundary conditions (essential or natural) can be specified. For example, if the transverse deflection is specified at a boundary point, then one cannot specify the shear force V at the same point, and vice versa. Similar comments apply to the slope dw/dx and the bending moment M. Note that in the present case, w and dw/dx are the primary variables, and V and M are the secondary variables.

The next example is concerned with the weak-form development of a pair of second-order differential equations in one dimension. The approach used for a single equation is followed for each equation of the pair. However, writing the variational problem and the associated functional is a bit tricky.

Example 2.4.3 (Timoshenko Beam)

Consider the following pair of coupled differential equations governing bending of the Timoshenko beam:

$$-\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right] + c_f w = q \tag{2.4.32a}$$

$$-\frac{d}{dx}\left(D\frac{d\phi_x}{dx}\right) + S\left(\frac{dw}{dx}+\phi_x\right) = 0 \tag{2.4.32b}$$

where S is the shear stiffness ($S = K_s GA$; K_s is the shear correction coefficient, G is the shear modulus, and A is the area of the cross section), $D = EI$ is the bending stiffness, w is the transverse deflection, ϕ_x is the rotation, k is the foundation modulus, and q is the distributed transverse load. We shall develop the weak form of the above equations using the three-step procedure.

Steps 1 and 2: Multiply the first equation with weight function v_1 and the second one with weight function v_2 and integrate over the length of the beam:

$$\textit{Step 1a:}\quad 0 = \int_0^L v_1 \left\{ -\frac{d}{dx}\left[S\left(\frac{dw}{dx}+\phi_x\right)\right] + c_f w - q \right\} dx$$

$$\begin{aligned}\textit{Step 2a:}\quad 0 &= \int_0^L \left[\frac{dv_1}{dx} S\left(\frac{dw}{dx}+\phi_x\right) + c_f v_1 w - v_1 q\right] dx - \left[v_1 S\left(\frac{dw}{dx}+\phi_x\right)\right]_0^L \\ &= \int_0^L \left[\frac{dv_1}{dx} S\left(\frac{dw}{dx}+\phi_x\right) + c_f v_1 w - v_1 q\right] dx - v_1(L)\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]_{x=L} \\ &\quad + v_1(0)\left[S\left(\frac{dw}{dx}+\phi_x\right)\right]_{x=0}\end{aligned} \tag{2.4.33a}$$

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$$\textit{Step 1b:} \quad 0 = \int_0^L v_2 \left[-\frac{d}{dx}\left(D\frac{d\phi_x}{dx} \right) + S\left(\frac{dw}{dx} + \phi_x \right) \right] dx$$

$$\textit{Step 2b:} \quad 0 = \int_0^L \left[D\frac{dv_2}{dx}\frac{d\phi_x}{dx} + v_2 S\left(\frac{dw}{dx} + \phi_x \right) \right] dx - \left[v_2 D\frac{d\phi_x}{dx} \right]_0^L$$

$$= \int_0^L \left[D\frac{dv_2}{dx}\frac{d\phi_x}{dx} + v_2 S\left(\frac{dw}{dx} + \phi_x \right) \right] dx$$

$$- v_2(L)\left[D\frac{d\phi_x}{dx} \right]_{x=L} + v_2(0)\left[D\frac{d\phi_x}{dx} \right]_{x=0} \tag{2.4.33b}$$

Note that integration-by-parts was used such that the expression $w_{,x} + \phi_x$ is preserved, as it enters the boundary term representing the shear force. Such considerations can only be used by knowing the mechanics of the problem at hand. Also, note that the pair of weight functions (v_1, v_2) satisfy the homogeneous form of specified essential boundary conditions on the pair (w, ϕ_x) (with the correspondence $v_1 \sim w$ and $v_2 \sim \phi_x$).

Steps 3: An examination of the boundary terms shows that $w \sim v_1$ and $\phi_x \sim v_2$ are the primary variables, and the secondary variables are given by [cf. Eq. (2.4.28*a*)]

$$S\left(\frac{dw}{dx} + \phi_x \right) \quad \text{(shear force)} \tag{2.4.34a}$$

$$D\frac{d\phi_x}{dx} \quad \text{(bending moment)} \tag{2.4.34b}$$

To finalize the weak forms, we must take care of the boundary terms by considering a specific beam problem. Using the beam of Fig. 2.4.1, we see that

$$w(0) = 0, \quad \phi_x(0) = 0, \quad \left[S\left(\frac{dw}{dx} + \phi_x \right) \right]_{x=L} = F_0, \quad \left[D\frac{d\phi_x}{dx} \right]_{x=L} = M_0 \tag{2.4.35}$$

and hence, $v_1(0) = 0$ and $v_2(0) = 0$. Consequently, the weak forms (2.4.33*a*) and (2.4.33*b*) become

$$0 = \int_0^L \left[\frac{dv_1}{dx} S\left(\frac{dw}{dx} + \phi_x \right) + c_f v_1 w - v_1 q \right] dx - v_1(L) F_0 \tag{2.4.36a}$$

$$0 = \int_0^L \left[\frac{dv_2}{dx} D\frac{d\phi_x}{dx} + v_2 S\left(\frac{dw}{dx} + \phi_x \right) \right] dx - v_2(L) M_0 \tag{2.4.36b}$$

Variational Problem and Quadratic Functional: To write the variational problem of finding (w, ϕ_x) such that

$$B((v_1, v_2), (w, \phi_x)) = l((v_1, v_2)) \tag{2.4.37}$$

holds for all (v_1, v_2), we must combine the two weak forms into a single expression

$$0 = \int_0^L \left[\left(\frac{dv_1}{dx} + v_2 \right) S\left(\frac{dw}{dx} + \phi_x \right) + \frac{dv_2}{dx} D\frac{d\phi_x}{dx} + c_f v_1 w - v_1 q \right] dx - v_1(L) F_0 - v_2(L) M_0 \tag{2.4.38}$$

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Thus, the bilinear and linear forms of the problem are given by

$$B((v_1, v_2), (w, \phi_x)) = \int_0^L \left[S\left(\frac{dv_1}{dx} + v_2\right)\left(\frac{dw}{dx} + \phi_x\right) + D\frac{dv_2}{dx}\frac{d\phi_x}{dx} + c_f v_1 w \right] dx \quad (2.4.39a)$$

$$l((v_1, v_2)) = \int_0^L v_1 q\, dx + v_1(L) F_0 + v_2(L) M_0 \quad (2.4.39b)$$

Clearly, $B((v_1, v_2), (w, \phi_x))$ is symmetric in its arguments (i.e. interchange of v_1 with w *and* v_2 with ϕ_x yields the same expression). Hence the functional is given by

$$I((w, \phi_x)) = \frac{1}{2}\int_0^L \left[S\left(\frac{dw}{dx} + \phi_x\right)^2 + D\left(\frac{d\phi_x}{dx}\right)^2 + \frac{c_f}{2} w^2 \right] dx - \left(\int_0^L w q\, dx + w(L) F_0 + \phi_x(L) M_0 \right) \quad (2.4.40)$$

The last example of this section is concerned with a second-order differential equation in two dimensions. The equation arises in a number of fields, including heat transfer, stream function or velocity potential formulation of inviscid flows, transverse deflections of a membrane, torsion of a cylindrical member, and others.

Example 2.4.4 (Poisson's Equation in Two Dimensions)

Consider the problem of determining the solution $u(x, y)$ to the partial differential equation,

$$-\frac{\partial}{\partial x}\left(a_1 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_2 \frac{\partial u}{\partial y}\right) + a_0 u = f \quad \text{in } \Omega \quad (2.4.41)$$

in a closed two-dimensional domain Ω with boundary Γ, as shown in Fig. 2.4.2. Here a_0, a_1, a_2, and f are known functions of position (x, y) in Ω. The function u is required to satisfy, in addition to the differential equation (2.4.41), certain boundary conditions on the boundary Γ of Ω. The weak formulation to be discussed next will reveal the precise form of the essential and natural boundary conditions of the equation.

Figure 2.4.2 A two-dimensional domain Ω with boundary Γ.

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The three step procedure applied to Eq. (2.4.41) results in the following equations:

$$\textit{Step 1:} \quad 0 = \int_\Omega w\left[-\frac{\partial}{\partial x}\left(a_1\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a_2\frac{\partial u}{\partial y}\right) + a_0 u - f\right] dxdy \tag{2.4.42}$$

$$\textit{Step 2:} \quad 0 = \int_\Omega \left(a_1\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2\frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 wu - wf\right) dxdy - \oint_\Gamma w\left(a_1\frac{\partial u}{\partial x}n_x + a_2\frac{\partial u}{\partial y}n_y\right) ds \tag{2.4.43}$$

where we used integration by parts [or the gradient and divergence theorems, Eqs. (2.2.27*b*) and (2.2.28*b*)] to transfer differentiation from u to w so that both u and w have the same order derivatives in Ω. The boundary term shows that u is the primary variable while

$$a_1\frac{\partial u}{\partial x}n_x + a_2\frac{\partial u}{\partial y}n_y$$

is the secondary variable.

Step 3: The last step in the procedure is to impose the specified boundary conditions. Suppose that u is specified on portion Γ_1 and the natural boundary condition is specified on the remaining portion Γ_2 of the boundary as (see Fig. 2.4.2):

$$u = \hat{u} \text{ on } \Gamma_1, \quad a_1\frac{\partial u}{\partial x}n_x + a_2\frac{\partial u}{\partial y}n_y = \hat{g} \text{ on } \Gamma_2 \tag{2.4.44}$$

Then w is arbitrary on Γ_2 and equal to zero on Γ_1. Consequently, Eq. (2.4.43) simplifies to

$$0 = \int_\Omega \left(a_1\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2\frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 wu - wf\right) dxdy - \int_{\Gamma_2} w\hat{g}\, ds \tag{2.4.45}$$

Variational Problem and Quadratic Functional: The weak statement (2.4.45) can be expressed as $B(w, u) = \ell(w)$, where the bilinear form and linear form are

$$B(w, u) = \int_\Omega \left(a_1\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + a_2\frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + a_0 wu\right) dxdy, \tag{2.4.46a}$$

$$\ell(w) = \int_\Omega wf\, dxdy + \int_{\Gamma_2} w\hat{g}\, ds, \tag{2.4.46b}$$

The associated quadratic functional is

$$I(u) = \frac{1}{2}\int_\Omega \left[a_1\left(\frac{\partial u}{\partial x}\right)^2 + a_2\left(\frac{\partial u}{\partial y}\right)^2 + a_0 u^2\right] dxdy - \int_\Omega uf\, dxdy - \int_{\Gamma_2} \hat{g}u\, ds. \tag{2.4.46c}$$

In the case of transverse deflections of a membrane, $I(u)$ represents the total potential energy.

As a specific example of the Poisson equation, consider steady heat conduction in a two-dimensional domain Ω, enclosed by lines AB, BC, CD, DE, EF, FG, GH, and HA, as indicated

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Figure 2.4.3 A two-dimensional heat transfer problem with a variety of boundary conditions.

in Fig. 2.4.3. The governing equation is

$$-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = g_0 \quad \text{in} \quad \Omega \tag{2.4.47}$$

where g_0 is the uniform heat generation, k is the conductivity of the isotropic material of the domain, and T is the temperature. At the second step of the weak-form development, we have

$$0 = \int_\Omega \left[k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right) - wg_0\right] dx\,dy - \oint_\Gamma wk\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial y}n_y\right) ds \tag{2.4.48}$$

In order to simplify the boundary expression, we first recognize the fact that the boundary Γ of the domain consists of several line segments, and they are subject to different types of boundary conditions as shown in Fig. 2.4.3:

$$\begin{aligned}
&\Gamma_1 = \text{AB}: n_x = -1,\ n_y = 0; && \text{heat flux is specified } q = \hat{q}(y)\\
&\Gamma_2 = \text{BC}: n_x = 0,\ \ n_y = -1; && \text{temperature is specified } T = T_0(x)\\
&\Gamma_3 = \text{CD}: n_x = 1,\ \ n_y = 0; && \text{boundary exposed to ambient temperature } T_\infty\\
& && \text{(convection) } k\frac{\partial T}{\partial n} + \beta(T - T_\infty) = 0 \qquad (2.4.49)\\
&\Gamma_4 = \text{DEFGHA}: && \text{insulated boundary } \frac{\partial T}{\partial n} = 0
\end{aligned}$$

Using the boundary information, the boundary integral in (2.4.48) can be simplified as follows (note that $w = 0$ on portion of the boundary wherever T is specified, Γ_2):

$$\begin{aligned}
\oint_\Gamma w\left(k\frac{\partial T}{\partial n}\right) ds &= \int_{\Gamma_1} wq_n\,ds + \int_{\Gamma_2} 0\left(k\frac{\partial T}{\partial n}\right) ds\\
&\quad - \int_{\Gamma_3} w[\beta(T - T_\infty)]\,ds + \int_{\Gamma_4} w\,0\,ds\\
&= -\int_0^b w(0, y)\hat{q}(y)\,dy - \beta\int_0^b w(a, y)[T(a, y) - T_\infty]\,dy
\end{aligned} \tag{2.4.50}$$

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Substituting (2.4.50) into (2.4.48), we obtain the weak form

$$0 = \int_\Omega \left[k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y} \right) - w g_0 \right] dx\, dy + \int_0^b w(0, y)\hat{q}(y) dy$$
$$+ \beta \int_0^b w(a, y)[T(a, y) - T_\infty]\, dy \tag{2.4.51}$$

Collecting terms involving both w and T into $B(\cdot, \cdot)$, and those involving only w into $l(\cdot)$, we can write (2.4.51) as

$$B(w, T) = l(w) \tag{2.4.52a}$$

where

$$B(w, T) = \int_\Omega k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial T}{\partial y} \right) dx\, dy + \beta \int_0^b w(a, y) T(a, y) dy$$
$$l(w) = \int_\Omega w g_0\, dx\, dy - \int_0^b w(0, y)\hat{q}(y)\, dy + \beta \int_0^b w(a, y) T_\infty\, dy \tag{2.4.52b}$$

The quadratic functional is given by

$$I(T) = \frac{k}{2}\int_\Omega \left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 \right] dx\, dy + \frac{\beta}{2}\int_0^b T^2(a, y)\, dy$$
$$- \int_\Omega T g_0\, dx\, dy + \int_0^b T(0, y)\hat{q}(y)\, dy - \beta \int_0^b T(a, y) T_\infty\, dy \tag{2.4.53}$$

Note that the boundary integrals in this example are defined along the y or x axes, because the boundaries are parallel to either the x- or y-axis.

2.5 VARIATIONAL METHODS

2.5.1 Introduction

Our objective in this section is to study the variational methods of approximation because they provide a background for the development of finite element models. The methods to be discussed include the Ritz and weighted-residual (e.g., Galerkin, Petrov–Galerkin, least-squares, and collocation) methods. In all these methods, we seek an approximate solution in the form of a linear combination of suitable approximation functions ϕ_j and undetermined parameters c_j: $\sum_j c_j \phi_j$. The Ritz method uses the weak form, whereas the weighted-residual methods use the weighted-integral form (Step 1 of the weak-form development) to determine the parameters c_j. Various methods differ from each other in the choice of weight function w and approximation functions ϕ_j. As we shall see in the coming chapters, the finite element method is a elementwise application of the classical variational methods.

2.5.2 The Ritz Method

In the Ritz method, the coefficients c_j of the approximation are determined using the weak form of the problem, and hence, the choice of weight functions is restricted to the

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approximation functions, $w = \phi_j$ $(w \sim u)$. Recall that the weak form contains both the governing differential equation and the natural boundary conditions of the problem, and hence it places weaker continuity requirements on the approximate solution than the original differential equation or its weighted-integral form. The method is described here for a linear variational problem (which is the same as the weak form).

Consider the variational problem resulting from the weak form: Find the solution u such that

$$B(w, u) = l(w) \tag{2.5.1}$$

for all sufficiently differentiable functions w that satisfy the homogeneous form of specified essential boundary conditions on u. In general, $B(\cdot, \cdot)$ can be unsymmetric in w and u, and it can be even nonlinear in u [$B(\cdot, \cdot)$ is always linear in w]. When $B(\cdot, \cdot)$ is bilinear and symmetric in w and u and l is linear, the problem in (2.5.1) is equivalent to minimization of the quadratic functional [see Eq. (2.4.19)]

$$I(u) = \frac{1}{2} B(u, u) - l(u) \tag{2.5.2}$$

In the Ritz method, we seek an approximate solution to (2.5.1) in the form of a finite series [see Eq. (2.4.2)]

$$U_N(x) = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x) \tag{2.5.3}$$

where the constants c_j, called the *Ritz coefficients*, are determined such that (2.5.1) holds for each $w = \phi_i$ $(i = 1, 2, \ldots, N)$, i.e., (2.5.1) holds for N different choices of w, so that N independent algebraic relations among c_j are obtained. The functions ϕ_j and ϕ_0, called *approximation functions*, are chosen such that U_N satisfies the specified essential boundary conditions [recall that the specified natural boundary conditions are already included in the variational problem (2.5.1) and hence in the functional $I(u)$]. More details on this will be discussed shortly. The ith algebraic equation is obtained from Eq. (2.5.1) by substituting $w = \phi_i$ and U_N from (2.5.2) for u:

$$B\left(\phi_i, \sum_{j=1}^{N} c_j \phi_j + \phi_0\right) = l(\phi_i) \quad (i = 1, 2, \cdots, N)$$

Since $B(\cdot, \cdot)$ is linear in u, we have

$$\sum_{j=1}^{N} B(\phi_i, \phi_j) c_j = l(\phi_i) - B(\phi_i, \phi_0)$$

or

$$\sum_{j=1}^{N} K_{ij} c_j = F_i, \quad i = 1, 2, \ldots, N \tag{2.5.4a}$$

$$K_{ij} = B(\phi_i, \phi_j), \quad F_i = l(\phi_i) - B(\phi_i, \phi_0) \tag{2.5.4b}$$

The algebraic equations in (2.5.4a) can be expressed in matrix form as

$$[K]\{c\} = \{F\} \quad \text{or} \quad \mathbf{Kc} = \mathbf{F} \tag{2.5.5}$$

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As stated earlier, for symmetric bilinear forms the Ritz method can also be viewed as seeking a solution of the form in (2.5.3) in which the parameters c_j are determined by minimizing the quadratic functional $I(u)$ in (2.5.2). After substituting U_N from (2.5.3) for u into (2.5.2) and integrating, the functional $I(u)$ becomes an ordinary function of the parameters $c_1, c_2, \ldots c_N$. Then the necessary condition for the minimization of $I(c_1, c_2, \ldots, c_N)$ is that its partial derivatives with respect to each of the parameters be zero:

$$\frac{\partial I}{\partial c_1} = 0, \quad \frac{\partial I}{\partial c_2} = 0, \quad \cdots, \quad \frac{\partial I}{\partial c_N} = 0 \tag{2.5.6}$$

Thus, there are N linear algebraic equations in N unknowns, c_j $(j = 1, 2, \cdots, N)$. These equations are exactly the same as those in (2.5.4) for all problems for which the variational problem (2.5.1) is equivalent to $\delta I = 0$. Of course, when $B(\cdot, \cdot)$ is not symmetric, we do not have a quadratic functional. In other words, (2.5.4a) is more general than (2.5.6), and they are the same when $B(\cdot, \cdot)$ is bilinear and symmetric. In all problems of interest in the present study, we shall have a symmetric bilinear form.

2.5.3 Approximation Functions

Returning to the approximation U_N in (2.5.3), we wish to discuss the selection of the approximation functions ϕ_j and ϕ_0 for the Ritz method. First, we note that U_N must satisfy only the specified essential boundary conditions of the problem, since the specified natural boundary conditions are included in the variational problem (2.5.1). The particular form of U_N in (2.5.3) facilitates satisfaction of specified boundary conditions. To see this, suppose that the approximate solution is sought in the form

$$U_N(x) = \sum_{j=1}^{N} c_j \phi_j(x)$$

and suppose that the specified essential boundary condition is $u(x_0) = u_0$. Then U_N must also satisfy the condition $U_N(x_0) = u_0$ at a boundary point $x = x_0$:

$$\sum_{j=1}^{N} c_j \phi_j(x_0) = u_0$$

Since c_j are unknown parameters to be determined, it is not easy to choose $\phi_j(x)$ such that the above relation holds. If $u_0 = 0$, then we can select all ϕ_j such that $\phi_j(x_0) = 0$ and satisfy the condition $U_N(x_0) = 0$. By writing the approximate solution U_N in the form (2.5.3), a sum of a homogeneous part $\sum c_j \phi_j(x)$ and a nonhomogeneous part $\phi_0(x)$, we require $\phi_0(x)$ to satisfy the specified essential boundary conditions while the homogeneous part vanishes at the same boundary point where the essential boundary condition is specified. This follows from

$$U_N(x_0) = \sum_{j=1}^{N} c_j \phi_j(x_0) + \phi_0(x_0)$$

$$u_0 = \sum_{j=1}^{N} c_j \phi_j(x_0) + u_0 \rightarrow \sum_{j=1}^{N} c_j \phi_j(x_0) = 0$$

which is satisfied, for arbitrary c_j, by choosing $\phi_j(x_0) = 0$.

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If all specified essential boundary conditions are homogeneous (i.e., the specified value u_0 is zero), then ϕ_0 is taken to be zero and ϕ_j must still satisfy the same conditions, $\phi_j(x_0) = 0$, $j = 1, 2, \ldots, N$. Note that the requirement that w be zero at the boundary points where the essential boundary conditions are specified is satisfied by the choice $w = \phi_j(x)$.

In summary, the approximation functions $\phi_i(x)$ and $\phi_0(x)$ are required to satisfy the following conditions:

1. (a) ϕ_i must be such that $B(\phi_i, \phi_j)$ is defined and nonzero, i.e., ϕ_i are sufficiently differentiable and integrable as required in the evaluation of $B(\phi_i, \phi_j)$.
   (b) ϕ_i must satisfy the homogeneous form of the specified essential boundary conditions of the problem.
2. For any N, the set $\{\phi_i\}_{i=1}^N$ along with the columns (and rows) of $B(\phi_i, \phi_j)$ must be *linearly independent*.
3. The set $\{\phi_i\}$ must be *complete*. For example, when ϕ_i are algebraic polynomials, completeness requires that the set $\{\phi_i\}$ contains all terms of the lowest order admissible, up to the highest order desired.
4. The only requirement on ϕ_0 is that it satisfy the specified essential boundary conditions. When the specified essential boundary conditions are zero, then ϕ_0 is identically zero. Also, for completeness reasons, ϕ_0 must be the lowest-order function that satisfies the specified essential boundary conditions.

2.5.4 Examples

Here, we consider a few examples of the application of the Ritz method to equilibrium, eigenvalue, and time-dependent problems.

Example 2.5.1

Consider the differential equation (see Example 2.4.1; set $a = 1$, $c = -1$, $L = 1$, and $f = -x^2$)

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{for} \quad 0 < x < 1 \tag{2.5.7}$$

We consider two sets of boundary conditions:

$$\textit{Set 1:}\ \ u(0) = 0, \quad u(1) = 0 \tag{2.5.8}$$

$$\textit{Set 2:}\ \ u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=1} = 1 \tag{2.5.9}$$

Boundary Conditions Set 1. The bilinear form and the linear functional associated with Eqs. (2.5.7) and (2.5.8) are [see (2.4.23c)]

$$B(w, u) = \int_0^1 \left(\frac{dw}{dx}\frac{du}{dx} - wu\right) dx, \quad l(w) = -\int_0^1 w\, x^2\, dx \tag{2.5.10}$$

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and K_{ij} and F_i are given by

$$K_{ij} = B(\phi_i, \phi_j) = \int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i \phi_j \right) dx, \quad l(\phi_i) = -\int_0^1 \phi_i x^2 \, dx \tag{2.5.11}$$

The same result as in (2.5.11) can be obtained by minimizing the quadratic functional [set $a = 1, c = -1, f = -x^2, L = 1$, and $u(1) = 0$ in Eq. (2.4.24)]

$$I(u) = \frac{1}{2}\int_0^1 \left[\left(\frac{du}{dx}\right)^2 - u^2 + 2x^2 u \right] dx$$

Substituting for $u \approx U_N$ from (2.5.3), with $\phi_0 = 0$, into the above functional, we obtain

$$I(c_1, c_2, \ldots, c_N) = \frac{1}{2}\int_0^1 \left[\left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx} \right)^2 - \left(\sum_{j=1}^{N} c_j \phi_j \right)^2 + 2x^2 \left(\sum_{j=1}^{N} c_j \phi_j \right) \right] dx$$

The necessary condition for the minimum of I is that

$$\begin{aligned} \frac{\partial I}{\partial c_i} = 0 &= \int_0^1 \left[\frac{d\phi_i}{dx} \left(\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx} \right) - \phi_i \left(\sum_{j=1}^{N} c_j \phi_j \right) + \phi_i x^2 \right] dx \\ &= \sum_{j=1}^{N} \left[\int_0^1 \left(\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} - \phi_i \phi_j \right) dx \right] c_j + \int_0^1 \phi_i x^2 \, dx \\ &= \sum_{j=1}^{N} K_{ij} c_j - F_i \quad \text{for } i = 1, 2, \ldots, N \end{aligned} \tag{2.5.12}$$

Clearly, K_{ij} and F_i are the same as those defined in Eq. (2.5.11). Equations (2.5.12) hold for any choice of admissible approximation functions ϕ_i.

Next, we discuss the choice of ϕ_i and ϕ_0. Since both boundary conditions $u(0) = u(1) = 0$ are homogeneous and of the essential type, we take $\phi_0 = 0$ and select ϕ_i in the N-parameter Ritz approximation to satisfy the conditions $\phi_i(0) = \phi_i(1) = 0$. Clearly, the function $(0 - x)(1 - x)$ vanishes at $x = 0$ and $x = 1$, and its first derivative is nonzero. Hence, we take $\phi_1 = x(1 - x)$. The next function in the sequence is obviously $\phi_2 = x^2(1 - x)$(or $\phi_2 = x(1 - x)^2$). Thus, the following set of functions are admissible:

$$\phi_1 = x(1 - x), \ \phi_2 = x^2(1 - x), \ldots, \ \phi_i = x^i(1 - x), \ldots, \ \phi_N = x^N(1 - x) \tag{2.5.13}$$

The approximation

$$U_N = c_1 x(1 - x) + c_2 x^2(1 - x) + \ldots + c_N x^N(1 - x) \tag{2.5.14a}$$

is equivalent to

$$U_N = \hat{c}_1 x(1 - x) + \hat{c}_2 x(1 - x)^2 + \ldots + \hat{c}_N x(1 - x)^N \tag{2.5.14b}$$

It should be noted that if we select, for example, the functions $\phi_1 = x^2(1 - x)$, $\phi_2 = x^3(1 - x)$, and so on [not including $x(1 - x)$], the completeness requirement is violated, because the set cannot be used to generate the linear term x of the exact solution, if the solution has such a term. As a rule, we must start with the lowest-order admissible function and include all admissible, higher-order functions up to the desired degree.

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For the choice of approximation functions in (2.5.13), the matrix coefficients $K_{ij} = B(\phi_i, \phi_j)$ and vector coefficients $F_i = l(\phi_i)$ can be computed as follows:

$$K_{ij} = \int_0^1 \{[ix^{i-1} - (i+1)x^i][jx^{j-1} - (j+1)x^j] - (x^i - x^{i+1})(x^j - x^{j+1})\}\, dx$$

$$= \frac{2ij}{(i+j)[(i+j)^2 - 1]} - \frac{2}{(i+j+1)(i+j+2)(i+j+3)} \tag{2.5.15a}$$

$$F_i = -\int_0^1 x^2(x^i - x^{i+1})dx = -\frac{1}{(i+3)(i+4)} \tag{2.5.15b}$$

for $i, j = 1, 2, \ldots, N$. We consider the one-, two- and three-parameter approximations to illustrate how the Ritz solution converges to the exact solution of the problem

$$u(x) = \frac{\sin x + 2\sin(1-x)}{\sin 1} + x^2 - 2 \tag{2.5.16}$$

For $N = 1$, we have

$$K_{11} = \frac{3}{10}, \quad F_1 = -\frac{1}{20} \quad \rightarrow \quad c_1 = -\frac{1}{6} = -0.1667$$

The one-parameter Ritz solution is given by

$$U_1 = c_1\phi_1 = -\frac{1}{6}(x - x^2)$$

For $N = 2$, we have

$$\frac{1}{420}\begin{bmatrix} 126 & 63 \\ 63 & 52 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = -\frac{1}{60}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$$

Solving the linear equations using Cramer's rule, we obtain

$$c_1 = -\frac{10}{123} = -0.0813, \quad c_2 = -\frac{21}{123} = -0.1707$$

The two-parameter Ritz solution is given by

$$U_2 = c_1\phi_1 + c_2\phi_2 = -\frac{10}{123}(x - x^2) - \frac{21}{123}(x^2 - x^3)$$
$$= -0.0813x - 0.0894x^2 + 0.1707x^3$$

For $N = 3$, we have

$$\frac{1}{2520}\begin{bmatrix} 756 & 378 & 228 \\ 378 & 312 & 237 \\ 228 & 237 & 206 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = -\frac{1}{420}\begin{Bmatrix} 21 \\ 14 \\ 10 \end{Bmatrix}$$

Note that the previously computed coefficients K_{ij} and F_i for $i, j = 1, 2$ remain unchanged and we only need to compute K_{3i}, $i = 1, 2, 3$, and F_3. The solution of the above equations is

$$c_1 = -0.0952, \quad c_2 = -0.1005, \quad c_3 = -0.0702$$

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Table 2.5.1 Comparison of the Ritz and exact solutions of the equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \text{ for } 0 < x < 1; \quad u(0) = u(1) = 0$$

Ritz coefficients†		x	Ritz solution, $-10u$ $N=1$	$N=2$	$N=3$	Exact solution
$N=1$:	$c_1 = -0.1667$	0.0	0.0	0.0	0.0	0.0
		0.1	0.1500	0.0885	0.0954	0.0955
		0.2	0.2667	0.1847	0.1890	0.1890
$N=2$:	$c_1 = -0.0813$	0.3	0.3500	0.2783	0.2766	0.2764
	$c_2 = -0.1707$	0.4	0.4000	0.3590	0.3520	0.3518
		0.5	0.4167	0.4167	0.4076	0.4076
$N=3$:	$c_1 = -0.0952$	0.6	0.4000	0.4410	0.4340	0.4342
	$c_2 = -0.1005$	0.7	0.3500	0.4217	0.4200	0.4203
	$c_3 = -0.0702$	0.8	0.2667	0.3486	0.3529	0.3530
		0.9	0.1500	0.2115	0.2183	0.2182
		1.0	0.0	0.0	0.0	0.0

†The four-parameter Ritz solution coincides with the exact solution up to four decimal places.

The three-parameter Ritz solution is given by

$$\begin{aligned} U_3 &= c_1\phi_1 + c_2\phi_2 + c_3\phi_3 = -0.0952(x - x^2) - 0.1005(x^2 - x^3) - 0.0702(x^3 - x^4) \\ &= -0.0952x - 0.0053x^2 + 0.0303x^3 + 0.0702x^4 \end{aligned}$$

The values of the Ritz coefficients c_i, $i = 1, 2, \ldots, N$ for various values of N can be obtained by solving the matrix equation $[K]\{c\} = \{F\}$ with the coefficients of K_{ij} and F_i given by (2.5.15a) and (2.5.15b). The Ritz coefficients and a comparison of the Ritz solution with the exact solution (2.5.16) is presented in Table 2.5.1 and Fig. 2.5.1. If the exact solution (2.5.16) is expanded in a series in terms of powers of x, we note that it is an infinite series. However, the three-parameter Ritz solution is already a good approximation of the exact solution, as can be seen from Fig. 2.5.1 and Table 2.5.1.

Boundary Conditions Set 2. For the second set of boundary conditions (2.5.9), the bilinear form is the same as that given in (2.5.10). The linear form is given by

$$l(w) = -\int_0^1 w\, x^2\, dx + w(1) \tag{2.5.17}$$

Therefore, we have

$$F_i = -\int_0^1 x^2 \phi_i\, dx + \phi_i(1) \tag{2.5.18}$$

As for the approximation functions, ϕ_0 is still zero and the ϕ_i must be selected to satisfy the condition $\phi_i(0) = 0$. Clearly, $\phi_1(x) = x$, $\phi_2 = x^2$, and so on meet the requirement. Thus, we have

$$U_N = c_1 x + c_2 x^2 + \ldots + c_i x^i + \ldots + c_N x^N \tag{2.5.19}$$

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Figure 2.5.1 Comparison of the Ritz solution with the exact solution of Eqs. (2.5.7) and (2.5.8). The three-parameter Ritz solution and the exact solution do not differ on the scale of the plot.

The coefficients K_{ij} and F_i can be computed using

$$K_{ij} = \int_0^1 (ijx^{i+j-2} - x^{(i+j)})\,dx = \frac{ij}{i+j-1} - \frac{1}{i+j+1}$$

$$F_i = -\int_0^1 x^{i+2}\,dx + 1 = -\frac{1}{i+3} + 1 \tag{2.5.20}$$

For example, for $N = 2$ we have

$$\frac{1}{60}\begin{bmatrix} 40 & 45 \\ 45 & 68 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{1}{20}\begin{Bmatrix} 15 \\ 16 \end{Bmatrix}$$

Solving the linear equations using Cramer's rule, we obtain

$$c_1 = \frac{180}{139} = 1.2950, \quad c_2 = -\frac{21}{139} = -0.1511$$

The two-parameter Ritz solution is given by

$$U_2 = c_1\phi_1 + c_2\phi_2 = 1.2950x - 0.1511x^2$$

The exact solution for this case is

$$u(x) = \frac{2\cos(1-x) - \sin x}{\cos 1} + x^2 - 2 \tag{2.5.21}$$

A comparison of the Ritz solutions with the exact solution is presented in Table 2.5.2.

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Table 2.5.2 Comparison of the Ritz and exact solutions of the equation

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \text{ for } 0 < x < 1; \quad u(0) = 0, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 1$$

Ritz coefficients†		x	Ritz solution, u $N=1$	$N=2$	$N=3$	Exact solution
$N=1$	$c_1 = 1.1250$	0.0	0.0	0.0	0.0	0.0
		0.1	0.1125	0.1280	0.1271	0.1262
		0.2	0.2250	0.2530	0.2519	0.2513
$N=2$	$c_1 = 1.2950$	0.3	0.3375	0.3749	0.3740	0.3742
	$c_2 = -0.1511$	0.4	0.4500	0.4938	0.4934	0.4944
		0.5	0.5625	0.6097	0.6099	0.6112
$N=3$	$c_1 = 1.2831$	0.6	0.6750	0.7226	0.7234	0.7244
	$c_2 = -0.1142$	0.7	0.7875	0.8325	0.8337	0.8340
	$c_3 = -0.0246$	0.8	0.9000	0.9393	0.9407	0.9402
		0.9	1.0125	1.0431	1.0443	1.0433
		1.0	1.1250	1.1439	1.1442	1.1442

†The four-parameter Ritz solution coincides with the exact solution up to four decimal places.

Example 2.5.2

Consider the problem of finding the transverse deflection of a cantilever beam under a uniform transverse load of intensity q_0 per unit length and subjected to point load F_0 and bending moment M_0 at the free end (see Example 2.4.2). The governing equations according to the Euler-Bernoulli beam theory are

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - q_0 = 0 \quad \text{for} \quad \begin{cases} 0 < x < L \\ EI > 0 \end{cases} \tag{2.5.22}$$

$$w(0) = \left(\frac{dw}{dx}\right)\bigg|_{x=0} = 0, \quad \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=L} = -M_0, \quad \left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]\bigg|_{x=L} = F_0 \tag{2.5.23}$$

The weak form of (2.5.22) and (2.5.23) (which includes the specified NBC) was derived in Example 2.4.2, and is given by (2.4.30a).

We now construct an N-parameter Ritz solution using the variational form, (2.4.30b). Since the specified EBCs $w(0) = 0$ and $(dw/dx)|_{x=0}$ are homogeneous, we set $\phi_0 = 0$. Next, we select algebraic approximation functions ϕ_i that satisfy the continuity conditions and homogeneous form of the specified EBCs. The lowest-order algebraic function that meets these conditions is $\phi_1 = x^2$. The next function in the sequence is $\phi_2 = x^3$. Thus, we have

$$\phi_1 = x^2, \ \phi_2 = x^3, \ldots, \ \phi_i = x^{i+1}, \ldots, \ \phi_N = x^{N+1} \tag{2.5.24}$$

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The N-parameter Ritz approximation is

$$W_N(x) = \sum_{j=1}^{N} c_j \phi_j, \quad \phi_j = x^{j+1} \tag{2.5.25}$$

Substituting (2.5.25) for w and $v = \phi_i$ into (2.4.30a) with $c_f = 0$, we obtain

$$[K]\{c\} = \{F\} \tag{2.5.26a}$$

$$K_{ij} = B(\phi_i, \phi_j) = \int_0^L EI(i+1)ix^{i-1}(j+1)jx^{j-1}\,dx$$

$$= EI(L)^{i+j-1}\frac{ij(i+1)(j+1)}{(i+j-1)} \tag{2.5.26b}$$

$$F_i = l(\phi_i) = \int_0^L q_0 x^{i+1}\,dx + (L)^{i+1}F_0 - (i+1)(L)^i M_0$$

$$= q_0(L)^{i+2}\frac{1}{i+2} + (L)^{i+1}F_0 - (i+1)(L)^i M_0 \tag{2.5.26c}$$

For $N = 1$, Eq. (2.5.26a) gives

$$c_1 = \left(\frac{q_0 L^4}{12EI} + \frac{F_0 L^3}{4EI} - \frac{M_0 L}{2EI}\right)$$

and the one-parameter Ritz solution is

$$W_1(x) = \left(\frac{q_0 L^4}{12EI} + \frac{F_0 L^3}{4EI} - \frac{M_0 L}{2EI}\right)\frac{x^2}{L^2}$$

For $N = 2$, we have

$$EI\begin{bmatrix} 4L & 6L^2 \\ 6L^2 & 12L^3 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{q_0 L^3}{12}\begin{Bmatrix} 4 \\ 3L \end{Bmatrix} + F_0 L^2\begin{Bmatrix} 1 \\ L \end{Bmatrix} - M_0 L\begin{Bmatrix} 2 \\ 3L \end{Bmatrix} \tag{2.5.27}$$

Solving for c_1 and c_2, we obtain

$$c_1 = \frac{1}{24EI}\left(5q_0 L^2 + 12F_0 L - 12M_0\right), \quad c_2 = -\frac{1}{12EI}(q_0 L + 2F_0)$$

and solution (2.5.25) becomes

$$W_2(x) = \frac{q_0 L^4}{24EI}\left(5\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right) + \frac{F_0 L^3}{6EI}\left(3\frac{x^2}{L^2} - \frac{x^3}{L^3}\right) - \frac{M_0 L^2}{2EI}\frac{x^2}{L^2} \tag{2.5.28}$$

For $N = 3$, we obtain the matrix equation

$$EI\begin{bmatrix} 4 & 6L & 8L^2 \\ 6L & 12L^2 & 18L^3 \\ 8L^2 & 18L^2 & \frac{144}{5}L^4 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{q_0 L^3}{60}\begin{Bmatrix} 20 \\ 15L \\ 12L^2 \end{Bmatrix} + F_0 L^2\begin{Bmatrix} 1 \\ L \\ L^2 \end{Bmatrix} - M_0 L\begin{Bmatrix} 2 \\ 3L \\ 4L^2 \end{Bmatrix} \tag{2.5.29}$$

The solution of these equations when substituted into (2.5.25) for $N = 3$, gives

$$W_3(x) = \frac{q_0 L^4}{24EI}\frac{x^2}{L^2}\left(6 - 4\frac{x}{L} + \frac{x^2}{L^2}\right) + \frac{FL^3}{6EI}\left(3\frac{x^2}{L^2} - \frac{x^3}{L^3}\right) - \frac{M_0 L^2}{2EI}\frac{x^2}{L^2} \tag{2.5.30}$$

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which coincides with the exact solution of Eqs. (2.5.22) and (2.5.23). Note that the one-parameter solution is exact when the beam is subjected to end moment M_0 only; the two-parameter solution is exact when the beam is subjected to both F_0 and M_0. If we try to compute the four-parameter solution without knowing that the three-parameter solution is exact for a beam subjected to distributed load q_0, point load F_0, and bending moment M_0, the parameters c_j $(j > 3)$ will be zero.

The next example deals with two-dimensional heat conduction in a square region. Note that the dependent variable, namely the temperature, is denoted by T, consistent with the standard notation used in heat transfer books.

Example 2.5.3

Consider the Poisson equation governing two-dimensional heat transfer in a square region

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=g_0 \quad \text{in} \quad \Omega=\{(x,y):0<(x,y)<1\} \tag{2.5.31}$$

with the following boundary conditions:

$$T=0 \quad \text{on sides } x=1 \quad \text{and} \quad y=1 \tag{2.5.32a}$$

$$\frac{\partial T}{\partial n}=0 \quad \text{on sides } x=0 \quad \text{and} \quad y=0 \tag{2.5.32b}$$

where g_0 is the rate of uniform heat generation in the region. Equation (2.5.31) is called Poisson's equation $(-k\nabla^2 T=g_0)$.

The variational problem is of the form (see Example 2.4.4; set $u=T$, $a_1=a_2=k$, $a_0=0$, and $f=g_0$)

$$B(w,T)=l(w) \tag{2.5.33a}$$

where the bilinear and linear functionals are

$$\begin{aligned} B(w,T)&=\int_0^1\int_0^1 k\left(\frac{\partial w}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right)dx\,dy \\ l(w)&=\int_0^1\int_0^1 w g_0\,dx\,dy \end{aligned} \tag{2.5.33b}$$

We consider an N-parameter approximation of the form

$$T_N=\sum_{i,j=1}^{N} c_{ij}\cos\alpha_i x\,\cos\alpha_j y, \quad \alpha_i=\frac{1}{2}(2i-1)\pi \tag{2.5.34}$$

Note that (2.5.34) involves a double summation. Since the boundary conditions are homogeneous, we have $\phi_0=0$. Incidentally, ϕ_i also satisfies the natural boundary conditions of the

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problem but that is not necessary to be admissible. While the choice $\hat{\phi}_{ij} = \sin i\pi x \sin j\pi y$ meets the essential boundary conditions, $\hat{\phi}_{ij}$ is not complete because it cannot be used to generate the solution that *does not* vanish on the sides $x = 0$ and $y = 0$. Hence, $\hat{\phi}_{ij}$ are not admissible.

The coefficients K_{ij} and F_i can be computed by substituting (2.5.34) into (2.5.33*b*). Since the double Fourier series has two summations, we introduce the notation

$$\begin{aligned} K_{(ij)(kl)} &= k\int_0^1\int_0^1 [(\alpha_i \sin\alpha_i x \cos\alpha_j y)(\alpha_k \sin\alpha_k x \cos\alpha_l y) \\ &\quad + (\alpha_j \cos\alpha_i x \sin\alpha_j y)(\alpha_l \cos\alpha_k x \sin\alpha_l y)]\, dx\, dy \\ &= \begin{cases} 0 & \text{if } i \neq k \text{ or } j \neq l \\ \frac{1}{4}k(\alpha_i^2 + \alpha_j^2) & \text{if } i = k \text{ and } j = l \end{cases} \end{aligned} \tag{2.5.35a}$$

$$F_{ij} = g_0 \int_0^1\int_0^1 \cos\alpha_i x \cos\alpha_j y\, dx\, dy = \frac{g_0}{\alpha_i\alpha_j}\sin\alpha_i \sin\alpha_j \tag{2.5.35b}$$

In evaluating the integrals, the following orthogonality conditions were used

$$\int_0^1 \sin\alpha_i x \sin\alpha_j x\, dx = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{2} & \text{if } i = j \end{cases}$$

$$\int_0^1 \cos\alpha_i x \cos\alpha_j x\, dx = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{2} & \text{if } i = j \end{cases}$$

Owing to the diagonal form of the coefficient matrix (2.5.35*a*), we can readily solve for the coefficients c_{ij}:

$$c_{ij} = \frac{F_{ij}}{K_{(ij)(ij)}} = \frac{4g_0}{k}\frac{\sin\alpha_i \sin\alpha_j}{(\alpha_i^2 + \alpha_j^2)\alpha_i\alpha_j} \tag{2.5.36}$$

The one- and two-parameter Ritz solutions are (the one-parameter solution has one term but the two-parameter solution has four terms)

$$T_1 = \frac{32g_0}{k\pi^4}\cos\frac{1}{2}\pi x \cos\frac{1}{2}\pi y \tag{2.5.37}$$

$$\begin{aligned} T_2 = \frac{g_0}{k}[&0.3285\cos\frac{1}{2}\pi x\cos\frac{1}{2}\pi y - 0.0219(\cos\frac{1}{2}\pi x\cos\frac{3}{2}\pi y \\ &+ \cos\frac{3}{2}\pi x\cos\frac{1}{2}\pi y) + 0.0041\cos\frac{3}{2}\pi x\cos\frac{3}{2}\pi y] \end{aligned} \tag{2.5.38}$$

If algebraic polynomials are to be used in the approximation of T, we can choose $\phi_1 = (1-x)(1-y)$ or $\phi_1 = (1-x^2)(1-y^2)$, both of which satisfy the (homogeneous) essential boundary conditions. However, the choice $\phi_1 = (1-x^2)(1-y^2)$ also meets the natural boundary conditions of the problem. The one-parameter Ritz solution for the choice $\phi_1 = (1-x^2)(1-y^2)$ is

$$T_1(x, y) = \frac{5g_0}{16k}(1-x^2)(1-y^2) \tag{2.5.39}$$

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The exact solution of Eqs. (2.5.31), (2.5.32*a*), and (2.5.32*b*) is

$$T(x, y) = \frac{g_0}{2k}\left[(1 - y^2) + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos\alpha_n y \cosh\alpha_n x}{\alpha_n^3 \cosh\alpha_n}\right] \tag{2.5.40}$$

where $\alpha_n = \frac{1}{2}(2n - 1)\pi$. The Ritz solutions (2.5.37), (2.5.38), and (2.5.39) are compared with the exact solution (2.5.40) in Fig. 2.5.2. The analytical solution is evaluated using 50 terms of the series in Eq. (2.5.40).

Figure 2.5.2 Comparison of the Ritz solutions with the analytical solution of the Poisson equations (2.5.31), (2.5.32*a*) and (2.5.32*b*) in two dimensions.

The next example deals with an eigenvalue problem, namely axial vibration of a bar [see Example 2.3.4 and Reddy (2002)].

Example 2.5.4

Consider a uniform cross-section bar of length L, with the left end fixed and the right end connected to a rigid support via a linear elastic spring (with spring constant k), as shown in Fig. 2.5.3. We wish to determine the first two natural axial frequencies of the bar using the Ritz method.

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Figure 2.5.3 A uniform bar with an end spring.

The starting point is to construct the Lagrangian function $L = K - (U + V)$ (see Section 2.3.6). The kinetic energy K and the strain energy U associated with the axial motion of a bar are

$$K = \int_0^L \frac{\rho A}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx, \quad U = \int_0^L \frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{k}{2}[u(L,t)]^2 \tag{2.5.41}$$

For potential energy due to applied loads, $V = 0$.

Substituting for K and U from Eq. (2.5.41) and $V = 0$ in Hamilton's principle (2.3.60), we obtain $[\delta u(x, t_1) = \delta u(x, t_2) = 0$ and $\delta u(0, t) = 0]$

$$\begin{aligned}
0 &= \int_{t_1}^{t_2} \delta(K - U)dt \\
&= \int_{t_1}^{t_2} \delta\left\{\frac{1}{2}\int_0^L \left[\rho A\left(\frac{\partial u}{\partial t}\right)^2 - EA\left(\frac{\partial u}{\partial x}\right)^2\right]dx - \frac{k}{2}[u(L,t)]^2\right\}dt \\
&= \int_{t_1}^{t_2}\left[\int_0^L \left(\rho A\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} - EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right)dx - ku(L,t)\delta u(L,t)\right]dt \\
&= \int_{t_1}^{t_2}\left[\int_0^L \left(-\rho A\frac{\partial^2 u}{\partial t^2}\delta u - EA\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right)dx - ku(L,t)\delta u(L,t)\right]dt
\end{aligned} \tag{2.5.42}$$

The Euler–Lagrange equations associated with (2.5.42) are given by

$$\begin{aligned}
&-\rho A\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = 0, \quad 0 < x < L \\
&\left[\left(EA\frac{\partial u}{\partial x}\right) + ku(x,t)\right]_{x=L} = 0
\end{aligned} \tag{2.5.43}$$

Natural vibration is nothing but the periodic motion of the form

$$u(x,t) = u_0(x)e^{i\omega t}, \quad i = \sqrt{-1} \tag{2.5.44}$$

where ω is the frequency of natural vibration and $u_0(x)$ is the amplitude. Substituting Eq. (2.5.44) into Eq. (2.5.42), we obtain

$$0 = \int_0^L \left(\rho A\omega^2 u_0 \delta u_0 - EA\frac{du_0}{dx}\frac{d\delta u_0}{dx}\right)dx - ku_0(L)\delta u_0(L) \tag{2.5.45}$$

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where $(i\omega)^2 = -\omega^2$, and $\int_{t_1}^{t_2} e^{2i\omega t}\,dt$, being nonzero, is factored out. Equation (2.5.45) is the weak form of the equations governing natural vibration of a bar with left end fixed and the right end spring supported. The Euler equation and natural boundary condition associated with Eq. (2.5.45) are

$$-\frac{d}{dx}\left(EA\frac{du_0}{dx}\right) - \rho A\omega^2 u_0 = 0, \quad 0 < x < L \tag{2.5.46a}$$

$$EA\frac{du_0}{dx} + ku_0 = 0 \quad \text{at } x = L \tag{2.5.46b}$$

The essential boundary condition is $u_0(0) = 0$. Conversely, we can use the three step procedure to obtain the weak form (2.5.45) from Eqs. (2.5.46a) and (2.5.46b).

Next, we seek a N–parameter Ritz approximation (obviously, $\phi_0 = 0$)

$$u_0(x) \approx U_N(x) = \sum_{i=1}^{N} c_i\phi_i(x)$$

Substituting into Eq. (2.5.45), we obtain

$$0 = \sum_{i=1}^{N}\left\{\sum_{j=1}^{N}\left[\lambda\int_0^L \rho A\phi_i\phi_j\,dx - \left(\int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\phi_i(L)\phi_j(L)\right)\right]c_j\right\}\delta c_i$$

where $\lambda = \omega^2$. Because of the independent nature of δc_i, we obtain

$$0 = \sum_{j=1}^{N}\left[\lambda\int_0^L \rho A\phi_i\phi_j\,dx - \left(\int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\phi_i(L)\phi_j(L)\right)\right]c_j \tag{2.5.47a}$$

and in matrix form

$$([A] - \lambda[M])\{c\} = \{0\} \quad \text{or} \quad (\mathbf{A} - \lambda\mathbf{M})\mathbf{c} = 0 \tag{2.5.47b}$$

where

$$a_{ij} = \int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\phi_i(L)\phi_j(L), \quad m_{ij} = \int_0^L \rho A\phi_i\phi_j\,dx \tag{2.5.47c}$$

Equation (2.5.47b) represents a matrix eigenvalue problem, and we obtain N eigenvalues, λ_i, $i = 1, 2, \cdots, N$.

For the problem at hand, the approximation functions $\phi_i(x)$ are required to be differentiable once with respect to x and vanish at $x = 0$. Hence, we choose

$$\phi_i(x) = \left(\frac{x}{L}\right)^i \tag{2.5.48a}$$

Substituting ϕ_i from (2.5.48a) into (2.5.47c), we obtain

$$m_{ij} = \int_0^L \rho A\phi_i\phi_j\,dx = \rho AL\frac{1}{i+j+1}$$

$$a_{ij} = \int_0^L EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx + k\phi_i(L)\phi_j(L) = \frac{EA}{L}\frac{ij}{i+j-1} + k \tag{2.5.48b}$$

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Since we wish to determine two eigenvalues, we take $N = 2$ and obtain

$$m_{11} = \frac{\rho AL}{3}, \quad m_{12} = \frac{\rho AL}{4}, \quad m_{22} = \frac{\rho AL}{5}, \quad a_{11} = \frac{EA}{L} + k, \quad a_{12} = \frac{EA}{L} + k, \quad a_{22} = \frac{4EA}{3L} + k$$

and the matrix eigenvalue problem (2.5.47*b*) becomes

$$\left(\frac{EA}{3L}\begin{bmatrix} 3+3\alpha & 3+3\alpha \\ 3+3\alpha & 4+3\alpha \end{bmatrix} - \lambda\frac{\rho AL}{60}\begin{bmatrix} 20 & 15 \\ 15 & 12 \end{bmatrix}\right)\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{2.5.49}$$

where $\alpha = kL/EA$. The algebraic eigenvalue problem (2.5.49) must be solved for $\lambda = \omega^2$ and c_i [hence, for the mode shape $u_0(x)$].

To carry out the remaining steps to obtain the numerical values for the natural frequencies, we take $\alpha = kL/EA = 1$. Then, for a nontrivial solution (i.e., $c_1 \neq 0$, $c_2 \neq 0$), we set the determinant of the coefficient matrix in Eq. (2.5.49) to zero:

$$\begin{vmatrix} 2-\dfrac{\bar{\lambda}}{3} & 2-\dfrac{\bar{\lambda}}{4} \\ 2-\dfrac{\bar{\lambda}}{4} & \dfrac{7}{3}-\dfrac{\bar{\lambda}}{5} \end{vmatrix} = 0, \quad \bar{\lambda} = \frac{\lambda\rho L^2}{E} = \frac{\omega^2\rho L^2}{E} \tag{2.5.50a}$$

or

$$15\bar{\lambda}^2 - 640\bar{\lambda} + 2400 = 0$$

The quadratic equation has two roots

$$\bar{\lambda}_1 = 4.1545, \quad \bar{\lambda}_2 = 38.512 \rightarrow \omega_1 = \frac{2.038}{L}\sqrt{\frac{E}{\rho}}, \quad \omega_2 = \frac{6.206}{L}\sqrt{\frac{E}{\rho}} \tag{2.5.50b}$$

The eigenvectors (or mode shapes) are given by

$$U_2^{(i)} = c_1^{(i)}\frac{x}{L} + c_2^{(i)}\frac{x^2}{L^2}$$

where $c_1^{(i)}$ and $c_2^{(i)}$ are calculated from the equations [see Eq. (2.5.49)]

$$\begin{bmatrix} 2-\dfrac{\bar{\lambda}_i}{3} & 2-\dfrac{\bar{\lambda}_i}{4} \\ 2-\dfrac{\bar{\lambda}_i}{4} & \dfrac{7}{3}-\dfrac{\bar{\lambda}_i}{5} \end{bmatrix}\begin{Bmatrix} c_1^{(i)} \\ c_2^{(i)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The above pair of equations is linearly dependent. Hence, one of the two equations can be used to determine c_2 in terms of c_1 (or vice versa) for each value of λ. We obtain

$$\bar{\lambda}_1 = 4.1545: \quad c_1^{(1)} = 1.0000, \quad c_2^{(1)} = -0.6399 \rightarrow U_2^{(1)}(x) = \frac{x}{L} - 0.6399\frac{x^2}{L^2}$$

$$\bar{\lambda}_2 = 38.512: \quad c_1^{(2)} = 1.0000, \quad c_2^{(2)} = -1.4207 \rightarrow U_2^{(2)}(x) = \frac{x}{L} - 1.4207\frac{x^2}{L^2}$$

Plots of the two mode shapes are shown in Fig. 2.5.4.

The exact values of λ are the roots of the transcendental equation (the reader may verify this)

$$\lambda + \tan\lambda = 0 \tag{2.5.51a}$$

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Figure 2.5.4 First two mode shapes obtained by the Ritz method for the natural longitudinal vibrations of a spring-supported bar.

whose first two roots are ($\omega^2 = \lambda$)

$$\omega_1 = \frac{2.02875}{L}\sqrt{\frac{E}{\rho}}, \quad \omega_2 = \frac{4.91318}{L}\sqrt{\frac{E}{\rho}} \tag{2.5.51b}$$

Note that the first approximate frequency is closer to the exact than the second.

If we select ϕ_0 and ϕ_i to satisfy the natural boundary condition also, the degree of polynomials will inevitably go up. For example, the lowest-order function that satisfies the homogeneous form (we still have $\phi_0 = 0$) of the natural boundary condition $u'(1) + u(1) = 0$ is

$$\hat{\phi}_1 = 3x - 2x^2$$

The one-parameter solution with the choice of $\hat{\phi}_1 = 3x - 2x^2$ gives $\lambda_1 = 50/12 = 4.1667$, which is no better than the two-parameter solution computed using $\phi_1 = x$ and $\phi_2 = x^2$. Of course, solution $c_1\hat{\phi}_1$ would yield a more accurate value for λ_1 than the solution $c_1\phi_1$. Although $c_1\hat{\phi}_1$ and $c_1\phi_1 + c_2\phi_2$ are of the same degree (polynomials), the latter gives better accuracy for λ_1 because of the number of parameters is greater, which provides greater freedom to adjust the parameters c_i in satisfying the weak form.

The eigenvalues and mode shapes determined through eigenvalue analysis are also useful in determining the transient response. The general homogeneous solution to the transient problem is [see Eq. (2.5.44)]

$$U_2(x, t) = (c_{11}\cos\omega_1 t + c_{12}\sin\omega_1 t)\,x + (c_{21}\cos\omega_2 t + c_{22}\sin\omega_2 t)\,x^2$$

where c_{1j} and c_{2j} are constants to be determined using the initial conditions.

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2.5.5 The Method of Weighted Residuals

As noted in Section 2.4.2, one can always write the weighted-integral form of a differential equation, whether the equation is linear or nonlinear (in the dependent variables). The weak form can be developed if the equations are second-order or higher, even if they are nonlinear. However, it is not always possible to construct a functional $I(u)$ whose first variation is equal to the variational form, $\delta I = B(\delta u, u) - l(\delta u) = 0$. The Ritz method can be applied to all problems, including nonlinear problems, that have weak forms.

The weighted-residual method is a generalization of the Ritz method in that the weight functions can be chosen from an independent set of functions, and it requires only the weighted-integral form to determine the parameters. Since the latter form does not include any of the specified boundary conditions of the problem, the approximation functions must be selected such that the approximate solution satisfies both the natural and essential boundary conditions. In addition, the weight functions can be selected independently of the approximation functions, but are required to be linearly independent (so that the resulting algebraic equations are linearly independent).

We discuss the general method of weighted residuals first, and then consider certain special cases that are known by specific names (e.g., the Galerkin method, the collocation method, the least-squares method, and so on). Although a limited use of the weighted-residual method is made in this book (see Chapter 14), it is informative to have a knowledge of this class of methods for use in the formulation of certain nonlinear problems and non-self-adjoint problems (i.e., which do not admit a functional formulation).

The method of weighted residuals can be described in its generality by considering the operator equation

$$A(u) = f \quad \text{in} \quad \Omega \tag{2.5.52}$$

where A is an operator (linear or nonlinear), often a differential operator, acting on the dependent variable u, and f is a known function of the independent variables. Some examples of such operators are given below.

$$\begin{aligned}
&(1) \quad A(u) = -\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu \\
&(2) \quad A(u) = \frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right) \\
&(3) \quad A(u) = -\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y\frac{\partial u}{\partial y}\right)\right] \\
&(4) \quad A(u) = -\frac{d}{dx}\left(u\frac{du}{dx}\right) \\
&(5) \quad A(u, v) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned} \tag{2.5.53}$$

For an operator A to be *linear* in its arguments, it must satisfy the relation

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v) \tag{2.5.54}$$

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for any scalars α and β and dependent variables u and v. It can be easily verified that all operators in (2.5.53), except for 4 and 5, are linear. When an operator does not satisfy the condition (2.5.54), it is said to be *nonlinear*.

The function u is not only required to satisfy the operator equation (2.5.52), it is also required to satisfy the boundary conditions associated with the operator equation. From the examples considered so far, the boundary conditions associated with the operators defined in 1, 2, and 3 of (2.5.53) are obvious (see Examples 2.4.1–2.4.3).

In the weighted-residual method, the solution u is approximated, in much the same way as in the Ritz method, by the expression

$$U_N(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}) + \phi_0(\mathbf{x}) \tag{2.5.55}$$

except that the requirements on ϕ_0 and ϕ_j for the weighted-residual method are more stringent than those for the Ritz method. Substitution of the approximate solution U_N into the left-hand side of (2.5.52) gives a function $A(U_N)$ that, in general, is not equal to the specified function f. The difference $A(U_N) - f$, called the *residual* of the approximation, is nonzero:

$$R \equiv A(U_N) - f = A\left(\sum_{j=1}^{N} c_j \phi_j + \phi_0\right) - f \neq 0 \tag{2.5.56a}$$

Note that the residual R is a function of position as well as of the parameters c_j. In the weighted-residual method, as the name suggests, the parameters c_j are determined by requiring the residual R to vanish in the weighted-integral sense:

$$\int_{\Omega} \psi_i(\mathbf{x}) R(\mathbf{x}, c_j)\, dx\, dy = 0 \quad (i = 1, 2, \cdots, N) \tag{2.5.56b}$$

where Ω is a two-dimensional domain and ψ_i are *weight functions*, which, in general, are not the same as the approximation functions ϕ_i. The set $\{\psi_i\}$ must be a linearly independent set; otherwise, the equations provided by (2.5.56*b*) will not be linearly independent and hence will not be solvable.

The requirements on ϕ_0 and ϕ_j for the weighted-residual method are different from those for the Ritz method, which is based on the weak (integral) form of the differential equation. The differentiability requirement on ϕ_j in the weighted-residual method is dictated by the integral statement (2.5.56*b*), as opposed to the weak form in the Ritz method. Thus, ϕ_j must have nonzero derivatives up to the order appearing in the operator equation (2.5.52). Since the weighted-integral form (2.5.56*b*) does not include any of the specified (either essential or natural) boundary conditions, we must also require U_N in (2.5.55) to satisfy all specified boundary conditions of the problem. Consequently, ϕ_0 is required to satisfy the homogeneous form of all specified boundary conditions of the problem. These requirements on ϕ_0 and ϕ_j will increase the order of the polynomial expressions used for the weighted-residual method. In general, the ϕ_j used in this method are higher-order functions than those used in the Ritz method, and the functions used in the latter may not satisfy the continuity (i.e., differentiability) requirements of the weighted-residual method. Various special cases of the weighted-residual method are discussed in the following paragraphs.

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The Petrov–Galerkin Method. The weighted-residual method is referred to as the *Petrov–Galerkin method* when $\psi_i \neq \phi_i$. When the operator A is linear, (2.5.56*b*) can be simplified to the form

$$\sum_{j=1}^{N} \left[\int_\Omega \psi_i A(\phi_j)\, d\mathbf{x} \right] c_j = \int_\Omega \psi_i [f - A(\phi_0)]\, d\mathbf{x}$$

or

$$\sum_{j=1}^{N} A_{ij} c_j = F_i \quad (\mathbf{Ac} = \mathbf{F}) \tag{2.5.57}$$

Note that the coefficient matrix $[A]$ is not symmetric:

$$A_{ij} = \int_\Omega \psi_i A(\phi_j)\, d\mathbf{x} \neq A_{ji} \tag{2.5.58}$$

The Galerkin Method. If the weight function ψ_i is chosen to be equal to the approximation function ϕ_i, the weighted-residual method is better known as the Galerkin method. The algebraic equations of the Galerkin approximation are

$$\mathbf{Ac} = \mathbf{F} \tag{2.5.59a}$$

where

$$A_{ij} = \int_\Omega \phi_i A(\phi_j)\, d\mathbf{x} \quad F_i = \int_\Omega \phi_i [f - A(\phi_0)]\, d\mathbf{x} \tag{2.5.59b}$$

Once again, we note that A_{ij} is not symmetric.

In general, the Galerkin method is *not* the same as the Ritz method. This should be clear from the fact that the former uses the weighted-integral form whereas the latter uses the weak (or variational) form to determine the coefficients c_j. Consequently, the approximation functions used in the Galerkin method are required to be of higher order than those in the Ritz method.

If the equation permits, and one wishes, differentiation from the dependent variable(s) can be transferred to the weight function $w = \phi_i$, thereby obtaining the weak form. Then there is no difference between the Galerkin method and the Ritz method. Thus, Ritz and Galerkin methods yield the same solutions in two cases: (*a*) when the specified boundary conditions of the problem are all of the essential type, and therefore the requirements on ϕ_i in the two methods become the same and the weighted-integral form reduces to the weak form; and (*b*) when the approximation functions of the Galerkin method are used in the Ritz method.

The Least-Squares Method. In least-squares method, we determine the parameters c_j by minimizing the integral of the square of the residual (2.5.56*a*):

$$\frac{\partial}{\partial c_i} \int_\Omega R^2(\mathbf{x}, c_j)\, d\mathbf{x} = 0$$

or

$$\int_\Omega \frac{\partial R}{\partial c_i} R\, d\mathbf{x} = 0 \tag{2.5.60a}$$

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Comparison of (2.5.60a) with (2.5.56b) shows that $\psi_i = \partial R/\partial c_i$. If A is a linear operator, $\psi_i = A(\phi_i)$, and (2.5.60a) becomes

$$\sum_{j=1}^{N}\left[\int_\Omega A(\phi_i)A(\phi_j)\,d\mathbf{x}\right]c_j = \int_\Omega A(\phi_i)[f - A(\phi_0)]\,d\mathbf{x}$$

or

$$\mathbf{Ac} = \mathbf{F} \tag{2.5.60b}$$

where

$$A_{ij} = \int_\Omega A(\phi_i)A(\phi_j)\,d\mathbf{x}, \quad F_i = A(\phi_i)[f - A(\phi_0)]\,d\mathbf{x} \tag{2.5.60c}$$

Note that the coefficient matrix A_{ij} is symmetric, but it involves the same order of differentiation as in the governing differential equation $A(u) - f = 0$.

The Collocation Method. In the collocation method, we seek an approximate solution U_N to (2.5.52) in the form of (2.5.55) by requiring the residual to vanish identically at N selected points $\mathbf{x}^i \equiv (x^i, y^i, z^i)$ $(i = 1, 2, \ldots, N)$ in the domain Ω

$$R(\mathbf{x}^i, c_j) = 0 \quad (i = 1, 2, \ldots, N) \tag{2.5.61}$$

The selection of the points $\mathbf{x}^i$ is crucial in obtaining a well-conditioned system of equations and ultimately in obtaining an accurate solution. The collocation method can be shown to be a special case of (2.5.56b) with $\psi_i = \delta(\mathbf{x} - \mathbf{x}^i)$, where $\delta(\mathbf{x})$ is the Dirac delta function, which is defined by

$$\int_\Omega f(\mathbf{x})\delta(\mathbf{x} - \xi)\,d\mathbf{x} = f(\xi) \tag{2.5.62}$$

With this choice of weight functions, the weighted-residual statement (2.5.56b) becomes

$$\int_\Omega \delta(\mathbf{x} - \mathbf{x}^i)R(\mathbf{x}, c_j)\,d\mathbf{x} = 0$$

or

$$R(\mathbf{x}^i, c_j) = 0 \tag{2.5.63}$$

We consider an example to illustrate the use of various types of weighted-residual methods.

Example 2.5.5

Consider the differential equation (see Example 2.5.1 with Set 2 boundary conditions):

$$-\frac{d^2u}{dx^2} - u + x^2 = 0, \quad u(0) = 0, \quad u'(1) = 1 \tag{2.5.64}$$

For a weighted-residual method, ϕ_0 and ϕ_i should satisfy the following conditions:

$\phi_0(0) = 0, \quad \phi_0'(1) = 1$ (satisfy actual boundary conditions)
$\phi_i(0) = 0, \quad \phi_i'(1) = 0$ (satisfy homogeneous form of the specified boundary conditions)

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For a choice of algebraic polynomials, we assume $\phi_0(x) = a + bx$ and use the two conditions on ϕ_0 to determine the constants a and b. We obtain

$$\phi_0(x) = x$$

Since there are two homogeneous conditions, we must assume at least a three-parameter polynomial to obtain a nonzero function, $\phi_1 = a + bx + cx^2$. Using the conditions on ϕ_i, we obtain

$$\phi_1 = cx(2 - x)$$

The constant c can be set equal to unity because it will be absorbed into the parameter c_1. For ϕ_2, we can assume one of the forms

$$\phi_2 = a + bx + dx^3 \quad \text{or} \quad \phi_2 = a + cx^2 + dx^3$$

with $d \neq 0$; ϕ_2 does not contain all-order terms in either case, but the approximate solution is complete because $\{\phi_1, \phi_2\}$ contains all terms up to degree three. For the second choice of ϕ_2, we obtain

$$\phi_2 = x^2\left(1 - \frac{2}{3}x\right)$$

The residual in the approximation of the equation is

$$\begin{aligned} R &= -\left(0 + \sum_{i=1}^{N} c_i \frac{d^2\phi_i}{dx^2}\right) - \left(\phi_0 + \sum_{i=1}^{N} c_i\phi_i\right) + x^2 \\ &= c_1(2 - 2x + x^2) + c_2\left(-2 + 4x - x^2 + \frac{2}{3}x^3\right) - x + x^2 \end{aligned} \tag{2.5.65}$$

We next consider various methods.

The Petrov–Galerkin Method. Let the weight functions be

$$\psi_1 = x, \quad \psi_2 = x^2 \tag{2.5.66}$$

Then

$$\int_0^1 xR\,dx = 0, \quad \int_0^1 x^2 R\,dx = 0$$

or

$$\frac{7}{12}c_1 + \frac{13}{60}c_2 - \frac{1}{12} = 0, \quad \frac{11}{30}c_1 + \frac{11}{45}c_2 - \frac{1}{20} = 0 \tag{2.5.67}$$

Solving for c_i, we obtain $c_1 = \frac{103}{682}$ and $c_2 = -\frac{15}{682}$; the solution becomes

$$U_{PG} = 1.302053x - 0.173021x^2 - 0.014663x^3 \tag{2.5.68}$$

The Galerkin Method. Taking $\psi_i = \phi_i$, we have

$$\int_0^1 x(2 - x)R\,dx = 0, \quad \int_0^1 x^2\left(1 - \frac{2}{3}x\right)R\,dx = 0$$

or

$$\frac{4}{5}c_1 + \frac{28}{45}c_2 - \frac{7}{60} = 0, \quad \frac{17}{90}c_1 + \frac{29}{315}c_2 - \frac{1}{36} = 0 \tag{2.5.69}$$

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Hence, the solution becomes (with $c_1 = \frac{623}{4306}$, $c_2 = \frac{21}{4306}$),

$$U_G = 1.2894x - 0.1398x^2 - 0.00325x^3 \tag{2.5.70}$$

The Least-Squares Method. Taking $\psi_i = \frac{\partial R}{\partial c_i}$, we have

$$\int_0^1 (2 - 2x + x^2) R\, dx = 0, \quad \int_0^1 \left(-2 + 4x - x^2 + \frac{2}{3}x^3 \right) R\, dx = 0$$

or

$$\frac{28}{15}c_1 - \frac{47}{90}c_2 - \frac{13}{60} = 0, \quad -\frac{47}{90}c_1 + \frac{253}{315}c_2 - \frac{1}{36} = 0 \tag{2.5.71}$$

The least-squares solution is given by (with $c_1 = \frac{1292}{9935}$, $c_2 = \frac{991}{19870}$),

$$U_{LS} = 1.2601x - 0.08017x^2 - 0.03325x^3 \tag{2.5.72}$$

The Collocation Method. Choosing the points $x = \frac{1}{3}$ and $x = \frac{2}{3}$ as the collocation points, we evaluate the residuals at these points and set them equal to zero:

$$\begin{aligned} R\left(\frac{1}{3}\right) = 0: &\quad 117c_1 - 61c_2 = 18 \\ R\left(\frac{2}{3}\right) = 0: &\quad 90c_1 - 34c_2 = 18 \end{aligned} \tag{2.5.73}$$

The solution is given by ($c_1 = 1710/9468$, $c_2 = 486/9468$)

$$U_C = 1.3612x - 0.12927x^2 - 0.03422x^3 \tag{2.5.74}$$

The four approximate solutions are compared in Table 2.5.3 with the exact solution (2.5.21). For this problem, the Petrov–Galerkin method gives the most accurate solution.

Table 2.5.3 Comparison of the Ritz, weighted-residual, and exact solutions[1] of the boundary value problem in Example 2.5.5.

x	u_{exact}	U_R	U_{PG}	U_G	U_{LS}	U_C
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.1262	0.1280	0.1285	0.1275	0.1252	0.1348
0.2	0.2513	0.2529	0.2536	0.2523	0.2485	0.2668
0.3	0.3742	0.3749	0.3754	0.3741	0.3699	0.3958
0.4	0.4943	0.4938	0.4941	0.4932	0.4891	0.5216
0.5	0.6112	0.6097	0.6096	0.6093	0.6058	0.6440
0.6	0.7244	0.7226	0.7221	0.7226	0.7200	0.7628
0.7	0.8340	0.8324	0.8317	0.8329	0.8314	0.8778
0.8	0.9402	0.9393	0.9384	0.9404	0.9397	0.9887
0.9	1.0433	1.0431	1.0424	1.0448	1.0449	1.0954
1.0	1.1442	1.1439	1.1437	1.1463	1.1467	1.1977

[1] Subscripts are as follows: R, Ritz; PG, Petrov–Galerkin; G, Galerkin; LS, least-squares; C, collocation.

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2.6 SUMMARY

In this chapter, we have studied two major topics that are of immediate interest in the study of finite element method in the forthcoming chapters:

1. Weighted-integral and weak formulations of differential equations
2. Solution of boundary value problems by the Ritz and weighted-residual (e.g., the Galerkin, least-squares, and collocation) methods

The weighted-integral statements are required in order to generate the necessary and sufficient number of algebraic equations to solve for the parameters c_i in the approximate solution. Thus, the algebraic equations are equivalent to minimizing, in a weighted-integral sense, the error introduced in the approximation of the differential equation.

In studying the two topics, a three-step procedure for developing the weak form of differential equations is presented, and methods for obtaining algebraic equations in terms of the unknown parameters in the approximate solution are developed. These topics are immediately applicable in the finite element method, which is an elementwise application of a variational method. Thus, the material covered in this chapter forms the core of the finite element method. A few remarks are in order on the classical variational methods studied here.

- The traditional variational methods (e.g., Ritz, Galerkin, and least-squares) presented in Section 2.5 provide a simple means of finding spatially continuous approximate solutions to physical problems. The approximate solutions obtained via these methods are continuous functions of position in the domain.
- The main limitation of classical variational methods that prevents them from being competitive with traditional finite difference methods is the difficulty encountered in constructing the approximation functions. The construction process becomes more difficult when the domain is geometrically complex. The so-called "meshless methods" are a return to the classical variational methods but with a procedure to construct the approximation functions.
- From the preceding discussion, it is apparent that the variational methods can provide a powerful means of finding approximate solutions, provided one can find a way to systematically construct approximation functions, for almost any geometry, that depend only on the differential equation being solved and not on the boundary conditions of the problem. This property enables one to develop a computer program for a particular class of problem (each problem in the class differs from the others only in the data), i.e., a *general-purpose* computer program. Since the functions must be constructed for a geometrically complex domain, it seems that (recall the discussion of the method of composites for the determination of the center of mass of an irregular shape from Chapter 1) the region must be represented (or approximated if required) as an assemblage of simple geometric shapes for which the construction of approximation functions becomes simpler. The finite element method to be discussed in the forthcoming chapters is based on these ideas.
- In the finite element method, a given domain is represented (discretized) by a collection of geometrically simple shapes (elements), and on each element of the collection, the governing equation is *formulated* using any one of the variational methods. The approximation functions are systematically generated for each (typical) element using the

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essential boundary conditions. The elements are *connected* together by imposing the continuity of the dependent variables across the interelement boundaries.

The remaining chapters of this book are devoted to the introduction of the finite element method and its use in the analysis of several model differential equations representing mathematical models for many physical processes.

PROBLEMS

In Problems 2.1–2.5, construct the weak forms and, whenever possible, quadratic functionals.

2.1 A nonlinear equation:

$$-\frac{d}{dx}\left(u\frac{du}{dx}\right)+f=0 \quad \text{for} \quad 0<x<L$$

$$\left(u\frac{du}{dx}\right)\bigg|_{x=0}=0 \quad u(1)=\sqrt{2}$$

2.2 The Euler-Bernoulli-von Kármán nonlinear theory of beams:

$$-\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}=f \quad \text{for} \quad 0<x<L$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)-\frac{d}{dx}\left\{a\frac{dw}{dx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}=q$$

$$u=w=0 \quad \text{at} \quad x=0,\ L; \quad \left(\frac{dw}{dx}\right)\bigg|_{x=0}=0; \quad \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=L}=M_0$$

where EA, EI, f, and q are functions of x, and M_0 is a constant. Here u denotes the axial displacement and w the transverse deflection of the beam.

2.3 A second-order equation:

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+f=0 \quad \text{in} \quad \Omega$$

$$u=u_0 \ \text{on}\ \Gamma_1, \quad \left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)n_x+\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)n_y=t_0 \ \text{on}\ \Gamma_2$$

where $a_{ij}=a_{ji}$ $(i, j=1, 2)$ and f are given functions of position (x, y) in a two-dimensional domain Ω, and u_0 and t_0 are known functions on portions Γ_1 and Γ_2 of the boundary Γ: $\Gamma_1+\Gamma_2=\Gamma$.

2.4 Navier-Stokes equations for two-dimensional flow of viscous, incompressible fluids:

$$\left.\begin{aligned}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\
u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&=0
\end{aligned}\right\} \quad \text{in} \quad \Omega \qquad (1)$$

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$$u = u_0, \quad v = v_0 \quad \text{on} \quad \Gamma_1 \tag{2}$$

$$\left.\begin{aligned} \nu\left(\frac{\partial u}{\partial x}n_x + \frac{\partial u}{\partial y}n_y\right) - \frac{1}{\rho}Pn_x = \hat{t}_x \\ \nu\left(\frac{\partial v}{\partial x}n_x + \frac{\partial v}{\partial y}n_y\right) - \frac{1}{\rho}Pn_y = \hat{t}_y \end{aligned}\right\} \quad \text{on} \quad \Gamma_2 \tag{3}$$

2.5 Two-dimensional flow of viscous, incompressible fluids (stream function–vorticity formulation):

$$\left.\begin{aligned} -\nabla^2\psi - \zeta = 0 \\ -\nabla^2\zeta + \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} = 0 \end{aligned}\right\} \quad \text{in} \quad \Omega$$

Assume that all essential boundary conditions are specified to be zero.

2.6 Compute the coefficient matrix and the right-hand side of the N-parameter Ritz approximation of the equation

$$-\frac{d}{dx}\left[(1+x)\frac{du}{dx}\right] = 0 \quad \text{for} \quad 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 1$$

Use algebraic polynomials for the approximation functions. Specialize your result for $N = 2$ and compute the Ritz coefficients. *Answer:* $c_1 = \frac{55}{131}$ and $c_2 = -\frac{20}{131}$.

2.7 Use trigonometric functions for the two-parameter approximation of the equation in Problem 2.6 and obtain the Ritz coefficients.

2.8 A steel rod of diameter $d = 2$ cm, length $L = 25$ cm, and thermal conductivity $k = 50$ W/(m · °C) is exposed to ambient air $T_\infty = 20$°C with a heat-transfer coefficient $\beta = 64$ W/(m^2 · °C). Given that the left end of the rod is maintained at a temperature of $T_0 = 120$°C and the other end is exposed to the ambient temperature, determine the temperature distribution in the rod using a two-parameter Ritz approximation with polynomial approximation functions. The equation governing the problem is given by

$$-\frac{d^2\theta}{dx^2} + c\theta = 0 \quad \text{for} \quad 0 < x < 25 \text{ cm}$$

where $\theta = T - T_\infty$, T is the temperature, and c is given by

$$c = \frac{\beta P}{Ak} = \frac{\beta\pi D}{\frac{1}{4}\pi D^2 k} = \frac{4\beta}{kD} = 256/\text{m}^2$$

P being the perimeter and A the cross-sectional area of the rod. The boundary conditions are

$$\theta(0) = T(0) - T_\infty = 100°\text{C}, \quad \left(k\frac{d\theta}{dx} + \beta\theta\right)\bigg|_{x=L} = 0$$

Answer: For $L = 0.25$ m, $\phi_0 = 100$, $\phi_i = x^i$, the Ritz coefficients are $c_1 = -1,033.385$ and $c_2 = 2,667.261$.

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2.9 Set up the equations for the N-parameter Ritz approximation of the following equations associated with a simply supported beam and subjected to a uniform transverse load $q = q_0$:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q_0 \quad \text{for} \quad 0 < x < L$$

$$w = EI\frac{d^2w}{dx^2} = 0 \quad \text{at} \quad x = 0,\ L$$

(*a*) Use algebraic polynomials.

(*b*) Use trigonometric functions.

Compare the two-parameter Ritz solutions with the exact solution. *Answer:* (*a*) $c_1 = q_0L^2/(24EI)$ and $c_2 = 0$.

2.10 Repeat Problem 2.9 for $q = q_0 \sin(\pi x/L)$. *Answer:* $N = 2$: $c_1 = c_2L = 2q_0L^2/(3EI\pi^3)$.

2.11 Repeat Problem 2.9 for $q = Q_0\delta(x - \frac{1}{2}L)$, where $\delta(x)$ is the Dirac delta function (i.e., a point load Q_0 is applied at the center of the beam).

2.12 Develop the N-parameter Ritz solution for a simply supported beam under uniform transverse load using Timoshenko beam theory. The governing equations are given in Eqs. (2.4.32*a*) and (2.4.32*b*). Use Trigonometric functions to approximate w and Ψ.

2.13 Solve the Poisson equation governing heat conduction in a square region:

$$-k\nabla^2 T = g_0$$

$$T = 0 \quad \text{on sides} \quad x = 1 \quad \text{and} \quad y = 1$$

$$\frac{\partial T}{\partial n} = 0 \quad \text{(insulated) on sides} \quad x = 0 \quad \text{and} \quad y = 0$$

using a one-parameter Ritz approximation of the form

$$T_1(x, y) = c_1(1 - x^2)(1 - y^2)$$

Answer: $c_1 = \frac{5g_0}{16k}$.

2.14 Determine ϕ_i for a two-parameter Galerkin approximation with algebraic approximation functions for Problem 2.8.

2.15 Consider the (Neumann) boundary value problem

$$-\frac{d^2u}{dx^2} = f \quad \text{for} \quad 0 < x < L$$

$$\left(\frac{du}{dx}\right)\bigg|_{x=0} = \left(\frac{du}{dx}\right)\bigg|_{x=L} = 0$$

Find a two-parameter Galerkin approximation of the problem using trigonometric approximation functions, when (*a*) $f = f_0\cos(\pi x/L)$ and (*b*) $f = f_0$. *Answer:* (*a*) $\phi_i = \cos(i\pi x/L)$, $c_1 = f_0L^2/\pi^2$, $c_i = 0$ for $i \neq 1$.

2.16 Find a one-parameter approximate solution of the nonlinear equation

$$-2u\frac{d^2u}{dx^2} + \left(\frac{du}{dx}\right)^2 = 4 \quad \text{for} \quad 0 < x < 1$$

subject to the boundary conditions $u(0) = 1$ and $u(1) = 0$, and compare it with the exact solution $u_0 = 1 - x^2$. Use (*a*) the Galerkin method, (*b*) the least-squares method, and (*c*) the Petrov–Galerkin method with weight function $w = 1$. *Answer:* (*a*) $(c_1)_1 = 1$, and $(c_1)_2 = -3$.

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2.17 Give a one-parameter Galerkin solution of the equation

$$-\nabla^2 u = 1 \quad \text{in} \quad \Omega \quad (= \text{unit square})$$
$$u = 0 \quad \text{on} \quad \Gamma$$

Use (*a*) algebraic and (*b*) trigonometric approximation functions.
Answer: (*b*) $c_{ij} = \frac{16}{\pi^4}\frac{1}{ij(i^2+j^2)}$ $(i, j$ odd), $\phi_{ij} = \sin i\pi x \sin j\pi y$

2.18 Repeat Problem 2.17(*a*) for an equilateral triangular domain. *Hint:* Use the product of equations of the lines representing the sides of the triangle for the approximation function.
Answer: $c_1 = -\frac{1}{2}$.

2.19 Consider the differential equation

$$-\frac{d^2u}{dx^2} = \cos \pi x \quad \text{for} \quad 0 < x < 1$$

subject to the following three sets of boundary conditions:

(1) $u(0) = 0, \quad u(1) = 0$

(2) $u(0) = 0, \quad \left(\frac{du}{dx}\right)\Big|_{x=1} = 0$

(3) $\left(\frac{du}{dx}\right)\Big|_{x=0} = 0, \quad \left(\frac{du}{dx}\right)\Big|_{x=1} = 0$

Determine a three-parameter solution, with trigonometric functions, using (*a*) the Ritz method, (*b*) the least-squares method, and (*c*) collocation at $x = \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$, and compare with the exact solutions:

(1) $u_0 = \pi^{-2}(\cos \pi x + 2x - 1)$

(2) $u_0 = \pi^{-2}(\cos \pi x - 1)$

(3) $u_0 = \pi^{-2} \cos \pi x$

Answer: (1*a*) $c_i = \frac{4}{\pi^5 i(i^2-1)}$.

2.20 Consider a cantilever beam of variable flexural rigidity, $EI = a_0[2 - (x/L)^2]$ and carrying a distributed load, $q = q_0[1 - (x/L)]$. Find a three-parameter solution using $\phi_i = X^{(j+1)}$ and the collocation method. *Answer:* $c_1 = -\frac{q_0 L^2}{4a_0}$, $c_2 = \frac{q_0 L}{12a_0}$, and $c_3 = 0$.

2.21 Consider the problem of finding the fundamental frequency of a circular membrane of radius a, fixed at its edge. The governing equation for axisymmetric vibration is

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) - \lambda u = 0, \quad 0 < r < a$$

where λ is the frequency parameter and u is the deflection of the membrane. (*a*) Determine the trigonometric approximation functions for the Galerkin method, (*b*) use one-parameter Galerkin approximation to determine λ, and (*c*) use two-parameter Galerkin approximation to determine λ. *Answer:* $\lambda = 5.832/a^2$.

2.22 Find the first two eigenvalues associated with the differential equation

$$-\frac{d^2u}{dx^2} = \lambda u, \; 0 < x < 1$$
$$u(0) = 0, \; u(1) + u'(1) = 0$$

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Use the least-squares method with algebraic polynomials. Use the operator definition to be $A = -(d^2/dx^2)$ to avoid increasing the degree of the characteristic polynomial for λ. *Answer:* $\lambda_1 = 4.212$ and $\lambda_2 = 34.188$.

2.23 Repeat Problem 2.22 using the Ritz method with algebraic polynomials. *Answer:* $\lambda_1 = 4.1545$ and $\lambda_2 = 38.512$.

2.24 Consider the Poisson equation

$$-\nabla^2 u = 0, \quad 0 < x < 1, \quad 0 < y < \infty$$

$$u(0, y) = u(1, y) = 0 \quad \text{for} \quad y > 0$$

$$u(x, 0) = x(1 - x), \quad u(x, \infty) = 0, \quad 0 \le x \le 1$$

Assuming an approximation of the form

$$u(x, y) = c_1(y)x(1 - x)$$

find the differential equation for $c_1(y)$ and solve it exactly. *Answer:* $U_1(x, y) = (x - x^2)e^{-\sqrt{10}y}$.

REFERENCES FOR ADDITIONAL READING

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Chapter 3

SECOND-ORDER DIFFERENTIAL EQUATIONS IN ONE DIMENSION: FINITE ELEMENT MODELS

3.1 BACKGROUND

The traditional variational methods (e.g., the Ritz, Galerkin, and least-squares) described in Chapter 2 cease to be effective because of a serious shortcoming, namely, the difficulty in constructing the approximation functions. The approximation functions, apart from satisfying continuity, linear independence, completeness, and essential boundary conditions, are arbitrary; the selection becomes even more difficult when the given domain is geometrically complex. Since the quality of the approximation is directly affected by the choice of the approximation functions, it is discomforting to know that there exists no systematic procedure to construct them. Because of this shortcoming, despite the simplicity in obtaining approximate solutions, the traditional variational methods of approximation were never regarded as competitive computationally when compared with traditional finite difference schemes. The finite element method overcomes the shortcomings of the traditional variational methods by providing a systematic way of constructing the approximation functions.

Ideally speaking, an effective computational method should have the following features:

1. It should have a sound mathematical as well as physical basis (i.e., yield convergent solutions and be applicable to practical problems).
2. It should not have limitations with regard to the geometry, the physical composition of the domain, or the nature of the "loading."
3. The formulative procedure should be independent of the shape of the domain and the specific form of the boundary conditions.
4. It should be flexible enough to allow different degrees of approximation without reformulating the entire problem.
5. It should involve a systematic procedure that can be automated for use on digital computers.

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The finite element method is a technique in which a given domain is represented as a collection of simple domains, called *finite elements*, so that it is possible to systematically construct the approximation functions needed in a variational or weighted-residual approximation of the solution of a problem over each element. Thus, the finite element method differs from the traditional Ritz, Galerkin, least-squares, collocation, and other weighted-residual methods in the manner in which the approximation functions are constructed. But this difference is responsible for the following three basic features of the finite element method:

1. *Division of whole domain into subdomains* that enable a systematic derivation of the approximation functions as well as representation of complex domains.
2. *Derivation of approximation functions* over each element. The approximation functions are often algebraic polynomials that are derived using interpolation theory. However, approximation functions need not be polynomials (like in meshless form of the finite element method).
3. *Assembly of elements* is based on continuity of the solution and balance of internal fluxes; the assemblage of elements results in a numerical analog of the mathematical model of the problem being analyzed.

These three features, which constitute three major steps of the finite element formulation, are closely related. The geometry of the elements used to represent the domain of a problem should be such that the approximation functions can be uniquely derived. The approximation functions depend not only on the geometry but also on the number and location of points, called nodes, in the element and the quantities to be interpolated (e.g., solution, or solution and its derivatives). Once the approximation functions have been derived, the procedure to obtain algebraic relations among the unknown coefficients (which give the values of the dependent variable at the nodes) is exactly the same as that used in the Ritz and weighted-residual methods. Hence, a study of Chapter 2, especially the weak-form development and the Ritz method, makes the present study easier.

The finite element method not only overcomes the shortcomings of the traditional variational methods, but it is also endowed with the features of an effective computational technique. The basic steps involved in the finite element analysis of a problem are given in Table 3.1.1.

In the sections that follow, our objective is to introduce many fundamental ideas that form the basis of the finite element method. In doing so, we postpone some issues of practical and theoretical complexity to later sections of this chapter and to Chapters 4–14. The basic steps of a finite element analysis are introduced via a model second-order differential equation.

3.2 BASIC STEPS OF FINITE ELEMENT ANALYSIS

3.2.1 Model Boundary Value Problem

Consider the problem of finding the function $u(x)$ that satisfies the differential equation

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0 \quad \text{for} \quad 0<x<L \tag{3.2.1}$$

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Table 3.1.1 Steps involved in the finite element analysis of a typical problem.

1. Discretization (or representation) of the given domain into a collection of preselected finite elements. (This step can be postponed until the finite element formulation of the equation is completed.)
   (*a*) Construct the finite element mesh of preselected elements.
   (*b*) Number the nodes and the elements.
   (*c*) Generate the geometric properties (e.g., coordinates and cross-sectional areas) needed for the problem.
2. Derivation of element equations for all typical elements in the mesh.
   (*a*) Construct the variational formulation of the given differential equation over the typical element.
   (*b*) Assume that a typical dependent variable u is of the form

$$u = \sum_{i=1}^{n} u_i \psi_i$$

   and substitute it into Step 2*a* to obtain element equations in the form

$$[K^e]\{u^e\} = \{F^e\}$$

   (*c*) Select, if already available in the literature, or derive element interpolation functions ψ_i and compute the element matrices.
3. Assembly of element equations to obtain the equations of the whole problem.
   (*a*) Identify the interelement continuity conditions among the primary variables (relationship between the local degrees of freedom and the global degrees of freedom—connectivity of elements) by relating element nodes to global nodes.
   (*b*) Identify the "equilibrium" conditions among the secondary variables (relationship between the local source or force components and the globally specified source components).
   (*c*) Assemble element equations using Steps 3*a* and 3*b*.
4. Imposition of the boundary conditions of the problem.
   (*a*) Identify the specified global primary degrees of freedom.
   (*b*) Identify the specified global secondary degrees of freedom (if not already done in Step 3*b*).
5. Solution of the assembled equations.
6. Postprocessing of the results.
   (*a*) Compute the gradient of the solution or other desired quantities from the primary degrees of freedom computed in Step 5.
   (*b*) Represent the results in tabular and/or graphical form.

and the boundary conditions

$$u(0) = u_0, \qquad \left(a\frac{du}{dx}\right)\bigg|_{x=L} = Q_0 \tag{3.2.2}$$

where $a = a(x)$, $c = c(x)$, $f = f(x)$, and u_0, and Q_0 are the *data* (i.e., known quantities) of the problem. Equation (3.2.1) arises in connection with the analytical description of many physical processes. For example, conduction and convection heat transfer in a plane wall or fin [see Fig. 3.2.1(*a*)], flow through channels and pipes, transverse deflection of cables, axial deformation of bars [see Fig. 3.2.1(*b*)], and many other physical processes are described by Eq. (3.2.1). A list of field problems described by Eq. (3.2.1) when $c(x) = 0$ are presented in Table 3.2.1 [see Reddy (2004)]. Thus, if we can develop a numerical procedure by which Eq. (3.2.1) can be solved for all possible boundary conditions, the procedure can be used to solve all field problems listed in Table 3.2.1, as well as many others. This fact provides us with the motivation to use (3.2.1) as the model second-order equation in one dimension. A step-by-step procedure for the formulation and solution of (3.2.1) by the

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Figure 3.2.1 (*a*) Heat transfer in a fin. (*b*) Axial deformation of a bar. (*c*) Mathematical idealization of the problem in (*a*) or (*b*).

finite element method is summarized in Table 3.1.1. The mathematical problem consists of solving the differential equation (3.2.1) in one-dimensional domain $\Omega = (0, L)$ subject to a suitable set of specified boundary conditions at the boundary points $x = 0$ and $x = L$, as shown in Fig. 3.2.1(*c*). As already shown in Chapter 2, the type of boundary conditions associated with a differential equation emerges in a natural way during the weak-form development of the differential equation. A detailed discussion of these ideas is presented next.

3.2.2 Discretization of the Domain

In the finite element method, the domain Ω of the problem [Fig. 3.2.2(*a*)] is divided into a set of subintervals, i.e., line elements, called *finite elements*. A typical element is denoted Ω_e and it is located between points A and B with coordinates x_a and x_b (i.e., of length $h_e = x_b - x_a$). The collection of finite elements in a domain is called the *finite element mesh* of the domain [see Fig. 3.2.2(*b*)].

The reason for dividing a domain into a set of subdomains, i.e., finite elements, is two-fold. First, domains of most systems, by design, are a composite of geometrically and/or

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Table 3.2.1 Some examples of engineering problems governed by the second-order equation (3.2.1) (see the footnote for the meaning of some parameters*).

Field of study	Primary variable u	Problem data a	c	f	Secondary variable Q
Heat transfer	Temperature $T - T_\infty$	Thermal conductance kA	Surface convection $AP\beta$	Heat generation f	Heat Q
Flow through porous medium	Fluid head ϕ	Permeability μ	0	Infiltraction f	Point source Q
Flow through pipes	Pressure P	Pipe resistance $1/R$	0	0	Point source Q
Flow of viscous fluids	Velocity v_x	Viscosity μ	0	Pressure gradient $-dP/dx$	Shear stress σ_{xz}
Elastic cables	Displacement u	Tension T	0	Transverse force f	Point force P
Elastic bars	Displacement u	Axial stiffness EA	0	Axial force f	Point force P
Torsion of bars	Angle of twist θ	Shear stiffness GJ	0	0	Torque T
Electrostatics	Electrical potential ϕ	Dielectric constant ϵ	0	Charge density ρ	Electric flux E

*k = thermal conductance; β = convective film conductance; p = perimeter; P = pressure or force; T_∞ = ambient temperature of the surrounding fluid medium; $R = 128\mu h/(\pi d^4)$ with μ being the viscosity, h the length, and d the diameter of the pipe; E = Young's modulus; A = area of cross section; J = polar moment of inertia.

materially different parts, and the solution on these subdomains is represented by different functions that are continuous at the interfaces of these subdomains. Therefore, it is appropriate to seek approximation of the solution over each subdomain. Second, approximation of the solution over each element of the mesh is simpler than its approximation over the entire domain. Approximation of the geometry of the domain in the present case is not a

Figure 3.2.2 (*a*) Whole domain. (*b*) Finite element discretization (mesh).

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concern, since it is a straight line. We must, however, seek a suitable approximation of the solution over each subdomain (i.e., finite element).

The number of elements into which the total domain is divided in a problem depends mainly on the geometry of the domain and on the desired accuracy of the solution. In the one-dimensional problem at hand, geometry is simple enough to represent it exactly. Since the exact solution is not known *a priori*, we may begin with a number of elements that are considered to be reasonable. Most often, the analyst has knowledge of the qualitative behavior of the solution, and this helps to choose a starting mesh. Whenever a problem is solved by the finite element method for the first time, we are required to investigate the convergence of the finite element solution by gradually *refining the mesh* (i.e., increasing the number of elements) and comparing the solution with those obtained by *higher-order* elements. The *order* of an element refers to the degree of polynomial used to represent the solution over the element. This is made clearer in the sequel.

3.2.3 Derivation of Element Equations

Next, we develop the algebraic equations among the unknown parameters, much the same way as we did in the Ritz and Galerkin methods discussed in Chapter 2 [see Eq. (2.5.5)]. The main difference here is that we work with a finite element (i.e., subdomain) as opposed to the total domain. This step results in a matrix equation of the form $[K^e]\{c^e\} = \{F^e\}$, which is called *the finite element model* of the original equation. Since the element is physically connected to its neighbors, the resulting algebraic equations will contain more unknowns (c^es as well as F^es are unknown in the element equations) than the number of algebraic equations. Then it becomes necessary to put the elements together (i.e., assembly) to eliminate the extra unknowns. This process is discussed in Section 3.2.5.

The derivation of finite element equations, i.e., algebraic equations among the unknown parameters of the finite element approximation, involves the following three steps:

1. Construct the weighted-residual or weak form of the differential equation.
2. Assume the form of the approximate solution over a typical finite element.
3. Derive the finite element equations by substituting the approximate solution into the weighted-residual or weak form.

A typical element $\Omega_e = (x_a, x_b)$ [see Fig. 3.2.3(*a*)], whose endpoints have the coordinates $x = x_a$ and $x = x_b$, is isolated from the mesh. We seek an approximate solution to the governing differential equation over the element. In principle, any method that allows the derivation of necessary algebraic relations among the nodal values of the dependent variable can be used. In this book we develop the algebraic equations using the Ritz method, which is based on the weak form of the differential equation. Other methods, such as the least-squares method, may also be used to construct the finite element equations. Apart from the method used to derive the algebraic equations, the steps presented in this book for the Ritz (or weak-form Galerkin) finite element models are the same for other methods. The three steps in the derivation of finite element equations associated with the model differential equation over a *typical element* of the mesh are discussed next.

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Figure 3.2.3 Finite element discretization of a one-dimensional domain for the model problem in (3.2.1). (*a*) A typical finite element from the finite element mesh. (*b*) A typical element, with the definition of the primary (u) and secondary (Q) variables at the element nodes.

Step 1. Weak Form and Minimum of a Quadratic Functional

In the finite element method, we seek an approximate solution to (3.2.1) over each finite element. The polynomial approximation of the solution within a typical finite element Ω_e is assumed to be of the form

$$u_h^e = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{3.2.3}$$

where u_j^e are the values of the solution $u(x)$ at the nodes of the finite element Ω_e, and ψ_j^e are the approximation functions over the element. The particular form in (3.2.3) will be derived in the next section. Note that the approximation in (3.2.3) differs from the one used in the Ritz method in that $c_j\phi_j(x)$ is now replaced with $u_j^e\psi_j^e$ (and $\phi_0 = 0$), and u_j^e plays the role of undetermined parameters and ψ_j^e the role of approximation functions. As we shall see later, writing the approximation in terms of the nodal values of the solution is necessitated by the fact that the continuity of $u(x)$ between elements can be readily imposed. The coefficients u_j^e are determined such that (3.2.1) is satisfied in a weighted-integral sense. As discussed in Chapter 2, the necessary and sufficient number of algebraic relations among the u_j^e can be obtained by recasting the differential equation (3.2.1) in a weighted-integral form:

$$0 = \int_{x_a}^{x_b} w\left[-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f\right] dx \tag{3.2.4}$$

where $w(x)$ denotes the weight function and $\Omega_e = (x_a, x_b)$ is the domain of a typical element [see Fig. 3.2.3(*a*)]. For $u \approx u_h^e$ and each independent choice of w, we obtain an independent

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algebraic equation relating all u_j^e of the element. A total of n independent equations are required to solve for the parameters u_j^e, $j = 1, 2, \ldots, n$. When w is selected to be ψ_i^e and (3.2.4) is used to obtain the ith equation of the required n equations, the resulting finite element model (i.e., system of algebraic equations among the nodal values) is termed the *Galerkin finite element model*. Since (3.2.4) contains the second derivative of u, the approximation functions ψ_j^e must be twice differentiable. In addition, if the secondary variables are to be included in the model, ψ_i^e must be at least cubic. Similar arguments apply for cases of the weighted-residual methods discussed in Chapter 2. For details of the weighted-residual finite element models, see Chapter 14 and Reddy (1986).

To weaken the continuity required of the functions $\psi_j^e(x)$, we trade the differentiation in (3.2.4) from u to w such that both u and w are differentiated equally—once each in the present case. The resulting integral form is termed the *weak form* of (3.2.1). This form is not only equivalent to (3.2.1) but it also contains the natural boundary conditions of the problem. The three-step procedure of constructing the weak form of (3.2.1) was presented in Chapter 2 and is revisited in the next few paragraphs.

The first step is to multiply the governing differential equation with a weight function w and integrate over a *typical element*, as given in (3.2.4). The second step is to trade differentiation from u to w using integration by parts:

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw}{dx}\frac{du}{dx} + cwu - wf \right) dx - \left[wa \frac{du}{dx} \right]_{x_a}^{x_b} \tag{3.2.5}$$

The third and last step is to identify the primary and secondary variables of the weak form. This requires us to classify the boundary conditions of each differential equation into *essential* (or geometric) and *natural* (or force) boundary conditions. The classification is made uniquely by examining the boundary term appearing in the weak form (3.2.5),

$$\left[wa \frac{du}{dx} \right]_{x_a}^{x_b}$$

As a rule, the coefficient of the weight function w in the boundary expression is called a *secondary variable*, and its specification constitutes the *natural* or Neumann boundary condition. The dependent unknown u *in the same form as the weight function* w appearing in the boundary expression is termed a *primary variable*, and its specification constitutes the *essential* or Dirichlet boundary condition. For the model equation at hand, the primary and secondary variables are

$$u \quad \text{and} \quad a\frac{du}{dx} \equiv Q$$

The primary and secondary variables at the nodes are shown on the typical element in Fig. 3.2.3(b).

In writing the final form of the weighted-integral statement (i.e., weak form) we wish to use, we must address the fate of the boundary terms. For a typical line element, the end points (nodes 1 and 2) are the boundary points. At these points we have the following four conditions (none of them specified at the moment)

$$u_h^e(x_a) = u_1^e, \left(-a\frac{du}{dx} \right)_{x=x_a} = Q_1^e, u_h^e(x_b) = u_2^e, \left(a\frac{du}{dx} \right)_{x=x_b} = Q_2^e$$

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If we select $u_h^e(x)$ in (3.2.3) such that it automatically satisfies the end conditions $u_h^e(x_a) = u_1^e$ and $u_h^e(x_b) = u_2^e$, then it remains that we include the remaining conditions

$$Q_1 = \left(-a\frac{du}{dx}\right)\bigg|_{x_a}, \quad Q_2 = \left(a\frac{du}{dx}\right)\bigg|_{x_b} \tag{3.2.6}$$

in the weak form (3.2.5). With the notation in (3.2.6), the weak form becomes

$$0 = \int_{x_a}^{x_b} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right) dx - w(x_a)Q_1 - w(x_b)Q_2 \tag{3.2.7}$$

This completes the three-step procedure of constructing the weak form of the model equation (3.2.1). The finite element model based on the weak form (3.2.7) is called the *weak form Galerkin finite element model.* It is clear that the weak form (3.2.7) admits approximation functions that are lower order than the weighted-residual statement (3.2.4).

Students of engineering recognize that Fig. 3.2.3(*b*) is the *free-body diagram* of a typical element. For axial deformation of bars, u denotes displacement, du/dx is the strain ε, $E\varepsilon$ is the stress σ, and $A\sigma$ denotes the force, where E is Young's modulus and A is the area of cross section of the bar; hence, $Q = EA(du/dx) = a(du/dx)$ has the meaning of force. The quantities Q_1^e and Q_2^e are the reaction forces at the left and right ends of the member; Q_1^e is a compressive force while Q_2^e is a tensile force [algebraically, both are positive, as shown in Fig. 3.2.3(*b*)]. For heat conduction problems, u denotes temperature, du/dx is the temperature gradient, $-k(du/dx)$ is the heat flux q, and Aq denotes the heat, where k is the thermal conductivity and A is the area of cross section of the bar; hence, $Q = kA(du/dx) = a(du/dx)$ has the meaning of heat; $Q_1^e = -kA(du/dx)_a$ is the heat *input* at node 1, while $Q_2^e = kA(du/dx)_b$ denote the heat *input* at node 2. Thus, the arrow on the second node should be reversed for heat transfer problems. For additional details on heat transfer, see Section 3.3.1.

The weak form in (3.2.7) contains two types of expressions: those containing both w and u (bilinear form) and those containing only w (linear form):

$$\begin{aligned} B^e(w,u) &= \int_{x_a}^{x_b} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu\right) dx \\ l^e(w) &= \int_{x_a}^{x_b} w\, f\, dx + w(x_a)Q_1 + w(x_b)Q_2 \end{aligned} \tag{3.2.8}$$

The weak form can be expressed as

$$B^e(w,u) = l^e(w) \tag{3.2.9}$$

which is called the *variational problem* associated with (3.2.1). As will be seen later, the bilinear form results directly in the element coefficient matrix, and the linear form leads to the right-hand-side column vector of the finite element equations. Derivation of the variational problem of the type in (3.2.9) is possible for all problems described by differential equations. However, the bilinear form $B^e(w,u)$ may not be linear in u, and it may not be symmetric in its arguments w and u.

Those who have a background in applied mathematics or solid and structural mechanics will appreciate the fact that the variational problem (3.2.9), when $B^e(w,u)$ is symmetric $B^e(w,u) = B^e(u,w)$ and $l^e(w)$ is linear in w, is the same as the statement of the minimum

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of the quadratic functional $I^e(u)$, $\delta I^e = 0$, where

$$\begin{aligned} I^e(u) &= \frac{1}{2} B^e(u, u) - l^e(u) \\ &= \frac{1}{2} \int_{x_a}^{x_b} \left[a \left(\frac{du}{dx} \right)^2 + cu^2 \right] dx \\ &\quad - \int_{x_a}^{x_b} u\, f\, dx - u(x_a) Q_1 - u(x_b) Q_2 \end{aligned} \tag{3.2.10}$$

Thus, the relationship between the weak form and the minimum of quadratic functional I^e is obvious [cf. (3.2.9)]:

$$0 = \delta I^e = B^e(w, u) - l^e(w), \quad w = \delta u$$

The statement $\delta I^e = 0$ in solid and structural mechanics is also known as the *principle of minimum total potential energy*. When (3.2.1) describes the axial deformation of a bar, $\frac{1}{2} B^e(u, u)$ represents the elastic strain energy stored in the bar element, $l^e(u)$ represents the work done by applied forces, and $I^e(u)$ is the total potential energy (Π^e) of the bar element. Thus, the finite element model can be developed using either the statement of the principle of minimum total potential energy of an element or the weak form of the governing equations of an element. However, this choice is restricted to those problems where the minimum of a quadratic finctional $I^e(u)$ corresponds to the governing equations. On the other hand, we can always construct a weak form of any set of differential equations, linear or not, of order 2 and higher. Finite element formulations do not require the existence of the functional $I^e(u)$; they only need weighted-integral statements or weak forms. However, when the functional $I^e(u)$ exists with an extremum (i.e., minimum or maximum principle), existence and uniqueness of solution to the variational problem and its discrete analog can be established. In all problems discussed in this book, the variational problem is derivable from a quadratic functional.

Step 2. Approximate Solution

Recall that in the traditional variational methods, approximate solutions are sought over the total domain $\Omega = (0, L)$ at once. Consequently, the approximate solution $[u(x) \approx U_N(x) = \sum c_j \phi_j(x) + \phi_0(x)]$ is required to satisfy the boundary conditions of the problem. This places severe restrictions on the derivation of the approximation functions $\phi_j(x)$ and $\phi_0(x)$, especially when discontinuities exist in the geometry, material properties, and/or loading of the problem (see Chapter 2 for details). The finite element method overcomes this shortcoming by seeking approximate solution (3.2.3) over each element. Obviously, geometry of the element should be simpler than that of the whole domain, and the geometry should allow a systematic derivation of the approximation functions, as we shall see shortly.

To put the elements back together into their original positions, i.e., connect the approximate solution from each element to form a continuous solution over the whole domain, we require the solution to be the same at points common to the elements. Therefore, we identify the end points of each line element as the *element nodes*, which play the role of interpolation points (or base points) in constructing the approximation functions over an element. Depending on the degree of polynomial approximation used to represent the solution, additional nodes may be identified inside the element.

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Since the weak form over an element is equivalent to the differential equation and the natural boundary conditions (3.2.6), i.e., conditions on Q_i^e of the element, the approximate solution u_h^e of (3.2.3) is required to satisfy only the end conditions $u_h^e(x_a) = u_1^e$ and $u_h^e(x_b) = u_2^e$. We seek the approximate solution in the form of algebraic polynomials, although this is not always the case. The reason for this choice is two-fold: First, the interpolation theory of numerical analysis can be used to develop the approximation functions systematically over an element; second, numerical evaluation of integrals of algebraic polynomials is easy.

As in classical variational methods, the approximation solution u_h^e must fulfill certain conditions in order that it be convergent to the actual solution u as the number of elements is increased. These are:

1. It should be continuous over the element and differentiable, as required by the weak form.
2. It should be a *complete* polynomial, i.e., include all lower-order terms up to the highest order used.
3. It should be an interpolant of the primary variables at the nodes of the finite element (at least the nodes on the boundary of the element so that the continuity of the solution can be imposed across the interelement boundary).

The reason for the first condition is obvious; it ensures a nonzero coefficient matrix. The second condition is necessary in order to capture all possible states, i.e., constant, linear, and so on, of the actual solution. For example, if a linear polynomial without the constant term is used to represent the temperature distribution in a one-dimensional system, the approximate solution can never be able to represent a uniform state of temperature in the element should such a state occur. The third condition is necessary in order to enforce continuity of the primary variables at points common to the elements.

For the weak form in (3.2.7), the minimum polynomial order of u_h^e is linear. A complete linear polynomial is of the form

$$u_h^e(x) = c_1^e + c_2^e x \tag{3.2.11}$$

where c_1^e and c_2^e are constants. The phrase "complete polynomial" refers to the inclusion of all terms up to the order desired; omission of c_1^e would make it an incomplete linear polynomial. Similarly, $c_1^e + c_3^e x^2$ is an incomplete quadratic polynomial because the linear term is missing. The expression in (3.2.11) meets the first two conditions of an approximation. The third condition is satisfied if c_1^e and c_2^e meet the conditions

$$u_h^e(x_a) = c_1^e + c_2^e x_a \equiv u_1^e, \quad u_h^e(x_b) = c_1^e + c_2^e x_b \equiv u_2^e \tag{3.2.12}$$

Equation (3.2.12) provides two relations between (c_1^e, c_2^e) and (u_1^e, u_2^e), which can be expressed in matrix form as

$$\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{bmatrix} 1 & x_a \\ 1 & x_b \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \end{Bmatrix} \tag{3.2.13}$$

Inverting (3.2.13), we obtain

$$\begin{aligned} c_1^e &= \frac{1}{h_e}(u_1^e x_b - u_2^e x_a) \equiv \frac{1}{h_e}(\alpha_1^e u_1^e + \alpha_2^e u_2^e) \\ c_2^e &= \frac{1}{h_e}(u_2^e - u_1^e) \equiv \frac{1}{h_e}(\beta_1^e u_1^e + \beta_2^e u_2^e) \end{aligned} \tag{3.2.14}$$

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where $h_e = x_b - x_a$ and ($\alpha_1^e = x_b$, $\alpha_2^e = -x_a$, $\beta_1^e = 1$, and $\beta_2^e = -1$)

$$\alpha_i^e = (-1)^j x_j^e, \quad \beta_i^e = (-1)^j; \quad x_1^e = x_a, \quad x_2^e = x_b \tag{3.2.15}$$

In (3.2.15), i and j permute in a natural order:

$$\text{if } i = 1 \quad \text{then } j = 2; \quad \text{if } i = 2 \quad \text{then } j = 1$$

The α_i^e and β_i^e are introduced to show the typical form of the interpolation functions. Substituting c_1^e and c_2^e from (3.2.15) into (3.2.11), we obtain

$$\begin{aligned} u_h^e(x) &= \frac{1}{h_e}[(\alpha_1^e u_1^e + \alpha_2^e u_2^e) + (\beta_1^e u_1^e + \beta_2^e u_2^e)x] \\ &= \frac{1}{h_e}(\alpha_1^e + \beta_1^e x)u_1^e + \frac{1}{h_e}(\alpha_2^e + \beta_2^e x)u_2^e \\ &= \psi_1^e(x)u_1^e + \psi_2^e(x)u_2^e = \sum_{j=1}^{2} \psi_j^e(x)u_j^e \end{aligned} \tag{3.2.16}$$

where

$$\psi_1^e(x) = \frac{1}{h_e}(\alpha_1^e + \beta_1^e x) = \frac{x_b - x}{x_b - x_a}, \quad \psi_2^e(x) = \frac{1}{h_e}(\alpha_2^e + \beta_2^e x) = \frac{x - x_a}{x_b - x_a} \tag{3.2.17}$$

which are called the linear *finite element approximation functions.*

The approximation functions $\psi_i^e(x)$ have some interesting properties. First, note that

$$u_1^e \equiv u_h^e(x_a) = \psi_1^e(x_a)u_1^e + \psi_2^e(x_a)u_2^e$$

implies $\psi_1^e(x_a) = 1$ and $\psi_2^e(x_a) = 0$. Similarly,

$$u_2^e \equiv u_h^e(x_b) = \psi_1^e(x_b)u_1^e + \psi_2^e(x_b)u_2^e$$

gives $\psi_1^e(x_b) = 0$ and $\psi_2^e(x_b) = 1$. In other words, ψ_i^e is unity at the ith node and zero at the other node. This property is known as the interpolation property (i.e., u_h^e is an interpolant of $u(x)$ through nodes 1 and 2) of $\psi_i^e(x)$ and they are also called *interpolation functions.* When they are derived to interpolate function values only and not the derivatives of the function, they are known as the *Lagrange interpolation functions.* When the function and its derivatives are interpolated, the resulting interpolation functions are known as the *Hermite family of interpolation functions.* These will be discussed in connection with beam finite elements in Chapter 5.

Another property of $\psi_i^e(x)$ is that their sum is unity. To see this, consider a constant state of $u_h^e = c_0$. Then the nodal values u_1^e and u_2^e should be equal to each other and both equal to the constant c_0. Hence, we have

$$u_h^e(x) = \psi_1^e(x)u_1^e + \psi_2^e(x)u_2^e = c_0 \rightarrow 1 = \psi_1^e(x) + \psi_2^e(x)$$

This property of $\psi_i^e(x)$ is known as the *partition of unity.* In summary, we have

$$\psi_i^e(x_j^e) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \tag{3.2.18a}$$

$$\sum_{i=1}^{n} \psi_i^e(x) = 1 \tag{3.2.18b}$$

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Figure 3.2.4 (*a*) Linear interpolation functions. (*b*) Linear finite element approximation.

where $x_1^e = x_a$ and $x_2^e = x_b$ [see Eq. (3.2.15)]. The functions ψ_i^e are shown in Fig. 3.2.4(*a*) and $u_h^e(x)$ is shown in Fig. 3.2.4(*b*). Although properties (3.2.18*a*) and (3.2.18*b*) are verified for the linear Lagrange interpolation functions, they hold for Lagrange interpolation functions of all degrees.

The element interpolation functions ψ_i^e in (3.2.17) were derived in terms of the *global coordinate* x [i.e., the coordinate appearing in the governing differential equation (3.2.1)], but they are defined only on the element domain $\Omega_e = (x_a, x_b)$. If we choose to express them in terms of a coordinate $\bar{x}$ (convenient in evaluating integrals of ψ_i^e), with the origin fixed at node 1 of the element Ω_e, the functions ψ_i^e of (3.2.17) in terms of $\bar{x}$ are

$$\psi_1^e(\bar{x}) = 1 - \frac{\bar{x}}{h_e}, \quad \psi_2^e(\bar{x}) = \frac{\bar{x}}{h_e} \tag{3.2.19}$$

The coordinate $\bar{x}$ is termed the *local* or *element coordinate* [see Fig. 3.2.3(*a*)].

The interpolation functions ψ_i^e are derived systematically: Starting with an assumed degree of a complete algebraic polynomial for the primary variable u and determining the coefficients c_i^e of the polynomial in terms of the values u_i^e of the primary variable u at the element nodes, the primary variable is finally expressed as a linear combination of approximation functions $\psi_i^e(x)$ and the nodal values u_i^e, which are called the element *nodal degrees of freedom*. The key point in this procedure is the selection of the number and the location of nodes in the element so that the geometry of the element is uniquely defined and interelement continuity may be easily imposed. The number of nodes must be sufficient to allow the rewriting of the coefficients in the assumed polynomial in terms of the primary variables. For a linear polynomial approximation, two nodes with one primary variable per node are sufficient to rewrite the polynomial in terms of the values of the primary variable at the two nodes and also define the geometry of the element, provided the two nodes are the end points of the element.

The degree (or order) of the polynomial approximation can be increased to improve the accuracy (see Fig. 3.2.5). To illustrate the derivation of the interpolation functions of higher order, we consider the quadratic approximation of $u(x)$

$$u_h^e(x) = c_1^e + c_2^e x + c_3^e x^2 \tag{3.2.20}$$

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Figure 3.2.5 (*a*) Linear approximation. (*b*) Quadratic approximation.

Since there are three parameters c_i^e $(i = 1, 2, 3)$, we must identify three nodes in the element so that the three parameters can be expressed in terms of the three nodal values u_i^e. Two of the nodes are identified as the endpoints of the element to define its geometry, and the third node is taken interior to the element. In theory, the third node can be placed at any interior point. However, the midpoint of the element, being equidistant from the end nodes, is the best choice. Other choices are dictated by special considerations (e.g., to have a certain degree of singularity in the derivative of the solution), and we will not discuss such special cases here. Thus, we identify three nodes, two at the ends and one in the middle, of the element of length h_e [see Fig. 3.2.5(*b*)]. Following the procedure outlined for the linear polynomial, we eliminate c_i^e by rewriting $u_h^e(x)$ in terms of the three nodal values, (u_1^e, u_2^e, u_3^e). The three relations among c_i^e and u_i^e are

$$\begin{aligned} u_1^e &\equiv u_h^e(x_1^e) = c_1^e + c_2^e x_1^e + c_3^e (x_1^e)^2 \\ u_2^e &\equiv u_h^e(x_2^e) = c_1^e + c_2^e x_2^e + c_3^e (x_2^e)^2 \\ u_3^e &\equiv u_h^e(x_3^e) = c_1^e + c_2^e x_3^e + c_3^e (x_3^e)^2 \end{aligned} \tag{3.2.21a}$$

or, in matrix form,

$$\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \begin{bmatrix} 1 & x_1^e & (x_1^e)^2 \\ 1 & x_2^e & (x_2^e)^2 \\ 1 & x_3^e & (x_3^e)^2 \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \\ c_3^e \end{Bmatrix} \tag{3.2.21b}$$

where x_i^e $(i = 1, 2, 3)$ is the global coordinate of the ith node of the element Ω_e $(x_1^e = x_a$, $x_2^e = x_a + 0.5h_e$, and $x_3^e = x_b)$. Inverting (3.2.21*b*), we obtain

$$\begin{aligned} c_1^e &= \frac{1}{D^e} \sum_{i=1}^{3} \alpha_i^e u_i^e, \quad \alpha_i^e = x_j^e (x_k^e)^2 - x_k^e (x_j^e)^2 \\ c_2^e &= \frac{1}{D^e} \sum_{i=1}^{3} \beta_i^e u_i^e, \quad \beta_i^e = (x_j^e)^2 - (x_k^e)^2 \\ c_3^e &= \frac{1}{D^e} \sum_{i=1}^{3} \gamma_i^e u_i^e, \quad \gamma_i^e = -(x_j^e - x_k^e), \quad D^e = \sum_{i=1}^{3} \alpha_i^e \end{aligned} \tag{3.2.22}$$

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where D^e denotes the determinant of the matrix in (3.2.21*b*), and α_i^e, β_i^e, and γ_i^e are defined in (3.2.22) in terms of the nodal coordinates. The subscripts used in (3.2.22) permute in a natural order:

$$\begin{array}{llllll} \text{if} & i=1, & \text{then} & j=2 & \text{and} & k=3 \\ \text{if} & i=2, & \text{then} & j=3 & \text{and} & k=1 \\ \text{if} & i=3, & \text{then} & j=1 & \text{and} & k=2 \end{array} \tag{3.2.23}$$

For example, α_2^e, β_3^e, and γ_1^e are given by

$$\alpha_2^e = x_3^e (x_1^e)^2 - x_1^e (x_3^e)^2, \quad \beta_3^e = (x_1^e)^2 - (x_2^e)^2, \quad \gamma_1^e = x_3^e - x_2^e$$

Note that if x_2^e is very close to either x_1^e or x_3^e (i.e., node 2 is placed close to node 1 or node 3), making the rows of the coefficient matrix in (3.2.21) nearly the same, which makes the matrix nearly singular and not invertable.

Substituting for c_i^e from (3.2.22) into (3.2.20) and collecting the coefficients of u_1^e, u_2^e, and u_3^e separately, we obtain

$$u_h^e(x) = \psi_1^e(x) u_1^e + \psi_2^e(x) u_2^e + \psi_3^e(x) u_3^e = \sum_{j=1}^{3} \psi_j^e(x) u_j^e \tag{3.2.24}$$

where ψ_j^e are the *quadratic Lagrange interpolation functions,*

$$\psi_i^e(x) = \frac{1}{D^e}(\alpha_i^e + \beta_i^e x + \gamma_i^e x^2) \quad (i = 1, 2, 3) \tag{3.2.25}$$

Once again, the quadratic interpolation functions can be expressed in terms of the *local coordinate* $\bar{x}$. When the interior node, node 2, is placed at a distance $\bar{x} = \alpha h_e$, $0 < \alpha < 1$, the quadratic interpolation functions are given by

$$\begin{aligned} \psi_1^e(\bar{x}) &= \left(1 - \frac{\bar{x}}{h}\right)\left(1 - \frac{1}{\alpha}\frac{\bar{x}}{h}\right) \\ \psi_2^e(\bar{x}) &= \frac{1}{\alpha(1-\alpha)}\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right) \\ \psi_3^e(\bar{x}) &= -\frac{\alpha}{(1-\alpha)}\frac{\bar{x}}{h}\left(1 - \frac{1}{\alpha}\frac{\bar{x}}{h}\right) \end{aligned} \tag{3.2.26}$$

For $\alpha = \frac{1}{2}$, i.e., when node 2 is placed at the midpoint of the element, the interpolation functions in (3.2.26) become

$$\begin{aligned} \psi_1^e(\bar{x}) &= \left(1 - \frac{\bar{x}}{h}\right)\left(1 - \frac{2\bar{x}}{h}\right) \\ \psi_2^e(\bar{x}) &= 4\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right) \\ \psi_3^e(\bar{x}) &= -\frac{\bar{x}}{h}\left(1 - \frac{2\bar{x}}{h}\right) \end{aligned} \tag{3.2.27}$$

Plots of the quadratic interpolation functions are given in Fig. 3.2.6. The function ψ_i^e is equal to 1 at node i and zero at the other two nodes, but varies quadratically between the nodes.

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(a)

(b)

Figure 3.2.6 One-dimensional Lagrange quadratic element and its interpolation functions: (*a*) geometry of the element; (*b*) interpolation functions.

The interpolation properties (3.2.18*a*) and (3.2.18*b*) can be used to construct the Lagrange interpolation functions of any degree. For example, the quadratic interpolation functions (3.2.27) can be derived using the interpolation property (3.2.18*a*). Since $\psi_1^e(\bar{x})$ must vanish at nodes 2 and 3, i.e., at $\bar{x} = \frac{1}{2}h_e$ and $x = h_e$, respectively, $\psi_1^e(\bar{x})$ is of the form

$$\psi_1^e(\bar{x}) = C_1(\bar{x} - \frac{1}{2}h_e)(\bar{x} - h_e)$$

The constant C_1 is determined such that ψ_1^e is equal to 1 at $\bar{x} = 0$:

$$1 = C_1(0 - \frac{1}{2}h_e)(0 - h_e) \quad \text{or} \quad C_1 = 2/h_e^2$$

This gives

$$\psi_1^e(\bar{x}) = \frac{2}{h_e^2}(\bar{x} - \frac{1}{2}h_e)(\bar{x} - h_e) = \left(1 - \frac{\bar{x}}{h_e}\right)\left(1 - \frac{2\bar{x}}{h_e}\right)$$

which is identical to that in (3.2.27). The other two interpolations functions can be derived in a similar manner.

Although a detailed discussion is presented here on how to construct the Lagrange interpolation functions for one-dimensional elements, they are readily available in books on numerical analysis, and their derivation is independent of the physics of the problem to be solved. Their derivation depends only on the geometry of the element and the number and location of the nodes. The number of nodes must be equal to the number of terms in the polynomial. Thus, the interpolation functions derived above are useful not only in the finite element approximation of the problem at hand but also in all problems that admit

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Lagrange interpolation of the variables, i.e., all problems for which the primary variables are the dependent unknowns—not their derivatives.

Step 3. Finite Element Model

The weak form (3.2.7) or (3.2.9) is equivalent to the differential equation (3.2.1) over the element Ω_e, and it also contains the natural boundary conditions (3.2.6). Further, the finite element approximations (3.2.16) and (3.2.24) are the interpolants of the solution. The substitution of (3.2.16) or (3.2.24) into (3.2.7) will give the necessary algebraic equations among the nodal values u_i^e and Q_i^e of the element Ω_e. In order to develop the finite element model based on the weak form (3.2.7), it is *not* necessary to decide *a priori* the degree of approximation of u_h^e. The finite element model can be developed for an arbitrary degree of interpolation:

$$u \approx u_h^e = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{3.2.28}$$

where ψ_j^e are the Lagrange interpolation functions of degree $n-1$. For $n > 2$, the weak form in (3.2.7) must be modified to include nonzero secondary variables, if any, at interior nodes. This modification is discussed next.

The integration by parts in Step 2 [see Eq. (3.2.5)] of the weak-form development for an element with interior nodes is carried out by intervals $(x_1^e, x_2^e), (x_2^e, x_3^e), \ldots, (x_{n-1}^e, x_n^e)$:

$$\begin{aligned}
0 &= \sum_{i=1}^{n-1} \left\{ \int_{x_i^e}^{x_{i+1}^e} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf \right) dx - \left[w(x) a \frac{du}{dx} \right]_{x_i^e}^{x_{i+1}^e} \right\} \\
&= \int_{x_1^e}^{x_n^e} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf \right) dx - w(x_1^e)\left(-a\frac{du}{dx}\right)_{x_1^e} - w(x_2^e)\left(a\frac{du}{dx}\right)_{x_2^e} \\
&\quad - w(x_2^e)\left(-a\frac{du}{dx}\right)_{x_2^e} - w(x_3^e)\left(a\frac{du}{dx}\right)_{x_3^e} - \cdots \\
&\quad - w(x_{n-1}^e)\left(-a\frac{du}{dx}\right)_{x_{n-1}^e} - w(x_n^e)\left(a\frac{du}{dx}\right)_{x_n^e} \\
&= \int_{x_1^e}^{x_n^e} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf \right) dx - w(x_1^e)\left(-a\frac{du}{dx}\right)_{x_1^e} \\
&\quad - w(x_2^e)\left[\left(a\frac{du}{dx}\right)_{x_2^{e-}} + \left(-a\frac{du}{dx}\right)_{x_2^{e+}}\right] - w(x_3^e)\left[\left(a\frac{du}{dx}\right)_{x_3^{e-}} + \left(-a\frac{du}{dx}\right)_{x_3^{e+}}\right] \\
&\quad \cdots - w(x_{n-1}^e)\left[\left(a\frac{du}{dx}\right)_{x_{n-1}^{e-}} + \left(-a\frac{du}{dx}\right)_{x_{n-1}^{e+}}\right] - w(x_n^e)\left(a\frac{du}{dx}\right)_{x_n^e} \\
&= \int_{x_1^e}^{x_n^e} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu - wf \right) dx - w(x_1^e)Q_1^e - w(x_2^e)Q_2^e - w(x_3^e)Q_3^e \\
&\quad - \cdots - w(x_{n-1}^e)Q_{n-1}^e - w(x_n^e)Q_n^e
\end{aligned} \tag{3.2.29a}$$

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where x_i^{e-} and x_i^{e+} denote the left and right sides, respectively, of node i, and

$$Q_1^e = \left(-a\frac{du}{dx}\right)_{x_1^e}, \quad Q_2^e = \left[\left(a\frac{du}{dx}\right)_{x_2^{e-}} + \left(-a\frac{du}{dx}\right)_{x_2^{e-}}\right]$$

$$\vdots \tag{3.2.29b}$$

$$Q_{n-1}^e = \left[\left(a\frac{du}{dx}\right)_{x_{n-1}^{e-}} + \left(-a\frac{du}{dx}\right)_{x_{n-1}^{e-}}\right], \quad Q_n^e = \left(a\frac{du}{dx}\right)_{x_n^e}$$

Thus, Q_i^e, $i = 2, 3, \ldots, n-1$, denotes the jump in the value of the secondary variable in going from the left to the right of the ith node. This value is zero if no external source is applied at the node. Thus, for an element with n nodes, the weak form becomes

$$0 = \int_{x_a}^{x_b} \left(a\frac{dw}{dx}\frac{du}{dx} + cwu\right) dx - \int_{x_a}^{x_b} wq\, dx - \sum_{j=1}^{n} w(x_j^e) Q_j^e \tag{3.2.30}$$

Note that $Q_1^e = Q_a^e$ and $Q_n^e = Q_b^e$ represent the unknown point sources at the end nodes, and all other Q_i^e $(i = 2, 3, \ldots, n-1)$ are the specified point sources, if any, at the interior nodes.

Next, we develop the finite element model of Eq. (3.2.1) when the $(n-1)$st-degree Lagrange polynomials are used to approximate $u(x)$. Following the Ritz procedure developed in Section 2.5.2, we substitute (3.2.28) for u and $w = \psi_1^e$, $w = \psi_2^e$, $\ldots$, $w = \psi_n^e$ into the weak form (3.2.30) to obtain n algebraic equations:

$$0 = \int_{x_a}^{x_b} \left[a\frac{d\psi_1^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx}\right) + c\psi_1^e\left(\sum_{j=1}^{n} u_j^e \psi_j^e(x)\right) - \psi_1^e f\right] dx - \sum_{j=1}^{n} \psi_1^e(x_j^e) Q_j^e$$

(1st equation)

$$0 = \int_{x_a}^{x_b} \left[a\frac{d\psi_2^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx}\right) + c\psi_2^e\left(\sum_{j=1}^{n} u_j^e \psi_j^e(x)\right) - \psi_2^e f\right] dx - \sum_{j=1}^{n} \psi_2^e(x_j^e) Q_j^e$$

(2nd equation)

$$\vdots$$

$$0 = \int_{x_a}^{x_b} \left[a\frac{d\psi_i^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx}\right) + c\psi_i^e\left(\sum_{j=1}^{n} u_j^e \psi_j^e(x)\right) - \psi_i^e f\right] dx - \sum_{j=1}^{n} \psi_i^e(x_j^e) Q_j^e$$

(ith equation)

$$\vdots$$

$$0 = \int_{x_a}^{x_b} \left[a\frac{d\psi_n^e}{dx}\left(\sum_{j=1}^{n} u_j^e \frac{d\psi_j^e}{dx}\right) + c\psi_n^e\left(\sum_{j=1}^{n} u_j^e \psi_j^e(x)\right) - \psi_n^e f\right] dx - \sum_{j=1}^{n} \psi_n^e(x_j^e) Q_j^e$$

(nth equation)

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The ith algebraic equation of the system of n equations can be written as

$$0 = \sum_{j=1}^{n} K_{ij}^e u_j^e - f_i^e - Q_i^e \quad (i = 1, 2, \ldots, n) \tag{3.2.31a}$$

where

$$K_{ij}^e = B^e(\psi_i^e, \psi_j^e) = \int_{x_a}^{x_b} \left(a \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c\psi_i^e \psi_j^e \right) dx$$

$$f_i^e = \int_{x_a}^{x_b} f \psi_i^e \, dx \tag{3.2.31b}$$

Note that the interpolation property (3.2.18a) is used to write

$$\sum_{j=1}^{n} \psi_i^e(x_j^e) Q_j^e = Q_i^e \tag{3.2.32}$$

Equations (3.2.31a) can be expressed in terms of the coefficients K_{ij}^e, f_i^e, and Q_i^e as

$$\begin{aligned} K_{11}^e u_1^e + K_{12}^e u_2^e + \cdots + K_{1n}^e u_n^e &= f_1^e + Q_1^e \\ K_{21}^e u_1^e + K_{22}^e u_2^e + \cdots + K_{2n}^e u_n^e &= f_2^e + Q_2^e \\ &\vdots \\ K_{n1}^e u_1^e + K_{n2}^e u_2^e + \cdots + K_{nn}^e u_n^e &= f_n^e + Q_n^e \end{aligned} \tag{3.2.33a}$$

In matrix notation, the linear algebraic equations (3.2.33a) can be written as

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\} \quad \text{or} \quad \mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e + \mathbf{Q}^e \tag{3.2.33b}$$

The matrix $\mathbf{K}^e$ is called the *coefficient matrix,* or *stiffness matrix* in structural mechanics applications. The column vector $\mathbf{f}^e$ is the *source vector,* or *force vector* in structural mechanics problems. Note that Eq. (3.2.33) contains $2n$ unknowns: $(u_1^e, u_2^e, \ldots, u_n^e)$ and $(Q_1^e, Q_2^e, \ldots, Q_n^e)$, called primary and secondary element *nodal degrees of freedom*; hence, it cannot be solved without having an additional n conditions. Some of these are provided by the boundary conditions and the remainder by "balance" or equilibrium of the secondary variables Q_i^e at nodes common to several elements. This balance can be implemented by putting the elements together (i.e., assembling the element equations). Upon assembly and imposition of boundary conditions, we shall obtain exactly the same number of algebraic equations as the number of unknown primary and secondary degrees of freedom. The ideas underlying the assembly procedure are discussed in the next section.

The coefficient matrix $[K^e]$, which is symmetric, and source vector $\{f^e\}$ can be evaluated for a given element and data (a, c, and f). For elementwise-constant values of a, c, and f (say, a_e, c_e, and f_e) the coefficients K_{ij}^e and f_i^e can easily be evaluated for a typical element, as discussed next.

Linear Element. For a mesh of linear elements, the element Ω_e is located between the global nodes $x_a = x_e$ and $x_b = x_{e+1}$ [see Fig. 3.2.2(b)]. Hence,

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} \left(a_e \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e \right) dx, \quad f_i^e = \int_{x_e}^{x_{e+1}} f_e \psi_i^e \, dx$$

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or, in the local coordinate system $\bar{x}$,

$$K_{ij}^e = \int_0^{h_e} \left(a_e \frac{d\psi_i^e}{d\bar{x}} \frac{d\psi_j^e}{d\bar{x}} + c_e \psi_i^e \psi_j^e \right) d\bar{x}, \quad f_i^e = \int_0^{h_e} f_e \psi_i^e \, d\bar{x}$$

where $x = x_1^e + \bar{x}$ and

$$dx = d\bar{x}, \quad \frac{d\psi_i^e}{dx} = \frac{d\psi_i^e}{d\bar{x}}$$

The ψ_i^e can be expressed in terms of $\bar{x}$ as [see (3.2.19)]

$$\psi_1^e(\bar{x}) = 1 - \bar{x}/h_e, \quad \psi_2^e(\bar{x}) = \bar{x}/h_e$$

We can compute K_{ij}^e and f_i^e by evaluating the integrals. We have

$$\begin{aligned} K_{11}^e &= \int_0^{h_e} \left[a_e \left(-\frac{1}{h_e}\right)\left(-\frac{1}{h_e}\right) + c_e \left(1 - \frac{\bar{x}}{h_e}\right)\left(1 - \frac{\bar{x}}{h_e}\right) \right] d\bar{x} \\ &= \frac{a_e}{h_e} + \frac{1}{3} c_e h_e \\ K_{12}^e &= \int_0^{h_e} \left[a_e \left(-\frac{1}{h_e}\right) \frac{1}{h_e} + c_e \left(1 - \frac{\bar{x}}{h_e}\right) \frac{\bar{x}}{h_e} \right] d\bar{x} \\ &= -\frac{a_e}{h_e} + \frac{1}{6} c_e h_e = K_{21}^e \quad \text{(by symmetry)} \\ K_{22}^e &= \int_0^{h_e} \left(a_e \frac{1}{h_e}\frac{1}{h_e} + c_e \frac{\bar{x}}{h_e}\frac{\bar{x}}{h_e} \right) d\bar{x} = \frac{a_e}{h_e} + \frac{1}{3} c_e h_e \end{aligned}$$

Similarly,

$$f_1^e = \int_0^{h_e} f_e \left(1 - \frac{\bar{x}}{h_e}\right) d\bar{x} = \frac{1}{2} f_e h_e, \quad f_2^e = \int_0^{h_e} f_e \frac{\bar{x}}{h_e} \, d\bar{x} = \frac{1}{2} f_e h_e$$

Thus, for constant f_e, the total source $f_e h_e$ is equally distributed to the two nodes. The coefficient matrix and column vector are

$$[K^e] = \frac{a_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \{f^e\} = \frac{f_e h_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{3.2.34}$$

If $a = a_e x$ and $c = c_e$, the coefficient matrix $[K^e]$ can be evaluated as

$$[K^e] = \frac{a_e}{h_e} \left(\frac{x_e + x_{e+1}}{2} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{3.2.35}$$

The reader should verify this. Note that when a is a linear function of x, this is equivalent to replacing a in the coefficient matrix with its average value [compare (3.2.34) with (3.2.35)]:

$$a_{\text{avg}} = \frac{1}{2}(x_e + x_{e+1}) a_e$$

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Figure 3.2.7 Approximation of an element with linearly varying cross section by an equivalent element of constant cross section.

For example, in the study of bars with linearly varying cross section (see Fig. 3.2.7)

$$a = EA(x) = E\left(A_e + \frac{A_{e+1} - A_e}{h_e}\bar{x}\right)$$

this amounts to replacing the linearly varying cross section with a constant cross section within each element, the cross-sectional area of the constant section being the average area of the linearly varying element. Here A_e denotes the cross-sectional area at x_e and A_{e+1} is the area at $x = x_{e+1}$.

Quadratic Element. For a mesh of quadratic elements, element Ω_e is located between global nodes $x_a = x_{2e-1}$ and $x_b = x_{2e+1}$. Hence,

$$\begin{aligned} K_{ij}^e &= \int_{x_{2e-1}}^{x_{2e+1}} \left(a_e \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c_e \psi_i^e \psi_j^e\right) dx \\ &= \int_0^{h_e} \left(a_e \frac{d\psi_i^e}{d\bar{x}}\frac{d\psi_j^e}{d\bar{x}} + c_e \psi_i^e \psi_j^e\right) d\bar{x} \qquad (3.2.36a) \\ f_i^e &= \int_{x_{2e-1}}^{x_{2e-1}} \psi_i^e f_e \, dx = \int_0^{h_e} \psi_i^e f_e \, d\bar{x} \qquad (3.2.36b) \end{aligned}$$

where $\psi_i^e(\bar{x})$ $(i = 1,2,3)$ are the Lagrange quadratic interpolation functions in (3.2.27). Evaluating the integrals in (3.2.36a) and (3.2.36b), we obtain

$$\begin{aligned} K_{11}^e &= \int_0^{h_e} \left\{a_e\left(-\frac{3}{h_e} + \frac{4\bar{x}}{h_e^2}\right)\left(-\frac{3}{h_e} + \frac{4\bar{x}}{h_e^2}\right) \right. \\ &\quad \left. + c_e\left[1 - \frac{3\bar{x}}{h_e} + 2\left(\frac{\bar{x}}{h_e}\right)^2\right]\left[1 - \frac{3\bar{x}}{h_e} + 2\left(\frac{\bar{x}}{h_e}\right)^2\right]\right\} d\bar{x} \\ &= \frac{7}{3}\frac{a_e}{h_e} + \frac{2}{15}c_e h_e \end{aligned}$$

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$$K_{12}^e = \int_0^{h_e} \left\{ a_e \left(-\frac{3}{h_e} + \frac{4\bar{x}}{h_e^2} \right) \left(\frac{4}{h_e} - \frac{8\bar{x}}{h_e} \right) \right.$$

$$\left. + c_e \left[1 - \frac{3\bar{x}}{h_e} + 2\left(\frac{\bar{x}}{h_e}\right)^2 \right] \left[4\frac{\bar{x}}{h_e}\left(1 - \frac{\bar{x}}{h_e}\right) \right] \right\} d\bar{x}$$

$$= -\frac{8}{3}\frac{a_e}{h_e} + \frac{2}{30} c_e h_e = K_{21}^e$$

$$K_{22}^e = \int_0^{h_e} \left\{ a_e \left(\frac{4}{h_e} - \frac{8\bar{x}}{h_e} \right) \left(\frac{4}{h_e} - \frac{8\bar{x}}{h_e} \right) \right.$$

$$\left. + c_e \left[4\frac{\bar{x}}{h_e}\left(1 - \frac{\bar{x}}{h_e}\right) \right] \left[4\frac{\bar{x}}{h_e}\left(1 - \frac{\bar{x}}{h_e}\right) \right] \right\} d\bar{x}$$

$$= \frac{7}{3}\frac{a_e}{h_e} + \frac{2}{15} c_e h_e$$

and so on. Similarly,

$$f_1^e = \int_0^{h_e} f_e \left[1 - \frac{3\bar{x}}{h_e} + 2\left(\frac{\bar{x}}{h_e}\right)^2 \right] d\bar{x} = \frac{1}{6} f_e h_e$$

$$f_2^e = \int_0^{h_e} f_e \left[4\frac{\bar{x}}{h_e}\left(1 - \frac{\bar{x}}{h_e}\right) \right] d\bar{x} = \frac{4}{6} f_e h_e$$

$$f_3^e = f_1^e \quad \text{(by symmetry)}$$

Thus, the element coefficient matrix and source vector for a quadratic element are

$$[K^e] = \frac{a_e}{3h_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{3.2.37a}$$

$$\{f^e\} = \frac{f_e h_e}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} \tag{3.2.37b}$$

Note that, for quadratic elements, the total value of the source $f_e h_e$ is not distributed equally between the three nodes. The distribution is *not* equivalent to that of two linear elements of length $\frac{1}{2}h_e$. The computation of f_i^e should be based on the interpolation functions of that element. The sum of f_i^e for *any element* should always be equal to the integral of $f(x)$ over the element:

$$\sum_{i=1}^{n} f_i^e = \int_{x_a}^{x_b} f(x)\, dx \tag{3.2.38}$$

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In summary, for elementwise-constant values of the data a, c, and f, the element matrices of a linear and quadratic elements are

Linear Element

$$\left(\frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{3.2.39}$$

Quadratic Element

$$\left(\frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \frac{f_e h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix} \tag{3.2.40}$$

When the coefficient $c_e = 0$, the corresponding contribution to the above equations should be omitted.

When a, c, and f are algebraic polynomials in x, the evaluation of K_{ij}^e and f_j^e is straightforward. When they are complicated functions of x, the integrals in $[K^e]$ and $\{f^e\}$ are evaluated using numerical integration. A complete discussion of numerical integration will be presented in Chapter 7.

3.2.4 Connectivity of Elements

In deriving the element equations, we isolated a typical element (the eth element) from the mesh and formulated the variational problem (or weak form) and developed its finite element model. The finite element model of a typical element contains n equations among $2n$ unknowns, $(u_1^e, u_2^e, \ldots, u_n^e)$ and $(Q_1^e, Q_2^e, \ldots, Q_n^e)$. Hence, they cannot be solved without using the equations from other elements to get rid of extra unknowns. From a physical point of view, this makes sense because we should not be able to solve the element equations without considering the complete set of elements and the boundary conditions of the total problem.

To obtain the finite element equations of the total problem, we must put the elements back into their original positions (undoing of what was done before formulating the discrete problem). In putting the elements with their nodal degrees of freedom back into their original positions, we must require that the solution $u(x)$ is uniquely defined (i.e., u is continuous) and their source terms Q_i^e are "balanced" at the points where elements are connected to each other. Of course, if the variable u is not continuous, we do not impose its continuity; but in all problems studied in this book, unless otherwise stated explicitly (like in the case of an internal hinge in the case of beam bending), the primary variables are assumed to be continuous. The continuity of the primary variables refers to the single-valued nature of the

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solution; balance of secondary variables refers to the equilibrium of point sources. Thus, the assembly of elements is carried out by imposing the following two conditions:

1. If the node i of element Ω^e is connected to the node j of element Ω^f and node k of element Ω^g, the continuity of the primary variable u requires

$$u_i^{(e)} = u_j^{(f)} = u_k^{(g)} \tag{3.2.41}$$

When elements are connected in series, as shown in Fig. 3.2.8, the continuity of u requires

$$u_n^e = u_1^{e+1} \tag{3.2.42}$$

2. For the same three elements, the balance of secondary variables at connecting nodes requires

$$Q_i^{(e)} + Q_j^{(f)} + Q_k^{(g)} = Q_I \tag{3.2.43}$$

where I is the global node number assigned to the nodal point that is common to the three elements, and Q_I is the value of externally applied source, if any (otherwise zero), at this node [the sign of Q_I must be consistent with the sign of Q_i^e in Fig. 3.2.3(b)]. For the case shown in Fig. 3.2.8, we have

$$Q_n^e + Q_1^{e+1} = \begin{cases} 0 & \text{if no external point source is applied} \\ Q_I & \text{if an external point source of magnitude } Q_I \text{ is applied} \end{cases} \tag{3.2.44}$$

Figure 3.2.8 Assembly of two Lagrange elements: (a) continuity of the primary variable; (b) balance of the secondary variables.

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The balance of secondary variables can be interpreted as the continuity of $a(du/dx)$ [not $a(du_h^e/dx)$] at the node (say, global node I) common to elements Ω_e and Ω_{e+1} when no change in adu/dx is imposed externally:

$$\left(a\frac{du}{dx}\right)_I^e = \left(a\frac{du}{dx}\right)_I^{e+1}$$

or

$$\left(a\frac{du}{dx}\right)_I^e + \left(-a\frac{du}{dx}\right)_I^{e+1} = 0 \rightarrow Q_n^e + Q_1^{e+1} = 0 \tag{3.3.45a}$$

If there is a discontinuity of magnitude Q_I in $a\frac{du}{dx}$ in going from one side of the node to the other side (in the positive x direction), we impose

$$\left(a\frac{du}{dx}\right)_I^e + \left(-a\frac{du}{dx}\right)_I^{e+1} = Q_I \rightarrow Q_n^e + Q_1^{e+1} = Q_I \tag{3.3.45b}$$

The interelement continuity of the primary variables can be imposed by simply renaming the variables of all elements connected to the common node. For the connection in Eq. (3.2.41), we simply use the name

$$u_i^{(e)} = u_j^{(f)} = u_k^{(g)} \equiv U_I \tag{3.2.46}$$

where I is the global node number at which the three elements are connected. For example, for a mesh of N linear finite elements ($n = 2$) connected in series [see Fig. 3.2.2(b)], we have

$$u_1^1 = U_1,\ u_2^1 = u_1^2 = U_2,\ u_2^2 = u_1^3 = U_3, \ldots, u_2^{N-1} = u_1^N = U_N,\ u_2^N = U_{N+1}$$

Figure 3.2.9 shows the connected finite element solution, u_h, composed of the element solutions u_h^e. From the connected solution, we can identify the *global interpolation functions* Φ_I, which can be defined in terms of the element interpolation functions ψ_i^e corresponding to the global node I, as shown in Fig. 3.2.9.

To enforce balance of the secondary variables, it is clear that we can set [see Eq. (3.2.44)] $Q_i^{(e)} + Q_j^{(f)} + Q_k^{(g)}$ equal to zero or to a specified value Q_I only if we have such an expression in the finite element equations. To obtain such expressions, it is clear that we must add the ith equation of the element Ω_e, the jth equation of element Ω_f, and the kth equation of element Ω_e. For the case shown in Fig. 3.2.8, the nth equation of element Ω_e must be added to the first equation of the element Ω_{e+1}, i.e., we add

$$\sum_{j=1}^{n} K_{nj}^e u_j^e = f_n^e + Q_n^e$$

and

$$\sum_{j=1}^{n} K_{1j}^{e+1} u_j^{e+1} = f_1^{e+1} + Q_1^{e+1}$$

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Figure 3.2.9 Global interpolation functions for the (*a*) linear elements and (*b*) quadratic elements.

to obtain

$$\sum_{j=1}^{n} \left(K_{nj}^e u_j^e + K_{1j}^{e+1} u_j^{e+1}\right) = f_n^e + f_1^{e+1} + \left(Q_n^e + Q_1^{e+1}\right)$$
$$= f_n^e + f_1^{e+1} + Q_I \tag{3.2.47}$$

This process reduces the number of equations from $2N$ to $N + 1$ in a mesh of N linear elements. The first equation of the first element and the last equation of the last element

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will remain unchanged, except for renaming of the primary variables. The left-hand side of (3.2.47) can be written in terms of the global nodal values as

$$\begin{aligned}
&\left(K_{n1}^e\, u_1^e + K_{n2}^e u_2^e + \cdots + K_{nn}^e u_n^e\right) + \left(K_{11}^{e+1} u_1^{e+1} + K_{12}^{e+1} u_2^{e+1} + \cdots + K_{1n}^{e+1} u_n^{e+1}\right) \\
&\quad = \left(K_{n1}^e U_N + K_{n2}^e U_{N+1} + \cdots + K_{nn}^e U_{N+n-1}\right) \\
&\qquad + \left(K_{11}^{e+1} U_{N+n-1} + K_{12}^{e+1} U_{N+n} + \ldots + K_{1n}^{e+1} U_{N+2n-2}\right) \\
&\quad = K_{n1}^e U_N + K_{n2}^e U_{N+1} + \cdots + K_{n(n-1)}^e U_{N+n-2} + \left(K_{nn}^e + K_{11}^{e+1}\right) U_{N+n-1} \\
&\qquad + K_{12}^{e+1} U_{N+n} + \ldots + K_{1n}^{e+1} U_{N+2n-2}
\end{aligned} \tag{3.2.48}$$

where $N = (n-1)e + 1$. For a mesh of N linear elements ($n = 2$), we have

$$\begin{aligned}
K_{11}^1 U_1 + K_{12}^1 U_2 &= f_1^1 + Q_1^1 \quad \text{(unchanged)} \\
K_{21}^1 U_1 + \left(K_{22}^1 + K_{11}^2\right) U_2 + K_{12}^2 U_3 &= f_2^1 + f_1^2 + Q_2^1 + Q_1^2 \\
K_{21}^2 U_2 + \left(K_{22}^2 + K_{11}^3\right) U_3 + K_{12}^3 U_4 &= f_2^2 + f_1^3 + Q_2^2 + Q_1^3 \\
&\vdots \\
K_{21}^{N-1} U_{N-1} + \left(K_{22}^{N-1} + K_{11}^N\right) U_N + K_{12}^N U_{N+1} &= f_2^{N-1} + f_1^N + Q_2^{N-1} + Q_1^N \\
K_{21}^N U_N + K_{22}^N U_{N+1} &= f_2^N + Q_2^N \quad \text{(unchanged)}
\end{aligned} \tag{3.2.49}$$

These are called the *assembled equations*. They contain the sum of coefficients and source terms at nodes common to two elements. Note that the numbering of the global equations corresponds to the numbering of the global primary degrees of freedom, U_I. This correspondence carries the symmetry of element matrices to the global matrix. Equations (3.2.49) can be expressed in matrix form as

$$\begin{bmatrix}
K_{11}^1 & K_{12}^1 & & & \\
K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & \mathbf{0} & \\
 & K_{21}^2 & K_{22}^2 + K_{11}^3 & & \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
 & \mathbf{0} & & K_{22}^{N-1} + K_{11}^N & K_{12}^N \\
 & & & K_{21}^N & K_{22}^N
\end{bmatrix}
\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_N \\ U_{N+1} \end{Bmatrix}
= \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \vdots \\ f_2^{N-1} + f_1^N \\ f_2^N \end{Bmatrix}
+ \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ \vdots \\ Q_2^{N-1} + Q_1^N \\ Q_2^N \end{Bmatrix} \tag{3.2.50}$$

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Figure 3.2.10 The geometry and finite element mesh of a three-bar structure.

Note that the discussion of assembly in Eqs. (3.2.49) and (3.2.50) is based on the assumption that elements are connected in series. In general, several elements can be connected at a global node, and the elements do not have to be consecutively numbered. In that case, the coefficients coming from all elements connected at that global node will add up. For example, consider the structure consisting of three bar elements shown in Fig. 3.2.10. Suppose that the connecting bar is rigid (i.e., not deformable) and is constrained to remain horizontal at all times. Then the continuity and force balance conditions for the structure are

$$u_2^1 = u_1^3 = u_2^2 \equiv U_3, \quad Q_2^1 + Q_1^3 + Q_2^2 = 2P \tag{3.2.51}$$

To enforce these conditions, we must add the second equation of element 1, the first equation of element 3, and the second equation of element 2:

$$\begin{aligned}(K_{21}^1 u_1^1 + K_{22}^1 u_2^1) + (K_{11}^3 u_1^3 + K_{12}^3 u_2^3) + (K_{21}^2 u_1^2 + K_{22}^2 u_2^2) \\ = f_2^1 + f_1^3 + f_2^2 + Q_2^1 + Q_1^3 + Q_2^2\end{aligned} \tag{3.2.52}$$

We note the following correspondence of local and global nodal values (see Fig. 3.2.10):

$$u_1^1 = U_1, \quad u_1^2 = U_2, \quad u_2^1 = u_1^3 = u_2^2 = U_3, \quad u_2^3 = U_4$$

Hence, (3.2.52) becomes

$$\begin{aligned}K_{21}^1 U_1 + K_{21}^2 U_2 + (K_{22}^1 + K_{11}^3 + K_{22}^2) U_3 + K_{12}^3 U_4 \\ = f_2^1 + f_1^3 + f_2^2 + Q_2^1 + Q_1^3 + Q_2^2 \\ = f_2^1 + f_1^3 + f_2^2 + 2P\end{aligned}$$

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The other equations remain unchanged, except for renaming of the primary variables. The assembled equations are

$$\begin{bmatrix} K_{11}^1 & 0 & K_{12}^1 & 0 \\ 0 & K_{11}^2 & K_{12}^2 & 0 \\ K_{21}^1 & K_{21}^2 & \hat{K} & K_{12}^3 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_1^2 \\ f_2^1 + f_1^3 + f_2^2 \\ f_2^3 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 + Q_1^3 + Q_2^2 \\ Q_2^3 \end{Bmatrix} \tag{3.2.53}$$

where $\hat{K} = K_{22}^1 + K_{11}^3 + K_{22}^2$. Note that all f_i^e are zero for the problem in Fig. 3.2.10 because there is no distributed axial force ($f = 0$ for all elements).

The coefficients of the assembled matrix can be obtained directly. We note that the global coefficient K_{IJ} is a physical property of the system, relating global node I to global node J. For axial deformation of bars, K_{IJ} denotes the force required at node I to induce a unit displacement at node J, while the displacements at all other nodes are zero. Therefore, K_{IJ} is equal to the sum of all K_{ij}^e for which i corresponds to I and j corresponds to J, and i and j are the local nodes of the element Ω_e. Thus, if we have a correspondence between element node numbers and global node numbers, then the assembled global coefficients can readily be written in terms of the element coefficients. The correspondence can be expressed through a matrix $[B]$, called the *connectivity matrix,* whose coefficient b_{ij} has the following meaning: b_{ij} is the global node number corresponding to the jth node of element i. For example, for the structure shown in Fig. 3.2.10, the matrix $[B]$ is of order 3×2 (3 elements and 2 nodes per element):

$$[B] = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

This array can be used in a variety of ways—not only for assembly, but also in the computer implementation of finite element computations. The matrix $[B]$ is used to assemble coefficient matrices as follows:

$K_{11}^1 = K_{11}$ because local node 1 of element 1 corresponds to global node 1.
$K_{12}^1 = K_{13}$ because local nodes 1 and 2 of element 1 correspond to global nodes 1 and 3, respectively

and so on. When more than one element is connected at a global node, the element coefficients are to be added. For example, global node 3 appears in all three rows (i.e., elements) of the matrix $[B]$, implying that all three elements are connected to a global node 3. More specifically, it indicates that node 2 of element 1, node 2 of element 2, and node 1 of element 3 are the same as global node 3. Hence,

$$K_{22}^1 + K_{22}^2 + K_{11}^3 = K_{33}$$

Assembly by hand can be carried out by examining the finite element mesh of the problem. For the mesh shown in Fig. 3.2.10, we have

$K_{23} = K_{12}^2$ because global node 2 is the same as node 1 and global node 3 is the same as node 2 of element 2.

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$K_{24} = 0$ because global nodes 2 and 4 do not belong to the same element
$K_{33} = K_{22}^1 + K_{22}^2 + K_{11}^3$

and so on. The problem can also be solved by numbering global nodes 1 and 2 to be the same node. This leads to a 3×3 global system of equations.

In summary, assembly of finite elements is carried out by imposing interelement continuity of primary variables and balance of secondary variables. Renaming the primary variables of an element in terms of the global primary variables and using the correspondence between the local and global nodes allows the assembly. When certain primary nodal values are not required to be continuous across elements (as dictated by physics or the variational formulation of the problem), such variables may be *condensed* out at the element level before assembling the elements.

3.2.5 Imposition of Boundary Conditions

The discussion in Section 3.2.1–3.2.4 is valid for any differential equation that is a special case of the model equation (3.2.1). Each problem differs from the other in the specification of the data (a, c, f) and boundary conditions on the primary and secondary variables (u, Q). Here we discuss how to impose the boundary conditions of a problem on the assembled set of finite element (algebraic) equations. To this end, we use the problem in Fig. 3.2.10. The boundary conditions of the problem are evident, at least for an engineering student, from the structure shown. The known primary degrees of freedom (i.e., displacements) are

$$u_1^1 = U_1 = 0, \quad u_1^2 = U_2 = 0, \quad u_2^3 = U_4 = 0 \tag{3.3.54a}$$

The known secondary degrees of freedom (i.e., forces) are

$$Q_2^1 + Q_2^2 + Q_1^3 = 2P \tag{3.3.54b}$$

The forces Q_1^1, Q_1^2, and Q_2^3 are unknown (reaction forces), and they can be determined after the primary degrees of freedom are determined.

Imposing the boundary conditions (3.2.54*a*) and (3.2.54*b*) on the assembled system of equations (3.2.53) with $f_i^e = 0$, we obtain

$$\begin{bmatrix} K_{11}^1 & 0 & K_{12}^1 & 0 \\ 0 & K_{11}^2 & K_{12}^2 & 0 \\ K_{21}^1 & K_{21}^2 & K_{22}^1 + K_{22}^2 + K_{11}^3 & K_{12}^3 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} U_1 = 0 \\ U_2 = 0 \\ U_3 \\ U_4 = 0 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ 2P \\ Q_2^3 \end{Bmatrix} \tag{3.2.55}$$

This contains four equations in four unknowns: U_3, Q_1^1, Q_1^2, and Q_2^3.

3.2.6 Solution of Equations

As a standard procedure in finite element analysis, the unknown primary degrees of freedom are determined first by *considering the algebraic equations corresponding to the unknown primary variables*. Thus, in the present case, we consider the third equation in (3.2.55) to solve for U_3:

$$K_{21}^1 U_1 + K_{21}^2 U_2 + (K_{22}^1 + K_{22}^2 + K_{11}^3) U_3 + K_{12}^3 U_4 = 2P$$

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or

$$(K_{22}^1 + K_{22}^2 + K_{11}^3)U_3 = 2P - (K_{21}^1 U_1 + K_{21}^2 U_2 + K_{12}^3 U_4) \tag{3.2.56}$$

Equation (3.2.56) is called the *condensed equation* for the unknown U_3. The term in parentheses on the right-hand side is zero because all specified displacements are zero in the present problem. Hence, the solution is given by

$$U_3 = \frac{2P}{K_{22}^1 + K_{22}^2 + K_{11}^3} \tag{3.2.57}$$

The unknown secondary variables are determined by considering the remaining equations of (3.2.55), i.e., those that contain the unknown secondary variables:

$$\begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^3 \end{Bmatrix} = \begin{bmatrix} K_{11}^1 & 0 & K_{12}^1 & 0 \\ 0 & K_{11}^2 & K_{12}^2 & 0 \\ 0 & 0 & K_{21}^3 & K_{22}^3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} K_{12}^1 U_3 \\ K_{12}^2 U_3 \\ K_{21}^3 U_3 \end{Bmatrix} \tag{3.2.58}$$

because U_1, U_2, and U_4 are zero.

It is possible, although not common with commercial finite element programs, to move all the unknowns to the left-hand side of (3.2.53) and solve for them all at once. But this process destroys the symmetry of the coefficient matrix and requires more computational time in practical problems.

To obtain numerical values, we use the geometric and material data shown in Fig. 3.2.10. We obtain

$$K_{11}^1 = \frac{E_b A_b}{h_1} = \frac{(14 \times 10^6)(\pi d_b^2/4)}{8 \times 12} = 87,693 \text{ lb-in.}$$

$$K_{11}^2 = \frac{E_s A_s}{h_2} = \frac{(30 \times 10^6)(\pi d_s^2/4)}{4 \times 12} = 122,718 \text{ lb-in.}$$

$$K_{11}^3 = \frac{E_a A_a}{h_3} = \frac{(10 \times 10^6)(\pi d_a^2/4)}{6 \times 12} = 109,083 \text{ lb-in.}$$

The displacement of node 3 is $U_3 = 0.01878$ in., reaction at node 1 is $Q_1^1 = -1,647$ lbs, $Q_1^2 = -2,305$ lbs and reaction at node 4 is $Q_2^3 = -2,049$ lbs. Hence, the stresses in elements 1, 2, and 3 are $\sigma^1 = 2,739$ psi, $\sigma^2 = 11,739$ psi, and $\sigma^3 = -2,609$ psi, respectively.

In general, the assembled finite element equations can be partitioned according to the sets of specified and unspecified displacements into the following form:

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^1 \\ \mathbf{U}^2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \end{Bmatrix} \tag{3.2.59}$$

where $\mathbf{U}^1$ is the column of known (i.e., specified) primary variables, $\mathbf{U}^2$ is the column of unknown primary variables, $\mathbf{F}^1$ is the column of unknown secondary variables, and $\mathbf{F}^2$ is the column of known secondary variables. Writing (3.2.59) as two matrix equations, we obtain

$$\mathbf{K}^{11}\mathbf{U}^1 + \mathbf{K}^{12}\mathbf{U}^2 = \mathbf{F}^1 \tag{3.2.60a}$$

$$\mathbf{K}^{21}\mathbf{U}^1 + \mathbf{K}^{22}\mathbf{U}^2 = \mathbf{F}^2 \tag{3.2.60b}$$

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From (3.2.60*b*), we have

$$\mathbf{U}^2 = \left(\mathbf{K}^{22}\right)^{-1}\left(\mathbf{F}^2 - \mathbf{K}^{21}\mathbf{U}^1\right) \tag{3.2.60c}$$

Once $\mathbf{U}^2$ is known from (3.2.60*c*), the vector of unknown secondary variables, $\mathbf{F}^1$, can be computed using Eq. (3.2.60*a*).

In most finite element computer programs, element matrices are assembled as soon as they are generated and they are not stored in the memory of the computer. Thus, element equations are not available for postcomputation of the secondary variables. Also, due to the fact that the assembled coefficient matrix is modified (when operations are performed to invert a matrix) during the solution of equations, Eq. (3.2.60*a*) cannot be used. Therefore, secondary variables can only be computed using their definitions, as discussed next.

3.2.7 Postcomputation of the Solution

The solution of the finite element equations gives the nodal values U_I of the primary unknowns (e.g., displacement, velocity, or temperature). Once the nodal values of the primary variables are known, we can use the finite element approximation $u_h^e(x)$ to compute the desired quantities. The process of computing desired quantities in numerical form or graphical form from the known finite element solution is termed *postcomputation* or *postprocessing;* these phrases are meant to indicate that further computations are made after obtaining the solution of the finite element equations for the nodal values of the primary variables.

Postprocessing of the solution includes one or more of the following tasks:

1. Computation of the primary and secondary variables at points of interest; primary variables are known at nodal points.
2. Interpretation of the results to check whether the solution makes sense (an understanding of the physical process and experience are the guides when other solutions are not available for comparison).
3. Tabular and/or graphical presentation of the results.

To determine the solution u as a continuous function of position x, we return to the approximation (3.2.28) over each element:

$$u(x) \approx \begin{cases} u_h^1(x) = \displaystyle\sum_{j=1}^{n} u_j^1 \psi_j^1(x) \\ u_h^2(x) = \displaystyle\sum_{j=1}^{n} u_j^2 \psi_j^2(x) \\ \vdots \\ u_h^N(x) = \displaystyle\sum_{j=1}^{n} u_j^N \psi_j^N(x) \end{cases} \tag{3.2.61}$$

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where N is the number of elements in the mesh. Depending on the value of x, the corresponding element equation from (3.2.61) is used. The derivative of the solution is obtained by differentiating (3.2.61):

$$\frac{du}{dx} \approx \begin{cases} \dfrac{du_h^1}{dx} = \displaystyle\sum_{j=1}^{n} u_j^1 \frac{d\psi_j^1}{dx} \\ \dfrac{du_h^2}{dx} = \displaystyle\sum_{j=1}^{n} u_j^2 \frac{d\psi_j^2}{dx} \\ \vdots \\ \dfrac{du_h^N}{dx} = \displaystyle\sum_{j=1}^{n} u_j^N \frac{d\psi_j^N}{dx} \end{cases} \tag{3.2.62}$$

Note that the derivative du_h^e/dx of the finite element approximation u_h^e based on Lagrange interpolation is discontinuous, for any order element, at the nodes connecting different elements because the continuity of the derivative of the finite element solution at the connecting nodes is not imposed. In the case of linear elements, the derivative of the solution is constant within each element (see Fig. 3.2.11).

The secondary variables Q_j^e can be computed in two different ways. In Eq. (3.2.58), we determined the unknown secondary variables Q_1^1, Q_1^2, and Q_2^3 from the assembled equations of the problem in Fig. 3.2.10. Since the assembled equations often represent the equilibrium relations of a system, the Q_I^e computed from them will be denoted by $(Q_I^e)_{\text{equil}}$. The Q_I^e can also be determined using the definitions in (3.2.6), replacing u with U. We shall denote Q_I^e computed in this way by $(Q_I^e)_{\text{def}}$. Since $(Q_I^e)_{\text{def}}$ are calculated using the approximate solution u_h^e, they are not as accurate as $(Q_I^e)_{\text{equil}}$. However, in finite element computer codes, $(Q_I^e)_{\text{def}}$ are calculated instead of $(Q_I^e)_{\text{equil}}$. This is primarily due to computational reasons. Recall that, in arriving at the results in (3.2.58), we used part of the assembled coefficient matrix. In the numerical solution of simultaneous algebraic equations in a computer, the original assembled coefficient matrix is often modified, and therefore the coefficients needed for the determination of the secondary variables are not available, unless they are saved in an additional array. For the problem in Fig. 3.2.10, we

Figure 3.2.11 Gradient of the finite element solution.

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have

$$
\begin{aligned}
(Q_1^1)_{\text{def}} &= -\left(EA\frac{du_h^1}{dx}\right)\bigg|_{x=0} = -EA\frac{U_3-U_1}{h_1} = -\frac{EA}{h_1}U_3 = K_{12}^1 U_3 \\
(Q_1^2)_{\text{def}} &= -\left(EA\frac{du_h^2}{dx}\right)\bigg|_{x=0} = K_{12}^2 U_3 \\
(Q_2^3)_{\text{def}} &= \left(EA\frac{du_h^3}{dx}\right)\bigg|_{x=h_1+h_3} = EA\frac{U_4-U_3}{h_3} = -\frac{EA}{h_3}U_3 = K_{21}^3 U_3
\end{aligned}
\tag{3.2.63}
$$

where h_1 and h_3 are the lengths of elements 1 and 3, respectively.

The secondary variables computed using the definitions (3.2.63) are the same as those derived from the assembled equations for the problem in Fig. 3.2.10. This equality is *not to be expected in general*. In fact, when the source vector f is not zero, the secondary variables computed from the definitions (3.2.6) will be in error compared with those computed from the assembled equations. The error decreases as the number of elements or the degree of interpolation is increased.

This completes the basic steps involved in the finite element analysis of the model equation (3.2.1). Next, we consider an example of application of the finite element method. Additional applications of the method to one-dimensional problems of heat transfer, fluid mechanics, and solid mechanics, are presented in Chapter 4.

Example 3.2.1

We wish to use the finite element method to solve the problem described by the following differential equation and boundary conditions (see Example 2.5.1):

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \quad \text{for} \quad 0 < x < 1 \tag{3.2.64}$$

$$u(0) = 0, \quad u(1) = 0 \tag{3.2.65}$$

The differential equation in (3.2.64) is a special case of the model equation (3.2.1) for the following data: $a = 1$, $c = -1$, and $f(x) = -x^2$. Hence, the coefficient matrix is given by

$$K_{ij}^e = \int_{x_b}^{x_a} \left(\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} - \psi_i^e\psi_j^e\right) dx$$

$$f_i^e = \int_{x_b}^{x_a} (-x^2)\psi_i^e \, dx$$

First we consider a mesh of four linear elements and next a mesh of two quadratic elements to solve the problem.

Linear Elements. The element coefficient matrix is given by [see Eq. (3.2.34), with $a_e = 1$, $c_e = -1$, $h_e = \frac{1}{4}$]

$$[K^e] = \frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix} = \begin{bmatrix} 3.9167 & -4.0417 \\ -4.0417 & 3.9167 \end{bmatrix}$$

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The coefficients f_i^e are evaluated as

$$f_1^e = -\frac{1}{h_e}\left[\frac{x_b}{3}\left(x_b^3 - x_a^3\right) - \frac{1}{4}\left(x_b^4 - x_a^4\right)\right]$$

$$f_2^e = -\frac{1}{h_e}\left[\frac{1}{4}\left(x_b^4 - x_a^4\right) - \frac{x_a}{3}\left(x_b^3 - x_a^3\right)\right]$$

Element 1. $(h_1 = \frac{1}{4}, x_a = 0, x_b = h_1 = \frac{1}{4})$:

$$f_1^1 = -0.001302, \quad f_2^1 = -0.003906$$

Element 2. $(h_2 = \frac{1}{4}, x_a = h_1 = \frac{1}{4}, x_b = h_1 + h_2 = \frac{1}{2})$:

$$f_1^2 = -0.014323, \quad f_2^2 = -0.022135$$

Element 3. $(h_3 = \frac{1}{4}, x_a = h_1 + h_2 = \frac{1}{2}, x_b = h_1 + h_2 + h_3 = \frac{3}{4})$:

$$f_1^3 = -0.042969, \quad f_2^3 = -0.05599$$

Element 4. $(h_4 = \frac{1}{4}, x_a = h_1 + h_2 + h_3 = \frac{3}{4}, x_b = h_1 + h_2 + h_3 + h_4 = 1)$:

$$f_1^4 = -0.08724, \quad f_2^4 = -0.10547$$

The assembled set of equations is

$$\begin{bmatrix} 3.9167 & -4.0417 & 0 & 0 & 0 \\ -4.0417 & 7.8333 & -4.0417 & 0 & 0 \\ 0 & -4.0417 & 7.8333 & -4.0417 & 0 \\ 0 & 0 & -4.0417 & 7.8333 & -4.0417 \\ 0 & 0 & 0 & -4.0417 & 3.9167 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= -\begin{Bmatrix} 0.00130 \\ 0.01823 \\ 0.06510 \\ 0.14323 \\ 0.10547 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

Since $U_1 = 0$ and $U_5 = 0$, the condensed equations are obtained by omitting the first and fifth rows and columns of the assembled equations. The condensed equations are

$$\begin{bmatrix} 7.8333 & -4.0417 & 0 \\ -4.0417 & 7.8333 & -4.0417 \\ 0 & -4.0417 & 7.8333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\begin{Bmatrix} 0.01823 \\ 0.06510 \\ 0.14323 \end{Bmatrix}$$

The solution is (obtained using a computer)

$$U_1 = 0.0, \; U_2 = -0.02323, \; U_3 = -0.04052, \; U_4 = -0.03919, \; U_5 = 0.0$$

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The secondary variables can be computed using either the definition or from the element equations. We have

$$(Q_1^1)_{def} \equiv \left(-a\frac{du_h^1}{dx}\right)\bigg|_{x=0} = \frac{U_1 - U_2}{h} = 0.09293$$

$$(Q_2^4)_{def} \equiv \left(a\frac{du_h^4}{dx}\right)\bigg|_{x=1} = \frac{U_5 - U_4}{h} = 0.15676$$

$$(Q_1^1)_{equil} = K_{11}^1 U_1 + K_{12}^1 U_2 - f_1^1 = 0.09520$$

$$(Q_2^4)_{equil} = K_{21}^4 U_4 + K_{22}^4 U_5 - f_2^4 = 0.26386$$

Quadratic Elements. The element coefficient matrix is given by [see Eq. (3.2.37a), with $a_e = 1,\ c_e = -1,\ h_e = \frac{1}{2}$]

$$[K^e] = \frac{1}{60}\begin{bmatrix} 276 & -322 & 41 \\ -322 & 624 & -322 \\ 41 & -322 & 276 \end{bmatrix} = \begin{bmatrix} 4.6000 & -5.3667 & 0.6833 \\ -5.3667 & 10.4000 & -5.3667 \\ 0.6833 & -5.3667 & 4.6000 \end{bmatrix}$$

The coefficients f_i^e are evaluated as

$$f_1^e = -\frac{h_e}{60}\left(-h_e^2 + 10x_a^2\right)$$

$$f_2^e = -\frac{h_e}{15}\left(3h_e^2 + 10x_a^2 + 10x_a^2 h_e\right)$$

$$f_3^e = -\frac{h_e}{60}\left(9h_e^2 + 20x_a^2 + 20x_a h_e\right)$$

Element 1. ($h_1 = \frac{1}{2}, x_a = 0, x_b = h_1 = \frac{1}{2}$):

$$f_1^1 = 0.00208,\ f_2^1 = -0.02500,\ f_3^1 = -0.01875$$

Element 2. ($h_2 = \frac{1}{2}, x_a = h_1 = \frac{1}{2}, x_b = h_1 + h_2 = 1$):

$$f_1^2 = -0.01875,\ f_2^2 = -0.19167,\ f_3^2 = -0.08125$$

The assembled set of equations are

$$\begin{bmatrix} 4.6000 & -5.3667 & 0.6833 & 0 & 0 \\ -5.3667 & 10.4000 & -5.3667 & 0 & 0 \\ 0.6833 & -5.3667 & 9.2000 & -5.3667 & 0.6833 \\ 0 & 0 & -5.3667 & 10.4000 & -5.3667 \\ 0 & 0 & 0.6833 & -5.3667 & 4.6000 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= -\begin{Bmatrix} -0.00208 \\ 0.02500 \\ 0.03750 \\ 0.19167 \\ 0.08125 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_2^2 \\ Q_3^2 \end{Bmatrix}$$

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Again, using $U_1 = 0$ and $U_5 = 0$, the condensed equations are obtained as

$$\begin{bmatrix} 10.4000 & -5.3667 & 0 \\ -5.3667 & 9.2000 & -5.3667 \\ 0 & -5.3667 & 10.4000 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\begin{Bmatrix} 0.02500 \\ 0.03750 \\ 0.19167 \end{Bmatrix}$$

The solution is

$$U_1 = 0.0,\ U_2 = -0.02345,\ U_3 = -0.04078,\ U_4 = -0.03947,\ U_5 = 0.0$$

The secondary variables can be computed using either the definition or from the element equations. We have

$$(Q_1^1)_{def} \equiv \left(-a\frac{du_h^1}{dx}\right)\bigg|_{x=0} = -\left[\frac{U_1}{h}\left(-3+4\frac{x}{h}\right) + \frac{U_2}{h}\left(4-8\frac{x}{h}\right) + \frac{U_3}{h}\left(-1+4\frac{x}{h}\right)\right]_{x=0}$$

$$= \left(\frac{1}{h}U_1 - \frac{4}{h}U_2 + \frac{1}{h}U_3\right) = 0.10602$$

Table 3.2.1 Comparison of the finite element results with the exact solution $(-10u)$ of eqs. (3.2.64) and (3.2.65).

x	FEM* solution 4L	FEM* solution 2Q	FEM* solution 4Q	Ritz† Solution	Exact Solution
0.0000	**0.0000**‡	**0.0000**	**0.0000**	0.0000	0.0000
0.0625	0.0581	0.0644	0.0602	0.0596	0.0598
0.1250	0.1162	0.1249	**0.1193**	0.1191	0.1192
0.1875	0.1743	0.1816	0.1771	0.1776	0.1775
0.2500	**0.2323**	**0.2345**	**0.2337**	0.2339	0.2337
0.3125	0.2756	0.2835	0.2876	0.2868	0.2866
0.3750	0.3188	0.3288	**0.3345**	0.3347	0.3345
0.4375	0.3620	0.3702	0.3745	0.3757	0.3755
0.5000	**0.4052**	**0.4078**	**0.4076**	0.4076	0.4076
0.5625	0.4019	0.4403	0.4298	0.4282	0.4283
0.6250	0.3986	0.4490	**0.4350**	0.4347	0.4350
0.6875	0.3952	0.4338	0.4231	0.4244	0.4246
0.7500	**0.3919**	**0.3947**	**0.3942**	0.3940	0.3942
0.8125	0.2939	0.3318	0.3421	0.3401	0.3402
0.8750	0.1960	0.2451	**0.2591**	0.2592	0.2590
0.9375	0.0980	0.1345	0.1450	0.1472	0.1470
1.0000	**0.0000**	**0.0000**	**0.0000**	0.0000	0.0000

* FEM, finite element method

† Three-parameter Ritz solution from Example 2.5.1.

‡ Numbers in bold are the nodal values; others are the interpolated values (4L = 4 linear elements; 2Q = 2 quadratic elements; 4Q = 4 quadratic elements).

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Figure 3.2.12 Comparison of the finite element solutions u_h with the exact and Ritz solutions.

Figure 3.2.13 Comparison of the finite element solutions du_h/dx with the exact and Ritz solutions.

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$$(Q_3^2)_{def} \equiv \left(a\frac{du_h^2}{dx}\right)\bigg|_{x=1} = \left[\frac{U_3}{h}\left(-3+4\frac{\bar{x}}{h}\right)+\frac{U_4}{h}\left(4-8\frac{\bar{x}}{h}\right)+\frac{U_5}{h}\left(-1+4\frac{\bar{x}}{h}\right)\right]_{\bar{x}=h}$$

$$= \left(\frac{1}{h}U_3 - \frac{4}{h}U_4 + \frac{3}{h}U_5\right) = 0.23442$$

$$(Q_1^1)_{equil} = K_{11}^1 U_1 + K_{12}^1 U_2 + K_{13}^1 U_3 - f_1^1 = 0.10006$$

$$(Q_3^2)_{equil} = K_{13}^2 U_3 + K_{23}^2 U_4 + K_{33}^2 U_5 - f_3^2 = 0.26521$$

Table 3.2.1 contains a comparison of the finite element results against the three-parameter Ritz solution and the exact solution (see Example 2.5.1 and Table 2.5.1). The four linear-element and two quadratic-element solutions are not as accurate as the three-parameter Ritz solution. However, the four quadratic-element solution at the nodes (details are not given here) is virtually the same as the exact solution. Plots of u_h and du_h/dx are shown in Figs. 3.2.12 and 3.2.13, respectively. Note that the finite element solutions for the derivative [postcomputed from $u_h(x)$] are discontinuous between the elements; the value of the difference in du_h/dx between elements reduces as we move from linear to quadratic elements and from coarse to refined meshes.

3.3 SOME REMARKS

A few remarks are in order on the steps described for the model equation.

Remark 1. Although the Ritz method was used to set up the element equations, any other method, such as a weighted-residual (i.e., the least-squares or Galerkin) method could be used.

Remark 2. Steps 1–6 (see Table 3.1.1) are common for any problem. The derivation of interpolation functions depends only on the element geometry and on the number and position of nodes in the element. The number of nodes in the element and the degree of approximation used are related.

Remark 3. The finite element equations (3.2.33*b*) are derived for the linear operator equation

$$A(u) = f, \quad \text{where} \quad A = -\frac{d}{dx}\left(a\frac{d}{dx}\right) + c \tag{3.3.1}$$

Hence, they are valid for any physical problem that is described by the operator equation $A(u) = f$ or its special cases. One need only interpret the quantities appropriately. Examples of problems described by this operator are listed in Table 3.2.1. Thus, a computer program written for the finite element analysis of (3.2.1) can be used to analyze any of the problems in Table 3.2.1. Also, note that the data $a = a(x)$, $c = c(x)$, and $f = f(x)$ can be different in different elements.

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Remark 4. Integration of the element matrices in (3.2.31*b*) can be implemented on a computer using numerical integration. When these integrals are algebraically complicated, we have no other choice but numerical integration (see Chapter 7).

Remark 5. As discussed in (3.2.45*a*) and (3.2.45*b*) the point sources at the nodes are included in the finite element model via the balance of sources at the nodes. Thus, in constructing finite element meshes, we should include nodes at the locations of point sources. If a point source does not occur at a node, it is possible to "distribute" it to the element nodes, consistent with the finite element approximation. Let Q_0 denote a point source at a point $\bar{x} = \bar{x}_0$, $\bar{x}_1^e \le \bar{x}_0 \le \bar{x}_n^e$, where $\bar{x}$ is the local coordinate. The point source Q_0 can be represented as a "function" with the help of the Dirac delta function

$$f(\bar{x}) = Q_0 \delta(\bar{x} - \bar{x}_0) \tag{3.3.2}$$

where the Dirac delta function $\delta(\cdot)$ is defined by

$$\int_{-\infty}^{\infty} F(\bar{x})\delta(\bar{x} - \bar{x}_0)\, d\bar{x} = F(\bar{x}_0) \tag{3.3.3}$$

The contribution of the function $f(\bar{x})$ to the nodes of the element $\Omega_e = (\bar{x}_1, \bar{x}_n)$ is computed from [see Eq. (3.2.31*b*)]

$$f_i^e = \int_0^{h_e} f(\bar{x})\psi_i^e(\bar{x})\, d\bar{x} = \int_0^{h_e} Q_0 \delta(\bar{x} - \bar{x}_0)\psi_i^e(\bar{x})\, d\bar{x} = Q_0 \psi_i^e(\bar{x}_0) \tag{3.3.4}$$

where ψ_i^e are the interpolation functions of the element Ω_e. Thus, the point source Q_0 is distributed to the element node i by the value $Q_0\psi_i^e(\bar{x}_0)$. Equation (3.3.4) holds for any element, irrespective of the degree of the interpolation, the nature of the interpolation (i.e., Lagrange or Hermite polynomials), or the dimension (i.e., one-dimensional, two-dimensional, or three-dimensional) of the elements. For one-dimensional linear Lagrange interpolation functions, Eq. (3.3.4) yields

$$f_1^e = Q_0\left(1 - \frac{\bar{x}_0}{h_e}\right), \quad f_2^e = Q_0\left(\frac{\bar{x}_0}{h_e}\right)$$

Note that $f_1^e + f_2^e = Q_0$. When $\bar{x}_0 = h_e/2$, we have $f_1^e = f_2^e = Q_0/2$, as expected.

Remark 6. There are three sources of error that may contribute to the inaccuracy of the finite element solution of a problem:

1. *Domain approximation error,* which is due to the approximation of the domain.
2. *Computational errors,* which are due to inexact evaluation of the coefficients K_{ij}^e and f_i^e, or are introduced owing to the finite arithmetic in a computer.
3. *Approximation error,* which is due to approximation of the solution by piecewise polynomials.

For the structure shown in Fig. 3.2.10, the geometry of the problem is exactly represented and the linear approximation is able to represent the exact solution at the nodes when a is a constant, $c = 0$, and f is arbitrary [see Reddy (1986), p. 403; also, see Sec. 14.5]. Therefore, the first and third type of errors are zero. The only error that can be introduced into the final numerical results is possibly due to the computer evaluation of the coefficients K_{ij}^e and f_i^e and the solution of algebraic equations. However, in general, computational as well

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as approximation errors exist even in one-dimensional problems. Additional discussion of the errors in the finite element approximation can be found in Oden and Carey (1983) and Reddy (1986); also, see Sec. 14.5.

Remark 7. The approach used in matrix methods of structural analysis to solve the problem in Fig. 3.2.10 is not much different than that presented here. The difference lies only in the derivation of the element equations (3.2.39) for the case $c_e = 0$. In matrix methods of structural analysis, the element equations are obtained directly from the definitions of stress and strain and their relationship. For example, consider the free-body diagram of a bar element [see Fig. 3.2.3(*b*)]. From a course on mechanics of deformable solids, we have

$$\begin{aligned} \text{strain, } \varepsilon &= \text{elongation/original length} \\ \text{stress, } \sigma &= \text{Young's modulus} \times \text{strain} \\ \text{load, } P &= \text{stress} \times \text{area of cross section} \end{aligned}$$

The strain defined above is the average (or engineering) strain. Mathematically, strain for one-dimensional problems is defined as $\varepsilon = du/dx$, u being displacement, which includes rigid body motion as well as elongation of the bar. Hence, the compressive force at the left end of the bar element is

$$P_1^e = A_e\sigma_1^e = A_e E_e \varepsilon_1^e = A_e E_e \frac{u_1^e - u_2^e}{h_e} = \frac{A_e E_e}{h_e}(u_1^e - u_2^e)$$

where E_e is Young's modulus of the bar element. Similarly, the force at the right end is

$$P_2^e = \frac{A_e E_e}{h_e}(u_2^e - u_1^e)$$

In matrix form, these relations can be expressed as

$$\frac{A_e E_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} P_1^e \\ P_2^e \end{Bmatrix} \tag{3.3.5a}$$

which is the same as (3.2.39) obtained for a linear finite element with $a_e = A_e E_e$, $c_e = 0$, and $P_i^e = Q_i^e + f_i^e$. Note that in deriving the element equations, we have used knowledge of the mechanics of materials and the assumption that the strain is constant (or the displacement is linear) over the length of the element. If a higher-order representation of the displacement is required, we cannot write the force-displacement relations (3.3.5*a*) directly, i.e., the element equations of a quadratic finite elements cannot be derived using the arguments presented above.

A similar approach can be used to develop the relations between the temperatures and heats at the ends of an insulated fin. From a course on basic heat transfer, we have

$$\begin{aligned} \text{temperature gradient} &= \text{difference in temperature/length} \\ \text{heat flux, } q &= \text{conductivity} \times (-\text{temperature gradient}) \\ \text{heat, } Q &= \text{heat flux} \times \text{area of cross section} \end{aligned}$$

Then, if the temperature is assumed to vary linearly between the ends of the fin and there is no internal heat generation, the heat input at the left end of the fin is

$$Q_1^e = A_e q_1^e = A_e k_e \frac{T_1^e - T_2^e}{h_e} = \frac{A_e k_e}{h_e}(T_1^e - T_2^e)$$

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where k_e is thermal conductivity of the fin. Similarly, the heat *input* at the right end is

$$Q_2^e = A_e q_2^e = A_e k_e \frac{T_2^e - T_1^e}{h_e} = \frac{A_e k_e}{h_e}(T_2^e - T_1^e)$$

In matrix form, we have

$$\frac{A_e k_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{3.3.5b}$$

which are the same as (3.3.5*a*) with E_e replaced by k_e, T_i^e by u_i^e, and P_i^e by Q_i^e [and reverse the arrow at node 2 in Fig. 3.2.3(*b*)].

Equations of the type (3.3.5*a*) can also be derived for discrete systems (as opposed to a continuum) consisting of spring elements, pipe-flow elements, electrical resistor elements, and so on. These elements will be discussed in Chapter 4 using physical principles. However, such a direct approach cannot be used when the element data is not constant or when higher-order approximation of the dependent unknowns is used.

Remark 8. Another interpretation of (3.2.39) for $c_e = 0$ can be given in terms of the finite difference approximation. The axial force at any point x is given by $P(x) = EA\, du/dx$. Using the forward difference approximation, we approximate the derivative du/dx and write

$$-P_1^e \equiv P(x)|_{x_e} = E_e A_e[u(x_{e+1}) - u(x_e)]/h_e \tag{3.3.6a}$$

$$P_2^e \equiv P(x)|_{x_{e+1}} = E_e A_e[u(x_{e+1}) - u(x_e)]/h_e \tag{3.3.6b}$$

which are the same as (3.3.5*a*), with $u_1^e = u(x_e)$ and $u_2^e = u(x_{e+1})$. Note that no explicit approximation of $u(x)$ itself is assumed in writing (3.3.6*a*) and (3.3.6*b*), but the fact that we used values of the function from two consecutive points to define its slope implies that we assumed a linear approximation of the function. Thus, to compute the value of u at a point other than the nodes (or mesh points), linear interpolation must be used.

Remark 9. For the model problem considered, the element matrices $[K^e]$ in (3.2.31*b*) are symmetric: $K_{ij}^e = K_{ji}^e$. This enables one to compute K_{ij}^e $(I = 1, 2, \ldots, n)$ for $j \le i$ only. In other words, we need compute only the diagonal terms and the upper or lower diagonal terms. Because of the symmetry of the element matrices, the assembled global matrix will also be symmetric. Thus, we need to store only the upper triangle, including the diagonal, of the assembled matrix in a finite element computer program. Another property characteristic of the finite element method is the *sparseness* of the assembled matrix. Since $K_{IJ} = 0$, if global nodes I and J do not belong to the same finite element, the global coefficient matrix is *banded*, i.e., all coefficients beyond a certain distance from the diagonal are zero. The maximum of the distances between the diagonal element, including the latter, of a row and the last nonzero coefficient in that row is called the *half-bandwidth*. When a matrix is banded and symmetric, we need to store only entries in the upper or lower band of the matrix. Equation solvers written for the solution of banded symmetric equations are available for use in such cases (see Chapter 7 for additional discussion). The symmetry of the coefficient matrix depends on the type of the differential equation, its variational form, and the numbering of the finite element equations. The sparseness of the matrix is a result of the finite element interpolation functions, which have nonzero values only over an element of the domain (i.e., so-called "compactness" property of the approximation functions).

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Remark 10. The balance (or "equilibrium") of the secondary variables Q_i^e at the interelement boundaries is expressed by (3.2.43). This amounts to imposing the condition that the secondary variable adu/dx at the node, where u is the actual solution, be continuous. However, this does not imply continuity of adu_h^e/dx, where u_h^e is the finite element solution. Thus, we have

$$Q_2^e + Q_1^{e+1} = 0 \quad \text{or} \quad Q_0 \tag{3.3.7}$$

but

$$\left(a\frac{du_h^e}{dx}\right)\bigg|_{x_e} + \left(-a\frac{du_h^{e+1}}{dx}\right)\bigg|_{x_e} \neq 0 \quad \text{or} \quad Q_0 \tag{3.3.8}$$

In most books on the finite element method, this point is not made clear to the reader. These books consider the quadratic form or weak form of the total problem and omit the sum of the interelement contributions (for linear elements),

$$\sum_{e=1}^{N}\left(\sum_{i=1}^{2} Q_i^e u_i^e\right) \tag{3.3.9}$$

in the quadratic form (or functional) of the problem. However, this amounts to imposing equilibrium conditions of the form (3.3.8). When the secondary variable is specified to be nonzero (say, Q_0) at an interelement boundary (say, at global node 2), we have

$$Q_2^1 + Q_1^2 = Q_0$$

In other books, Q_0 is included in the functional as Q_0U_2, where U_2 is the value of u at global node 2.

To fully understand the difference between the direct use of the global statement versus using the assembly of element equations, consider the model problem described by Eqs. (3.2.1) and (3.2.2) with $c = 0$. The variational form of these equations over the entire domain is given by

$$0 = \int_0^L \left(a\frac{dv}{dx}\frac{du}{dx} - vq\right) dx - v(L)Q_0 \tag{3.3.10}$$

When u is approximated by functions that are defined only on a local interval (which is the case in the finite element method), use of the above variational form implies the omission of the sum of the interelement contributions of (3.3.9).

Consider a mesh of three linear elements. Since $\psi_i^e (e = 1, 2, 3)$ is zero in any element Ω_f for $e \neq f$, the (global) finite element solution for the entire domain is given by

$$u_h(x) = \sum_{e=1}^{3}\left(\sum_{i=1}^{2} u_i^e \psi_i^e\right) \equiv \sum_{I=1}^{4} U_I \Phi_I(x) \tag{3.3.11}$$

where $\Phi_I(x)$ $(I = 1, 2, 3, 4)$ are the piecewise-continuous *global interpolation functions* (see Fig. 3.2.9),

$$\Phi_I(x) = \begin{cases} \psi_2^{(I-1)}(x) & \text{for } x_{I-1} \le x \le x_I \\ \psi_1^{(I)}(x) & \text{for } x_I \le x \le x_{I+1} \end{cases} \tag{3.3.12}$$

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Substituting (3.3.11) for u and $v = \Phi_I$ into (3.3.10), we obtain

$$0 = \int_0^L \left[a \frac{d\Phi_I}{dx} \left(\sum_{J=1}^{4} U_J \frac{d\Phi_J}{dx} \right) - \Phi_I q \right] dx - \Phi_I(L) Q_0 \tag{3.3.13}$$

Since Φ_I is nonzero only between x_{I-1} and x_{I+1}, the integral becomes

$$0 = \int_{x_{I-1}}^{x_{I+1}} \left[a \frac{d\Phi_I}{dx} \left(U_{I-1} \frac{d\Phi_{I-1}}{dx} + U_I \frac{d\Phi_I}{dx} + U_{I+1} \frac{d\Phi_{I+1}}{dx} \right) - \Phi_I q \right] dx - \Phi_I(L) Q_0 \tag{3.3.14}$$

and we have

$$\begin{aligned}
I = 1 : 0 &= \int_{x_1=0}^{x_2} \left[a \frac{d\Phi_1}{dx} \left(U_1 \frac{d\Phi_1}{dx} + U_2 \frac{d\Phi_2}{dx} \right) - \Phi_1 q \right] dx - \Phi_1(L) Q_0 \\
I = 2 : 0 &= \int_{x_1=0}^{x_3} \left[a \frac{d\Phi_2}{dx} \left(U_1 \frac{d\Phi_1}{dx} + U_2 \frac{d\Phi_2}{dx} + U_3 \frac{d\Phi_3}{dx} \right) - \Phi_2 q \right] dx \\
&\quad - \Phi_2(L) Q_0 \\
I = 3 : 0 &= \int_{x_2}^{x_4=L} \left[a \frac{d\Phi_3}{dx} \left(U_2 \frac{d\Phi_2}{dx} + U_3 \frac{d\Phi_3}{dx} + U_4 \frac{d\Phi_4}{dx} \right) - \Phi_3 q \right] dx \\
&\quad - \Phi_3(L) Q_0 \\
I = 4 : 0 &= \int_{x_3}^{x_4=L} \left[a \frac{d\Phi_4}{dx} \left(U_3 \frac{d\Phi_3}{dx} + U_4 \frac{d\Phi_4}{dx} \right) - \Phi_4 q \right] dx - \Phi_4(L) Q_0
\end{aligned} \tag{3.3.15}$$

These equations, upon performing the integration, yield (3.2.50), with the last column (containing Qs) in the latter replaced by

$$\begin{Bmatrix} 0 \\ Q_0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.3.16}$$

Although this procedure, known as the *direct stiffness method* in structural mechanics, gives the assembled equations directly, it is algebraically complicated (especially for two-dimensional problems) and not amenable to simple computer implementation.

3.4 AXISYMMETRIC PROBLEMS

3.4.1 Model Equation

The equations governing physical processes in a cylindrical geometry are described analytically in terms of cylindrical coordinates (see Fig. 3.4.1; also see Fig. 1.4.5). When the geometry, loading, and boundary conditions are independent of the circumferential direction (i.e., θ-coordinate in Fig. 3.4.1), the problem is said to be axisymmetric and the governing equations become two-dimensional in terms of r and z. The equations are functions of only

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Figure 3.4.1 Volume element and computational domain of an axisymmetric problem.

the radial coordinate r if the problem geometry and data are independent of z. Here, we consider a model second-order equation in a single variable and formulate its finite element model.

Consider the differential equation [an analogue of (3.2.1)]

$$-\frac{1}{r}\frac{d}{dr}\left[a(r)\frac{du}{dr}\right] = f(r) \quad \text{for} \quad R_i < r < R_0 \tag{3.4.1}$$

where r is the radial coordinate, a and f are known functions of r, and u is the dependent variable. Such equations arise, for example, in connection with radial heat flow in a long circular cylinder of inner radius R_i and outer radius R_0. The radially symmetric conditions require that both $a = kr$ (k is the conductivity) and f (internal heat generation) be functions of only r. Since the cylinder is long, the temperature distribution at any section along its length (except perhaps at the ends) is the same, and it is sufficient to consider any cross section away from the ends, i.e., the problem is reduced from a three-dimensional problem to a two-dimensional one. Since a and f are independent of the circumferential direction θ, the temperature distribution along any radial line is the same, reducing the two-dimensional problem to a one-dimensional one, as described by (3.4.1).

3.4.2 Weak Form

In developing the weak form of (3.4.1), we multiply it with a weight function $w(r)$ and integrate over the volume of the cylinder of unit length (see Fig. 3.4.1)

$$\begin{aligned} 0 &= \int_V w\left[-\frac{1}{r}\frac{d}{dr}\left(a\frac{du}{dr}\right) - f\right] dv \\ &= \int_0^1 \int_0^{2\pi} \int_{r_a}^{r_b} w\left[-\frac{1}{r}\frac{d}{dr}\left(a\frac{du}{dr}\right) - f\right] r\,dr\,d\theta\,dz \\ &= 2\pi \int_{r_a}^{r_b} w\left[-\frac{1}{r}\frac{d}{dr}\left(a\frac{du}{dr}\right) - f\right] r\,dr \end{aligned} \tag{3.4.2}$$

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where (r_a, r_b) is the domain of a typical element along the radial direction. Next, we carry out the remaining two steps of the weak formulation:

$$0 = 2\pi \int_{r_a}^{r_b} \left(a \frac{dw}{dr}\frac{du}{dr} - rwf \right) dr - 2\pi \left[wa\frac{du}{dr} \right]_{r_a}^{r_b}$$

$$0 = 2\pi \int_{r_a}^{r_b} \left(a \frac{dw}{dr}\frac{du}{dr} - rwf \right) dr - w(r_a)Q_1^e - w(r_b)Q_2^e \tag{3.4.3a}$$

where

$$Q_1^e \equiv -2\pi \left(a\frac{du}{dr} \right)\bigg|_{r_a}, \quad Q_2^e \equiv 2\pi \left(a\frac{du}{dr} \right)\bigg|_{r_b} \tag{3.4.3b}$$

3.4.3 Finite Element Model

The finite element model is obtained by substituting the approximation

$$u(r) \approx \sum_{j=1}^{n} u_j^e \psi_j^e(r) \tag{3.4.4}$$

and $w = \psi_1, \psi_2, \ldots, \psi_n$ into (3.4.3a). The finite-element model is given by

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\} \tag{3.4.5a}$$

where

$$K_{ij}^e = 2\pi \int_{r_a}^{r_b} a \frac{d\psi_i^e}{dr}\frac{d\psi_j^e}{dr} dr, \quad f_i^e = 2\pi \int_{r_a}^{r_b} \psi_i^e f r\, dr \tag{3.4.5b}$$

and ψ_i^e are the interpolation functions expressed in terms of the radial coordinate r. For example, the linear interpolation functions are of the form $(h_e = r_b - r_a)$ [see Eq. (3.2.17)]

$$\psi_1^e(r) = \frac{r_b - r}{h_e}, \quad \psi_2^e(r) = \frac{r - r_a}{h_e} \tag{3.4.6}$$

The explicit forms of the coefficients K_{ij}^e and f_i^e for $a = a_e r$ and $f = f_e$ are given below (r_a denotes the global coordinate of node 1 of the element).

Linear Element

$$[K^e] = \frac{2\pi a_e}{h_e}(r_a + \frac{1}{2}h_e)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{f^e\} = \frac{2\pi f_e h_e}{6}\begin{Bmatrix} 3r_a + h_e \\ 3r_a + 2h_e \end{Bmatrix} \tag{3.4.7}$$

Quadratic Element

$$[K^e] = \frac{2\pi a_e}{6h_e}\begin{bmatrix} 3h_e + 14r_a & -(4h_e + 16r_a) & h_e + 2r_a \\ -(4h_e + 16r_a) & 16h_e + 32r_a & -(12h_e + 16r_a) \\ h_e + 2r_a & -(12h_e + 16r_a) & 11h_e + 14r_a \end{bmatrix}$$

$$\{f^e\} = \frac{2\pi f_e h_e}{6}\begin{Bmatrix} r_a \\ 4r_a + 2h_e \\ r_a + h_e \end{Bmatrix} \tag{3.4.8}$$

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Example 3.4.1

Equation (3.4.1) governs, for example, temperature distribution $u(r)$ in a *long* solid cylindrical bar of radius R and thermal conductivity k (i.e., $a(r) = rk$) that is heated by the passage of an electric current, which generates heat energy f_0. Heat is dissipated from the surface of the bar by convection into the surrounding medium at an ambient temperature of u_∞. We wish to determine the temperature distribution as a function of the radial distance. The boundary conditions for this case are

$$\frac{du}{dr} = 0 \text{ at } r = 0 \text{ (symmetry)}, \quad k\frac{du}{dr} + \beta(u - u_\infty) = 0 \text{ at } r = R_0 \text{ (convection)} \tag{3.4.9}$$

Let us consider the following data for numerical computations:

$$R = 0.05 \text{ m}, \quad k = 40 \text{ W/(m}\cdot{}^\circ\text{C)}, \quad f_0 = 4 \times 10^6 \text{ W/m}^3, \quad \beta = 400 \text{ W/(m}^2\cdot{}^\circ\text{C)}, \quad u_\infty = 20^\circ\text{C} \tag{3.4.10}$$

For the uniform mesh of four linear elements ($h = 0.0125$ m), we have (omitting the common factor 2π) the following coefficient matrices.

Element 1. ($r_a = 0$ and $r_b = h$):

$$[K^1] = \frac{40}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{f^1\} = \frac{4 \times 10^6 (0.0125)^2}{6}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

Element 2. ($r_a = h$ and $r_b = 2h$):

$$[K^2] = \frac{40}{2}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}, \quad \{f^2\} = \frac{4 \times 10^6 (0.0125)^2}{6}\begin{Bmatrix} 4 \\ 5 \end{Bmatrix}$$

Element 3. ($r_a = 2h$ and $r_b = 3h$):

$$[K^3] = \frac{40}{2}\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}, \quad \{f^3\} = \frac{4 \times 10^6 (0.0125)^2}{6}\begin{Bmatrix} 7 \\ 8 \end{Bmatrix}$$

Element 4. ($r_a = 3h$ and $r_b = 4h$):

$$[K^2] = \frac{40}{2}\begin{bmatrix} 7 & -7 \\ -7 & 7 \end{bmatrix}, \quad \{f^4\} = \frac{4 \times 10^6 (0.0125)^2}{6}\begin{Bmatrix} 10 \\ 11 \end{Bmatrix}$$

The assembled equations are

$$\begin{bmatrix} 20 & -20 & 0 & 0 & 0 \\ -20 & 80 & -60 & 0 & 0 \\ 0 & -60 & 160 & -100 & 0 \\ 0 & 0 & -100 & 240 & -140 \\ 0 & 0 & 0 & -140 & 140 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = 104.166\begin{Bmatrix} 1 \\ 6 \\ 12 \\ 18 \\ 11 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

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The boundary and balance conditions require

$$Q_1^1 = 0,\ \ Q_2^1 + Q_1^2 = 0,\ \ Q_2^2 + Q_1^3 = 0,\ \ Q_2^3 + Q_1^4 = 0,\ \ Q_2^4 = -R_0\beta(U_5 - u_\infty) = -20(U_5 - 20)$$

The condensed equations are (here, no equations are eliminated)

$$\begin{bmatrix} 20 & -20 & 0 & 0 & 0 \\ -20 & 80 & -60 & 0 & 0 \\ 0 & -60 & 160 & -100 & 0 \\ 0 & 0 & -100 & 240 & -140 \\ 0 & 0 & 0 & -140 & 160 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = 104.166 \begin{Bmatrix} 1 \\ 6 \\ 12 \\ 18 \\ 11 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 400 \end{Bmatrix}$$

The solution of these equations (with the help of a computer) is

$$U_1 = 334.68,\ U_2 = 329.47,\ U_3 = 317.32,\ U_4 = 297.53,\ U_5 = 270.00$$

The uniform mesh of two quadratic elements yields the nodal values

$$U_1 = 332.50,\ U_2 = 328.59,\ U_3 = 316.87,\ U_4 = 297.34,\ U_5 = 270.00$$

which coincide with the exact solution at the nodes

$$u(r) = u_\infty + \frac{f_0 R_0}{2\beta} + \frac{f_0 R_0^2}{4k}\left(1 - \frac{r^2}{R_0^2}\right) \tag{3.4.11}$$

3.5 SUMMARY

A systematic study of the steps involved in the finite element formulation of a model second-order differential equation in single variable is presented. The basic steps of the formulation and analysis of a typical equation are outlined in Table 3.1.1. The model equation is representative of the equations arising in various fields of engineering (see Table 3.2.1). Couple of numerical examples are presented to illustrate the steps in the finite element analysis of second-order differential equations. Additional examples, especially those arising in heat transfer, fluid mechanics, and solid mechanics will be presented in Chapter 4.

The finite element model is developed following three steps:

1. Weak (or variational) formulation of the differential equation over an element.
2. Finite element interpolation of the primary variables of the weak formulation.
3. Finite element model (i.e., a set of algebraic equations relating the "primary" and "secondary" variables) development over a typical element.

The weak formulation itself involves a three-step procedure, which enables identification of primary and secondary variables. The primary variables are required to be continuous throughout the domain, including the nodes at which elements are connected. The secondary variables are included in the weak form. The finite element interpolation functions have been developed systematically on the basis of continuity, completeness, and linear independence. The finite element model has been developed by substituting appropriate interpolation of the primary variable into the weak form of the differential equation.

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PROBLEMS

For Problems 3.1–3.4, carry out the following tasks:

(a) Develop the *weak forms* of the given differential equation(s) over a typical finite element, which is a geometric subdomain located between $x = x_a$ and $x = x_b$. Note that there are no "specified" boundary conditions at the element level. Therefore, in going from Step 2 to Step 3 of the weak-form development, one must identify the secondary variable(s) at the two ends of the domain by some symbols (like Q_1^e and Q_2^e for the first problem) and complete the weak form.

(b) Assume an approximation(s) of the form

$$u(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{1}$$

where u is a primary variable of the formulation, $\psi_j^e(x)$ are the interpolation functions, and u_j^e are the values of the primary variable(s) at the jth node of the element. Substitute the expression in (1) for the primary variable and ψ_i^e for the weight function into the weak form(s) and derive the finite element model. Be sure to define all coefficients of the model in terms of the problem data and ψ_i^e.

3.1 Develop the weak form and the finite element model of the following differential equation over an element:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + \frac{d^2}{dx^2}\left(b\frac{d^2u}{dx^2}\right) + cu = f \quad \text{for} \quad x_a < x < x_b$$

where a, b, c, and f are known functions of position x. Ensure that the element coefficient matrix $[K^e]$ is symmetric. What is the nature of the interpolation functions for the problem?

3.2 Construct the weak form and the finite element model of the differential equation

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) - b\frac{du}{dx} = f \quad \text{for} \quad 0 < x < L$$

over a typical element $\Omega_e = (x_a, x_b)$. Here a, b, and f are known functions of x, and u is the dependent variable. The natural boundary condition should *not* involve the function $b(x)$. What type of interpolation functions may be used for u?

3.3 Develop the weak forms of the following pair of coupled second-order differential equations over a typical element (x_a, x_b):

$$-\frac{d}{dx}\left[a(x)\left(u + \frac{dv}{dx}\right)\right] = f(x) \tag{1a}$$

$$-\frac{d}{dx}\left(b(x)\frac{du}{dx}\right) + a(x)\left(u + \frac{dv}{dx}\right) = q(x) \tag{1b}$$

where u and v are the dependent variables, and a, b, f, and q are known functions of x. Also identify the primary and secondary variables of the formulation.

3.4 Consider the following weak forms of a pair of coupled differential equations:

$$0 = \int_{x_a}^{x_b}\left(\frac{dw_1}{dx}\frac{dv}{dx} - w_1 f\right)dx - P_a w_1(x_a) - P_b w_1(x_b) \tag{1a}$$

$$0 = \int_{x_a}^{x_b}\left(\frac{dw_2}{dx}\frac{du}{dx} + c\, w_2 v - w_2 q\right)dx - Q_a w_2(x_a) - Q_b w_2(x_b) \tag{1b}$$

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where $c(x)$ is a known function, w_1 and w_2 are weight functions, u and v are dependent variables (primary variables), and P_a, P_b, Q_a, and Q_b are the secondary variables of the formulation. Use the finite element approximations of the form

$$u(x) = \sum_{j=1}^{m} u_j^e \psi_j^e(x), \quad v(x) = \sum_{j=1}^{n} v_j^e \varphi_j^e(x) \tag{2}$$

and $w_1 = \psi_i$ and $w_2 = \varphi_i$ and derive the finite element equations from the weak forms. The finite element equations should be in the form

$$0 = \sum_{j=1}^{m} K_{ij}^{11} u_j^e + \sum_{j=1}^{n} K_{ij}^{12} v_j^e - F_i^1 \tag{3a}$$

$$0 = \sum_{j=1}^{m} K_{ij}^{21} u_j^e + \sum_{j=1}^{n} K_{ij}^{22} v_j^e - F_i^2 \tag{3b}$$

Define the coefficients K_{ij}^{11}, K_{ij}^{12}, K_{ij}^{21}, K_{ij}^{22}, F_i^1, and F_i^2 in terms of the interpolation functions, known data, and secondary variables.

3.5 Derive the Lagrange cubic interpolation functions for a four-node (one-dimensional) element (with equally spaced nodes) using the alternative procedure based on interpolation properties (3.2.18*a*) and (3.2.18*b*). Use the local coordinate $\bar{x}$ for simplicity.

3.6 Evaluate the element matrices $[K^{11}]$, $[K^{12}]$, and $[K^{22}]$ for the linear interpolation of $u(x)$ and $v(x)$ in Problem 3.4.

3.7 Evaluate the following coefficient matrices and source vector using the linear Lagrange interpolation functions:

$$K_{ij}^e = \int_{x_a}^{x_b} (a_0^e + a_1^e x) \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} dx, \quad M_{ij}^e = \int_{x_a}^{x_b} (c_0^e + c_1^e x) \psi_i^e \psi_j^e dx$$

$$f_i^e = \int_{x_a}^{x_b} (f_0^e + f_1^e x) \psi_i^e dx$$

where a_0^e, a_1^e, c_0^e, c_1^e, f_0^e, and f_1^e are constants.

3.8 (*Heat transfer in a rod*) The governing differential equation and convection boundary condition are of the form:

$$-\frac{d^2\theta}{dx^2} + c\theta = 0, \quad 0 < x < L \tag{1}$$

$$\theta(0) = T_0 - T_\infty, \quad \left[k\frac{d\theta}{dx} + \beta\theta\right]_{x=L} = 0 \tag{2}$$

where $\theta = T - T_\infty$, $c = \beta P/(Ak)$, β is the heat transfer coefficient, P is the perimeter, A is the area of cross section, and k is the conductivity. For a mesh of two linear elements (of equal length), give (*a*) the boundary conditions on the nodal variables (primary as well as secondary variables) and (*b*) the final condensed finite element equations for the unknowns (both primary and secondary nodal variables). Use the following data: $T_0 = 120°\text{C}$, $T_\infty = 20°\text{C}$, $L = 0.25$ m, $c = 256$, $\beta = 64$, and $k = 50$ (with proper units).

3.9 (*Axial deformation of a bar*) The governing differential equation is of the form (E and A are constant):

$$-\frac{d}{dx}\left[EA\frac{du}{dx}\right] = 0, \quad 0 < x < L \tag{1}$$

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For the minimum number of linear elements, give (*a*) the boundary conditions on the nodal variables (primary as well as secondary variables) and (*b*) the final condensed finite element equations for the unknowns.

Figure P3.9

3.10 Resolve the problem in Example 3.2.1 using the uniform mesh of three linear finite elements. *Answer:* $U_2 = -0.02999$, $U_3 = -0.04257$, $(Q_1^1)_{def} = 0.08998$, $(Q_2^3)_{def} = 0.12771$.

3.11 Solve the differential equation in Example 3.2.1 for the mixed boundary conditions

$$u(0) = 0, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 1$$

Use the uniform mesh of three linear elements. The exact solution is

$$u(x) = \frac{2\cos(1-x) - \sin x}{\cos(1)} + x^2 - 2$$

Answer: $U_2 = 0.4134$, $U_3 = 0.7958$, $U_4 = 1.1420$, $(Q_1^1)_{\text{def}} = -1.2402$.

3.12 Solve the differential equation in Example 3.2.1 for the *natural* (or Neumann) boundary conditions

$$\left(\frac{du}{dx}\right)\bigg|_{x=0} = 1, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1} = 0$$

Use the uniform mesh of three linear finite elements to solve the problem. Verify your solution with the analytical solution

$$u(x) = \frac{\cos(1-x) + 2\cos x}{\sin(1)} + x^2 - 2$$

Answer: $U_1 = 1.0280$, $U_2 = 1.3002$, $U_4 = 1.4447$, $U_5 = 1.4821$.

3.13 Solve the problem described by the following equations

$$-\frac{d^2u}{dx^2} = \cos \pi x, \quad 0 < x < 1; \quad u(0) = 0, \quad u(1) = 0$$

Use the uniform mesh of three linear elements to solve the problem and compare against the exact solution

$$u(x) = \frac{1}{\pi^2}(\cos \pi x + 2x - 1)$$

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3.14 Solve the differential equation in Problem 3.13 using the mixed boundary conditions

$$u(0)=0, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1}=0$$

Use the uniform mesh of three linear elements to solve the problem and compare against the exact solution

$$u(x)=\frac{1}{\pi^2}(\cos \pi x-1)$$

3.15 Solve the differential equation in Problem 3.13 using the Neumann boundary conditions

$$\left(\frac{du}{dx}\right)\bigg|_{x=0}=0, \quad \left(\frac{du}{dx}\right)\bigg|_{x=1}=0$$

Use the uniform mesh of three linear elements to solve the problem and compare against the exact solution

$$u(x)=\frac{\cos \pi x}{\pi^2}$$

Note: For Neumann boundary conditions, none of the primary dependent variables are specified, and therefore the solution can be determined within an arbitrary constant for this equation (i.e., when the cu term is not present, the coefficient matrix is singular and cannot be inverted). In such cases, one of the U_i should be set equal to a constant to remove the "rigid-body" mode (i.e., to determine the arbitrary constant in the solution).

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Chapter 4

SECOND-ORDER DIFFERENTIAL EQUATIONS IN ONE DIMENSION: APPLICATIONS

4.1 PRELIMINARY COMMENTS

In Chapter 3 we developed weak forms and finite element models of continuum problems described by a fairly general second-order differential equation. For discrete systems, such as a network of springs or electrical circuits, no differential equations exist and the weak form concept is not applicable. Therefore, an alternate approach based on the laws of physics must be used to develop finite element models (i.e., relations between the cause and effect) of such systems. Physical principles can also be used to develop finite element models of continuum problems (as discussed in Remark 7 of Chapter 3) but the approach cannot be used to derive finite element models with higher-order approximation of the field variable.

The objective of this chapter is two-fold. First, we derive finite element models of some typical discrete systems. Finite element models of discrete systems are developed using physical laws familiar to most engineering and applied science majors, and the approach requires no concept of weak form. Second, we present numerical examples of application of finite element models developed for both discrete systems and continuum systems. We will consider several examples to illustrate the steps involved in the finite element analysis of one-dimensional second-order differential equations arising in heat transfer, fluid mechanics, and solid mechanics. The examples presented here make use of the element equations already developed in Chapter 3. While the notation used for the dependent variables, independent coordinates, and data of problems from field to field is different, the reader should keep the common mathematical structure in mind and not get confused with the change of notation from problem to problem and field to field.

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4.2 DISCRETE SYSTEMS

4.2.1 Linear Elastic Spring

A linear elastic spring is a discrete element (i.e., not a continuum) whose load-displacement relationship can be expressed as

$$F = k\delta \tag{4.2.1}$$

where F is the force (N) at the right end, δ is the displacement (m) of the right end of the spring relative to the left end in the direction of the force, and k is the constant known as the *spring constant* (N/m). The spring constant depends on the elastic modulus, area of cross section, and number of turns in the coil of the spring. Often a spring is used to characterize the elastic behavior of complex physical systems.

A relationship between the end forces (F_1^e, F_2^e) and end displacements (δ_1^e, δ_2^e) of a typical spring element [see Fig. 4.2.1(a)] can be developed as discussed in Remark 7 for a bar element. The force F_1^e at node 1 is equal to the spring constant multiplied by the relative displacement of node 1 with respect to node 2, $\delta_1^e - \delta_2^e$:

$$F_1^e = k_e(\delta_1^e - \delta_2^e) = k_e\delta_1^e - k_e\delta_2^e$$

Similarly, the force at node 2 is equal to

$$F_2^e = k_e(\delta_2^e - \delta_1^e) = -k_e\delta_1^e + k_e\delta_2^e$$

Note that the force equilibrium, $F_1^e + F_2^e = 0$, is automatically satisfied. The above equations can be written in matrix form as

$$k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1^e \\ \delta_2^e \end{Bmatrix} = \begin{Bmatrix} F_1^e \\ F_2^e \end{Bmatrix} \tag{4.2.2}$$

Equation (4.2.2) is applicable to any spring element whose force-displacement relation is linear. Thus a typical spring in a network of springs of different spring constants obey Eq. (4.2.2).

Example 4.2.1

Consider the spring assemblage shown in Fig. 4.2.1(b). We wish to determine the displacement of the rigid block and forces in the springs. Assume that the rigid block is required to remain vertical (i.e., no tilting from its vertical position). Since the rigid block must remain vertical, all points on it will move horizontally by the same amount; hence, all global nodes on the block must have the same node number, say 2. Each spring in the assemblage has the same force-displacement relationship as in Eq. (4.2.2), except that element number will be different for different elements.

The three elements are connected at node 2 through the rigid block. Hence, the continuity and equilibrium conditions at node 2 require

$$\delta_2^{(1)} = \delta_1^{(2)} = \delta_1^{(3)} = U_2 \tag{4.2.3a}$$

$$F_2^{(1)} + F_1^{(2)} + F_1^{(3)} = F_2 \tag{4.2.3b}$$

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The equilibrium condition (4.2.3*b*) suggests that we must add the second equation of element 1, the first equation of element 2, and the first equation of element 3 together to replace the sum of three unknowns forces $F_2^{(1)} + F_1^{(2)} + F_1^{(3)}$ with the known force F_2. Thus, we have four equations: the first equation of element 1, the sum of the three equations stated above, the second equation of element 2, and the second equation of element 3:

$$\begin{aligned} k_1U_1 - k_1U_2 &= F_1 \\ -k_1U_1 + (k_1 + k_2 + k_3)\,U_2 - k_2U_3 - k_3U_4 &= F_2 \\ -k_2U_2 + k_2U_3 &= F_3 \\ -k_3U_2 + k_3U_4 &= F_4 \end{aligned} \tag{4.2.4a}$$

Figure 4.2.1 (*a*) A spring finite element. (*b*) Three-spring assemblage.

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In matrix form, the above equations can be expressed as

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \tag{4.2.4b}$$

Next, we identify the boundary conditions and impose them on Eq. (4.2.4*b*). From Fig. 4.2.1, it is clear that the displacements of nodes 1, 3, and 4 are zero, and the force at node 2 is specified to be P:

$$U_1 = U_3 = U_4 = 0, \qquad F_2 = P \tag{4.2.5}$$

Thus, there are four unknowns ($F_1 = F_1^{(1)}$, U_2, $F_3 = F_2^{(2)}$, $F_4 = F_2^{(3)}$) and four equations. Using the second equation in (4.2.4*a*), we determine U_2 (condensed equation for the displacement)

$$(k_1 + k_2 + k_3)\, U_2 = P \text{ or } U_2 = \frac{P}{k_1 + k_2 + k_3} \tag{4.2.6}$$

The forces F_1, F_3, and F_4 can be calculated using equations 1, 3, and 4 of (4.2.4*a*). The condensed equations for forces are

$$\begin{aligned} F_1 &= F_1^{(1)} = -k_1 U_2 = -\frac{P k_1}{k_1 + k_2 + k_3} \\ F_3 &= F_2^{(2)} = -k_2 U_2 = -\frac{P k_2}{k_1 + k_2 + k_3} \\ F_4 &= F_2^{(3)} = -k_3 U_2 = -\frac{P k_3}{k_1 + k_2 + k_3} \end{aligned} \tag{4.2.7}$$

4.2.2 Torsion of Circular Shafts

Another problem that can be directly formulated as a discrete element is the torsion of circular shafts. From a course on mechanics of deformable solids, the angle of twist θ of a constant cross-section circular cylindrical member is related to the torque T (about the axis of the member) by [see Fig. 4.2.2(*a*)]

$$T = \frac{GJ}{L}\theta \tag{4.2.8}$$

where J is the polar moment of area, L is the length, and G is the shear modulus of the material of the shaft. The above equation can be used to write a relationship between the end torques (T_1^e, T_2^e) and the end twists (θ_1^e, θ_2^e) of a circular cylindrical member of length h_e [see Fig. 4.2.2(*b*)]:

$$T_1^e = \frac{G_e J_e}{h_e}\left(\theta_1^e - \theta_2^e\right), \; T_2^e = \frac{G_e J_e}{h_e}\left(\theta_2^e - \theta_1^e\right)$$

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Figure 4.2.2 Torsion of a circular shaft.

or

$$\frac{G_e J_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1^e \\ \theta_2^e \end{Bmatrix} = \begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} \tag{4.2.9}$$

4.2.3 Electrical Resistor Circuits

There is a direct analogy between a network of mechanical springs and a direct current electric resistor network. Ohm's law provides the relationship between flow of electric current I (amperes) through an ideal resistor and voltage drop V (volts) between the ends of the resistor

$$V = IR \tag{4.2.10}$$

where R denotes the electric resistance (ohms) of the wire.

Kirchhoff's voltage rule states that the algebraic sum of the voltage changes in any loop must be equal to zero. Applied to a single resistor, the rule gives [see Fig. 4.2.3(*a*)]

$$I_1^e R_e + V_2^e - V_1^e = 0, \quad I_2^e R_e + V_1^e - V_2^e = 0$$

Figure 4.2.3 (*a*) Direct current electric element (current flows from high to low voltage). (*b*) A resistor circuit. (*c*) Finite element mesh of the resistor circuit.

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or in matrix form

$$\frac{1}{R_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} V_1^e \\ V_2^e \end{Bmatrix} = \begin{Bmatrix} I_1^e \\ I_2^e \end{Bmatrix} \tag{4.2.11}$$

Thus, once again we have the same form of relationship between voltages and currents as in the case of springs. The quantity $1/R_e$ is known as the *electrical conductance.*

The assembly of resistor equations is based on the following rules:

1. Voltage is single-valued.
2. *Kirchhoff current rule:* The sum of all currents entering a node is equal to zero.

Example 4.2.2

Consider the resistor circuit shown in Fig. 4.2.3(*b*). We wish to determine the currents in the loops and voltage at the nodes. The element numbering, element node numbering, and global node numbering are indicated in Fig. 4.2.3(*c*). The element node numbering is important in assembling the element equations. From the element notation indicated in Fig. 4.2.3(*a*), it is clear that current flows from node 1 to node 2 of the element. The element node numbering in Fig. 4.2.3(*c*) indicates the assumed direction of the currents.

The assembled coefficient matrix is given by

$$[K] = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & 0 & 0 \\ & K_{22}^1 + K_{22}^2 + K_{22}^5 & K_{21}^2 & 0 & K_{21}^5 & 0 \\ & & K_{11}^2 + K_{22}^3 & K_{21}^3 & 0 & 0 \\ & & & K_{11}^3 + K_{11}^4 & K_{12}^4 & 0 \\ \text{symm.} & & & & K_{22}^4 + K_{11}^5 + K_{11}^6 & K_{12}^6 \\ & & & & & K_{22}^6 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$= \begin{bmatrix} 0.2 & -0.2 & 0 & 0 & 0 & 0 \\ & 0.2+0.1+0.1 & -0.1 & 0 & -0.1 & 0 \\ & & 0.1+0.2 & -0.2 & 0 & 0 \\ & & & 0.2+0.0667 & -0.0667 & 0 \\ \text{symm.} & & & & 0.0667+0.1+0.05 & -0.05 \\ & & & & & 0.05 \end{bmatrix} \tag{4.2.12}$$

The boundary conditions are $V_1 = 200$ V and $V_6 = 0$ V. The condensed equations for the nodal voltages are obtained by omitting the first row and last row of the system, and then moving

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the terms involving V_1 and V_6 to the right. We obtain

$$\begin{bmatrix} 0.4 & -0.1 & 0.0 & -0.1 \\ -0.1 & 0.3 & -0.2 & 0.0 \\ 0.0 & -0.2 & 0.2667 & -0.0667 \\ -0.1 & 0.0 & -0.0667 & 0.2167 \end{bmatrix} \begin{Bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \end{Bmatrix} = \begin{Bmatrix} 0.2V_1 \\ 0 \\ 0 \\ 0.05V_6 \end{Bmatrix}$$

The solution of these equations is given by (obtained with the aid of a computer)

$$V_2 = 169.23 \text{ V}, \; V_3 = 153.85 \text{ V}, \; V_4 = 146.15 \text{ V}, \; V_5 = 123.08 \text{ V}$$

The condensed equations for the unknown currents at nodes 1 and 6 can be calculated from equations 1 and 6 of the system. We have

$$I_1^{(1)} = 0.2V_1 - 0.2V_2 = 40 - 33.846 = 6.154 \text{ A}$$

$$I_2^{(6)} = -0.05V_5 + 0.05V_6 = -6.154 \text{ A}$$

The negative sign on $I_2^{(6)}$ indicates that the current is flowing out of global node 6.

The currents through each element can be calculated using the element equations (4.2.11). For example, the nodal currents in resistor 5 are given by

$$\begin{Bmatrix} I_1^{(5)} \\ I_2^{(5)} \end{Bmatrix} = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} \begin{Bmatrix} 123.08 \\ 169.23 \end{Bmatrix} = \begin{Bmatrix} -4.615 \\ 4.615 \end{Bmatrix}$$

which indicates that the net current flow in resistor 5 is from its node 2 to node 1 (or global node 2 to global node 5), and its value is 4.615 amps (which is the same as 6.154 − 1.539). The finite element solutions for the voltages and currents are shown in Figure 4.2.4.

Figure 4.2.4 The solution for currents and voltages obtained with the finite element method.

4.2.4 Fluid Flow through Pipes

Another example of a discrete element is provided by steady, fully developed, flows of viscous incompressible fluids through circular pipes. The velocity of fully developed laminar

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Figure 4.2.5 Flow of viscous fluids through pipes.

flow of viscous fluids through circular pipes is given by

$$v_x = -\frac{1}{4\mu}\frac{dP}{dx}\left[1 - \left(\frac{2r}{d}\right)^2\right] \tag{4.2.13a}$$

where dP/dx is the pressure gradient, d is the diameter of the pipe, and μ is the viscosity of the fluid [see Fig. 4.2.5(a)]. The volume rate of flow, Q, is obtained by integrating v_x over the pipe cross section. Thus, the relationship between Q and the pressure gradient dP/dx is given by the equation

$$Q = -\frac{\pi d^4}{128\mu}\frac{dP}{dx} \tag{4.2.13b}$$

The negative sign indicates that the flow is in the direction of negative pressure gradient.

Equation (4.2.13b) can be used to develop a relationship between the nodal values of the volume rate of flow, (Q_1^e, Q_2^e) and the pressure, (P_1^e, P_2^e), of a pipe element of length h_e and diameter d_e. The volume rate of flow entering node 1 is given by [see Fig. 4.2.5(b)]

$$Q_1^e = -\frac{\pi d_e^4}{128\mu h_e}\left(P_2^e - P_1^e\right)$$

Similarly, the volume rate of flow entering node 2 is

$$Q_2^e = -\frac{\pi d^4}{128\mu h_e}\left(P_1^e - P_2^e\right)$$

Thus, we have

$$\frac{\pi d^4}{128\mu h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} P_1^e \\ P_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{4.2.14}$$

The constant, $R_e = 128\mu h_e/\pi d_e^4$ is called the *pipe resistance*, in analogy with the electrical resistance [see Eq. (4.2.11)].

4.3 HEAT TRANSFER

4.3.1 Governing Equations

The equations governing conduction heat transfer were discussed in Example 1.2.2. Here we briefly-review the pertinent equations for our use.

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The Fourier heat conduction law for one-dimensional systems states that the heat flow $q(x)$ is relate the temperature gradient $\partial T/\partial x$ by the relation (with heat flow in the positive direction of x)

$$q = -kA\frac{\partial T}{\partial x} \tag{4.3.1}$$

where k is the thermal conductivity of the material, A the cross-sectional area, and T the temperature. The negative sign in (4.3.1) indicates that heat flows downhill on the temperature scale. The balance of energy requires that

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + Ag = \rho cA\frac{\partial T}{\partial t} \tag{4.3.2}$$

where g is the heat energy generated per unit volume, ρ is the density, c is the specific heat of the material, and t is time. Equation (4.3.2) governs the transient heat conduction in a slab or fin (i.e., a one-dimensional system) when the heat flow in the normal to the x-direction is zero. For a plane wall, we take $A = 1$.

In the case of radially symmetric problems with cylindrical geometries, (4.3.2) takes a different form. Consider a long cylinder of inner radius R_i, outer radius R_o, and length L. When L is very large compared with the diameter, it is assumed that heat flows in the radial direction r. The transient radially symmetric heat flow in a cylinder is governed by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + g = \rho c\frac{\partial T}{\partial t} \tag{4.3.3}$$

A cylindrical fuel element of a nuclear reactor, a current-carrying electrical wire, and a thick-walled circular tube provide examples of one-dimensional radial systems.

The boundary conditions for heat conduction involve specifying either the temperature T or the heat flow Q at a point:

$$T = T_0 \quad \text{or} \quad Q \equiv -kA\frac{\partial T}{\partial x} = Q_0 \tag{4.3.4}$$

It is known that when a heated surface is exposed to a cooling medium, such as air or liquid, the surface will cool faster. We say that the heat is convected away. The *convection heat transfer* between the surface and the medium in contact is given by *Newton's law of cooling*:

$$Q = \beta A(T_s - T_\infty) \tag{4.3.5}$$

where T_s is the surface temperature, T_∞ is the temperature of the surrounding medium, called the *ambient temperature*; and β is the *convection heat transfer coefficient* or *film conductance* (or film coefficient). The heat flow due to conduction and convection at a boundary point must be in balance with the applied flow Q_0:

$$\pm kA\frac{\partial T}{\partial x} + \beta A(T - T_\infty) + Q_0 = 0 \tag{4.3.6}$$

The sign of the first term in (4.3.6) is negative when the heat flow is from the fluid at T_∞ to the surface at the left end of the element, and it is positive when the heat flow is from the fluid at T_∞ to the surface at the right end.

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Convection of heat from a surface to the surrounding fluid can be increased by attaching thin strips of conducting metal to the surface. The metal strips are called *fins*. For a fin with heat flow along its length, heat can convect across the lateral surface of the fin [see Fig. 4.3.1(*a*)]. To account for the convection of heat through the surface, we must add the rate of heat loss by convection to the right-hand side of (4.3.2):

$$\frac{\partial}{\partial x}\left(Ak\frac{\partial T}{\partial x}\right) + Aq = \rho c A\frac{\partial T}{\partial t} + P\beta(T - T_\infty) \tag{4.3.7a}$$

where P is the perimeter and β is the film coefficient. Equation (4.3.7*a*) can be expressed in the alternative form

$$\rho c A\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + P\beta T = Ag + P\beta T_\infty \tag{4.3.7b}$$

The units of various quantities (in metric system) are as follows:

T	°C (celsius)	k	$W/(m \cdot °C)$
g	W/m^3	ρ	kg/m^3
c	$J/(kg \cdot °C)$	β	$W/(m^2 \cdot °C)$

For a steady state, we set the time derivatives in (4.3.2), (4.3.3), (4.3.7*a*), and (4.3.7*b*) equal to zero. The steady-state equations for various one-dimensional systems are summarized below [see Fig. 4.3.1(*b*) and (*c*); see Eqs. (1.2.14) and (1.2.17)].

Plane Wall $[Q = k(dT/dx)]$

$$-\frac{d}{dx}\left(k\frac{dT}{dx}\right) = Ag \tag{4.3.8}$$

Fin $[Q = kA(dT/dx)]$

$$-\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + cT = Ag + cT_\infty, \quad c = P\beta \tag{4.3.9}$$

Cylindrical System $[Q = k(dT/dr)]$

$$-\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = g(r) \tag{4.3.10}$$

The essential and natural boundary conditions associated with these equations are of the form

$$T = T_0, \quad Q + \beta A(T - T_\infty) + Q_0 = 0$$

Equations (4.3.8)–(4.3.10) are a special case of the model equation (3.2.1) discussed in Section 3.2, with $a = kA$, $c = P\beta$, and $f \to Ag + P\beta T_\infty$. We immediately have the finite element model of Eqs. (4.3.8) and (4.3.9) from (3.2.31*a*) and (3.2.31*b*):

$$[K^e]\{T^e\} = \{f^e\} + \{Q^e\} \tag{4.3.11a}$$

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Figure 4.3.1 Heat transfer in (*a*) fins, (*b*) plane wall, and (*c*) radially symmetric system.

Here

$$K_{ij}^e = \int_{x_a}^{x_b} \left(kA \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + P\beta \psi_i^e \psi_j^e \right) dx, \quad f_i^e = \int_{x_a}^{x_b} \psi_i^e (Ag + P\beta T_\infty)\, dx$$

$$Q_1^e = \left(-kA \frac{dT}{dx} \right) \bigg|_{x_a}, \quad Q_2^e = \left(kA \frac{dT}{dx} \right) \bigg|_{x_b} \tag{4.3.11b}$$

where Q_1^e and Q_2^e denote heat flow *into* the element at the nodes.

Equation (4.3.10) is also a special case of the model boundary value problem. However, in developing the weak forms of (4.3.10), integration must be carried over a typical volume element of each system, as discussed in Section 3.4 [see Eq. (3.4.2)].

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Figure 4.3.2 One-dimensional heat transfer through composite walls and their thermal circuits.

4.3.2 Finite Element Models

It is interesting to note the analogy between Eq. (4.2.11) of an electric resistor and Eq. (3.3.5*b*) of one-dimensional heat transfer (see Remark 7 of Section 3.3):

$$\frac{A_e k_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{4.3.12}$$

If we identify *thermal resistance* R_{th}^e by

$$R_{th}^e = \frac{h_e}{k_e A_e} \tag{4.3.13}$$

Equations (4.2.11) and (4.3.12) are the same with the following correspondence:

$$R_e \sim R_{th}^e, \quad I_i^e \sim Q_i^e, \quad V_i^e \sim T_i^e \tag{4.3.14}$$

This allows us to model complicated problems involving both series and parallel thermal resistances. Typical problems and their electrical analogies are shown in Figure 4.3.2.

4.3.3 Numerical Examples

Example 4.3.1

A composite wall consists of three materials, as shown in Fig. 4.3.3. The inside wall temperature is 200°C and the outside air temperature is 50°C with a convection coefficient of $\beta =$ 10 W (m^2·K). We wish to determine the temperature along the composite wall.

First, we note that the problem is governed by the equation

$$-kA\frac{d^2T}{dx^2} = 0, \quad 0 < x < L \tag{4.3.15}$$

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$k_1 = 70$ W/(m·K)
$k_2 = 40$ W/(m·K)
$k_3 = 20$ W/(m·K)
$h_1 = 2$ cm
$h_2 = 2.5$ cm
$h_3 = 4$ cm
$T_\infty = 50°C$
$\beta = 10$ W/(m^2·K)

Figure 4.3.3 Heat transfer problem discussed in Example 4.3.1.

and is subjected to the boundary conditions

$$T(0) = T_0, \quad \left[kA\frac{dT}{dx} + \beta A(T - T_\infty)\right]_{x=L} = 0 \tag{4.3.16}$$

where A is the area of cross section (can be taken to be equal to 1) and k is the conductivity.

The data is discontinuous (i.e., different values of $a_e = k_e A_e$ and h_e for different elements). The source term $f = 0$ and therefore $\{f\}^{(e)} = \{0\}$ for $e = 1, 2, 3$. For a mesh of three linear elements (minimum needed), we have

$$[\mathbf{K}]^{(1)} = \frac{k_1 A}{h_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{70 \times 1}{0.02}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3500 & -3500 \\ -3500 & 3500 \end{bmatrix}$$

$$[\mathbf{K}]^{(2)} = \frac{k_2 A}{h_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{40 \times 1}{0.025}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1600 & -1600 \\ -1600 & 1600 \end{bmatrix}$$

$$[\mathbf{K}]^{(3)} = \frac{k_3 A}{h_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{20 \times 1}{0.04}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

The assembled equations are

$$\begin{bmatrix} 3500 & -3500 & 0 & 0 \\ -3500 & 3500 + 1600 & -1600 & 0 \\ 0 & -1600 & 1600 + 500 & -500 \\ 0 & 0 & -500 & 500 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 \end{Bmatrix}$$

The boundary and balance conditions are

$$U_1 = 200,\ Q_2^1 + Q_1^2 = 0,\ Q_2^2 + Q_1^3 = 0,\ Q_2^3 = -\beta A(U_4 - T_\infty) = -10 \times 1(U_4 - 50)$$

Hence, the condensed equations (obtained by omitting first row and column and modifying the right-hand side) for the unknown temperatures are

$$\begin{bmatrix} 3500 + 1600 & -1600 & 0 \\ -1600 & 1600 + 500 & -500 \\ 0 & -500 & 500 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 3500 \times 200 \\ 0 \\ -10U_4 + 500 \end{Bmatrix}$$

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or

$$\begin{bmatrix} 5100 & -1600 & 0 \\ -1600 & 2100 & -500 \\ 0 & -500 & 510 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 7 \times 10^5 \\ 0 \\ 500 \end{Bmatrix}$$

The solution is given by

$$U_1 = 200^\circ\text{C}, U_2 = 199.58^\circ\text{C}, U_3 = 198.67^\circ\text{C}, \ U_4 = 195.76^\circ\text{C}$$

The condensed equations for the unknown heats (per unit area) are

$$Q_1^{(1)} = 3500U_1 - 3500U_2 = 3500 \times 200 - 3500 \times 199.58354 = 1457.6 \text{ W/m}^2 \text{(heat in)}$$

$$Q_2^{(3)} = -10 \times 1\,(U_4 - 50) = -10(195.76 - 50) = -1457.6 \text{ W/m}^2 \text{ (heat out)}$$

It can be shown that for this problem the finite element solution for the nodal temperatures and heats coincides with the exact solution (see Remark 6 of Chapter 3), which is given by

$$T_{\text{exact}}(x) = \begin{cases} A_1 + A_2 x, & 0 < x < h_1 \\ B_1 + B_2 x, & h_1 < x < h_1 + h_2 \\ C_1 + C_2 x, & h_1 + h_2 < x < h_1 + h_2 + h_3 = L \end{cases} \tag{4.3.17a}$$

where

$$A_1 = T_0, \ A_2 = \frac{T_\infty - T_0}{\Delta}, \ B_1 = T_0 + h_1\left(1 - \frac{k_1}{k_2}\right) A_2$$

$$B_2 = \frac{k_1}{k_2} A_2, \ C_1 = T_\infty - k_1\left(\frac{1}{\beta} + \frac{L}{k_3}\right) A_2, \ C_2 = \frac{k_2}{k_3} B_2$$

$$\Delta = k_1\left(\frac{h_1}{k_1} + \frac{h_2}{k_2} + \frac{h_3}{k_3} + \frac{1}{\beta}\right) \tag{4.3.17b}$$

Example 4.3.2

Rectangular fins are used to remove heat from a heated surface [see Fig. 4.3.1(*a*)]. The fins are exposed to ambient air at T_∞. The heat transfer coefficient associated with fin material and the air is β. Assuming that heat is conducted along the length of the fin and uniform along the width and thickness directions, we wish to determine the temperature distribution along the fin and heat loss per fin for two different sets of boundary conditions.

The governing differential equation and boundary conditions of the problem are

$$-kA\frac{d^2T}{dx^2} + \beta P\,(T - T_\infty) = 0, \quad 0 < x < L \tag{4.3.18}$$

$$\textbf{Set 1:} \quad T(0) = T_0, \quad \left[kA\frac{dT}{dx} + \beta A\,(T - T_\infty)\right]_{x=L} = 0 \tag{4.3.19a}$$

$$\textbf{Set 2:} \quad T(0) = T_0, \quad T(L) = T_L \tag{4.3.19b}$$

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where T is the temperature, k the conductivity, β is the heat transfer coefficient, P is the perimeter, and A is the area of cross section. Compared to the model equation (3.2.1), we have the following data:

$$a = kA,\ c = P\beta,\ f = P\beta T_\infty$$

For this case, $c \neq 0$, and the finite element solution at the nodes does not coincide with the exact solution.

We consider a uniform mesh (i.e., $h_1 = h_2 = h_3 = h_4 = L/4$) of four linear elements (see Fig. 4.3.4). The element matrices are

$$[K]^{(e)} = \frac{kA}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{P\beta h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ \{f\}^{(e)} = \frac{P\beta T_\infty}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

The assembled system of equations is given by ($h_1 = h_2 = h_3 = h_4 = h = L/4$)

$$\left(\frac{kA}{h}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} + \frac{\beta P h}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix} + \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ f_2^3 + f_1^4 \\ f_2^4 \end{Bmatrix}$$

Set 1 Boundary Conditions

The boundary and balance conditions are

$$U_1 = T_0,\ Q_2^1 + Q_1^2 = 0,\ Q_2^2 + Q_1^3 = 0,\ Q_2^3 + Q_1^4 = 0,\ Q_2^4 = -\beta A U_5 + \beta A T_\infty$$

Figure 4.3.4 Finite element mesh of a rectangular fin.

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Hence, the condensed equations are

$$\left(\frac{kA}{h}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} + \frac{\beta P h}{6}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2+\alpha \end{bmatrix}\right)\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(\frac{kA}{h} - \frac{\beta P h}{6}\right) T_0 \\ 0 \\ 0 \\ \beta A T_\infty \end{Bmatrix} + \frac{P\beta T_\infty h}{2}\begin{Bmatrix} 1+1 \\ 1+1 \\ 1+1 \\ 1 \end{Bmatrix}$$

$$Q_1^1 = -f_1^1 + \left(\frac{kA}{h} + \frac{\beta P h}{3}\right) T_0 + \left(-\frac{kA}{h} + \frac{\beta P h}{6}\right) U_2$$

$$Q_2^4 = -\beta A U_5 + \beta A T_\infty$$

where $\alpha = \beta A/(\beta P h/6) = 6A/Ph$.

For the choice of the following data (material of the fin is copper),

$$k = 385\ \mathrm{W(m \cdot {}^\circ C)},\ \beta = 25\ \mathrm{W(m^2 \cdot {}^\circ C)},\ T_0 = 100^\circ\mathrm{C},\ T_\infty = 20^\circ\mathrm{C}$$

$$L = 100\ \mathrm{mm},\ t = 1\ \mathrm{mm},\ b = 5\ \mathrm{mm}$$

we have

$$\frac{kA}{h} = \frac{385 \times 5 \times 10^{-6}}{25 \times 10^{-3}} = 0.077,\quad \frac{\beta P h}{6} = \frac{25 \times 12 \times 10^{-3} \times 25 \times 10^{-3}}{6} = 0.00125$$

$$\beta P T_\infty h = 25 \times 12 \times 10^{-3} \times 20 \times 25 \times 10^{-3} = 0.15,\ \beta A T_\infty = 25 \times 5 \times 10^{-6} \times 20 = 0.0025$$

$$\alpha = 6\frac{A}{Ph} = \frac{6 \times 5}{12 \times 25} = 0.1$$

The condensed equations for the unknown nodal temperatures become

$$\left(\begin{bmatrix} 0.154 & -0.077 & 0 & 0 \\ -0.077 & 0.154 & -0.077 & 0 \\ 0 & -0.077 & 0.154 & -0.077 \\ 0 & 0 & -0.077 & 0.077 \end{bmatrix} + \begin{bmatrix} 0.005 & 0.00125 & 0 & 0 \\ 0.00125 & 0.005 & 0.00125 & 0 \\ 0 & 0.00125 & 0.005 & 0.00125 \\ 0 & 0 & 0.00125 & 0.002625 \end{bmatrix}\right)$$

$$\times \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} (0.077 - 0.0025)\,100 \\ 0 \\ 0 \\ 25 \times 5 \times 10^{-6} \times 20 \end{Bmatrix} + \begin{Bmatrix} 0.15 \\ 0.15 \\ 0.15 \\ 0.075 \end{Bmatrix}$$

The solution of these equations is

$$U_1 = 100.0^\circ\mathrm{C},\ U_2 = 82.283^\circ\mathrm{C},\ U_3 = 70.732^\circ\mathrm{C},\ U_4 = 64.204^\circ\mathrm{C},\ U_5 = 62.053^\circ\mathrm{C}$$

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The heat input at node 1 is

$$Q_1^1 = -0.075 + (0.077 + 0.0025)\,100 + (-0.077 + 0.00125)\,82.283 = 1.642 \text{ W}$$

The total heat loss from the surface of the fin can be calculated using

$$\begin{aligned}
Q &= \sum_{e=1}^{4} Q^e + \beta A(T(L) - T_\infty) = \sum_{e=1}^{4} \int_{x_a}^{x_b} P\beta(T - T_\infty)dx + kA(T(L) - T_\infty) \\
&= \sum_{e=1}^{4} \int_{x_a}^{x_b} P\beta\left(T_1^e\psi_1^e + T_2^e\psi_2^e - T_\infty\right)dx + \beta A(U_5 - T_\infty) \\
&= \sum_{e=1}^{4} \beta P h_e\left(\frac{T_1^e + T_2^e}{2} - T_\infty\right) + \beta A(U_5 - T_\infty) \\
&= 0.0075\,(0.5U_1 + U_2 + U_3 + U_4 + 0.5U_5 - 4T_\infty) + 25 \times 5 \times 10^{-6}(62.053 - 20) \\
&= 1.6368 + 0.00526 = 1.642 \text{ W}
\end{aligned}$$

The exact solution of Eqs. (4.3.18) and (4.3.19*a*) is

$$\frac{T(x) - T_\infty}{T_0 - T_\infty} = \left[\frac{\cosh m(L - x) + (\beta/mk)\sinh m(L - x)}{\cosh mL + (\beta/mk)\sinh mL}\right], \; m^2 = \frac{\beta P}{Ak} \tag{4.3.20a}$$

$$Q(0) = -kA\frac{dT}{dx} = (T_0 - T_\infty)M\left[\frac{\sinh mL + (\beta/mk)\cosh mL}{\cosh mL + (\beta/mk)\sinh mL}\right], \; M^2 = \beta PAk \tag{4.3.20b}$$

Evaluating the exact solution at the nodes, we obtain

$$T(0.025) = 82.414^\circ\text{C}, \; T(0.05) = 70.958^\circ\text{C}, \; T(0.075) = 64.505^\circ\text{C}, \; T(0.1) = 62.422^\circ\text{C}$$

and $Q_1^1 = 1.62995$ W.

Next, we consider a mesh of two quadratic elements to analyze the problem described by Eqs. (4.3.18) and (4.3.19*a*). The assembled equations of two quadratic elements ($h_1 = h_2 = L/2$) is

$$\left(\frac{kA}{3h}\begin{bmatrix} 7 & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 7+7 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix} + \frac{\beta P h}{30}\begin{bmatrix} 4 & 2 & -1 & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 \\ -1 & 2 & 4+4 & 2 & -1 \\ 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & -1 & 2 & 4 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_2^2 \\ Q_3^2 \end{Bmatrix} + \frac{\beta P T_\infty h}{6}\begin{Bmatrix} 1 \\ 4 \\ 1+1 \\ 4 \\ 1 \end{Bmatrix}$$

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The boundary and balance conditions are [for boundary conditions in Eq. (4.3.19a)]

$$U_1 = T_0,\ Q_2^1 = 0,\ Q_3^1 + Q_1^2 = 0,\ Q_2^2 = 0,\ Q_3^2 = -\beta A U_5 + \beta A T_\infty$$

Hence, the condensed equations are

$$\left(\frac{kA}{3h}\begin{bmatrix} 16 & -8 & 0 & 0 \\ -8 & 14 & -8 & 1 \\ 0 & -8 & 16 & -8 \\ 0 & 1 & -8 & 7 \end{bmatrix} + \frac{\beta P h}{30}\begin{bmatrix} 16 & 2 & 0 & 0 \\ 2 & 8 & 2 & -1 \\ 0 & 2 & 16 & 2 \\ 0 & -1 & 2 & 4+\alpha \end{bmatrix}\right)\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(\frac{7kA}{3h} - \frac{4\beta P h}{30}\right) T_0 \\ 0 \\ 0 \\ \beta A T_\infty \end{Bmatrix} + \frac{\beta P T_\infty h}{6}\begin{Bmatrix} 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix}$$

where $\alpha = \beta A/(\beta P h/30) = 30A/Ph$.

The solution of the condensed equations for the unknown temperatures is

$$U_1 = 100.0^\circ\text{C},\ U_2 = 82.374^\circ\text{C},\ U_3 = 70.884^\circ\text{C},\ U_4 = 64.380^\circ\text{C},\ U_5 = 62.240^\circ\text{C}$$

The heat input at node 1 is

$$Q_1^1 = -0.05 + 0.091833 \times 100 - 0.10167 \times 82.374 + 0.012333 \times 70.884 = 1.633\ \text{W}$$

The total heat loss from the surface of the fin can be calculated using

$$Q = \sum_{e=1}^{4} \int_{x_a}^{x_b} P\beta\left(T_1^e \psi_1^e + T_2^e \psi_2^e + T_3^e \psi_3^e - T_\infty\right) dx + \beta A(U_5 - T_\infty)$$

$$= \sum_{e=1}^{4} \beta P h_e \left(\frac{T_1^e + 4T_2^e + T_3^e}{6} - T_\infty\right) + \beta A(U_5 - T_\infty)$$

$$= \frac{\beta P h}{6}(U_1 + 4U_2 + 2U_3 + 4U_4 + U_5 - 12T_\infty) = 1.62756 + 0.00528 = 1.633\ \text{W}$$

Set 2 Boundary Conditions

For Set 2 boundary conditions (4.3.19b), the finite element boundary and balance conditions for the mesh of four linear elements are

$$U_1 = T_0,\ Q_2^1 + Q_1^2 = 0,\ Q_2^2 + Q_1^3 = 0,\ Q_2^3 + Q_1^4 = 0,\ U_5 = T_L = 20^\circ\text{C}$$

and the finite element solution is

$$U_1 = 100.0^\circ\text{C},\ U_2 = 73.955^\circ\text{C},\ U_3 = 53.252^\circ\text{C},\ U_4 = 35.842^\circ\text{C},\ U_5 = 20^\circ\text{C}$$

The heats at nodes 1 and 5 are

$$Q_1^1 = -0.075 + (0.077 + 0.0025)\,100 + (-0.077 + 0.00125)\,73.955 = 2.273\ \text{W}$$

$$Q_2^4 = -0.075 + (-0.077 + 0.00125)\,35.842 + (0.077 + 0.0025)\,20 = -1.2\ \text{W}$$

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Thus, the heat loss from the end of the fin is overestimated by assuming that the end $x = L$ is at the ambient temperature. The total heat loss from the lateral surface of the fin is given by

$$Q = \sum_{e=1}^{4} Q^e = 0.0075\,(0.5U_1 + U_2 + U_3 + U_4 + 0.5U_5 - 4T_\infty) = 1.073 \text{ W}$$

The exact solution of Eqs. (4.3.18) and (4.3.19*b*) is ($\theta_L = T_L - T_\infty$ and $\theta_0 = T_0 - T_\infty$)

$$T(x) - T_\infty = \left[\frac{\theta_L \sinh mx + \theta_0 \sinh m(L-x)}{\sinh mL}\right], \quad m^2 = \frac{\beta P}{Ak} \tag{4.3.21a}$$

$$Q(0) = -kA\frac{dT}{dx} = M\left[\frac{\theta_0 \cosh mL - \theta_L}{\sinh mL}\right], \quad M^2 = \beta PAk \tag{4.3.21b}$$

Evaluating the exact solution at the nodes, we obtain

$$T(0.025) = 74^\circ\text{C}, \quad T(0.05) = 53.3^\circ\text{C}, \quad T(0.075) = 35.87^\circ\text{C}, \quad T(0.1) = 20^\circ\text{C}$$

and $Q_1^1 = 2.268$ W.

The next example is concerned with heat transfer in a rod and comparison of finite difference and finite element analysis steps.

Example 4.3.3

A steel rod of diameter $D = 0.02$ m, length $L = 0.05$m, and thermal conductivity $k = 50$ W/(m·°C) is exposed to ambient air at $T_\infty = 20$°C with a heat transfer coefficient $\beta = 100$ W/(m^2·°C). The left end of the rod is maintained at temperature $T_0 = 320$°C and the other end is insulated. We wish to determine the temperature distribution and the heat input at the left end of the rod [see Fig. 4.3.5(*a*)].

The governing equation of the problem is the same as in Eq. (4.3.18). We rewrite Eq. (4.3.18) in the nondimensional form

$$-\frac{d^2\theta}{dx^2} + m^2\theta = 0 \quad \text{for} \quad 0 < x < L \tag{4.3.22}$$

Figure 4.3.5 (*a*) Heat transfer in a rod. (*b*) Finite difference mesh.

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where $\theta = T - T_\infty$, T being the temperature, and m^2 is given by

$$m^2 = \frac{\beta P}{Ak} = \frac{\beta \pi D}{\frac{1}{4}\pi D^2 k} = \frac{4\beta}{kD} \tag{4.3.23}$$

The boundary conditions of the problem become

$$\theta(0) = T(0) - T_\infty = 300^\circ\text{C}, \quad \left(\frac{d\theta}{dx}\right)\bigg|_{x=L} = 0 \tag{4.3.24}$$

The exact solution is given by [see Eqs. (4.3.20a) and (4.3.20b)]

$$\theta(x) = \theta(0)\frac{\cosh m(L-x)}{\cosh mL}, \; Q(0) = \theta(0)\sqrt{\beta PAk}\,\frac{\sinh mL}{\cosh mL} \tag{4.3.25}$$

For the sake of comparison, we will consider both finite difference and finite element solutions of the problem.

Finite Difference Solutions

The second-derivative may be approximated with the centered finite difference formula (see Example 1.3.2)

$$\frac{d^2\theta}{dx^2} \approx \frac{1}{h^2}(\theta_{i+1} - 2\theta_i + \theta_{i-1}) \tag{4.3.26}$$

Substituting the above formula for the second derivative into Eq. (4.3.22), we arrive at

$$-\theta_{i+1} + (2 + m^2h^2)\theta_i - \theta_{i-1} = 0 \tag{4.3.27}$$

which is valid for any point where θ is not specified.

We choose a mesh of three points ($h = 0.025$), two end points and one in the middle [note that there is no concept of elements in the finite difference method; instead, we identify the number of mesh points (or nodes) in the domain at which we apply the formula (4.3.27), i.e., we arrive at the global equations directly]. Applying the formula (4.3.27) at nodes 2 and 3, we obtain ($m^2 = 400$)

$$(2 + 400h^2)\theta_2 - \theta_3 = \theta_1, \quad -\theta_2 + (2 + 400h^2)\theta_3 - \theta_4 = 0 \tag{4.3.28}$$

where $\theta_1 = \theta(0) = 300^\circ$C. Note that θ_4 is the value of θ at the fictitious node 4, which is considered to be a mirror image because of the boundary condition $d\theta/dx = 0$ [see Fig. 4.3.5(b)]. To eliminate θ_4, we can use one of the following formulas:

$$\left(\frac{d\theta}{dx}\right)_{x=0} = \frac{\theta_4 - \theta_3}{h} = 0 \quad \text{(forward)} \tag{4.3.29a}$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = \frac{\theta_4 - \theta_2}{2h} = 0 \quad \text{(centered)} \tag{4.3.29b}$$

The latter is of order $O(h^2)$, consistent with the centered difference formula (4.3.26). Using (4.3.29b), we set $\theta_4 = \theta_2$ in Eq. (4.3.28). Equations (4.3.28) can be written in matrix form as

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(the coefficient matrix is not symmetric)

$$\begin{bmatrix} 2.25 & -1 \\ -2 & 2.25 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 300 \\ 0 \end{Bmatrix}$$

The solution of these equations is

$$\theta_2 = 220.41^\circ\text{C}, \quad \theta_3 = 195.92^\circ\text{C}$$

The heat at node 1 ($x = 0$) can be computed using the definition

$$Q(0) = kA\left(-\frac{d\theta}{dx}\right)_{x=0} = kA\frac{\theta_1 - \theta_2}{h} = 50.01 \text{ W}$$

Next, we use a mesh of five points. Applying Eq. (4.3.27) at nodes 2, 3, 4, and 5, and using $\theta_6 = \theta_4$, we obtain ($h = 0.0125$)

$$\begin{bmatrix} 2.0625 & -1 & 0 & 0 \\ -1 & 2.0625 & -1 & 0 \\ 0 & -1 & 2.0625 & -1 \\ 0 & 0 & -2 & 2.0625 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{Bmatrix} = \begin{Bmatrix} 300 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The solution of these equations is

$$\theta_2 = 251.89^\circ\text{C}, \quad \theta_3 = 219.53^\circ\text{C}, \quad \theta_4 = 200.89^\circ\text{C}, \quad \theta_5 = 194.80^\circ\text{C}$$

The heat at node 1 ($x = 0$) is

$$Q(0) = kA\left(-\frac{d\theta}{dx}\right)_{x=0} = kA\frac{\theta_1 - \theta_2}{h} = 60.46 \text{ W}$$

Finite Element Solutions

For the choice of two linear elements, the assembled system of equations is (U_I denotes the temperature $\theta(x)$ at the Ith global node)

$$\begin{bmatrix} 43.333 & -38.333 & 0 \\ -38.333 & 86.667 & -38.333 \\ 0 & -38.333 & 43.333 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} + Q_1^{(2)} \\ Q_2^{(2)} \end{Bmatrix} \tag{4.3.30}$$

The boundary conditions are

$$U_1 = 300^\circ\text{C}, \quad Q_2^{(1)} + Q_1^{(2)} = 0, \quad Q_2^{(2)} = 0$$

Hence, we have

$$\begin{bmatrix} 86.667 & -38.333 \\ -38.333 & 43.333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 38.333 \times 300 \\ 0 \end{Bmatrix}$$

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The solution of these equations is

$$U_2 = 217.98^\circ\text{C}, \quad U_3 = 192.83^\circ\text{C}$$

The heat at node 1 ($x = 0$) can be computed using the first equation of the first element

$$Q(0) = kAQ_1^{(1)} = kA(43.333U_1 - 38.333U_2) = 72.95 \text{ W}$$

Once we have the nodal values U_1, U_2, and U_3, values at other points (intermediate to the nodes) can be computed using the interpolation

$$\theta^e(\bar{x}) = \sum_{j=1}^{2} \theta_j^e \psi_j^e(\bar{x}), \quad 0 \le \bar{x} \le h_e$$

where $\theta_1^1 = U_1$, $\theta_2^1 = U_2 = \theta_1^2$, and $\theta_2^2 = U_3$.

For the choice of four linear elements, the assembled system of equations is

$$\begin{bmatrix} 81.667 & -79.167 & 0 & 0 & 0 \\ -79.167 & 163.333 & -79.167 & 0 & 0 \\ 0 & -79.167 & 163.333 & -79.167 & 0 \\ 0 & 0 & -79.167 & 163.333 & -79.167 \\ 0 & 0 & 0 & -79.167 & 81.667 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} + Q_1^{(2)} \\ Q_2^{(2)} + Q_1^{(3)} \\ Q_2^{(3)} + Q_1^{(4)} \\ Q_2^{(4)} \end{Bmatrix} \tag{4.3.31}$$

The boundary conditions are ($e = 1, 2, 3$):

$$U_1 = 300^\circ\text{C}, \quad Q_2^{(e)} + Q_1^{(e+1)} = 0, \quad Q_2^{(4)} = 0$$

Hence, we have

$$\begin{bmatrix} 163.333 & -79.167 & 0 & 0 \\ -79.167 & 81.667 & -79.167 & 0 \\ 0 & -79.167 & 81.667 & -79.167 \\ 0 & 0 & -79.167 & 81.667 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 79.167 \times 300 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The solution of these equations is

$$U_2 = 251.52^\circ\text{C}, \quad U_3 = 218.92^\circ\text{C}, \quad U_4 = 200.16^\circ\text{C}, \quad U_5 = 194.03^\circ\text{C}$$

The heat at node 1 ($x = 0$) is

$$Q(0) = kAQ_1^{(1)} = kA(81.667U_1 - 79.167U_2) = 72.07 \text{ W}$$

A comparison of the numerical results with the exact solution is presented in Table 4.3.1. The finite element solution is the best solution in the sence that it gives the minimum value of the integral of the square of $|u - u_n|$ over the domain. The exact value of $Q(0)$ is 71.78 W.

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Table 4.3.1 Comparison of finite difference and finite element solutions with the exact solution of $-\frac{d^2\theta}{dx^2}+400\theta=0,\ \theta(0)=300,\ \frac{d\theta}{dx}(0.05)=0.$

x	Exact solution	FEM† solution $N=2$	FEM† solution $N=4$	FDM solution $N=2$	FDM solution $N=4$
0.0000	300.00	300.00	300.00	300.00	300.00
0.0125	251.71	258.99*	251.52	260.21*	251.89
0.0250	219.23	217.98	218.92	220.41	219.53
0.0375	200.52	205.41*	200.16	208.17*	200.89
0.0500	194.42	192.83	194.03	195.92	194.80

*Values computed by interpolation.
† FEM, finite element method; FDM, finite difference method.

The last example of heat transfer deals with radially symmetric heat transfer in a cylinder.

Example 4.3.4

Consider a *long*, homogeneous, solid cylinder of radius R_0 [see Fig. 4.3.1(*c*)] in which energy is generated at a constant rate g_0 (W/m^3). The boundary surface at $r=R_0$ is maintained at a constant temperature T_0. We wish to calculate the temperature distribution $T(r)$ and heat flux $q(r)=-kdT/dr$ (or heat $Q=AkdT/dr$).

The governing equation for this problem is given by (4.3.10) with $g=g_0$. The boundary conditions are

$$T(R_0)=T_0,\quad \left(2\pi kr\frac{dT}{dr}\right)\bigg|_{r=0}=0 \tag{4.3.32}$$

The zero-flux boundary condition at $r=0$ is a result of the radial symmetry at $r=0$. If the cylinder is hollow with inner radius R_i then the boundary condition at $r=R_i$ can be a specified temperature, specified heat flux, or convection boundary condition, depending on the situation. The finite element solution will not be exact at nodes because a is not constant.

The finite element model of the governing equation is [see Eqs. (3.4.5*a*) and (3.4.5*b*)]

$$[K^e]\{T^e\}=\{f^e\}+\{Q^e\} \tag{4.3.33}$$

where

$$K_{ij}^e=2\pi\int_{r_a}^{r_b} kr\frac{d\psi_i^e}{dr}\frac{d\psi_j^e}{dr}\,dr,\quad f_i^e=2\pi\int_{r_a}^{r_b}\psi_i^e g_0 r\,dr \tag{4.3.34a}$$

$$Q_1^e=-2\pi k\left(r\frac{dT}{dr}\right)\bigg|_{r_a},\quad Q_2^e=2\pi k\left(r\frac{dT}{dr}\right)\bigg|_{r_b} \tag{4.3.34b}$$

and (r_a, r_b) are the global coordinates of typical element $\Omega_e=(r_a, r_b)$.

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For linear interpolation of $T(r)$, the element equations for a typical element are given by Eq. (3.4.7). The element equations for individual elements are obtained from these by giving the element length h_e and the global coordinates of the element nodes, $r_e = r_a$ and $r_{e+1} = r_b$.

For the mesh of one linear element, we have $r_a = r_1 = 0$ and $r_2 = h_e = R_0$, and

$$\pi k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \frac{\pi g_0 R_0}{3} \begin{Bmatrix} R_0 \\ 2R_0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$

The boundary conditions imply $U_2 = T_0$ and $Q_1^1 = 0$. Hence the temperature at node 1 is

$$U_1 = \frac{g_0 R_0^2}{3k} + T_0$$

and the heat at $r = R_0$ is

$$Q_2^1 = \pi k(U_2 - U_1) - \frac{2}{3}\pi g_0 R_0^2 = -\pi g_0 R_0^2$$

The negative sign indicates that heat is removed from the body (because $dT/dr < 0$). The one-element solution as a function of the radial coordinate r is

$$T_h(r) = U_1 \psi_1^1(r) + U_2 \psi_2^1(r) = \frac{g_0 R_0^2}{3k}\left(1 - \frac{r}{R_0}\right) + T_0$$

and the heat flux is

$$q(r) \equiv -k\frac{dT_h}{dr} = \frac{1}{3} g_0 R_0$$

For a mesh of two linear elements, we take $h_1 = h_2 = \frac{1}{2}R_0$, $r_1 = 0$, $r_2 = h_1 = \frac{1}{2}R_0$, and $r_3 = h_1 + h_2 = R_0$. The two-element assembly gives

$$\pi k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \frac{\pi g_0 R_0^2}{6} \begin{Bmatrix} \frac{1}{2} \\ 1+2 \\ \frac{1}{2}+2 \end{Bmatrix} \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

Imposing the boundary conditions $U_3 = T_0$, $Q_2^1 + Q_1^2 = 0$, and $Q_1^1 = 0$, the condensed equations for the unknown temperatures are

$$\pi k \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \frac{\pi g_0 R_0^2}{12} \begin{Bmatrix} 1 \\ 6 \end{Bmatrix} + \pi k \begin{Bmatrix} 0 \\ 3T_0 \end{Bmatrix}$$

The nodal values are

$$U_1 = \frac{5}{18}\frac{g_0 R_0^2}{k} + T_0, \quad U_2 = \frac{7}{36}\frac{g_0 R_0^2}{k} + T_0$$

From equilibrium, Q_2^2 is computed as

$$Q_2^2 = -\frac{5}{12}\pi g_0 R_0^2 + 3\pi k(U_3 - U_2) = -\pi g_0 R_0^2$$

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The finite element solution becomes

$$T_h(r) = \begin{cases} U_1\psi_1^1(r) + U_2\psi_2^1(r) = \left(\dfrac{5}{18}\dfrac{g_0R_0^2}{k} + T_0\right)\left(1 - 2\dfrac{r}{R_0}\right) + 2\left(\dfrac{7}{36}\dfrac{g_0R_0^2}{k} + T_0\right)\dfrac{r}{R_0} \\ U_2\psi_1^2 + U_3\psi_2^2 = 2\left(\dfrac{7}{36}\dfrac{g_0R_0^2}{k} + T_0\right)\left(1 - \dfrac{r}{R_0}\right) + T_0\left(2\dfrac{r}{R_0} - 1\right) \end{cases}$$

$$= \begin{cases} \dfrac{1}{18}\dfrac{g_0R_0^2}{k}\left(5 - 3\dfrac{r}{R_0}\right) + T_0, & \text{for } 0 \le r \le \dfrac{1}{2}R_0 \\ \dfrac{7}{18}\dfrac{g_0R_0^2}{k}\left(1 - \dfrac{r}{R_0}\right) + T_0, & \text{for } \dfrac{1}{2}R_0 \le r \le R_0 \end{cases}$$

The exact solution of the problem is

$$T(r) = \frac{g_0R_0^2}{4k}\left[1 - \left(\frac{r}{R_0}\right)^2\right] + T_0(^\circ\text{C}) \tag{4.3.35}$$

$$q(r) = \frac{1}{2}g_0r \ (\text{W/m}^2), \quad Q(R_0) = \left(2\pi kr\frac{dT}{dr}\right)\bigg|_{R_0} = -\pi g_0R_0^2 \ (\text{W}) \tag{4.3.36}$$

The temperature at the center of the cylinder according to the exact solution is $T(0) = g_0R_0^2/4k + T_0$, whereas it is $g_0R_0^2/3k + T_0$ and $5g_0R_0^2/18k + T_0$ according to the one- and two-element models, respectively.

The finite element solutions obtained using one-, two-, four-, and eight-element meshes of linear elements are compared with the exact solution in Table 4.3.2. Convergence of the finite element solutions, $\bar{T} = (T - T_0)k/g_0R_0^2$, to the exact solution with an increasing number of elements is clear (see Fig. 4.3.6). Figure 4.3.7 shows plots of $\bar{Q}(r) = Q(r)/2\pi R_0g_0$ and $Q(r) = 2\pi krdT/dr$, versus $\bar{r} = r/R_0$, as computed in the finite element analysis and the exact solution.

Table 4.3.2 Comparison of the finite element and exact solutions for heat transfer in a radially symmetric cylinder $R_0 = 0.01$ m, $g_0 = 2 \times 10^8$ W/m^3, $k = 20$ W/(m·°C), $T_0 = 100$°C.

	Temperature $T(r)$[†]				
r/R_0	One element	Two elements	Four elements	Eight elements	Exact
0.000	433.33	377.78	358.73	352.63	350.00
0.125	391.67	356.24	348.31	347.42	346.09
0.250	350.00	335.11	337.90	335.27	334.38
0.375	308.33	315.28	313.59	315.48	314.84
0.500	266.67	294.44	289.29	287.95	287.50
0.625	225.00	245.83	249.70	252.65	252.34
0.750	183.33	197.22	210.12	209.56	209.38
0.875	141.67	148.61	155.06	158.68	158.59
1.000	100.00	100.00	100.00	100.00	100.00

[†] The underlined terms are nodal values and others are interpolated values.

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Figure 4.3.6 Comparison of the finite element solutions with the exact solution for heat transfer in a radially symmetric problem with cylindrical geometry.

Figure 4.3.7 Comparison of the finite element solution with the exact solution for the temperature gradient in a radially symmetric problem with cylindrical geometry.

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4.4 FLUID MECHANICS

4.4.1 Governing Equations

All bulk matter in nature exists in one of two forms: solid or fluid. A solid body is characterized by the relative immobility of its molecules whereas a fluid state is characterized by relative mobility of its molecules. Fluids can exist either as gases or liquids. The field of fluid mechanics is concerned with the motion of fluids and the conditions affecting the motion (see Reddy and Gartling, 2001).

The basic equations of fluid mechanics are derived from the global laws of conservation of mass, momentum, and energy. Conservation of mass gives the continuity equation, while the conservation of momentum results in the equations of motion. The conservation of energy, considered in the last section, is the first law of thermodynamics, and it results in Eqs. (4.3.8)–(4.3.10) for one-dimensional systems when thermal-fluid coupling is omitted. For additional details, see Schlichting (1979), Bird et al. (1960), and Reddy and Gartling (2001). More details are provided in Chapter 10, which is dedicated to finite element models of two-dimensional flows of viscous incompressible fluids.

Here, we consider so-called parallel flow, where only one velocity component is different from zero resulting in all the fluid particles moving in one direction, i.e., $u = u(x, y, z)$, where u is the velocity component along the x coordinate. We assume that there are no body forces. The z-momentum equation requires that $u = u(x, y)$. The conservation of mass in this case reduces to

$$\frac{\partial u}{\partial x} = 0$$

which implies that $u = u(y)$. The y-momentum equation simplifies to

$$\frac{\partial P}{\partial y} = 0$$

which implies that $P = P(x)$, where P is the pressure. The x-momentum equation simplifies to

$$\mu \frac{d^2 u}{dy^2} = \frac{dP}{dx} \tag{4.4.1}$$

The energy equation for this problem reduces to

$$\rho c u \frac{\partial T}{\partial x} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\left(\frac{du}{dy}\right)^2 \tag{4.4.2}$$

Here we are primarily interested in the finite element analysis of Eq. (4.4.1).

4.4.2 Finite Element Model

Equation (4.4.1) is a special case of the model equation (3.2.1) with the following correspondence:

$$f = -\frac{dP}{dx}, \quad a = \mu = \text{constant}, \quad c = 0, \quad x = y \tag{4.4.3}$$

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Therefore, the finite element model in (3.2.31a) and (3.2.31b) is valid for this problem:

$$[\boldsymbol{K}^e]\{\boldsymbol{u}^e\} = \{\boldsymbol{f}^e\} + \{\boldsymbol{Q}^e\} \quad \text{or} \quad \boldsymbol{K}^e\boldsymbol{u}^e = \boldsymbol{f}^e + \boldsymbol{Q}^e \tag{4.4.4a}$$

where

$$K_{ij}^e = \int_{y_a}^{y_b} \mu \frac{d\psi_i^e}{dy}\frac{d\psi_j^e}{dy}\,dy, \quad f_i^e = \int_{y_a}^{y_b} \left(-\frac{dP}{dx}\right)\psi_i^e\,dy$$

$$Q_1^e = -\left(\mu\frac{du}{dy}\right)\bigg|_a, \quad Q_2^e = -\left(\mu\frac{du}{dy}\right)\bigg|_b \tag{4.4.4b}$$

Next, we consider an example.

Example 4.4.1

Consider parallel flow between two long flat walls separated by a distance $2L$ [see Fig. 4.4.1(a)]. We wish to determine the velocity distribution $u(y)$, $-L < y < L$, for a given pressure gradient $-dP/dx$, using the finite element method.

For a two-element mesh of linear elements ($h = L$), we have

$$\frac{\mu}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \frac{f_0 h}{2}\begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

We consider two sets of boundary conditions [see Fig. 4.4.1(b)]

Set 1: $u(-L) = 0,\ u(L) = 0$ (two stationary walls)

Set 2: $u(-L) = 0,\ u(L) = U_0$ (bottom wall stationary and top wall moving)

(4.4.5)

For the first case, we may use symmetry and model domain $0 < x < L$. Here we consider the full domain for the two sets of boundary conditions.

For Set 1, we have $U_1 = U_3 = 0$. The finite element solution is given by

$$U_2 = \frac{f_0 L^2}{2\mu}, \quad u_h(y) = \frac{f_0 L^2}{2\mu}\frac{y}{L} \tag{4.4.6}$$

For set 2, we have $U_1 = 0$ and $U_3 = U_0$. The finite element solution is

$$U_2 = \frac{f_0 L^2}{2\mu} + \frac{1}{2}U_0, \quad u_h(y) = \left(\frac{f_0 L^2}{2\mu} + \frac{1}{2}U_0\right)\frac{y}{L} \tag{4.4.7}$$

For a one-element mesh of the quadratic element ($h = 2L$), we have

$$\frac{\mu}{6L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \frac{f_0 L}{3}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ 0 \\ Q_3^1 \end{Bmatrix} \tag{4.4.8}$$

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Figure 4.4.1 (*a*) Flow between parallel plates. (*b*) Finite element mesh; $u(L) = 0$ for set 1 and $u(L) = U_0$ for set 2.

and the finite element solutions for the two sets of boundary conditions are

$$U_2 = \frac{f_0 L^2}{2\mu} \qquad \text{for Set 1}$$
$$U_2 = \frac{f_0 L^2}{2\mu} + \frac{1}{2}U_0 \qquad \text{for Set 2} \tag{4.4.9}$$

Although the nodal values predicted in the linear- and quadratic-element meshes are the same, they vary linearly and quadratically between nodes of linear and quadratic elements, respectively.

The exact solutions for the two sets of boundary conditions in (4.4.5) are $(-L \le y \le L)$

$$u(y) = \frac{f_0 L^2}{2\mu}\left(1 - \frac{y^2}{L^2}\right) \qquad \text{for Set 1}$$
$$u(y) = U_0\frac{1}{2}\left(1 + \frac{y}{L}\right) + \frac{f_0 L^2}{2\mu}\left(1 - \frac{y^2}{L^2}\right) \qquad \text{for Set 2} \tag{4.4.10}$$

Note that the finite element solutions at the nodes are exact, as expected. The quadratic-element solution agrees with the exact solutions (4.4.10) for every value of y.

4.5 SOLID AND STRUCTURAL MECHANICS

4.5.1 Preliminary Comments

Solid mechanics is that branch of mechanics dealing with the motion and deformation of solids. The Lagrangian description of motion is used to express the global conservation laws. The conservation of mass for solid bodies is trivially satisfied because of the fixed material viewpoint used in the Lagrangian description. The conservation of momentum is nothing but Newton's second law of motion. Under isothermal conditions, the energy equation uncouples from the momentum equations, and we need only consider the equations of motion or equilibrium (see Example 2.3.3).

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Unlike in fluid mechanics, the equations governing solid bodies undergoing different forms of deformations are derived directly without specializing the three-dimensional elasticity equations to one dimension. Various types of load-carrying members are called by different names, e.g., bars, beams, and plates. A *bar* is a structural member that is subjected to only axial loads (see Examples 1.2.3 and 2.3.2), while a *beam* is a member that is subjected to loads that tend to bend it about an axis perpendicular to the axis of the member (see Example 2.4.2). The equations governing the motion of such structural elements are not deduced directly from (2.3.52), but they are derived either by considering the equilibrium of an element of the member with all its proper forces and using Newton's second law (Example 1.2.3), or by using an energy principle (Examples 2.3.2 and 2.4.2).

4.5.2 Finite Element Model of Bars and Cables

The equation of motion governing axial deformation of a bar is (see Example 1.2.3)

$$\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = f(x,t) \tag{4.5.1}$$

For static problems, Eq. (4.5.1) reduces to

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f(x) \tag{4.5.2}$$

It should be recalled that Eq. (4.5.2) is derived under the assumption that all material points on the line x = constant (i.e., all points on any cross section) move by the same amount $u(x)$. This is equivalent to the assumption that the stress on any cross section is uniform. Equation (4.5.2) is the same as the model equation (3.2.1), with $a = EA$ and $c = 0$. Hence, the finite element model in (3.2.31a) and (3.2.31b) is valid for bars.

The average transverse deflection $u(x)$ of a cable made of elastic material is also governed by an equation of the form:

$$-\frac{d}{dx}\left(T\frac{du}{dx}\right) = f(x) \tag{4.5.3}$$

where T is the uniform tension in the cable and f is the distributed transverse force. Again, Eq. (4.5.3) is a special case of Eq. (3.2.1), with $a = T$ and $c = 0$.

In structural mechanics problems, the quadratic functional of (3.2.10) takes the special meaning of total potential energy, Π^e, which can be expressed in the form

$$\Pi^e = \frac{1}{2}\int_{x_a}^{x_b} \varepsilon^{\mathrm{T}}\sigma A_e \;\; dx - \int_{x_a}^{x_b} uf\, dx - \sum_i u_i^e Q_i^e$$

where u is the displacement, ε the strain and σ the stress. The finite element approximation (3.2.24) of u can be written as

$$u_h = \left\{\psi_1^e \;\; \psi_2^e \;\; \ldots \;\; \psi_n^e\right\} \begin{Bmatrix} u_1^e \\ u_2^e \\ \vdots \\ u_n^e \end{Bmatrix} = \mathbf{N}^e \mathbf{u}^e$$

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and the strains and stresses take the form

$$\varepsilon = \frac{du}{dx} = \frac{d}{dx}\left(\mathbf{N}^e\mathbf{u}^e\right) \equiv \mathbf{B}^e\mathbf{u}^e$$
$$\sigma = E\varepsilon = E_e\mathbf{B}^e\mathbf{u}^e$$

and the expression for the total potential energy becomes

$$\Pi^e = \frac{1}{2}\int_{x_a}^{x_b} A_e E_e \mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u}\,dx - \int_{x_a}^{x_b} \mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} f\,dx - \mathbf{u}^{\mathrm{T}}\mathbf{Q}$$

where the element label on $\mathbf{B}$ and $\mathbf{u}$ is omitted for brevity. Then the principle of minimum total potential energy, $\delta\Pi^e = 0$, yields the finite element model

$$\delta\Pi^e = \delta\mathbf{u}^{\mathrm{T}}\left[\left(\int_{x_a}^{x_b} A_e E_e \mathbf{B}^{\mathrm{T}}\mathbf{B}\,dx\right)\mathbf{u} - \int_{x_a}^{x_b} \mathbf{B}^{\mathrm{T}} f\,dx - \mathbf{Q}\right] = 0$$

or

$$\mathbf{K}^e\mathbf{u}^e = \mathbf{f}^e + \mathbf{Q}^e \tag{4.5.4a}$$

where

$$\mathbf{K}^e = \int_{x_a}^{x_b} A_e E_e \mathbf{B}^{\mathrm{T}}\mathbf{B}\,dx, \quad \mathbf{f}^e = \int_{x_a}^{x_b} \mathbf{B}^{\mathrm{T}} f\,dx, \quad \mathbf{B} = \frac{d\mathbf{N}}{dx} \tag{4.5.4b}$$

These are just matrix form of the equations already presented in (3.2.31*a*) and (3.2.31*b*).

4.5.3 Numerical Examples

In this section we consider a number of examples of finite element analysis of bars.

Example 4.5.1

A bridge is supported by several concrete piers, and the geometry and loads of a typical pier are shown in Fig. 4.5.1. The load 20 kN/m^2 represents the weight of the bridge and an assumed distribution of the traffic on the bridge. The concrete weighs approximately 25 kN/m^3 and its modulus is $E = 28 \times 10^6$ kN/m^2. We wish to analyze the pier for displacements and stresses using the finite element method. The pier is indeed a three-dimensional structure. However, we wish to approximate the deformation and stress fields in the pier as one-dimensional.

We represent the distributed force at the top of the pier as a point force

$$F = (0.5 \times 0.5)20 = 5 \text{ kN}$$

The weight of the concrete is represented as the body force per unit length. The total force at any distance x is equal to the weight of the concrete above that point. The weight at a distance x is equal to the product of the volume of the body above x and the specific weight of the concrete:

$$W(x) = 0.5\frac{0.5 + (0.5 + 0.5x)}{2}x \times 25.0 = 6.25(1 + 0.5x)x$$

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Figure 4.5.1 The geometry and loading in the concrete pier problem of Example 4.5.1.

The body force per unit length is computed from

$$f = \frac{dW}{dx} = 6.25(1 + x)$$

This completes the load representation of the problem.

The governing differential equation for the problem is given by (4.5.2), with $E = 28 \times 10^6$ kN/m^2 and cross-sectional area $A(x)$:

$$A(x) = (0.5 + 0.5x)0.5 = \frac{1}{4}(1 + x)$$

Thus, the concrete pier problem is idealized as a one-dimensional problem whose axial displacement u is governed by the equation

$$-\frac{d}{dx}\left[\frac{1}{4}E(1+x)\frac{du}{dx}\right] = 6.25(1 + x) \tag{4.5.5a}$$

subject to the boundary conditions

$$\left[\frac{1}{4}E(1+x)\frac{du}{dx}\right]\bigg|_{x=0} = -5, \quad u(2) = 0 \tag{4.5.5b}$$

Equation (4.5.5a) is a special case of the model equation (3.2.1) with the following correspondance:

$$a(x) = 0.25E(1 + x), \quad f(x) = 6.25(1 + x) \tag{4.5.6}$$

For a typical linear element, the stiffness matrix and force vector can be computed using Eq. (3.2.31b). We have

$$\begin{aligned} [K^e] &= \frac{E}{4h_e}\left[1 + \frac{1}{2}(x_e + x_{e+1})\right]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \{f^e\} &= 6.25\frac{h_e}{2}\left(\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{1}{3}\begin{Bmatrix} x_{e+1} + 2x_e \\ 2x_{e+1} + x_e \end{Bmatrix}\right) \end{aligned} \tag{4.5.7}$$

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Let us consider a two-element mesh with $h_1 = h_2 = 1$ m. We have

$$[K^1] = \frac{E}{4}\begin{bmatrix} 1.5 & -1.5 \\ -1.5 & 1.5 \end{bmatrix}, \quad \{f^1\} = \frac{6.25}{6}\begin{Bmatrix} 3+1 \\ 3+2 \end{Bmatrix} = \begin{Bmatrix} 4.167 \\ 5.208 \end{Bmatrix}$$

$$[K^2] = \frac{E}{4}\begin{bmatrix} 2.5 & -2.5 \\ -2.5 & 2.5 \end{bmatrix}, \quad \{f^2\} = \frac{6.25}{6}\begin{Bmatrix} 3+4 \\ 3+5 \end{Bmatrix} = \begin{Bmatrix} 7.292 \\ 8.333 \end{Bmatrix}$$

The assembled equations are

$$E\begin{bmatrix} 0.375 & -0.375 & 0.000 \\ -0.375 & 1.000 & -0.625 \\ 0.000 & -0.625 & 0.625 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}\begin{Bmatrix} 4.167 \\ 12.500 \\ 8.333 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

The boundary and equilibrium conditions require

$$U_3 = 0, \quad Q_2^1 + Q_1^2 = 0, \quad Q_1^1 = 5 \text{ kN}$$

The condensed equations for the unknown displacements and forces are

$$E\begin{bmatrix} 0.375 & -0.375 \\ -0.375 & 1.000 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 9.167 \\ 12.500 \end{Bmatrix}, \quad Q_2^2 = -0.6255U_2 - 8.333$$

The solution is given by (positive displacements, because of the coordinate system used, indicate that the pier is in compression)

$$U_1 = 2.111 \times 10^{-6}\text{m}, \quad U_2 = 1.238 \times 10^{-6}\text{m}, \quad Q_2^2 = -30 \text{ kN}$$

Hence, the stress at the fixed end is (compressive)

$$\sigma_x = Q_2^2/A = -30/0.75 = -40 \text{ kN/m}^2$$

The four-element model gives 2.008×10^{-6} m and 1.228×10^{-6}m, respectively.

The exact solution of Eqs. (4.5.5*a*) and (4.5.5*b*) is

$$u(x) = \frac{1}{E}\left[56.25 - 6.25(1+x)^2 - 7.5\ln\left(\frac{1+x}{3}\right)\right] \tag{4.5.8}$$

The exact values of u at nodes 1 and 2 are

$$u(0) = 2.08 \times 10^{-6}\text{m}, \quad u(1) = 1.225 \times 10^{-6}\text{m}$$

The finite element solution at the nodes is not exact because $a = EA$ is not a constant in this problem.

Example 4.5.2

Consider the composite bar consisting of a tapered steel bar fastened to an aluminum rod of uniform cross section and subjected to loads as shown in Figure 4.5.2. We wish to determine the displacement field in the bar using the finite element method.

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Figure 4.5.2 Axial deformation of a composite member (*a*) Geometry and loading. (*b*) Finite element representation.

The governing equations are given by

$$
\begin{aligned}
-\frac{d}{dx}\left(E_s A_s \frac{du}{dx}\right) &= 0, \quad 0 < x < h_1 \\
-\frac{d}{dx}\left(E_a A_a \frac{du}{dx}\right) &= 0, \quad h_1 < x < h_1 + h_2 = L
\end{aligned}
\tag{4.5.9}
$$

where the subscript "s" refers to steel and "a" to aluminum. The boundary conditions are obvious. We consider the following data:

$$
\begin{aligned}
&E_s = 30 \times 10^6 \text{ psi}, \; A_s = (c_1 + c_2 x)^2, \; E_a = 10^7 \text{ psi} \\
&A_a = 1 \text{ in.}^2, \; h_1 = 96 \text{ in.}, \; L = 216 \text{ in.}, \; P_0 = 10,000 \text{ lb}
\end{aligned}
\tag{4.5.10}
$$

The finite element equations for a uniform bar element with constant $E_e A_e$ and $f(x) = 0$ are given by [see Eq. (3.3.5*a*)]

$$
\frac{E_e A_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}
\tag{4.5.11a}
$$

where Q_i^e are the end forces

$$
Q_1^e = \left[-EA\frac{du}{dx}\right]_{x_a}, \quad Q_2^e = \left[EA\frac{du}{dx}\right]_{x_b}
\tag{4.5.11b}
$$

For the present problem, A_e is not constant, but Eq. (4.6.11a) is still valid with

$$
A_e = (c_1^e)^2 + \frac{1}{3}(c_2^e)^2(x_b^2 + x_a^2 + x_a x_b) + c_1^e c_2^e (x_b + x_a)
$$

To see this, we calculate K_{ij}^e for the problem using the local coordinate $\bar{x}$ ($x = \bar{x} + x_a$). We have

$$
K_{ij}^e = \int_{x_a}^{x_b} E_e (c_1^e + c_2^e x)^2 \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}dx = \int_0^{h_e} E_e \left[c_1^e + c_2^e(\bar{x} + x_a)\right]^2 \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}d\bar{x}
$$

$$
K_{11}^e = \frac{E_e}{h_e^2}\int_{x_a}^{x_b} (c_1^e + c_2^e x)^2 dx
$$

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$$= \frac{E_e}{h_e^2}\left[(c_1^e)^2 h_e + \frac{1}{3}(c_2^e)^2(x_b^3 - x_a^3) + c_1^e c_2^e (x_b^2 - x_a^2)\right]$$

$$= \frac{E_e}{h_e}\left[(c_1^e)^2 + \frac{1}{3}(c_2^e)^2(x_b^2 + x_a^2 + x_a x_b) + c_1^e c_2^e (x_b + x_a)\right] = \frac{E_e A_e}{h_e}$$

$K_{12}^e = K_{21}^e = -K_{11}^e$ and $K_{22}^e = K_{11}^e$, where we have used the algebraic equalities, $a^2 - b^2 = (a-b)(a+b)$ and $a^3 - b^3 = (a-b)(a^2+b^2+ab)$ to simplify the expressions.

For two linear elements of lengths $h_1 = 96$ in. and $h_2 = 120$ in., we have $c_1^1 = 1.5$, $c_2^1 = -0.5/96$, $c_1^2 = 1$, $c_2^2 = 0$, and the element stiffness matrices become

$$[K^1] = \frac{4.75}{96} \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^4 \begin{bmatrix} 49.479 & -49.479 \\ -49.479 & 49.479 \end{bmatrix}$$

$$[K^2] = \frac{1}{120} \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^4 \begin{bmatrix} 8.333 & -8.333 \\ -8.333 & 8.333 \end{bmatrix}$$

The assembled equations are

$$10^4 \begin{bmatrix} 49.479 & -49.479 & 0.000 \\ -49.479 & 57.812 & -8.333 \\ 0.000 & -8.333 & 8.333 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

The boundary conditions imply

$$U_1 = 0, \quad Q_2^1 + Q_1^2 = 2P_0, \quad Q_2^2 = P_0$$

Hence, the condensed equations are

$$10^4 \begin{bmatrix} 57.812 & -8.333 \\ -8.333 & 8.333 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 2P_0 \\ P_0 \end{Bmatrix}, \quad Q_1^1 = -10^4 \times 49.479\, U_2$$

whose solution is

$$U_2 = 0.06063 \text{ in.}, \quad U_3 = 0.18063 \text{ in.}, \quad Q_1^1 = -30,000 \text{ lb}$$

The negative sign indicates that the reaction is acting away from the end (i.e., tensile force). The magnitude of Q_1^1 is consistent with the static equilibrium of the forces:

$$Q_1^1 + 2P_0 + P_0 = 0 \text{ or } Q_1^1 = -3P_0 = -30,000 \text{ lb}$$

The axial displacement at any point x along the bar is given by

$$u_h(x) = \begin{cases} u_1^{(1)}\psi_1^{(1)} + u_2^{(1)}\psi_2^{(1)} = 0.06063x/96, & 0 \le x \le 96 \\ u_1^{(2)}\psi_1^{(2)} + u_2^{(2)}\psi_2^{(2)} = -0.03537 + 0.001x, & 96 \le x \le 216 \end{cases}$$

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and its first derivative is given by

$$\frac{du_h}{dx} = \begin{cases} \dfrac{0.06063}{96}, & 0 \le x \le 96 \\ 0.0001, & 96 \le x \le 216 \end{cases}$$

The exact solution of Eqs. (4.5.9) subject to the boundary conditions

$$u(0) = 0, \left[\left(a\frac{du}{dx}\right)_{x=96^+} - \left(a\frac{du}{dx}\right)_{x=96^-}\right] = 2P_0, \left(a\frac{du}{dx}\right)_{x=216} = P_0$$

is given by

$$u(x) = \begin{cases} 0.128[x/(288-x)], & 0 \le x \le 96 \\ 0.001(x-32), & 96 \le x \le 216 \end{cases}$$

$$\frac{du}{dx} = \begin{cases} 36.864/(288-x)^2, & 0 \le x \le 96 \\ 0.001, & 96 \le x \le 216 \end{cases}$$

In particular, the exact solution at nodes 2 and 3 is given by

$$u(96) = 0.064 \text{ in.}, \quad u(216) = 0.1840 \text{ in.}$$

Thus, the two-element solution is about 1.8 percent off from the maximum displacement.

Next consider a two-element mesh of quadratic elements. The element matrix and force vector for element 1 are

$$[K^1] = 10^4 \begin{bmatrix} 142.19 & -159.37 & 17.18 \\ -159.37 & 266.67 & -107.29 \\ 17.18 & -107.29 & 90.10 \end{bmatrix}, \quad \{F^1\} = \begin{Bmatrix} Q_1^1 \\ 0 \\ Q_3^1 \end{Bmatrix}$$

The assembled stiffness matrix is of order 5×5, and it is of the form

$$[K] = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & 0 & 0 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & K_{12}^2 & K_{13}^2 \\ 0 & 0 & K_{21}^2 & K_{22}^2 & K_{23}^2 \\ 0 & 0 & K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix}$$

After imposing boundary conditions ($U_1 = 0$, $Q_3^1 + Q_1^2 = 20,000$, $Q_3^2 = 10,000$), and solving the resulting 4×4 equations, we obtain

$$U_2 = 0.02572 \text{ in.}, U_3 = 0.06392 \text{ in.}, U_4 = 0.12392 \text{ in.}, U_5 = 0.18392 \text{ in.}$$

The two-element solution obtained using the quadratic element is very accurate, as can be seen from a comparison of the finite element solution with the exact solution presented in Table 4.5.1.

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Table 4.5.1 Comparison of the finite element solutions with the exact solution of the bar problem in Example 4.5.2.

		Linear					Quadratic	
x (in.)	Exact	(1, 1)†	(2, 1)	(3, 2)	(4, 2)	(6, 2)	(1, 1)	(2, 1)
16	0.00753	—	—	—	—	0.00752	—	—
24	0.01600	—	—	—	0.01161	—	—	0.01164
32	0.01600	—	—	0.01593	—	0.01598	—	—
48	0.02560	—	0.02532	—	0.02553	0.02557	0.02572	0.02560
64	0.03657	—	—	0.03638	—	0.03652	—	—
72	0.04267	—	—	—	0.04253	—	—	0.04268
80	0.04923	—	—	—	—	0.04916	—	—
96	0.06400	0.06063	0.06309	0.06359	0.06377	0.06390	0.06392	0.06399
156	0.12392	—	—	—	0.12377	0.12390	0.12392	0.12399
216	0.18400	0.18063	0.18309	0.18359	0.18377	0.18390	0.18392	0.18399

† (m, n) means m elements in the interval (0,96) and n elements in the interval (96,216); all elements in each interval are of the same size.

Example 4.5.3

Consider a hollow circular cylinder with inner radius a, outer radius b, and length L. The outer surface of the hollow cylinder is assumed to be fixed and its inner surface ideally bonded to a rigid circular cylindrical core of radius a and length L, as shown in Fig. 4.5.3. Suppose that an axial force P is applied to the rigid core along its centroidal axis. We wish to find the axial displacement δ of the rigid core by assuming that the displacement field in the hollow cylinder is of the form

$$u_r = u_\theta = 0, \quad u_z = U(r) \tag{4.5.12}$$

where (u_r, u_θ, u_z) are the displacements along (r, θ, z) coordinates.

Figure 4.5.3 Axisymmetric deformation of a hollow cylinder fixed at the outer surface and pulled by a rigid core at the inner surface.

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The equation governing $U(r)$ can be determined as follows. First, we note that

$$\varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{zz} = \varepsilon_{r\theta} = \varepsilon_{\theta z} = 0, \quad 2\varepsilon_{rz} = \frac{dU}{dr} \tag{4.5.13a}$$

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{\theta r} = \sigma_{\theta z} = 0; \quad \sigma_{rz} = G\frac{dU}{dr} \tag{4.5.13b}$$

where G is the shear modulus. Of the three stress-equilibrium equations in the cylindrical coordinates, the two equilibrium equations associated with r and θ directions are trivially satisfield (in the absence of body forces) by the stress field. The third equilibrium equation

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{\partial \sigma_{\theta z}}{\partial \theta} + r\frac{\partial \sigma_{zz}}{\partial z}\right] = 0$$

yields the equation

$$\frac{1}{r}\frac{d}{dr}\left(rG\frac{dU}{dr}\right) = 0 \tag{4.5.14}$$

The boundary conditions on $U(r)$ are

$$u_z(b) = 0 \rightarrow U(b) = 0, \quad -2\pi L(r\sigma_{rz})|_{r=a} = P \rightarrow -\left(rG\frac{dU}{dr}\right)_{r=a} = \frac{P}{2\pi L} \tag{4.5.15}$$

This completes the theoretical formulation of the problem. We wish to analyze the problem using the finite element method.

The finite element model of Eq. (4.5.14) [cf. Eq. (3.4.1)] is given by Eqs. (3.4.5a) and (3.4.5b) with $a(r) = rG$ and $f = 0$. One linear element in the domain yields ($r_a = a$ and $h = b - a$)

$$\frac{2\pi G}{h}\left(r_a + \frac{1}{2}h\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$

and with $U_2 = 0$ and $Q_1^1 = P/L$, we obtain

$$U_1 = \frac{P}{\pi LG}\frac{(b-a)}{(b+a)} \tag{4.5.16}$$

The exact solution of Eqs. (4.5.14) and (4.5.15) is given by

$$U(r) = -\frac{P}{2\pi LG}\log(r/b), \quad U(a) \equiv \delta = \frac{P}{2\pi LG}\log(b/a) \tag{4.5.17}$$

The one element solution in Eq. (4.5.16) corresponds to the first term in the logarithmic series of δ [i.e., $\log(a/b)$].

Use of one quadratic element gives ($h = b - a$)

$$\frac{2\pi G}{6h}\begin{bmatrix} 3h + 14a & -(4h + 16a) & h + 2a \\ -(4h + 16a) & 16h + 32a & -(12h + 16a) \\ h + 2a & -(12h + 16a) & 11h + 14a \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ 0 \\ Q_3^1 \end{Bmatrix} \tag{4.5.18}$$

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Using the boundary conditions $U_3 = 0$ and $Q_1^1 = P/L$, we obtain the following condensed equations in U_1 and U_2:

$$\frac{2\pi G}{6h}\begin{bmatrix} 3h+14a & -(4h+16a) \\ -(4h+16a) & 16h+32a \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} P/L \\ 0 \end{Bmatrix} \tag{4.5.19}$$

whose solution is

$$U_1 = \frac{3hP}{\pi GL}\frac{h+2a}{3h^2+20ah+28a^2} \tag{4.5.20}$$

$$U_2 = \frac{3hP}{\pi GL}\frac{h+4a}{12h^2+80ah+112a^2}$$

The finite element solution becomes

$$U_h(r) = U_1(1-r/h)(1-2r/h) + 4U_2(r/h)(1-r/h)$$

A comparison of the finite element solutions obtained with linear and quadratic elements with the analytical solution is shown in Fig. 4.5.4. The four-element mesh of quadratic elements virtually gives the exact solution.

Figure 4.5.4 Comparison of finite element solutions with the analytical solution for axisymmetric deformation of a hollow cylinder fixed at the outer surface and pulled by a rigid core at the inner surface.

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4.6 PLANE TRUSSES

4.6.1 Introduction

Consider a structure consisting of several bar elements connected to each other by pins, as shown in Figure 4.6.1. The members may rotate freely about the axis of the pin. Consequently, each member carries only axial forces. The planar structure with pin-connected members (i.e., all members lie in the same plane) is called a *plane truss*. Since each member is oriented differently with respect to a global coordinate system (x, y), it is necessary to transform the force displacement relations that were derived in element coordinate system $(\bar{x}, \bar{y})$ to the global coordinate system (x, y) so that the structure stiffness can be assembled from the element stiffness referred to the same global coordinate system.

4.6.2 Basic Truss Element

First, we consider a uniform bar element with constant EA and oriented at an angle θ_e, measured counterclockwise, from the positive x-axis. If the member coordinate system $(\bar{x}_e, \bar{y}_e)$ is taken as shown in Figure 4.6.2(a), and $(\bar{u}_i^e, \bar{v}_i^e)$ and $(\bar{F}_i^e, 0)$ denote the displacements and forces at node i with respect to the member coordinate system $(\bar{x}_e, \bar{y}_e)$, respectively, the element equations (4.5.11a) can be expressed as (we now use the notation $\bar{Q}_i^e + \bar{f}_i^e = \bar{F}_i^e$)

$$\frac{E_e A_e}{h_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1^e \\ 0 \\ \bar{F}_2^e \\ 0 \end{Bmatrix} \quad \text{or} \quad [\bar{K}^e]\{\bar{\Delta}^e\} = \{\bar{F}^e\} \tag{4.6.1}$$

We wish to write the force-deflection relations (4.6.1) in terms of the corresponding global displacements and forces. Toward this end, we first write the transformation relations between the two sets of coordinate systems (x, y) and $(\bar{x}_e, \bar{y}_e)$ (see Fig. 4.6.2)

$$\bar{x}_e = x\ \cos\theta_e + y\ \sin\theta_e, \quad \bar{y}_e = -x\ \sin\theta_e + y\ \cos\theta_e$$

$$x = \bar{x}_e\ \cos\theta_e - \bar{y}_e\ \sin\theta_e, \quad y = \bar{x}_e\ \sin\theta_e + \bar{y}_e\ \cos\theta_e$$

Figure 4.6.1 A plane truss structure.

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Figure 4.6.2 A bar element oriented at an angle with respect to the global coordinate system (x, y). (a) Forces and displacements in the element coordinates. (b) Forces and displacements in the global coordinates.

or, in matrix form, we have

$$\begin{Bmatrix} \bar{x}_e \\ \bar{y}_e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e \\ -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}, \quad \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & -\sin\theta_e \\ \sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} \bar{x}_e \\ \bar{y}_e \end{Bmatrix} \tag{4.6.2}$$

where θ_e is the angle between the positive x-axis and positive $\bar{x}_e$-axis, measured in the counterclockwise direction. Note that all quantities with a bar over them refer to the member (or local) coordinate system $(\bar{x}_e, \bar{y}_e)$, while the quantities without a bar refer to the global coordinate system (x, y).

The above relationship also holds for the displacements and forces of the two coordinate systems. We have

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{Bmatrix} \tag{4.6.3a}$$

or

$$\{\bar{\Delta}^e\} = [T^e]\{\Delta^e\} \tag{4.6.3b}$$

where $\{\bar{\Delta}^e\}$ and $\{\Delta^e\}$ denote the nodal displacement vectors in the member and structure coordinate systems, respectively. Similarly, we have

$$\{\bar{F}^e\} = [T^e]\{F^e\} \tag{4.6.4}$$

Here $\{\bar{F}^e\}$ and $\{F^e\}$ denote the nodal force vectors in the member and structure coordinate systems, respectively (see Figures 4.6.2).

4.6.3 General Truss Element

Next, we derive the relationship between the global displacements and global forces. Using Eqs. (4.6.3b) and (4.6.4) in Eq. (4.6.1), we obtain

$$[\bar{K}^e][T^e]\{\Delta^e\} = [T^e]\{F^e\} \tag{4.6.5}$$

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Premultiplying both sides of the above equation with $[T^e]^{\mathrm{T}}$ and noting that $[T^e]^{-1} = [T^e]^{\mathrm{T}}$, we obtain

$$[T^e]^{\mathrm{T}}[\bar{K}^e][T^e]\{\Delta^e\} = \{F^e\} \quad \text{or} \quad [K^e]\{\Delta^e\} = \{F^e\} \tag{4.6.6}$$

where

$$[K^e] = [T^e]^{\mathrm{T}}[\bar{K}^e][T^e], \qquad \{F^e\} = [T^e]^{\mathrm{T}}\{\bar{F}^e\} \tag{4.6.7}$$

Carrying out the indicated matrix multiplications, we obtain

$$[K^e] = \frac{E_e A_e}{h_e}\begin{bmatrix} \cos^2\theta_e & \frac{1}{2}\sin 2\theta_e & -\cos^2\theta_e & -\frac{1}{2}\sin 2\theta_e \\ \frac{1}{2}\sin 2\theta_e & \sin^2\theta_e & -\frac{1}{2}\sin 2\theta_e & -\sin^2\theta_e \\ -\cos^2\theta_e & -\frac{1}{2}\sin 2\theta_e & \cos^2\theta_e & \frac{1}{2}\sin 2\theta_e \\ -\frac{1}{2}\sin 2\theta_e & -\sin^2\theta_e & \frac{1}{2}\sin 2\theta_e & \sin^2\theta_e \end{bmatrix} \tag{4.6.8}$$

$$\{F^e\} = \begin{Bmatrix} F_1^e \\ F_2^e \\ F_3^e \\ F_4^e \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1^e\cos\theta_e \\ \bar{F}_1^e\sin\theta_e \\ \bar{F}_2^e\cos\theta_e \\ \bar{F}_2^e\sin\theta_e \end{Bmatrix} + \begin{Bmatrix} \bar{f}_1^e\cos\theta_e \\ \bar{f}_1^e\sin\theta_e \\ \bar{f}_2^e\cos\theta_e \\ \bar{f}_2^e\sin\theta_e \end{Bmatrix} \tag{4.6.9}$$

where $\bar{f}_i^e$ are computed using Eq. (3.2.31*b*) [also see Eq. (3.2.34)]

$$\bar{f}_i^e = \int_0^{h_e} f(\bar{x})\psi_i^e(\bar{x})\,d\bar{x} \tag{4.6.10}$$

Equations (4.6.8) and (4.6.9) provide the means to compute the element stiffness matrix $[K^e]$ and force vector $\{F^e\}$, respectively, both referred to the global coordinate system, of a bar element oriented at an angle θ_e. The assembly of elements with their stiffness matrix and force vector in the global coordinates follows the same ideas as discussed before except that we must note that each node now has two displacement degrees of freedom. These ideas are illustrated in the following example.

Example 4.6.1

Consider a three-member truss shown in Figure 4.6.3(*a*). All members of the truss have identical areas of cross section A and modulus E. The hinged supports at joints A, B, and C allow free rotation of the members about the z-axis (taken positive out of the plane of the paper). We wish to determine the horizontal and vertical displacements at the joint C and forces in each member of the structure.

Finite Element Mesh

We use three finite elements to model the structure. Any further subdivision of the members does not add to the accuracy because the finite element solutions for displacements and forces at the nodes are exact for all truss problems. The global node numbers and element numbers are shown in Figure 4.6.3(*b*). There are two displacement degrees of freedom, horizontal and

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Figure 4.6.3 Geometry and finite element representation of a plane truss. (*a*) Geometry and applied loads. (*b*) Element numbering and reaction forces.

vertical displacements, at each node of the element. The element stiffness matrix in the local coordinate system is given by Eq. (4.6.1) while the stiffness matrix and force vector (with $\bar{f}_i^e = 0$) in the global coordinate system is given by Eqs. (4.6.8) and (4.6.9), respectively. The element data and connectivity is given in the following table.

Element number	Global nodes	Geometric properties	Material property	Orientation
1	1 2	$A,\ h_1 = L$	E	$\theta_1 = 0°$
2	2 3	$A,\ h_2 = L$	E	$\theta_2 = 90°$
3	1 3	$A,\ h_3 = \sqrt{2}L$	E	$\theta_3 = 45°$

Element Matrices

The element stiffness matrices are given by $[1/(2\sqrt{2}) = 0.3536]$

$$[K^1] = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [K^2] = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[K^3] = \frac{EA}{L}\begin{bmatrix} 0.3536 & 0.3536 & -0.3536 & -0.3536 \\ 0.3536 & 0.3536 & -0.3536 & -0.3536 \\ -0.3536 & -0.3536 & 0.3536 & 0.3536 \\ -0.3536 & -0.3536 & 0.3536 & 0.3536 \end{bmatrix} \tag{4.6.11}$$

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Assembly of Elements

The assembled stiffness matrix is given by (subscripts of K_{ij} refer to the local degrees of freedom)

$$
[K] = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} K_{11}^1 + K_{11}^3 & K_{12}^1 + K_{12}^3 & K_{13}^1 & K_{14}^1 & K_{13}^3 & K_{14}^3 \\ & K_{22}^1 + K_{22}^3 & K_{23}^1 & K_{24}^1 & K_{23}^3 & K_{24}^3 \\ & & K_{33}^1 + K_{11}^2 & K_{34}^1 + K_{12}^2 & K_{13}^2 & K_{14}^2 \\ & \text{symm.} & & K_{44}^1 + K_{22}^2 & K_{23}^2 & K_{24}^2 \\ & & & & K_{33}^2 + K_{33}^3 & K_{34}^2 + K_{34}^3 \\ & & & & & K_{44}^2 + K_{44}^3 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \tag{4.6.12}
$$

The element stiffness matrices of (4.6.11) can be used in Eq. (4.6.12) to obtain the assembled global stiffness matrix

$$
[K] = \frac{EA}{L} \begin{bmatrix} 1.3536 & 0.3536 & -1.0 & 0.0 & | -0.3536 & -0.3536 \\ & 0.3536 & 0.0 & 0.0 & | -0.3536 & -0.3536 \\ & & 1.0 & 0.0 & | \ 0.0 & 0.0 \\ & & & 1.0 & | \ 0.0 & -1.0 \\ \text{symm.} & --- & --- & --- & | --- & --- \\ & & & & | \ 0.3536 & 0.3536 \\ & & & & | & 1.3536 \end{bmatrix} \tag{4.6.13}
$$

The displacement continuity conditions are

$$
\begin{aligned} u_1^1 &= u_1^3 = U_1, \quad v_1^1 = v_1^3 = V_1 \\ u_2^1 &= u_1^2 = U_2, \quad v_2^1 = v_1^2 = V_2 \\ u_2^2 &= u_2^3 = U_3, \quad v_2^2 = v_2^3 = V_3 \end{aligned} \tag{4.6.14}
$$

and the force equilibrium conditions are

$$
\begin{aligned} F_1^1 + F_1^3 &= F_{1x}, \quad F_2^1 + F_2^3 = F_{1y} \\ F_3^1 + F_1^2 &= F_{2x}, \quad F_4^1 + F_2^2 = F_{2y} \\ F_3^2 + F_3^3 &= F_{3x}, \quad F_4^2 + F_4^3 = F_{3y} \end{aligned} \tag{4.6.15}
$$

which can be used to write the global displacement and force vectors as

$$
\{\Delta\} = \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}, \quad \{F\} = \begin{Bmatrix} F_1^1 + F_1^3 \\ F_2^1 + F_2^3 \\ F_3^1 + F_1^2 \\ F_4^1 + F_2^2 \\ F_3^2 + F_3^3 \\ F_4^2 + F_4^3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} \tag{4.6.16}
$$

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where (U_I, V_I) and (F_{Ix}, F_{Iy}) denote the x and y components of the displacement and external forces, respectively, at the global node I.

Boundary Conditions

The specified displacement and force degrees of freedom are

$$U_1 = V_1 = U_2 = V_2 = 0, \quad F_{3x} = P, \quad F_{3y} = -2P \tag{4.6.17}$$

The first two boundary conditions correspond to the horizontal and vertical displacements at node 1, the next two correspond to the horizontal and vertical displacements at node 2, and the last correspond to the force boundary conditions at node 3. The unknowns are: the displacements (U_3, V_3) of node 3 and forces (F_{1x}, F_{1y}) at node 1 and forces (F_{2x}, F_{2y}) at node 2.

Condensed Equations

The condensed equations for the unknown displacements (U_3, V_3) are obtained from the last two equations of the system, as indicated by the dotted lines in (4.6.13)

$$\frac{EA}{L}\begin{bmatrix} 0.3536 & 0.3536 \\ 0.3536 & 1.3536 \end{bmatrix}\begin{Bmatrix} U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \end{Bmatrix} \tag{4.6.18}$$

and the condensed equations for the unknown reactions are (from the first four equations of the system)

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} -0.3536 & -0.3536 \\ -0.3536 & -0.3536 \\ 0.0 & 0.0 \\ 0.0 & -1.0 \end{bmatrix}\begin{Bmatrix} U_3 \\ V_3 \end{Bmatrix} \tag{4.6.19}$$

Solution of the Finite Element Equations

Solving Eqs. (4.6.18) for U_5 and U_6, we obtain

$$U_3 = (3 + 2\sqrt{2})\frac{PL}{EA} = 5.828\,\frac{PL}{EA}, \quad V_3 = -\frac{3PL}{EA} \tag{4.6.20}$$

and the reaction forces are computed using Eq. (4.6.19)

$$F_{1x} = -P, \quad F_{1y} = -P, \quad F_{2x} = 0.0, \quad F_{2y} = 3P \tag{4.6.21}$$

Postcomputation

The stress in each member can be computed from the relation

$$\sigma^e = -\frac{\bar{P}_1^e}{A_e} = \frac{\bar{P}_2^e}{A_e} \tag{4.6.22}$$

where $\bar{P}_1^e$ and $\bar{P}_2^e$ can be determined from the element equations

$$\begin{Bmatrix} \bar{P}_1^e \\ \bar{P}_2^e \end{Bmatrix} = \frac{A_e E_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \end{Bmatrix} \tag{4.6.23}$$

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and the element displacements $(\bar{u}_1^e, \bar{u}_2^e)$ are determined from the transformation relation (4.6.3*a*)

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{Bmatrix} \tag{4.6.24}$$

By definition [see Eqs. (4.6.14)], we have

$$u_1^1 = v_1^1 = u_2^1 = v_2^1 = 0, \quad u_1^2 = v_1^2 = u_1^3 = v_1^3 = 0$$

$$u_2^2 = u_2^3 = U_3 = (3 + 2\sqrt{2})\frac{PL}{EA}, \; v_2^2 = v_2^3 = V_3 = -\frac{3PL}{EA} \tag{4.6.25}$$

Hence, Eqs. (4.6.22)–(4.6.25) give

$$\sigma^e = \frac{E_e}{h_e}(\bar{u}_2^e - \bar{u}_1^e) \tag{4.6.26}$$

$$\begin{aligned} \bar{u}_1^1 &= u_1^1 \cos\theta_1 + v_1^1 \sin\theta_1 = 0 \\ \bar{u}_2^1 &= u_2^1 \cos\theta_1 + v_2^1 \sin\theta_1 = 0 \\ \bar{u}_1^2 &= u_1^2 \cos\theta_2 + v_1^2 \sin\theta_2 = 0 \\ \bar{u}_2^2 &= U_3 \cos\theta_2 + V_3 \sin\theta_2 = V_3 = -\frac{3PL}{EA} \\ \bar{u}_1^3 &= u_1^3 \cos\theta_3 + v_1^3 \sin\theta_3 = 0 \\ \bar{u}_2^3 &= U_3 \cos\theta_3 + V_3 \sin\theta_3 = \frac{1}{\sqrt{2}}(U_3 + V_3) = \frac{2PL}{EA} \end{aligned} \tag{4.6.27}$$

Thus, the member forces are

$$\bar{P}_1^1 = -\bar{P}_2^1 = 0, \quad \bar{P}_1^2 = -\bar{P}_2^2 = 3P, \quad \bar{P}_1^3 = -\bar{P}_2^3 = -\sqrt{2}P \tag{4.6.28}$$

The axial stresses in the members are

$$\sigma^{(1)} = 0, \quad \sigma^{(2)} = -\frac{3P}{A}, \quad \sigma^{(3)} = \sqrt{2}\,\frac{P}{A} \tag{4.6.29}$$

Interpretation and Verification of the Results

An examination of the structure and the sense of loads applied indicate that the displacements (U_3, V_3) are qualitatively correct (positive U_3 and negative V_3). Also, the geometry of the structure indicates that it has relatively more stiffness in the vertical direction (member 2 takes much of the load directly) compared to the horizontal direction, which explains the relatively large displacement in the horizontal direction.

The forces in Eq. (4.6.21) can be verified by applying the method of sections to the free-body-diagram in Figure 4.6.3(*b*). Sections AA, BB, and CC (see Figure 4.6.4) yield

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Figure 4.6.4 Method of sections to determine the member forces.

the relations ($H = 0$)

$$\frac{1}{\sqrt{2}}R + F_y^1 = 0, \quad F_x^1 + H + \frac{1}{\sqrt{2}}R = 0 \tag{4.6.30a}$$

$$F_x^2 - H = 0, \quad F_y^2 + Q = 0 \tag{4.6.30b}$$

$$-\frac{1}{\sqrt{2}}R + P = 0, \quad \frac{1}{\sqrt{2}}R + Q + 2P = 0 \tag{4.6.30c}$$

which yield

$$Q = -3P, \quad R = \sqrt{2}P, \quad F_y^2 = 3P, \quad F_y^1 = -P, \quad F_x^1 = -P, \quad H = F_x^2 = 0 \tag{4.6.30d}$$

Note that the member forces computed using the method of sections agree with those computed in the finite element method ($\bar{P}_1^1 = H$, $\bar{P}_1^2 = -Q$, and $\bar{P}_1^3 = -R$). It should be noted that this exercise is only for checking purposes, and the finite element method can be used to determine the member forces Q, R, and H, as discussed in the postcomputation. This completes the example.

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Figure 4.6.5 Transformation of specified boundary conditions from a local coordinate system to the global coordinate system (for an inclined support).

4.6.4 Constraint Equations: Penalty Approach

It is not uncommon in structural systems to find that the displacement components at a point are related. For example, when the plane of a roller support is at an angle to the global coordinate system (see Fig. 4.6.5), the boundary conditions on displacements and forces at the roller are known only in terms of the normal (to the support) component of the displacement and the tangential component of the force

$$u_n^e = 0, \qquad Q_t^e = Q_0 \tag{4.6.31}$$

where u_n^e is the normal component of displacement and Q_t^e is the tangential component of the force at node 1 of the element Ω^e; Q_0 is any specified tangential force. These conditions, when expressed in terms of the global components of displacements and forces by means of the transformation of the form (4.6.3*b*) and (4.6.4), become

$$u_n^e = -u_1^e \sin\alpha + u_2^e \cos\alpha = 0 \tag{4.6.32a}$$

$$Q_t^e = Q_1^e \cos\alpha + Q_2^e \sin\alpha = Q_0 \tag{4.6.32b}$$

where (u_1^e, u_2^e) and (Q_1^e, Q_2^e) are the x and y components of the displacements and forces, respectively, at the support. Equations (4.6.32*a*) can be viewed as constraint equations among the global displacements, which have a companion relation among the associated forces, namely Eq. (4.6.32*b*). Here, we present the penalty function method through which constraint equations of the type in (4.6.32*a*) and (4.6.32*b*) can be included in the finite element equations.

The penalty function method allows us to reformulate a problem with constraints as one without constraints. The basic idea of the method can be described by considering an algebraic constrained problem:

minimize the function $f(x, y)$ *subject to the constraint* $G(x, y) = 0$

In the Lagrange multiplier method the problem is reformulated as one of determining the stationary (or critical) points of the modified function $F_L(x, y)$,

$$F_L(x, y) = f(x, y) + \lambda G(x, y) \tag{4.6.33}$$

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subject to no constraints. Here λ denotes the Lagrange multiplier. The solution to the problem is obtained by setting partial derivatives of F_L with respect to x, y and λ to zero:

$$\frac{\partial F_L}{\partial x}=0, \quad \frac{\partial F_L}{\partial y}=0, \quad \frac{\partial F_L}{\partial \lambda}=0 \tag{4.6.34}$$

which gives three equations in the three unknowns (x, y, λ).

In the penalty function method, the problem is reformulated as one of finding the minimum of the modified function F_P,

$$F_P(x, y)=f(x, y)+\frac{\gamma}{2}[G(x, y)]^2 \tag{4.6.35}$$

where γ is a preassigned weight parameter, called the *penalty parameter*. The factor $\frac{1}{2}$ in Eq. (4.6.35) is used for convenience: When F_p is differentiated with respect to its arguments, the factor will be cancelled by the power on $G(x, y)$. The solution to the modified problem is given by the following two equations:

$$\frac{\partial F_P}{\partial x}=0, \quad \frac{\partial F_P}{\partial y}=0 \tag{4.6.36a}$$

The solution of Eqs. (4.6.36*a*) will be a function of the penalty parameter, (x_γ, y_γ). The larger the value of γ, the more exactly the constraint is satisfied (in a least-squares sense), and (x_γ, y_γ) approaches the actual solution (x, y) as $\gamma \to \infty$. An approximation to the Lagrange multiplier is computed from the equation,

$$\lambda_\gamma=\gamma G(x_\gamma, y_\gamma) \tag{4.6.36b}$$

We consider a specific example to illustrate the ideas presented above.

Example 4.6.2

Minimize the quadratic function

$$f(x, y)=4x^2-3y^2+2xy+6x-3y+5$$

subject to the constraint

$$G(x, y)\equiv 2x+3y=0$$

Geometrically, we seek the inflection point of the surface $f(x, y)$ that is on the line $2x+3y=0$. We solve the problem using the Lagrange multiplier method and the penalty function method.

Lagrange Multiplier Method. The modified functional is

$$F_L(x, y)=f(x, y)+\lambda(2x+3y)$$

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where λ is the Lagrange multiplier to be determined. We have

$$\frac{\partial F_L}{\partial x} = 8x + 2y + 6 + 2\lambda = 0$$

$$\frac{\partial F_L}{\partial y} = -6y + 2x - 3 + 3\lambda = 0$$

$$\frac{\partial F_L}{\partial \lambda} = 2x + 3y = 0$$

Solving the three algebraic equations, we obtain

$$x = -3,\ y = 2,\ \lambda = 7$$

Penalty Function Method. The modified functional is

$$F_P(x, y) = f(x, y) + \frac{\gamma}{2}(2x + 3y)^2$$

and we have

$$\frac{\partial F_P}{\partial x} = 8x + 2y + 6 + 2\gamma(2x + 3y) = 0$$

$$\frac{\partial F_P}{\partial y} = -6y + 2x - 3 + 3\gamma(2x + 3y) = 0$$

The solution of these equations is

$$x_\gamma = \frac{15 - 36\gamma}{-26 + 12\gamma},\ y_\gamma = \frac{18 + 24\gamma}{-26 + 12\gamma}$$

The Lagrange multiplier is given by

$$\lambda_\gamma = \gamma G(x_\gamma, y_\gamma) = \frac{84\gamma}{-26 + 12\gamma}$$

Clearly, in the limit $\gamma \to \infty$, the penalty function solution approaches the exact solution:

$$\lim_{\gamma\to\infty} x_\gamma = -3,\ \lim_{\gamma\to\infty} y_\gamma = 2,\ \lim_{\gamma\to\infty} \lambda_\gamma = 7$$

An approximate solution to the problem can be obtained, within a desired accuracy, by selecting a finite value of the penalty parameter (see Table 4.6.1). This completes the example.

Table 4.6.1 Convergence of penalty function solution with increasing penalty parameter.

γ	x_γ	y_γ	λ_γ	$E_\gamma = 2x_\gamma + 3y_\gamma$
0	−0.5769	−0.6923	0.0000	−3.2308
1	1.5000	−3.0000	−6.0000	−6.0000
10	−3.6702	2.7447	8.9362	0.8936
100	−3.0537	2.0596	7.1550	0.0716
1,000	−3.0053	2.0058	7.0152	0.0068
10,000	−3.0005	2.0006	7.0015	0.0008
∞	−3.0000	2.0000	7.0000	0.0000

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Now we turn our attention to constraint equations of the form used with bar elements

$$\beta_m u_m + \beta_n u_n = \beta_{mn} \tag{4.6.37}$$

where β_m, β_n and β_{mn} are known constants, and u_m and u_n are the mth and nth displacement degrees of freedom in the mesh, respectively. The functional that must be minimized subject to the constraint in (4.6.37) in this case is the total potential energy of the system (see Section 4.5.2)

$$\Pi = \frac{1}{2}\int_\Omega AE\mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u}\,dx - \int_\Omega \mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} f\,dx - \mathbf{u}^{\mathrm{T}}\mathbf{Q} \tag{4.6.38}$$

The penalty functional is given by

$$\begin{aligned}\Pi_p = \frac{1}{2}\int_\Omega AE\mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u}\,dx - \int_\Omega \mathbf{u}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} f\,dx - \mathbf{u}^{\mathrm{T}}\mathbf{Q}\\ + \frac{\gamma}{2}\left(\beta_m u_m + \beta_n u_n - \beta_{mn}\right)^2\end{aligned} \tag{4.6.39}$$

The functional Π_p attains a minimum only when $\beta_m u_m + \beta_n u_n - \beta_{mn}$ is very small, i.e., approximately satisfying the constraint (4.6.37). Setting $\delta\Pi_p = 0$ yields

$$\left(\mathbf{K} + \mathbf{K}_p\right)\mathbf{u} = \mathbf{f} + \mathbf{Q} + \mathbf{Q}_p \tag{4.6.40a}$$

where [see Eq. (4.5.4a)]

$$\mathbf{K} = \int_\Omega AE\mathbf{B}^{\mathrm{T}}\mathbf{B}\,dx, \quad \mathbf{K}_p = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \gamma\beta_m^2 & \cdots & \gamma\beta_m\beta_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \gamma\beta_m\beta_n & \cdots & \gamma\beta_n^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\mathbf{f} = \int_\Omega \mathbf{B}^{\mathrm{T}} f\,dx, \quad \mathbf{Q}_p = \begin{Bmatrix} \cdots \\ \gamma\beta_{mn}\beta_m \\ \cdots \\ \gamma\beta_{mn}\beta_n \\ \cdots \end{Bmatrix} \tag{4.6.40b}$$

Thus, a modification of the stiffness and force coefficients associated with the constrained degrees of freedom will provide the desired solution to the constrained problem. As illustrated in Example 4.6.2, the value of the penalty parameter γ dictates the degree to which the constraint condition (4.6.37) is met. An analysis of the discrete problem shows that the following value of γ may be used:

$$\gamma = \max|K_{ij}| \times 10^4, \quad 1 \le i, j \le N \tag{4.6.41}$$

where N is the order of the global coefficient matrix. The reaction forces associated with the constrained displacement degrees of freedom are obtained from

$$F_{mp} = -\gamma\beta_m\left(\beta_m u_m + \beta_n u_n - \beta_{mn}\right), \quad F_{np} = -\gamma\beta_n\left(\beta_m u_m + \beta_n u_n - \beta_{mn}\right) \tag{4.6.42}$$

Because of the large magnitudes of the penalty terms, it is necessary to carry out computations in double precision (hand calculations do not give accurate results).

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Example 4.6.3

Consider the structure shown in Fig. 4.6.6(*a*). The rigid bar ABE is supported by deformable bars AC and BD. Bar AC is made of aluminum ($E_a = 70$ GPa) and has cross-sectional area of $A_a = 500$ mm^2; bar BD is made of steel ($E_s = 200$ GPa) and has a cross-sectional area of $A_s = 600$ mm^2. The rigid bar carries loads of $F_1 = 10$ kN and $F_2 = 30$ kN at points O and E, respectively. We wish to determine the displacements of points A, B, and E, and the stresses in the aluminum and steel bars.

One may note that this is a statically determinate problem, i.e., the forces at points A and B can be readily determined from statics. Using the free-body diagram of the rigid bar ABE [see Fig. 4.6.6(*b*)], we obtain

$$F_{AC} + F_{BD} = F_1 + F_2, \quad 0.3F_{AC} + 0.5F_{BD} - 0.9F_2 = 0$$

Figure 4.6.6 (*a*) Given structure. (*b*) Free-body diagram. (*c*) Finite element mesh. (*d*) Modified structure. (*e*) Free-body diagram of the modified structure.

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which yield the values

$$F_{AC} = 2.5F_1 - 2F_2 = -35\text{kN}, \quad F_{BD} = -1.5F_1 + 3F_2 = 75\text{kN}$$

If we use two linear finite elements to represent the bars AC and BD, the assembled matrix of the structure is given by [see Fig. 4.6.6(*c*)]

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & 0 & -k_2 \\ -k_1 & 0 & k_1 & 0 \\ 0 & -k_2 & 0 & k_2 \end{bmatrix} \end{array} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 \\ Q_2^2 \end{Bmatrix} \tag{4.6.43}$$

where

$$k_1 = \frac{E_a A_a}{h_1} = 116.6667 \times 10^6 \text{ N/m}, \quad k_2 = \frac{E_s A_s}{h_2} = 300 \times 10^6 \text{ N/m}$$

The boundary conditions of the problem are

$$U_3 = U_4 = 0; \quad Q_1^1 = -F_{AC} = 35 \text{ kN}, \quad Q_1^2 = -F_{BD} = -75 \text{ kN}$$

Hence, the condensed equations are given by

$$10^6 \begin{bmatrix} 116.6667 & 0 \\ 0 & 300 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 10^3 \begin{Bmatrix} 35 \\ -75 \end{Bmatrix}$$

whose solution is

$$U_1 = 0.30 \times 10^{-3} \text{ m} = 0.30 \text{ mm}, \quad U_2 = -0.25 \times 10^{-3} \text{ m} = -0.25 \text{ mm}$$

Using the principle of similar triangles, we can determine the displacement of point E. We have

$$\frac{U_E - U_1}{0.6} = \frac{U_2 - U_1}{0.2} \quad \rightarrow \quad U_E = 3U_2 - 2U_1 = -1.35 \text{ mm}$$

Thus, end A of the bar AC moves up by 0.3 mm, end B of BD moves down by 0.25 mm, and point E moves down by 1.35 mm. The stresses in bars AC and BD are

$$\sigma_{AC} = \frac{F_{AC}}{A_a} = -\frac{35}{500 \times 10^{-6}} = -70 \text{ MPa}, \quad \sigma_{BD} = \frac{F_{BD}}{A_s} = \frac{75}{600 \times 10^{-6}} = 125 \text{ MPa}$$

Next, consider the case in which point O is pin connected to a fixed, immovable part, as shown in Fig. 4.6.6(*d*). Then the problem becomes a statically indeterminate one. Of course, the finite element method can still be used to solve the problem. The assembled equations (4.6.43) are still valid for this case. However, forces Q_1^1 and Q_1^2 are not known (because we cannot solve for F_{AC} and F_{BD}). In addition, points A and B are constrained to move as the rigid member ABE is rotated about point O. This geometric constraint is equivalent to the following conditions among the displacements U_1, U_2, and U_5:

$$\frac{U_1}{0.3} = \frac{U_5}{0.9} \rightarrow 3U_1 - U_5 = 0, \quad \frac{U_2}{0.5} = \frac{U_5}{0.9} \rightarrow 1.8U_2 - U_5 = 0 \tag{4.6.44}$$

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These constraints bring in an additional degree of freedom, namely U_5, into the equations. Hence, the assembled equations before including the constraint conditions are

$$10^6 \begin{bmatrix} 116.67 & 0 & -116.67 & 0 & 0 \\ 0 & 300 & 0 & -300 & 0 \\ -116.67 & 0 & 116.67 & 0 & 0 \\ 0 & -300 & 0 & 300 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 \\ Q_2^2 \\ F_2 \end{Bmatrix} \tag{4.6.45}$$

The last row and column of the above equation are added to facilitate the addition of penalty terms. Without the addition of the penalty contributions the last equation is nonsensical.

Using the procedure developed in this section, we can include the constraints, $3U_1 - U_5 = 0$ and $1.8U_2 - U_5 = 0$ into assembled equations (4.6.43). The value of the penalty parameter is selected to be $\gamma = (300 \times 10^6)10^4$. The stiffness additions due to the two constraints are ($\beta_1 = 3$, $\beta_2 = 1.8$, $\beta_5 = -1$, and $\beta_{15} = \beta_{25} = 0$)

$$\begin{array}{cc} & \begin{array}{cc} 1 & 5 \end{array} \\ \begin{array}{c} 1 \\ 5 \end{array} & \begin{bmatrix} (3)^2\gamma & 3(-1)\gamma \\ (-1)3\gamma & (-1)^2\gamma \end{bmatrix} = 10^{10} \begin{bmatrix} 2700.00 & -900.00 \\ -900.00 & 300.00 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} 2 & 5 \end{array} \\ \begin{array}{c} 2 \\ 5 \end{array} & \begin{bmatrix} (1.8)^2\gamma & (-1)1.8\gamma \\ (-1)1.8\gamma & (-1)^2\gamma \end{bmatrix} = 10^{10} \begin{bmatrix} 972.00 & -540.00 \\ -540.00 & 300.00 \end{bmatrix} \end{array} \tag{4.6.46}$$

The force additions are zero on account of $\beta_{15} = \beta_{25} = 0$. Hence, the modified finite element equations become

$$10^6 \begin{bmatrix} 27{,}000{,}117 & 0 & -117 & 0 & -9{,}000{,}000 \\ 0 & 9{,}720{,}300 & 0 & -300 & -5{,}400{,}000 \\ -116.67 & 0 & 117 & 0 & 0 \\ 0 & -300 & 0 & 300 & 0 \\ -9{,}000{,}000 & -5{,}400{,}000 & 0 & 0 & 6{,}000{,}000 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 \\ Q_2^2 \\ F_2 \end{Bmatrix} \tag{4.6.47}$$

The condensed equations are

$$10^6 \begin{bmatrix} 27{,}000{,}117 & 0 & -9{,}000{,}000 \\ 0 & 9{,}720{,}300 & -5{,}400{,}000 \\ -9{,}000{,}000 & -5{,}400{,}000 & 6{,}000{,}000 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 30 \times 10^3 \end{Bmatrix} \tag{4.6.48}$$

whose solution (using an equation solver) is (deflections are in the direction of the force)

$$U_1 = 0.09474 \text{ (mm)}, \quad U_2 = 0.15789 \text{ (mm)}, \quad U_5 = 0.28422 \text{ (mm)} \tag{4.6.49a}$$

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The forces in the bars AC and BD can be calculated using Eq. (4.6.45)

$$\begin{Bmatrix} Q_1^1 \\ Q_1^2 \end{Bmatrix} = 10^6 \begin{bmatrix} 116.6667 & 0 \\ 0 & 300 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 10^3 \begin{Bmatrix} 11.053 \\ 47.367 \end{Bmatrix} \text{N}$$

The stresses are $\sigma_{AC} = 22.11$ MPa and $\sigma_{BD} = 79$ MPa. Alternatively, from Eq. (4.6.42) we have (one must include significant number of decimal points in the computation)

$$\begin{aligned} (Q_1^1)_p &= -900 \times 10^7 \left(3 \times 0.094739 \times 10^{-3} - 0.28422 \times 10^{-3}\right) = 11.053 \text{ kN} \\ (Q_1^2)_p &= -540 \times 10^7 \left(1.8 \times 0.15789 \times 10^{-3} - 0.28422 \times 10^{-3}\right) = 47.368 \text{ kN} \end{aligned} \tag{4.6.49b}$$

Next, we consider a plane truss with an inclined support. The penalty approach is used to include the constraint condition among the displacement components at the support.

Example 4.6.4

Consider the truss shown in Fig. 4.6.7(*a*). We wish to determine the unknown displacements of nodes 2 and 3 and the reactions associated with these displacements.

The element stiffness matrices are

$$[K^1] = 10^9 \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.126 & 0.000 & -0.126 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.126 & 0.000 & 0.126 \end{bmatrix}$$

$E = 210$ GPa for all members

$L = 1$ m, $A_1 = A_2 = A_0 = 6 \times 10^{-4}$ m^2

$A_3 = \sqrt{2}\, A_0$ m^2, $P = 10^3$ kN

Figure 4.6.7 (*a*) Given structure. (*b*) Reaction forces.

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$$[K^2] = 10^9 \begin{bmatrix} 0.126 & 0.000 & -0.126 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ -0.126 & 0.000 & 0.126 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$

$$[K^3] = 0.63 \times 10^8 \begin{bmatrix} 1.0 & 1.0 & -1.0 & -1.0 \\ 1.0 & 1.0 & -1.0 & -1.0 \\ -1.0 & -1.0 & 1.0 & 1.0 \\ -1.0 & -1.0 & 1.0 & 1.0 \end{bmatrix}$$

The assembled equations before including the constraint conditions are

$$10^8 \begin{bmatrix} 0.63 & 0.63 & 0.00 & 0.00 & -0.63 & -0.63 \\ 0.63 & 1.89 & 0.00 & -1.26 & -0.63 & -0.63 \\ 0.00 & 0.00 & 1.26 & 0.00 & -1.26 & 0.00 \\ 0.00 & -1.26 & 0.00 & 1.26 & 0.00 & 0.00 \\ 0.00 & -0.63 & -1.26 & 0.00 & 1.89 & 0.63 \\ -0.63 & -0.63 & 0.00 & 0.00 & 0.63 & 0.63 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 + Q_1^3 \\ Q_2^1 + Q_2^3 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^2 + Q_4^3 \end{Bmatrix} \tag{4.6.50}$$

The constraint condition at node 3 is

$$u_n \equiv -u \sin\alpha + v \cos\alpha = 0 \rightarrow -0.7071u + 0.7071v = 0$$

Comparing this constraint equation to the general constraint equation (4.6.37), we find that $\beta_1 = -0.7071$, $\beta_2 = 0.7071$, and $\beta_{12} = 0$.

The value of the penalty parameter is selected to be $\gamma = (1.89 \times 10^8)10^4$. The stiffness additions due to the constraint are

$$\begin{array}{c c} & \begin{array}{cc} 5 & 6 \end{array} \\ \begin{array}{c} 5 \\ 6 \end{array} & \gamma \begin{bmatrix} (-0.7071)^2 & -(0.7071)^2 \\ -(0.7071)^2 & (0.7071)^2 \end{bmatrix} \end{array} = 1.89 \times 10^{12} \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

The assembled equations after including the constraint conditions are (the numbers shown are truncated but more accurate numbers are used in actual computation in a computer)

$$10^8 \begin{bmatrix} 0.63 & 0.63 & 0.00 & 0.00 & -0.63 & -0.63 \\ 0.63 & 1.89 & 0.00 & -1.26 & -0.63 & -0.63 \\ 0.00 & 0.00 & 1.26 & 0.00 & -1.26 & 0.00 \\ 0.00 & -1.26 & 0.00 & 1.26 & 0.00 & 0.00 \\ 0.00 & -0.63 & -1.26 & 0.00 & 6301.8 & 6299.2 \\ -0.63 & -0.63 & 0.00 & 0.00 & 6299.2 & 6300.5 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 + Q_1^3 \\ Q_2^1 + Q_2^3 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^2 + Q_4^3 \end{Bmatrix}$$

Imposing the boundary and force equilibrium conditions $U_1 = U_2 = U_4 = 0$ and $Q_3^1 + Q_1^2 = P$, we obtain the condensed equations

$$10^8 \begin{bmatrix} 1.26 & -1.26 & 0.00 \\ -1.26 & 6301.8 & 6299.2 \\ 0.00 & 6299.2 & 6300.5 \end{bmatrix} \begin{Bmatrix} U_3 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix}$$

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The solution to these equations (as computed in double precision in a computer) is

$$U_3 = 11.905 \times 10^{-3}\ \text{m}, \quad U_5 = 3.9688 \times 10^{-3}\ \text{m}, \quad U_6 = 3.9680 \times 10^{-3}\ \text{m}$$

and the reactions at node 3, as computed using (4.6.50) are

$$F_{3x} = -500\ \text{kN}, \quad F_{3y} = 500\ \text{kN}$$

The reactions of the whole structure are shown in Fig. 4.6.7(*b*).

4.6.5 Constraint Equations: A Direct Approach

Here, we present an exact method by which constraint equations of the type in (4.6.32*a*) and (4.6.32*b*) can be included in the assembled equations for the unknowns. The method involves expressing the global displacement degrees of freedom at the node with a constraint in terms of the local displacement degrees of freedom so that the boundary conditions can be readily imposed.

Recall from Eqs. (4.6.2) and (4.6.3*a*) that the displacements $(\hat{u}, \hat{v})$, referred to the local coordinate system $(\hat{x}, \hat{y})$ at a point, are related to the displacements (u, v), referred to the global coordinates system (x, y), by

$$\begin{Bmatrix} \hat{u} \\ \hat{v} \end{Bmatrix}_c = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}_c \rightarrow \quad \hat{\mathbf{u}}_c = \mathbf{A}\mathbf{u}_c \tag{4.6.51a}$$

and the inverse relation is given by

$$\mathbf{u}_c = \mathbf{A}^{\mathrm{T}}\hat{\mathbf{u}}_c, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \tag{4.6.51b}$$

where the subscript "c" refers to the constrained degrees of freedom. Since we wish to express the global displacements at a given node in terms of the local displacements at a specific node, we construct the transformation of the whole (global) system as

$$[T] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [A] & [0] \\ [0] & [0] & [I] \end{bmatrix} \tag{4.6.52}$$

Thus, all displacement degrees of freedom that are not constrained are unaffected and only global displacements that are constrained are transformed to the local displacements. Further, note that the transformation matrix $[A]$ is placed in $[T]$ in such a way that only the constrained degrees of freedom are transformed. We have

$$\begin{Bmatrix} \Delta^1 \\ \mathbf{u}_c \\ \Delta^2 \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \Delta^1 \\ \hat{\mathbf{u}} \\ \Delta^2 \end{Bmatrix} \quad \text{or } \boldsymbol{\Delta} = \mathbf{T}^{\mathrm{T}}\hat{\boldsymbol{\Delta}} \tag{4.6.53}$$

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where $\boldsymbol{\Delta}^1$ and $\boldsymbol{\Delta}^2$ denote vectors of the global displacement components ahead and behind (in terms of numbering) the constrained displacement degrees of freedom $\mathbf{u}_c$ in the mesh. We also note that the transformed displacement vector $\hat{\boldsymbol{\Delta}}$ contains the global displacement vectors $\boldsymbol{\Delta}^1$ and $\boldsymbol{\Delta}^2$ and the local (constrained) displacement vector $\hat{\mathbf{u}}_c$.

The remaining steps of the procedure are the same as that described in Sections 4.6.2 and 4.6.3. Thus, we obtain

$$\hat{\mathbf{K}}\hat{\boldsymbol{\Delta}} = \hat{\mathbf{F}} \tag{4.6.54}$$

where the transformed global stiffness matrix $\hat{\mathbf{K}}$ and global force vector $\hat{\mathbf{F}}$ are known in terms of the assembled global stiffness matrix $\mathbf{K}$ and force vector $\mathbf{F}$ as [see Eq. (4.6.7)]

$$\hat{\mathbf{K}} = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}, \quad \hat{\mathbf{F}} = \mathbf{T}^{\mathrm{T}}\mathbf{F} \tag{4.6.55}$$

Since the constrained displacements are a part of the global system of equations, we may impose the boundary conditions on them directly (such as $\hat{u}_n = 0$ at an inclined roller support, where n denotes the coordinate normal to the roller).

In summary, we may introduce a transformation of the displacements that facilitates the imposition of boundary conditions or inclusion of constraints on the displacements. Once the transformation $\mathbf{T}$ is identified, we may use Eq. (4.6.7) or (4.6.55) to obatin the modified equations that have the desired effect. We revisit the problems of Examples 4.6.3 and 4.6.4 to illustrate the ideas described here.

Example 4.6.5

As an example to illustrate the direct approach, consider the structure shown in Fig. 4.6.6(d). The assembled equations are

$$10^3 \begin{bmatrix} 116.67 & 0 & -116.67 & 0 \\ 0 & 300.00 & 0 & -300.00 \\ -116.67 & 0 & 116.67 & 0 \\ 0 & -300.00 & 0 & 300.00 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_1^2 \\ Q_2^1 \\ Q_2^2 \end{Bmatrix} \tag{4.6.56}$$

The geometric constraint between the displacements U_1 and U_2 is

$$\frac{U_1}{0.3} = \frac{U_2}{0.5} \rightarrow U_1 = 0.6U_2 \tag{4.6.57}$$

In this case, we can introduce a transformation $\mathbf{T}$ between (U_1, U_2, U_3, U_4) and (U_2, U_3, U_4) (i.e., eliminate U_1 by means of the constraint equation $U_1 = 0.6U_2$) as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} 0.6 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} \quad \text{or } \boldsymbol{\Delta} = \mathbf{T}\hat{\boldsymbol{\Delta}}$$

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Hence, the transformed equations are

$$\hat{\mathbf{K}}\hat{\mathbf{\Delta}} = \hat{\mathbf{F}}, \quad \mathbf{\Delta} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}, \quad \hat{\mathbf{\Delta}} = \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix}$$

or, in explicit form,

$$10^3 \begin{bmatrix} 342.0 & -70.0 & -300.0 \\ -70.0 & 116.67 & 0 \\ -300.0 & 0 & 300.0 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0.6Q_1^1 + Q_1^2 \\ Q_2^1 \\ Q_2^2 \end{Bmatrix}$$

From the free-body diagram of the bar OABE, we find that $0.6F_{AC} + F_{BD} = 1.8F_2$; therefore, we have $0.6Q_1^1 + Q_1^2 = 1.8F_2 = 54$ kN and the condensed equation for the unknown U_2 is ($U_3 = U_4 = 0.0$)

$$342U_2 = 54, \quad U_2 = 0.15789 \text{ (mm)}, \quad U_1 = 0.6U_2 = 0.09474 \text{ (mm)}$$

The forces in the bars AC and BD are calculated using Eq. (4.6.56) ($U_3 = U_4 = 0.0$)

$$\begin{Bmatrix} Q_1^1 \\ Q_1^2 \end{Bmatrix} = 10^3 \begin{bmatrix} 116.6667 & 0 \\ 0 & 300 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = 10^3 \begin{Bmatrix} 11.053 \\ 47.368 \end{Bmatrix}$$

The stresses are $\sigma_{AC} = 22.11$ MPa and $\sigma_{BD} = 79$ MPa.

Example 4.6.6

Here we consider the truss in Figure 4.6.7(a). The assembled global equations are given by Eq. (4.6.50). The transformation between the global degrees of freedom ($U_1 = u_1$, $U_2 = v_1$, $U_3 = u_2$, $U_4 = v_2$, $U_5 = u_3$, $U_6 = v_3$) and ($U_1 = u_1$, $U_2 = v_1$, $U_3 = u_2$, $U_4 = v_2$, $\hat{u}_3 = u_t$, $\hat{v}_3 = u_n$) is given by [i.e., Eq. (4.6.53) for the problem at hand takes the form]

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \hat{u}_3 \\ \hat{v}_3 \end{Bmatrix} \tag{4.6.58a}$$

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where $\alpha = 45°$. Using Eq. (4.6.55), we obtain the transformed equations

$$10^8 \begin{bmatrix} 0.630 & 0.630 & 0.000 & 0.00 & -0.891 & 0.000 \\ 0.630 & 1.890 & 0.000 & -1.26 & -0.891 & 0.000 \\ 0.000 & 0.000 & 1.260 & 0.00 & -0.891 & 0.891 \\ 0.000 & -1.260 & 0.000 & 1.26 & 0.000 & 0.000 \\ -0.891 & -0.891 & -0.891 & 0.00 & 1.890 & -0.630 \\ 0.000 & 0.000 & 0.891 & 0.00 & -0.630 & 0.630 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \hat{u}_3 \\ \hat{v}_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 + Q_1^3 \\ Q_2^1 + Q_2^3 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ \hat{F}_{3t} \\ \hat{F}_{3n} \end{Bmatrix} \tag{4.6.58b}$$

where $\hat{F}_{3t}$ and $\hat{F}_{3n}$ are the reaction forces at node 3 in the tangential and normal directions, respectively.

Now applying the boundary conditions

$$U_1 = U_2 = U_4 = \hat{v}_3 = 0, \quad Q_3^1 + Q_1^2 = P = 10^6, \quad \hat{F}_{3t} = 0$$

we obtain the following condensed equations:

$$10^8 \begin{bmatrix} 1.26000 & -0.89095 \\ -0.89095 & 1.88990 \end{bmatrix} \begin{Bmatrix} U_3 \\ \hat{u}_3 \end{Bmatrix} = \begin{Bmatrix} 10^6 \\ 0 \end{Bmatrix} \tag{4.6.59}$$

The x component of displacement at node 2 and the tangential displacement $\hat{u}_3$ of node 3 are

$$U_3 = 11.905 \times 10^{-3} \text{ m}, \quad \hat{u}_3 = 5.6122 \times 10^{-3} \text{ m} \tag{4.6.60a}$$

The unknown reactions can be calculated from Eq. (4.6.58b) as

$$F_{1x} = -500 \text{ kN}, \quad F_{1y} = -500 \text{ kN}, \quad F_{2y} = 0 \text{ kN}, \quad \hat{F}_{3n} = 707.1 \text{ kN} \tag{4.6.60b}$$

These results are the same as those obtained by the penalty method in Example 4.6.4.

4.7 SUMMARY

In this chapter, finite element models of discrete systems have been developed and applications of finite element models to the solution of problems of heat transfer, fluid mechanics, and solid mechanics have been presented. To aid the reader, a brief review of the basic terminology and governing equations of each of the three fields has also been given. Analysis of plane trusses and inclusion of constraint relations between primary variables by two different methods is also discussed.

It has been shown that the secondary variables of a problem can be computed using either the global algebraic equations of the finite element mesh (i.e., condensed equations for the secondary variables) or by their original definition through finite element interpolation.

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The former method gives more accurate results, which will satisfy the equilibrium at inter-element nodes, whereas the latter gives less accurate results that are discontinuous at the nodes. The secondary variables computed using the linear elements are elementwise constant, while they are elementwise linear for the Lagrange quadratic elements. The magnitude of discontinuity can be reduced by refining the mesh (h or p refinement). The discontinuity of the secondary variables at nodes is due to the fact that the secondary variables are not made continuous across the elements.

In closing this chapter, the reader is reminded that the finite element method is a powerful tool for engineering analysis. The power of the method lies in transforming the traditional variational methods (e.g., Ritz, Galerkin, least-squares, and other weighted-residual methods) into a powerful computer-based technique by developing suitable approximation functions for complex problems. Without a good understanding of the method as well as the engineering background behind each problem, one is ill-equipped to analyze the problem.

PROBLEMS

Many of the following problems are designed for hand calculation while some are intended specifically for computer calculations using the program FEM1D (see Chapter 7 for details on how to use the program). The problem set should give the student deeper understanding of what is involved in the setting up of the finite element equations, imposition of boundary conditions, and identifying the condensed equations for the unknown primary and secondary variables of a given problem. When the number of equations to be solved is greater than three, the student should opt for a computer solution of the equations. The calculations can be verified, in most cases, by solving the same problem using FEM1D.

Discrete Elements

4.1 Consider the system of linear elastic springs shown in Fig. P4.1. Assemble the element equations to obtain the force-displacement relations for the entire system. Use the boundary conditions to write the condensed equations for the unknown displacements and forces.

Figure P4.1

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4.2 Repeat Problem 4.1 for the system of linear springs shown in Fig. P4.2.

Figure P4.2

4.3 Consider the direct current electric network shown in Fig. P4.3. We wish to determine the voltages V and currents I in the network using the finite element method. Set up the algebraic equations (i.e., condensed equations) for the unknown voltages and currents.

Figure P4.3

4.4 Repeat Problem 4.3 for the direct current electric network shown in Fig. P4.4.

Figure P4.4

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4.5 Write the condensed equations for the unknown pressures and flows (use the minimum number of elements) for the hydraulic pipe network shown in Fig. P4.5. *Answer:* $P_1 = \frac{39}{14}Qa$, $P_2 = \frac{12}{7}Qa$, and $P_3 = \frac{15}{14}Qa$.

Figure P4.5

4.6 Consider the hydraulic pipe network (the flow is assumed to be laminar) shown in Fig. P4.6. Write the condensed equations for the unknown pressures and flows (use minimum number of elements.)

Figure P4.6

4.7 Determine the maximum shear stresses in the solid steel ($G_s = 12$ Msi) and aluminum ($G_a = 4$ Msi) shafts shown in Fig. P4.7.

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Figure P4.7

Figure P4.8

4.8 A steel shaft and an aluminum tube are connected to a fixed support and to a rigid disk, as shown in Fig. P4.8. If the torque applied at the end is equal to $T = 6,325$ N·m, determine the shear stresses in the steel shaft and aluminum tube. Use $G_s = 77$ GPa and $G_a = 27$ GPa.

Heat Transfer

4.9 Consider heat transfer in a plane wall of total thickness L. The left surface is maintained at temperature T_0 and the right surface is exposed to ambient temperature T_∞ with heat transfer coefficient β. Determine the temperature distribution in the wall and heat input at the left surface of the wall for the following data: $L = 0.1$ m, $k = 0.01$ W/(m · °C), $\beta = 25$ W/(m^2 · °C), $T_0 = 50$°C, and $T_\infty = 5$°C. Solve for nodal temperatures and the heat at the left wall using (*a*) two linear finite elements and (*b*) one quadratic element. *Answer:* (*a*) $U_2 = 27.59$°C, $U_3 = 5.179$°C, $Q_1^1 = 4.482$ W/m$^2 = -Q_2^2$.

4.10 An insulating wall is constructed of three homogeneous layers with conductivities k_1, k_2, and k_3 in intimate contact (see Fig. P4.10). Under steady-state conditions, the temperatures of the media in contact at the left and right surfaces of the wall are at ambient temperatures of T_∞^L and T_∞^R, respectively, and film coefficients β_L and β_R, respectively. Determine the temperatures on the left and right surfaces as well as at the interfaces. Assume that there is no internal heat generation and that the heat flow is one-dimensional ($\partial T/\partial y = 0$). *Answer:* $U_1 = 61.582$°C, $U_2 = 61.198$°C, $U_3 = 60.749$°C, $U_4 = 60.612$°C.

Figure P4.10

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4.11 Rectangular fins are used to remove heat from the surface of a body by conduction along the fins and convection from the surface of the fins into the surroundings. The fins are 100 mm long, 5 mm wide, and 1 mm thick, and made of aluminum with thermal conductivity $k = 170$ W/(m · °C). The natural convection heat transfer coefficient associated with the surrounding air is $\beta = 35$ W/(m^2 · °C) and the ambient temperature is $T_\infty = 20$°C. Assuming that the heat transfer is one dimensional along the length of the fins and that the heat transfer in each fin is independent of the others, determine the temperature distribution along the fins and the heat removed from each fin by convection. Use (*a*) four linear elements, and (*b*) two quadratic elements.

4.12 Find the heat transfer per unit area through the composite wall shown in Fig. P4.12. Assume one-dimensional heat flow.

Figure P4.12

4.13 A steel rod of diameter $D = 2$ cm, length $L = 5$ cm, and thermal conductivity $k = 50$ W/(m·°C) is exposed to ambient air at $T_\infty = 20$°C with a heat transfer coefficient $\beta = 100$ W/(m^2·°C). If the left end of the rod is maintained at temperature $T_0 = 320$°C, determine the temperatures at distances 25 mm and 50 mm from the left end, and the heat at the left end. The governing equation of the problem is

$$-\frac{d^2\theta}{dx^2} + m^2\theta = 0 \quad \text{for} \quad 0 < x < L$$

where $\theta = T - T_\infty$, T is the temperature, and $m^2 = \beta P/Ak$. The boundary conditions are

$$\theta(0) = T(0) - T_\infty = 300°\text{C}, \qquad \left(\frac{d\theta}{dx} + \frac{\beta}{k}\theta\right)\bigg|_{x=L} = 0$$

Use (*a*) two linear elements and (*b*) one quadratic element to solve the problem by the finite element method. Compare the finite element nodal temperatures against the exact values. *Answer:* (*a*) $U_1 = 300$°C, $U_2 = 211.97$°C, $U_3 = 179.24$°C, $Q_1^1 = 3{,}521.1$ W/m^2. (*b*) $U_1 = 300$°C, $U_2 = 213.07$°C, $U_3 = 180.77$°C, $Q_1^1 = 4{,}569.9$ W/m^2.

4.14 Find the temperature distribution in the tapered fin shown in Fig. P4.14. Assume that the temperature at the root of the fin is 250°F, the conductivity $k = 120$ Btu/(h·ft·°F), and the film coefficient $\beta = 15$ Btu/(h·ft^2·°F); use three linear elements. The ambient temperature at the top and bottom of the fin is $T_\infty = 75$°F. *Answer:* T_1(tip) = 166.23°F, $T_2 = 191.1$°F, $T_3 = 218.89$°F.

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Figure P4.14

4.15 Consider steady heat conduction in a wire of circular cross section with an electrical heat source. Suppose that the radius of the wire is R_0, its electrical conductivity is K_e (Ω^{-1}/cm), and it is carrying an electric current density of I (A/cm^2). During the transmission of an electric current, some of the electrical energy is converted into thermal energy. The rate of heat production per unit volume is given by $q_e = I^2/K_e$. Assume that the temperature rise in the wire is sufficiently small that the dependence of the thermal or electric conductivity on temperature can be neglected. The governing equations of the problem are

$$-\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = q_e \quad \text{for} \quad 0 \le r \le R_0, \qquad \left(rk\frac{dT}{dr}\right)\bigg|_{r=0} = 0, \qquad T(R_0) = T_0$$

Determine the distribution of temperature in the wire using (*a*) two linear elements and (*b*) one quadratic element, and compare the finite element solution at eight equal intervals with the exact solution

$$T(r) = T_0 + \frac{q_e R_0^2}{4k}\left[1 - \left(\frac{r}{R_0}\right)\right]$$

Also, determine the heat flow, $Q = -2\pi R_0 k(dT/dr)|_{R_0}$, at the surface using (i) the temperature field and (ii) the balance equations.

4.16 Consider a nuclear fuel element of spherical form, consisting of a sphere of "fissionable" material surrounded by a spherical shell of aluminum "cladding" as shown in Fig. P4.16. Nuclear fission is a source of thermal energy, which varies nonuniformly from the center of the sphere to the interface of the fuel element and the cladding. We wish to determine the temperature distribution in the nuclear fuel element and the aluminum cladding.

The governing equations for the two regions are the same, with the exception that there is no heat source term for the aluminum cladding. We have

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 k_1 \frac{dT_1}{dr}\right) = q \quad \text{for} \quad 0 \le r \le R_F$$

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 k_2 \frac{dT_2}{dr}\right) = 0 \quad \text{for} \quad R_F \le r \le R_C$$

where subscripts 1 and 2 refer to the nuclear fuel element and cladding, respectively. The heat generation in the nuclear fuel element is assumed to be of the form

$$q_1 = q_0\left[1 + c\left(\frac{r}{R_F}\right)^2\right]$$

where q_0 and c are constants depending on the nuclear material. The boundary conditions are

$$kr^2\frac{dT_1}{dr} = 0 \quad \text{at} \quad r = 0$$

$$T_1 = T_2 \quad \text{at} \quad r = R_F, \quad \text{and} \quad T_2 = T_0 \quad \text{at} \quad r = R_C$$

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Use two linear elements to determine the finite element solution for the temperature distribution, and compare the nodal temperatures with the exact solution

$$T_1 - T_0 = \frac{q_0 R_F^2}{6k_1}\left\{\left[1-\left(\frac{r}{R_F}\right)^2\right]+\frac{3}{10}c\left[1-\left(\frac{r}{R_F}\right)^4\right]\right\}+\frac{q_0 R_F^2}{3k_2}\left(1+\frac{3}{5}c\right)\left(1-\frac{R_F}{R_C}\right)$$

$$T_2 - T_0 = \frac{q_0 R_F^2}{3k_2}\left(1+\frac{3}{5}c\right)\left(\frac{R_F}{r}-\frac{R_F}{R_C}\right)$$

Figure P4.16

Fluid Mechanics

4.17 Consider the flow of a Newtonian viscous fluid on an inclined flat surface, as shown in Fig. P4.17. Examples of such flow can be found in wetted-wall towers and the application of coatings to wallpaper rolls. The momentum equation, for a fully developed steady laminar flow along the z coordinate, is given by

$$-\mu\frac{d^2w}{dx^2} = \rho g \cos\beta$$

where w is the z component of the velocity, μ is the viscosity of the fluid, ρ is the density, g is the acceleration due to gravity, and β is the angle between the inclined surface and the vertical.

The boundary conditions associated with the problem are that the shear stress is zero at $x = 0$ and the velocity is zero at $x = L$:

$$\left(\frac{dw}{dx}\right)\bigg|_{x=0} = 0, \quad w(L) = 0$$

Use (a) two linear finite elements of equal length and (b) one quadratic finite element in the domain $(0, L)$ to solve the problem and compare the two finite element solutions at four points

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Figure P4.17

$x = 0$, $\frac{1}{4}L$, $\frac{1}{2}L$, and $\frac{3}{4}L$ of the domain with the exact solution

$$w_e = \frac{\rho g L^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{L}\right)^2\right]$$

Evaluate the shear stress ($\tau_{xz} = -\mu\, dw/dx$) at the wall using (i) the velocity fields and (ii) the equilibrium equations, and compare with the exact value. *Answer:* (*a*) $U_1 = \frac{1}{2}f_0$, $U_2 = \frac{3}{8}f_0$, $f_0 = (\rho g \cos\beta)L^2/\mu$.

4.18 Consider the steady laminar flow of a viscous fluid through a long circular cylindrical tube. The governing equation is

$$-\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{dw}{dr}\right) = \frac{P_0 - P_L}{L} \equiv f_0$$

where w is the axial (i.e., z) component of velocity, μ is the viscosity, and f_0 is the gradient of pressure (which includes the combined effect of static pressure and gravitational force). The boundary conditions are

$$\left(r\frac{dw}{dr}\right)\bigg|_{r=0} = 0, \quad w(R_0) = 0$$

Using the symmetry and (*a*) two linear elements and (*b*) one quadratic element, determine the velocity field and compare with the exact solution at the nodes:

$$w_e(r) = \frac{f_0 R_0^2}{4\mu}\left[1 - \left(\frac{r}{R_0}\right)^2\right]$$

4.19 In the problem of the flow of a viscous fluid through a circular cylinder (Problem 4.18), assume that the fluid slips at the cylinder wall; i.e., instead of assuming that $w = 0$ at $r = R_0$, use the boundary condition that

$$kw = -\mu\frac{dw}{dr} \quad \text{at} \quad r = R_0$$

in which k is the "coefficient of sliding friction." Solve the problem with two linear elements.

4.20 Consider the steady laminar flow of a Newtonian fluid with constant density in a long annular region between two coaxial cylinders of radii R_i and R_0 (see Fig. P4.20). The differential

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equation for this case is given by

$$-\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{dw}{dr}\right)=\frac{P_1-P_2}{L}\equiv f_0$$

where w is the velocity along the cylinders (i.e., the z component of velocity), μ is the viscosity, L is the length of the region along the cylinders in which the flow is fully developed, and P_1 and P_2 are the pressures at $z=0$ and $z=L$, respectively (P_1 and P_2 represent the combined effect of static pressure and gravitational force).

The boundary conditions are

$$w=0 \quad \text{at} \quad r=R_0 \quad \text{and} \quad R_i$$

Solve the problem using (*a*) two linear elements and (*b*) one quadratic element, and compare the finite element solutions with the exact solution at the nodes:

$$w_e(r)=\frac{f_0R_0^2}{4\mu}\left[1-\left(\frac{r}{R_0}\right)^2+\frac{1-k^2}{\ln(1/k)}\ln\left(\frac{r}{R_0}\right)\right]$$

Figure P4.20

where $k=R_i/R_0$. Determine the shear stress $\tau_{rz}=-\mu\, dw/dr$ at the walls using (i) the velocity field and (ii) the equilibrium equations, and compare with the exact values. (Note that the steady laminar flow of a viscous fluid through a long cylinder or a circular tube can be obtained as a limiting case of $k\to 0$.)

4.21 Consider the steady laminar flow of two immiscible incompressible fluids in a region between two parallel stationary plates under the influence of a pressure gradient. The fluid rates are adjusted such that the lower half of the region is filled with fluid I (the denser and more viscous fluid) and the upper half is filled with fluid II (the less dense and less viscous fluid), as shown in Fig. P4.21. We wish to determine the velocity distributions in each region using the finite element method.

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The governing equations for the two fluids are

$$-\mu_1 \frac{d^2u_1}{dx^2} = f_0, \quad -\mu_2 \frac{d^2u_2}{dx^2} = f_0$$

where $f_0 = (P_0 - P_L)/L$ is the pressure gradient. The boundary conditions are

$$u_1(-b) = 0 \quad u_2(b) = 0, \quad u_1(0) = u_2(0)$$

Solve the problem using four linear elements, and compare the finite element solutions with the exact solution at the nodes

$$u_i = \frac{f_0 b^2}{2\mu_i}\left[\frac{2\mu_i}{\mu_1 + \mu_2} + \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\frac{y}{b} - \left(\frac{y}{b}\right)^2\right] \quad (i = 1, 2)$$

Figure P4.21

4.22 The governing equation for an unconfined aquifer with flow in the radial direction is given by the differential equation

$$-\frac{1}{r}\frac{d}{dr}\left(rk\frac{du}{dr}\right) = f$$

where k is the coefficient of permeability, f the recharge, and u the piezometric head. Pumping is considered to be a negative recharge. Consider the following problem. A well penetrates an aquifer and pumping is performed at $r = 0$ at a rate $Q = 150$ m^3/h. The permeability of the aquifer is $k = 25$ m^3/h. A constant head $u_0 = 50$ m exists at a radial distance $L = 200$ m. Determine the piezometric head at radial distances of 0, 10, 20, 40, 80, and 140 m (see Fig. P4.22). You are required to set up the finite element equations for the unknowns using a nonuniform mesh of six linear elements.

Figure P4.22

4.23 Consider a slow, laminar flow of a viscous substance (for example, glycerin solution) through a narrow channel under controlled pressure drop of 150 Pa/m. The channel is 5 m long (flow

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direction), 10 cm high, and 50 cm wide. The upper wall of the channel is maintained at 50°C while the lower wall is maintained at 25°C. The viscosity and density of the substance are temperature dependent, as given in Table P4.23. Assuming that the flow is essentially one dimensional (justified by the dimensions of the channel), determine the velocity field and mass flow rate of the fluid through the channel.

Table P4.23: Properties of the viscous substance of Problem 4.23.

y (m)	Temp. (°C)	Viscosity [kg/(m· s)]	Density (kg/m^3)
0.00	50	0.10	1233
0.02	45	0.12	1238
0.04	40	0.20	1243
0.06	35	0.28	1247
0.08	30	0.40	1250
0.10	25	0.65	1253

Solid and Structural Mechanics

4.24 The equation governing the axial deformation of an elastic bar in the presence of applied mechanical loads f and P and a temperature change T is

$$-\frac{d}{dx}\left[EA\left(\frac{du}{dx}-\alpha T\right)\right]=f \quad \text{for} \quad 0<x<L$$

where α is the thermal expansion coefficient, E the modulus of elasticity, and A the cross-sectional area. Using three linear finite elements, determine the axial displacements in a nonuniform rod of length 30 in., fixed at the left end and subjected to an axial force P = 400 lb and a temperature change of 60°F. Take $A(x)=6-\frac{1}{10}x$ in.^{2} $E=30\times10^6$ lb/in.^{2}, and $\alpha=12\times10^{-6}$/(in. ·°F).

4.25 Find the stresses and compressions in each section of the composite member shown in Fig. P4.25. Use $E_s=30\times10^6$ psi, $E_a=10^7$ psi, $E_b=15\times10^6$ psi, and the minimum number of linear elements.

Figure P4.25

4.26 Find the three-element finite element solution to the stepped-bar problem. See Fig. P4.26 for the geometry and data. *Hint:* Solve the problem to see if the end displacement exceeds the gap. If it does, resolve the problem with modified boundary condition at $x=24$ in.

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Steel, $E_s = 30 \times 10^6$ psi, Aluminum, $E_a = 10 \times 10^6$ psi

Figure P4.26

4.27 Analyze the stepped bar with its right end supported by a linear axial spring (see Fig. P4.27). The boundary condition at $x = 24$ in. is

$$EA\frac{du}{dx} + ku = 0$$

Steel, $E_s = 30 \times 10^6$ psi, Aluminum, $E_a = 10 \times 10^6$ psi

Figure P4.27

4.28 A solid circular brass cylinder $E_b = 15 \times 10^6$ psi, $d_b = 0.25$ in.) is encased in a hollow circular steel ($E_s = 30 \times 10^6$ psi, $d_s = 0.21$ in). A load of $P = 1330$ lb compresses the assembly, as shown in Fig. P4.28. Determine (a) the compression, and (b) compressive forces and stresses in the steel shell and brass cylinder. Use the minimum number of linear finite elements. Assume that the Poisson effect is negligible.

Figure P4.28

4.29 A rectangular steel bar ($E_s = 30 \times 10^6$ psi) of length 24 in. has a slot in the middle half of its length, as shown in Fig. 4.29. Determine the displacement of the ends due to the axial loads $P = 2000$ lb. Use the minimum number of linear elements.

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Figure P4.29

Figure P4.30

4.30 Repeat Problem 4.29 for the steel bar shown in Fig. P4.30.

4.31 The aluminum and steel pipes shown in Fig. P4.31 are fastened to rigid supports at ends A and B and to a rigid plate C at their junction. Determine the displacement of point C and stresses in the aluminum and steel pipes. Use the minimum number of linear finite elements.

4.32 A steel bar ABC is pin-supported at its upper end A to an immovable wall and loaded by a force F_1 at its lower end C, as shown in Fig. P4.32. A rigid horizontal beam BDE is pinned to the vertical bar at B, supported at point D, and carries a load F_2 at end E. Determine the displacements u_B and u_C at points B and C.

4.33 Repeat Problem 4.32 when point C is supported vertically by a spring ($k = 1000$ lb/in.).

Figure P4.31

Figure P4.32

4.34 Consider the steel column (a typical column in a multi-storey building structure) shown in Fig. P4.34. The loads shown are due to the loads of different floors. The modulus of elasticity is $E = 30 \times 10^6$ psi and cross-sectional area of the column is $A = 40$ in^2. Determine the vertical displacements and axial stresses in the column at various floor-column connection points.

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Figure P4.34

4.35 The bending moment (M) and transverse deflection (w) in a beam according to the Euler-Bernoulli beam theory are related by

$$-EI\frac{d^2w}{dx^2} = M(x)$$

For statically determinate beams, we can readily obtain the expression for the bending moment in terms of the applied loads. Thus, $M(x)$ is a known function of x. Determine the maximum deflection of the simply supported beam under uniform load (see Fig. P4.35) using the finite element method.

Figure P4.35

Figure P4.36

4.36 Repeat Problem 4.35 for the cantilever beam shown in Fig. P4.36.

4.37 Turbine disks are often thick near their hub and taper down to a smaller thickness at the periphery. The equation governing a variable-thickness $t = t(r)$ disk is

$$\frac{d}{dr}(rt\sigma_r) - t\sigma_\theta + t\rho\omega^2 r^2 = 0$$

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where ω^2 is the angular speed of the disk and

$$\sigma_r = c\left(\frac{du}{dr} + \nu\frac{u}{r}\right), \quad \sigma_\theta = c\left(\frac{u}{r} + \nu\frac{du}{dr}\right), \quad c = \frac{E}{1-\nu^2}$$

(*a*) Construct the weak integral form of the governing equation such that the bilinear form is symmetric and the natural boundary condition involves specifying the quantity $tr\sigma_r$.

(*b*) Develop the finite element model associated with the weak form derived in part (*a*).

4.38–4.44 For the plane truss structures shown in Figs. P4.38–P4.44, give (*a*) the transformed element matrices, (*b*) the assembled element matrices, and (*c*) the condensed matrix equations for the unknown displacements and forces.

Figure P4.38

Figure P4.39

Figure P4.40

Figure P4.41

Figure P4.42

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Figure P4.43

Figure P4.44

4.45 Determine the forces and elongations of each bar in the structure shown in Fig. P4.45. Also, determine the vertical displacements of points A and D.

Figure P4.45

4.46 Determine the forces and elongations of each bar in the structure shown in Fig. P4.45 when end A is pinned to a rigid wall (and P_1 is removed).

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Chapter 5
BEAMS AND FRAMES

5.1 INTRODUCTION

Here we consider the finite element formulation of the one-dimensional fourth-order differential equation that arises in the Euler–Bernoulli beam theory and the pair of one-dimensional second-order equations associated with the Timoshenko beam theory. The superposition of the beam and bar elements give rise to frame elements that can be used to analyze plane frame structures. The formulations of a fourth-order equation as well as coupled second-order equations (see Problems 3.1 and 3.3) involve the same steps as described in Section 3.2 for a second-order equation, but the mathematical details are somewhat different, especially in the finite element formulation of the equations.

5.2 EULER–BERNOULLI BEAM ELEMENT

5.2.1 Governing Equation

In the Euler-Bernoulli beam theory, it is assumed that plane cross sections perpendicular to the axis of the beam remain plane and perpendicular to the axis after deformation. In this theory, the transverse deflection w of the beam is governed by the fourth-order differential equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w=q(x) \quad \text{for} \quad 0<x<L \tag{5.2.1}$$

where $EI=E(x)I(x)$, $c_f=c_f(x)$, $q=q(x)$ are given functions of x (i.e., data), and w is the dependent variable; E denotes the modulus of elasticity, I the second moment of area about the y axis of the beam, q is the distributed transverse load, c_f the elastic foundation modulus (if any), and w is the transverse deflection of the beam. The sign convention used in the derivation of (5.2.1) is shown in Fig. 5.2.1. In addition to satisfying the differential equation (5.2.1), w must also satisfy appropriate boundary conditions; since the equation is of fourth order, four boundary conditions are needed to solve it. The weak formulation of the equation will provide the form of these four boundary conditions. A step-by-step procedure for the finite element analysis of (5.2.1) is presented next (see Example 2.4.2).

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Figure 5.2.1 (*a*) Typical beam with distributed load $q(x)$ and point force F_0 and moment M_0. (*b*) The shear force-bending moment-deflection relations and the sign convention.

5.2.2 Discretization of the Domain

The domain $\Omega_e = (x_a, x_b) = (x_e, x_{e+1})$ of the straight beam [see Fig. 5.2.2(*a*)] is divided into a set of N line elements, each element having at least the two end nodes [see Fig. 5.2.2(*b*)]. Although the element is geometrically the same as that used for bars, the number and form of the primary and secondary unknowns at each node are dictated by the variational formulation of the differential equation (5.2.1). In most practical problems, the discretization of a given structure into a minimum number of elements is often dictated by the geometry, loading, and material properties.

5.2.3 Derivation of Element Equations

We isolate a typical element $\Omega_e = (x_e, x_{e+1})$ [see Fig. 5.2.2(*b*)] and construct the weak form of (5.2.1) over the element. The variational formulation provides the primary and secondary variables of the problem. Then suitable approximations for the primary variables are selected, interpolation functions are developed, and the element equations are derived.

Weak Form. The weak forms of problems in solid mechanics can be developed either from the principle of virtual work (i.e., the principle of virtual displacements or virtual forces) or from the governing differential equations. Here we start with a given differential equation and use the three-step procedure to obtain the weak form. Following the three-step procedure

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Figure 5.2.2 (*a*) Geometry and loads on a beam. (*b*) Finite element discretization. (*c*) Generalized displacements and generalized forces on a typical element.

illustrated in Chapter 2 (see Example 2.4.2) and revisited in Section 3.2.3, we write

$$
\begin{aligned}
0 &= \int_{x_e}^{x_{e+1}} v\left[\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + c_f w - q\right] dx \\
&= \int_{x_e}^{x_{e+1}} \left[-\frac{dv}{dx}\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) + c_f vw - vq\right] dx + \left[v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]_{x_e}^{x_{e+1}} \\
&= \int_{x_e}^{x_{e+1}} \left(EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f vw - vq\right) dx + \left[v\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) - \frac{dv}{dx}EI\frac{d^2w}{dx^2}\right]_{x_e}^{x_{e+1}}
\end{aligned}
\tag{5.2.2}
$$

where $v(x)$ is a weight function that is twice differentiable with respect to x. Note that, in the present case, the first term of the equation is integrated twice by parts, to yield two differentiations to the weight function v while retaining two derivatives of the dependent variable, w; the result is that the differentiation is distributed equally between the weight function v and the dependent variable w. Because of the two integration by parts, there appear two boundary expressions (see Example 2.4.2), which are to be evaluated at the two boundary points

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Figure 5.2.3 Deformation of a beam element.

$x = x_a = x_e$ and $x = x_b = x_{e+1}$. Examination of the boundary terms indicates that the essential boundary conditions involve the specification of the deflection w and slope dw/dx, and the natural boundary conditions involve the specification of the bending moment $EI\,d^2w/dx^2$ and shear force $(d/dx)(EI\,d^2w/dx^2)$ at the endpoints of the element. Thus, there are two essential boundary conditions and two natural boundary conditions; therefore, we must identify w and dw/dx as the primary variables at each node (so that the essential boundary conditions are included in the interpolation). The natural boundary conditions always remain in the weak form and end up on the right-hand side, $\{F^e\}$, of the matrix equation.

We introduce the following notation for the secondary variables that is consistent with the sign convention in Fig. 5.2.1(b) ($\theta = -dw/dx$; see Fig. 5.2.3):

$$
\begin{aligned}
Q_1^e &\equiv \left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]\bigg|_{x_e} = -V(x_e)\\
Q_2^e &\equiv \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x_e} = -M(x_e)\\
Q_3^e &\equiv -\left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]\bigg|_{x_{e+1}} = V(x_{e+1})\\
Q_4^e &\equiv -\left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x_{e+1}} = M(x_{e+1})
\end{aligned}
\tag{5.2.3}
$$

where Q_1^e and Q_3^e denote the shear forces, and Q_2^e and Q_4^e denote the bending moments [see Fig. 5.2.2c]. The set $\{Q_1^e, Q_2^e, Q_3^e, Q_4^e\}$ is often referred to as the *generalized forces*. The corresponding displacements and rotations are called the *generalized displacements*.

With the notation in (5.2.3), the weak form (5.2.2) can be expressed as

$$
\begin{aligned}
0 &= \int_{x_e}^{x_{e+1}} \left(EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f vw - vq\right) dx\\
&\quad - v(x_e)Q_1^e - \left(-\frac{dv}{dx}\right)\bigg|_{x_e} Q_2^e - v(x_{e+1})Q_3^e - \left(-\frac{dv}{dx}\right)\bigg|_{x_{e+1}} Q_4^e\\
&\equiv B(v, w) - l(v)
\end{aligned}
\tag{5.2.4}
$$

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We can identify the bilinear and linear forms of the problem as

$$B(v,w) = \int_{x_e}^{x_{e+1}} \left(EI \frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f vw \right) dx$$

$$l(v) = \int_{x_e}^{x_{e+1}} vq\, dx + v(x_e)Q_1^e + \left(-\frac{dv}{dx}\right)\bigg|_{x_e} Q_2^e \tag{5.2.5}$$

$$+ v(x_{e+1})Q_3^e + \left(-\frac{dv}{dx}\right)\bigg|_{x_{e+1}} Q_4^e$$

Equation (5.2.4) is a statement of the principle of virtual displacements (where v denotes virtual displacement δw) for the Euler–Bernoulli beam theory. The quadratic functional, known as the *total potential energy* of the isolated beam element, is given by [see Eq. (2.4.31)]

$$\Pi_e(w) = \int_{x_e}^{x_{e+1}} \left[\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 + \frac{1}{2}c_f w^2 - wq \right] dx - w(x_e)Q_1^e - w(x_{e+1})Q_3^e$$

$$- \left(-\frac{dw}{dx}\right)\bigg|_{x_e} Q_2^e - \left(-\frac{dw}{dx}\right)\bigg|_{x_{e+1}} Q_4^e \tag{5.2.6}$$

The first term in the square brackets represents the elastic strain energy due to bending, the second is the strain energy stored in the elastic foundation, and the third is the work done by the distributed load; the remaining terms account for the work done by the generalized forces Q_i^e in moving through the respective generalized displacements of the element. Conversely, we may go from the total potential energy functional (5.2.6) to the weak form (5.2.4) by using the principle of minimum potential energy, $\delta\Pi^e = 0$.

Interpolation Functions. The variational form (5.2.4) requires that the interpolation functions of an element be continuous with nonzero derivatives up to order two. The approximation $w_h^e(x)$ of $w(x)$ over a finite element should be such that it is twice differentiable and satisfies the interpolation properties, i.e., satisfies the following essential "boundary conditions" of the element [see Fig. 5.2.2(*c*)]:

$$w_h^e(x_e) = w_1^e, \quad w_h^e(x_{e+1}) = w_2^e, \quad \theta_h^e(x_e) = \theta_1^e, \quad \theta_h^e(x_{e+1}) = \theta_2^e \tag{5.2.7}$$

In satisfying the essential boundary conditions (5.2.7), the approximation automatically satisfies the continuity conditions. Hence, we pay attention to the satisfaction of (5.2.7), which forms the basis for the derivation of the interpolation functions of the Euler–Bernoulli beam element.

Since there are a total of four conditions in an element (two per node), a four-parameter polynomial must be selected for w_h^e:

$$w(x) \approx w_h^e(x) = c_1^e + c_2^e x + c_3^e x^2 + c_4^e x^3 \tag{5.2.8}$$

Note that the continuity conditions (i.e., the existence of a nonzero second derivative of w_h^e in the element) are automatically met. The next step involves expressing c_i^e in terms of the primary nodal variables (i.e., generalized displacements)

$$\Delta_1^e \equiv w_h^e(x_e), \quad \Delta_2^e \equiv -\frac{dw_h^e}{dx}\bigg|_{x=x_e}, \quad \Delta_3^e \equiv w_h^e(x_{e+1}), \quad \Delta_4^e \equiv -\frac{dw_h^e}{dx}\bigg|_{x=x_{e+1}}$$

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such that the conditions (5.2.7) are satisfied:

$$\begin{aligned}
\Delta_1^e &= w_h^e(x_e) &&= c_1^e + c_2^e x_e + c_3^e x_e^2 + c_4^e x_e^3 \\
\Delta_2^e &= -\left.\frac{dw_h^e}{dx}\right|_{x=x_e} &&= -c_2^e - 2c_3^e x_e - 3c_4^e x_e \\
\Delta_3^e &= w_h^e(x_{e+1}) &&= c_1^e + c_2^e x_{e+1} + c_3^e x_{e+1}^2 + c_4^e x_{e+1}^3 \\
\Delta_4^e &= -\left.\frac{dw_h^e}{dx}\right|_{x=x_{e+1}} &&= -c_2^e - 2c_3^e x_{e+1} - 3c_4^e x_{e+1}
\end{aligned}$$

or

$$\begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{bmatrix} 1 & x_e & x_e^2 & x_e^3 \\ 0 & -1 & -2x_e & -3x_e^2 \\ 1 & x_{e+1} & x_{e+1}^2 & x_{e+1}^3 \\ 0 & -1 & -2x_{e+1} & -3x_{e+1}^2 \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \\ c_3^e \\ c_4^e \end{Bmatrix} \tag{5.2.9}$$

Inverting this matrix equation to express c_i^e in terms of Δ_1^e, Δ_2^e, Δ_3^e, and Δ_4^e, and substituting the result into (5.2.8), we obtain

$$w_h^e(x) = \Delta_1^e \phi_1^e + \Delta_2^e \phi_2^e + \Delta_3^e \phi_3^e + \Delta_4^e \phi_4^e = \sum_{j=1}^{4} \Delta_j^e \phi_j^e \tag{5.2.10}$$

where (with $x_{e+1} = x_e + h_e$)

$$\begin{aligned}
\phi_1^e &= 1 - 3\left(\frac{x - x_e}{h_e}\right)^2 + 2\left(\frac{x - x_e}{h_e}\right)^3 \\
\phi_2^e &= -(x - x_e)\left(1 - \frac{x - x_e}{h_e}\right)^2 \\
\phi_3^e &= 3\left(\frac{x - x_e}{h_e}\right)^2 - 2\left(\frac{x - x_e}{h_e}\right)^3 \\
\phi_4^e &= -(x - x_e)\left[\left(\frac{x - x_e}{h_e}\right)^2 - \frac{x - x_e}{h_e}\right]
\end{aligned} \tag{5.2.11}$$

Note that the cubic interpolation functions in (5.2.11) are derived by interpolating w as well as its derivative dw/dx at the nodes. Such polynomials are known as the *Hermite family of interpolation functions*, and ϕ_i^e in (5.2.11) are called the *Hermite cubic interpolation* (or *cubic spline*) *functions*. Plots of the Hermite cubic interpolation functions are shown in Fig. 5.2.4.

Recall that the Lagrange cubic interpolation functions are derived to interpolate a function, but not its derivatives, at the nodes. Hence, a Lagrange cubic element will have four nodes, with the dependent variable, not its derivative, as the nodal degree of freedom. Since the slope (or derivative) of the dependent variable is also required by the weak form to be continuous at the nodes for the Euler–Bernoulli beam theory, the Lagrange cubic

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Figure 5.2.4 Hermite cubic interpolations functions used in the Euler–Bernoulli beam element.

interpolation of w, although it meets the continuity requirement for w, is *not admissible* in the finite element approximation of the Euler–Bernoulli beam theory.

The interpolation functions ϕ_i^e can be expressed in terms of the local coordinate $\bar{x} = x - x_e$:

$$\begin{aligned} \phi_1^e &= 1 - 3\left(\frac{\bar{x}}{h_e}\right)^2 + 2\left(\frac{\bar{x}}{h_e}\right)^3, \quad \phi_2^e = -\bar{x}\left(1 - \frac{\bar{x}}{h_e}\right)^2 \\ \phi_3^e &= 3\left(\frac{\bar{x}}{h_e}\right)^2 - 2\left(\frac{\bar{x}}{h_e}\right)^3, \quad \phi_4^e = -\bar{x}\left[\left(\frac{\bar{x}}{h_e}\right)^2 - \frac{\bar{x}}{h_e}\right] \end{aligned} \tag{5.2.12}$$

The first, second, and third derivatives of ϕ_i^e with respect to $\bar{x}$ are

$$\frac{d\phi_1^e}{d\bar{x}} = -\frac{6}{h_e}\frac{\bar{x}}{h_e}\left(1 - \frac{\bar{x}}{h_e}\right), \quad \frac{d\phi_2^e}{d\bar{x}} = -\left[1 + 3\left(\frac{\bar{x}}{h_e}\right)^2 - 4\frac{\bar{x}}{h_e}\right] \tag{5.2.13a}$$

$$\frac{d\phi_3^e}{d\bar{x}} = -\frac{d\phi_1^e}{d\bar{x}} = \frac{6}{h_e}\frac{\bar{x}}{h_e}\left(1 - \frac{\bar{x}}{h_e}\right), \quad \frac{d\phi_4^e}{d\bar{x}} = -\frac{\bar{x}}{h_e}\left(3\frac{\bar{x}}{h_e} - 2\right)$$

$$\frac{d^2\phi_1^e}{d\bar{x}^2} = -\frac{6}{h_e^2}\left(1 - 2\frac{\bar{x}}{h_e}\right), \quad \frac{d^2\phi_2^e}{d\bar{x}^2} = -\frac{2}{h_e}\left(3\frac{\bar{x}}{h_e} - 2\right) \tag{5.2.13b}$$

$$\frac{d^2\phi_3^e}{d\bar{x}^2} = -\frac{d^2\phi_1^e}{d\bar{x}^2} = \frac{6}{h_e^2}\left(1 - 2\frac{\bar{x}}{h_e}\right), \quad \frac{d^2\phi_4^e}{d\bar{x}^2} = -\frac{2}{h_e}\left(3\frac{\bar{x}}{h_e} - 1\right)$$

$$\frac{d^3\phi_1^e}{d\bar{x}^3} = \frac{12}{h_e^3}, \quad \frac{d^3\phi_2^e}{d\bar{x}^3} = -\frac{6}{h_e^2}$$

$$\frac{d^3\phi_3^e}{d\bar{x}^3} = -\frac{12}{h_e^3}, \quad \frac{d^3\phi_4^e}{d\bar{x}^3} = -\frac{6}{h_e^2} \tag{5.2.13c}$$

Plots of $d\phi_i^e/dx$ are shown in Fig. 5.2.5.

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Figure 5.2.5 Plots of the first derivatives, $d\phi_i^e/d\bar{x}$, of the Hermite cubic interpolations functions.

The Hermite cubic interpolation functions (5.2.11) or (5.2.12) satisfy the following interpolations properties (see Figs. 5.2.4 and 5.2.5):

$$\phi_1^e(x_e) = 1, \qquad \phi_i^e(x_e) = 0 \quad (i \neq 1)$$

$$\phi_3^e(x_{e+1}) = 1, \qquad \phi_i^e(x_{e+1}) = 0 \quad (i \neq 3)$$

$$\left(-\frac{d\phi_2^e}{dx}\right)\bigg|_{x_e} = 1, \quad \left(-\frac{d\phi_i^e}{dx}\right)\bigg|_{x_e} = 0 \quad (i \neq 2) \tag{5.2.14a}$$

$$\left(-\frac{d\phi_4^e}{dx}\right)\bigg|_{x_{e+1}} = 1, \left(-\frac{d\phi_i^e}{dx}\right)\bigg|_{x_{e+1}} = 0 \quad (i \neq 4)$$

These can be stated in compact form as $(i, j = 1, 2)$

$$\phi_{2i-1}^e(\bar{x}_j) = \delta_{ij}, \quad \phi_{2i}^e(\bar{x}_j) = 0, \quad \sum_{i=1}^{2} \phi_{2i-1}^e = 1$$

$$\left(\frac{d\phi_{2i-1}^e}{dx}\right)\bigg|_{\bar{x}_j} = 0, \quad \left(-\frac{d\phi_{2i}^e}{dx}\right)\bigg|_{\bar{x}_j} = \delta_{ij} \tag{5.2.14b}$$

where $\bar{x}_1 = 0$ and $\bar{x}_2 = h_e$ are the local coordinates of nodes 1 and 2 of the element $\Omega^e = (x_e, x_{e+1})$. The finite element solution

$$w_h^e(x) = \Delta_1^e\phi_1^e(x) + \Delta_2^e\phi_2^e(x) + \Delta_3^e\phi_3^e(x) + \Delta_4^e\phi_4^e(x)$$

is a linear combination of four terms, which is shown in Fig. 5.2.6 along with the actual function.

It should be noted that the order of the interpolation functions derived above is the minimum required for the variational formulation (5.2.4). If a higher-order (i.e., higher than cubic) approximation of w is desired, we must either identify additional primary

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Figure 5.2.6 Finite element solution over an element.

unknowns at each of the two nodes or add additional nodes with the two degrees of freedom $(w, -dw/dx)$. For example, if we add d^2w/dx^2 as the primary unknown at each of the two nodes or add a third node with $(w, -dw/dx)$ at each node, there will be a total of six conditions, and a fifth-order polynomial is required to interpolate the end conditions (see Problems 5.4 and 5.5). However, interelement continuity of d^2w/dx^2 (curvature) is not required by the weak form (5.2.4).

Finite Element Model. The finite element model of the Euler–Bernoulli beam is obtained by substituting the finite element interpolation (5.2.10) for w and the ϕ_j^e for the weight function v into the weak form (5.2.4). Since there are four nodal variables Δ_i^e, four different choices are used for v: $v = \phi_1^e$, ϕ_2^e, ϕ_3^e, and ϕ_4^e, allowing us to obtain a set of four algebraic equations. The ith algebraic equation of the finite element model is (for $v = \phi_i^e$)

$$0 = \sum_{j=1}^{4}\left[\int_{x_e}^{x_{e+1}}\left(EI\frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2} + c_f\,\phi_i^e\phi_j^e\right)dx\right]u_j^e - \int_{x_e}^{x_{e+1}}\phi_i^e q\,dx - Q_i^e \tag{5.2.15a}$$

or

$$\sum_{j=1}^{4} K_{ij}^e\Delta_j^e - F_i^e = 0 \text{ or } [K^e]\{\Delta^e\} = \{F^e\} \tag{5.2.15b}$$

where

$$K_{ij}^e = \int_{x_e}^{x_{e+1}}\left(EI\frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2} + c_f\,\phi_i^e\phi_j^e\right)dx \tag{5.2.16a}$$

$$F_i^e = \int_{x_e}^{x_{e+1}}\phi_i^e q\,dx + Q_i^e \tag{5.2.16b}$$

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Note that the coefficients K_{ij}^e are symmetric: $K_{ij}^e = K_{ji}^e$. In matrix notation, (5.2.15*b*) can be written as

$$\begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e & K_{14}^e \\ K_{21}^e & K_{22}^e & K_{23}^e & K_{24}^e \\ K_{31}^e & K_{32}^e & K_{33}^e & K_{34}^e \\ K_{41}^e & K_{42}^e & K_{43}^e & K_{44}^e \end{bmatrix} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix} \tag{5.2.17}$$

This represents the finite element model of (5.2.1). For the case in which EI and q are constant over an element, the element stiffness matrix $[K^e]$ and force vector $\{F^e\}$ have the following specific forms [see Fig. 5.2.2(*c*) for the element displacement and force degrees of freedom]

$$[K^e] = \frac{2E_eI_e}{h_e^3} \begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix} + \frac{c_f^e h_e}{420} \begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & 22h_e \\ 13h & -3h_e^2 & 22h_e & 4h_e^2 \end{bmatrix}$$

$$\{F^e\} = \frac{q_e h_e}{12} \begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} \tag{5.2.18}$$

It can be verified that the generalized force vector in (5.2.18) represents the "statically equivalent" forces and moments at nodes 1 and 2 due to the uniformly distributed load of intensity q_e over the element (see Fig. 5.2.7). For any given function $q(x)$, (5.2.16*b*) provides a straightforward way of computing the components of the generalized force vector $\{q^e\}$

$$q_i^e = \int_{x_e}^{x_{e+1}} \phi_i^e q \, dx \tag{5.2.19}$$

Figure 5.2.7 Generalized nodal forces due to uniformly distributed load.

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When a transverse point force F_0^e is applied at a point x_0 inside the element, it is distributed to the element nodes by the relation [see Remark 5 in Chapter 3; Eq. (3.3.4)]:

$$q_i^e = \int_{x_e}^{x_{e+1}} \phi_i^e(x) F_0^e \delta(x - x_0)\, dx = F_0^e \phi_i^e(x_0), \quad x_e \le x_0 \le x_{e+1} \tag{5.2.20}$$

which contains both transverse forces (q_1^e and q_3^e) and bending moments (q_2^e and q_4^e).

5.2.4 Assembly of Element Equations

The assembly procedure for beam elements is the same as that used for bar elements except that we must take into account the two degrees of freedom at each node. Recall that the assembly of elements is based on: (a) interelement continuity of the primary variables (deflection and slope) and (b) interelement equilibrium of the secondary variable (shear force and bending moment) at the nodes common to elements. To demonstrate the assembly procedure, we select a two-element model (see Fig. 5.2.8). There are three global nodes and a total of six global generalized displacements and six generalized forces in the problem. The continuity of the primary variables implies the following relation between the element degrees of freedom Δ_i^e and the global degrees of freedom U_i (see Fig. 5.2.8):

$$\begin{aligned} &\Delta_1^1 = U_1, \quad \Delta_2^1 = U_2, \quad \Delta_3^1 = \Delta_1^2 = U_3 \\ &\Delta_4^1 = \Delta_2^2 = U_4, \quad \Delta_3^2 = U_5, \quad \Delta_4^2 = U_6 \end{aligned} \tag{5.2.21}$$

Figure 5.2.8 Assembly of two Euler–Bernoulli beam finite elements. (*a*) Continuity of generalized displacements. (*b*) Balance of the generalized forces.

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In general, the equilibrium of the generalized forces at a node between two connecting elements Ω_e and Ω_f requires that

$$\begin{aligned} Q_3^e + Q_1^f &= \text{applied external point force} \\ Q_4^e + Q_2^f &= \text{applied external bending moment} \end{aligned} \tag{5.2.22}$$

If no external applied forces are given, the sum should be equated to zero. In equating the sums to the applied generalized forces (i.e., force or moment), the sign convention for the element force degrees of freedom [see Fig. 5.2.2(*c*)] should be followed. Forces are taken as positive when they act in the direction of positive z-axis, and moments are taken as positive when they follow the right-hand screw rule (i.e., when the thumb is along the positive y-axis, the four fingers show the direction of the moment). With respect to the coordinate system used in Figs. 5.2.1 and 5.2.2, *forces acting downward are positive and counterclockwise moments are positive*.

To impose the equilibrium of forces in (5.2.22), it is necessary to add the third and fourth equations (corresponding to the second node) of element Ω^e to the first and second equations (corresponding to the first node) of element Ω^f. Consequently, the global stiffness parameters K_{33}, K_{34}, K_{43}, and K_{44} associated with global node 2 are the superposition of the element stiffnesses:

$$K_{33} = K_{33}^1 + K_{11}^2, \quad K_{34} = K_{34}^1 + K_{12}^2, \quad K_{43} = K_{43}^1 + K_{21}^2, \quad K_{44} = K_{44}^1 + K_{22}^2 \tag{5.2.23}$$

In general, the assembled stiffness matrix and force vector for beam elements connected in series have the forms given in Eqs. (5.2.24*a*) and (5.2.24*b*):

$$[K] = \begin{array}{c} \begin{array}{cccc} \overbrace{\qquad\qquad}^{\text{Global node 1}} & \overbrace{\qquad\qquad\qquad\qquad}^{\text{Global node 2}} & \overbrace{\qquad\qquad}^{\text{Global node 3}} & \end{array} \\ \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & K_{14}^1 & & \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & K_{24}^1 & & \\ K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & K_{34}^1 + K_{12}^2 & K_{13}^2 & K_{14}^2 \\ K_{41}^1 & K_{42}^1 & K_{43}^1 + K_{21}^2 & K_{44}^1 + K_{22}^2 & K_{23}^2 & K_{24}^2 \\ & & K_{31}^2 & K_{32}^2 & K_{33}^2 & K_{34}^2 \\ & & K_{41}^2 & K_{42}^2 & K_{43}^2 & K_{44}^2 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} 1 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} 2 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} 3 \end{array} \tag{5.2.24a}$$

$$\{F\} = \begin{Bmatrix} q_1^1 \\ q_2^1 \\ q_3^1 + q_1^2 \\ q_4^1 + q_2^2 \\ q_3^2 \\ q_4^2 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^2 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix} \tag{5.2.24b}$$

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5.2.5 Imposition of Boundary Conditions

At this step of the analysis, we must impose the particular boundary conditions of the problem being analyzed. The type of essential (also know as *geometric*) boundary conditions for a specific beam problem depend on the nature of the geometric support. Table 5.2.1 contains a list of commonly used geometric supports for beams. The natural (also called *force*) boundary conditions involve the specification of generalized forces when the corresponding primary variables are not constrained. One must bear in mind that one and only one element of each of the following pairs must be specified at every node of the finite element mesh:

$$\left(w, \frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right]\right), \quad \left(\frac{dw}{dx}, EI\frac{d^2w}{dx^2}\right) \tag{5.2.25}$$

At an interior node, we impose the continuity of generalized displacements and balance of generalized forces as discussed in Eqs. (5.2.21) and (5.2.22).

Table 5.2.1 Types of commonly used support conditions for beams and frames.

Type of support	Displacement boundary conditions	Force boundary conditions
Free	None	All, as specified
Pinned	$u=0$ $w=0$	Moment is specified
Roller (vertical)	$u=0$	Transverse force and moment are specified
Roller (horizontal)	$w=0$	Horizontal force and bending moment are specified
Fixed (or clamped)	$u=0$ $w=0$ $dw/dx=0$	None specified
Elastically restrained	$EI(d^2w/dx^2)+\mu\theta=M_0$, M_0 specified $EI(d^3w/dx^3)+kw=Q_0$, Q_0 specified	

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There are two alternative ways to include the effect of a linear elastic spring (extensional as well as torsional). (1) Include the spring through the boundary condition for the appropriate degree of freedom (see Table 5.2.1). (2) Include the spring as another finite element, whose element equations are given by Eq. (4.2.2). In the former case, after assembly of the element equations, the secondary variable in the direction of the spring action is replaced by the negative of the spring constant times the associated primary variable. Let Q^v, and Q^θ denote the secondary variables associated with the transverse and rotational degrees of freedom at a node. Then, we have, respectively

$$Q^v + kw = 0 \text{ or } Q^v = -kw \quad \text{for vertical spring with spring constant } k$$

$$Q^\theta + \mu\theta = 0 \text{ or } Q^\theta = -\mu\theta \quad \text{for torsional spring with spring constant } \mu$$

Note that Q^v is the shear force and Q^θ the bending moment. In the second case, the spring element may be assembled along with beam elements by noting that the axial displacement of the spring is the same as the transverse displacement of the beam.

For example, consider the case of a beam clamped at the left end and supported by a spring at the right end, as shown in Fig. 5.2.9(a). Using one-element model of the beam, we obtain [set $c_f = 0$ in Eq. (5.2.18)]

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

Figure 5.2.9 (a) A spring supported cantilever beam. (b) Spring action. (c) Finite element mesh of beam and spring elements.

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The obvious boundary conditions are $U_1 = U_2 = Q_4 = 0$. The effect of the spring is that [see Fig. 5.2.9(*b*)] it exerts a force of kU_3 upward on the beam. Hence, $Q_3 = -kU_3$. Thus, we have

$$\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ U_3 \\ U_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ -kU_3 \\ 0 \end{Bmatrix}$$

and the condensed equations for the unknown displacements U_3 (deflection) and U_4 (rotation) become

$$\begin{bmatrix} \frac{12EI}{L^3} + k & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix}$$

whose solution is

$$U_3 = w(L) = \frac{q_0 L^4}{8EI}\frac{1}{\left(1 + \frac{kL^3}{3EI}\right)}, \quad U_4 = \theta(L) = -\frac{q_0 L^3}{6EI}\frac{\left(EI - \frac{kL^3}{24}\right)}{\left(EI + \frac{kL^3}{3}\right)}$$

Note that when $k = 0$, we obtain the deflection $U_3 = q_0 L^4/8EI$ and rotation $U_4 = -q_0 L^3/6EI$ at the free end of a cantilever beam under uniformly distributed load of intensity q_0. When $k \to \infty$, we obtain the deflection $U_3 = 0$ and rotation $U_4 = -q_0 L^3/48EI$ at $x = L$ (where it is simply supported).

Alternatively, the assembly of the beam and spring elements [see Fig. 5.2.9(*c*)] yields the result

$$\begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 \\ -\frac{6EI}{L^2} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 \\ -\frac{12EI}{L^3} & \frac{6EI}{L^2} & \frac{12EI}{L^3} + k & \frac{6EI}{L^2} & -k \\ -\frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{L} & \frac{4EI}{L} & 0 \\ 0 & 0 & -k & 0 & k \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 \\ Q_2^2 \end{Bmatrix}$$

Using the boundary conditions, $U_1 = U_2 = U_5 = 0$ and $Q_4^1 = 0$, and the equilibrium condition $Q_3^1 + Q_1^2 = 0$, we obtain the condensed equations

$$\begin{bmatrix} \frac{12EI}{L^3} + k & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \frac{q_0 L}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix}$$

which are identical to those obtained earlier.

5.2.6 Postprocessing of the Solution

Once the boundary conditions are imposed, the resulting equations are solved for the unknown nodal displacements and forces. The solution is then given by Eq. (5.2.10) in each

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element. The slope of the deflection at an interior point is

$$-\frac{dw_h^e}{dx} = -\sum_{j=1}^{4} \Delta_j^e \frac{d\phi_j^e}{dx} \tag{5.2.26}$$

Similarly, the bending moment at any point in the element Ω^e of the beam can be computed from the formula

$$M(x) = \int_A \sigma_x \, z \, dA = B \int_{-H/2}^{H/2} \sigma_x \, z \, dz = -BE \int_{-H/2}^{H/2} z^2 \frac{d^2 w}{dx^2} \, dz = -EI \frac{d^2 w}{dx^2}$$

$$= -EI \sum_{j=1}^{4} \Delta_j^e \frac{d^2 \phi_j^e}{dx^2} \tag{5.2.27}$$

where B is the width and H is the height of the beam (for a rectangular cross-section beam). The bending stress is given by

$$\sigma_x(x, z) = -\frac{M(x)z}{I} = Ez\frac{d^2 w}{dx^2} = Ez \sum_{j=1}^{4} \Delta_j^e \frac{d^2 \phi_j^e(x)}{dx^2} \tag{5.2.28}$$

The maximum tensile stress occurs at the bottom ($z = H/2$) and the maximum compressive stress at the top ($z = -H/2$) of the beam.

We close this section with a note that whenever the flexural rigidity EI is a constant in each element, the finite element solution for the generalized displacements at the nodes is exact for any applied transverse load q. Further, the solution is exact at all points if the distributed load is such that the exact solution is a cubic.

5.2.7 Numerical Examples

Example 5.2.1

Consider a cantilever beam of length L and subjected to linearly varying distributed load $q(x)$, point load F_0, and moment M_0, as shown in Fig. 5.2.10. We wish to determine the displacement field and bending moment in the beam using two elements ($h = L/2$).

First we note that $q(x) = q_0(1 - x/L)$. There, we must evaluate its contribution to the element load vector by computing [see Eq. (5.2.19)]

$$q_i^e = \int_{x_e}^{x_{e+1}} q_0 \left(1 - \frac{x}{L}\right) \phi_i^e(x) \, dx = \int_0^{h_e} q_0 \left(1 - \frac{\bar{x} + x_e}{L}\right) \phi_i^e(\bar{x}) \, d\bar{x}$$

where $\phi_i^e(\bar{x})$ are given in Eq. (5.2.12). Evaluating the integrals,

$$\{q^e\} = \frac{q_0 h_e}{12} \begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \frac{q_0 h_e}{60L} \begin{Bmatrix} -(9h_e + 30x_e) \\ h_e(2h_e + 5x_e) \\ -(21h_e + 30x_e) \\ -h_e(3h_e + 5x_e) \end{Bmatrix}$$

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Figure 5.2.10 The cantilever beam problem considered in Example 5.2.1.

and specializing the above vectors to element 1 ($x_e = 0$) and element 2 ($x_e = h_e$), we obtain

$$\{q^1\} = \frac{q_0 L}{48}\begin{Bmatrix} 12 \\ -L \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L^2}{480}\begin{Bmatrix} -18 \\ 2L \\ -42 \\ -3L \end{Bmatrix}; \{q^2\} = \frac{q_0 L}{48}\begin{Bmatrix} 12 \\ -L \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L^2}{480}\begin{Bmatrix} -78 \\ 7L \\ -102 \\ -8L \end{Bmatrix}$$

The assembled system of equations for a mesh of two elements with constant EI (and foundation modulus $c_f = 0$) is

$$\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}$$

$$= \frac{q_0 L}{48}\begin{Bmatrix} 12 \\ -L \\ 24 \\ 0 \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L}{480}\begin{Bmatrix} -18 \\ 2L \\ -120 \\ 4L \\ -102 \\ -8L \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

At global node 1, Q_1^1 and Q_2^1 (the shear force and the bending moment, respectively), i.e., the reactions at the fixed end are not known. Hence, the corresponding generalized displacements,

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U_1 and U_2 must be known. Since the end $x = 0$ is fixed, we have $U_1 = U_2 = 0$. At global node 2, there are no externally applied shear forces and bending moment, i.e.,

$$Q_3^1 + Q_1^2 = 0, \quad Q_4^1 + Q_2^2 = 0$$

At global node 3, the shear force is given as F_0, and the bending moment as M_0 [note the sign convention for F_0 and M_0 from Fig. 5.2.2(c)]:

$$Q_3^2 = F_0, \quad Q_4^2 = -M_0$$

Hence, the assembled equations take the form

$$\frac{4EI}{L^3}\begin{bmatrix} 24 & -6L & -24 & -6L & 0 & 0 \\ -6L & 2L^2 & 6L & L^2 & 0 & 0 \\ -24 & 6L & 48 & 0 & -24 & -6L \\ -6L & L^2 & 0 & 4L^2 & 6L & L^2 \\ 0 & 0 & -24 & 6L & 24 & 6L \\ 0 & 0 & -6L & L^2 & 6L & 2L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}$$

$$= \frac{q_0 L}{48}\begin{Bmatrix} 12 \\ -L \\ 24 \\ 0 \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L}{480}\begin{Bmatrix} -18 \\ 2L \\ -120 \\ 4L \\ -102 \\ -8L \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ 0 \\ 0 \\ F_0 \\ -M_0 \end{Bmatrix}$$

Thus, the condensed equations for the unknown generalized displacements and generalized forces are

$$\frac{4EI}{L^3}\begin{bmatrix} 48 & 0 & -24 & -6L \\ 0 & 4L^2 & 6L & L^2 \\ -24 & 6L & 24 & 6L \\ -6L & L^2 & 6L & 2L^2 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \frac{q_0 L}{48}\begin{Bmatrix} 24 \\ 0 \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L}{480}\begin{Bmatrix} -120 \\ 4L \\ -102 \\ -8L \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ F_0 \\ -M_0 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix} = \frac{4EI}{L^3}\begin{bmatrix} -24 & -6L & 0 & 0 \\ 6L & L^2 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} - \frac{q_0 L}{48}\begin{Bmatrix} 12 \\ -L \end{Bmatrix} - \frac{q_0 L}{480}\begin{Bmatrix} -18 \\ 2L \end{Bmatrix}$$

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The solution of the condensed equations for the displacements yields

$$\begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \frac{L^3}{4EI}\begin{bmatrix} 48 & 0 & -24 & -6L \\ 0 & 4L^2 & 6L & L^2 \\ -24 & 6L & 24 & 6L \\ -6L & L^2 & 6L & 2L^2 \end{bmatrix}^{-1} \begin{Bmatrix} \frac{1}{4}q_0L \\ \frac{1}{120}q_0L^2 \\ F_0 + \frac{3}{80}q_0L \\ -M_0 + \frac{1}{240}q_0L^2 \end{Bmatrix}$$

$$= \frac{L}{48EI}\begin{bmatrix} 2L^2 & -6L & 5L^2 & -6L \\ -6L & 24 & -18L & 24 \\ 5L^2 & -18L & 16L^2 & -24L \\ -6L & 24 & -24L & 48 \end{bmatrix} \begin{Bmatrix} \frac{1}{4}q_0L \\ \frac{1}{120}q_0L^2 \\ F_0 + \frac{3}{80}q_0L \\ -M_0 + \frac{1}{240}q_0L^2 \end{Bmatrix}$$

$$= \frac{L}{48EI}\begin{Bmatrix} 5L^2F_0 + 6LM_0 + \frac{49}{80}q_0L^3 \\ -18LF_0 - 24M_0 - \frac{15}{8}q_0L^2 \\ 16L^2F_0 + 24LM_0 + \frac{16}{10}q_0L^3 \\ -24LF_0 - 48M_0 - 2q_0L^2 \end{Bmatrix} \tag{5.2.29}$$

The reactions Q_1^1 and Q_2^1 are

$$\begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix} = \frac{4EI}{L^3}\begin{bmatrix} -24 & -6L \\ 6L & L^2 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} - \frac{q_0L}{480}\begin{Bmatrix} 102 \\ -8L \end{Bmatrix} = \begin{Bmatrix} -\left(F_0 + \frac{1}{2}q_0L\right) \\ L\left(F_0 + \frac{1}{6}q_0L\right) + M_0 \end{Bmatrix} \tag{5.2.30}$$

It is clear that the reactions Q_1^1 and Q_2^1 computed above satisfy the static equilibrium equations of the beam:

$$Q_1^1 + F_0 + \frac{1}{2}q_0L = 0, \quad Q_2^1 - \left(F_0L + \frac{1}{6}q_0L^2 + M_0\right) = 0$$

The reactions Q_1^1 and Q_2^1 can also be computed using the definitions (5.2.3):

$$(Q_1^1)_{\text{def}} \equiv \frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=0}, \quad (Q_2^1)_{\text{def}} \equiv \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=0}$$

From (5.2.13*b*) and (5.2.13*c*), we note that the second derivative of the Hermite cubic interpolation functions is linear over the element and the third derivative is constant over the element. Therefore, the bending moment and shear force computed using the definition (5.2.3) are elementwise linear and constant, respectively. Further, at nodes connecting two elements, they yield discontinuous values because the second and third derivatives of w are not made continuous across the interelement nodes. Thus, we have

$$(Q_1^1)_{\text{def}} = EI\left(U_3\frac{d^3\phi_3^1}{dx^3} + U_4\frac{d^3\phi_4^1}{dx^3}\right)\bigg|_{x=0} = EI\left[U_3\left(-\frac{96}{L^3}\right) + U_4\left(-\frac{24}{L^2}\right)\right]$$

$$= -\left(F_0 + \frac{23}{80}q_0L\right)$$

$$(Q_2^1)_{\text{def}} = EI\left(U_3\frac{24}{L^2} + U_4\frac{4}{L}\right) = \left(M_0 + F_0L + \frac{3}{20}q_0L^2\right)$$

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which are in error by $q_1^e = -\frac{17}{80}q_0L$ and $q_2^e = \frac{1}{60}q_0L^2$ compared with those computed using the condensed (i.e., equilibrium) equations [see Eq. (5.2.30)].

The finite element solution as a function of position x is given by

$$w_h^e(x) = \begin{cases} U_3\phi_3^1 + U_4\phi_4^1 & \text{for } 0 \le x \le h \\ U_3\phi_1^2 + U_4\phi_2^2 + U_5\phi_3^2 + U_6\phi_4^2 & \text{for } h \le x \le 2h \end{cases} \tag{5.2.31a}$$

where

$$\begin{aligned} \phi_3^1 &= 3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3, & \phi_4^1 &= h\left[\left(\frac{x}{h}\right)^2 - \left(\frac{x}{h}\right)^3\right] \\ \phi_1^2 &= 1 - 3\left(1-\frac{x}{h}\right)^2 - 2\left(1-\frac{x}{h}\right)^3, & \phi_2^2 &= h\left(1-\frac{x}{h}\right)\left(2-\frac{x}{h}\right)^2 \\ \phi_3^2 &= 3\left(1-\frac{x}{h}\right)^2 + 2\left(1-\frac{x}{h}\right)^3, & \phi_4^2 &= h\left[\left(1-\frac{x}{h}\right)^3 + \left(1-\frac{x}{h}\right)^2\right] \end{aligned} \tag{5.2.31b}$$

The exact solution of the problem in Fig. 5.2.9 can be obtained by direct integration and is given by

$$\begin{aligned} w(x) &= \frac{q_0L^4}{120EI}\left[10\left(\frac{x}{L}\right)^2 - 10\left(\frac{x}{L}\right)^3 + 5\left(\frac{x}{L}\right)^4 - \left(\frac{x}{L}\right)^5\right] \\ &\quad + \frac{F_0L^3}{6EI}\left[-\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right] + \frac{M_0L^2}{2EI}\left(\frac{x}{L}\right)^2 \\ \theta(x) &= -\frac{q_0L^3}{24EI}\left[4\left(\frac{x}{L}\right) - 6\left(\frac{x}{L}\right)^2 + 4\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^4\right] \\ &\quad + \frac{F_0L^2}{2EI}\left[\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)\right] - \frac{M_0}{EI}\left(\frac{x}{L}\right) \\ M(x) &= -\frac{q_0L^2}{6}\left[1 - 3\left(\frac{x}{L}\right) + 3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right] + F_0L\left[\left(\frac{x}{L}\right) - 1\right] - M_0 \end{aligned} \tag{5.2.32}$$

The finite element solution (5.2.31) and the exact solution (5.2.32) are compared in Table 5.2.2 for the data

$$\begin{aligned} &q_0 = 24\ \text{kN/m}, \quad F_0 = 60\ \text{kN}, \quad L = 3\ \text{m}, \quad M_0 = 0\ \text{kN-m} \\ &E = 200 \times 10^6\ \text{kN/m}^2, \quad I = 29 \times 10^6\ \text{mm}^4, \quad (EI = 5800\ \text{kN-m}^2) \end{aligned} \tag{5.2.33}$$

As expected, the finite element solution for w and θ coincides with the exact solution at the nodes. At points other than the nodes, the difference between the finite element and exact solutions is virtually negligible.

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Table 5.2.2 Comparison of the finite element method (FEM) solution with the exact solution of the cantilever beam of Example 5.2.1 (2 elements).

	$w(m)$		$-\theta = dw/dx$		$-M \times 10^{-6}$ (kN m)	
x (m)	FEM	Exact	FEM	Exact	FEM	Exact
0.0000	0.0000†	0.0000	0.0000†	0.0000	0.2124†	0.2160
0.1875	0.0006	0.0006	0.0066	0.0067	0.1973	0.1984
0.3750	0.0025	0.0025	0.0128	0.0128	0.1821	0.1816
0.5625	0.0054	0.0054	0.0184	0.0185	0.1670	0.1656
0.7500	0.0093	0.0094	0.0235	0.0235	0.1519	0.1502
0.9375	0.0142	0.0142	0.0282	0.0282	0.1367	0.1354
1.1250	0.0199	0.0199	0.0324	0.0323	0.1216	0.1213
1.3125	0.0263	0.0263	0.0361	0.0360	0.1065	0.1077
1.5000	0.0333†	0.0334	0.0393†	0.0393	0.0913*	0.0945
1.6875	0.0410	0.0410	0.0421	0.0421	0.0814	0.0818
1.8750	0.0491	0.0491	0.0445	0.0446	0.0696	0.0694
2.0625	0.0577	0.0577	0.0466	0.0466	0.0579	0.0574
2.2500	0.0666	0.0666	0.0483	0.0483	0.0461	0.0456
2.4375	0.0758	0.0758	0.0496	0.0496	0.0344	0.0340
2.6250	0.0852	0.0852	0.0505	0.0505	0.0226	0.0226
2.8025	0.0947	0.0947	0.0510	0.0510	0.0109	0.0113
3.0000	0.1043†	0.1042	0.0512†	0.0512	−0.0009	0.0000

* 0.0932 from the second element.
† Nodal values; all others are computed by interpolation;

Example 5.2.2

Consider the indeterminate beam shown in Fig. 5.2.11(a). We assume the following discontinuous data ($c_f = 0$):

$$EI = \begin{cases} 2 \times 10^8 \text{ lb-ft.}^2 & \text{for } 0 \le x \le 10 \text{ ft.} \\ 10^8 \text{ lb-ft.}^2 & \text{for } 10 \le x \le 28 \text{ ft.} \end{cases} \tag{5.2.34a}$$

$$q(x) = \begin{cases} 2400 \text{ lb/ft.} & \text{for } 0 \le x \le 10 \text{ ft.} \\ 0 & \text{for } 10 \le x \le 28 \text{ ft.} \end{cases} \tag{5.2.34b}$$

The boundary conditions are

$$[EI(d^3w/dx^3) + kw]_{x=0} = 0, \quad [EI(d^2w/dx^2)]_{x=0} = 0, \quad w(28) = 0, \quad [dw/dx]_{x=28} = 0$$

We shall use three elements to analyze the problem. There are four nodes and eight global degrees of freedom in the nonuniform mesh of three elements.

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Figure 5.2.11 (*a*) Physical problem. (*b*) Finite element mesh of three elements. (*c*) Equilibrium of generalized forces.

Since EI and q are elementwise-constant, the element stiffness matrix and the force vector are given by (5.2.18), with $q_e = 2400$ lb/ft. in element 1 and $q_e = 0$ in elements 2 and 3. The assembled equations are

$$10^7 \begin{bmatrix} 0.2400 & -1.2000 & -0.2400 & -1.2000 & & & & \\ & 8.0000 & 1.2000 & 4.0000 & & & \mathbf{0} & \\ & & 0.3094 & 0.7833 & -0.0694 & -0.4167 & & \\ & & & 11.3333 & 0.4167 & 1.6667 & & \\ & & & & 0.6250 & -1.2500 & -0.5556 & -1.6667 \\ & & & & & 10.0000 & 1.6667 & 3.3333 \\ & \text{symmetric} & & & & & 0.5556 & 1.6667 \\ & & & & & & & 6.6667 \end{bmatrix}$$

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$$\times \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = 10^3 \begin{Bmatrix} 12 \\ -20 \\ 12 \\ 20 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 + Q_1^3 \\ Q_4^2 + Q_2^3 \\ Q_3^3 \\ Q_4^3 \end{Bmatrix}$$

The boundary conditions and equilibrium of internal forces and moments are given by

$$Q_2^1 = 0, \quad Q_3^1 + Q_1^2 = 0, \quad Q_4^1 + Q_2^2 = 0, \quad Q_3^2 + Q_1^3 = F_0, \quad Q_4^2 + Q_2^3 = -aF_0$$

Note that the forces Q_1^1 and Q_3^3 and the moment Q_4^3 (the reactions at the supports) are not known. The boundary conditions on the generalized displacements are

$$\left[EI\frac{d^3w}{dx^3} + kw\right]_{x=0} = 0 \Rightarrow Q_1^1 + kU_1 = 0; \; w(28) = 0 \Rightarrow U_7 = 0; \; \left(\frac{dw}{dx}\right)\bigg|_{x=28} = 0 \Rightarrow U_8 = 0$$

Using the boundary and equilibrium conditions listed above, we can write the condensed equations for the unknown generalized displacements and forces. The condensed equations for the unknown generalized displacements can be obtained by deleting the last two rows and two columns, which correspond to the known U_i ($F_0 = 10^3$ lb):

$$10^7 \begin{bmatrix} 0.24+\alpha & -1.200 & -0.240 & -1.200 & 0.000 & 0.000 \\ & 8.000 & 1.200 & 4.000 & 0.000 & 0.000 \\ & & 0.309 & 0.783 & -0.069 & -0.417 \\ & & & 11.333 & 0.417 & 1.667 \\ & & \text{symmetric} & & 0.625 & -1.250 \\ & & & & & 10.000 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = 10^3 \begin{Bmatrix} 12 \\ -20 \\ 12 \\ 20 \\ 10 \\ -10 \end{Bmatrix}$$

where $\alpha = 10^{-7}k$. The unknown reactions can be computed from ($Q_1^1 = -kU_1$)

$$\begin{Bmatrix} Q_3^3 \\ Q_4^3 \end{Bmatrix} = 10^7 \begin{bmatrix} -0.5556 & 1.6667 \\ -1.6667 & 3.3333 \end{bmatrix} \begin{Bmatrix} U_5 \\ U_6 \end{Bmatrix}$$

This completes the finite element analysis of the problem.

The solution (with the help of a computer) of the condensed equations for the generalized displacements, when $k = 10^{11}$ lb/in. (a hard spring), gives $U_1 = 0.18356 \times 10^{-6} \approx 0$, and (rounded to six decimal points)

$$U_2 = -0.003686, \; U_3 = 0.026560 \text{ ft.}, \; U_4 = -0.001097, \; U_5 = 0.010215 \text{ ft.}, \; U_6 = 0.002466$$

The reaction forces, from the element equilibrium equations, are

$$(Q_1^1)_{\text{equil}} = -kU_1 = -18{,}356 \text{ lb}, \quad (Q_3^3)_{\text{equil}} = -15{,}644 \text{ lb}, \quad (Q_4^3)_{\text{equil}} = -88{,}040 \text{ ft-lb}$$

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Based on the definitions, we have

$$(Q_3^3)_{\text{def}} = -15{,}644 \text{ lb}, \quad (Q_4^3)_{\text{def}} = -88{,}038 \text{ ft-lb}$$

For $k = 10^5$ lb/in. (a soft spring or no spring), we obtain

$$U_1 = 0.16104 \text{ ft.}, \quad U_2 = 0.004579, \quad U_3 = 0.10682 \text{ ft.}, \quad U_4 = 0.006605$$
$$U_5 = 0.020755 \text{ ft.}, \quad U_6 = 0.005844$$

Because of the discontinuity in the loading and the flexural rigidity, the exact solution $w(x)$ is also defined by three separate expressions for the three regions.

Example 5.2.3

This example deals with the indeterminate beam shown in Fig. 5.2.12. The beam is made of steel ($E = 30 \times 10^6$ psi) and the cross-sectional dimensions are 2×3 in. ($I = 4.5$ in.^{4}). We are interested in finding the transverse deflection w using the Euler–Bernoulli beam finite element.

Because of the discontinuity in loading, the beam should be divided into three elements: $\Omega_1 = (0, 16)$, $\Omega_2 = (16, 36)$, and $\Omega_3 = (36, 48)$; the element lengths are: $h_1 = 16$ in., $h_2 = 20$ in., and $h_3 = 12$ in. The load variation in each element is given by

$$q^{(1)}(x) = \left(30 - \frac{10}{16}x\right), \quad q^{(2)}(x) = 20, \quad q^{(3)}(x) = 0$$

Using Eq. (5.2.19), we obtain the following load vectors due to the distributed loads:

$$\{q^1\} = \begin{Bmatrix} 216.00 \\ -554.67 \\ 184.00 \\ 512.00 \end{Bmatrix}, \quad \{q^2\} = \begin{Bmatrix} 200.00 \\ -666.77 \\ 200.00 \\ 666.67 \end{Bmatrix}, \{q^3\} = \{0\}$$

The element stiffness matrices can be computed from Eq. (5.2.18) by substituting appropriate values of h, E, I, and $c_f = 0$. For Element 1, for example, we have ($h_1 = 16$ in., $EI = 135 \times 10^6$ lb-in.^{2}),

$$[K^1] = 10^6 \begin{bmatrix} 0.3955 & -3.1641 & -0.3955 & -3.1641 \\ -3.1641 & 33.7500 & 3.1641 & 16.8750 \\ -0.3955 & 3.1641 & 0.3955 & 3.1641 \\ -3.1641 & 16.8750 & 3.1641 & 33.750 \end{bmatrix}$$

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Figure 5.2.12 Finite element modeling of an indeterminate beam.

The assembled stiffness matrix and load vector for this mesh is of the same form as given in Eqs. (5.2.24*a*) and (5.2.24*b*). The boundary conditions for the problem are

$$w(0) = 0 \to U_1 = 0, \quad \frac{dw}{dx}(0) = 0 \to U_2 = 0, \quad w(36) = 0 \to U_5 = 0$$

$$Q_3^1 + Q_1^2 = 0, \quad Q_4^1 + Q_2^2 = 0, \quad Q_4^2 + Q_2^3 = 0, \quad Q_3^3 = 500, \quad Q_4^3 = 0$$

Note that Q_1^1, Q_2^1, and $Q_3^2 + Q_1^3$ are the reactions that are not known and are to be calculated in the postcomputation. Since the specified boundary conditions on the primary variables are homogeneous, we can delete the rows and columns corresponding to the specified displacements (i.e., delete rows and columns 1, 2, and 5) and solve the remaining five equations for U_3, U_4, U_6, U_7, and U_8:

$$U_3 = -0.000322 \text{ in.}, \quad U_4 = 0.0000593 \text{ rad.}, \quad U_6 = -0.0002513 \text{ rad.}$$

$$U_7 = 0.00515 \text{ in.}, \quad U_8 = -0.000518 \text{ rad.}$$

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The exact deflection in the three intervals of the beam is given by (large numbers were rounded to whole numbers)

$$w(x) = \begin{cases} \dfrac{1}{EI}\left(268.543x^2 - \dfrac{691}{15}x^3 + \dfrac{5}{4}x^4 - \dfrac{1}{192}x^5\right) & 0 \le x \le 16 \\ \dfrac{1}{EI}\left(-\dfrac{16{,}384}{3} + \dfrac{5120}{3}x + 55.2097x^2 - \dfrac{491}{15}x^3 + \dfrac{5}{6}x^4\right) & 16 \le x \le 36 \\ \dfrac{1}{EI}\left(6{,}554{,}368 - 506{,}066x + 12{,}000x^2 - \dfrac{250}{3}x^3\right) & 36 \le x \le 48 \end{cases}$$

A comparison of the finite element solutions for deflections (w), slopes (θ), and bending moments (M) (calculated in the postcomputation) at points other than the nodes are compared with the exact values in Table 5.2.3 for three different meshes. The values of deflections, slopes, and bending moments were computed using the finite element solution and its derivatives.

Table 5.2.3 Comparison of the finite element solution with the exact solution (units of the quantities should be obvious) of the beam problem considered in Example 5.2.3.

		$w_0 \times 10^6$		$(-dw_0/dx) \times 10^6$		$-(d^2w_0/dx^2) \times 10^6$	
x	N†	FEM	Exact	FEM	Exact	FEM	Exact
	3	−0.8570		1.1553		−1.0250	
2.0	5	4.1405	5.3739	−3.3386	−4.1552	0.4663	0.3219
	6	5.2319		−4.1558		0.4638	
	3	−18.4510		8.8348		−2.8147	
6.0	5	8.3936	9.6048	4.4199	5.2328	−4.3456	−4.4727
	6	9.4751		5.2323		−4.3431	
	3	−138.2300		33.7760		−5.4992	
12.0	5	−122.5900	−120.8100	39.6630	39.6720	−5.4794	−5.9238
	6	−122.5900		39.6630		−5.4794	
	3	−674.3600		71.5620		3.5507	
21.0	5	−643.4900	−639.6300	62.3030	62.3020	3.5507	2.9335
	6	−643.4900		62.3030		3.5507	
	3	−812.9300		−83.7970		27.5210	
31.0	5	−782.0700	−778.1900	−74.5370	−74.5390	27.5210	26.9040
	6	−782.0700		−74.5370		27.5210	
	3	2174.9000		−451.3600		22.2220	
42.0	5	2174.9000	2174.8000	−451.3600	−451.3600	22.2220	22.2220
	6	2174.9000		−451.3600		22.2220	

†3-elements: $h_1 = 16$, $h_2 = 20$, $h_3 = 12$. 5-Elements: $h_1 = 8$, $h_2 = 8$, $h_3 = 10$, $h_4 = 10$, $h_5 = 12$. 6-Elements: $h_1 = 4$, $h_2 = 4$, $h_3 = 8$, $h_4 = 10$, $h_5 = 10$, $h_6 = 12$.

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Example 5.2.4 (Beam Element with Nodal Hinge)

The last example of this section deals with a beam with an internal hinge (see Fig. 5.2.13). It is not uncommon to find beams with an internal hinge about which the beam is free to rotate. Thus, at the hinge there cannot be any moment and the rotation is *not* continuous across the hinge. To account for the discontinuity in the rotation at a hinge, we must eliminate the rotation associated with the hinge.

Consider a uniform beam element with a hinge at node 2. The element equation is given by Eq. (5.2.17) with the stiffness matrix given in (5.2.18) (with zero foundation modulus, $c_f = 0$)

$$\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & \vdots & -3h \\ -3h & 2h^2 & 3h & \vdots & h^2 \\ -6 & 3h & 6 & \vdots & 3h \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ -3h & h^2 & 3h & \vdots & 2h^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \cdots \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \cdots \\ F_4 \end{Bmatrix} \tag{5.2.35}$$

Since the moment at a hinge is zero, we have $Q_4 = 0$ and $F_4 = q_4 + Q_4 = q_4$, which is known. This allows us to eliminate θ_2 (rotation at node 2) using the method discussed in Eqs. (3.2.59)–(3.2.60). Comparing Eq. (5.2.35) with Eq. (3.2.59), we have the following definitions:

$$\mathbf{K}^{11} = \frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 \\ -3h & 2h^2 & 3h \\ -6 & 3h & 6 \end{bmatrix}, \quad \mathbf{K}^{12} = \frac{2EI}{h^3}\begin{Bmatrix} -3h \\ h^2 \\ 3h \end{Bmatrix}, \quad \mathbf{K}^{21} = (\mathbf{K}^{12})^{\mathrm{T}}$$

$$\mathbf{K}^{22} = \frac{4EI}{h}, \quad \mathbf{U}^1 = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \end{Bmatrix}, \quad \mathbf{U}^2 = \theta_2$$

Figure 5.2.13 Compound beam with an internal hinge.

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Substituting (3.2.60c) for $\mathbf{U}^2 = \theta_2$ into (3.2.60a), we obtain

$$\begin{aligned}
\hat{\mathbf{K}} &= \mathbf{K}^{11} - \mathbf{K}^{12}\left(\mathbf{K}^{22}\right)^{-1}\mathbf{K}^{21} \\
&= \frac{EI}{h^3}\begin{bmatrix} 12 & -6h & -12 \\ -6h & 4h^2 & 6h \\ -12 & 6h & 12 \end{bmatrix} - \frac{2EI}{h^3}\begin{Bmatrix} -3h \\ h^2 \\ 3h \end{Bmatrix}\frac{h}{4EI}\frac{2EI}{h^3}\left\{-3h\ h^2\ 3h\right\} \\
&= \frac{3EI}{h^3}\begin{bmatrix} 1 & -h & -1 \\ -h & h^2 & h \\ -1 & h & 1 \end{bmatrix}
\end{aligned}$$

Thus, the equations for a beam element with a hinge at node 2 can be written as

$$\frac{3EI}{h^3}\begin{bmatrix} 1 & -h & -1 & 0 \\ -h & h^2 & h & 0 \\ -1 & h & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \end{Bmatrix} \tag{5.2.36}$$

Similarly, we can derive the element equations for a beam element with hinge at node 1

$$\frac{3EI}{h^3}\begin{bmatrix} 1 & 0 & -1 & -h \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & h \\ -h & 0 & h & h^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ F_3 \\ F_4 \end{Bmatrix} \tag{5.2.37}$$

As a specific example of application of the equations derived, consider the beam in Fig. 5.2.13. We use a two-element mesh, with element 1 having the hinge at its node 2, while element 2 is the usual beam element. The assembled system of equations is

$$EI\begin{bmatrix} \frac{3}{a^2} & -\frac{3}{a^2} & -\frac{3}{a^2} & 0 & 0 & 0 \\ -\frac{3}{a^2} & \frac{3}{a} & \frac{3}{a^2} & 0 & 0 & 0 \\ -\frac{3}{a^2} & \frac{3}{a^2} & \frac{3}{a^3}+\frac{12}{b^3} & -\frac{6}{b^2} & -\frac{12}{b^3} & -\frac{6}{b^2} \\ 0 & 0 & -\frac{6}{b^2} & \frac{4}{b} & \frac{6}{b^2} & \frac{2}{b} \\ 0 & 0 & -\frac{12}{b^3} & \frac{6}{b^2} & \frac{12}{b^3} & \frac{6}{b^2} \\ 0 & 0 & -\frac{6}{b^2} & \frac{2}{b} & \frac{6}{b^2} & \frac{4}{b} \end{bmatrix}\begin{Bmatrix} w_1^1 \\ \theta_1^1 \\ w_2^1 = w_1^2 \\ \theta_1^2 \\ w_2^2 \\ \theta_2^2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix} + \frac{q_0 b}{12}\begin{Bmatrix} 0 \\ 0 \\ 6 \\ -b \\ 6 \\ b \end{Bmatrix} \tag{5.2.38}$$

Using the boundary conditions

$$w_1^1 = 0,\ \ w_2^2 = 0,\ \ \theta_2^2 = 0,\ \ Q_2^1 = 0,\ \ Q_3^1 + Q_1^2 = 0,\ \ Q_4^1 + Q_2^2 = 0$$

we obtain the condensed equations

$$EI\begin{bmatrix} \frac{3}{a} & \frac{3}{a^2} & 0 \\ \frac{3}{a^2} & \frac{3}{a^3}+\frac{12}{b^3} & -\frac{6}{b^2} \\ 0 & -\frac{6}{b^2} & \frac{4}{b} \end{bmatrix}\begin{Bmatrix} \theta_1^1 \\ w_2^1 = w_1^2 \\ \theta_1^2 \end{Bmatrix} = \frac{q_0 b}{12}\begin{Bmatrix} 0 \\ 6 \\ -b \end{Bmatrix} \tag{5.2.39}$$

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and the solution is given by

$$\theta_1^1 = -\frac{q_0 b^4}{8aEI}, \quad w_2^1 = w_1^2 = \frac{q_0 b^4}{8EI}, \quad \theta_2^2 = \frac{q_0 b^3}{2EI}$$

5.3 TIMOSHENKO BEAM ELEMENTS

5.3.1 Governing Equations

Recall that the Euler–Bernoulli beam theory is based on the assumption that plane cross sections remain plane and *normal* to the longitudinal axis after bending [see Fig. 5.3.1(*a*)]. This assumption results in zero transverse shear strain. When the normality assumption is not used, i.e., plane sections remain plane but not necessarily normal to the longitudinal axis after deformation, the transverse shear strain $\gamma_{xz} = 2\varepsilon_{xz}$ is not zero. Therefore, the rotation

Figure 5.3.1 (*a*) Kinematics of the Euler–Bernoulli beam theory. (*b*) Kinematics of the Timoshenko beam theory where normals before deformation no longer remain normal after deformation.

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of a transverse normal plane about the y-axis is not equal to $-dw/dx$ [see Fig. 5.3.1(b)]. Beam theory based on these relaxed assumptions is known as a *shear deformation beam theory*, most commonly known as the *Timoshenko beam theory*. We denote the rotation about the y-axis by an independent function $\Psi(x)$.

The equilibrium equations of the Timoshenko beam theory [see Example 2.4.3; also see Reddy (2002)] are the same as those of the Euler–Bernoulli beam theory (see Fig. 5.2.1), but the kinematic relations are different:

$$-\frac{dV}{dx}+c_f w=q,\quad -\frac{dM}{dx}+V=0,\quad M=EI\frac{d\Psi}{dx},\quad V=GAK_s\left(\frac{dw}{dx}+\Psi\right) \tag{5.3.1}$$

Thus, the governing equations in terms of the deflection w and rotation Ψ become

$$-\frac{d}{dx}\left[GAK_s\left(\Psi+\frac{dw}{dx}\right)\right]+c_f w=q \tag{5.3.2a}$$

$$-\frac{d}{dx}\left(EI\frac{d\Psi}{dx}\right)+GAK_s\left(\Psi+\frac{dw}{dx}\right)=0 \tag{5.3.2b}$$

where G is the shear modulus and K_s is the *shear correction coefficient*, which is introduced to account for the difference in the constant state of shear stress in this theory and the parabolic variation of the shear stress predicted by the elasticity theory through the beam thickness. All other quantities have the same meaning as before. For short and chubby beams (i.e., length-to-height ratio less than 20), $\Psi \neq -(dw/dx)$ and the difference, $\Psi+(dw/dx)$, is the transverse shear strain. When the shear strain is zero (i.e., for long slender beams), substituting the second equation into the first for $GAK_s(\Psi+dw/dx)$ and replacing Ψ with $-dw/dx$, we obtain governing equation (5.2.1) of the Euler–Bernoulli beam theory.

5.3.2 Weak Form

The weak form of Eqs. (5.3.2a) and (5.3.2b) over an element $\Omega^e=(x_a,x_b)$ can be developed using the usual three-step procedure, as already discussed in Example 2.4.3. At the end of the second step (i.e., after integration by parts), we obtain

$$0=\int_{x_a}^{x_b}\left[\frac{dv_1}{dx}GAK_s\left(\Psi+\frac{dw}{dx}\right)+c_f v_1 w-v_1 q\right]dx-\left[v_1 GAK_s\left(\Psi+\frac{dw}{dx}\right)\right]_{x_a}^{x_b}$$

$$0=\int_{x_a}^{x_b}\left[\frac{dv_2}{dx}EI\frac{d\Psi}{dx}+v_2 GAK_s\left(\Psi+\frac{dw}{dx}\right)\right]dx-\left[v_2 EI\frac{d\Psi}{dx}\right]_{x_a}^{x_b}$$

The coefficients of the weight functions v_1 and v_2 in the boundary integrals are, respectively,

$$GAK_s\left(\Psi+\frac{dw}{dx}\right)\equiv V \quad\text{and}\quad EI\frac{d\Psi}{dx}\equiv M \tag{5.3.3}$$

where V is the shear force and M is the bending moment; these coefficients constitute the secondary variables of the weak form. The weight functions v_1 and v_2 must have the physical interpretations such that the products $v_1 V$ and $v_2 M$ have the units of work. Clearly, v_1 must be equivalent to (the variation of) the transverse deflection w, and v_2 must be equivalent to

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(the variation of) the rotation function Ψ:

$$v_1 \sim w, \quad v_2 \sim \Psi$$

Hence, the primary variable of the formulation are w and Ψ. Denoting the shear forces and bending moments at the endpoints of the element by the expressions [cf. Eq. (5.2.3)]

$$\begin{aligned}
Q_1^e &\equiv -\left[GAK_s\left(\Psi + \frac{dw}{dx}\right)\right]\bigg|_{x_a} = -V(x_a) \\
Q_2^e &\equiv -\left(EI\frac{d\Psi}{dx}\right)\bigg|_{x_a} = -M(x_a) \\
Q_3^e &\equiv \left[GAK_s\left(\Psi + \frac{dw}{dx}\right)\right]\bigg|_{x_b} = V(x_b) \\
Q_4^e &\equiv \left(EI\frac{d\Psi}{dx}\right)\bigg|_{x_b} = M(x_b)
\end{aligned} \tag{5.3.4}$$

we arrive at the final weak statements of Eqs. (5.3.2a) and (5.3.2b):

$$\begin{aligned}
0 &= \int_{x_a}^{x_b}\left[GAK_s\frac{dv_1}{dx}\left(\Psi + \frac{dw}{dx}\right) + c_f\, v_1 w - v_1 q\right]dx - v_1(x_a)Q_1^e - v_1(x_b)Q_3^e \\
0 &= \int_{x_a}^{x_b}\left[EI\frac{dv_2}{dx}\frac{d\Psi}{dx} + GAK_s v_2\left(\Psi + \frac{dw}{dx}\right)\right]dx - v_2(x_a)Q_2^e - v_2(x_b)Q_4^e
\end{aligned} \tag{5.3.5}$$

We note that Q_i^e have the same meaning as well as sense as in the Euler–Bernoulli beam theory.

We can identify the bilinear and linear forms from the weak forms as

$$\begin{aligned}
B((v_1, v_2), (w, \Psi)) &= \int_{x_a}^{x_b}\left[GAK_s\left(\frac{dv_1}{dx} + v_2\right)\left(\Psi + \frac{dw}{dx}\right) + EI\frac{dv_2}{dx}\frac{d\Psi}{dx}\right]dx \\
l((v_1, v_2)) &= \int_{x_a}^{x_b} v_1 q\, dx + v_1(x_a)Q_1^e + v_2(x_a)Q_2^e + v_1(x_b)Q_3^e + v_2(x_b)Q_4^e
\end{aligned} \tag{5.3.6}$$

Equations (5.3.5) are equivalent to the statement of the principle of virtual displacements for the Timoshenko beam theory. The total potential energy functional of the isolated beam finite element is given by [see Eq. (2.4.40)]

$$\begin{aligned}
\Pi_e(w, \Psi) = \int_{x_e}^{x_{e+1}}&\left[\frac{EI}{2}\left(\frac{d\Psi}{dx}\right)^2 + \frac{GAK_s}{2}\left(\frac{dw}{dx} + \Psi\right)^2 + \frac{1}{2}c_f\, w^2 - wq\right]dx \\
&- w(x_a)Q_1^e - \Psi(x_a)Q_2^e - w(x_b)Q_3^e - \Psi(x_b)Q_4^e
\end{aligned} \tag{5.3.7}$$

The first term in the square brackets represents the elastic strain energy due to bending, the second term represents the elastic energy due to the transverse shear deformation, the third is the strain energy stored in the elastic foundation, and the fourth is the work done by the distributed load; the remaining terms account for the work done by the generalized forces Q_i^e in moving through the respective generalized displacements (w, Ψ) at the ends of the element. Once again, the principle of minimum total potential energy, $\delta\Pi_e = 0$, gives the weak forms in Eq. (5.3.5).

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5.3.3 General Finite Element Model

A close examination of the terms in (5.3.5) shows that both w and Ψ are differentiated only once with respect to x. Since the primary variables are the dependent unknowns themselves (and do not include their derivatives), the Lagrange interpolation of both w and Ψ is admissible here. The minimum admissible degree of interpolation is linear, so that $dw/dx \neq 0$ and $d\Psi/dx \neq 0$. The variables w and Ψ do not have the same physical units; they can be interpolated, in general, with different degrees of interpolation.

Let us consider Lagrange interpolation of w and Ψ in the form

$$w = \sum_{j=1}^{m} w_j \psi_j^{(1)}, \quad \Psi = \sum_{j=1}^{n} S_j \psi_j^{(2)} \tag{5.3.8}$$

where $\psi_j^{(1)}$ and $\psi_j^{(2)}$ are the Lagrange interpolation functions of degree $m-1$ and $n-1$, respectively. In general, m and n are independent of each other, although $m = n$ is most common. However, when $m = n = 2$ (i.e., linear interpolation of both w and Ψ is used; see Fig. 5.3.2), the derivative of w is

$$\left(\frac{dw}{dx}\right)^e = \frac{w_2^e - w_1^e}{h_e}$$

which is elementwise-constant. The rotation Ψ, being linear, is not consistent with that predicted by $w(x)$. For thin beams, the transverse shear deformation is negligible, and we must have $\Psi = -dw/dx$, which requires

$$S_1^e \frac{x_b - x}{h_e} + S_2^e \frac{x - x_a}{h_e} = -\frac{w_2^e - w_1^e}{h_e}$$

or, equivalently (by equating like coefficients on both sides),

$$S_1^e x_b - S_2^e x_a = -(w_2^e - w_1^e), \quad S_2^e - S_1^e = 0$$

which in turn requires

$$S_1^e = S_2^e = -\frac{w_2^e - w_1^e}{h_e} \tag{5.3.9}$$

This implies that $\Psi(x)$ is a constant, i.e., $S_1^e = S_2^e = S^e$:

$$\Psi(x) = S_1^e \frac{x_b - x}{h_e} + S_2^e \frac{x - x_a}{h_e} = S^e \tag{5.3.10}$$

Figure 5.3.2 Linear Timoshenko beam finite element. (*a*) Generalized displacements. (*b*) Generalized forces.

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However, a constant state of $\Psi(x)$ is not admissible because the bending energy of the element,

$$\int_{x_a}^{x_b} \frac{EI}{2}\left(\frac{d\Psi}{dx}\right)^2 dx \tag{5.3.11}$$

would be zero. This numerical problem is known in the finite element literature as *shear locking*.

To overcome shear locking, two alternative procedures have been used in the literature:

1. *Consistent interpolation.* Use an approximation of w and Ψ such that dw/dx and Ψ are polynomials of the same degree (i.e., $m = n + 1$).
2. *Reduced integration.* Use equal interpolation (i.e., $m = n$) for w and Ψ and evaluate the bending stiffness coefficients associated with (5.3.11) using a numerical integration rule (to be discussed in Chapter 7) consistent with the actual interpolation of Ψ. However, the stiffness coefficients associated with the shear energy

$$\int_{x_a}^{x_b} \frac{GAK_s}{2}\left(\frac{dw}{dx} + \Psi\right)^2 dx \tag{5.3.12}$$

must be evaluated using a numerical integration rule that treats Ψ as if it is the same order polynomial as dw/dx. Thus, if w and Ψ are approximated with linear polynomials, dw/dx is a constant and Ψ is linear. In evaluating the stiffness terms coming from the shear energy (5.3.12), we must use one-point integration, as dictated by dw/dx and not Ψ. Note that one-point integration in this case is sufficient to evaluate the bending energy exactly but not the shear energy because it is quadratic in Ψ. Thus, it amounts to underintegrating the term. This is known as the *reduced integration* technique.

For illustrative purposes, we take a detailed look at the expression

$$\frac{GAK_s}{2}\int_{x_a}^{x_b}\left(\frac{dw}{dx} + \Psi\right)^2 dx = \frac{GAK_s}{2}\left[\left(\frac{dw}{dx} + \Psi\right)^2\Bigg|_{x=x_a+h_e/2}\right]h_e$$

where $x = x_a + \frac{1}{2}h_e$ is the midpoint of the element and h_e is its length. Substituting (5.3.8) into this expression (with $m = n = 2$) and equating to zero for thin beams, we obtain

$$\begin{aligned}
&\frac{GAK_s h_e}{2}\left(\frac{w_2^e - w_1^e}{h_e} + S_1^e\frac{x_b - x}{h_e} + S_2^e\frac{x - x_a}{h_e}\right)^2\Bigg|_{x=x_a+h_e/2} \\
&\quad = \frac{GAK_s h_e}{2}\left(\frac{w_2^e - w_1^e}{h_e} + \frac{S_1^e + S_2^e}{2}\right)^2 = 0
\end{aligned} \tag{5.3.13}$$

which is a weaker requirement than (5.3.9), i.e., if (5.3.9) holds, then (5.3.13) also holds, but (5.3.13) does not imply (5.3.9). We note that (5.3.9) must hold only for problems for which the transverse shear energy (5.3.12) is negligible.

In summary, we use either consistent interpolation ($m = n + 1$) or equal interpolation with reduced integration in the evaluation of the transverse shear stiffness coefficients in the Timoshenko beam element. We consider both forms of elements here. First, we complete the finite element model development using the general interpolations given in Eq. (5.3.8) and then discuss the specific choice of interpolations functions used for w and Ψ.

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Substitution of (5.3.8) for w and Ψ, and $v_1 = \psi_i^{(1)}$ and $v_2 = \psi_i^{(2)}$ into the weak forms (5.3.5), we obtain the following finite element equations of the Timoshenko beam element:

$$
\begin{aligned}
0 &= \sum_{j=1}^{m} K_{ij}^{11} w_j + \sum_{j=1}^{n} K_{ij}^{12} s_j - F_i^1 \quad (i = 1, 2, \ldots, m) \\
0 &= \sum_{j=1}^{m} K_{ij}^{21} w_j + \sum_{j=1}^{n} K_{ij}^{22} s_j - F_i^2 \quad (i = 1, 2, \ldots, n)
\end{aligned}
\tag{5.3.14}
$$

where

$$
\begin{aligned}
K_{ij}^{11} &= \int_{x_a}^{x_b} \left(GAK_s \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx} + c_f \psi_i^{(1)} \psi_j^{(1)} \right) dx \\
K_{ij}^{12} &= \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^{(1)}}{dx} \psi_j^{(2)}\, dx = K_{ji}^{21} \quad (\text{i.e., } [K^{21}] = [K^{12}]^{\mathrm{T}}) \\
K_{ij}^{22} &= \int_{x_a}^{x_b} \left(EI \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} + GAK_s \psi_i^{(2)} \psi_j^{(2)} \right) dx \\
F_i^1 &= \int_{x_a}^{x_b} q \psi_i^{(1)}\, dx + Q_{2i-1}, \qquad F_i^2 = Q_{2i}
\end{aligned}
\tag{5.3.15}
$$

In the interest of clarity, the element label e on the quantities is omitted. Equations (5.3.14) can be written in matrix form as

$$
\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{s\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}
\tag{5.3.16}
$$

The finite element model in (5.3.16) with the coefficients given in (5.3.15) is the most general displacement finite element model of the Timoshenko beam theory. It can be used to obtain a number of special finite element models, as discussed next.

5.3.4 Consistent Interpolation Elements

The first consistent interpolation element (CIE) that we will consider is the one in which quadratic interpolation is used for w and linear interpolation for Ψ so that dw/dx and Ψ are of the same order polynomial (and hence, no shear locking occurs). That is, we select $\psi_i^{(1)}$ to be quadratic polynomials and $\psi_i^{(2)}$ to be linear polynomials. For this choice of interpolation, $[K^{11}]$ is 3×3, $[K^{12}]$ is 3×2, and $[K^{22}]$ is 2×2. The explicit forms of the matrices, when EI and GAK_s are constant, are (note that some of the matrix coefficients are readily available from Chapter 3)

$$
[K^{11}] = \frac{G_e A_e K_s}{3h_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_f^e h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}
$$

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$$[K^{12}] = \frac{G_e A_e K_s}{6}\begin{bmatrix} -5 & -1 \\ 4 & -4 \\ 1 & 5 \end{bmatrix} = [K^{21}]^{\mathrm{T}}$$

$$[K^{22}] = \frac{EI}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{G_e A_e K_s h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{5.3.17}$$

When $c_f = 0$, the finite element equations for this choice of interpolation are given by (see Fig. 5.3.3)

$$\frac{GAK_s}{6h_e}\begin{bmatrix} 14 & -16 & 2 & -5h_e & -h_e \\ -16 & 32 & -16 & 4h_e & -4h_e \\ 2 & -16 & 14 & h_e & 5h_e \\ -5h_e & 4h_e & h_e & 2h_e^2\Sigma_e & h_e^2\Theta_e \\ -h_e & -4h_e & 5h_e & h_e^2\Theta_e & 2h_e^2\Sigma_e \end{bmatrix}\begin{Bmatrix} w_1^e \\ w_c^e \\ w_2^e \\ S_1^e \\ S_2^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_c^e \\ q_2^e \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ \hat{Q}_c^e \\ Q_3^e \\ Q_2^e \\ Q_4^e \end{Bmatrix} \tag{5.3.18}$$

where

$$\Lambda_e = \frac{E_e I_e}{G_e A_e K_s h_e^2}, \quad \mu_0 = 12\Lambda_e, \quad \Theta_e = 1 - 6\Lambda_e, \quad \Sigma_e = 1 + 3\Lambda_e \tag{5.3.19}$$

$(Q_1^e, Q_2^e, Q_3^e, Q_4^e)$ are the generalized forces defined in Eq. (5.3.4), w_c^e and $\hat{Q}_c^e$ are the deflection and applied external load, respectively, at the center node of the quadratic element, and

$$q_i^e = \int_{x_a}^{x_b} \psi_i^{(1)} q \, dx, \quad (i = 1, 2, c), \quad \psi_i^{(1)} = \text{quadratic} \tag{5.3.20}$$

This element is designated as CIE-1 (see Fig. 5.3.3).

Note that node c, which is the center node of the element, is not connected to other elements, and the only degree of freedom there is the transverse deflection. Thus, there are different number of degrees of freedom at different nodes of the element, and this therefore complicates the assembly of elements and its implementation on a computer. Hence, we eliminate the node c dependence in the system of element equations by condensing out w_c^e. The second equation of (5.3.18) can be used to express w_c^e in terms of w_1^e, w_2^e, S_1^e, S_2^e, q_c^e,

Figure 5.3.3 Consistent interpolation Timoshenko beam element, CIE-1. (*a*) Generalized displacements. (*b*) Generalized forces.

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and $\hat{Q}_c^e$:

$$w_c^e = \frac{6h_e}{32G_eA_eK_s}\left(q_c^e + \hat{Q}_c^e\right) + \left(\frac{w_1^e + w_2^e}{2}\right) + h_e\left(\frac{S_2^e - S_1^e}{8}\right) \tag{5.3.21}$$

Substituting for w_c^e from Eq. (5.3.21) into the remaining equations of (5.3.18) [i.e., eliminate w_c^e from Eq. (5.3.18)] and rearranging the equations, we obtain

$$\left(\frac{2E_eI_e}{\mu_0 h_e^3}\right)\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2(1.5+6\Lambda_e) & 3h_e & h_e^2(1.5-6\Lambda_e) \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2(1.5-6\Lambda_e) & 3h_e & h_e^2(1.5+6\Lambda_e) \end{bmatrix}\begin{Bmatrix} w_1^e \\ S_1^e \\ w_2^e \\ S_2^e \end{Bmatrix}$$

$$= \begin{Bmatrix} q_1^e + \frac{1}{2}\hat{q}_c^e \\ -\frac{1}{8}\hat{q}_c^e h_e \\ q_2^e + \frac{1}{2}\hat{q}_c^e \\ \frac{1}{8}\hat{q}_c^e h_e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix} \tag{5.3.22}$$

where $\hat{q}_c^e = q_c^e + \hat{Q}_c^e$. For simplicity, but without loss of generality, we will assume that $\hat{Q}_c^e = 0$ (i.e., no external point force is placed at the center of the element) so that $\hat{q}_c^e = q_c^e$. Note that the load vector is equivalent to that of the Euler–Bernoulli beam element [see Eq. (5.2.18)]. In fact, for uniform load q, the load vector in Eq. (5.3.22) is identical to the one in (5.2.18). Thus, the CIE-1 element, for all analysis steps, is exactly the same as that shown in Fig. 5.3.2 with the element equations given by (5.3.22). However, one must keep in mind that w and q_i^e are determined by quadratic interpolations functions.

The second consistent interpolation element, denoted CIE-2, is based on Lagrange cubic interpolation of $w(x)$ and quadratic interpolation of $\Psi(x)$. However, this element leads to 7×7 element stiffness matrix with seven degrees of freedom ($w_1^e, w_e^2, w_3^e, w_4^e, S_1^e, S_2^e, S_3^e$) per element. Elimination of the internal nodal degrees of freedom will result in a 4×4 matrix. Alternatively, the same element can be derived directly by assuming Hermite cubic interpolation of $w(x)$ and a *dependent* quadratic interpolation of $\Psi(x)$. This approach leads to an element that yields, as a special case, the Euler–Bernoulli beam element [see Reddy (1997, 2000)]. The derivation is presented here for the case $c_f^e = 0$.

The exact solution of Eqs. (5.3.2*a*) and (5.3.2*b*) for the homogeneous case (i.e., $q = 0$) and when $c_f = 0$ is

$$w(x) = -\frac{1}{EI}\left(c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4\right) + \frac{1}{GAK_s}(c_1x) \tag{5.3.23a}$$

$$EI\Psi(x) = c_1\frac{x^2}{2} + c_2x + c_3 \tag{5.3.23b}$$

where c_1 through c_4 are the constants of integration. Note that the constants c_1, c_2, and c_3 appearing in (5.3.23*b*) are the same as those in Eq. (5.3.23*a*). This suggests that one may use cubic approximation of $w(x)$ and an *interdependent* quadratic approximation of $\Psi(x)$. The resulting finite element model is termed an *interdependent interpolation element* (IIE).

Since the solutions in (5.3.23*a*) and (5.3.23*b*) are valid also for a typical finite element (replace c_i with c_i^e), we proceed to express c_i^e in terms of the nodal values of w and Ψ.

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The resulting four relations among $(w_1^e, w_2^e, S_1^e, S_2^e)$ and $(c_1^e, c_2^e, c_3^e, c_4^e)$, when inverted and substituted back into Eqs. (5.3.23*a*) and (5.3.24*b*), give

$$w(x) \approx w_h^e(x) = \sum_{j=1}^{m} \bar{\phi}_j^e \Delta_j^e, \quad \Psi(x) \approx \Psi_h^e(x) = \sum_{j=1}^{n} \varphi_j^e \Delta_j^e \tag{5.3.24a}$$

$$\Delta_1^e = w_1^e, \quad \Delta_2^e = S_1^e, \quad \Delta_3^e = w_2^e, \quad \Delta_4^e = S_2^e \tag{5.3.24b}$$

where $\bar{\phi}_i^e$ and φ_i^e are the approximation functions

$$\begin{aligned}
\bar{\phi}_1^e &= \frac{1}{\mu_e}\left[\mu_e - 12\Lambda_e\bar{x} - (3-2\bar{x})\bar{x}^2\right] \\
\bar{\phi}_2^e &= -\frac{h_e}{\mu_e}\left[(1-\bar{x})^2\bar{x} + 6\Lambda_e(1-\bar{x})\bar{x}\right] \\
\bar{\phi}_3^e &= \frac{1}{\mu_e}\left[(3-2\bar{x})\bar{x}^2 + 12\Lambda_e\bar{x}\right] \\
\bar{\phi}_4^e &= \frac{h_e}{\mu_e}\left[(1-\bar{x})\bar{x}^2 + 6\Lambda_e(1-\bar{x})\bar{x}\right]
\end{aligned} \tag{5.3.25a}$$

$$\begin{aligned}
\varphi_1^e &= \frac{6}{h_e\mu_e}(1-\bar{x})\bar{x} \\
\varphi_2^e &= \frac{1}{\mu_e}(\mu_e - 4\bar{x} + 3\bar{x}^2 - 12\Lambda_e\bar{x}) \\
\varphi_3^e &= -\frac{6}{h_e\mu_e}(1-\bar{x})\bar{x} \\
\varphi_4^e &= \frac{1}{\mu_e}(3\bar{x}^2 - 2\bar{x} + 12\Lambda_e\bar{x})
\end{aligned} \tag{5.3.25b}$$

Here, $\bar{x}$ is the nondimensional local coordinate

$$\bar{x} = \frac{x - x_a}{h_e}, \quad \mu_e = 1 + 12\Lambda_e, \quad \Lambda_e = \frac{E_e I_e}{G_e A_e K_s h_e^2} \tag{5.3.26}$$

Note that the Hermite cubic interpolation functions $\phi_i^e(x)$ of Eq. (5.2.11) are a special case of $\bar{\phi}_i^e$, i.e., ϕ_i^e can be obtained from $\bar{\phi}_i^e$ by setting $\Lambda_e = 0$ (hence, $\mu_e = 1$).

Substitution of Eq. (5.3.24*a*) into the total potential energy functional in (5.3.7) and differentiating it with respect to Δ_i^e yields the finite element model

$$[K^e]\{\Delta^e\} = \{q^e\} + \{Q^e\} \tag{5.3.27a}$$

where

$$\begin{aligned}
K_{ij}^e &= \int_{x_a}^{x_b} \left[E_e I_e \frac{d\varphi_i^e}{dx}\frac{d\varphi_j^e}{dx} + G_e A_e K_s \left(\varphi_i^e + \frac{d\bar{\phi}_i^e}{dx}\right)\left(\varphi_j^e + \frac{d\bar{\phi}_j^e}{dx}\right)\right] dx \\
q_i^e &= \int_{x_a}^{x_b} \bar{\phi}_i^e q(x)\, dx
\end{aligned} \tag{5.3.27b}$$

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and Q_i^e have the same meaning as before [see Eq. (5.3.4)]. Equation (5.3.27a) has the explicit form

$$\left(\frac{2E_eI_e}{\mu_e h_e^3}\right)\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2\Sigma_e & 3h_e & h_e^2\Theta_e \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2\Theta_e & 3h_e & 2h_e^2\Sigma_e \end{bmatrix}\begin{Bmatrix} w_1^e \\ S_1^e \\ w_2^e \\ S_2^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix} \tag{5.3.28a}$$

where

$$\Lambda_e = \frac{E_eI_e}{G_eA_eK_sh_e^2}, \quad \mu_e = 1 + 12\Lambda_e, \quad \Theta_e = 1 - 6\Lambda_e, \quad \Sigma_e = 1 + 3\Lambda_e \tag{5.3.28b}$$

For uniformly distributed load, $q(x) = q_0$, the load vector in (5.3.27b) yields

$$\{q^e\} = \frac{q_0h_e}{12}\begin{Bmatrix} 6 \\ -h_e \\ 12 \\ h_e \end{Bmatrix} \tag{5.3.28c}$$

which is exactly the same as that in the Euler–Bernoulli beam element (EBE). Of course, for nonuniform loads the load vectors of the two elements are different.

In the thin beam limit (i.e., $\Lambda_e \to 0$), Eq. (5.3.28a) reduces to the EBE equations (5.2.17) and (5.2.18) (with $c_f^e = 0$). The IIE element leads to the exact nodal values for any distribution of the transverse load $q(x)$ provided that the bending stiffness EI and shear stiffness K_sGA are elementwise constant (and $c_f = 0$). Such an element is said to be *superconvergent*. Note that the bar element and EBE are also superconvergent elements (i.e., bar element, IIE, and EBE yield exact nodal values of the respective theories).

5.3.5 Reduced Integration Element

When equal interpolation of $w(x)$ and $\Psi(x)$ is used ($m = n$), all submatrices in (5.3.16) are of the same order: $n \times n$, where n is the number of terms in the polynomial (or $n - 1$ is the degree of interpolation). The element coefficient matrices K_{ij}^{11}, K_{ij}^{12} as well as the first part of K_{ij}^{22} must be evaluated exactly. The second part of K_{ij}^{22} is to be evaluated using reduced integration. For the choice of linear interpolation functions, and for elementwise constant values of GAK_s and EI, the matrices have the following explicit values (when $c_f^e = 0$):

$$\begin{aligned} [K^{11}] &= \frac{G_eA_eK_s}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [K^{12}] = \frac{G_eA_eK_s}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \\ [K^{22}] &= \frac{E_eI_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{G_eA_eK_sh_e}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned} \tag{5.3.29}$$

where one-point integration is used to evaluate the second part of $[K^{22}]$. Note that $[K^{11}]$, $[K^{12}]$, and the first part of $[K^{22}]$ can be evaluated exactly with one-point quadrature (i.e., numerical integration) when EI and GAK_s are constant because the integrands of these coefficients are constant. Hence, one-point integration for $[K^{\alpha\beta}]$ satisfies all the requirements. The resulting beam element is termed the reduced integration element (RIE).

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The element equations of the RIE are ($\alpha_e = 4E_eI_e/G_eA_eK_s$)

$$\frac{G_eA_eK_s}{4h_e}\begin{bmatrix} 4 & -2h_e & -4 & -2h_e \\ -2h_e & h_e^2+\alpha_e & 2h_e & h_e^2-\alpha_e \\ -4 & 2h_e & 4 & 2h_e \\ -2h_e & h_e^2-\alpha_e & 2h_e & h_e^2+\alpha_e \end{bmatrix}\begin{Bmatrix} w_1^e \\ S_1^e \\ w_2^e \\ S_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix} + \begin{Bmatrix} q_1^e \\ 0 \\ q_2^e \\ 0 \end{Bmatrix} \tag{5.3.30a}$$

or, in alternative form (to resemble those of the EBE)

$$\frac{2E_eI_e}{\mu_0 h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2(1.5+6\Lambda_e) & 3h_e & h_e^2(1.5-6\Lambda_e) \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2(1.5-6\Lambda_e) & 3h_e & h_e^2(1.5+6\Lambda_e) \end{bmatrix}\begin{Bmatrix} w_1^e \\ S_1^e \\ w_2^e \\ S_2^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ 0 \\ q_2^e \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix} \tag{5.3.30b}$$

where

$$q_i^e = \int_{x_a}^{x_b} \psi_i^e q\, dx, \quad (i = 1, 2) \tag{5.3.31}$$

$$\Lambda_e = \frac{E_eI_e}{G_eA_eK_sh_e^2}, \quad \mu_0 = 12\Lambda_e \tag{5.3.32}$$

and ψ_i are the linear interpolation functions.

It is interesting to note that the element stiffness matrix (5.3.30*b*) of the linear RIE is the same as that of the CIE-1 in (5.3.22) obtained using quadratic approximation of w and linear approximation of Ψ. The only difference is the load representation. In CIE-1, the load vector is equivalent to that of the Euler–Bernoulli beam theory, whereas in RIE it is based on (5.3.31), which contributes only to the force degrees of freedom and not to the moment degrees of freedom.

The quadratic interpolation of both w and Ψ with full integration of the element coefficient matrices also suffers slightly from the shear-locking phenomenon. A uniform two-point quadrature rule has the desired effect on $[K^{11}]$, $K^{12}]$, and $[K^{22}]$, i.e., $[K^{11}]$, $[K^{12}]$, and the first term of $[K^{22}]$ will be evaluated exactly and the second term of $[K^{22}]$ approximately. As the degree of approximation and/or the number of elements in the mesh is increased, shear locking will disappear and reduced integration is not necessary.

5.3.6 Numerical Examples

Example 5.3.1

Consider a simply supported beam under distributed transverse load of intensity q_0. The following data is used in computing the numerical values:

$$E = 10^6,\ \nu = 0.25,\ K_s = \frac{5}{6},\ q_0 = 1,\ I = \frac{bH^3}{12},\ A = bH,\ b = 1$$

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Table 5.3.1 Comparison of the finite element solutions with the exact maximum deflection and rotation of a simply supported isotropic beam of Example 5.3.1.

Element	$N^*=1$	$N=2$	$N=4$	$N=1$	$N=2$	$N=4$
	$w(L/2)\times(EH^3/q_0L^4)$			$-\Psi(0)\times 10^3$		
			Uniform load $(L/H=10)$			
RIE	0.09750	0.14438	0.15609	0.37500	0.46875	0.49219
CIE-1	0.12875	0.15219	0.15805	0.50000	0.50000	0.50000
IIE†	0.16000	0.16000	0.16000	0.50000	0.50000	0.50000
EBE†	0.15625	0.15625	0.15625	0.50000	0.50000	0.50000
			Uniform load $(L/H=100)$			
RIE	0.09379	0.14066	0.15238	0.37500	0.46875	0.49219
CIE-1	0.12504	0.14847	0.15433	0.50000	0.50000	0.50000
IIE†	0.15629	0.15629	0.15629	0.50000	0.50000	0.50000
EBE†	0.15625	0.15625	0.15625	0.50000	0.50000	0.50000
			Sinusoidal load $(L/H=100)$			
RIE	0.07639	0.11079	0.12007	0.30543	0.36702	0.38204
CIE-1	0.09679	0.11682	0.12163	0.38705	0.38702	0.38702
IIE†	0.12322	0.12322	0.12322	0.38702	0.38702	0.38702
EBE†	0.12319	0.12319	0.12319	0.38702	0.38702	0.38702

*N is the number of elements used in the half beam.
†Exact values compared to the respective beam theories.

Two different beam length-to-height ratios, $L/H=10$ and $L/H=100$, are considered. Table 5.3.1 shows a comparison of the finite element solutions obtained with one, two, and four elements in half beam with the exact beam solutions for two different types of loads, namely, uniform load and sinusoidal load. The exact solutions according to the two theories are given below [see Wang et al. (2000)].

Euler–Bernoulli Beam Theory

Uniform Load

$$w^E(x)=\frac{q_0L^4}{24EI}\left(\bar{x}-2\bar{x}^3+\bar{x}^4\right) \tag{5.3.33}$$

Sinusoidal Load

$$w^E(x)=\frac{16q_0L^4}{\pi EI}\sin\pi\bar{x},\ \bar{x}=\frac{x}{L}$$

Timoshenko Beam Theory

Uniform Load

$$w^T(x)=\left[w^E(x)+\frac{1}{GAK_s}M^E(x)\right]=\frac{q_0L^4}{24EI}\left(\bar{x}-2\bar{x}^3+\bar{x}^4\right)+\frac{q_0L^2}{2GAK_s}\left(\bar{x}-\bar{x}^3\right)$$

$$\Psi(x)=-\frac{dw^E}{dx}=-\frac{q_0L^3}{24EI}\left(1-6\bar{x}^2+4\bar{x}^3\right),\quad \bar{x}=\frac{x}{L} \tag{5.3.34}$$

Sinusoidal Load

$$w^T(x)=\left[w^E(x)+\frac{1}{GAK_s}M^E(x)\right]=\left(\frac{16q_0L^4}{\pi EI}+\frac{16\pi q_0L^2}{GAK_s}\right)\sin\pi\bar{x}$$

$$\Psi(x)=-\frac{dw^E}{dx}=-\frac{16q_0L^3}{EI}\cos\pi\bar{x},\quad \bar{x}=\frac{x}{L} \tag{5.3.35}$$

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where the superscripts E and T refer to the Euler–Bernoulli and Timoshenko beam theories, respectively. Clearly, more than two elements of CIE-1 and RIE are required to obtain acceptable solutions. On the other hand, IIE yields exact nodal values with one element (because it is a superconvergent element).

Example 5.3.2

Consider the cantilever beam of Example 5.2.1. We wish to analyze the problem using various Timoshenko beam finite elements. The problem data is given in (5.2.33); $\nu = 0.25$, $K_s = 5/6$. The exact solution of the problem according to the Timoshenko beam theory is given by [see Wang et al. (2000)]

$$w^T(x) = w^E(x) + GAK_s\left[M^E(x) - M^E(0)\right], \quad \Psi(x) = -\frac{dw^E}{dx}$$

where again the superscripts E and T refer to the Euler–Bernoulli and Timoshenko beam theories, respectively. Table 5.3.2 contains the end deflection $w(L)$ and end rotation $\Psi(L)$ for two, four, and eight elements in full beam.

To further understand the effect of shear locking, we consider Timoshenko beam elements with equal interpolation of w and Ψ. Linear as well as quadratic elements with and without reduced integration are tested and the results are included in Table 5.3.3. It is clear that the *full integration elements* (FIEs) produce erroneous results, implying that shear locking is present. Shear locking gradually vanishes as the mesh is refined with either linear elements or quadratic elements.

The following general observations can be made about various finite element models based on the Timoshenko beam theory:

1. The RIE exhibits less locking compared with the FIE.
2. As the number of elements in the mesh is increased or the degree of approximation is increased (i.e., higher-order elements are used), the finite element solutions obtained by both RIE and FIE improve; i.e., the effect of locking is reduced with mesh refinements and with the use of higher-order elements.

Table 5.3.2 Comparison of the finite element solutions obtained with various types of finite elements for the cantilever beam of Example 5.3.2.

	$w(L)$ (m)				$-\Psi(L)$			
N	EBE	RIE	CIE-1	IIE	EBE	RIE	CIE-1	IIE
Thin beam, $H = 0.0703$ (w and Ψ)								
2	0.1043	0.0995	0.0974	0.1043	0.0512	0.0524	0.0512	0.0512
4	0.1043	0.1031	0.1026	0.1043	0.0512	0.0515	0.0512	0.0512
8	0.1043	0.1040	0.1039	0.1043	0.0512	0.0513	0.0512	0.0512
Thick beam, $H = 0.3$ ($w \times 10^2$ and $\Psi \times 10^3$)								
2	0.1344	0.1293	0.1265	0.1355	0.6600	0.6750	0.6600	0.6600
4	0.1344	0.1340	0.1332	0.1355	0.6600	0.6638	0.6600	0.6600
8	0.1344	0.1351	0.1349	0.1355	0.6600	0.6609	0.6600	0.6600

† For EBE, $\Psi = \theta \equiv -dw/dx$.

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Table 5.3.3 Effect of reduced integration of transverse shear coefficients on the deflections of the cantilever beam of Example 5.3.2.

	Linear			Quadratic		
Element	$N=2$	$N=4$	$N=8$	$N=1$	$N=2$	$N=4$
Thin beam, $H=0.0703$ (exact $w=0.1043$)						
RIE $(2\times 1)^{\dagger}$	0.0995	0.1031	0.1040	0.1048	0.1043	0.1043
FIE (2×2)	0.0007	0.0027	0.0100	0.0771	0.0976	0.1028
Thick beam, $H=0.3$ (exact $w\times 10^2=0.1355$)						
RIE (3×2)	0.1293	0.1340	0.1351	0.1361	0.1355	0.1355
FIE (3×3)	0.0148	0.0442	0.0892	0.1048	0.1299	0.1348

† Gauss integration rule used to evaluate bending and shear terms.

3. The IIE (i.e., Hermite cubic approximation of w and a dependent quadratic approximation of Ψ) with full integration yields exact nodal values.
4. The element with quadratic approximation of both w and Ψ and reduced integration of the coefficients yields more accurate results than the CIE with quadratic approximation of w and the linear approximation of Ψ and with full integration of the coefficients.
5. The IIE is the best element for the analysis of bending beams as it contains the EBE as a special case; IIE automatically takes into account the transverse shear deformation if it is significant.

5.4 PLANE FRAME ELEMENTS

5.4.1 Introductory Comments

Recall that a bar element oriented arbitrarily in a plane is called a truss element. A plane truss element contains two degrees of freedom per node (u, v), an axial (u) and transverse displacement (v). By definition, a bar element can carry only axial loads and deform axially, whereas beams can take transverse loads and bending moments about an axis perpendicular to the plane of the member. A superposition of bar and beam degrees of freedom gives a finite element that is known as a frame element (see Fig. 5.4.1). Members of a frame structure are connected by rigid connections (e.g., welded or riveted), and therefore axial and transverse forces and bending moments are developed in the members. The objective of this section is to formulate a frame finite element, with the help of the developments from Sections 4.6, 5.2, and 5.3.

5.4.2 Frame Element

In many truss and frame structures, the bar and beam structural elements are found in many different orientations (see, for example, Fig. 4.6.1). Analysis of such structures for displacements and stresses requires the setting up of a global coordinate system and referencing all quantities (i.e., displacements, forces, and stiffnesses) of individual elements to the common (global) coordinate system in order to assemble the elements and impose boundary conditions on the whole structure.

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Figure 5.4.1 Superposition of bar and beam element to obtain a frame element [degrees of freedom are referred to the element coordinate system $(\bar{x}, \bar{y}, \bar{z})$].

A superposition of the bar element of Section 4.6 with the EBE of Section 5.2 or the Timoshenko beam element (RIE, CIE, or IIE) of Section 5.3 gives a frame element with three primary degrees of freedom (u, w, S) per node (note that the transverse displacement v of Section 4.6 is now denoted by w to be consistent with Sections 5.2 and 5.3). When the axial stiffness EA and bending stiffness EI are elementwise constant, the superposition of the linear bar element with the IIE gives the following element equations (see Fig. 5.4.1):

$$[\bar{K}]^e\{\bar{\Delta}\}^e = \{\bar{F}\}^e \tag{5.4.1a}$$

or, in explicit form,

$$\frac{2EI}{\mu_0 h^3}\begin{bmatrix} \mu & 0 & 0 & -\mu & 0 & 0 \\ 0 & 6 & -3h & 0 & -6 & -3h \\ 0 & -3h & h^2(1.5+6\Lambda) & 0 & 3h & h^2(1.5-6\Lambda) \\ -\mu & 0 & 0 & \mu & 0 & 0 \\ 0 & -6 & 3h & 0 & 6 & 3h \\ 0 & -3h & h^2(1.5-6\Lambda) & 0 & 3h & h^2(1.5+6\Lambda) \end{bmatrix}^e \begin{Bmatrix} \bar{u}_1 \\ \bar{w}_1 \\ \bar{S}_1 \\ \bar{u}_2 \\ \bar{w}_2 \\ \bar{S}_2 \end{Bmatrix}^e = \begin{Bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \\ \bar{F}_4 \\ \bar{F}_5 \\ \bar{F}_6 \end{Bmatrix}^e \tag{5.4.1b}$$

where

$$\begin{Bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \\ \bar{F}_4 \\ \bar{F}_5 \\ \bar{F}_6 \end{Bmatrix}^e = \begin{Bmatrix} \bar{f}_1 \\ \bar{q}_1 \\ \bar{q}_2 \\ \bar{f}_2 \\ \bar{q}_3 \\ \bar{q}_4 \end{Bmatrix}^e + \begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \\ \bar{Q}_3 \\ \bar{Q}_4 \\ \bar{Q}_5 \\ \bar{Q}_6 \end{Bmatrix}^e \tag{5.4.2a}$$

$$\bar{f}_i = \int_0^{h_e} f^e(\bar{x})\psi_i^e(\bar{x})\,d\bar{x} \quad (i=1,2), \quad \bar{q}_i = \int_0^{h_e} q^e(\bar{x})\phi_i^e(\bar{x})\,d\bar{x} \quad (i=1,2,3,4) \tag{5.4.2b}$$

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and

$$\mu = \frac{A\mu_0 h^2}{2I}, \quad \Lambda = \frac{EI}{GAK_s h^2}, \quad \mu_0 = 12\Lambda \tag{5.4.3}$$

In Eq. (5.4.2*b*), f^e denotes the distributed axial force, q^e the distributed transverse force, ψ_i^e the linear interpolation functions, and ϕ_i^e the Hermite cubic interpolation functions. In the following paragraphs, we develop transformation relations to express the element equations (5.4.1*b*)—valid in the element coordinate system $(\bar{x}, \bar{y}, \bar{z})$—to the global coordinate system (x, y, z).

The local coordinates $(\bar{x}_e, \bar{y}_e, \bar{z}_e)$ of a typical element Ω_e are related to the global coordinates (x, y, z) by [cf. Eq. (4.6.2)]

$$\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix}^e = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}^e \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{5.4.4}$$

where the angle α_e is measured clockwise from the global x-axis to the element $\bar{x}_e$-axis. Note that the y and $\bar{y}_e$ coordinates are parallel to each other, and they are *out of* the plane of the paper (see Fig. 5.4.2). The same transformation relations hold for displacements (u, w) along the global coordinates (x, z) and displacements $(\bar{u}, \bar{w})$ in the local coordinates

Figure 5.4.2 (*a*) Generalized displacements. (*b*) Generalized forces. (*c*) Generalized displacements in the element coordinates. (*d*) Generalized displacements in the global coordinates.

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$(\bar{x}, \bar{z})$. Note that there is no displacement in the direction of the coordinate y (i.e., $v = 0$). However, there is a rotation about the y-axis, and it remains the same in both coordinate systems because $y = \bar{y}$. Note that rotation θ is equal to $-dw/dx$ in Euler–Bernoulli beam theory and it is equal to Ψ in Timoshenko beam theory. Hence, the relationship between (u, w, θ) and $(\bar{u}, \bar{w}, \bar{\theta})$ can be written as

$$\begin{Bmatrix} \bar{u} \\ \bar{w} \\ \bar{\theta} \end{Bmatrix}^e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^e \begin{Bmatrix} u \\ w \\ \theta \end{Bmatrix}^e \tag{5.4.5}$$

Therefore, the three nodal degrees of freedom $(\bar{u}_i^e, \bar{w}_i^e, \bar{S}_i^e)$ at the ith node $(i = 1, 2)$ in the $(\bar{x}, \bar{y}, \bar{z})$ system are related to the three degrees of freedom (u_i^e, w_i^e, S_i^e) in the (x, y, z) system by

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{w}_1 \\ \bar{S}_1 \\ \bar{u}_2 \\ \bar{w}_2 \\ \bar{S}_2 \end{Bmatrix}^e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & & & \\ -\sin\alpha & \cos\alpha & 0 & & \mathbf{0} & \\ 0 & 0 & 1 & & & \\ & & & \cos\alpha & \sin\alpha & 0 \\ & \mathbf{0} & & -\sin\alpha & \cos\alpha & 0 \\ & & & 0 & 0 & 1 \end{bmatrix}^e \begin{Bmatrix} u_1 \\ w_1 \\ S_1 \\ u_2 \\ w_2 \\ S_2 \end{Bmatrix}^e \tag{5.4.6a}$$

or

$$\{\bar{\Delta}^e\} = [T^e]\{\Delta^e\} \tag{5.4.6b}$$

Analogously, the element force vectors in the local and global coordinate systems are related according to

$$\{\bar{F}\}^e = [T]^e\{F\}^e \tag{5.4.7}$$

Returning to Eq. (5.4.1a), we substitute the transformation equations (5.4.6b) and (5.4.7) into (5.4.1a) and obtain

$$[\bar{K}]^e[T]^e\{\Delta\}^e = [T]^e\{F\}^e$$

Premultiplying both sides with $[T]^{-1} = [T]^{\mathrm{T}}$, we obtain

$$[T]^{\mathrm{T}}[\bar{K}]^e[T]^e\{\Delta\}^e = \{F\}^e \quad \text{or} \quad [K]^e\{\Delta\}^e = \{F\}^e \tag{5.4.8}$$

where

$$[K^e] = [T]^{\mathrm{T}}[\bar{K}]^e[T]^e, \quad \{F\}^e = [T]^{\mathrm{T}}\{\bar{F}\}^e \tag{5.4.9}$$

Thus, if we know the element matrices $[\bar{K}]^e$ and $\{\bar{F}\}^e$ of an element Ω_e in the local coordinate system $(\bar{x}, \bar{y}, \bar{z})$, the element matrices in the global coordinate system are obtained by (5.4.9).

Using $[\bar{K}]^e$ and $\{\bar{F}\}^e$ from Eq. (5.4.1b) in (5.4.9) and carrying out the indicated matrix multiplications, we arrive at the following element stiffness matrix $[K^e]$ referred to the

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global coordinates:

$$[K]^e = \frac{2EI}{\mu_0 h^3}\begin{bmatrix} \mu\cos^2\alpha + 6\sin^2\alpha & (\mu-6)\cos\alpha\sin\alpha & 3h\sin\alpha & -(\mu\cos^2\alpha + 6\sin^2\alpha) & -(\mu-6)\cos\alpha\sin\alpha & 3h\sin\alpha \\ (\mu-6)\cos\alpha\sin\alpha & \mu\sin^2\alpha + 6\cos^2\alpha & -3h\cos\alpha & -(\mu-6)\sin\alpha\cos\alpha & -(\mu\sin^2\alpha + 6\cos^2\alpha) & -3h\cos\alpha \\ 3h\sin\alpha & -3h\cos\alpha & h^2(1.5+6\Lambda) & -3h\sin\alpha & 3h\cos\alpha & h^2(1.5-6\Lambda) \\ -(\mu\cos^2\alpha + 6\sin^2\alpha) & -(\mu-6)\sin\alpha\cos\alpha & -3h\sin\alpha & (\mu\cos^2\alpha + 6\sin^2\alpha) & (\mu-6)\cos\alpha\sin\alpha & -3h\sin\alpha \\ -(\mu-6)\cos\alpha\sin\alpha & -(\mu\sin^2\alpha + 6\cos^2\alpha) & 3h\cos\alpha & (\mu-6)\cos\alpha\sin\alpha & \mu\sin^2\alpha + 6\cos^2\alpha & 3h\cos\alpha \\ 3h\sin\alpha & -3h\cos\alpha & h^2(1.5-6\Lambda) & -3h\sin\alpha & 3h\cos\alpha & h^2(1.5+6\Lambda) \end{bmatrix} \tag{5.4.10a}$$

$$\{F\}^e = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}^e = \begin{Bmatrix} \bar{F}_1\cos\alpha - \bar{F}_2\sin\alpha \\ \bar{F}_1\sin\alpha + \bar{F}_2\cos\alpha \\ \bar{F}_3 \\ \bar{F}_4\cos\alpha - \bar{F}_5\sin\alpha \\ \bar{F}_4\sin\alpha + \bar{F}_5\cos\alpha \\ \bar{F}_6 \end{Bmatrix}^e \tag{5.4.10b}$$

which is the element force vector referred to the global coordinates.

Example 5.4.1

The frame structure shown in Fig. 5.4.3 is to be analyzed for displacements and forces. Both members of the structure are made of the same material (E) and have the same geometric properties (A, I). The element stiffness matrices and force vectors in the global coordinate system (x, y, z) can be computed from (5.4.10a) and (5.4.10b). The geometric and material properties of each element are as follows (f is the axial and q is the transverse distributed load).

Element 1.

$$L = 144 \text{ in.}, \quad A = 10 \text{ in.}^2, \quad I = 10 \text{ in.}^2, \quad \cos\alpha_1 = 0.0, \quad \sin\alpha_1 = -1.0$$
$$E = 10^6 \text{ psi}, \quad f^{(1)} = 0, \quad q^{(1)} = \frac{P}{72} \text{ lb/in.} \tag{5.4.11}$$

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Figure 5.4.3 (*a*) Geometry and loading, and (*b*) member forces in the plane frame structure of Example 5.4.1.

$$[K^1] = 10^5 \begin{bmatrix} 0.0004 & 0.0000 & -0.0289 & -0.0004 & 0.0000 & -0.0289 \\ 0.0000 & 0.6944 & 0.0000 & 0.0000 & -0.6944 & 0.0000 \\ -0.0289 & 0.0000 & 2.7778 & 0.0289 & 0.0000 & 1.3889 \\ -0.0004 & 0.0000 & 0.0289 & 0.0004 & 0.0000 & 0.0289 \\ 0.0000 & -0.6944 & 0.0000 & 0.0000 & 0.6944 & 0.0000 \\ -0.0289 & 0.0000 & 1.3889 & 0.0289 & 0.0000 & 2.7778 \end{bmatrix}$$

$$\{f^1\} = P\{1.0 \quad 0.0 \quad -24.0 \quad 1.0 \quad 0.0 \quad 24.0\}^{\mathrm{T}}$$

Element 2.

$$L = 180 \text{ in.}, \quad A = 10 \text{ in.}^2, \quad I = 10 \text{ in.}^2, \quad \cos\alpha_2 = 0.8, \quad \sin\alpha_2 = -0.6$$
$$E = 10^6 \text{ psi}, \quad f^{(2)} = 0, \quad q^{(2)} = 0 \tag{5.4.12}$$

The load $F_0 = 4P$ at the center of the element ($\cos\alpha = -0.6$ and $\sin\alpha = 0.8$) is distributed to the nodes according to Eq. (5.2.20).

$$[K^2] = 10^5 \begin{bmatrix} 0.3556 & -0.2666 & -0.0111 & -0.3556 & 0.2666 & -0.0111 \\ -0.2666 & 0.2001 & -0.0148 & 0.2666 & -0.2001 & -0.0148 \\ -0.0111 & 0.0148 & 2.2222 & 0.0111 & 0.0148 & 1.1111 \\ -0.3556 & 0.2666 & 0.0111 & 0.3556 & -0.2666 & 0.0111 \\ 0.2666 & -0.2001 & 0.0148 & -0.2666 & 0.2001 & 0.0148 \\ -0.0111 & -0.0148 & 1.1111 & 0.0111 & 0.0148 & 2.2222 \end{bmatrix}$$

$$\{f^2\} = P\{0.0 \quad 2.0 \quad -72.0 \quad 0.0 \quad 2.0 \quad 72.0\}^{\mathrm{T}}$$

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The assembled stiffness matrix and force vectors are obtained by superposing the last three rows and columns of element 1 to the first three rows and columns of element 2, i.e., the 3×3 submatrix associated with rows and columns 4, 5, and 6 of element 1, and the 3×3 submatrix associated with rows and columns 1, 2, and 3 of element 2 overlap in the global stiffness matrix.

The known geometric boundary conditions are

$$U_1 = 0, \quad U_2 = 0, \quad U_3 = 0, \quad U_7 = 0, \quad U_8 = 0, \quad U_9 = 0 \tag{5.4.13a}$$

The force boundary conditions are

$$Q_4^1 + Q_1^2 = 0, \quad Q_5^1 + Q_2^2 = 2P, \quad Q_6^1 + Q_3^2 = 0 \tag{5.4.13b}$$

Since all specified values of the known boundary conditions on the primary variables are zero, the condensed equations for the unknown generalized displacement degrees of freedom are

$$10^5 \begin{bmatrix} 0.3560 & -0.2666 & 0.0178 \\ -0.2666 & 0.8946 & -0.0148 \\ 0.0178 & -0.0148 & 5.0000 \end{bmatrix} \begin{Bmatrix} U_4 \\ U_5 \\ U_6 \end{Bmatrix} = P \begin{Bmatrix} 1.0 \\ 2.0 \\ -4.8 \end{Bmatrix} \tag{5.4.14}$$

The solution is

$$U_4 = 0.8390 \times 10^{-4} P \text{ (in.)}, \quad U_5 = 0.6812 \times 10^{-4} P \text{ (in.)}, \quad U_6 = -0.9610 \times 10^{-4} P \text{ (rad.)} \tag{5.4.15}$$

The reactions and forces in each member in the global coordinates can be computed from the element equations

$$\{Q^e\} = [K^e]\{u^e\} - \{f^e\} \tag{5.4.16a}$$

The forces $\{Q^e\}$ can be transformed to those in the element coordinate system by means of (5.4.7):

$$\{\bar{Q}^e\} = [T^e]\{Q^e\} \tag{5.4.16b}$$

We obtain

$$\{\bar{Q}^1\} = \begin{Bmatrix} 4.731 \\ -0.725 \\ 10.900 \\ -4.731 \\ -1.275 \\ -50.450 \end{Bmatrix} P, \quad \{\bar{Q}^2\} = \begin{Bmatrix} 2.658 \\ -1.420 \\ 50.45 \\ -0.258 \\ -1.780 \\ -82.87 \end{Bmatrix} P \tag{5.4.17}$$

Table 5.4.1 contains the displacements obtained by various types of elements at point B. As noted earlier, one EBE or IIE per member of a structure gives exact displacements, whereas at least two RIE or CIE per member are needed to obtain acceptable results. The forces in each element are included in Table 5.4.2. The forces calculated from the element equations are also exact for EBE and IIE.

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Table 5.4.1 Comparison of the generalized displacements [$\bar{v} = (v/P) \times 10^4$ where v is a typical displacement] at point B of the frame structure shown in Fig. 5.4.3.

Displ.	RIE(1)*	RIE (2)	RIE (4)	CIE (1)	CIE (2)	CIE (4)	IIE†	EBE†
$\bar{u}_B$	0.27092	0.84765	0.84107	0.28435	0.84146	0.83959	0.83898	0.83904
$\bar{w}_B$	0.46609	0.68056	0.68108	0.44315	0.68083	0.68114	0.68123	0.68124
$-\bar{\phi}_B$	−0.0016	0.86647	0.94501	0.00036	0.77029	0.91640	0.96206	0.96098

*Number in the parenthesis denotes the number of elements per member.
†Values are independent of the number of elements (and coincide with the exact values predicated by the respective beam theories).

Table 5.4.2 Comparison of the generalized forces (divided by P) in each member of the frame structure.

Element*	$\bar{Q}_1$	$-\bar{Q}_2$	$\bar{Q}_3$	$-\bar{Q}_4$	$-\bar{Q}_5$	$-\bar{Q}_6$
RIE(1)	3.237	1.865	62.24	3.237	0.136	62.26
	0.850	0.908	62.26	1.550	2.292	62.28
RIE(2)	4.723	0.671	0.332	4.723	1.329	47.70
	2.699	1.384	47.70	0.299	1.816	86.67
RIE(4)	4.730	0.713	8.362	4.730	1.288	49.76
	2.668	1.411	49.76	0.268	1.789	83.74
CIE(1)	3.077	1.575	65.39	3.077	0.425	17.38
	0.987	0.607	17.38	1.413	2.593	161.4
CIE(2)	4.728	0.708	8.327	4.728	1.292	50.37
	2.670	1.407	50.37	0.270	1.793	85.07
CIE(4)	4.730	0.721	10.30	4.730	1.279	50.43
	2.661	1.417	50.43	0.261	1.783	83.39
IIE†	4.731	0.725	10.92	4.731	1.275	50.45
	2.658	1.420	50.45	0.258	1.780	82.87
EBE†	4.731	0.725	10.90	4.731	1.275	50.45
	2.658	1.420	50.45	0.258	1.780	82.87

*Number in the parenthesis denotes the number of elements per member, and the two rows correspond to the two members of the structure.
† Values are independent of the number of elements (and coincide with the exact values predicated by the respective beam theories).

5.5 SUMMARY

In this chapter, finite element models of the classical (i.e., Euler–Bernoulli) and Timoshenko beam theories have been developed. The classical beam theory is governed by a fourth-order differential equation and therefore results in a weak form whose primary variables contain the transverse deflection and its first derivative. Therefore, Hermite interpolation of the transverse deflection is required in order to impose the continuity of the deflection and its derivative at the nodes between elements. In the case of the Timoshenko beam theory, there

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are two, coupled, second-order equations governing the transverse deflection and the rotation. The weak forms of these equations require Lagrange interpolation of the transverse deflection and rotation. Since the rotation function is like the (negative of) the derivative of the transverse deflection, the degree of the interpolation used for the rotation should be one less than that used for the transverse deflection. Such selective interpolation of the variables is called *consistent interpolation*. When the same interpolation functions are used to approximate the transverse deflection and the rotation, the resulting stiffness matrix is often too stiff to yield good solutions—especially when the number of elements used is small. This is due to the inconsistency of interpolation of the variables, and the phenomenon is known as *shear locking*. It is overcome by the use of reduced integration to evaluate the stiffness coefficients associated with transverse shear strains. The elements developed here are RIE, CIE-1, which has quadratic interpolation of the transverse deflection and linear interpolation of the rotation, and IIE.

The plane frame element based on the unified beam element (i.e., IIE) that contains the classical beam theory and Timoshenko beam theory have also been discussed. The frame element is a superposition of the beam and bar elements and has three degrees of freedom (axial displacement, transverse deflection, and rotation about an axis perpendicular to the plane of axial and transverse coordinates) per node. The general plane frame element is oriented at an angle from the horizontal position, and its equations are obtained by transforming the equations of the frame element in local coordinates.

PROBLEMS

5.1 The natural vibration of a beam under applied axial compressive load N_0 is governed by the differential equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+N_0\frac{d^2w}{dx^2}=\rho A\omega^2 w$$

where ω denotes nondimensional frequency of natural vibration, EI is the bending stiffness, and ρA is the mass (mass density times cross-sectional area) of the beam. Develop (*a*) the weak form and (*b*) finite element model of the equation.

5.2 The differential equation governing axisymmetric bending of circular plates on elastic foundation is given by

$$-\frac{1}{r}\frac{d}{dr}\left[\frac{d}{dr}(rM_{rr})-M_{\theta\theta}\right]+kw=q(r)$$

where k is the modulus of the elastic foundation, q is the transverse distributed load, and

$$M_{rr}=-D\left(\frac{d^2w}{dr^2}+\nu\frac{1}{r}\frac{dw}{dr}\right),\quad M_{\theta\theta}=-D\left(\nu\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)$$

Develop (*a*) the weak form and identify the primary and secondary variables, and (*b*) the finite element model. Note that the shear force is defined by

$$Q_r=\frac{1}{r}\left[\frac{d}{dr}(rM_{rr})-M_{\theta\theta}\right]$$

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5.3 The differential equations governing axisymmetric bending of circular plates according to the shear deformation plate theory are

$$-\frac{1}{r}\frac{d}{dr}(rQ_r) - q = 0 \tag{1}$$

$$-\frac{1}{r}\left[\frac{d}{dr}(rM_{rr}) - M_{\theta\theta}\right] + Q_r = 0 \tag{2}$$

where

$$M_{rr} = D\left(\frac{d\Psi}{dr} + \upsilon\frac{\Psi}{r}\right), \; M_{\theta\theta} = D\left(\upsilon\frac{d\Psi}{dr} + \frac{\Psi}{r}\right)$$

$$Q_r = K_s GH\left(\Psi + \frac{dw}{dr}\right)$$

where $D = EH^3/[12(1-\nu^2)]$ and H is the plate thickness. Develop (*a*) the weak form of the equations over an element and (*b*) the finite element model of the equations.

5.4 Consider the fourth-order equation (5.2.1) and its weak form (5.2.4). Suppose that a two-node element is employed, with *three* primary variables at each node: w, θ, and κ, where $\theta = dw/dx$ and $\kappa = d^2w/dx^2$. Show that the associated Hermite interpolation functions are given by

$$\phi_1 = 1 - 10\frac{\bar{x}^3}{h^3} + 15\frac{\bar{x}^4}{h^4} - 6\frac{\bar{x}^5}{h^5}, \qquad \phi_2 = \bar{x}\left(1 - 6\frac{\bar{x}^2}{h^2} + 8\frac{\bar{x}^3}{h^3} - 3\frac{\bar{x}^4}{h^4}\right)$$

$$\phi_3 = \frac{\bar{x}^2}{2}\left(1 - 3\frac{\bar{x}}{h} + 3\frac{\bar{x}^2}{h^2} - \frac{\bar{x}^3}{h^3}\right), \qquad \phi_4 = 10\frac{\bar{x}^3}{h^3} - 15\frac{\bar{x}^4}{h^4} + 6\frac{\bar{x}^5}{h^5}$$

$$\phi_5 = -\bar{x}\left(4\frac{\bar{x}^2}{h^2} - 7\frac{\bar{x}^3}{h^3} + 3\frac{\bar{x}^4}{h^4}\right), \qquad \phi_6 = \frac{\bar{x}^2}{2}\left(\frac{\bar{x}}{h} - 2\frac{\bar{x}^2}{h^2} + \frac{\bar{x}^3}{h^3}\right)$$

where $\bar{x}$ is the element coordinate with the origin at node 1.

5.5 Consider the weak form (5.2.4) of the EBE. Use a three-node element with two degrees of freedom (w, θ), where $\theta \equiv -dw/dx$. Derive the Hermite interpolation functions for the element. Compute the element stiffness matrix and force vector. *Partial answer:*

$$\phi_1 = 1 - 23\frac{\bar{x}^2}{h^2} + 66\frac{\bar{x}^3}{h^3} - 68\frac{\bar{x}^4}{h^4} + 24\frac{\bar{x}^5}{h^5}$$

5.6–5.20 Use the minimum number of Euler–Bernoulli beam finite elements to analyze the beam problems shown in Figs. P5.6–P5.20. In particular, give:

(a) The assembled stiffness matrix and force vector.

(b) The specified global displacements and forces, and the equilibrium conditions.

(c) The condensed matrix equations for the primary unknowns (i.e., generalized forces) separately. Exploit symmetries, if any, in analyzing the problems. The positive convention used for the generalized displacements and forces is the same as that given in Fig. 5.2.2.

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Figure P5.6

Figure P5.7

Figure P5.8

Figure P5.9

Figure P5.10

Figure P5.11

Figure P5.12

Figure P5.13

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Figure P5.14

Figure P5.15

Figure P5.16

Figure P5.17

Figure P5.18

Figure P5.19

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Figure P5.20

Figure P5.24

Figure P5.28

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5.21 Analyze Problem 5.8 using the reduced-integration Timoshenko beam finite element (i.e., RIE). Use a value of $\frac{5}{6}$ for the shear correction factor and $\nu = 0.25$.

5.22 Analyze Problem 5.8 using the consistent interpolation (quadratic w and linear Ψ) Timoshenko beam element (i.e., CIE-1). Use a value of $\frac{5}{6}$ for the shear correction factor and $\nu = 0.25$.

5.23 Analyze Problem 5.8 using the consistent interpolation (cubic w and quadratic Ψ) Timoshenko beam element (i.e., CIE-2). Use a value of $\frac{5}{6}$ for the shear correction factor and $\nu = 0.25$.

5.24 Analyze the problem in Fig. P5.24 using the consistent interpolation (quadratic w and linear Ψ) Timoshenko beam element (i.e., CIE-1). Use a value of $\frac{5}{6}$ for the shear correction factor and $\nu = 0.25$.

5.25 Analyze the problem in Fig. P5.24 using the consistent interpolation (cubic w and quadratic Ψ) Timoshenko beam element (i.e., CIE-2). Use a value of $\frac{5}{6}$ for the shear correction factor and $\nu = 0.25$.

5.26 Consider a thin isotropic circular plate of radius R_0 and suppose that the plate is clamped at $r = R_0$. If two finite elements (see Problem 5.2) are used in the domain $(0 \le r \le R_0)$, give the boundary conditions on the primary and secondary variables of the mesh if the plate is subjected to (*a*) a uniformly distributed transverse load of intensity q_0 and (*b*) point load Q_0 at the center.

5.27 Repeat the circular plate problem of Problem 5.26 when a two-element mesh of Timoshenko elements is used.

5.28–5.35 For the frame problems shown in Figs. P5.28–P5.35, give (*a*) the transformed element matrices; (*b*) the assembled element matrices; and (*c*) the condensed matrix equations for the unknown generalized displacements and forces. Use the sign conventions shown in the figure below for global and element displacement (and force) degrees of freedom. The angle between the +ve x-axis and +ve $\bar{x}$-axis is measured in the clockwise sense.

EI, *EA* are the same for the two members
$E = 30 \times 10^6$ lb/in.^{2}, $\nu = 0.3$
$A = 100$ in.^{2}, $I = 100$ in.^{4}

Figure P5.29

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Figure P5.30

Figure P5.31

Figure P5.32

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Figure P5.33

Figure P5.34

Figure P5.35

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Chapter 6

EIGENVALUE AND TIME-DEPENDENT PROBLEMS

6.1 EIGENVALUE PROBLEMS

6.1.1 Introduction

An *eigenvalue problem* is defined to be one in which we seek the values of the parameter λ such that the equation

$$A(u) = \lambda B(u) \tag{6.1.1}$$

is satisfied for nontrivial values of u. Here A and B denote either matrix operators or differential operators, and values of λ for which Eq. (6.1.1) is satisfied are called *eigenvalues*. For each value of λ there is a vector u, called an *eigenvector* or *eigenfunction*. For example, the equation

$$-\frac{d^2u}{dx^2} = \lambda u(x), \quad \text{with } A = -\frac{d^2}{dx^2}, \quad B = 1$$

which arises in connection with natural axial vibrations of a bar or the transverse vibration of a cable, constitutes an eigenvalue problem. Here λ denotes the square of the frequency of vibration, ω.

In general, the determination of the eigenvalues is of engineering as well as mathematical importance. In structural problems, the eigenvalues denote either natural frequencies or buckling loads. In fluid mechanics and heat transfer, eigenvalue problems arise in connection with the determination of the homogeneous parts of the transient solution, as will be shown shortly. In these cases, eigenvalues often denote amplitudes of the Fourier components making up the solution. Eigenvalues are also useful in determining the stability characteristics of temporal schemes, as discussed in Section 6.2.

In this section, we develop finite element models of eigenvalue problems described by differential equations. In view of the close similarity between the differential equations governing eigenvalue and boundary value problems, the steps involved in the construction of their finite element models are entirely analogous. Differential eigenvalue problems are reduced to algebraic eigenvalue problems (i.e., $[A]\{X\} = \lambda[B]\{X\}$) by means of the finite element approximation. The methods of solution of algebraic eigenvalue problems are then used to solve for the eigenvalues and eigenvectors.

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6.1.2 Formulation of Eigenvalue Problems

Parabolic Equation

Consider the partial differential equation

$$\rho c A \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(kA\frac{\partial u}{\partial x}\right) = q(x,t) \tag{6.1.2}$$

which arises in connection with transient heat transfer in one-dimensional systems (e.g., a plane wall or a fin). Here u denotes the temperature, k the thermal conductivity, ρ the density, A the cross-sectional area, c the specific heat, and q the heat generation per unit length. Equations involving the first-order time derivative are called *parabolic equations.*

The homogenous solution (i.e., the solution when $q = 0$) of Eq. (6.1.2) is often sought in the form of a product of a function of x and a function of t (i.e., through the *separation-of-variables* technique):

$$u^h(x,t) = U(x)T(t) \tag{6.1.3}$$

Substitution of this assumed form of solution into the homogeneous form of (6.1.2) gives

$$\rho c A U \frac{dT}{dt} - \frac{d}{dx}\left(kA\frac{dU}{dx}\right)T = 0$$

Separating variables of t and x (assuming that $\rho c A$ and kA are functions of x only), we arrive at

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{\rho c A}\frac{1}{U}\frac{d}{dx}\left(kA\frac{dU}{dx}\right) \tag{6.1.4}$$

Note that the left-hand side of this equation is a function of t only while the right-hand side is a function of x only. For two functions of two independent variables to be equal for all values of the independent variables, both functions must be equal to the same constant, say $-\lambda$ ($\lambda > 0$):

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{\rho c A}\frac{1}{U}\frac{d}{dx}\left(kA\frac{dU}{dx}\right) = -\lambda$$

or

$$\frac{dT}{dt} = -\lambda T \tag{6.1.5}$$

$$-\frac{d}{dx}\left(kA\frac{dU}{dx}\right) - \lambda \rho c A U = 0 \tag{6.1.6}$$

The negative sign of the constant λ is based on the physical requirement that the solution $U(x)$ be harmonic in x while $T(t)$ decay exponentially with increasing t.

The solution of (6.1.5) is

$$T(t) = K e^{-\lambda t} \tag{6.1.7}$$

where K is a constant of integration. The values of λ are determined by solving (6.1.6), which also gives $U(x)$. With $T(t)$ and $U(x)$ known, we have the complete homogeneous solution (6.1.3) of Eq. (6.1.2). The problem of solving (6.1.6) for λ and $U(x)$ is termed an *eigenvalue problem,* and λ is called the eigenvalue and $U(x)$ the eigenfunction. When k,

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A, ρ, and c are constants, the solution of Eq. (6.1.6) is

$$U(x) = C \sin \alpha x + D \cos \alpha x, \quad \alpha^2 = \frac{\rho c}{k}\lambda \tag{6.1.8}$$

where C and D are constants of integration. Boundary conditions of the problem are used to find algebraic relations among C and D. The algebraic relations can be expressed in matrix form as $([A] - \alpha[B])\{V\} = \{0\}$, where $[A]$ and $[B]$ depend, in general, on k, c, and ρ, and $\{V\}$ is the vector of integration constants C and D. Since C and D both cannot be zero (otherwise, we obtain the trivial solution), we require the determinant of the coefficient matrix $[A] - \alpha[B]$ to be zero. This results in an algebraic eigenvalue problem whose solution yields α (hence, λ) and $\{V\} = (C, D)$.

To fix the ideas, consider Eq. (6.1.6) subject to the boundary conditions (e.g., a fin with specified temperature at $x = 0$ and insulated at $x = L$)

$$U(0) = 0, \quad \left[kA\frac{dU}{dx}\right]_{x=L} = 0 \tag{6.1.9}$$

We note that nonhomogeneous boundary conditions can be converted to homogeneous boundary conditions by a change of variables. Using the above boundary conditions in (6.1.8), we obtain

$$0 = C \cdot 0 + D \cdot 1, \quad 0 = \alpha\,(C \cos \alpha L - D \sin \alpha L)$$

or

$$\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \alpha \begin{bmatrix} 0 & 0 \\ \cos \alpha L & -\sin \alpha L \end{bmatrix}\right) \begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.1.10}$$

For nontrivial solution (i.e., not both C and D are equal to zero), we set the determinant of the coefficient matrix in (6.1.10) to zero and obtain (since α cannot be zero)

$$\cos \alpha L = 0 \;\rightarrow\; \alpha_n L = \frac{(2n-1)\pi}{2}$$

Hence, the homogeneous solution becomes [note that the constant K of Eq. (6.1.7) is absorbed into C_n]

$$u^h(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} \sin \alpha_n x, \quad \lambda_n = \alpha_n^2 \left(\frac{k}{\rho c}\right), \quad \alpha_n = \frac{(2n-1)\pi}{2L} \tag{6.1.11}$$

The constants C_n are determined using the initial condition of the problem, $u(x, 0) = u_0(x)$:

$$u^h(x, 0) = \sum_{n=1}^{\infty} C_n \sin \alpha_n x = u_0(x)$$

Multiplying both sides with $\sin \alpha_m x$, integrating over the interval $(0, L)$, and making use of the orthogonality condition

$$\int_0^L \sin \alpha_n x \, \sin \alpha_m x \, dx = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{L}{2}, & \text{if } m = n \end{cases} \tag{6.1.12}$$

we obtain

$$C_n = \frac{2}{L}\int_0^L u_0(x) \sin \alpha_n x \, dx \tag{6.1.13}$$

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The complete solution of Eq. (6.1.2) is given by the sum of the homogeneous solution and the particular solution $u(x,t) = u^h(x,t) + u^p(x,t)$.

The example discussed above provides the need for determining eigenvalues (α_n) in the context of finding the transient response of a parabolic equation. Next, we consider the transient response of a second-order equation in time, known as a *hyperbolic equation*.

Hyperbolic Equation

The axial motion of a bar, for example, is described by the equation [see Reddy (2002), pp. 185–187]

$$\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = f(x,t) \tag{6.1.14}$$

where u denotes the axial displacement, E the modulus of elasticity, A the cross-sectional area, ρ the density, and f the axial force per unit length. The solution of (6.1.14) consists of two parts: homogeneous solution u^h (i.e., when $f = 0$) and particular solution u^p. The homogeneous part is determined by the separation-of-variables technique, as we discussed for the parabolic equation.

The homogenous solution of Eq. (6.1.14) is also assumed to be of the form in (6.1.3). Substitution of (6.1.3) into the homogeneous form of (6.1.14) gives

$$\rho AU\frac{d^2T}{dt^2} - \frac{d}{dx}\left(EA\frac{dU}{dx}\right)T = 0$$

Assuming that ρA and EA are functions of x only, we arrive at

$$\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{\rho A}\frac{1}{U}\frac{d}{dx}\left(kA\frac{dU}{dx}\right) = -\alpha^2$$

or

$$\frac{d^2T}{dt^2} + \alpha^2 T = 0 \tag{6.1.15}$$

$$-\frac{d}{dx}\left(EA\frac{dU}{dx}\right) - \alpha^2\rho AU = 0 \tag{6.1.16}$$

The negative sign of the constant α is based on the physical requirement that the solution $u(x,t)$ be harmonic in x and t.

The solution of (6.1.15) is

$$T(t) = Ke^{-i\alpha t} = K_1\cos\alpha t + K_2\sin\alpha t \tag{6.1.17}$$

where K_1 and K_2 are constants of integration. The solution of (6.1.16), when E, A, and ρ, are constants, is

$$U(x) = C\sin\bar{\alpha}x + D\cos\bar{\alpha}x, \quad \bar{\alpha}^2 = \frac{\rho}{E}\alpha^2 \tag{6.1.18}$$

where again C and D are constants of integration. In the process of determining the constants C and D using the boundary conditions of the problem, once again we are required to solve an eigenvalue problem (the steps are analogous to those described for a parabolic equation). Unlike in the case of heat conduction problem, the quantities α_n in the case of bars have direct physical meaning, namely, they represent natural frequencies of the system. Thus, in

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structural mechanics we may be interested in determining only the natural frequencies of the system but not the transient response.

When we are interested only in natural frequencies of the system, the eigenvalue problem may be formulated directly from the equation of motion (6.1.14) by assuming a solution form that is periodic in time t

$$u(x,t) = U(x)e^{-i\omega t} \tag{6.1.19}$$

where ω denotes the frequency of natural vibration, i equals $\sqrt{-1}$, and $U(x)$ denotes the configuration of the structure at that frequency, called the *mode shape* (i.e., for each value of ω, there is an associated mode shape). Substitution of (6.1.19) into the homogeneous form of (6.1.14) gives

$$\left[-\rho A\omega^2 U - \frac{d}{dx}\left(EA\frac{dU}{dx}\right)\right]e^{-i\omega t} = 0$$

or

$$-\frac{d}{dx}\left(EA\frac{dU}{dx}\right) - \omega^2 \rho A U = 0$$

which is identical to Eq. (6.1.16) with $\alpha = \omega$. Similar equations can be formulated for natural vibration of beams. Another eigenvalue problem that arises directly from the governing equilibrium equation is that of buckling of beam-columns. We shall return to this topic shortly.

In summary, eigenvalue problems associated with parabolic equations are obtained from the corresponding equations of motion by assuming solution of the form

$$u(x,t) = U(x)e^{-\alpha t}, \quad \lambda = \alpha \tag{6.1.20a}$$

whereas those associated with hyperbolic equations are obtained by assuming solution of the form

$$u(x,t) = U(x)e^{-i\omega t}, \quad \lambda = \omega^2 \tag{6.1.20b}$$

where λ denotes the eigenvalue. Finite element formulations of both problems are presented next.

6.1.3 Finite Element Formulation

Comparison of Eqs. (6.1.6) and (6.1.16) with the model equation (3.2.1) reveals that the equations governing eigenvalue problems are special cases of the model equations studied in Chapters 3 and 5. Here we summarize the steps in the finite element formulation of eigenvalue problems for the sake of completeness and ready reference. We will consider eigenvalue problems described by (*a*) a single equation in a single unknown (e.g., heat transfer, bar, and Euler–Bernoulli beam problems), and (*b*) a pair of equations in two variables (e.g., Timoshenko beam theory).

Heat Transfer and Bar-Like Problems

Consider the problem of solving the equation

$$-\frac{d}{dx}\left[a(x)\frac{dU}{dx}\right] + c(x)U(x) = \lambda c_0(x)U(x) \tag{6.1.21}$$

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for λ and $U(x)$. Here a, c, and c_0 are known quantities that depend on the physical problem (i.e., data), λ is the eigenvalue, and U is the eigenfunction. Special cases of Eq. (6.1.21) are given below.

$$\text{Heat transfer:} \quad a = kA, \quad c = P\beta, \quad c_0 = \rho c A \tag{6.1.22}$$

$$\text{Bars:} \quad a = EA, \quad c = 0, \quad c_0 = \rho A \tag{6.1.23}$$

Over a typical element Ω_e, we seek a finite element approximation of U in the form

$$U_h^e(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{6.1.24}$$

The weak form of (6.1.21) is

$$0 = \int_{x_a}^{x_b} \left(a \frac{dw}{dx}\frac{dU}{dx} + cwU(x) - \lambda c_0 wU \right) dx - Q_1^e w(x_a) - Q_n^e w(x_b) \tag{6.1.25}$$

where w is the weight function, and Q_1^e and Q_n^e are the secondary variables at node 1 and node n, respectively (assume that $Q_i^e = 0$ when $i \neq 1$ and $i \neq n$)

$$Q_1^e = -\left[a \frac{dU}{dx} \right]_{x_a}, \quad Q_n^e = \left[a \frac{dU}{dx} \right]_{x_b} \tag{6.1.26}$$

Substitution of the finite element approximation into the weak form gives the finite element model of the eigenvalue equation (6.1.21):

$$[K^e]\{u^e\} - \lambda [M^e]\{u^e\} = \{Q^e\} \tag{6.1.27a}$$

where

$$K_{ij}^e = \int_{x_a}^{x_b} \left[a(x) \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx} + c(x)\psi_i^e \psi_j^e \right] dx, \quad M_{ij}^e = \int_{x_a}^{x_b} c_0(x)\psi_i^e \psi_j^e \, dx \tag{6.1.27b}$$

Equation (6.1.27a) contains the finite element models of the eigenvalue equations (6.1.6) and (6.1.16) as special cases.

The assembly of element equations and imposition of boundary conditions on the assembled equations remain the same as in static problems of Chapter 3. However, the solution of the condensed equations for the unknown primary nodal variables is reduced to an algebraic eigenvalue problem in which the determinant of the coefficient matrix is set to zero to determine the values of λ and subsequently the nodal values of the eigenfunction $U(x)$. These ideas are illustrated through examples.

Example 6.1.1

Consider a plane wall, initially at a uniform temperature T_0, which has both surfaces suddenly exposed to a fluid at temperature T_∞. The governing differential equation is (for constant data)

$$k \frac{\partial^2 T}{\partial x^2} = \rho c_0 \frac{\partial T}{\partial t} \tag{6.1.28a}$$

and the initial condition is

$$T(x, 0) = T_0 \tag{6.1.28b}$$

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where k is the thermal conductivity, ρ the density, and c_0 the specific heat at constant pressure. Equation (6.1.28a) is also known as the *diffusion equation* with diffusion coefficient $\alpha = k/\rho c_0$.

We consider two sets of boundary conditions, each being representative of a different scenario for $x = L$. It amounts to solving Eq. (6.1.28a) and (6.1.28b) for two different sets of boundary conditions.

Set 1. If the heat transfer coefficient at the surfaces of the wall is assumed to be infinite, the boundary conditions can be expressed as

$$T(0,t) = T_\infty, \quad T(L,t) = T_\infty \quad \text{for } t > 0 \tag{6.1.29a}$$

Set 2. If we assume that the wall at $x = L$ is subjected to ambient temperature, we have

$$T(0,t) = T_\infty, \quad \left[k\frac{\partial T}{\partial x} + \beta(T - T_\infty)\right]\bigg|_{x=L} = 0 \tag{6.1.29b}$$

Equation (6.1.28a) can be normalized to make the boundary conditions homogeneous. Let

$$\alpha = \frac{k}{\rho c_0}, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{\alpha t}{L^2}, \quad u = \frac{T - T_\infty}{T_0 - T_\infty}$$

The differential equation (6.1.28a), initial condition (6.1.28b), and boundary conditions (6.1.29a) and (6.1.29b) become

$$-\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0 \tag{6.1.30a}$$

$$u(0,t) = 0, \quad u(1,t) = 0 \quad u(x,0) = 1 \tag{6.1.30b}$$

$$u(0,t) = 0, \quad \left(\frac{\partial u}{\partial x} + Hu\right)\bigg|_{x=1} = 0, \quad H = \frac{\beta L}{k} \tag{6.1.30c}$$

where the bars over x and t are omitted in the interest of brevity.

Solution of (6.1.30a) and (6.1.30b) by separation-of-variables technique (or substitute $u = Ue^{-\lambda t}$) leads to the solution of the eigenvalue problem

$$-\frac{d^2U}{dx^2} - \lambda U = 0, \quad U(0) = 0, \quad U(1) = 0 \tag{6.1.31a}$$

whereas solution of (6.1.30a) and (6.1.30c) results in

$$-\frac{d^2U}{dx^2} - \lambda U = 0, \quad U(0) = 0, \quad \left(\frac{dU}{dx} + HU\right)\bigg|_{x=1} = 0 \tag{6.1.31b}$$

This differential equation is a special case of Eq. (6.1.21) with $a = 1$, $c = 0$, and $c_0 = 1$.

For a linear element, the element equations (6.1.27a) have the explicit form [see Eq. (3.2.34) for element matrices]

$$\left(\frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\frac{h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{6.1.32a}$$

For a quadratic element, we have [see Eq. (3.2.37a) for element matrices]

$$\left(\frac{1}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} - \lambda\frac{h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_1^e \\ u_2^e \\ u_3^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ 0 \\ Q_3^e \end{Bmatrix} \tag{6.1.32b}$$

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Solution for Set 1. For a mesh of two linear elements (the minimum number needed for Set 1 boundary conditions), with $h_1 = h_2 = 0.5$, the assembled equations are

$$\left(2\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \lambda\frac{1}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

The boundary conditions $U(0) = 0$, $Q_2^1 + Q_1^2 = 0$, and $U(1) = 0$ require $U_1 = U_3 = 0$. Hence, the eigenvalue problem reduces to the single equation

$$\left(4 - \lambda\frac{4}{12}\right)U_2 = 0, \quad \text{or} \quad \lambda_1 = 12.0, \quad U_2 \neq 0$$

The mode shape is given (within an arbitrary constant; take $U_2 = 1$) by

$$U(x) = U_2\Phi_2(x) = \begin{cases} U_2\psi_2^1(x) = x/h = 2x & 0 \le x \le 0.5 \\ U_2\psi_1^2(x) = (2h - x)/h = 2(1 - x) & 0.5 \le x \le 1.0 \end{cases}$$

For a mesh of one quadratic element, we have ($h = 1.0$)

$$\frac{16}{3} - \lambda\frac{16}{30} = 0, \quad \text{or} \quad \lambda_1 = 10.0, \quad U_2 \neq 0$$

The corresponding eigenfunction is

$$U(x) = U_2\Phi_2(x) = U_2\psi_2^1 = 4\frac{x}{h}\left(1 - \frac{x}{h}\right), \quad 0 \le x \le 1.0$$

The exact eigenvalues are $\lambda_n = (n\pi)^2$ and $\lambda_1 = \pi^2 = 9.8696$. The exact eigenfunctions for Set 1 boundary conditions are $U_n(x) = \sin n\pi x$ and $U_1(x) = \sin \pi x$. Clearly, one quadratic element gives more accurate solution than two linear elements.

The eigenvalues and eigenfunctions can be used to construct the solution of the transient problem. For example, the solution of (6.1.30*a*) and (6.1.30*b*) is

$$u(x, t) = \sum_{n=1}^{\infty} C_n U_n(x) e^{-\lambda_n t} = \sum_{n=1}^{\infty} C_n \sin n\pi x \, e^{-(n\pi)^2 t} \tag{6.1.33}$$

where C_n are constants to be determined using the initial condition of the problem [see Eq. (6.1.13)]. The finite element solution of the same problem, when one quadratic element is used, is given by

$$u_h(x, t) = 4x(1 - x)e^{-10t}T_0$$

which is very close to the one-term solution in (6.1.33).

Solution for Set 2. For a mesh of two linear elements, the Set 2 boundary conditions translate into $U_1 = 0$ and $Q_2^2 + HU_3 = 0$. The condensed equations are

$$\left(\frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\frac{h}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -HU_3 \end{Bmatrix}$$

or

$$\left(\begin{bmatrix} 4 & -2 \\ -2 & 2 + H \end{bmatrix} - \frac{\lambda}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6.1.34}$$

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For nontrivial solution (i.e., at least one of the U_i is nonzero), the determinant of the coefficient matrix should be zero:

$$\begin{vmatrix} 4-4\bar{\lambda} & -(2+\bar{\lambda}) \\ -(2+\bar{\lambda}) & 2+H-2\bar{\lambda} \end{vmatrix} = 0, \quad \bar{\lambda} = \frac{\lambda}{12}$$

or

$$7\bar{\lambda}^2 - 4(5+H)\bar{\lambda} + 4(1+H) = 0$$

The above equation is known as the *characteristic equation* of the eigenvalue problem. The two roots of this quadratic equation for $H = 1$ are

$$\bar{\lambda}_{1,2} = \frac{12 \pm \sqrt{(12)^2 - 7 \times 8}}{7} \rightarrow \bar{\lambda}_1 = 0.374167, \quad \bar{\lambda}_2 = 3.0544$$

and the eigenvalues are ($\lambda = 12\bar{\lambda}$)

$$\lambda_1 = 4.49, \quad \lambda_2 = 36.6529$$

The eigenvectors associated with each eigenvalue can be computed from equations (6.1.34)

$$\begin{bmatrix} 4-4\bar{\lambda}_i & -(2+\bar{\lambda}_i) \\ -(2+\bar{\lambda}_i) & 3-2\bar{\lambda}_i \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}^{(i)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad i = 1, 2$$

For example, for $\bar{\lambda}_1 = 0.374167$ ($\lambda_1 = 4.4900$), we have

$$2.5033U_2^{(1)} - 2.3742U_3^{(1)} = 0$$

Taking $U_2^{(1)} = 1$, we obtain

$$\{U^{(1)}\} = \begin{Bmatrix} U_2^{(1)} \\ U_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ 1.0544 \end{Bmatrix}$$

On the other hand, if we normalize the components, $(U_2^{(1)})^2 + (U_3^{(1)})^2 = 1$, we obtain

$$\{U^{(1)}\} = \begin{Bmatrix} U_2^{(1)} \\ U_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0.6881 \\ 0.7256 \end{Bmatrix}$$

In other words, we obtain a mode shape whose amplitude is not unique. Hence, the eigenfunction corresponding to $\lambda_1 = 4.4900$ ($h = 0.5$) is

$$U^{(1)}(x) = \begin{cases} 0.6881\, x/h & \text{for } 0 \le x \le 0.5 \\ 0.6881\,(1-x)/h + 0.7256(2x-1)/2h & \text{for } 0.5 \le x \le 1.0 \end{cases}$$

Similarly, the eigenvector components corresponding to $\lambda_2 = 36.6529$ is (not normalized)

$$\{U^{(2)}\} = \begin{Bmatrix} U_2^{(2)} \\ U_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ -1.6258 \end{Bmatrix}$$

For a mesh of one quadratic element, the condensed equations are

$$\left(\frac{1}{3} \begin{bmatrix} 16 & -8 \\ -8 & 7+3H \end{bmatrix} - \lambda \frac{1}{30} \begin{bmatrix} 16 & 2 \\ 2 & 4 \end{bmatrix} \right) \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and the characteristic equation is

$$15\bar{\lambda}^2 - 4(13+3H)\bar{\lambda} + 12(1+H) = 0, \quad \bar{\lambda} = \frac{\lambda}{10}$$

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The two roots of this quadratic equation for $H = 1$ are

$$\bar{\lambda}_{1,2} = \frac{32 \pm \sqrt{(32)^2 - 15 \times 24}}{15} \rightarrow \bar{\lambda}_1 = 0.41545, \quad \bar{\lambda}_2 = 3.85121$$

and the eigenvalues are ($\lambda = 12\bar{\lambda}$)

$$\lambda_1 = 4.1545, \quad \lambda_2 = 38.5121$$

The corresponding eigenvector components are (not normalized)

$$\{U^{(1)}\} = \begin{Bmatrix} U_2^{(1)} \\ U_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ 1.0591 \end{Bmatrix}$$

$$\{U^{(2)}\} = \begin{Bmatrix} U_2^{(2)} \\ U_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1.0000 \\ -2.9052 \end{Bmatrix}$$

The exact eigenfunctions for Set 2 boundary conditions are

$$U_n(x) = \sin \sqrt{\lambda_n}\, x \tag{6.1.35a}$$

and the eigenvalues λ_n are computed from the equation

$$H \sin \sqrt{\lambda_n} + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} = 0 \tag{6.1.35b}$$

The first two roots of the transcendental equation (6.1.35*b*) are (for $H = 1$)

$$\sqrt{\lambda_1} = 2.0288 \rightarrow \lambda_1 = 4.1160; \quad \sqrt{\lambda_2} = 4.9132 \rightarrow \lambda_2 = 24.1393$$

A comparison of the eigenvalues obtained using meshes of linear and quadratic elements with the exact values is presented in Table 6.1.1. Note that the number of eigenvalues obtained in

Table 6.1.1 Eigenvalues of the heat conduction equations (6.1.31*a*) and (6.1.31*b*) for two sets of boundary conditions.

Mesh	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
2L	12.0000						
	(4.4900)*	(36.6529)					
4L	10.3866	48.0000	126.756				
	(4.2054)	(27.3318)	(85.7864)	(177.604)			
8L	9.9971	41.5466	99.4855	192.000	328.291	507.025	686.512
	(4.1380)	(24.9088)	(69.1036)	(143.530)	(257.580)	(417.701)	(607.022)
1Q	10.000						
	(4.1545)	(38.5121)					
2Q	9.9439	40.000	128.723				
	(4.1196)	(24.8995)	(81.4446)	(207.653)			
4Q	9.8747	39.7754	91.7847	160.000	308.253	514.891	794.794
	(4.1161)	(24.2040)	(64.7704)	(129.261)	(240.540)	(405.254)	(658.133)
Exact	9.8696	39.4784	88.8264	157.9137	246.740	355.306	483.611
	(4.1160)	(24.1393)	(63.6597)	(122.889)	(201.851)	(300.550)	(418.987)

* The second line for each mesh corresponds to set 2 boundary conditions.

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Figure 6.1.1 The first three mode shapes as predicted by a mesh of eight linear elements and a mesh of four quadratic elements for the heat transfer problem of Example 6.1.1 (Set 1 boundary conditions).

Figure 6.1.2 The first three mode shapes of the heat transfer problem of Example 6.1.1 (Set 2 boundary conditions).

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the finite element method is always equal to the number of unknown nodal values. As the mesh is refined, not only do we increase the number of eigenvalues but we also improve the accuracy of the preceding eigenvalues. Note also that the convergence of the numerical eigenvalues to the exact ones is from the above, i.e., the finite element solution provides an upper bound to the exact eigenvalues. For structural systems, this can be interpreted as follows: According to the principle of minimum total potential energy, any approximate displacement field would overestimate the total potential energy of the system. This is equivalent to approximating the stiffness of the system with a larger value than the actual one. A stiffer system will have larger eigenvalues (or frequencies). The first three mode shapes of the system are shown in Figs. 6.1.1 and 6.1.2.

We note that the eigenvalue equations (6.1.31*a*) and (6.1.31*b*) can also be interpreted as those arising in connection with the axial vibrations of a constant cross sectional member. In that case, U denotes the axial displacement $[u(x,t) = U(x)e^{-i\omega t}]$ and $\lambda = \omega^2\rho/E$, ω being the frequency of natural vibration. For example, the boundary conditions in (6.1.31*b*) can be interpreted as those of a bar fixed at the left end and the right end being connected to a linear elastic spring (see Fig. 6.1.3). The constant H is equal to k/EA, k being the spring constant. Thus, the results presented in Table 6.1.1 can be interpreted as $\omega^2\rho/E$ of an uniform bar for the two different boundary conditions shown in Fig. 6.1.3.

Natural Vibration of Beams

Euler–Bernoulli Beam Theory

For the Euler–Bernoulli beam theory, the equation of motion is of the form [see Reddy (2002), pp. 193–196]

$$\rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = q(x,t) \tag{6.1.36}$$

where ρ denotes the mass density per unit length, A the area of cross section, E the modulus, and I the second moment of area (see Chapter 5). The expression involving ρI is called

Figure 6.1.3 Uniform bar with two sets of boundary conditions.

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rotary inertia term. Equation (6.1.36) can be formulated as an eigenvalue problem in the interest of finding the frequency of natural vibration by assuming periodic motion

$$w(x,t) = W(x)e^{-i\omega t} \tag{6.1.37}$$

where ω is the frequency of natural transverse motion and $W(x)$ is the mode shape of the transverse motion. Substitution of Eq. (6.1.37) into (6.1.36) yields

$$\frac{d^2}{dx^2}\left(EI\frac{d^2W}{dx^2}\right) - \lambda\left(\rho AW - \rho I\frac{d^2W}{dx^2}\right) = 0 \tag{6.1.38}$$

where $\lambda = \omega^2$.

The weak form of Eq. (6.1.38) is given by

$$\begin{aligned} 0 = &\int_{x_a}^{x_b}\left(EI\frac{d^2v}{dx^2}\frac{d^2W}{dx^2} - \lambda\rho AvW - \lambda\rho I\frac{dv}{dx}\frac{dW}{dx}\right)dx \\ &+ \left\{v\left[\frac{d}{dx}\left(EI\frac{d^2W}{dx^2}\right) + \lambda\rho I\frac{dW}{dx}\right]\right\}_{x_a}^{x_b} + \left[\left(-\frac{dv}{dx}\right)EI\frac{d^2W}{dx^2}\right]_{x_a}^{x_b} \end{aligned} \tag{6.1.39}$$

where v is the weight function. Note that the rotary inertia term contributes to the shear force term, giving rise to an effective shear force that must be known at a boundary point when the deflection is unknown at the point.

To obtain the finite element model of Eq. (6.1.39), assume finite element approximation of the form

$$W(x) = \sum_{j=1}^{4}\Delta_j^e\phi_j^e(x) \tag{6.1.40}$$

where ϕ_j^e are the Hermite cubic polynomials [see Eqs. (5.2.11) and (5.2.12)], and obtain the finite element model

$$\left([K^e] - \omega^2[M^e]\right)\{\Delta^e\} = \{Q^e\} \tag{6.1.41a}$$

where

$$K_{ij}^e = \int_{x_a}^{x_b} EI\frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2}\,dx, \quad M_{ij}^e = \int_{x_a}^{x_b}\left(\rho A\,\phi_i^e\,\phi_j^e + \rho I\,\frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx}\right)dx \tag{6.1.41b}$$

$$\begin{aligned} Q_1^e &\equiv \left[\frac{d}{dx}\left(EI\frac{d^2W}{dx^2}\right) + \lambda\rho I\frac{dW}{dx}\right]\Bigg|_{x_a}, \qquad Q_2^e \equiv \left(EI\frac{d^2W}{dx^2}\right)\Bigg|_{x_a} \\ Q_3^e &\equiv -\left[\frac{d}{dx}\left(EI\frac{d^2W}{dx^2}\right) + \lambda\rho I\frac{dW}{dx}\right]\Bigg|_{x_b}, \quad Q_4^e \equiv -\left(EI\frac{d^2W}{dx^2}\right)\Bigg|_{x_b} \end{aligned} \tag{6.1.41c}$$

For constant values of EI and ρA, the stiffness matrix $[K^e]$ and mass matrix $[M^e]$ are

$$[K^e] = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

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$$[M^e] = \frac{\rho^e A_e h_e}{420}\begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & 22h_e \\ 13h & -3h_e^2 & 22h_e & 4h_e^2 \end{bmatrix}$$
$$+ \frac{\rho^e I_e}{30h_e}\begin{bmatrix} 36 & -3h_e & -36 & -3h_e \\ -3h & 4h_e^2 & 3h_e & -h_e^2 \\ -36 & 3h_e & 36 & 3h_e \\ -3h_e & -h_e^2 & 3h_e & 4h_e^2 \end{bmatrix} \tag{6.1.42}$$

When rotary inertia is neglected, we omit the second part of the mass matrix in (6.1.42).

Timoshenko Beam Theory

The equations of motion of the Timoshenko beam theory are [see Reddy (2002), pp. 193–196]

$$\rho A\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left[GAK_s\left(\frac{\partial w}{\partial x} + \Psi\right)\right] = 0 \tag{6.1.43a}$$

$$\rho I\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial}{\partial x}\left(EI\frac{\partial \Psi}{\partial x}\right) + GAK_s\left(\frac{\partial w}{\partial x} + \Psi\right) = 0 \tag{6.1.43b}$$

where G is the shear modulus ($G = E/[2(1+\nu)]$) and K_s is the shear correction factor ($K_s = 5/6$). Note that Eq. (6.1.43*b*) contains the rotary inertia term. Once again, we assume periodic motion and write

$$w(x,t) = W(x)e^{-i\omega t}, \quad \Psi(x,t) = S(x)e^{-i\omega t} \tag{6.1.44}$$

and obtain the eigenvalue problem from Eqs. (6.1.43*a*) and (6.1.43*b*)

$$-\frac{d}{dx}\left[GAK_s\left(\frac{dW}{dx} + S\right)\right] - \omega^2\rho AW = 0 \tag{6.1.45a}$$

$$-\frac{d}{dx}\left(EI\frac{dS}{dx}\right) + GAK_s\left(\frac{dW}{dx} + S\right) - \omega^2\rho I\, S = 0 \tag{6.1.45b}$$

For equal interpolation of $W(x)$ and $S(x)$

$$W(x) = \sum_{j=1}^{n} W_j^e\psi_j^e(x), \quad S(x) = \sum_{j=1}^{n} S_j^e\psi_j^e(x) \tag{6.1.46}$$

where ψ_j^e are the $(n-1)$ order Lagrange polynomials, the finite element model is given by

$$\left(\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} - \omega^2\begin{bmatrix} [M^{11}] & [0] \\ [0] & [M^{22}] \end{bmatrix}\right)\begin{Bmatrix} \{W\} \\ \{S\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix} \tag{6.1.47}$$

where $[K^e]$ is the stiffness matrix and $[M^e]$ is the mass matrix

$$K_{ij}^{11} = \int_{x_a}^{x_b} GAK_s\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} GAK_s\frac{d\psi_i^e}{dx}\psi_j^e\, dx = K_{ji}^{21}$$

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$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(EI \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + GAK_s \psi_i^e \psi_j^e \right) dx$$

$$M_{ij}^{11} = \int_{x_a}^{x_b} \rho A \psi_i^e \psi_j^e \, dx, \quad M_{ij}^{22} = \int_{x_a}^{x_b} \rho I \psi_i^e \psi_j^e \, dx$$

$$F_j^1 = Q_{2i-1}, \quad F_i^2 = Q_{2i} \tag{6.1.48a}$$

$$Q_1^e \equiv -\left[GAK_s \left(S + \frac{dW}{dx} \right) \right] \Bigg|_{x_a}, \quad Q_2^e \equiv -\left(EI \frac{dS}{dx} \right) \Bigg|_{x_a}$$

$$Q_3^e \equiv \left[GAK_s \left(S + \frac{dW}{dx} \right) \right] \Bigg|_{x_b}, \quad Q_4^e \equiv \left(EI \frac{dS}{dx} \right) \Bigg|_{x_b} \tag{6.1.48b}$$

For the choice of linear interpolation functions, Eq. (6.1.47) has the explicit form (after rearranging the nodal variables)

$$[K^e] = \left(\frac{2E_e I_e}{\mu_0 h_e^3} \right) \begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & h_e^2(1.5 + 6\Lambda_e) & 3h_e & h_e^2(1.5 - 6\Lambda_e) \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2(1.5 - 6\Lambda_e) & 3h_e & h_e^2(1.5 + 6\Lambda_e) \end{bmatrix}$$

$$[M^e] = \frac{\rho^e A_e}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2r_e & 0 & r_e \\ 1 & 0 & 2 & 0 \\ 0 & r_e & 0 & 2r_e \end{bmatrix}, \quad r_e = \frac{I_e}{A_e} \tag{6.1.49a}$$

$$\Lambda_e = \frac{E_e I_e}{G_e A_e K_s h_e^2}, \quad \mu_0 = 12\Lambda_e \tag{6.1.49b}$$

For Hermite cubic interpolation of $W(x)$ and related quadratic approximation of $S(x)$ [i.e., *interdependent interpolation element* (IIE)], the resulting mass matrix is cumbersome and depends on the stiffness parameters (E and G). We will not consider it here [see Reddy (1999*b*, 2000)].

Example 6.1.2

Consider a uniform beam of rectangular cross section ($B \times H$), fixed at $x = 0$ and free at $x = L$ (i.e., a cantilever beam). We wish to determine the first four flexural frequencies of the beam.

The boundary conditions of the structure are

$$W(0) = 0, \quad \frac{dW}{dx} = 0, \quad \left[EI \frac{d^2W}{dx^2} \right]_{x=L} = 0, \quad \left[EI \frac{d^3W}{dx^3} \right]_{x=L} = 0 \tag{6.1.50a}$$

in the Euler–Bernoulli beam theory and

$$W(0) = 0, \quad S(0) = 0, \quad \left[EI \frac{dS}{dx} \right]_{x=L} = 0, \quad \left[GAK_s \left(\frac{dW}{dx} + S \right) \right]_{x=L} = 0 \tag{6.1.50b}$$

in the Timoshenko beam theory. The first two terms in each case are the essential (or geometric) boundary conditions and the last two are the natural (or force) boundary conditions.

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The number of eigenvalues we wish to determine dictates the minimum number of elements to be used to analyze the structure. First, consider the Euler–Bernoulli beam element. Since two primary degrees of freedom are specified ($U_1 = U_2 = 0$), a one-element mesh will have only two unknown degrees of freedom (U_3 and U_4). Hence, we can determine only two natural frequencies. Thus, a minimum of two Euler–Bernoulli beam elements are needed to determine four natural frequencies. If we use the reduced integration Timoshenko beam element, a mesh of two linear elements can only represent the first two mode shapes of the cantilever beam (see Fig. 6.1.4) because there are only two independent deflection degrees of freedom that are unspecified. In order to represent the first four mode shapes using the *reduced integration element* (RIE), we must use at least four linear elements or two quadratic elements. However, the four computed frequencies may not be the lowest four.

For illustrative purpose, first we consider the one-element mesh of the Euler–Bernoulli beam element. We have

$$\left(\frac{2EI}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix} - \omega^2\frac{\rho AL}{420}\begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix}\right.$$

$$\left. -\omega^2\frac{\rho I}{30L}\begin{bmatrix} 36 & -3L & -36 & -3L \\ -3h & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix}\right)\begin{Bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

Figure 6.1.4 Two possible nonzero mode shapes of a cantilever beam that can be represented by a mesh of two linear Timoshenko beam elements (or one Euler–Bernoulli beam element).

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The specified displacement and force degrees of freedom are $W_1 = 0$, $\Theta_1 = 0$, $Q_3 = 0$, and $Q_4 = 0$. Hence, the condensed equations are

$$\left(\frac{2EI}{L^3}\begin{bmatrix} 6 & 3L \\ 3L & 2L^2 \end{bmatrix} - \omega^2 \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L \\ 22L & 4L^2 \end{bmatrix} - \omega^2 \frac{\rho I}{30L}\begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix}\right)\begin{Bmatrix} W_2 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Setting the determinant of the coefficient matrix in the above equation to zero, we obtain a quadratic characteristic equation in ω^2.

First, consider the case in which the rotary inertia is neglected. The characteristic equation for this case is

$$15120 - 1224\lambda + \lambda^2 = 0, \quad \lambda = \frac{\rho AL^4}{EI}\omega^2$$

whose roots are

$$\lambda_1 = 12.48 \quad \lambda_2 = 1211.52, \quad \text{or} \quad \omega_1 = \frac{3.5327}{L^2}\sqrt{\frac{EI}{\rho A}}, \quad \omega_2 = \frac{34.8069}{L^2}\sqrt{\frac{EI}{\rho A}}$$

The exact frequencies are $\bar{\omega}_1 = 3.5160$ and $\bar{\omega}_2 = 22.0345$, where $\bar{\omega}_i = \omega_i(L^2\sqrt{\rho A/EI})$. The eigenvector components are computed from the first of the condensed equations

$$W_2^{(i)} = -L\left(\frac{1260 - 11\lambda_i}{420 - 13\lambda_i}\right)\Theta_2^{(i)}, \quad i = 1, 2$$

Hence, the eigenvectors are

$$\begin{Bmatrix} W_2 \\ \Theta_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.7259L \\ -1.0000 \end{Bmatrix}, \quad \begin{Bmatrix} W_2 \\ \Theta_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.1312L \\ -1.0000 \end{Bmatrix}$$

For the case in which rotary inertia is not neglected, we write $\rho I = \rho A\,(H^2/12)$ and take $H/L = 0.01$. The characteristic equation is

$$15112 - 1223.406\lambda + \lambda^2 = 0, \quad \lambda = \frac{\rho AL^4}{EI}\omega^2$$

whose roots are

$$\lambda_1 = 12.4797, \quad \lambda_2 = 1210.926 \quad \text{or} \quad \omega_1 = \frac{3.5327}{L^2}\sqrt{\frac{EI}{\rho A}}, \quad \omega_2 = \frac{34.7984}{L^2}\sqrt{\frac{EI}{\rho A}}$$

The exact frequencies are $\bar{\omega}_1 = 3.5158$ and $\bar{\omega}_2 = 22.0226$, where $\bar{\omega}_i = \omega_i(L^2\sqrt{\rho A/EI})$.

Next, we consider the one-element mesh of the Timoshenko beam element (RIE). We have

$$\left(\frac{2EI}{\mu_0 L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & L^2\Xi_1 & 3L & L^2\Xi_2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2\Xi_2 & 3L & L^2\Xi_1 \end{bmatrix} - \omega^2\frac{\rho AL}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2r & 0 & r \\ 1 & 0 & 2 & 0 \\ 0 & r & 0 & 2r \end{bmatrix}\right)\begin{Bmatrix} W_1 \\ S_1 \\ W_2 \\ S_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

where $K_s = 5/6$, $r = I/A = H^2/12$, and

$$\Xi_1 = 1.5 + 6\Lambda, \quad \Xi_2 = 1.5 - 6\Lambda, \quad \Lambda = \frac{EI}{GAK_sL^2} = \frac{1+\nu}{5}\frac{H^2}{L^2}, \quad \mu_0 = 12\Lambda$$

The specified displacement and force degrees of freedom are $W_1 = 0$, $S_1 = 0$, $Q_3 = 0$, and $Q_4 = 0$; and the condensed equations are

$$\left(\frac{2EI}{\mu_0 L^3}\begin{bmatrix} 6 & 3L \\ 3L & L^2(1.5 + 6\Lambda) \end{bmatrix} - \omega^2\frac{\rho AL}{6}\begin{bmatrix} 2 & 0 \\ 0 & 2r \end{bmatrix}\right)\begin{Bmatrix} W_2 \\ S_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

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Setting the determinant of the coefficient matrix to zero, we obtain the characteristic equation

$$\bar{\lambda}^2 - 3\left(1 + 3s^2 + 12s^2\Lambda\right)\bar{\lambda} + 108s^2 = 0, \quad \bar{\lambda} = \frac{\rho A L^4 \Lambda}{EI}\omega^2$$

where s is the length-to-height ratio, $s = L/H$.

When the rotary inertia is neglected, we have

$$\bar{\lambda} = \frac{12\Lambda}{1 + 4\Lambda}$$

To further simplify the expression, we assume $\nu = 0.25$ and obtain

$$\omega^2 = \frac{12}{(1 + H^2/L^2)}\frac{EI}{\rho A L^4}$$

and

$$\frac{L}{H} = 100: \quad \omega = \frac{3.4639}{L^2}\sqrt{\frac{EI}{\rho A}}; \qquad \frac{L}{H} = 10: \quad \omega = \frac{3.4469}{L^2}\sqrt{\frac{EI}{\rho A}}$$

The exact frequency is $\bar{\omega} = 3.5158$ for $L/H = 100$ and $\bar{\omega} = 3.5092$ for $L/H = 10$, where $\bar{\omega}_i = \omega_i(L^2\sqrt{\rho A/EI})$. If the rotary inertia is included, we obtain

$$\frac{L}{H} = 100: \quad \omega = \frac{3.4639}{L^2}\sqrt{\frac{EI}{\rho A}}; \qquad \frac{L}{H} = 10: \quad \omega = \frac{3.4413}{L^2}\sqrt{\frac{EI}{\rho A}}$$

The frequencies obtained by the Euler–Bernoulli beam elements and Timoshenko beam elements (RIE) are shown in Table 6.1.2 for two different values of the length-to-height ratio L/H. The value $L/H = 100$ makes the effect of shear deformation negligible, and we obtain essentially the same result as in the Euler–Bernoulli beam theory. Since the frequencies are normalized, $\bar{\omega} = \omega L^2\sqrt{(\rho A/EI)}$, it is necessary only to select the values of ν and L/H. For

Table 6.1.2 Natural frequencies of a cantilever beam according to Timoshenko beam theory (TBT) and Euler–Bernoulli beam theory (EBT) $[\bar{\omega} = \omega L^2(\rho A/EI)^{1/2}]$.

	$L/H = 100$				$L/H = 10$			
Mesh	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$
4L	3.5406	25.6726	98.3953	417.1330	3.5137	24.1345	80.2244	189.9288
8L	3.5223	22.8851	68.8937	151.8431	3.4956	21.7004	60.6297	119.2798
16L	3.5174	22.2350	63.3413	127.5434	3.4908	21.1257	56.4714	104.6799
2Q	3.5214	23.3226	78.3115	328.3250	3.4947	22.0762	67.0884	181.0682
4Q	3.5161	22.1054	63.3271	133.9828	3.4895	21.0103	56.4572	108.6060
8Q	3.5158	22.0280	61.7325	121.4458	3.4892	20.4421	55.2405	100.7496
EBT†	3.5158	22.0226	61.6179	120.6152	3.4892	20.9374	55.1530	100.2116
TBT†	3.5158	22.0315	61.6774	120.8300	3.5092	21.7425	59.8013	114.2898
EBT‡	3.5160	22.0345	61.6972	120.9019	3.5160	22.0345	61.6972	120.9019

† Rotary inertia is included (Exact).

‡ Rotary inertia is neglected; the results are independent of the ratio L/H (Exact).

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computational purposes, we take $\nu = 0.25$; also, note that

$$GAK = \frac{E}{2(1+\nu)} BH\frac{5}{6} = \frac{4EI}{H^2}, \quad \rho I = \frac{H^2 \rho A}{12}$$

where B and H are the width and height, respectively, of the beam.

We note from Table 6.1.2 that the finite element results converge with h-refinement (i.e., when more of the same kind of elements are used) and also with p-refinement (i.e., when higher-order elements are used). The p-refinement shows more rapid convergence of the fundamental (i.e., first) frequency. Note that the effect of shear deformation is to reduce the magnitude of natural frequency when compared to the EBT. In other words, the assumed infinite shear rigidity makes the EBT over-predict the magnitude of frequencies. The first four mode shapes of the cantilever beam, as obtained using the 16-element mesh of linear elements, are shown in Fig. 6.1.5.

Figure 6.1.5 First four natural mode shapes of a cantilever beam, as predicted using a 16-element mesh of linear Timoshenko beam elements ($L/H = 10$).

In closing the discussion on natural vibration, it is noted that when the symmetry of the system is used to model the problem, only symmetric modes are predicted. It is necessary to model the whole system in order to obtain all the modes of vibration.

Stability (Buckling) of Beams

EBT

The study of buckling of beam-columns also leads to an eigenvalue problem. For example, the equation governing onset of buckling of a column subjected to an axial compressive

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Figure 6.1.6 Columns with different boundary conditions and subjected to axial compressive load, N^0. (*a*) Hinged-hinged. (*b*) Hinged-clamped. (*c*) Clamped-clamped. (*d*) Clamped-free.

force N^0 (see Fig. 6.1.6), according to EBT is [see Reddy (1999*b*)]

$$\frac{d^2}{dx^2}\left(EI\frac{d^2W}{dx^2}\right)+N^0\frac{d^2W}{dx^2}=0 \tag{6.1.51}$$

which describes an eigenvalue problem with $\lambda = N^0$. The smallest value of N^0 is called the *critical buckling load*.

The finite element model of Eq. (6.1.51) is

$$[K^e]\{\Delta^e\}-N^0[G^e]\{\Delta^e\}=\{Q^e\} \tag{6.1.52a}$$

where $\{\Delta^e\}$ and $\{Q^e\}$ are the columns of generalized displacement and force degrees of freedom at the two ends of the Euler–Bernoulli beam element:

$$\{\Delta\}^e=\begin{Bmatrix} W_1 \\ \left(-\dfrac{dW}{dx}\right)_1 \\ W_2 \\ \left(-\dfrac{dW}{dx}\right)_2 \end{Bmatrix}^e, \quad \{Q\}^e=\begin{Bmatrix} \left[\dfrac{d}{dx}\left(EI\dfrac{d^2W}{dx^2}\right)+N^0\dfrac{dW}{dx}\right]_1 \\ \left(EI\dfrac{d^2W}{dx^2}\right)_1 \\ \left[-\dfrac{d}{dx}\left(EI\dfrac{d^2W}{dx^2}\right)-N^0\dfrac{dW}{dx}\right]_2 \\ \left(-EI\dfrac{d^2W}{dx^2}\right)_2 \end{Bmatrix}^e \tag{6.1.52b}$$

where the subscripts 1 and 2 refer to element nodes 1 and 2 (at $x = x_a$ and $x = x_b$, respectively). The coefficients of the stiffness matrix $[K^e]$ and the *stability matrix* $[G^e]$ are

$$K_{ij}^e=\int_{x_a}^{x_b} EI\frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2}\,dx, \quad G_{ij}^e=\int_{x_a}^{x_b}\frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx}\,dx \tag{6.1.52c}$$

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where ϕ_i^e are the Hermite cubic interpolation functions. The explicit form of $[G^e]$ is [see the second part of the mass matrix in Eq. (6.1.42)]

$$[G^e] = \frac{1}{30h_e}\begin{bmatrix} 36 & -3h_e & -36 & -3h_e \\ -3h_e & 4h_e^2 & 3h_e & -h_e^2 \\ -36 & 3h_e & 36 & 3h_e \\ -3h_e & -h_e^2 & 3h_e & 4h_e^2 \end{bmatrix} \tag{6.1.52d}$$

TBT

For TBT, the equations governing buckling of beams are [see Reddy (1999*b*)]

$$-\frac{d}{dx}\left[GAK_s\left(\frac{dW}{dx}+S\right)\right] + N^0\frac{d^2W}{dx^2} = 0 \tag{6.1.53a}$$

$$-\frac{d}{dx}\left(EI\frac{dS}{dx}\right) + GAK_s\left(\frac{dW}{dx}+S\right) = 0 \tag{6.1.53b}$$

Here $W(x)$ and $S(x)$ denote the transverse deflection and rotation, respectively, at the onset of buckling.

The finite element model of Eqs. (6.1.53*a*) and (6.1.53*b*), with equal interpolation of W and S, is

$$\left(\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{12}]^{\mathrm{T}} & [K^{22}] \end{bmatrix} - N^0\begin{bmatrix} [G] & [0] \\ [0] & [0] \end{bmatrix}\right)\begin{Bmatrix} \{W^e\} \\ \{S^e\} \end{Bmatrix} = \begin{Bmatrix} \{V^e\} \\ \{M^e\} \end{Bmatrix} \tag{6.1.54a}$$

where

$$K_{ij}^{11} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\,dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} GAK_s \frac{d\psi_i^e}{dx}\psi_j^e\,dx$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(GAK_s\psi_i^e\psi_j^e + EI\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\right)dx$$

$$G_{ij} = \int_{x_a}^{x_b} \psi_i^e\psi_j^e\,dx$$

$$V_1^e \equiv Q_1^e = \left[-GAK_s\left(\frac{dW}{dx}+S\right) + N^0\frac{dW}{dx}\right]_{x_a}$$

$$V_n^e \equiv Q_3^e = \left[GAK_s\left(\frac{dW}{dx}+S\right) - N^0\frac{dW}{dx}\right]_{x_b}$$

$$M_1^e \equiv Q_2^e = \left(-EI\frac{dS}{dx}\right)_{x_a}$$

$$M_n^e \equiv Q_4^e = \left(-EI\frac{dS}{dx}\right)_{x_b} \tag{6.1.54b}$$

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For the linear RIE, the stiffness matrix $[K^e]$ is given in Eq. (6.1.49a). The stability matrix is given by

$$[G^e] = \frac{1}{h_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.1.55}$$

Example 6.1.3

Consider a uniform column (L, A, I, E, and ν) fixed at one end and pinned at the other, as shown in Fig. 6.1.7. We wish to determine the critical buckling load.

Let us consider a mesh of two Euler–Bernoulli beam elements. The assembled system of finite element equations is ($h = L/2$)

$$\left(\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix} - \frac{N^0}{30h}\begin{bmatrix} 36 & -3h & -36 & -3h & 0 & 0 \\ -3h & 4h^2 & 3h & -h^2 & 0 & 0 \\ -36 & 3h & 36+36 & 3h-3h & -36 & -3h \\ -3h & -h^2 & 3h-3h & 4h^2+4h^2 & 3h & -h^2 \\ 0 & 0 & -36 & 3h & 36 & 3h \\ 0 & 0 & -3h & -h^2 & 3h & 4h^2 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

where U_i denote the generalized displacements of the global nodes.

Figure 6.1.7 A column fixed at $x = 0$ and hinged at $x = L$, and subjected to axial compressive load N^0.

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The geometric supports and equilibrium conditions of the column require that (note that the axial compressive force N^0 enters the equilibrium equation and the boundary condition through Q_3^2)

$$U_1 = U_2 = U_5 = 0, \quad Q_3^1 + Q_1^2 = 0, \quad Q_4^1 + Q_2^2 = 0, \quad Q_4^2 = 0$$

Hence, the condensed equations become

$$\left(\frac{2EI}{h^3} \begin{bmatrix} 12 & 0 & -3h \\ 0 & 4h^2 & h^2 \\ -3h & h^2 & 2h^2 \end{bmatrix} - \frac{N^0}{30h} \begin{bmatrix} 72 & 0 & -3h \\ 0 & 8h^2 & -h^2 \\ -3h & -h^2 & 4h^2 \end{bmatrix} \right) \begin{Bmatrix} U_3 \\ U_4 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

which defines the eigenvalue problem for the determination of N^0 and U_i.

Setting the determinant of the coefficient matrix to zero, we obtain a cubic equation for N^0. The smallest value of N^0 is the critical buckling load, which is (obtained using FEM1D)

$$N_{\text{cri}}^0 = 20.7088 \frac{EI}{L^2}$$

whereas the "exact" solution is $20.187\, EI/L^2$ [see Reddy (1999*b*), pp. 155–166].

Next, consider a two-element mesh of the RIE. The assembled equations are

$$\left(\frac{2EI}{\mu_0 h^3} \begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & h^2\Xi_1 & 3h & h^2\Xi_2 & 0 & 0 \\ -6 & 3h & 12 & 0 & -6 & -3h \\ -3h & h^2\Xi_2 & 0 & 2h^2\Xi_1 & 3h & h^2\Xi_2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2\Xi_2 & 3h & h^2\Xi_1 \end{bmatrix} - \frac{N^0}{h} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}$$

where $K_s = 5/6$, and

$$\Xi_1 = 1.5 + 6\Lambda, \quad \Xi_2 = 1.5 - 6\Lambda, \quad \mu_0 = 12\Lambda, \quad \Lambda = \frac{EI}{GAK_s h^2} = \frac{1+\nu}{5}\frac{H^2}{h^2}$$

The condensed equations are

$$\left(\frac{2EI}{\mu_0 h^3} \begin{bmatrix} 12 & 0 & -3h \\ 0 & 2h^2\Xi_1 & h^2\Xi_2 \\ -3h & h^2\Xi_2 & h^2\Xi_1 \end{bmatrix} - \frac{N^0}{h} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} U_3 \\ U_4 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Note that because of the zero diagonal element in the stability matrix, we obtain a linear algebraic equation for N^0 ($\nu = 0.25$ and $\Lambda = H^2/L^2$)

$$N^0 = \frac{180 + 144\Lambda}{2.25 + 54\Lambda + 36\Lambda^2} \frac{EI}{L^2}$$

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which is almost four times the correct value! A mesh of eight linear RIEs yields $21.348\,EI/L^2$ and four quadratic elements yields $20.267\,EI/L^2$ for $L/H = 100$. Clearly, the RIE has slow convergence rate for buckling problems.

We close this section with a note that eigenvalue problems (for natural vibration and buckling analysis) of frame structures can be formulated using the ideas discussed in Section 5.4. The transformed element equations are of the form

$$[K^e]\{\Delta^e\} - \omega^2[M^e]\{\Delta^e\} = \{Q^e\}, \quad [K^e]\{\Delta^e\} - N^0[G^e]\{\Delta^e\} = \{Q^e\}$$
$$[K^e] = [T^e]^{\mathrm{T}}[\bar{K}^e][T^e], \quad [M^e] = [T^e]^{\mathrm{T}}[\bar{M}^e][T^e], \quad [G^e] = [T^e]^{\mathrm{T}}[\bar{G}^e][T^e] \tag{6.1.56}$$

6.2 TIME-DEPENDENT PROBLEMS

6.2.1 Introduction

In this section, we develop the finite element models of one-dimensional time-dependent problems and describe time approximation schemes to convert ordinary differential equations in time to algebraic equations. We consider finite element models of the time-dependent version of the differential equations studied in Chapters 3 and 5. These include the second-order (in space) parabolic (i.e., first time derivative) and hyperbolic (i.e., second time derivative) equations and fourth-order hyperbolic equations arising in connection with the bending of beams [see Eqs. (6.1.2), (6.1.14), (6.1.36), (6.1.43*a*), and (6.1.43*b*)].

Finite element models of time-dependent problems can be developed in two alternative ways: (*a*) coupled formulation in which the time t is treated as an additional coordinate along with the spatial coordinate x and (*b*) decoupled formulation where time and spatial variations are assumed to be separable. Thus, the approximation in the two formulations takes the form

$$u(x,t) \approx u_h^e(x,t) = \sum_{j=1}^{n} \hat{u}_j^e \hat{\psi}_j^e(x,t) \quad \text{(coupled formulation)} \tag{6.2.1a}$$

$$u(x,t) \approx u_h^e(x,t) = \sum_{j=1}^{n} u_j^e(t)\psi_j^e(x) \quad \text{(decoupled formulation)} \tag{6.2.1b}$$

where $\hat{\psi}_j^e(x,t)$ are time-space (two-dimensional) interpolation functions and $\hat{u}_j$ are the nodal values that are independent of x and t, where $\psi_j^e(x)$ are the usual one-dimensional interpolation functions in spatial coordinate x only and the nodal values $u_j^e(t)$ are functions of time t only. Derivation of the interpolation functions in two coordinates will be discussed in Chapters 8 and 9 in connection with the finite element analysis of two-dimensional problems. Space-time coupled finite element formulations are not common, and they have not been adequately studied. In this section, we consider the space-time decoupled formulation only.

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Figure 6.2.1 Deflections of a cable at different times.

The space-time decoupled finite element formulation of time-dependent problems involves two steps:

1. *Spatial approximation,* where the solution u of the equation under consideration is approximated by expressions of the form (6.2.1*b*), and the spatial finite element model of the equation is developed using the procedures of static or steady-state problems while carrying all time-dependent terms in the formulation. This step results in a set of ordinary differential equations (i.e., a semidiscrete system of equations) in time for the nodal variables $u_j^e(t)$ of the element. Equation (6.2.1*b*) represents the spatial approximation of u for any time t. When the solution is separable into functions of time only and space only, $u(x, t) = T(t)X(x)$, the approximation (6.2.1*b*) is clearly justified. Even when the solution is not separable, (6.2.1*b*) can represent a good approximation of the actual solution provided a sufficiently small time step is used.
2. *Temporal approximation,* where the system of ordinary differential equations are further approximated in time, often using finite difference formulae for the time derivatives. This step allows conversion of the system of ordinary differential equations into a set of algebraic equations among u_j^e at time $t_{s+1} = (s+1)\Delta t$, where Δt is the time increment and s is a nonnegative integer.

All time approximation schemes seek to find u_j at time t_{s+1} using the known values of u_j from previous times:

$$\text{compute} \quad \{u\}_{s+1} \quad \text{using} \quad \{u\}_s,\ \{u\}_{s-1},\ \ldots$$

Thus, at the end of the two-stage approximation, we have a continuous spatial solution at discrete intervals of time:

$$u(x, t_s) \approx u_h^e(x, t_s) = \sum_{j=1}^{n} u_j^e(t_s)\psi_j^e(x) \quad (s = 0, 1, \ldots) \tag{6.2.2}$$

Note that the approximate solution (6.2.2) has the same form as that in the separation-of-variables technique used to solve boundary value and initial value problems. By taking nodal values to be functions of time, we see that the spatial points in an element take on different values for different times (see Fig. 6.2.1).

We study the details of the two steps by considering a model differential equation that contains both second- and fourth-order spatial derivatives and first- and second-order time

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derivatives:

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0u+c_1\frac{\partial u}{\partial t}+c_2\frac{\partial^2 u}{\partial t^2}=f(x,t) \tag{6.2.3a}$$

The above equation is subject to appropriate boundary and initial conditions. The boundary conditions are of the form

$$\text{specify} \quad u(x,t) \quad \text{or} \quad -a\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)$$

$$\text{specify} \quad \frac{\partial u}{\partial x} \quad \text{or} \quad b\frac{\partial^2 u}{\partial x^2} \tag{6.2.3b}$$

at $x=0$, L, and the initial conditions involve specifying

$$c_2u(x,0) \quad \text{and} \quad c_2\dot{u}(x,0)+c_1u(x,0) \tag{6.2.3c}$$

where $\dot{u}\equiv\partial u/\partial t$. Equation (6.2.3$a$) describes, for example, the following physical problems:

(a) Heat transfer and fluid flow: $c_2=0$ and $b=0$

(b) Transverse motion of a cable: $a=T$, $c_0=0$, $b=0$, $c_1=\rho$, $c_2=0$

(c) The longitudinal motion of a rod: $a=EA$, $b=0$; if damping is not considered, $c_1=0$, $c_2=\rho A$

(d) The transverse motion of an Euler–Bernoulli beam: $a=0$, $b=EI$, $c_0=k$, $c_1=0$, $c_2=\rho A$

We will consider these special cases through examples.

6.2.2 Semidiscrete Finite Element Models

The semidiscrete formulation involves approximation of the spatial variation of the dependent variable. The formulation follows essentially the same steps as described in Section 3.2. The first step involves the construction of the weak form of the equation over a typical element. In the second step, we develop the finite element model by seeking approximation of the form in (6.2.1b).

Following the three-step procedure of constructing the weak form of a differential equation, we can develop the weak form of (6.2.3a) over an element. Integration by parts is used once on the first term and twice on the second term to distribute the spatial derivatives equally between the weight function w and the dependent variable u:

$$\begin{aligned}
0&=\int_{x_a}^{x_b} w\left[-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+\frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right)+c_0u+c_1\frac{\partial u}{\partial t}+c_2\frac{\partial^2 u}{\partial t^2}-f\right]dx\\
&=\int_{x_a}^{x_b}\left[\frac{\partial w}{\partial x}a\frac{\partial u}{\partial x}+\frac{\partial^2 w}{\partial x^2}b\frac{\partial^2 u}{\partial x^2}+c_0wu+c_1w\frac{\partial u}{\partial t}+c_2w\frac{\partial^2 u}{\partial t^2}-wf\right]dx\\
&\quad+\left[w\left[\left(-a\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)\right]+\frac{\partial w}{\partial x}\left(-b\frac{\partial^2 u}{\partial x^2}\right)\right]_{x_a}^{x_b}
\end{aligned}$$

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$$= \int_{x_a}^{x_b} \left(a \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + b \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 u}{\partial x^2} + c_0 w u + c_1 w \frac{\partial u}{\partial t} + c_2 w \frac{\partial^2 u}{\partial t^2} - w f \right) dx$$

$$- \hat{Q}_1 w(x_a) - \hat{Q}_3 w(x_b) - \hat{Q}_2 \left(-\frac{\partial w}{\partial x} \right) \bigg|_{x_a} - \hat{Q}_4 \left(-\frac{\partial w}{\partial x} \right) \bigg|_{x_b} \tag{6.2.4a}$$

where

$$\hat{Q}_1 = \left[-a \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \left(b \frac{\partial^2 u}{\partial x^2} \right) \right]_{x_a}, \qquad \hat{Q}_2 = \left[b \frac{\partial^2 u}{\partial x^2} \right]_{x_a}$$
$$\hat{Q}_3 = -\left[-a \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \left(b \frac{\partial^2 u}{\partial x^2} \right) \right]_{x_b}, \qquad \hat{Q}_4 = -\left[b \frac{\partial^2 u}{\partial x^2} \right]_{x_b} \tag{6.2.4b}$$

Next, we assume that u is interpolated by an expression of the form (6.2.1b). Equation (6.2.1b) implies that, at any arbitrarily fixed time $t > 0$, the function u can be approximated by a linear combination of the ψ_j^e and $u_j^e(t)$, with $u_j^e(t)$ being the value of u at time t at the jth node of the element Ω_e. In other words, the time and spatial variations of u are separable. This assumption is not valid, in general, because it may not be possible to write the solution $u(x, t)$ as the product of a function of time only and a function of space only. However, with sufficiently small time steps, it is possible to obtain accurate solutions to even those problems for which the solution is not separable in time and space. The finite element solution that we obtain at the end of the analysis is continuous in space but not in time. We only obtain the finite element solution in the form

$$u(x, t_s) = \sum_{j=1}^{n} u_j^e(t_s) \psi_j^e(x) = \sum_{j=1}^{n} (u_j^s)^e \psi_j^e(x) \quad (s = 1, 2, \ldots) \tag{6.2.5}$$

where $(u_j^s)^e$ is the value of $u(x, t)$ at time $t = t_s$ and node j of the element Ω_e.

Substituting $w = \psi_i(x)$ (to obtain the ith equation of the system) and (6.2.1b) into (6.2.4a), we obtain

$$0 = \int_{x_a}^{x_b} \left[a \frac{d\psi_i}{dx} \left(\sum_{j=1}^{n} u_j \frac{d\psi_j}{dx} \right) + b \frac{d^2\psi_i}{dx^2} \left(\sum_{j=1}^{n} u_j \frac{d^2\psi_j}{dx^2} \right) + c_0 \psi_i \left(\sum_{j=1}^{n} u_j \psi_j \right) \right.$$
$$\left. + c_1 \psi_i \left(\sum_{j=1}^{n} \frac{du_j}{dt} \psi_j \right) + c_2 \psi_i \left(\sum_{j=1}^{n} \frac{d^2 u_j}{dt^2} \psi_j \right) - \psi_i f \right] dx$$
$$- \hat{Q}_1 \psi_i(x_a) - \hat{Q}_3 \psi_i(x_b) - \hat{Q}_2 \left(-\frac{d\psi_i}{dx} \right) \bigg|_{x_a} - \hat{Q}_4 \left(-\frac{d\psi_i}{dx} \right) \bigg|_{x_b}$$
$$= \sum_{j=1}^{n} \left[(K_{ij}^1 + K_{ij}^2) u_j + M_{ij}^1 \frac{du_j}{dt} + M_{ij}^2 \frac{d^2 u_j}{dt^2} \right] - F_i \tag{6.2.6}$$

In matrix form, we have

$$[K]\{u\} + [M^1]\{\dot{u}\} + [M^2]\{\ddot{u}\} = \{F\} \tag{6.2.7a}$$

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where

$$[K] = [K^1] + [K^2] + [M^0] \tag{6.2.7b}$$

$$\begin{aligned} M_{ij}^0 &= \int_{x_a}^{x_b} c_0\, \psi_i \psi_j\, dx, & M_{ij}^1 &= \int_{x_a}^{x_b} c_1\, \psi_i \psi_j\, dx \\ M_{ij}^2 &= \int_{x_a}^{x_b} c_2\, \psi_i \psi_j\, dx, & K_{ij}^1 &= \int_{x_a}^{x_b} a \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\, dx \\ K_{ij}^2 &= \int_{x_a}^{x_b} b \frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}\, dx, & F_i &= \int_{x_a}^{x_b} \psi_i f\, dx + \hat{Q}_i \end{aligned} \tag{6.2.7c}$$

Equation (6.2.7a) is a hyperbolic equation, and it contains the parabolic equation as a special case (set $[M^2] = [0]$). The time approximation of (6.2.7a) for these two cases will be considered separately. We begin with the parabolic equation.

6.2.3 Parabolic Equations

Time Approximation

The time approximation is discussed with the help of a single first-order differential equation. Suppose that we wish to determine $u(t)$ for $t > 0$ such that $u(t)$ satisfies

$$a\frac{du}{dt} + bu = f(t), \quad 0 < t < T \quad \text{and} \quad u(0) = u_0 \tag{6.2.8}$$

where $a \neq 0$, b, and u_0 are constants, and f is a function of time t. The exact solution of the problem consists of two parts: the homogeneous and particular solutions. The homogeneous solution is

$$u^h(t) = Ae^{-kt}, \quad k = \frac{b}{a}$$

The particular solution is given by

$$u^p(t) = \frac{1}{a}e^{-kt}\left(\int_0^t e^{k\tau} f(\tau)\, d\tau\right)$$

The complete solution is given by

$$u(t) = e^{-kt}\left(A + \frac{1}{a}\int_0^t e^{k\tau} f(\tau)\, d\tau\right) \tag{6.2.9}$$

In the finite difference solution of (6.2.8), we replace the derivatives with their finite difference approximation. The most commonly used scheme for solving (6.2.8) is the *α-family of approximation* in which a weighted average of the time derivatives at two consecutive time steps is approximated by linear interpolation of the values of the variable at the two steps (see Fig. 6.2.2):

$$(1-\alpha)\dot{u}_s + \alpha\, \dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \quad \text{for } 0 \le \alpha \le 1 \tag{6.2.10a}$$

where u_s denotes the value of $u(t)$ at time $t = t_s = \sum_{i=1}^{s} \Delta t_i$, and $\Delta t_s = t_s - t_{s-1}$ is the sth time step. If the total time $[0,\ T]$ is divided into equal time steps, then $t_s = s\,\Delta t$. Equation (6.2.10a) can be expressed as

$$\begin{aligned} u_{s+1} &= u_s + \Delta t\, \dot{u}_{s+\alpha} \\ \dot{u}_{s+\alpha} &= (1-\alpha)\dot{u}_s + \alpha\, \dot{u}_{s+1} \quad \text{for } 0 \le \alpha \le 1 \end{aligned} \tag{6.2.10b}$$

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Figure 6.2.2 Approximation of the derivative of a function.

When $\alpha = 0$, Eq. (6.2.10a) gives

$$\dot{u}_s = \frac{u_{s+1} - u_s}{\Delta t_{s+1}}$$

which is nothing but the slope of the function $u(t)$ at time $t = t_s$ based on the values of the function at time t_s and t_{s+1}. Since the value of the function from a step in front is used, it is termed a *forward difference* approximation. When $\alpha = 1$, we obtain

$$\dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t_{s+1}} \quad \rightarrow \quad \dot{u}_s = \frac{u_s - u_{s-1}}{\Delta t_s}$$

which is termed, for obvious reason, the *backward difference* approximation.

Returning to Eq. (6.2.8), we note that it is valid for all times $t > 0$. In particular, it is valid at times $t = t_s$ and $t = t_{s+1}$. Hence, from Eq. (6.2.8) we have

$$\dot{u}_s = \frac{1}{a}(f_s - bu_s), \quad \dot{u}_{s+1} = \frac{1}{a}(f_{s+1} - bu_{s+1})$$

Substituting the above expressions into Eq. (6.2.10a), we arrive at

$$(1-\alpha)(f_s - bu_s) + \alpha(f_{s+1} - bu_{s+1}) = a\left(\frac{u_{s+1} - u_s}{\Delta t_{s+1}}\right)$$

Solving for u_{s+1}, we obtain

$$[a + \alpha\,\Delta t_{s+1}\,b]\,u_{s+1} = [a - (1-\alpha)\,\Delta t_{s+1}\,b]\,u_s + \Delta t_{s+1}\,[\alpha\,f_{s+1} + (1-\alpha)\,f_s] \tag{6.2.11a}$$

or

$$u_{s+1} = \frac{a - (1-\alpha)\,\Delta t_{s+1}\,b}{a + \alpha\,\Delta t_{s+1}\,b}u_s + \Delta t_{s+1}\frac{[\alpha\,f_{s+1} + (1-\alpha)\,f_s]}{a + \alpha\,\Delta t_{s+1}\,b} \tag{6.2.11b}$$

Thus, Eq. (6.2.11) can be used repeatedly to march in time and obtain the solution at times $t = t_{s+1}, t_{s+2}, \ldots, t_N$, where Ntime is the number of time steps required to reach the final

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time T (or the solution reaches a steady state). At the very beginning, i.e., $s = 0$, the solution u_1 is calculated using the initial value u_0:

$$u_1 = \frac{a - (1-\alpha)\,\Delta t_1\, b}{a + \alpha\,\Delta t_1\, b} u_0 + \Delta t_1 \frac{[\alpha\, f_1 + (1-\alpha) f_0]}{a + \alpha\,\Delta t_1\, b} \tag{6.2.12}$$

We may also develop a time approximation scheme using the finite element method. To this end, we consider the problem in (6.2.8). We wish to determine u_{s+1} in terms of u_s. The weighted-integral form of (6.2.8) over the time interval (t_s, t_{s+1}) is

$$0 = \int_{t_s}^{t_{s+1}} v(t) \left(a \frac{du}{dt} + bu - f \right) dt \tag{6.2.13}$$

where v is the weight function. Assuming a solution of the form

$$u(t) \approx \sum_{j=1}^{n} u_j \psi_j(t)$$

where $\psi_j(t)$ are interpolation functions of order $(n-1)$. The Galerkin finite element model is obtained by substituting the above approximation for u and $v = \psi_i$ into (6.2.13). We obtain

$$[A]\{u\} = \{F\} \tag{6.2.14}$$

where

$$A_{ij} = \int_{t_s}^{t_{s+1}} \psi_i(t) \left(a \frac{d\psi_j}{dt} + b\psi_j \right) dt, \quad F_i = \int_{t_s}^{t_{s+1}} \psi_i(t) f(t)\, dt \tag{6.2.15}$$

Equation (6.2.14) is valid with the time interval (t_s, t_{s+1}), and it represents a relationship between the values $u_1, u_2, \ldots, u_n$, which are the values of u at time $t_s, t_{s+\Delta t/(n-1)}, t_{s+2\Delta t/(n-1)}, \ldots, t_{s+1}$, respectively. This would yield a multistep approximation scheme.

To obtain a single-step approximation scheme, i.e., write u_{s+1} in terms of u_s only, we assume linear approximation (i.e., $n = 2$)

$$u(t) = u_s \psi_1(t) + u_{s+1} \psi_2(t)$$

where $\psi_1(t) = \frac{t_{s+1} - t}{\Delta t}$ and $\psi_2(t) = \frac{t - t_s}{\Delta t}$. In the same fashion, $f(t)$ can be represented in terms of its values at t_s and t_{s+1}:

$$f(t) = f_s \psi_1(t) + f_{s+1} \psi_2(t)$$

For this choice of approximation, Eq. (6.2.15) becomes

$$\left(\frac{a}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{b\Delta t}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_s \\ u_{s+1} \end{Bmatrix} = \frac{\Delta t}{6} \begin{Bmatrix} f_s \\ 2 f_{s+1} \end{Bmatrix} \tag{6.2.16}$$

Assuming that u_s is known, we solve for u_{s+1} from the second equation in (6.2.16)

$$\left(a + \frac{2b\Delta t}{3} \right) u_{s+1} = \left(a - \frac{b\Delta t}{3} \right) u_s + \Delta t \left(\frac{f_s}{3} + \frac{2 f_{s+1}}{3} \right) \tag{6.2.17}$$

Comparing Eq. (6.2.17) with Eq. (6.2.11a), we find that the Galerkin scheme is a special case of the α-family of approximation, with $\alpha = 2/3$.

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Stable and Conditionally Stable Schemes

Equation (6.2.11*b*) can be written in the form

$$u_{s+1} = A(u_s) + F_{s,s+1}, \quad A = \frac{a - (1-\alpha)\,\Delta t_{s+1}\, b}{a + \alpha\,\Delta t_{s+1}\, b}$$

$$F_{s,s+1} = \Delta t_{s+1} \frac{[\alpha\, f_{s+1} + (1-\alpha) f_s]}{a + \alpha\,\Delta t_{s+1}\, b} \tag{6.2.18}$$

The operator A is known as the *amplification operator*. Since u_s is an approximate solution, the error $E_s = u_a(t_s) - u_s$ at time t_s (where u_a is the exact solution) will influence the solution at t_{s+1}. The error will grow (i.e., E_s will be amplified) as we march in time if the magnitude of the operator is greater than 1, $|A| > 1$. When the error grows without bound, the computational scheme (6.2.11*b*) becomes unstable (i.e., solution u_{s+1} becomes unbounded with time). Therefore, in order for the scheme to be stable, it is necessary that $|A| \le 1$:

$$\left| \frac{a - (1-\alpha)\,\Delta t_{s+1}\, b}{a + \alpha\,\Delta t_{s+1}\, b} \right| \le 1 \tag{6.2.19}$$

The above equation places a restriction on the magnitude of the time step for certain values of α. When the error remains bounded for any time step [i.e., condition (6.2.19) is trivially satisfied for any value of Δt], the scheme is *stable*. If the error remains bounded only when the time step remains below certain value [in order to satisfy (6.2.19)], the scheme is said to be *conditionally stable*.

For different values of α, the time approximation scheme in (6.2.11*b*) yields a different scheme. The following well-known time-approximation schemes along with their order of accuracy and stability should be noted:

$$\alpha = \begin{cases} 0, & \text{the forward difference (or Euler) scheme (conditionally stable); order of accuracy } = O(\Delta t) \\ \frac{1}{2}, & \text{the Crank–Nicolson scheme (stable); } O(\Delta t)^2 \\ \frac{2}{3}, & \text{the Galerkin method (stable); } O(\Delta t)^2 \\ 1, & \text{the backward difference scheme (stable); } O(\Delta t) \end{cases} \tag{6.2.20}$$

Of these, the Crank–Nicolson method is the most commonly used scheme.

Fully Discretized Finite Element Equations

We now have the tools necessary to convert the set of ordinary differential equations (6.2.7*a*) to a set of algebraic equations in much the same way we converted a single differential equation (6.2.8) to an algebraic equation (6.2.11*b*). Here, we work with matrix equation [i.e., $[M^1] = [M]$ and $[M^2] = [0]$ in Eq. (6.2.7*a*)]

$$[M]\{\dot{u}\} + [K]\{u\} = \{F\} \tag{6.2.21a}$$

subject to the initial conditions

$$\{u\}_0 = \{u_0\} \tag{6.2.21b}$$

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where $\{u\}_0$ denotes the vector of nodal values of u at time $t=0$, whereas $\{u_0\}$ denotes the column of nodal values u_{j0}. As applied to a vector of time derivatives of the nodal values, the α-family of approximation reads as

$$\Delta t_{s+1}\,[(1-\alpha)\{\dot{u}\}_s + \alpha\{\dot{u}\}_{s+1}] = \{u\}_{s+1} - \{u\}_s \quad \text{for } 0 \le \alpha \le 1 \tag{6.2.22a}$$

or

$$\begin{aligned} \{u\}_{s+1} &= \{u\}_s + \Delta t\{\dot{u}\}_{s+\alpha} \\ \{\dot{u}\}_{s+\alpha} &= (1-\alpha)\{\dot{u}\}_s + \alpha\{\dot{u}\}_{s+1} \end{aligned} \tag{6.2.22b}$$

for $0 \le \alpha \le 1$. Equation (6.2.22a) can be used to reduce the ordinary differential equations (6.2.21a) to algebraic equations among the u_j at time t_{s+1}. Since (6.2.21a) is valid for any $t > 0$, we can write it for times $t = t_s$ and $t = t_{s+1}$:

$$[M]\{\dot{u}\}_s + [K]_s\{u\}_s = \{F\}_s \tag{6.2.23a}$$

$$[M]\{\dot{u}\}_{s+1} + [K]_{s+1}\{u\}_{s+1} = \{F\}_{s+1} \tag{6.2.23b}$$

where it is assumed that the matrix $[M]$ is independent of time. Premultiplying both sides of (6.2.22a) with $[M]$ we obtain

$$\Delta t_{s+1}\alpha[M]\{\dot{u}\}_{s+1} + \Delta t_{s+1}(1-\alpha)[M]\{\dot{u}\}_s = [M](\{u\}_{s+1} - \{u\}_s)$$

Substituting for $[M]\{\dot{u}\}_{s+1}$ and $[M]\{\dot{u}\}_s$ from (6.2.23a) and (6.2.23b), respectively, we arrive at

$$\Delta t_{s+1}\alpha(\{F\}_{s+1} - [K]_{s+1}\{u\}_{s+1}) + \Delta t_{s+1}(1-\alpha)(\{F\}_s - [K]_s\{u\}_s) = [M](\{u\}_{s+1} - \{u\}_s)$$

Solving for the vector $\{u\}_{s+1}$, we obtain

$$[\hat{K}]_{s+1}\{u\}_{s+1} = [\bar{K}]_s\{u\}_s + \{\bar{F}\}_{s,s+1} \tag{6.2.24a}$$

where

$$\begin{gathered} [\hat{K}]_{s+1} = [M] + a_1[K]_{s+1}, \quad [\bar{K}]_s = [M] - a_2[K]_s \\ \{\bar{F}\}_{s,s+1} = \Delta t_{s+1}[\alpha\{F\}_{s+1} + (1-\alpha)\{F\}_s] \\ a_1 = \alpha\Delta t_{s+1}, \quad a_2 = (1-\alpha)\Delta t_{s+1} \end{gathered} \tag{6.2.24b}$$

Note that, in deriving (6.2.24a) and (6.2.24b), it has been assumed that $[M]$ is independent of time and that the time step is nonuniform.

Equations (6.2.24a) and (6.2.24b) are valid for a typical finite element whose semidiscretized equations are of the form (6.2.21a). In other words, Eqs. (6.2.24a) and (6.2.24b) hold for any problem, independent of the dimension and method of spatial approximation, as long as the end result is Eq. (6.2.21a). The assembly, imposition of boundary conditions, and solution of the assembled equations are the same as described before for steady-state problems. Calculation of $[\hat{K}]$ and $\{\bar{F}\}$ at time $t=0$ requires knowledge of the initial conditions $\{u\}_0$ and the time variation of $\{F\}$. Note that for $\alpha = 0$ (the forward difference scheme), we obtain $[\hat{K}] = [M]$. If matrix $[M]$ is diagonal, (6.2.24a) becomes

$$u_i^{s+1} = \frac{1}{M_{(ii)}}\left(\sum_{j=1}^{n} \bar{K}_{ij}^s u_j^s + \hat{F}_i^{s,s+1}\right), \quad \text{(no sum on } i\text{)} \tag{6.2.25}$$

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Thus, no inversion of the coefficient matrix is required in solving for $\{u\}^{s+1}$. Such a scheme is called *explicit*. A scheme is said to be *implicit* when it is not explicit (i.e., an implicit scheme requires the inversion of a coefficient matrix). Thus, explicit schemes are computationally less expensive compared to implicit schemes; implicit schemes are more accurate and have larger critical time steps. In real-world problems, the cost of computation precludes the use of implicit schemes.

In conventional finite element formulations, $[M]$ is seldom diagonal. Therefore, explicit schemes in finite element analysis can exist only if the time-approximation scheme is such that $[\hat{K}] = [M]$ and $[M]$ is *diagonal*. The matrix $[M]$ computed according to the definition (6.2.7*c*) is called the *consistent (mass) matrix*, and it is not diagonal unless ψ_i are orthogonal functions over the element domain. There are several ways to diagonalize mass matrices; these will be discussed in Section 6.2.5.

Consistency, Accuracy, and Stability

Since (6.2.22*a*) represents an approximation, which is used to derive (6.2.24*a*), error is introduced into the solution $\{u\}_{s+1}$ at each time step. In addition to the truncation error introduced in approximating the derivative, round-off errors can be introduced because of the finite arithmetic used in the computations. Since the solution at time t_{s+1} depends on the solution at time t_s, the error can grow with time. As discussed earlier, if the error is bounded, the solution scheme is said to be stable. The numerical scheme (6.2.24*a*) is said to be *consistent* with the continuous problem (6.2.7*a*) if the round-off and truncation errors go to zero as $\Delta t \to 0$. *Accuracy* of a numerical scheme is a measure of the closeness between the approximate solution and the exact solution whereas *stability* of a solution is a measure of the boundedness of the approximate solution with time. As we might expect, the size of the time step can influence both accuracy and stability. When we construct an approximate solution, we like it to converge to the true solution when the number of elements or the degree of approximation is increased and the time step Δt is decreased. A time-approximation scheme is said to be *convergent* if, for fixed t_s, the numerical value $\{u\}_s$ converges to its true value $\{u(t_s)\}$ as $\Delta t \to 0$. Accuracy is measured in terms of the rate at which the approximate solution converges. If a numerical scheme is stable and consistent, it is also convergent.

For all numerical schemes in which $\alpha < \frac{1}{2}$, the α-family of approximations is stable only if the time step satisfies the following (stability) condition [follows from Eq. (6.2.19)]:

$$\Delta t < \Delta t_{\text{cri}} \equiv \frac{2}{(1-2\alpha)\lambda} \tag{6.2.26}$$

where λ is the largest eigenvalue associated with the problem and Δt_{cri} is (see below)

$$([M] - \lambda[K])\{u\} = \{Q\} \tag{6.2.27}$$

Note that the same mesh as that used for the transient analysis must be used to calculate the eigenvalues.

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6.2.4 Hyperbolic Equations

Time Approximation

Consider matrix equations of the form [set $[M^1] = [C]$ and $[M^2] = [M]$ in Eq. (6.2.7*a*)]

$$[K]\{u\} + [C]\{\dot{u}\} + [M]\{\ddot{u}\} = \{F\} \tag{6.2.28a}$$

subjected to initial conditions

$$\{u(0)\} = \{u_0\}, \quad \{\dot{u}(0)\} = \{v_0\} \tag{6.2.28b}$$

Such equations arise in structural dynamics, where $[M]$ denotes mass matrix, $[C]$ the damping matrix, and $[K]$ the stiffness matrix. The damping matrix $[C]$ is often taken to be a linear combination of the mass and stiffness matrices, $[C] = \beta_1[M] + \beta_2[K]$, where β_1 and β_2 are determined from physical experiments. In the present study, we will not consider damping (i.e., $[C] = [0]$) in the numerical examples, although the theoretical developments will account for it. Transient analysis of both bars and beams lead to equations of the type given in (6.2.28*a*) and (6.2.28*b*). The mass and stiffness matrices for these problems are discussed below.

Axial Motion of Bars

The equation of motion is given by Eq. (6.1.14). The semidiscretization results in Eq. (6.2.28*a*), with $[M^e]$ and $[K^e]$ given by Eq. (6.1.27*b*) and $[C] = [0]$.

Transverse Motion of Euler–Bernoulli Beams

The equation of motion is given by Eq. (6.1.36). The semidiscretized equation is the same as Eq. (6.2.28*a*), with $[M^e]$ and $[K^e]$ given by Eq. (6.1.41*b*) and $[C] = [0]$.

Transverse Motion of Timoshenko Beams

The equations of motion are given by Eqs. (6.1.43*a*) and (6.1.43*b*). The semidiscretization results in Eq. (6.2.28*a*), with $[M^e]$ and $[K^e]$ given by Eq. (6.1.48*a*). Note that $[M^e]$ and $[K^e]$ are the matrices given in Eq. (6.1.47).

There are several numerical methods available to approximate the second-order time derivatives and convert differential equations to algebraic equations. Among these, the Newmark family of time-approximation schemes is widely used in structural dynamics [Newmark (1959)]. Other methods, such as the Wilson method [Bathe and Wilson (1973)] and the Houbolt method [Houbolt (1950)], can also be used to develop the algebraic equations from the second-order differential equations (6.2.28*a*). Here we consider the Newmark family of approximation schemes.

Newmark's Scheme

In the Newmark method, the function and its first time derivative are approximated according to

$$\{u\}_{s+1} = \{u\}_s + \Delta t\{\dot{u}\}_s + \frac{1}{2}(\Delta t)^2\{\ddot{u}\}_{s+\gamma} \tag{6.2.29a}$$

$$\{\dot{u}\}_{s+1} = \{\dot{u}\}_s + \{\ddot{u}\}_{s+\alpha}\Delta t \tag{6.2.29b}$$

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where

$$\{\ddot{u}\}_{s+\theta} = (1-\theta)\{\ddot{u}\}_s + \theta\{\ddot{u}\}_{s+1} \tag{6.2.29c}$$

and α and γ $(=2\beta)$ are parameters that determine the stability and accuracy of the scheme. Equations (6.2.29a) and (6.2.29b) can be viewed as Taylor's series expansions of u and $\dot{u}$. The following schemes are special cases of (6.2.29a) and (6.2.29b):

$$\begin{array}{lll} \alpha=\frac{1}{2}, & \gamma=2\beta=\frac{1}{2} & \text{Constant-average acceleration method (stable)} \\ \alpha=\frac{1}{2}, & \gamma=2\beta=\frac{1}{3} & \text{Linear acceleration method (conditionally stable)} \\ \alpha=\frac{1}{2}, & \gamma=2\beta=0 & \text{Central difference method (conditionally stable)} \\ \alpha=\frac{3}{2}, & \gamma=2\beta=\frac{8}{5} & \text{Galerkin method (stable)} \\ \alpha=\frac{3}{2}, & \gamma=2\beta=2 & \text{Backward difference method (stable)} \end{array} \tag{6.2.30}$$

For all schemes in which $\gamma < \alpha$ and $\alpha \geq \frac{1}{2}$, the stability requirement is

$$\Delta t \leq \Delta t_{\text{cri}} = \left[\frac{1}{2}\omega_{\max}^2(\alpha-\gamma)\right]^{-1/2} \tag{6.2.31}$$

where $\omega_{\max}$ is the maximum natural frequency of the system (6.2.28a) without $[C]$:

$$\left([M]-\omega^2[K]\right)\{u\} = \{F\} \tag{6.2.32}$$

Fully Discretized Finite Element Equations

Eliminating $\{\ddot{u}\}_{s+1}$ from Eqs. (6.2.29a) and (6.2.29b) and writing the result for $\{\dot{u}\}_{s+1}$, we obtain

$$\{\dot{u}\}_{s+1} = a_6\left(\{u\}_{s+1} - \{u\}_s\right) - a_7\{\dot{u}\}_s - a_8\{\ddot{u}\}_s \tag{6.2.33}$$

where

$$a_6 = \frac{\alpha}{\beta\Delta t}, \quad a_7 = \frac{\alpha}{\beta} - 1, \quad a_8 = \left(\frac{\alpha}{\gamma} - 1\right)\Delta t \tag{6.2.34}$$

Now premultiplying Eq. (6.2.29a) with $[M]_{s+1}$ and substituting for $[M]_{s+1}\{\ddot{u}\}_{s+1}$ from Eq. (6.2.28a), we obtain

$$\begin{aligned} \left([M]_{s+1} + \beta(\Delta t)^2[K]_{s+1}\right)\{u\}_{s+1} &= [M]_{s+1}\{b\}_s + \beta(\Delta t)^2\{F\}_{s+1} \\ &\quad -\beta(\Delta t)^2[C]_{s+1}\{\dot{u}\}_{s+1} \end{aligned} \tag{6.2.35}$$

where

$$\{b\}_s = \{u\}_s + \Delta t\{\dot{u}\}_s + \frac{1}{2}(1-\gamma)(\Delta t)^2\{\ddot{u}\}_s \tag{6.2.36}$$

Now, multiplying throughout with $a_3 = 1/[\beta(\Delta t)^2]$, we arrive at

$$\left(a_3[M]_{s+1} + [K]_{s+1}\right)\{u\}_{s+1} = a_3[M]_{s+1}\{b\}_s + \{F\}_{s+1} - [C]_{s+1}\{\dot{u}\}_{s+1} \tag{6.2.37}$$

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Using Eq. (6.2.33) for $\{\dot{u}\}_{s+1}$ in Eq. (6.2.37) and collecting terms, we obtain the final result

$$[\hat{K}]_{s+1}\{u\}_{s+1} = \{\hat{F}\}_{s,s+1} \tag{6.2.38}$$

where

$$\begin{aligned}
[\hat{K}]_{s+1} &= [K]_{s+1} + a_3[M]_{s+1} + a_6[C]_{s+1} \\
\{\hat{F}\}_{s,s+1} &= \{F\}_{s+1} + [M]_{s+1}\{A\}_s + [C]_{s+1}\{B\}_s \\
\{A\}_s &= a_3\{b\}_s = a_3\{u\}_s + a_4\{\dot{u}\}_s + a_5\{\ddot{u}\}_s \\
\{B\}_s &= a_6\{u\}_s + a_7\{\dot{u}\}_s + a_8\{\ddot{u}\}_s
\end{aligned} \tag{6.2.39}$$

and

$$a_3 = \frac{1}{\beta(\Delta t)^2}, \quad a_4 = a_3\Delta t\ , \quad a_5 = \frac{1}{\gamma} - 1 \tag{6.2.40}$$

Note that the calculation of $[\hat{K}]$ and $\{\hat{F}\}$ requires knowledge of the initial conditions $\{u\}_0$, $\{\dot{u}\}_0$, and $\{\ddot{u}\}_0$. In practice, we do not know $\{\ddot{u}\}_0$. As an approximation, it can be calculated from (6.2.28*a*) (we often assume that the applied force is zero at $t = 0$):

$$\{\ddot{u}\}_0 = [M]^{-1}(\{F\}_0 - [K]\{u\}_0 - [C]\{\dot{u}\}_0) \tag{6.2.41}$$

At the end of each time step, the new velocity vector $\{\dot{u}\}_{s+1}$ and acceleration vector $\{\ddot{u}\}_{s+1}$ are computed using [from Eqs. (6.2.29*a*) and (6.2.29*b*)]

$$\begin{aligned}
\{\ddot{u}\}_{s+1} &= a_3(\{u\}_{s+1} - \{u\}_s) - a_4\{\dot{u}\}_s - a_5\{\ddot{u}\}_s \\
\{\dot{u}\}_{s+1} &= \{\dot{u}\}_s + a_2\{\ddot{u}\}_s + a_1\{\ddot{u}\}_{s+1} \\
a_1 &= \alpha\Delta t, \quad a_2 = (1-\alpha)\Delta t
\end{aligned} \tag{6.2.42}$$

The remaining procedure stays the same as in static problems.

The fully discretized model in (6.2.38) is based on the assumption that $\gamma \neq 0$. Obviously, for centered difference scheme ($\gamma = 0$), the formulation must be modified (see Problem 6.26).

6.2.5 Mass Lumping

Recall from the time approximation of parabolic equations that use of the forward difference scheme (i.e., $\alpha = 0$) results in the following time marching scheme [see (6.2.24*a*) and (6.2.24*b*)]:

$$[M^e]\{u\}_{s+1} = ([M^e] - \Delta t[K^e])\{u\}_s + \Delta t\{F^e\}_s \tag{6.2.43}$$

The mass matrix $[M^e]$ derived from the weighted-integral formulations of the governing equation is called the *consistent mass matrix*, and it is symmetric positive-definite and nondiagonal. Solution of the global equations associated with (6.2.43) requires inversion of the assembled mass matrix. If the mass matrix is diagonal, then the assembled equations can be solved directly (i.e., without inverting a matrix)

$$(U_I)_{s+1} = M_{II}^{-1}\left[M_{II}(U_I)_s - \Delta t\sum_{J=1}^{N} K_{IJ}(U_J)_s + \Delta t(F_I)_s\right] \tag{6.2.44}$$

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and thus saving computational time. The explicit nature of (6.2.44) has motivated analysts to find rational ways of diagonalizing the mass matrix.

There are several ways of constructing diagonal mass matrices, also known as *lumped mass matrices*. The *row-sum* and *proportional lumping* techniques are discussed here.

Row-Sum Lumping

The sum of the elements of each row of the consistent mass matrix is used as the diagonal element:

$$M_{ii}^e = \sum_{j=1}^{n} \int_{x_a}^{x_b} \rho \psi_i^e \psi_j^e \, dx = \int_{x_a}^{x_b} \rho \psi_i^e \, dx \tag{6.2.45}$$

where the property $\sum_{j=1}^{n} \psi_j^e = 1$ of the interpolation functions is used. When ρ is constant, (6.2.45) gives

$$[M^e]_L = \frac{\rho h_e}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(linear element)}$$

$$[M^e]_L = \frac{\rho h_e}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(quadratic element)} \tag{6.2.46}$$

Compare these lumped mass matrices with the consistent mass matrices

$$[M^e]_C = \frac{\rho h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{(linear element)}$$

$$[M^e]_C = \frac{\rho h_e}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \quad \text{(quadratic element)} \tag{6.2.47}$$

Here subscripts L and C refer to lumped and consistent mass matrices, respectively.

Proportional Lumping

Here the diagonal elements of the lumped mass matrix are computed to be proportional to the diagonal elements of the consistent mass matrix while conserving the total mass of the element:

$$M_{ii}^e = \alpha \int_{x_a}^{x_b} \rho \psi_i^e \psi_i^e \, dx, \quad \alpha = \int_{x_a}^{x_b} \rho \, dx \Big/ \sum_{i=1}^{n} \int_{x_a}^{x_b} \rho \psi_i^e \psi_i^e \, dx \tag{6.2.48}$$

For constant ρ, proportional lumping gives the same lumped mass matrices as those obtained in the row-sum technique for the Lagrange linear and quadratic elements.

The use of a lumped mass matrix in transient analyses can save computational time in two ways. First, for forward difference schemes, lumped mass matrices result in explicit algebraic equations not requiring matrix inversions. Second, the critical time step required for conditionally stable schemes is larger, and hence less computational time is required when lumped mass matrices are used. To see this, consider the stability criterion in (6.2.31) for the case $\alpha = \frac{1}{2}$ and $\beta = 0$. For a one linear element model of a uniform bar of stiffness

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EA and mass ρA, fixed at the left end, the eigenvalue problem with a consistent mass matrix is

$$\left(\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \omega^2\frac{\rho Ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}U_1\\ U_2\end{Bmatrix} = \begin{Bmatrix}Q_1^1\\ Q_2^1\end{Bmatrix} \tag{6.2.49}$$

Since $U_1 = 0$ and $Q_2^1 = 0$, we have

$$\omega^2 = \frac{EA}{h}\bigg/\frac{\rho Ah}{3} = \frac{3E}{\rho h^2}$$

Substituting this into the critical time step relation (6.2.31), we have,

$$(\Delta t_{\text{cri}})_C = 2/\omega_{\max} = h(4\rho/3E)^{1/2}$$

If we use the lumped matrix, ω is given by

$$\omega = (2E/\rho)^{1/2}/h$$

and the critical time step is

$$(\Delta t_{\text{cri}})_L = h(2\rho/E)^{1/2} > (\Delta t_{\text{cri}})_C \tag{6.2.50}$$

Thus, explicit schemes require larger time steps than implicit schemes.

6.2.6 Applications

Here we consider two examples of applications of finite element models of one-dimensional problems. Example 6.2.1 is taken from transient heat transfer (parabolic equation), and Example 6.2.2 is taken from solid mechanics (hyperbolic equation).

Example 6.2.1

Consider the transient heat conduction problem described by the differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } 0 < x < 1 \tag{6.2.51a}$$

with boundary conditions

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(1, t) = 0 \tag{6.2.51b}$$

and initial condition

$$u(x, 0) = 1.0 \tag{6.2.51c}$$

where u is the nondimensionalized temperature. The problem at hand is a special case of (6.2.3a) with $a = 1$, $b = 0$, $c_0 = 0$, $c_1 = 1$, $c_2 = 0$, and $f = 0$ [or Eq. (6.1.2) with $kA = 1$, $\rho cA = 1$, and $q = 0$]. The finite element model is

$$[M^e]\{\dot{u}\} + [K^e]\{u\} = \{Q^e\}$$

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where

$$M_{ij}^e = \int_{x_a}^{x_b} \psi_i^e \psi_j^e \, dx, \quad K_{ij}^e = \int_{x_a}^{x_b} \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} \, dx$$

For the choice of linear interpolation functions, the semidiscrete equations of a typical element are

$$\frac{h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \dot{u}_1^e \\ \dot{u}_2^e \end{Bmatrix} + \frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

where h_e is the length of the element. Use of the α-family of approximation (6.2.10a) results in the equation [see (6.2.24a)]

$$([M^e] + \Delta t\, \alpha [K^e])\{u^e\}_{s+1} = ([M^e] - \Delta t(1-\alpha)[K^e])\{u^e\}_s + \Delta t(\alpha\{Q^e\}_{s+1} + (1-\alpha)\{Q^e\}_s)$$

where Δt is the time step.

First, consider a one-element mesh. We have ($h = 1$)

$$\begin{bmatrix} \frac{1}{3}h + \alpha\frac{\Delta t}{h} & \frac{1}{6}h - \alpha\frac{\Delta t}{h} \\ \frac{1}{6}h - \alpha\frac{\Delta t}{h} & \frac{1}{3}h + \alpha\frac{\Delta t}{h} \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}_{s+1} = \begin{bmatrix} \frac{1}{3}h - (1-\alpha)\frac{\Delta t}{h} & \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} \\ \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} & \frac{1}{3}h - (1-\alpha)\frac{\Delta t}{h} \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}_s + \Delta t \begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{Bmatrix}$$

where $\bar{Q}_i = \alpha(Q_i^1)_{s+1} + (1-\alpha)(Q_i^1)_s$. The boundary conditions of the problem require

$$(U_1)_s = 0, \quad (Q_2^1)_s = 0 \quad \text{for all } s > 0 \text{ (i.e., } t > 0)$$

However, the initial condition (6.2.51c) requires

$$U_1(0)\psi_1(x) + U_2(0)\psi_2(x) = 1$$

Since the initial condition should be consistent with the boundary conditions, we take $(U_1)_0 = 0$. Then it follows that $(U_2)_0 = 1$.

Using the boundary conditions, we can write for the one-element model ($h = 1.0$)

$$\left(\frac{1}{3}h + \alpha\frac{\Delta t}{h}\right)(U_2)_{s+1} = \left[\frac{1}{3}h - (1-\alpha)\frac{\Delta t}{h}\right](U_2)_s \tag{6.2.52}$$

which can be solved repeatedly for U_2 at different times, $s = 0, 1, \ldots$.

Repeated use of (6.2.52) can cause the temporal approximation error to grow with time, depending on the value of α. As noted earlier, the forward difference scheme ($\alpha = 0$) is a conditionally stable scheme. To determine the critical time step for the one-element mesh, we first calculate the maximum eigenvalue of the associated system

$$\left(-\lambda\frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}, \quad U_1 = 0, \quad Q_2^1 = 0$$

The eigenvalue is (there is only one)

$$-\frac{1}{3}\lambda h U_2 + h^{-1}U_2 = 0, \quad \text{or} \quad \lambda = 3/h^2 = 3$$

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Hence, from Eq. (6.2.26) we have $\Delta t_{\text{cri}} = 2/\lambda = 0.66667$. Thus, in order for the forward difference scheme (6.2.52)

$$\frac{1}{3}h(U_2)_{s+1} = \left(\frac{1}{3}h - \frac{\Delta t}{h}\right)(U_2)_s$$

to be stable, the time step should be smaller than $\Delta t_{\text{cri}} = 0.6667$; otherwise, the solution will be unstable, as shown in Fig. 6.2.3.

For a two-element mesh, we have $(h_1 = h_2 = h = 0.5)$; the condensed equations of the time-marching scheme are given by

$$\begin{bmatrix} \frac{2}{3}h + 2\alpha\frac{\Delta t}{h} & \frac{1}{6}h - \alpha\frac{\Delta t}{h} \\ \frac{1}{6}h - \alpha\frac{\Delta t}{h} & \frac{1}{3}h + \alpha\frac{\Delta t}{h} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_{s+1} = \begin{bmatrix} \frac{2}{3}h - 2(1-\alpha)\frac{\Delta t}{h} & \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} \\ \frac{1}{6}h + (1-\alpha)\frac{\Delta t}{h} & \frac{1}{3}h - (1-\alpha)\frac{\Delta t}{h} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_s$$

with $(U_2)_0$ and $(U_3)_0$. The forward difference scheme yields

$$\frac{h}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_{s+1} = \frac{h}{6}\begin{bmatrix} 4-2\mu & 1+\mu \\ 1+\mu & 2-\mu \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}_s, \quad \mu = \frac{6\Delta t}{h^2}$$

Figure 6.2.3 Stability of the forward difference ($\alpha = 0.0$) and Crank–Nicolson ($\alpha = 0.5$) schemes as applied to a parabolic equation.

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The associated eigenvalue problem is

$$\left(-\lambda\frac{h}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}+\frac{1}{h}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\right)\begin{Bmatrix}U_2\\ U_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

The characteristic equation is

$$7\bar{\lambda}^2-10\bar{\lambda}+1=0,\quad \bar{\lambda}=\frac{\lambda h^2}{6}$$

whose roots are $\lambda_1=2.5967$ and $\lambda_2=31.6891$. Hence, the critical time step becomes $\Delta t_{cri}=2/31.6891=0.0631$. A time step of $\Delta t=0.065$ results in an unstable solution, as shown in Fig. 6.2.3.

For (unconditionally) stable schemes ($\alpha\geq\frac{1}{2}$), there is no restriction on the time step (e.g., Crank–Nicolson method with two linear elements and $\Delta t=0.065$ yield very smooth and stable solution, as shown in Fig. 6.2.3). However, to obtain a sufficiently accurate solution, the time step must be taken as a fraction of Δt_{cri}. Of course, the accuracy of the solution also depends on the mesh size h. As this is decreased (i.e., the number of elements is increased), Δt_{cri} decreases.

Plots of $u(1,t)$ versus time for $\alpha=0$ and $\alpha=0.5$ with $\Delta t=0.05$ are shown in Fig. 6.2.4. Solutions predicted by meshes of one, two, or four linear (L) or quadratic (Q) elements are compared. The convergence of the solution with increasing number of elements is clear. The finite element solutions obtained with different methods, time steps, and meshes are compared with the exact solution in Table 6.2.1.

Figure 6.2.4 Transient solution of a parabolic equation according to linear and quadratic finite elements.

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Table 6.2.1 A comparison of the finite element solutions obtained using various time approximation schemes and meshes with the analytical solution of a parabolic equation ($\Delta t = 0.05$).

	$\alpha=0$	$\alpha=1$	$\alpha=0.5$							
t	1L	1L	1L	2L	4L	4L	1Q	2Q	4Q	Exact
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	0.8500	0.8696	0.8605	1.0359	0.9951	0.9933	1.0870	0.9942	0.9928	0.9969
0.10	0.7225	0.7561	0.7404	0.9279	0.9588	0.9554	0.9819	0.9550	0.9549	0.9493
0.15	0.6141	0.6575	0.6371	0.8169	0.8639	0.8707	0.8693	0.8831	0.8725	0.8642
0.20	0.5220	0.5718	0.5482	0.7176	0.7557	0.7694	0.7679	0.7633	0.7731	0.7723
0.25	0.4437	0.4972	0.4717	0.6300	0.6759	0.6824	0.6780	0.6933	0.6855	0.6854
0.30	0.3771	0.4323	0.4059	0.5533	0.5906	0.6037	0.5987	0.6006	0.6070	0.6068
0.35	0.3206	0.3759	0.3492	0.4858	0.5250	0.5325	0.5286	0.5394	0.5358	0.5367
0.40	0.2725	0.3269	0.3005	0.4266	0.4608	0.4713	0.4668	0.4710	0.4741	0.4745
0.45	0.2316	0.2843	0.2586	0.3746	0.4083	0.4158	0.4121	0.4201	0.4188	0.4194
0.50	0.1969	0.2472	0.2225	0.3289	0.3592	0.3676	0.3639	0.3687	0.3701	0.3708
0.55	0.1673	0.2149	0.1914	0.2888	0.3176	0.3247	0.3213	0.3275	0.3273	0.3277
0.60	0.1422	0.1869	0.1647	0.2536	0.2798	0.2868	0.2837	0.2883	0.2890	0.2897
0.65	0.1209	0.1625	0.1418	0.2227	0.2472	0.2535	0.2505	0.2556	0.2556	0.2561
0.70	0.1028	0.1413	0.1220	0.1955	0.2180	0.2238	0.2212	0.2253	0.2258	0.2264
0.75	0.0874	0.1229	0.1050	0.1717	0.1924	0.1979	0.1953	0.1995	0.1996	0.2001
0.80	0.0743	0.1069	0.0903	0.1508	0.1697	0.1747	0.1725	0.1761	0.1764	0.1769
0.85	0.0631	0.0929	0.0777	0.1324	0.1498	0.1544	0.1523	0.1557	0.1559	0.1563
0.90	0.0536	0.0808	0.0669	0.1162	0.1322	0.1363	0.1345	0.1375	0.1378	0.1382
0.95	0.0456	0.0703	0.0575	0.1020	0.1166	0.1205	0.1187	0.1216	0.1218	0.1222
1.00	0.0388	0.0611	0.0495	0.0896	0.1029	0.1065	0.1048	0.1074	0.1076	0.1080

Example 6.2.2

We wish to determine the transverse motion of a beam clamped at both ends and subjected to initial deflection using EBT and TBT. The governing equations are

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0 \quad \text{for } 0 < x < 1 \tag{6.2.53a}$$

$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(0,t) = 0, \quad w(1,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = 0 \tag{6.2.53b}$$

$$w(x,0) = \sin \pi x - \pi x(1-x), \quad \frac{\partial w}{\partial t}(x,0) = 0 \tag{6.2.53c}$$

Note that the initial deflection of the beam is consistent with the boundary conditions. The initial slope is given by

$$\theta(x,0) = -\left(\frac{\partial w}{\partial x}\right)_{t=0} = -\pi \cos \pi x + \pi(1-2x) \tag{6.2.53d}$$

Because of symmetry about $x = 0.5$ (center of the beam), we consider only a half span of the beam for finite element modeling. Here we use the first half of the beam, $0 \le x \le 0.5$, as the

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computational domain. The boundary condition at $x = 0.5$ is $\theta(0.5, t) = -(\partial w/\partial x)(0.5, t) = 0$. The semidiscretized finite element model of a typical element is

$$\frac{h_e}{420}\begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & 22h_e \\ 13h_e & -3h_e^2 & 22h_e & 4h_e^2 \end{bmatrix}\begin{Bmatrix} \ddot{\Delta}_1^e \\ \ddot{\Delta}_2^e \\ \ddot{\Delta}_3^e \\ \ddot{\Delta}_4^e \end{Bmatrix} + \frac{2}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}\begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

We begin with a one-element mesh with the Euler–Bernoulli beam element. The semidiscretized model is

$$\frac{h}{420}\begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix}\begin{Bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \\ \ddot{U}_4 \end{Bmatrix} + \frac{2}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 \end{Bmatrix}$$

The boundary conditions for the one-element model translate into

$$U_1 = 0, \quad U_2 = 0, \quad U_4 = 0, \quad Q_3^1 = 0 \quad \text{for all } t > 0$$

The initial conditions can be computed from (6.2.53*c*) and (6.2.53*d*)

$$\left.\begin{matrix} U_1 = 0, & U_2 = 0, & U_3 = 0.2146, & U_4 = 0 \\ \dot{U}_1 = 0, & \dot{U}_2 = 0, & \dot{U}_3 = 0 & \dot{U}_4 = 0 \end{matrix}\right\} \quad \text{for } t = 0$$

The condensed equation of the time marching scheme for this case takes the form

$$(K_{33} + a_3 M_{33})(U_3)_{s+1} = (\hat{F}_3)_{s,s+1} \equiv M_{33}(a_3 U_3 + a_4 \dot{U}_3 + a_5 \ddot{U}_3)_s, \quad s = 0, 1, \ldots$$

where a_3, a_4, and a_5 are defined in (6.2.40). The second derivative $\ddot{U}_3$ for time $t = 0$ (i.e., when $s = 0$) is computed from the equation of motion:

$$(\ddot{U}_3)_0 = -\frac{K_{33}(U_3)_0}{M_{33}} = -\left(\frac{12}{h^3} \times 0.2146\right)\frac{420}{156h} = -110.932$$

For $\gamma < \frac{1}{2}$, we must compute the critical time step Δt_{cri}, which depends on the square of maximum natural frequency of the beam [see Eq. (6.2.31)]. For the present model, $\omega_{\max}$ is computed from the eigenvalue problem

$$(K_{33} - \omega^2 M_{33})U_3 = 0 \quad \text{or } \omega^2 = K_{33}/M_{33} = 516.923$$

Hence, the critical time step for $\alpha = 0.5$ and $\gamma = \frac{1}{3}$ (i.e., the linear acceleration scheme) is

$$\Delta t_{\text{cri}} = \sqrt{12}/\omega_{\max} = 0.15236$$

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Although there is no restriction on time integration schemes with $\alpha = 0.5$ and $\gamma > 0.5$, the critical time step provides an estimate of the time step to be used to obtain transient solution.

Figure 6.2.5 shows plots of $w(0.5, t)$ versus time for the scheme with $\alpha = 0.5$ and $\gamma = \frac{1}{3}$. Three different time steps, $\Delta t = 0.175$, 0.150, and 0.05, are used to illustrate the accuracy. For $\Delta t = 0.175 > \Delta t_{\text{cri}}$, the solution is unstable, whereas for $\Delta t < \Delta t_{\text{cri}}$, it is stable but inaccurate. The period of the solution is given by

$$T = 2\pi/\omega = 0.27635$$

For two- and four-element meshes of Euler–Bernoulli elements, the critical time steps are (details are not presented here)

$$(\Delta t_{\text{cri}})_2 = 0.00897, \quad (\Delta t_{\text{cri}})_4 = 0.00135$$

where the subscripts refer to the number of elements in the mesh. The transverse deflection obtained with the one and two Euler–Bernoulli elements ($\Delta t = 0.005$) in half beam for a complete period (0, 0.28) are shown in Fig. 6.2.6.

The problem can also be analyzed using the Timoshenko beam element (RIE). In writing the governing equations [see (6.1.43*a*) and (6.1.43*b*)] of the TBT as they apply to the present problem, we first identify the coefficients GAK_s, EI, ρA, and ρI consistent with those in the differential equation (6.2.53*a*). We have $EI = 1.0$ and $\rho A = 1.0$. Then GAK_s can be computed as

$$GAK_s = \frac{E}{2(1+\nu)} BHK_s = \frac{EI}{2(1+\nu)} \frac{12}{H^2} \frac{5}{6} = \frac{4}{H^2} EI \tag{6.2.54}$$

Figure 6.2.5 Center deflection $w(0.5, t)$ versus time t for a clamped beam.

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Figure 6.2.6 Center transverse deflection versus time for a clamped beam subjected to an initial transverse deflection ($\Delta t = 0.005$, $\alpha = 0.5$, and $\gamma = 0.5$).

where B is the width and H the height of the beam, and $I = \frac{1}{12}BH^3$, $\nu = 0.25$, and $K_s = \frac{5}{6}$ are used in arriving at the last expression. Similarly,

$$\rho I = \rho \frac{1}{12} BH^3 = \frac{1}{12} \rho A H^2 \tag{6.2.55}$$

Thus, the governing equations of the TBT for the problem at hand are

$$\frac{\partial^2 w}{\partial t^2} - \frac{4}{H^2} \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} + \Psi \right) = 0 \tag{6.2.56a}$$

$$\frac{H^2}{12} \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} + \frac{4}{H^2} \left(\frac{\partial w}{\partial x} + \Psi \right) = 0 \tag{6.2.56b}$$

The values of $w(0.5, t)$ as obtained using the Euler–Bernoulli and Timoshenko elements (both elements include the rotary inertia) for various numbers of elements are presented in Table 6.2.2. The time step is taken to be $\Delta t = 0.005$, which is smaller than the critical time step of the two-element mesh of the Euler–Bernoulli beam element when $\gamma = \frac{1}{3}$. Plots of $w(0.5, t)$ obtained with two and four linear Timoshenko beam elements, two quadratic Timoshenko beam elements for $L/H = 100$ (since $L = 1.0$, we take $H = 0.01$; for $L/H = 100$ the shear deformation effect is negligible) along with the two-element solution of the Euler–Bernoulli beam element are shown in Fig. 6.2.7. The two linear element mesh of TBT beam elements predicts transient response that differs significantly from the EBT solution, and the TBT solution converges to the EBT solution as the number of elements or the degree of approximation is increased. Should we use conditionally stable schemes, it can be shown that the Timoshenko beam element requires larger Δt_{cri} than the Euler–Bernoulli beam element. This is because, as L/H is decreased, the ω_{max} predicted by the TBT is smaller than that predicted by the EBT.

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Table 6.2.2 Effect of mesh on the transient response of a beam clamped at both ends ($\Delta t = 0.005$).

	EBT elements					TBT elements*		
	$\alpha = 0.5, \gamma = 1/3$		$\alpha = 0.5, \gamma = 0.5$			$\alpha = 0.5, \gamma = 0.5$		
Time t	1	2	1	2	4	2L	4L	2Q
0.00	0.2146	0.2146	0.2146	0.2146	0.2146	0.2146	0.2146	0.2146
0.01	0.2091	0.2098	0.2091	0.2098	0.2098	0.2124	0.2099	0.2116
0.02	0.1928	0.1951	0.1928	0.1951	0.1951	0.1938	0.1936	0.1951
0.03	0.1666	0.1696	0.1667	0.1696	0.1698	0.1550	0.1686	0.1690
0.04	0.1319	0.1346	0.1320	0.1348	0.1350	0.1145	0.1374	0.1427
0.05	0.0904	0.0930	0.0905	0.0931	0.0935	0.0695	0.0980	0.1067
0.06	0.0442	0.0481	0.0443	0.0482	0.0483	0.0093	0.0073	0.0657
0.07	−0.0043	0.0014	−0.0041	0.0014	0.0018	−0.0467	−0.0403	0.0234
0.08	−0.0525	−0.0462	−0.0523	−0.0459	−0.0455	−0.0917	−0.0844	−0.0267
0.09	−0.0980	−0.0926	−0.0978	−0.0923	−0.0916	−0.1422	−0.1254	−0.0706
0.10	−0.1385	−0.1345	−0.1383	−0.1342	−0.1336	−0.1833	−0.1588	−0.1100
0.11	−0.1719	−0.1685	−0.1717	−0.1685	−0.1682	−0.2010	−0.1875	−0.1461
0.12	−0.1964	−0.1933	−0.1963	−0.1933	−0.1932	−0.2154	−0.2069	−0.1717
0.13	−0.2108	−0.2088	−0.2108	−0.2088	−0.2087	−0.2205	−0.2063	−0.1969
0.14	−0.2144	−0.2153	−0.2144	−0.2150	−0.2148	−0.1986	−0.2138	−0.2110
0.15	−0.2070	−0.2113	−0.2071	−0.2112	−0.2111	−0.1696	−0.2134	−0.2146

* 2L = two linear elements; 4L = four linear elements; 2Q = two quadratic elements.

Figure 6.2.7 Transient response of a beam clamped at both ends, according to the TBT and EBT ($EI = 1$, $\rho A = 1$, $H = 0.01$, $\Delta t = 0.005$, $\alpha = 0.5$, and $\gamma = 0.5$).

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6.3 SUMMARY

In this chapter, finite element formulations of eigenvalue problems and time-dependent problems are developed. One-dimensional, second- and fourth-order equations (beams) have been discussed. The eigenvalue problems studied include problems of heat transfer (and the like), bars, and beams. In the case of bars and beams, the eigenvalue problems arise in connection with natural vibrations and buckling of columns. Except for the solution procedure, the finite element formulation of eigenvalue problems is entirely analogous to boundary value problems.

Finite element models of time-dependent problems described by parabolic and hyperbolic equations have also been presented. A two-step procedure to derive finite element models from differential equations has been described. In the first step, we seek spatial approximations of the dependent variables of the problem as linear combinations of nodal values that are functions of time and interpolation functions that are functions of space. This procedure is entirely analogous to the finite element formulation presented for boundary value problems in Chapters 3–5. The end result of this step is a set of ordinary differential equations (in time) among the nodal values. In the second step, the ordinary differential equations are further approximated using the finite difference approximation of the time derivatives. The resulting algebraic equations can be solved for repeatedly, marching in time. Examples using both transient heat transfer and beam bending are presented.

PROBLEMS

Section 6.1

Most problems are formulative in nature. For eigenvalue problems, we need to write the final characteristic equations for the eigenvalues.

6.1 Determine the first two eigenvalues associated with the heat transfer problem whose governing equations and boundary conditions are given by (Fig. P6.1)

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)+b\frac{\partial u}{\partial t}+cu=0 \quad \text{for } 0<x<L$$

$$u(0)=0, \quad \left(a\frac{\partial u}{\partial x}+\beta u\right)\bigg|_{x=L}=0$$

where a, b, c, and β are constants. Use (a) two linear finite elements and (b) one quadratic element in the domain to solve the problem.

Figure P6.1

Figure P6.2

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6.2 Determine the first two longitudinal frequencies of a rod (with Young's modulus E, area of cross section A, and length L) that is fixed at one end (say, at $x = 0$) and supported axially at the other end (at $x = L$) by a linear elastic spring (with spring constant k), as shown in Fig. P6.1:

$$-EA\frac{\partial^2 u}{\partial x^2} + \rho A\frac{\partial^2 u}{\partial t^2} = 0 \quad \text{for } 0 < x < L$$

$$u(0) = 0, \quad \left(EA\frac{du}{dx} + ku\right)\bigg|_{x=L} = 0$$

Use (a) two linear finite elements and (b) one quadratic element in the domain to solve the problem. *Answer:* (a) The characteristic equation is $7\lambda^2 - (10 + 4c)\lambda + (1 + 2c) = 0$, $c = kL/2EA$, $\lambda = (\rho h^2/6E)\omega^2$.

6.3 Determine the smallest natural frequency of a beam with clamped ends and of constant cross-sectional area A, moment of inertia I, and length L. Use the symmetry and two Euler–Bernoulli beam elements in the half beam.

6.4 Resolve the above problem with two RIEs in the half-beam.

6.5 Consider a beam (of Young's modulus E, shear modulus G, area of cross section A, second moment area about the axis of bending I, and length L) with its left end ($x = 0$) clamped and its right end ($x = L$) supported vertically by a linear elastic spring (see Fig. P6.5). Determine the fundamental natural frequency using (a) one Euler–Bernoulli beam element and (b) one Timoshenko beam element (IIE) (use the same mass matrix in both elements).

Figure P6.5

6.6 Consider a simply supported beam (of Young's modulus E, mass density ρ, area of cross section A, second moment of area about the axis of bending I, and length L) with an elastic support at the center of the beam (see Fig. P6.6). Determine the fundamental natural frequency using the minimum number of Euler–Bernoulli beam elements. *Answer:* The characteristic polynomial is $455\lambda^2 - 2(129 + c)\lambda + 3 + 2c = 0$.

Figure P6.6

6.7 Determine the critical buckling load of a cantilever beam (A, I, L, E) using (a) one Euler–Bernoulli beam element and (b) one Timoshenko beam element.

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6.8 The natural vibration of a beam under applied axial compressive load N^0 is governed by the differential equation

$$EI\frac{d^4w}{dx^4} + N^0\frac{d^2w}{dx^2} = \lambda w$$

where λ denotes nondimensional frequency of natural vibration and EI is the flexural stiffness of the beam. (*a*) Determine the fundamental (i.e., smallest) natural frequency ω of a cantilever beam (i.e., fixed at one end and free at the other end) of length L with axial compressive load N_0 using one beam element. (*b*) What is the buckling load of the beam? You are required to give the final characteristic equation in each case.

6.9 Determine the fundamental natural frequency of the truss shown in Fig. P6.9 (you are required only to formulate the problem).

Figure P6.9

Figure P6.10

6.10 Determine the fundamental natural frequency of the truss shown in Fig. P6.10 (you are required only to formulate the problem).

6.11 Determine the first two longitudinal natural frequencies of a rod (A, E, L, m), fixed at one end and with an attached mass m_2 at the other. Use two linear elements. *Hint:* Note that the boundary conditions for the problem are $u(0) = 0$ and $(EA\,\partial u/\partial x + m_2\,\partial^2 u/\partial t^2)|_{x=L} = 0$.

6.12 The equation governing torsional vibration of a circular rod is

$$-GJ\frac{\partial^2\phi}{\partial x^2} + mJ\frac{\partial^2\phi}{\partial t^2} = 0$$

where ϕ is the angular displacement, J the moment of inertia, G the shear modulus, and m the density. Determine the fundamental torsional frequency of a rod with disk (J_1) attached at each end. Use the symmetry and (*a*) two linear elements and (*b*) one quadratic element.

6.13 The equations governing the motion of a beam according to TBT can be reduced to the single equation

$$a^2\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - b^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{b^2 m}{kG}\frac{\partial^4 w}{\partial t^4} = 0$$

where $a^2 = EI/mA$ and $b^2 = I/A$. Here E is the Young's modulus, G is the shear modulus, m is the mass per unit length, A is the area of cross section, and I is the moment of inertia. Assuming that $(b^2 m/kG) << 1$ (i.e., neglect the last term in the governing equation), formulate the finite element model of the (*a*) eigenvalue problem for the determination of natural frequencies and (*b*) fully discretized problem for the determination of the transient response.

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6.14 Use the finite element model of Problem 6.13 to determine the fundamental frequency of a simply supported beam.

6.15 Find the critical buckling load P_{cri} by determining the eigenvalues of the equation

$$EI\frac{d^4w}{dx^4} + P_{cri}\frac{d^2w}{dx^2} = 0 \quad \text{for } 0 < x < L$$

$$w(0) = w(L) = 0, \quad \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=0} = \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=L} = 0$$

Use one Euler–Bernoulli element in the half-beam. *Answer:* $P_{cri} = 9.9439EI/L^2$.

Section 6.2

6.16 Consider the partial differential equation arising in connection with unsteady heat transfer in an insulated rod:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) = f \quad \text{for } 0 < x < L$$

$$u(0,t) = 0, \quad u(x,0) = u_0, \quad \left[a\frac{\partial u}{\partial x} + \beta(u - u_\infty) + \hat{q}\right]\bigg|_{x=L} = 0$$

Following the procedure outlined in Section 6.2, derive the semidiscrete variational form, the semidiscrete finite element model, and the fully discretized finite element equations for a typical element.

6.17 Using a two-element (linear) model and the semidiscrete finite element equations derived in Problem 6.16, determine the nodal temperatures as functions of time for the case in which $a = 1$, $f = 0$, $u_0 = 1$, and $\hat{q} = 0$. Use the Laplace transform technique [see Reddy (2002)] to solve the ordinary differential equations in time.

6.18 Consider a uniform bar of cross-sectional area A, modulus of elasticity E, mass density m, and length L. The axial displacement under the action of time-dependent axial forces is governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \quad a = \left(\frac{E}{m}\right)^{1/2}$$

Determine the transient response [i.e., find $u(x,t)$] of the bar when the end $x = 0$ is fixed and the end $x = L$ is subjected to a force P_0. Assume zero initial conditions. Use one linear element to approximate the spatial variation of the solution, and solve the resulting ordinary differential equation in time exactly to obtain

$$u_2(x,t) = \frac{P_0 L}{AE}\frac{x}{L}(1 - \cos\alpha t), \quad \alpha = \sqrt{3}\frac{a}{L}$$

6.19 Resolve Problem 6.18 with a mesh of two linear elements. Use the Laplace transform method to solve the two ordinary differential equations in time.

6.20 Solve Problem 6.18 when the right end is subjected to an axial force F_0 and supported by an axial spring of stiffness k. *Answer:*

$$u_2(t) = c(1 - \cos\beta t), \quad c = \frac{3F_0}{mAL\beta^2}, \quad \beta = \sqrt{3}\frac{a}{L}\left(1 + \frac{kL}{EA}\right)^{1/2}$$

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6.21 A bar of length L moving with velocity v_0 strikes a spring of stiffness k. Determine the motion $u(x, t)$ from the instant when the end $x = 0$ strikes the spring. Use one linear element.

6.22 A uniform rod of length L and mass m is fixed at $x = 0$ and loaded with a mass M at $x = L$. Determine the motion $u(x, t)$ of the system when the mass M is subjected to a force P_0. Use one linear element. *Answer:*

$$u_2(t) = c(1 - \cos \lambda t), \quad c = \frac{P_0 L}{AE}, \quad \lambda = \sqrt{3}\frac{a}{L}\left(\frac{3M}{AL} + m\right)^{-1}$$

6.23 The flow of liquid in a pipe, subjected to a surge-of-pressure wave (i.e., a water hammer), experiences a surge pressure p, which is governed by the equation

$$\frac{\partial^2 p}{\partial t^2} - c^2 \frac{\partial^2 p}{\partial x^2} = 0, \quad c^2 = \frac{1}{m}\left(\frac{1}{k} + \frac{D}{bE}\right)^{-1}$$

where m is the mass density and K the bulk modulus of the fluid, D is the diameter and b the thickness of the pipe, and E is the modulus of elasticity of the pipe material. Determine the pressure $p(x, t)$ using one linear finite element for the following boundary and initial conditions:

$$p(0, t) = 0, \quad \frac{\partial p}{\partial x}(L, t) = 0, \quad p(x, 0) = p_0, \quad \dot{p}(x, 0) = 0$$

6.24 Consider the problem of determining the temperature distribution of a solid cylinder, initially at a uniform temperature T_0 and cooled in a medium of zero temperature (i.e., $T_\infty = 0$). The governing equation of the problem is

$$\rho c \frac{\partial T}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) = 0$$

The boundary conditions are

$$\frac{\partial T}{\partial r}(0, t) = 0, \quad \left(rk\frac{\partial T}{\partial r} + \beta T\right)\bigg|_{r=R} = 0$$

The initial condition is $T(r, t) = T_0$. Determine the temperature distribution $T(r, t)$ using one linear element. Take $R = 2.5$ cm, $k = 215$ W/(m·°C), $\beta = 525$ W/(m·°C), $T_0 = 130$°C, $\rho = 2700$ kg/m^2, and $c = 0.9$ kJ/(kg·°C). What is the heat loss at the surface? Formulate the problem.

6.25 Determine the nondimensional temperature $\theta(r, t)$ in the region bounded by two long cylindrical surfaces of radii R_1 and R_2. The dimensionless heat conduction equation is

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + \frac{\partial \theta}{\partial t} = 0$$

with boundary and initial conditions

$$\frac{\partial \theta}{\partial r}(R_1, t) = 0, \quad \theta(R_2, t) = 1, \quad \theta(r, 0) = 0$$

6.26 Show that (6.2.28*a*), (6.2.28*b*), (6.2.29*a*), and (6.2.29*b*) can be reworked to match the form in Eq. (6.2.38) as

$$[H]\{\ddot{u}\}_{s+1} = \{\tilde{F}\}_{s+1}$$

and define $[H]$ and $\{\tilde{F}\}_{s+1}$.

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6.27 A uniform cantilever beam of length L, moment of inertia I, modulus of elasticity E, and mass m begins to vibrate with initial displacement

$$w(x, 0) = w_0 x^2 / L^2$$

and zero initial velocity. Find its displacement at the free end at any subsequent time. Use one Euler–Bernoulli beam element to determine the solution. Solve the resulting differential equations in time using the Laplace transform method.

6.28 Resolve Problem 6.27 using one Timoshenko beam element.

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Chapter 7

COMPUTER IMPLEMENTATION

7.1 NUMERICAL INTEGRATION

7.1.1 Background

The finite element method, like most numerical methods, converts a continuum problem to a discrete one (i.e., converting a system with an infinite number of degrees of freedom into one with a finite number of degrees of freedom). The finite element model of a system ultimately represents a set of algebraic equations among the values of the dependent variables of the system at the selected nodes of the domain. The coefficients of the algebraic equations are typically integrals of approximation functions multiplied by the data of the problem. Exact evaluation of these integrals is not always possible because of the algebraic complexity of the data $a(x)$, $b(x)$, $c(x)$, and $f(x)$ in the mathematical model. In such cases, it is natural to seek numerical evaluation of these integral expressions. Numerical evaluation of the coefficient matrices is also useful in problems with constraints, where reduced integration techniques are used (e.g., the reduced integration element of the Timoshenko beam theory in Section 5.3).

Numerical evaluation of integrals, called *numerical integration* or *numerical quadrature,* involves approximation of the integrand by a polynomial of sufficient degree, because the integral of a polynomial can be evaluated exactly. For example, consider the integral,

$$I = \int_{x_a}^{x_b} F(x)\, dx \tag{7.1.1}$$

We approximate the function $F(x)$ by a polynomial

$$F(x) \approx \sum_{I=1}^{N} F_I \psi_I(x) \tag{7.1.2}$$

where F_I denotes the value of $F(x)$ at the Ith point of the interval $[x_a, x_b]$ and $\psi_I(x)$ are polynomials of degree $N-1$. The representation (7.1.2) can be viewed as the finite element interpolation of $F(x)$, where F_I is the value of the function at the Ith node. The interpolation can be of the Lagrange type or the Hermite type.

Substitution of (7.1.2) into (7.1.1) and evaluation of the integral gives an approximate value of I. For example, suppose that we choose linear interpolation of $F(x)$. Then $N=2$, $\psi_1 = (x_b - x)/h$, $\psi_2 = (x - x_a)/h$, and

$$I = \frac{1}{2}h(F_1 + F_2), \quad F_1 = F(x_a), \quad F_2 = F(x_b) \tag{7.1.3}$$

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Figure 7.1.1 Approximate evaluation of an integral using the trapezoidal rule (*a*) two-point formula and (*b*) three-point formula.

Thus, the value of the integral is given by the area of a trapezoid used to approximate the area under the function $F(x)$ (see Fig. 7.1.1). Equation (7.1.3) is known as the *trapezoidal rule* of numerical integration. If we use the Lagrange quadratic interpolation of $F(x)$, we obtain

$$I = \frac{1}{6}h(F_1 + 4F_2 + F_3), \quad F_1 = F(x_a), \quad F_2 = F(x_a + 0.5h), \quad F_3 = F(x_b) \tag{7.1.4}$$

which is known as *Simpson's one-third rule*.

Equations (7.1.3) and (7.1.4) represent forms of numerical quadrature formulae. In general, a quadrature formula has the form

$$I = \int_{x_a}^{x_b} F(x)\,dx \approx \sum_{I=1}^{r} F(x_I)W_I \tag{7.1.5}$$

where x_I are called the *quadrature points* and W_I are the *quadrature weights*. These formulae require functional evaluations, multiplications, and additions to obtain the numerical value of the integral. They yield exact values of the integral whenever $F(x)$ is a polynomial of order $r - 1$.

In this section, we describe several numerical integration techniques and formulations in which the geometry as well as the dependent variables are approximated using different degrees of polynomials. We begin with the discussion of a local coordinate system.

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Figure 7.1.2 Global coordinate x, local coordinate $\bar{x}$, and normalized coordinate ξ.

7.1.2 Natural Coordinates

Of all the quadrature formulae, as will be discussed in the subsequent sections, the Gauss–Legendre one is the most commonly used. The details of the method itself will be discussed shortly. The formula requires the integral to be cast as one to be evaluated over the interval $[-1, 1]$. This requires the transformation of the problem coordinate x to a local coordinate ξ such that (see Fig. 7.1.2):

$$\text{when} \quad x = x_a, \quad \xi = -1; \quad \text{when} \quad x = x_b, \quad \xi = 1$$

The transformation between x and ξ can be represented by the linear "stretch" transformation

$$x = a + b\xi$$

where a and b are constants to be determined such that the above conditions hold:

$$x_a = a + b(-1), \quad x_b = a + b(1)$$

Solving for a and b, we obtain

$$b = \frac{1}{2}(x_b - x_a) = \frac{1}{2}h_e, \quad a = \frac{1}{2}(x_b + x_a) = x_a + \frac{1}{2}h_e$$

Hence, the transformation takes the form

$$x(\xi) = x_a + \frac{1}{2}h_e(1 + \xi) \tag{7.1.6}$$

where x_a denotes the global coordinate of the left-end node of the element Ω_e and h_e is the element length (see Fig. 7.1.2).

The local coordinate ξ is called the *normal coordinate* or *natural coordinate*, and its values always lie between -1 and 1, with its origin at the center of the element. The local coordinate ξ is useful in two ways: It is (*a*) convenient in constructing the interpolation functions and (*b*) required in numerical integration using Gauss–Legendre quadrature.

The derivation of the Lagrange family of interpolation functions in terms of the natural coordinate ξ is made easy by the following interpolation property of the approximation functions:

$$\psi_i(\xi_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{7.1.7}$$

where ξ_j is the ξ coordinate of the jth node in the element. For an element with n nodes, ψ_i $(i = 1, 2, \ldots, n)$ are polynomials of degree $n - 1$. To construct ψ_i satisfying (7.1.7), we

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$$\psi_1(\xi) = \tfrac{1}{2}(1-\xi), \qquad \psi_2(\xi) = \tfrac{1}{2}(1+\xi)$$

$$\psi_1(\xi) = -\tfrac{1}{2}\xi(1-\xi), \qquad \psi_2(\xi) = (1-\xi^2)$$
$$\psi_3(\xi) = \tfrac{1}{2}\xi(1+\xi)$$

$$\psi_1(\xi) = -\tfrac{9}{16}(1-\xi)(\tfrac{1}{9}-\xi^2), \quad \psi_2(\xi) = \tfrac{27}{16}(1-\xi^2)(\tfrac{1}{3}-\xi)$$
$$\psi_3(\xi) = \tfrac{27}{16}(1-\xi^2)(\tfrac{1}{3}+\xi), \quad \psi_4(\xi) = -\tfrac{9}{16}(1+\xi)(\tfrac{1}{9}-\xi^2)$$

Figure 7.1.3 Lagrange family of one-dimensional interpolation functions in terms of the normalized coordinate.

proceed as follows: For each ψ_i, we form the product of $n-1$ linear functions $\xi - \xi_j$ ($j = 1, 2, \ldots, i-1, i+1, \ldots, n;\ j \neq i$):

$$\psi_i = c_i(\xi-\xi_1)(\xi-\xi_2)\cdots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\cdots(\xi-\xi_n)$$

Note that ψ_i is zero at all nodes except the ith. Next, we determine the constant c_i such that $\psi_i = 1$ at $\xi = \xi_i$:

$$c_i = [(\xi_i-\xi_1)(\xi_i-\xi_2)\cdots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\cdots(\xi_i-\xi_n)]^{-1}$$

Thus, the interpolation function associated with node i is

$$\psi_i(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)\cdots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\cdots(\xi-\xi_n)}{(\xi_i-\xi_1)(\xi_i-\xi_2)\cdots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\cdots(\xi_i-\xi_n)} \tag{7.1.8}$$

The linear, quadratic, and cubic Lagrange interpolation functions in terms of the natural coordinate (for equally spaced nodes) are shown in Fig. 7.1.3.

7.1.3 Approximation of Geometry

We wish to use the Gauss–Legendre quadrature to numerically evaluate all integrals in the finite element method. The integrals are generally expressed in terms of the coordinate appearing in the problem description (like x or r). We shall call x and r as the *problem coordinates* or global coordinates. The Gauss–Legendre quadrature requires us to express the integral in terms of ξ over the interval -1 to $+1$. We assume a relation (or transformation) between the problem coordinate x and natural coordinate ξ in the form

$$x = f(\xi) \tag{7.1.9}$$

where f is assumed to be a one-to-one transformation. An example of $f(\xi)$ is provided by (7.1.6):

$$f(\xi) = x_a + \frac{1}{2}h_e(1+\xi)$$

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In this case, $f(\xi)$ is a linear function of ξ. Hence, a straight line is transformed into a straight line.

It is natural to think of approximating the geometry in the same way as we approximated a dependent variable. In other words, the transformation $x = f(\xi)$ can be written as

$$x = \sum_{i=1}^{m} x_i^e \hat{\psi}_i^e(\xi) \tag{7.1.10}$$

where x_i^e is the global coordinate of the ith node of the element Ω_e and $\hat{\psi}_i^e$ are the Lagrange interpolation functions of degree $m-1$. When $m=2$, we have a linear transformation, and Eq. (7.1.10) is exactly the same as (7.1.6). When $m=3$, Eq.(7.1.10) expresses a quadratic relation between x and ξ. The functions $\hat{\psi}_i^e$ are called *shape functions* because they are used to express the geometry or shape of the element. When the element is a straight line, the mapping is linear (because two points, x_1^e and x_2^e, are sufficient to define a line).

The transformation (7.1.10) allows us to rewrite integrals involving x as those in terms of ξ:

$$\int_{x_a}^{x_b} F(x)\,dx = \int_{-1}^{1} \hat{F}(\xi)\,d\xi, \quad \hat{F}(\xi)\,d\xi = F(x(\xi))\,dx \tag{7.1.11}$$

so that the Gauss–Legendre quadrature can be used to evaluate the integral over $[-1, 1]$. The differential element dx in the global coordinate system x is related to the differential element $d\xi$ in the natural coordinate system ξ by

$$dx = \frac{dx}{d\xi} d\xi = J_e\, d\xi$$

where J_e is called the *Jacobian* of the transformation. We have

$$J_e = \frac{dx}{d\xi} = \frac{d}{d\xi}\left(\sum_{i=1}^{m} x_i^e \hat{\psi}_i^e\right) = \sum_{i=1}^{m} x_i^e \frac{d\hat{\psi}_i^e}{d\xi} \tag{7.1.12}$$

For a linear transformation [i.e., when $m=2$ in (7.1.10)], we have

$$\hat{\psi}_1^e = \frac{1}{2}(1-\xi), \quad \hat{\psi}_2^e = \frac{1}{2}(1+\xi)$$

$$J_e = x_1^e\left(-\frac{1}{2}\right) + x_2^e\left(\frac{1}{2}\right) = \frac{1}{2}(x_2^e - x_1^e) = \frac{1}{2}h_e \tag{7.1.13}$$

It can be shown that $J_e = \frac{1}{2}h_e$ whenever the element is a straight line, irrespective of the degree of interpolation used in the transformation (7.1.10).

7.1.4 Isoparametric Formulations

Recall that a dependent variable u is approximated in an element Ω_e by expressions of the form

$$u(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x) \tag{7.1.14}$$

In general, the degree of approximation used to describe the coordinate transformation (7.1.10) is not equal to the degree of approximation (7.1.14) used to represent a dependable

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variable, i.e., $\hat{\psi}_i^e \neq \psi_i^e$. In other words, two independent meshes of elements may be used in the finite element formulation of a problem: one for the approximation of the geometry x and the other for the interpolation of the dependent variable u. Depending on the relationship between the degree of approximation used for the coordinate transformation and that used for the dependent variable, the finite element formulations are classified into three categories:

1. Subparametric formulations: $m < n$
2. Isoparametric formulations: $m = n$ (7.1.15)
3. Superparametric formulations: $m > n$

In subparametric formulations, the geometry is represented by lower-order elements than those used to approximate the dependent variable. An example of this category is provided by the Euler–Bernoulli beam element, where the Hermite cubic functions are used to approximate the deflection $w(x)$ and linear interpolation can be used, when straight beams are analyzed, to represent the geometry. In isoparametric formulations (which are the most common in practice), the same element is used to approximate the geometry as well as the dependent unknowns: $\psi_i(x) = \hat{\psi}_i(\xi)$. In the superparametric formulations, the geometry is represented with higher-order elements than those used to approximate the dependent variables. The superparametric formulation is seldom used in practice.

7.1.5 Numerical Integration

As discussed in the introduction, the evaluation of integrals of the form

$$\int_a^b F(x)\, dx \tag{7.1.16}$$

by exact means is either difficult or impossible owing to the complicated form of the integrand F. Numerical integration is also required when the integrand is to be evaluated inexactly (as in the Timoshenko beam element) or when the integrand is known only at discrete points (e.g., experimentally obtained data).

The basic idea behind all numerical integration techniques is to find a function $P(x)$, often a polynomial, that is both a suitable approximation of $F(x)$ and simple to integrate. The interpolating polynomials of degree n, denoted by P_n, which interpolate the integrand at $n+1$ points of the interval $[a, b]$, often produce a suitable approximation and possess the desired property of simple integrability. An illustration of the approximation of the function $F(x)$ by the polynomial $P_4(x)$ that exactly matches the function $F(x)$ at the indicated base points is given in Fig. 7.1.4(a). The exact value of (7.1.16) is given by the area under the solid curve, while the approximate value

$$\int_a^b P_4(x)\, dx$$

is given by the area under the dashed curve. It should be noted that the difference (i.e., the error in the approximation) $E = F(x) - P_4(x)$ is not always of the same sign, and therefore the overall integration error may be small (because positive errors in one part cancel negative errors in other parts), even when P_4 is not a good approximation of F.

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Figure 7.1.4 Numerical integration by the Newton–Cotes quadrature: (*a*) approximation of a function by $P_4(x)$; (*b*) the trapezoidal rule; and (*c*) Simpson's rule.

The commonly used numerical integration methods can be classified into two groups:

1. The Newton–Cotes formulae that employ values of the function at equally spaced points
2. The Gauss–Legendre quadrature formula that employs unequally spaced points

These two methods are described here.

The Newton–Cotes Quadrature. For r equally spaced base points, the *Newton–Cotes closed integration formula* is given by

$$\int_a^b F(x)\,dx = (b-a)\sum_{I=1}^{r} F(x_I)w_I \tag{7.1.17}$$

where w_I are the *weighting coefficients*, x_I are the *base points* that are equally spaced, and r is the number of base points (or $r-1$ is the number of intervals). Note that $r=1$ is a special case in which the number of base points as well as the number intervals is the same; in this case Eq. (7.1.17) gives the rectangle formula. For $r=2$, (7.1.17) gives the familiar *trapezoidal rule*, in which the required area under the solid curve in Fig. 7.1.4(*b*) is approximated by the area under the dotted straight line [i.e., $F(x)$ is approximated by $P_1(x)$]:

$$\int_{a=x_1}^{b=x_2} F(x)\,dx = \frac{b-a}{2}[F(x_1)+F(x_2)], \quad E = O(h^3), \quad h = b-a \tag{7.1.18}$$

where E denotes the error in the approximation and h is the uniform spacing between two base points. The notation $O(h)$, read as "order of h," is used to indicate the order of the

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Table 7.1.1 Weighing coefficients for the Newton–Cotes formula.

r	w_1	w_2	w_3	w_4	w_5	w_6	w_7
1	1						
2	$\frac{1}{2}$	$\frac{1}{2}$					
3	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$				
4	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$			
5	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$		
6	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$	
7	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$

error in terms of the spacing h. For $r = 3$ (i.e., two intervals), (7.1.17) gives the familiar *Simpson's one-third rule* [see Fig. 7.1.4(*c*)]:

$$\int_{a=x_1}^{b=x_3} F(x)\,dx = \frac{b-a}{6}[F(x_1) + 4F(x_2) + F(x_3)], \quad E = O(h^5), \quad h = 0.5(b-a) \tag{7.1.19}$$

The weighting coefficients for $r = 1, 2, \ldots, 7$ are given in Table 7.1.1. Note that $\sum_{l=1}^{r} w_l = 1$. The base point location for $r = 1$ is $x_1 = a + \frac{1}{2}(b-a) = \frac{1}{2}(a+b)$. For $r > 1$, the base point locations are

$$x_1 = a, \quad x_2 = a + \Delta x, \ldots, \; x_r = a + (r-1)\Delta x = b$$

and $\Delta x = (b-a)/(r-1)$. We note that when r is *odd* (i.e., when there is an even number of intervals or an odd number of base points), the formula is exact when $F(x)$ is a polynomial of degree r or less; when r is *even*, the formula is exact for a polynomial of degree $r-1$ or less. Odd-point formulas are frequently used because of their high order of accuracy [see Carnahan, Luther, and Wilkes (1969)].

The Gauss–Legendre Quadrature. In the Newton–Cotes quadrature, the base point locations have been specified. If the x_l are not specified, then there will be $2r+2$ undetermined parameters, $r+1$ weights w_l and $r+1$ base points x_l, which define a polynomial of degree $2r+1$. The Gauss–Legendre quadrature is based on the idea that the base points x_l and the weights w_l can be chosen so that the sum of the $r+1$ appropriately weighted values of the function yields the integral exactly when $F(x)$ is a polynomial of degree $2r+1$ or less. The Gauss–Legendre quadrature formula is given by

$$\int_a^b F(x)\,dx = \int_{-1}^{1} \hat{F}(\xi)\,d\xi \approx \sum_{l=1}^{r} \hat{F}(\xi_l) w_l \tag{7.1.20}$$

where w_l are the weight factors, ξ_l are the base points [roots of the Legendre polynomial $P_{r+1}(\xi)$], and $\hat{F}$ is the transformed integrand

$$\hat{F}(\xi) = F(x(\xi))J(\xi), \quad dx = J\,d\xi \tag{7.1.21}$$

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Table 7.1.2 Weights and Gauss points for the Gauss–Legendre quadrature.

$$\int_{-1}^{1} F(\xi)d\xi = \sum_{i=1}^{r} F(\xi_i)w_i$$

Points, ξ_i†	r	Weights, w_i
0.0000000000	1	2.0000000000
±0.5773502692	2	1.0000000000
0.0000000000	3	0.8888888889
±0.7745966692		0.5555555555
±0.3399810435	4	0.6521451548
±0.8611363116		0.3478548451
0.0000000000	5	0.5688888889
±0.5384693101		0.4786286705
±0.9061798459		0.2369268850
±0.2386191861	6	0.4679139346
±0.6612093865		0.3607615730
±0.9324695142		0.1713244924

†Note that $0.57735\ldots = 1/\sqrt{3}$, $0.77459\ldots = \sqrt{3/5}$, and $0.888\ldots = 8/9$, and $0.555\ldots = 5/9$.

where J is the Jacobian of the transformation between x anbd ξ. The weight factors and Gauss points for the Gauss–Legendre quadrature (7.1.20) are given for $r = 1, \ldots, 6$ in Table 7.1.2.

The Gauss–Legendre quadrature is more frequently used than the Newton–Cotes quadrature because it requires fewer base points (hence, a saving in computation) to achieve the same accuracy. The error in the approximation is zero if the $(2r+2)$th derivative of the integrand vanishes. In other words, a polynomial of degree p is integrated exactly by employing $r = \frac{1}{2}(p+1)$ Gauss points. When $p+1$ is odd, one should pick the nearest larger integer:

$$r = \left[\frac{1}{2}(p+1)\right] \tag{7.1.22}$$

In finite element formulations, we encounter integrals whose integrands F are functions of x, $\psi_i(x)$, and derivatives of $\psi_i(x)$ with respect to x. For the Gauss–Legendre quadrature, we must transform $F(x)dx$ to $\hat{F}(\xi)d\xi$ in order to use the formula (7.1.20). For example, consider the integral

$$K_{ij}^e = \int_{x_a}^{x_b} a(x)\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\,dx \tag{7.1.23}$$

Using the chain rule of differentiation, we have

$$\frac{d\psi_i^e(x)}{dx} = \frac{d\psi_i^e(\xi)}{d\xi}\frac{d\xi}{dx} = J^{-1}\frac{d\psi_i^e(\xi)}{d\xi} \tag{7.1.24}$$

Therefore, the integral in (7.1.23) can be written, with the help of (7.1.10), as

$$K_{ij}^e = \int_{-1}^{1} a(x(\xi))\frac{1}{J}\frac{d\psi_i^e}{d\xi}\frac{1}{J}\frac{d\psi_j^e}{d\xi}\,J\,d\xi \tag{7.1.25}$$

$$\approx \sum_{I=1}^{r} \hat{F}_{ij}^e(\xi_I)w_I \tag{7.1.26}$$

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where

$$\hat{F}_{ij}^e = a\frac{1}{J}\frac{d\psi_i^e}{d\xi}\frac{d\psi_j^e}{d\xi}, \quad J = \sum_{i=1}^{m} x_i^e \frac{d\hat{\psi}_i^e}{d\xi} \tag{7.1.27}$$

For the isoparametric formulation, we take $\psi_i^e = \hat{\psi}_i^e$. As noted earlier, the Jacobian matrix will be the same ($J_e = \frac{1}{2}h_e$) when the element is a straight line, even if the coordinate transformation is quadratic or cubic. However, when the element is curved, the Jacobian is a function of ξ for transformations other than linear. The transformation from x to ξ is not required in the Newton–Cotes quadrature.

It is possible to determine the exact number of Gauss points required to evaluate the following element coefficients:

$$\begin{aligned}
K_{ij}^e &= \int_{x_a}^{x_b} \frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\,dx = \int_{-1}^{+1} \frac{d\psi_i^e}{d\xi}\frac{d\psi_j^e}{d\xi}(J)^{-2}\,J\,d\xi \equiv \int_{-1}^{+1} G_{ij}^K(\xi)\,d\xi \\
&\approx \sum_{I=1}^{N^K} G_{ij}^K(\xi_I)\,W_I \\
M_{ij}^e &= \int_{x_a}^{x_b} \psi_i^e \psi_j^e\,dx = \int_{-1}^{+1} \psi_i^e(\xi)\,\psi_j^e(\xi)\,J\,d\xi \equiv \int_{-1}^{+1} G_{ij}^M(\xi)\,d\xi \\
&\approx \sum_{I=1}^{N^M} G_{ij}^M(\xi_I)\,W_I \\
f_i^e &= \int_{x_a}^{x_b} \psi_i^e\,dx = \int_{-1}^{+1} \psi_i^e(\xi)\,J\,d\xi \equiv \int_{-1}^{+1} G_{ij}^F(\xi)\,d\xi \\
&\approx \sum_{I=1}^{N^F} G_{ij}^F(\xi_I)\,W_I
\end{aligned} \tag{7.1.28}$$

when linear, quadratic, and cubic interpolation functions are used. For linear interpolation functions, the integrand of K_{ij}^e is constant, requiring only one-point Gauss–Legendre quadrature ($N^K = 1$). The integrand of the mass matrix M_{ij}^e is quadratic ($p = 2$), requiring $[r = \frac{1}{2}(p+1) = \frac{3}{2}]$, the two-point quadrature ($N^M = 2$). The coefficients f_i^e are evaluated exactly by one-point quadrature ($N^F = 1$). Similarly, for quadratic and cubic elements, we can estimate the number of Gauss points needed to evaluate K_{ij}^e, M_{ij}^e, and f_i^e exactly. The results are summarized in Table 7.1.3. Note that, in estimating the quadrature points, it is assumed that the Jacobian J is a constant, which holds true when the element is a straight line.

If the matrices in (7.1.28) have variable coefficients or the elements are curved [and hence $J_e = J_e(\xi)$], the degree of the variation of the integrands changes and the number of Gauss points needed to exactly evaluate the integral changes. If the elements are straight and the coefficients $a = a(x)$ and $c = c(x)$ together with $f = f(x)$ are no more than linear in x, then the number of Gauss points for evaluating the coefficients

$$K_{ij}^e = \int_{x_a}^{x_b} a\frac{d\psi_i^e}{dx}\frac{d\psi_j^e}{dx}\,dx, \quad M_{ij}^e = \int_{x_a}^{x_b} c\psi_i^e\psi_j^e\,dx \tag{7.1.29}$$

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Table 7.1.3 The number of Gauss–Legendre quadrature points required to evaluate K_{ij}^e, M_{ij}^e, and f_i^e of Eq. (7.1.28).

Element type	N^K	N^M	N^F
Linear	1	2	1
Quadratic	2	3	2
Cubic	3	4	2

remain the same as listed in the above table. However, the evaluation of f_i^e requires one point more than before. Conversely, the two-point quadrature for linear elements, three-point quadrature for quadratic elements, and four-point quadrature for cubic elements would exactly evaluate K_{ij}^e with a quadratic variation of $a(x)$, M_{ij}^e with linear variation of $c(x)$, and f_i^e with quadratic variation of $f(x)$.

The use of Gauss–Legendre quadrature in (7.1.28) yields the following values (exact up to the fifth decimal place) when the element is straight and the isoparametric formulation is used:

Quadratic Element (Three-Point Formula)

$$[K] = \frac{1}{h_e}\begin{bmatrix} 2.33333 & -2.66667 & 0.33333 \\ -2.66667 & 5.33333 & -2.66667 \\ 0.33333 & -2.66667 & 2.33333 \end{bmatrix}$$

$$[M] = \frac{h_e}{10}\begin{bmatrix} 1.33333 & 0.66667 & 0.33333 \\ 0.66667 & 5.33333 & 0.66667 \\ 0.33333 & 0.66667 & 1.333433 \end{bmatrix} \tag{7.1.30}$$

$$\{f\} = h_e\begin{Bmatrix} 0.166667 \\ 0.666667 \\ 0.166667 \end{Bmatrix}$$

Cubic Element (Four-Point Formula)

$$[K] = \frac{1}{h_e}\begin{bmatrix} 3.700 & -4.725 & 1.350 & -0.325 \\ -4.725 & 10.800 & -7.425 & 1.350 \\ 1.350 & -7.425 & 10.800 & -4.725 \\ -0.325 & 1.350 & -4.725 & 3.700 \end{bmatrix}$$

$$[M] = \frac{h_e}{10}\begin{bmatrix} 0.761905 & 0.589286 & -0.214286 & 0.113095 \\ 0.589286 & 3.857143 & -0.482143 & -0.214286 \\ -0.214286 & -0.482143 & 3.857143 & 0.589286 \\ 0.113095 & -0.214286 & 0.589286 & 0.761905 \end{bmatrix} \tag{7.1.31}$$

$$\{f\} = h_e\begin{Bmatrix} 0.125 \\ 0.375 \\ 0.375 \\ 0.125 \end{Bmatrix}$$

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Example 7.1.1

We wish to evaluate the following integrals using the Newton–Cotes and Gauss–Legendre quadratures:

$$K_{12} = \int_{x_a}^{x_b} (x_0 + x) \frac{d\psi_1}{dx} \frac{d\psi_2}{dx}\, dx, \quad G_{12} = \int_{x_a}^{x_b} (x_0 + x) \psi_1 \psi_2\, dx$$

where ψ_i are the linear interpolation functions

$$\psi_1 = \frac{x - x_a}{x_b - x_a} = \frac{1}{2}(1 - \xi), \quad \psi_2 = \frac{x_b - x}{x_b - x_a} = \frac{1}{2}(1 + \xi)$$

and x_0 is a constant.

Note that the integrand $F(x)$ in the integral of K_{12} is a linear polynomial (i.e., the degree is $r = 1$). Hence, we expect the one-point Newton–Cotes or Gauss–Legendre quadrature to yield the exact value of K_{12}. On the other hand, the integrand of G_{12} is a cubic polynomial (i.e., the degree is $r = 3$). Hence, we expect the three-point Newton–Cotes or two-point Gauss–Legendre quadrature to yield the exact value of G_{12}. The exact values are

$$K_{12} = -\frac{1}{x_b - x_a}\left[x_0 + \frac{1}{2}(x_b + x_a)\right], \quad G_{12} = \frac{x_b - x_a}{6}\left[x_0 + \frac{1}{2}(x_b + x_a)\right]$$

Newton–Cotes Quadrature: For various number of base points r, we have the following results:

$r = 1$, $[x_1 = 0.5(x_a + x_b)]$:

$$\begin{aligned}
K_{12} &= \int_{x_a}^{x_b} (x_0 + x) \frac{d\psi_1}{dx} \frac{d\psi_2}{dx}\, dx = \int_{x_a}^{x_b} (x_0 + x)\left(-\frac{1}{x_b - x_a}\right)\left(\frac{1}{x_b - x_a}\right) dx \\
&= (x_b - x_a)\left[-\frac{1}{(x_b - x_a)^2}(x_0 + x_1)\right] = -\frac{1}{x_b - x_a}\left[x_0 + \frac{1}{2}(x_b + x_a)\right] \\
G_{12} &= \int_{x_a}^{x_b} (x_0 + x)\psi_1\psi_2\, dx = \int_{x_a}^{x_b} (x_0 + x)\left(\frac{x - x_a}{x_b - x_a}\right)\left(\frac{x_b - x}{x_b - x_a}\right) dx \\
&= (x_b - x_a)(x_0 + x_1)\left(\frac{x_1 - x_a}{x_b - x_a}\right)\left(\frac{x_b - x_1}{x_b - x_a}\right) \\
&= \frac{x_b - x_a}{4}\left[x_0 + \frac{1}{2}(x_b + x_a)\right]
\end{aligned}$$

$r = 2$, $(x_1 = x_a, x_2 = x_b)$:

$$\begin{aligned}
K_{12} &= -\frac{1}{2(x_b - x_a)}\left[(x_0 + x_1) + (x_0 + x_2)\right] \\
&= -\frac{1}{2(x_b - x_a)}(2x_0 + x_a + x_b) = -\frac{1}{x_b - x_a}\left[x_0 + \frac{1}{2}(x_b + x_a)\right] \\
G_{12} &= \frac{x_b - x_a}{2}\left[(x_0 + x_1)\left(\frac{x_1 - x_a}{x_b - x_a}\right)\left(\frac{x_b - x_1}{x_b - x_a}\right)\right. \\
&\quad \left. + (x_0 + x_2)\left(\frac{x_2 - x_a}{x_b - x_a}\right)\left(\frac{x_b - x_2}{x_b - x_a}\right)\right] = 0
\end{aligned}$$

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$r = 3$, $[x_1 = x_a, x_2 = 0.5(x_a + x_b), x_3 = x_b]$:

$$K_{12} = -\frac{1}{6(x_b - x_a)}[(x_0 + x_1) + 4(x_0 + x_2) + (x_0 + x_3)]$$

$$= -\frac{1}{x_b - x_a}\left[x_0 + \frac{1}{2}(x_b + x_a)\right]$$

$$G_{12} = \frac{x_b - x_a}{6}\left[(x_0 + x_1)\left(\frac{x_1 - x_a}{x_b - x_a}\right)\left(\frac{x_b - x_1}{x_b - x_a}\right)\right.$$

$$+ 4(x_0 + x_2)\left(\frac{x_2 - x_a}{x_b - x_a}\right)\left(\frac{x_b - x_2}{x_b - x_a}\right)$$

$$\left. + (x_0 + x_3)\left(\frac{x_3 - x_a}{x_b - x_a}\right)\left(\frac{x_b - x_3}{x_b - x_a}\right)\right]$$

$$= \frac{x_b - x_a}{6}\left[x_0 + \frac{1}{2}(x_b + x_a)\right]$$

Gauss–Legendre Quadrature: The coefficient K_{12} for this case takes the form $[dx = 0.5(x_b - x_a)d\xi$, $\psi_1 = 0.5(1 - \xi)$, $\psi_2 = 0.5(1 + \xi)$, and $d\psi_i/dx = (1/J)d\psi_i/d\xi$ with $J = 0.5(x_b - x_a)]$

$$K_{12} = \int_{x_a}^{x_b} (x_0 + x)\frac{d\psi_1}{dx}\frac{d\psi_2}{dx}\,dx$$

$$= -\frac{1}{2(x_b - x_a)}\int_{-1}^{1}\left[x_0 + x_a + \frac{x_b - x_a}{2}(1 + \xi)\right]d\xi$$

The polynomial order is $p = 1$; hence, the one-point Gauss rule ($r = 1$) would yield the exact value:

$$K_{12} = -\frac{1}{2(x_b - x_a)}\left[x_0 + x_a + \frac{x_b - x_a}{2}(1 + 0.0)\right]\cdot 2$$

$$= -\frac{1}{x_b - x_a}\left[x_0 + \frac{1}{2}(x_b + x_a)\right]$$

The coefficient G_{12} takes the form

$$G_{12} = \int_{x_a}^{x_b} (x_0 + x)\psi_1\psi_2\,dx$$

$$= \frac{x_b - x_a}{8}\int_{-1}^{1}\left[x_0 + x_a + \frac{x_b - x_a}{2}(1 + \xi)\right](1 - \xi^2)d\xi$$

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The polynomial order is $p = 3$. Therefore, the two-point Gauss rule ($r = 2$) would yield the exact value

$$G_{12} = \frac{x_b - x_a}{8}\left\{\left[x_0 + x_a + \frac{x_b - x_a}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right]\left(1 - \frac{1}{3}\right) + \left[x_0 + x_a + \frac{x_b - x_a}{2}(1 + \frac{1}{\sqrt{3}})\right](1 - \frac{1}{3})\right\}$$

$$= \frac{x_b - x_a}{6}\left[x_0 + \frac{1}{2}(x_b + x_a)\right]$$

7.2 COMPUTER IMPLEMENTATION

7.2.1 Introductory Comments

Chapters 3–6 were devoted to the finite element formulations of two *classes* of boundary value, initial value, or eigenvalue problems in one dimension:

1. Second-order differential equations (e.g., heat transfer, fluid mechanics, one-dimensional elasticity, bars, and the Timoshenko beam theory)
2. Fourth-order differential equations governing the Euler–Bernoulli beam theory

The frame element, obtained by superposing the bar element and the beam element, was discussed in Chapter 5.

By now, it should be clear to the reader that the steps involved in the finite element analysis of a general class of problems (e.g., single second-order, single fourth-order, and a pair of second-order equations) are systematic, and once the finite element formulation is completed, the model can be implemented on a digital computer. Indeed, the success of the finite element method is largely due to the ease with which the analysis of a class of problems, without regard to the geometry and boundary conditions, can be implemented on a digital computer. A particular class of problems (say that described by the model equation of Chapter 3) can be solved by simply supplying the required input data to the program. For example, if we develop a general computer program to solve equations of the form

$$c_1\frac{\partial u}{\partial t} + c_2\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right) + cu = f \tag{7.2.1}$$

then all physical problems described by Eqs. (3.1.1) and (5.1.1) and their time-dependent versions can be solved for any compatible boundary and initial conditions.

The purpose of this section is to discuss the basic steps involved in the development of a computer program for second- and fourth-order one-dimensional differential equations studied in the preceding chapters. The ideas presented here are used in the development of the model program **FEM1D,** and they are meant to be illustrative of the steps used in a typical finite element program development. One can make use of the ideas presented here and implement them using **FEM1D** to develop a program of one's own. The discussion here focuses on the finite element computations, and no attempt is made to discuss the Gauss elimination

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procedure used to solve the resulting system of algebraic equations (a solver is provided with **FEM1D**; see Appendix 1 located on the book's website at www.mhhe.com/reddy3e).

7.2.2 General Outline

A typical finite element program consists of three basic units (see Fig. 7.2.1):

1. Preprocessor
2. Processor
3. Postprocessor

In the preprocessor part of the program, the input data of the problem are read in and/or generated. This includes the geometry (e.g., length of the domain and boundary conditions), the data of the problem (e.g., coefficients in the differential equation), finite element mesh information (e.g., element type, number of elements, element length, coordinates of the nodes, and connectivity matrix), and indicators for various options (e.g., print, no print, type of field problem analyzed, static analysis, eigenvalue analysis, transient analysis, and degree of interpolation).

In the processor part, all the steps in the finite element method discussed in the preceding chapter, except for postprocessing, are performed. The major steps of the processor are:

1. Generation of the element matrices using numerical integration
2. Assembly of element equations
3. Imposition of the boundary conditions
4. Solution of the algebraic equations for the nodal values of the primary variables

Figure 7.2.1 The three main functional units of a finite element program.

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In the postprocessor part of the program, the solution is computed by interpolation at points other than nodes, secondary variables that are derivable from the solution are computed, and the output data are processed in a desired format for printout and/or plotting.

The preprocessor and postprocessors can be a few Fortran statements to read and print pertinent information, simple subroutines (e.g., subroutines to generate mesh and compute the gradient of the solution), or complex programs linked to other units via disk and tape files. The processor, where typically large amounts of computing time are spent, can consist of several subroutines, each having a special purpose (e.g., a subroutine for the calculation of element matrices, a subroutine for the imposition of boundary conditions, and a subroutine for the solution of equations). The degree of sophistication and the complexity of a finite element program depend on the general class of problems being programmed, the generality of the data in the equation, and the intended user of the program. It is always desirable to describe, through comment statements, all variables used in the computer program. A flow chart of the computer program **FEM1D** is presented in Fig. 7.2.2. The objective of each of the subroutines listed in the flow chart is described below.

ASSEMBLE: Subroutine for the assembly of element equations. The equations are assembled in upper-banded form for static and transient problems, and in full matrix form for eigenvalue problems.

EIGNSLVR: Subroutine for the solution of the eigenvalue problem $[A]\{X\} = \lambda[B]\{X\}$.

Figure 7.2.2 Flow chart of the computer program **FEM1D**.

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BOUNDARY: Subroutine for imposition of specified boundary conditions (Dirichlet, Neumann, and Newton type).

COEFFCNT: Subroutine for computing element matrices $[K^e]$, $[M^e]$, and $\{f^e\}$ for all model problems except for truss and frame elements.

ECHODATA: Subroutine to echo the input to the program.

MESH1D: Subroutine to generate the mesh (coordinates of the global nodes and the connectivity array).

POSTPROC: Subroutine to postprocess the solution for all model problems except for truss and frame elements.

REACTION: Subroutine to calculate the reactions (i.e., generalized forces) for truss and frame elements.

SHAPE1D: Subroutine to compute the approximation functions and their derivatives.

EQNSOLVR: Subroutine for solving banded symmetric system of algebraic equations.

TRANSFRM: Subroutine to compute element stiffness matrix and force vector for truss and frame elements.

In the following sections, a discussion of the basic components of a typical finite element program is presented, and then the ideas are illustrated via FORTRAN statements.

7.2.3 Preprocessor

The preprocessor unit reads the input data, generates the finite element mesh, and prints the data and mesh information. The input data to a finite element program consist of element type IELEM (i.e., Lagrange element or Hermite element), number of elements used (NEM), specified boundary conditions on primary and secondary variables (number of boundary conditions, global node number and degree of freedom, and specified values of the degrees of freedom), the global coordinates of global nodes, and element properties (e.g., coefficients $a(x)$, $b(x)$, $c(x)$, $f(x)$, etc.). If a uniform mesh is used, the length of the domain should be read in, and global coordinates of the nodes can be generated in the program.

The preprocessor portion that deals with the generation of finite element mesh (when not supplied by the user) can be separated into a subroutine (**MESH1D**), depending on the convenience and complexity of the program. Mesh generation includes computation of the global coordinates X_I and the connectivity array $[B] = \text{NOD}$. Recall that the connectivity matrix describes the relationship between element nodes to global nodes as

$$\text{NOD}(I, J) = \textit{global node number} \text{ corresponding to the } J\text{th (local) node of element } I$$

This array is used in the assembly procedure as well as to transfer information from the element to the global system and vice versa. For example, to extract the vector ELX of global coordinates of element nodes from the vector GLX of global coordinates of global nodes, we can use the matrix NOD as follows: The global coordinate $x_i^{(n)}$ of the ith node of the nth element is the same as the global coordinate X_I of the global node I, where $I = \text{NOD}(n, i)$:

$$\{x_i^{(n)}\} = \{X_I\}, \quad I = \text{NOD}(n, i) \rightarrow \text{ELX}(i) = \text{GLX}(\text{NOD}(n, i))$$

The arrays ELX and GLX are used in **FEM1D** to denote $\{x^{(n)}\}$ and $\{X\}$, respectively.

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7.2.4 Calculation of Element Matrices (Processor)

The most significant function of a processor is to generate element matrices. The element matrices are computed in various subroutines (**COEFFCNT** and **TRANSFRM**), depending on the type of problem being solved. These subroutines typically involve numerical evaluations of the element matrices $[K^e]$ and $[M^e]$ (program variables ELK and ELM) and the element vector $\{f^e\}$ (program variable ELF) for various field problems. The Gauss–Legendre quadrature described in Section 7.1.5 is used to evaluate element matrices and vectors, and the arrays are assembled as soon as they are computed. Thus, a loop on the number of elements in the mesh (NEM) is used to compute element matrices and assemble them (subroutine **ASSEMBLE**). It is here that the connectivity array NOD plays a crucial role. By putting one element matrix into global locations at a time, we avoid the computation of all element matrices at once.

Element matrices for different model equations (MODEL) and type of problem (NTYPE) are generated by assigning values as discussed next. Governing equations are listed for the static case. The variables used have the following meaning: $H =$ thickness of the beam/plate; $B =$ width of a beam; $E =$ Young's modulus; $G = E/[2(1+\nu)] =$ shear modulus; $\nu =$ Poisson's ratio; $D =$ bending stiffness ($D = EI = EBH^3/12$ for beams and $D = EH^3/[12(1-\nu^2)]$ for plates); $A =$ cross-sectional area; $K_s =$ shear correction factor; and $c_f =$ foundation modulus.

1. MODEL = 1, NTYPE = 0 All field problems described by the model equations (3.2.1) and (3.4.1), including radially symmetric heat-transfer-type problems:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f = 0;\ AX = a(x),\ CX = c(x),\ FX = f(x)$$

$$-\frac{1}{r}\frac{d}{dr}\left(ra\frac{du}{dr}\right) + cu - f = 0;\ AX = ra(r),\ CX = rc(r),\ FX = rf(r)$$

2. MODEL = 1, NTYPE = 1 Radially symmetric deformation of polar orthotropic disks (see Problem 4.37):

$$-\frac{d}{dr}\left[H\left(c_{11}r\frac{du}{dr} + c_{12}u\right)\right] + H\left(c_{22}\frac{u}{r} + c_{12}\frac{du}{dr}\right) = rf(r)$$

$$c_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}},\quad c_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}},\quad c_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$

where $f(r)$ is the distributed force per unit volume (i.e., $Hf(r) = \hat{f}$ is the distributed force per unit area). For the isotropic case, $E_1 = E_2 = E$ and $\nu_{12} = \nu_{21} = \nu$.

3. MODEL = 1, NTYPE = 2 Radially symmetric deformation of cylinders:

$$-\frac{d}{dr}\left\{c\left[(1-\nu)r\frac{du}{dr} + \nu u\right]\right\} + c\left[(1-\nu)\frac{u}{r} + \nu\frac{du}{dr}\right] = rf(r)$$

$$c = \frac{E}{(1+\nu)(1-2\nu)}$$

4. MODEL = 2, NTYPE = 0 (reduced integration element; RIE) or MODEL = 2, NTYPE = 2 (consistent interpolation element; CIE) Bending of straight beams using

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the Timoshenko beam theory:

$$-\frac{d}{dx}\left[GAK_s\left(\Psi+\frac{dw}{dx}\right)\right]+c_f w=q$$

$$-\frac{d}{dx}\left(EI\frac{d\Psi}{dx}\right)+GAK_s\left(\Psi+\frac{dw}{dx}\right)=0$$

$$AX=GAK_s,\quad CX=c_f(x),\quad FX=q(x),\quad BX=EI$$

5. MODEL = 2, NTYPE = 1 (RIE) or MODEL = 2, NTYPE = 3 (CIE) Axisymmetric bending of circular plates using the shear deformation plate theory:

$$-\frac{1}{r}\left[\frac{d}{dr}(rM_{rr})-M_{\theta\theta}\right]+Q_r=0,\quad -\frac{1}{r}\frac{d}{dr}(rQ_r)-q=0$$

$$M_{rr}=D\left(\frac{d\Psi}{dr}+\nu\frac{\Psi}{r}\right),\quad M_{\theta\theta}=D\left(\nu\frac{d\Psi}{dr}+\frac{\Psi}{r}\right),\quad Q_r=K_sGh\left(\Psi+\frac{dw}{dr}\right)$$

6. MODEL = 3, NTYPE = 0 Bending of straight beams using the Euler–Bernoulli beam theory:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)+c_f w=q(x),\quad BX=EI,\ CX=c_f(x),\ FX=q(x)$$

7. MODEL = 3, NTYPE = 1 Axisymmetric bending of circular plates using the classical plate theory:

$$D\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}+c_f w=q$$

8. MODEL = 4, NTYPE = 0 Two-node truss element.
9. MODEL = 4, NTYPE = 1 Two-node Euler–Bernoulli frame element.
10. MODEL = 4, NTYPE = 2 Two-node Timoshenko frame element.

The time-dependent option is exercised through variable ITEM:

ITEM = 0 static analysis
ITEM = 1 first-order time derivative (i.e., parabolic) problems
ITEM = 2 second-order time derivative (i.e., hyperbolic) problems

The element matrices are evaluated using the Gauss–Legendre quadrature except for MODEL = 4, where the explicit forms of element coefficients are programmed in the interest of computational efficiency.

The element shape functions SF and their derivatives GDSF are evaluated at the Gauss points in subroutine **SHAPE1D**. The gaussian weights and points associated with two-, three-, four-, and five-point integration are stored in arrays GAUSWT and GAUSPT, respectively. The nth column of GAUSWT, for example, contains the weights corresponding to the n-point Gauss–Legendre quadrature rule:

GAUSPT (i, n) = ith Gauss point corresponding to the n-point Gauss rule

The variable NGP is used to denote the number of Gauss points, which is selected to achieve good accuracy. As noted earlier, the linear, quadratic and cubic interpolation

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functions require two, three, and four quadrature points, respectively, to evaluate the element coefficients exactly. Thus, if IELEM is the element type,

$$\text{IELEM} = \begin{cases} 1 & \text{linear} \\ 2 & \text{quadratic (Lagrange elements)} \\ 3 & \text{cubic} \end{cases}$$

then NGP = IELEM +1 would evaluate K_{ij}^e, M_{ij}^e, and f_i^e [see (7.1.29)] exactly when $c(x)$ is linear, and $a(x)$, $b(x)$, and $f(x)$ are quadratic functions. The Hermite cubic element is identified with IELEM = 0, in which case NGP is taken to be 4.

The coefficients $a(x) = \text{AX}$, $b(x) = \text{BX}$, and $c(x) = \text{CX}$, together with $f(x) = \text{FX}$ in the differential equation (7.2.1) are assumed to vary with x as follows:

$$\begin{array}{ll} \text{AX=AX0+AX1*X} & (a = a_0 + a_1 x) \\ \text{BX=BX0+BX1*X} & (b = b_0 + b_1 x) \\ \text{CX=CX0+CX1*X} & (c = c_0 + c_1 x) \\ \text{FX=FX0+FX1*X+FX2*X*X} & (f = f_0 + f_1 x + f_2 x^2) \end{array} \tag{7.2.2}$$

For radially symmetric elasticity problems, (AX0, AX1) [or (BX0, BX1) for circular plates] are used to input Young's modulus E and Poisson's ratio ν.

The Gauss–Legendre quadrature formula (7.1.20) can be implemented in the computer as follows: Consider K_{ij}^e of the form

$$K_{ij}^e = \int_{x_a}^{x_b} \left[a(x) \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c(x) \psi_i^e \psi_j^e \right] dx \tag{7.2.3}$$

We use the following program variables for the quantities in (7.2.3):

$$\text{ELK(I, J)} = K_{ij}^e, \quad \text{SF(I)} = \psi_i^e, \quad \text{GDSF(I)} = \frac{d\psi_i^e}{dx}$$

$$\text{AX} = a(x), \quad \text{CX} = c(x), \quad \text{ELX(I)} = x_i^e$$

$$\text{NPE} = n \quad \text{(the number of nodes in the element)}$$

After transforming x to ξ [or $x = x_a + 0.5h_e(1 + \xi)$]

$$x = \sum_{i=1}^{n} x_i^e \psi_i^e \tag{7.2.4}$$

the coefficients K_{ij}^e in (7.2.3) can be written as

$$K_{ij}^e = \int_{-1}^{1} \left[a(\xi) \frac{1}{J} \frac{d\psi_i^e}{d\xi} \frac{1}{J} \frac{d\psi_j^e}{d\xi} + c(\xi) \psi_i^e \psi_j^e \right] J \, d\xi \equiv \int_{-1}^{1} G_{ij}^e(\xi) J \, d\xi \tag{7.2.5a}$$

$$= \sum_{I=1}^{\text{NGP}} G_{ij}^e(\xi_I) J W_I \tag{7.2.5b}$$

where G_{ij}^e denotes the expression in the square brackets in (7.2.5a), J is the Jacobian, and (ξ_I, W_I) are the Ith Gauss point and weight.

Examination of (7.2.5b) shows that there are three free indices: i, j, and I. We take the Gauss-point loop on I as the outermost one. Inside this loop, we evaluate G_{ij}^e at the Gauss point ξ_I for each i and j, multiply with the Jacobian $J = \frac{1}{2}h_e$ and the weights W_I, and sum

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over the range of I:

$$\text{ELK}(i, j) = \text{ELK}(i, j) + G_{ij}^{e}(\xi_I) J W_I \tag{7.2.6}$$

Since $[K^e]$, $[M^e]$, $\{f^e\}$ are evaluated for $e = 1, 2, \ldots,$ NEM, where NEM denotes the number of elements in the mesh, we must initialize all arrays that are being evaluated using the Gauss–Legendre quadrature. The initialization must be made outside of the Gauss–Legendre quadrature loop.

The computation of coefficients K_{ij}^e in (7.2.5a) requires evaluation of a, c, ψ_i, and $d\psi_i/d\xi$ at the Gauss point ξ_I. Hence, inside the loop on I, we call subroutine **SHAPE1D** to evaluate ψ_i, $d\psi_i/dx = (d\psi_i/d\xi)/J$. Fortran statements to evaluate $[K^e]$ and $\{f^e\}$ are given below.

```
   DO 100 NI = 1,NGP
   XI = GAUSPT(NI,NGP)
C  Call subroutine SHP1D to evaluate the interpolation functions
C  (SF) and their global derivatives (GDSF) at the Gauss point XI
   CALL SHP1D(XI,NPE,SF,GDSF,GJ)
   CONST = GJ*GAUSWT(NI,NGP)
   DO 20 I = 1,NPE
20 X = X + SF(I)*ELX(I)
   AX = AX0 + AX1*X
   CX = CX0 + CX1*X
   DO 30 J = 1,NPE
   ELF(J) = ELF(J) + CONST*SF(J)*FX
   DO 30 I = 1,NPE
30 ELK(I,J) = ELK(I,J) + CONST*(AX*GDSF(I)*GDSF(J)
              + CX*SF(I)*SF(J))
```

In the same way, all the other coefficients (e.g., M_{ij}^e and f_i^e) can be evaluated. Recall that the element properties (i.e., K_{ij}^e, M_{ij}^e, and f_i^e) are calculated by calling a suitable subroutine (**COEFFCNT** or **TRANSFRM**) for the field problem being analyzed within a loop with a counter based on the number of elements in the mesh (NEM).

7.2.5 Assembly of Element Equations (Processor)

The assembly of element equations should be carried out as soon as the element matrices are computed, rather than waiting till the element coefficients of all the elements are computed. The latter requires storage of the element coefficients of each element. In the former case, we can perform the assembly in the same loop in which a subroutine is called to calculate element matrices.

A feature of the finite element equations that enables us to save storage and computing time is the assembly of element matrices in upper-banded form. When element matrices are symmetric, the resulting global (or assembled) matrix is also symmetric, with many zeros away from the main diagonal. Therefore, it is sufficient to store only the upper *half-band* of

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the assembled matrix. The half bandwidth of a matrix is defined as follows: Let N_i be the number of matrix elements between the diagonal element and the last nonzero element in the ith row, after which all elements in that row are zero; the half-bandwidth is the maximum of $(N_i + 1) \times$ NDF, where NDF is the number of degrees of freedom per node

$$b_I = \max_{1 \le i \le n} [(N_i + 1) \times \text{NDF}]$$

and n is the number of rows in the matrix (or equations in the problem). General-purpose equation solvers are available for such banded systems of equations.

The half-bandwidth NHBW of the assembled (i.e., global) finite element matrix can be determined in the finite element program itself. The local nature of the finite element interpolation functions (i.e., ψ_i^e are defined to be nonzero only over the element Ω_e) is responsible for the banded character of the assembled matrix. If two global nodes do not belong to the same element, then the corresponding entries in the global matrix are zeros:

$$K_{IJ} = 0 \quad \text{if global nodes } I \text{ and } J \text{ do not correspond to local nodes of the same element}$$

This property enables us to determine the half-bandwidth NHBW of the assembled matrix

$$\text{NHBW} = \max_{\substack{1 \le N \le \text{NEM} \\ 1 \le I,\ J \le \text{NPE}}} \{\text{abs}[\text{NOD}(N,\ I) - \text{NOD}(N,\ J)] + 1\} \times \text{NDF} \tag{7.2.7a}$$

where

$$\begin{aligned} \text{NEM} &= \text{number of elements in the mesh} \\ \text{NPE} &= \text{number of nodes per element} \\ \text{NDF} &= \text{number of degrees of freedom per element} \end{aligned} \tag{7.2.7b}$$

For example, for one-dimensional problems with elements connected in series, the maximum difference between the nodes of an element is equal to NPE − 1. Hence,

$$\text{NHBW} = [(\text{NPE} - 1) + 1] \times \text{NDF} = \text{NPE} \times \text{NDF} \tag{7.2.8}$$

Of course, NHBW is always less than or equal to the number of primary degrees of freedom in the mesh, i.e., the number of equations, NEQ.

The logic for assembling the element matrices K_{ij}^e into the upper-banded form of the global coefficients K_{ij} is that the assembly can be skipped whenever $J < I$ and $J >$ NHBW. The main diagonal, $I = J$, of the assembled square matrix (i.e., full storage form), becomes the first column of the assembled banded matrix (i.e., banded storage form), as shown in Fig. 7.2.3. The upper diagonals (parallel to the main diagonal) take the position of respective columns in the banded matrix. Thus, the banded matrix has dimension NEQ × NHBW, where NEQ denotes the total number of equations in the problem.

The element coefficients K_{ij}^n and f_i^n of a typical element Ω_n are to be assembled into the global coefficients matrix $[K]$ and source vector $\{F\}$, respectively. If the ith node of the element is equal to the Ith global node and the jth node of the element is equal to the Jth global node, we have

$$K_{IJ} = K_{ij}^n, \quad F_I = F_i^n \quad (\text{for NDF} = 1) \tag{7.2.9a}$$

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Figure 7.2.3 Finite element coefficient matrix storage in upper-half-banded form.

The values of I and J can be obtained with the help of array NOD:

$$I = \text{NOD}(n, i), \quad J = \text{NOD}(n, j) \tag{7.2.9b}$$

Recall that it is possible that the same I and J may correspond to a pair of i and j of some other element Ω^m. In that case, K_{ij}^m will be added to existing coefficients K_{IJ} during the assembly. For NDF > 1, the logic still holds, with the change

$$K_{(\text{NR})(\text{NC})} = K^n_{(i+p-1)(j+q-1)} \quad (p, q = 1, 2, \ldots, \text{NDF}) \tag{7.2.10a}$$

where

$$\text{NR} = (I - 1) \times \text{NDF} + p, \quad \text{NC} = (J - 1) \times \text{NDF} + q \tag{7.2.10b}$$

and I and J are related to i and j by (7.2.9b). These ideas are implemented in subroutine **ASSEMBLE**.

7.2.6 Imposition of Boundary Conditions (Processor)

Imposition of boundary conditions on the primary and secondary global degrees of freedom can be carried out through a subroutine **(BOUNDARY)**, which remains unchanged for two-dimensional or three-dimensional problems. There are three types of boundary conditions for any problem:

1. Essential boundary conditions, i.e., boundary conditions on primary variables (Dirichlet boundary conditions).
2. Natural boundary conditions, i.e., boundary conditions on secondary variables (Neumann boundary conditions).
3. Mixed boundary conditions (Newton boundary conditions).

The procedure for implementing the boundary conditions on the primary variables involves modifying the assembled coefficient matrix (GLK) and right-hand column vector (GLF) by three operations:

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Step 1. Moving the known products to the right-hand column of the matrix equation.
Step 2. Replacing the columns and rows of GLK corresponding to the known primary variable by zeros, and setting the coefficient on the main diagonal to unity.
Step 3. Replacing the corresponding component of the right-hand column by the specified value of the variable.

Consider the following N algebraic equations in full matrix form:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots \\ K_{21} & K_{22} & K_{23} & \cdots \\ K_{31} & K_{32} & K_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \end{Bmatrix}$$

where U_I and F_I are the global primary and secondary degrees of freedom, respectively, and K_{IJ} are the assembled coefficients. Suppose that $U_S = \hat{U}_S$ is specified. Recall that when the primary degree of freedom at a node is known, the corresponding secondary degree of freedom is unknown, and vice versa. Set $K_{SS} = 1$ and $F_S = \hat{U}_S$; further, set $K_{SI} = K_{IS} = 0$ for $I = 1, 2, \ldots, N$ and $I \neq S$. For $S = 2$, the modified equations are

$$\begin{bmatrix} K_{11} & 0 & K_{13} & K_{14} & \ldots & K_{1n} \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ K_{31} & 0 & K_{33} & K_{34} & \ldots & K_{3n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ K_{n1} & 0 & K_{n3} & K_{n4} & \ldots & K_{nn} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_n \end{Bmatrix} = \begin{Bmatrix} \hat{F}_1 \\ \hat{U}_2 \\ \hat{F}_3 \\ \vdots \\ \hat{F}_n \end{Bmatrix}$$

where

$$\hat{F}_i = F_i - K_{i2}\hat{U}_2 \quad (i = 1, 3, 4, 5, \ldots, n;\ i \neq 2)$$

Thus, in general, if $U_S = \hat{U}_S$ is known, we have

$$K_{SS} = 1, \quad F_S = \hat{U}_S; \quad \hat{F}_i = F_i - K_{iS}\hat{U}_S; \quad K_{Si} = K_{iS} = 0$$

where $i = 1, 2, \ldots, S-1, S+1, \ldots, n$ $(i \neq S)$. This procedure is repeated for every specified primary degree of freedom. It enables us to retain the original order of the matrix, and the specified boundary conditions on the primary degrees of freedom are printed as part of the solution. Of course, the logic should be implemented for a banded system of equations.

The specified secondary degrees of freedom (Q_i) are implemented directly by adding their specified values to the computed values. Suppose that the point source corresponding to the Rth secondary degree of freedom is specified to be $\hat{F}_R$. Then

$$F_R = f_R + \hat{F}_R$$

where f_R is the contribution due to the distributed source $f(x)$; f_R is computed as a part of the element computations and assembled.

Mixed-type boundary conditions are of the form

$$a\frac{du}{dx} + k(u - \bar{u}) = 0 \quad (\bar{u} \text{ and } k \text{ are known constants}) \tag{7.2.11}$$

which contains both the primary variable u and the secondary variable $a\ du/dx$. Thus, $a\ du/dx$ at the node p is replaced by $-k_p(u_p - \bar{u}_p)$:

$$Q_p = -k_p(U_p - \bar{U}_p)$$

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This amounts to modifying K_{pp} by adding k_p to its existing value,

$$K_{pp} \leftarrow K_{pp} + k_p$$

and adding $k_p \bar{U}_p$ to F_p,

$$F_p \leftarrow F_p + k_p \bar{U}_p$$

All three types of boundary conditions are implemented in the subroutine **BOUNDARY** for boundary, initial, and eigenvalue problems. The following variable names are used in the subroutine **BOUNDARY:**

NSPV	Number of specified primary variables
NSSV	Number of specified secondary variables
NNBC	Number of Newton boundary conditions
VSPV	Column of the specified values $\bar{U}_S$ of primary variables
VSSV	Column of the specified values $\bar{F}_R$ of secondary variables
VNBC	Column of the specified values k_p
UREF	Column of the specified values $\bar{U}_p$
ISPV	Array of the global node and degree of freedom at the node that is specified [ISPV(I,1)=global node of the Ith boundary condition, ISPV(I,2)=degree of freedom specified at the global node, ISPV(I,1)]

Similar definitions are used for ISSV and INBC arrays.

7.2.7 Solving Equations and Postprocessing

Subroutine **EQNSOLVR** is used to solve a banded system of equations, and the solution is returned in array GLF. The program performs the gaussian elimination and back-substitution to compute the solution. For a discussion of the gaussian elimination used to solve a set of linear algebraic equations, the reader is referred to the book by Carnhan, Luther, and Wilkes (1969).

Postprocessing involves computation of the solution and its gradient at preselected points of the domain. The preselected points are the end points of the element, the mid-point, and three evenly spaced points between the end points and the midpoint. Subroutine **POSTPROC** is used to evaluate the solution and its derivatives at the preselected points of an element:

$$u^e(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x), \quad \left(\frac{du^e}{dx}\right)\bigg|_x = \sum_{j=1}^{n} u_j^e \left(\frac{d\psi_j^e}{dx}\right)\bigg|_x \tag{7.2.12}$$

for the Lagrange elements and

$$w^e(x) = \sum_{j=1}^{4} u_j^e \phi_j^e(x), \quad \frac{d^m w^e}{dx^m} = \sum_{j=1}^{4} u_j^e \left(\frac{d^m \phi_j^e}{dx^m}\right) \quad (m = 1, 2, 3) \tag{7.2.13}$$

for the Hermite cubic elements. The nodal values u_j^e of the element Ω_e are deduced from the global nodal values U_I as follows:

$$u_j^e = U_I, \quad I = \text{NOD}(e, j)$$

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Table 7.3.1 Meaning of the program variables for various field problems. Subscripts 0, 1, and 2 on the variables denote the constant, linear, and quadratic coefficients of the variables, respectively, (e.g., f_0, f_1, f_2 denotes the coefficients in $f(x) = f_0 + f_1 \cdot x + f_2 \cdot x^2$).

	Field problem	MODEL	NTYPE	ITEM†	AX0	AX1	BX0	BX1	CX0	CX1	FX0	FX1	FX2	CT0‡	CT1‡
1.	Plane wall	1	0	1	k_0	k_1	0.0	0.0	0.0	0.0	f_0	f_1	f_2	ρ_0	ρ_1
2.	Heat Exchanger fin	1	0	1	$(kA)_0$	$(kA)_1$	0.0	0.0	c_0	c_1	f_0	f_1	f_2	ρ_0	ρ_1
3.	Radially symmetric heat transfer	1	0	1	0.0	k_1	0.0	0.0	0.0	0.0	0.0	f_1	f_2	0.0	ρ_1
4.	Viscous flow through channels	1	0	1	μ_0	μ_1	0.0	0.0	0.0	0.0	f_0	f_1	f_2	ρ_0	ρ_1
5.	Viscous flow through pipes	1	0	1	0.0	μ_1	0.0	0.0	0.0	0.0	0.0	f_1	f_2	0.0	ρ_1
6.	Unidirectional seepage	1	0	1	μ_0	μ_1	0.0	0.0	0.0	0.0	f_0	f_1	f_2	ρ_0	ρ_1
7.	Radially symmetric seepage (ground water flow)	1	0	1	0.0	μ_1	0.0	0.0	0.0	0.0	0.0	f_1	f_2	0.0	ρ_1
8.	Axial deformation of a bar	1	0	2	$(AE)_0$	$(AE)_1$	0.0	0.0	c_0	c_1	f_0	f_1	f_2	$(\rho A)_0$	$(\rho A)_1$
9.	Radially symmetric deformation of a disk (plane stress)	1	1	2	E_1	E_2	υ_{12}	H	c_0	c_1	f_0	f_1	f_2	$(\rho A)_0$	$(\rho A)_1$
10.	Radially symmetric deformation of a cylinder (plane strain)	1	2	2	E_1	E_2	υ_{12}	H	c_0	c_2	f_0	f_1	f_2	$(\rho A)_0$	$(\rho A)_1$
11.	Euler–Bernoulli beam theory	3	0	2	0.0	0.0	$(EI)_0$	$(EI)_1$	c_0	c_1	f_0	f_1	f_2	ρA	ρ_I

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Table 7.3.1 *Continued.*

	Field problem	MODEL	NTYPE	ITEM†	AX0	AX1	BX0	BX1	CX0	CX1	FX0	FX1	FX2	CT0‡	CT1‡
12.	Euler–Bernoulli theory for circular plates	3	1	2	E_1	E_2	ν_{12}	H	c_0	c_2	f_0	f_1	f_2	ρ_A	ρ_I
13.	Timoshenko beam theory (RIE)*	2	0	2	$(SK)_0$	$(SK)_1$	$(EI)_0$	$(EI)_1$	c_0	c_1	f_0	f_1	f_2	ρ_A	ρ_I
14.	Timoshenko beam theory (CIE)*	2	2	2	$(SK)_0$	$(SK)_1$	$(EI)_0$	$(EI)_1$	c_0	c_1	f_0	f_1	f_2	ρ_A	ρ_I
15.	Timoshenko theory of circular plates (RIE)	2	1	2	E_1	E_2	ν_{12}	H	c_0	c_1	f_0	f_1	KG_{13}	ρ_A	ρ_I
16.	Timoshenko theory of circular plates (CIE)	2	3	2	E_1	E_2	ν_{12}	H	c_0	c_1	f_0	f_1	KG_{13}	ρ_A	ρ_I
17.	Plane truss	4	0	0											
18.	The Euler–Bernoulli frame element	4	1	0	For field problems 17–19, these parameters are not read; instead, E =SE, A =SA, L =SL, and so on are read for each member of the structure, where SE=modulus E, SA=cross-sectional area A, SI=moment of inertia I, SL=length L of the member, CN= $\cos\alpha$, SN= $\sin\alpha$, etc. (see Table 7.3.2).										
19.	The Timoshenko frame element	4	2	0											

* $S = GA$ and K is the shear correction factor ($K = 5/6$).

† For time-dependent problems only; when steady-state solution is required, set ITEM=0.

‡ For transient analysis only; the transient analysis option is not available in **FEM1D** for truss and frame problems.

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when NDF = 1. For NDF > 1, I is given by $I = [\text{NOD}(e, j) - 1] \times \text{NDF}$ and

$$u^e_{j+p} = U_{I+p} \quad (p = 1, 2, \ldots, \text{NDF})$$

The values computed using the derivatives of the solution are often inaccurate because the derivatives of the approximate solution become increasingly inaccurate with increasing order of differentiation. For example, the shear force computed in the Euler–Bernoulli beam theory (EBT)

$$V = -\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right) = \sum_{j=1}^{n} u^e_j \frac{d}{dx}\left(b\frac{d^2\phi^e_j}{dx^2}\right) \tag{7.2.14}$$

will be in considerable error in addition to being discontinuous across the elements. The accuracy increases, rather slowly, with mesh refinement and higher-order elements. The derivatives computed using (7.2.14) are more accurate if they are computed at the Gauss points. When accurate values of the secondary variables are desired at the nodes, it is recommended that they be computed from the element equations:

$$Q^e_i = \sum_{j=1}^{n} K^e_{ij} u^e_j - f^e_i \quad (i = 1, 2, \ldots, n) \tag{7.2.15}$$

However, this requires recomputation or saving of element coefficients K^e_{ij} and f^e_i. Recall that the nodal values of generalized forces are exact at the nodes when computed using (7.2.15) for certain problems as discussed before.

7.3 APPLICATIONS OF PROGRAM FEM1D

7.3.1 General Comments

The computer program **FEM1D**, which embodies the ideas presented in the previous section, is intended to illustrate the use of the finite element models developed in Chapters 3–6 to a variety of one-dimensional field problems, some of them are not discussed in this book. The program **FEM1D** is developed as a learning computational tool for students of a first course on the finite element method (see Appendix 1 for a listing of the program located on the book's website at www.mhhe.com/reddy3e). In the interest of simplicity and ease of understanding, only the model equations discussed in this book and their immediate extensions are included in the program.

Table 7.3.1 contains a summary of the definitions of coefficients of various model problems and their corresponding program variables. The table can be used as a ready reference to select proper values of AX0, AX1, and so on for different problems.

7.3.2 Illustrative Examples

Here we revisit some of the example problems considered earlier to illustrate the use of **FEM1D** in their solution. Only certain key observations concerning the input data are made, but complete listings of input files for each problem are given. In the interest of brevity, the complete output files for most problems are not included.

A description of the input variables to program **FEM1D** is presented in Table 7.3.2. In Table 7.3.2, "skip" means that the input data is omitted (i.e., no data is required). In the "free

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Table 7.3.2 Description of the input variables to the program **FEM1D**

• **Data Card 1**	
TITLE	Title of the problem being solved (80 characters)
• **Data Card 2**	
MODEL	Model equation being solved (see below)
NTYPE	Type of problem solved (see below)
	MODEL = 1, NTYPE = 0: A problem of MODEL EQUATION (3.2.1)
	MODEL = 1, NTYPE = 1: A circular DISK (PLANE STRESS)
	MODEL = 1, NTYPE > 1: A circular DISK (PLANE STRAIN)
	MODEL = 2, NTYPE = 0: A Timoshenko BEAM (RIE) problem
	MODEL = 2, NTYPE = 1: A Timoshenko PLATE (RIE) problem
	MODEL = 2, NTYPE = 2: A Timoshenko BEAM (CIE†) problem
	MODEL = 2, NTYPE > 2: A Timoshenko PLATE (CIE) problem
	MODEL = 3, NTYPE = 0: A Euler–Bernoulli BEAM problem
	MODEL = 3, NTYPE > 0: A Euler–Bernoulli circular plate
	MODEL = 4, NTYPE = 0: A plane TRUSS problem
	MODEL = 4, NTYPE = 1: A Euler–Bernoulli FRAME problem
	MODEL = 4, NTYPE = 2: A Timoshenko (CIE) FRAME problem
ITEM	Indicator for transient analysis
	ITEM = 0, Steady-state solution
	ITEM = 1, Transient analysis of PARABOLIC equations
	ITEM = 2, Transient analysis of HYPERBOLIC equations
	ITEM = 3, Eigenvalue analysis
• **Data Card 3**	
IELEM	Type of finite element
	IELEM = 0, Hermite cubic finite element
	IELEM = 1, Linear Lagrange finite element
	IELEM = 2, Quadratic Lagrange finite element
NEM	Number of elements in the mesh
• **Data Card 4**	
ICONT	Indicator for continuity of data for the problem
	ICONT = 1, Data (AX,BX,CX,FX and mesh) is continuous
	ICONT = 0, Data is element dependent
NPRNT	Indicator for printing of element/global matrices
	NPRNT = 0, Not to print element or global matrices but postprocess the solution and print
	NPRNT = 1, Print Element 1 coefficient matrices only but postprocess the solution and print
	NPRNT = 2, Print Element 1 and global matrices but NOT postprocess the solution
	NPRNT > 2, Not to print element or global matrices and NOT postprocess the solution

Skip Cards 5–15 for TRUSS/FRAME problems (MODEL=4), and read Cards 5–15 only if MODEL $\neq$ 4. SKIP cards 5–9 if data is discontinuous (ICONT = 0).

• **Data Card 5**	
DX(I)	Array of element lengths. DX(1) denotes the global coordinate of Node 1 of the mesh; DX(I) (I = 2, NEM1) denotes the length of the (I - 1)st element, where NEM1 = NEM + 1, and NEM denotes the number of elements in the mesh.

Cards 6–9 define the coefficients in the model equations. All coefficients are expressed in terms of GLOBAL coordinate x. See Table 7.2.1 for the meaning of the coefficients.

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(**Table 7.3.2** continued)

- **Data Card 6**

AX0	Constant term of the coefficient $[a(x) =]$ AX
AX1	Linear term of AX

- **Data Card 7**

BX0	Constant term of the coefficient $[b(x) =]$ BX
BX1	Linear term of the coefficient BX

- **Data Card 8**

CX0	Constant term of the coefficient $[c(x) =]$ CX
CX1	Linear term of the coefficient CX

SKIP Card 9 for eigenvalue problems (i.e., when ITEM = 3)

- **Data Card 9**

FX0	Constant term of the source $[f(x) =]$ FX
FX1	Linear term of FX
FX2	Quadratic term of FX

SKIP Cards 10–15 if data is continuous (ICONT $\neq 0$). Cards 10–15 are read for each element (i.e., NEM times). All coefficients are with respect to the LOCAL coordinate $\bar{x}$.

- **Data Card 10**

NNM	Number of global nodes in the mesh

- **Data Card 11**

NOD	Connectivity of the element: NOD(N,I) = Global node number corresponding to the Ith node of Element N (I=1, NPE) where NPE denotes the Number of nodes Per Element
GLX(I)	Length of the Ith element

- **Data Card 12**

DCAX	Constant and linear terms of the coefficient AX

- **Data Card 13**

DCBX	Constant and linear terms of the coefficient BX

- **Data Card 14**

DCCX	Constant and linear terms of the coefficient CX

- **Data Card 15**

DCFX	Constant, linear and quadratic terms of FX

READ Cards 16–23 only for TRUSS/FRAME problems (MODEL = 4); otherwise SKIP.

- **Data Card 16**

NNM	Number of nodes in the finite element mesh

SKIP Cards 17–19 for TRUSS problems (NTYPE = 0)

- **Data Card 17 (Read for each element)**

PR	Poisson's ratio of the material (not used in EBT)
SE	Young's modulus of the material
SL	Length of the element
SA	Cross-sectional area of the element
SI	Moment of inertia of the element
CS	Cosine of the angle of orientation of the element
SN	Sine of the angle of orientation of the element; the angle is measured clockwise from the global x axis

- **Data Card 18 (Read for each element)**

HF	Intensity of the horizontal distributed force
VF	Intensity of the transversely distributed force
PF	Point load on the element
XB	Distance from node 1, along the length of the element to the point of load application, PF
CNT	Cosine of the angle of orientation of the load PF
SNT	Sine of the angle of orientation of the load PF; the angle is measured clockwise from the element x axis.

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(**Table 7.3.2** continued)

- **Data Card 19**

NOD	Connectivity of the element: NOD(N,I) = global node number corresponding to the Ith node of element N (I = 1,NPE)

READ Cards 20 and 21 only for TRUSS problems (NTYPE = 0).

- **Data Card 20 (Read for each element)**

SE	Young's modulus of the material
SL	Length of the element
SA	Cross-sectional area of the element
CS	Cosine of the angle of orientation of the element
SN	Sine of the angle of orientation of the element Angle is measured counterclockwise from x axis
HF	Intensity of the horizontal distributed force

- **Data Card 21**

NOD(N,I)	Connectivity of the element: NOD(N,I) = global node number corresponding to the Ith node of element N (I = 1,NPE)

- **Data Card 22**

NCON	Number of inclined support conditions

SKIP Card 23 if no inclined support conditions are specified (NCON=0).

- **Data Card 23 (I = 1 to NCON)**

ICON(I)	Global node number of the support
VCON(I)	Angle (in degrees) between the normal and the global x-axis

- **Data Card 24**

NSPV	Number of specified PRIMARY degrees of freedom

SKIP Card 25 if no primary variables is specified (NSPV=0).

- **Data Card 25 (I = 1 to NSPV)**

ISPV(I,1)	Node number at which the PV is specified
ISPV(I,2)	Specified local primary degree of freedom (DOF) at the node
VSPV(I)	Specified value of the primary variable (PV) (will not read for eigenvalue problems)

SKIP Card 26 for eigenvalue problems (i.e., when ITEM = 3).

- **Data Card 26**

NSSV	Number of specified (nonzero) SECONDARY variables

SKIP Card 27 if no secondary variables is specified (NSSV=0); repeat Card 27 NSSV times.

- **Data Card 27 (I = 1 to NSSV)**

ISSV(I,1)	Node number at which the SV is specified
ISSV(I,2)	Specified local secondary DOF at the node
VSSV(I)	Specified value of the secondary variable (SV)

- **Data Card 28**

NNBC	Number of the Newton (mixed) boundary conditions

SKIP Card 29 if no mixed boundary condition is specified (NNBC = 0). The mixed boundary condition is assumed to be of the form:

SV+VNBC *(PV − UREF) = 0. Repeat Card 29 NNBC times.

- **Data Card 29 (I = 1 to NNBC)**

INBC(I,1)	Node number at which the mixed B.C. is specified
INBC(I,2)	Local DOF of the PV and SV at the node
VNBC(I)	Value of the coefficient of the PV in the B.C.
UREF(I)	Reference value of the PV

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(**Table 7.3.2** continued)

- **Data Card 30**

NMPC	Number of multipoint constraints (solid mechanics)

SKIP Card 31 if no multipoint conditions are specified (NMPC = 0). The multipoint condition is assumed to be of the form:
VMPC(.,1)*PV1+VMPC(.,2)*PV2=VMPC(.,3). Repeat Card 31 NMPC times.

- **Data Card 31 (I = 1 to NMPC)**

IMC1(I,1)	Node number associated with PV1
IMC1(I,2)	Local DOF of PV1
IMC2(I,1)	Node number associated with PV2
IMC2(I,2)	Local DOF of PV2
VMPC(I)	Values of the coefficients of the constraint equation
VMPC(4)	Value of the force applied at the node of PV1 or PV2

Skip Card 32 if ITEM = 0 (read only for time-dependent or eigenvalue problems).

- **Data Card 32**

CT0	Constant part of CT = CT0 + CT1*X
CT1	Linear part of CT = CT0 + CT1*X

Skip remaining cards if steady-state or eigenvalue analysis is to be performed (ITEM = 0 or ITEM = 3).

- **Data Card 33**

DT	Time increment (uniform)
ALFA	Parameter in the time approximation scheme
GAMA	Parameter in the time approximation scheme*
	GAMA (not used when ITEM = 1: parabolic equation).

Give GAMA= 10^{-6} when centered difference is used (formulation in Problem 6.23 is the correct way to implement the centered difference scheme).

- **Data Card 34**

INCOND	Indicator for initial conditions
	INCOND = 0, Homogeneous (zero) initial conditions
	INCOND > 0, Nonhomogeneous initial conditions
NTIME	Number of time steps for which solution is sought
INTVL	Time step intervals at which solution is to be printed

Skip Cards 35 and 36 if initial conditions are zero (INCOND = 0).

- **Data Card 35**

GUO	Array of initial values of the primary variables

Skip Card 36 for parabolic equations (ITEM = 1).

- **Data Card 36**

GUI	Array of initial values of the first time derivatives of the primary variables.

format" used here, variables of each "data card" (we shall use this terminology to imply an input sequence in a single instruction) are read from the same line; if the values are not found on the same line, the computer will look for them on the next line(s). However, data required by different data cards cannot be put on single line; each data card must start with a new line. The space available after typing required data on a given line may be used to include any comments. For example, we may list the variable names on that line for ready reference but only after all of the required data are listed. The text included thereafter is *not* read by the computer (except to echo the input file).

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Example 7.3.1 (Steady Heat Transfer in a Rod)

Consider the heat transfer problem of Example 4.3.3. The problem is governed by

$$-\frac{d^2\theta}{dx^2} + m^2\theta = 0 \quad \text{for} \quad 0 < x < L$$

$$\theta(0) = \theta_0, \quad \left(\frac{d\theta}{dx}\right)\bigg|_{x=L} = 0$$

where θ is the nondimensional temperature, and L, m^2, θ_0, β, and k are

$$L = 0.05 \text{ m}, \quad m^2 = 400/\text{m}^2, \quad \theta_0 = 300^\circ\text{C},$$

$$\beta = 100 \text{ W/(m}^2\cdot{}^\circ\text{C)}, \quad k = 50 \text{ W/(m}\cdot{}^\circ\text{C)} \tag{7.3.1}$$

For this problem, we have MODEL = 1, NTYPE = 0, and ITEM = 0 (for a steady-state solution). Since $a = 1.0$ and $c = m^2 = P\beta/Ak = 400$ are the same for all elements, we set ICONT = 1, AX0 = 1.0, and CX0 = 400. All other coefficients are zero [$b = 0$ and $f = 0$] for this problem. For a uniform mesh of four linear elements (NEM = 4, IELEM = 1), the increments of the array DX(I) [where DX(1) is always the x coordinate of node 1 (see Figure 7.3.1) and $h = L/4 = 0.05/4 = 0.0125$] are: DX = {0.0, 0.0125, 0.0125, 0.0125, 0.0125}.

Figure 7.3.1 Finite element mesh of a rectangular fin.

The boundary conditions of the problem reduce to $U_1 = 0$ and $Q_2^4 = 0$. There is one specified boundary condition (BC) on the primary variable (NSPV = 1), and it is degree of freedom (DOF) (for heat transfer problems, there is only one DOF) at node 1 is: ISPV(1, 1) = 1 and ISPV(1, 2) = 1. The specified value is VSPV(1) = 300. Since the natural boundary condition ($Q_2^4 = 0$) is homogenous, there is no need to add a zero to the corresponding entry of the source vector (i.e., NSSV = 0). There are no mixed (i.e., convection) boundary conditions in this problem (NNBC = 0). The complete input data required to analyze the problem using **FEM1D** are presented in Box 7.3.1 and the output file is presented in Box 7.3.2. Input data and partial output for the same problem when a mesh of two quadratic elements are used are presented in Box 7.3.3.

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Box 7.3.1 Input file from **FEM1D** for the problem in Example 7.3.1.

```
Example 7.3.1a: Heat transfer in a rod (4 linear elements)
  1 0 0                                  MODEL, NTYPE, ITEM
  1 4                                    IELEM, NEM
  1 1                                    ICONT, NPRNT
  0.0  0.0125   0.0125   0.0125   0.0125  DX(1)=X0; DX(2), etc. Element lengths
  1.0  0.0                               AX0, AX1
  0.0  0.0                               BX0, BX1
400.0  0.0                               CX0, CX1
  0.0  0.0  0.0                          FX0, FX1,  FX2
  1                                      NSPV
  1  1   300.0                          ISPV(1, 1), ISPV(1,2), VSPV(1)
  0                                      NSSV
  0                                      NNBC
  0                                      NMPC
```

Box 7.3.2 Edited output from FEM1D for the problem in Example 7.3.1.

```
OUTPUT from program FEM1D by J. N. REDDY

Example 7.3.1a: Heat transfer in a rod (4 linear elements)

 *** ANALYSIS OF MODEL 1, AND TYPE 0 PROBLEM ***
        (see the code below)
 MODEL=1, NTYPE=0: A problem described by MODEL EQ. 1
 MODEL=1, NTYPE=1: A circular DISK (PLANE STRESS)
 MODEL=1, NTYPE>1: A circular DISK (PLANE STRAIN)
 MODEL=2, NTYPE=0: A Timoshenko BEAM (RIE) problem
 MODEL=2, NTYPE=1: A Timoshenko PLATE (RIE) problem
 MODEL=2, NTYPE=2: A Timoshenko BEAM (CIE) problem
 MODEL=2, NTYPE>2: A Timoshenko PLATE (CIE) problem
 MODEL=3, NTYPE=0: A Euler-Bernoulli BEAM problem
 MODEL=3, NTYPE>0: A Euler-Bernoulli Circular plate
 MODEL=4, NTYPE=0: A plane TRUSS problem
 MODEL=4, NTYPE=1: A Euler-Bernoulli FRAME problem
 MODEL=4, NTYPE=2: A Timoshenko (CIE) FRAME problem

     Element type (0, Hermit e,>0, Lagrange)  ..=  1
     No. of deg. of freedom per node, NDF   ...=  1
     No. of elements in the mesh, NEM  .......=  4
     No. of total DOF in the model, NEQ  ..... =  5
     No. of specified primary DOF, NSPV  ......=  1
     No. of specified secondary DOF, NSSV....=  0
     No. of specified Newton B. C.: NNBC .....=  0
```

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(**Box 7.3.2** is continued from the previous page)

```
Boundary information on primary variables:

   1   1   0.30000E+03

Global coordinates of the nodes, {GLX}:

 0.00000E+00 0.12500E-01  0.25000E-01  0.37500E-01  0.50000E-01

Coefficients of the differential equation:
     AX0 = 0.1000E+01     AX1 = 0.0000E+00
     BX0 = 0.0000E+00     BX1 = 0.0000E+00
     CX0 = 0.4000E+03     CX1 = 0.0000E+00
     FX0 = 0.6000E+01     FX1 = 0.0000E+00     FX2 =  0.0000E+00

Element coefficient matrix, [ELK]:
 0.81667E+02 -0.79167E+02
-0.79167E+02  0.81667E+02

Element source vector, {ELF}:
 0.00000E+00  0.00000E+00

SOLUTION (values of PVs) at the NODES:

 0.30000E+03  0.25152E+03  0.21892E+03  0.20016E+03  0.19403E+03
```

X	P. Variable	S. Variable
0.00000E+00	0.30000E+03	-0.38785E+04
0.12500E-01	0.25152E+03	-0.26076E+04
0.25000E-01	0.21893E+03	-0.15015E+04
0.37500E-01	0.20016E+03	-0.49018E+03

Box 7.3.3 Input and partial output using quadratic elements.

```
Example 7.3.1: Heat transfer in a rod (2 quadratic elements)

 1  0  0                    MODEL, NTYPE, ITEM
 2  2                       IELEM, NEM
 1  1                       ICONT, NPRNT
 0.0  0.025  0.025          DX(1)=X0; DX(2), etc. Ele. lengths
 1.0  0.0                   AX0,  AX1
 0.0  0.0                   BX0, BX1
 400.0  0.0                 CX0,  CX1
 0.0  0.0  0.0              FX0,  FX1,  FX2
 1                          NSPV
 1  1  300.0                ISPV(1,1), ISPV(1,2), VSPV(1)
 0                          NSSV
 0                          NNBC
 0                          NMPC
```

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(**Box 7.3.3** is continued from the previous page)

```
OUTPUT from program FEM1D by J. N. REDDY

Example 7.3.1: Heat transfer in a rod (2 quadratic elements)

    Element type (0, Hermite, > 0, Lagrange) ..= 2
    No. of deg. of freedom per node, NDF    ....= 1
    No. of elements in the mesh, NEM .......= 2
    No. of total DOF in the model, NEQ......= 5
    No. of specified primary DOF, NSPV ......= 1
    No. of specified secondary DOF, NSSV....= 0
    No. of specified Newton B. C.: NNBC .....= 0

Global coordinates of the nodes, {GLX}:
0.00000 E+00  0.12500E-01  0.25000E-01  0.37500E-01  0.50000E-01

Element coefficient matrix, [ELK]:
0.94667E+02   -0.10600E+03    0.13000E + 02
-0.10600E+03   0.21867E+03   -0.10600E + 03
0.13000E+02   -0.10600E+03    0.94667E + 02

Element source vector, {ELF}:
0.25000E-01  0.10000E+00  0.25000E-01

SOLUTION (values of PVs) at the NODES:
0.30000E+03  0.25170E+03  0.21923E+03  0.20052E+03  0.19442E+03
```

X	P.Variable	S. Variable
0.00000E+00	0.30000E+03	–0.44971E+04
0.25000E-01	0.21924E+03	–0.19642E+04
0.25000E-01	0.21924E+03	–0.20014E+04
0.50000E-01	0.19443E+03	0.16471E+02

Example 7.3.2 (Steady Heat Transfer in a Composite Wall)

Here we consider the composite wall problem of Example 4.3.1. The governing equations of the problem are

$$-\frac{d}{dx}\left(kA\frac{dT}{dx}\right)=0, \quad 0<x<L \tag{7.3.2a}$$

$$T(0)=T_0, \qquad \left[kA\frac{dT}{dx}+\beta A(T-T_\infty)\right]_{x=L}=0 \tag{7.3.2b}$$

This is a problem with *discontinuous data* (ICONT = 0) because $a = kA$ is discontinuous (as well as $h_1 \neq h_2 \neq h_3$). For a three-element (nonuniform) mesh of linear elements (see Figure 7.3.2), the input data and edited output are given in Box 7.3.4.

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Figure 7.3.2 Heat transfer in a composite wall.

Box 7.3.4 Input for heat transfer in a composite wall (Example 7.3.2).

```
Example 7.3.2: Heat transfer in a composite wall
  1  0  0                    MODEL, NTYPE, ITEM
  1  3                       IELEM, NEM
  0  0                       ICONT, NPRNT
  4                          NNM
  1     2     0.02           NOD(1,J), GLX(1)   }
  70.0  0.0                  AX0, AX1           }
  0.0   0.0                  BX0, BX1           }  Data for
  0.0   0.0                  CX0, CX1           }  Element 1
  0.0   0.0   0.0            FX0, FX1, FX2      }

  2     3     0.025          NOD(2,J) , GLX(2)  }
  40.0  0.0                  AX0, AX1           }
  0.0   0.0                  BX0, BX1           }  Data for
  0.0   0.0                  CX0, CX1           }  Element 2
  0.0   0.0   0.0            FX0, FX1, FX2      }

  3     4     0.04           NOD(2,J), G LX(3)  }
  20.0  0.0                  AX0, AX1           }
  0.0   0.0                  BX0, BX1           }  Data for
  0.0   0.0                  CX0, CX1           }  Element 3
  0.0   0.0   0.0            FX0, FX1, FX2      }
  1                          NSPV
  1  1  200.0                ISPV(1,1), ISPV(1,2), VSPV(1)
  0                          NSSV
  1                          NNBC
  4  1  10.0  50.0           INBC(1,1), INBC(1, 2), VNBC(1), UREF(1)
  0                          NMPC

  OUTPUT from program  FEM1D  by J. N. REDDY

  SOLUTION (values of PVs) at the NODES:
  0.20000E+03   0.19958E+03   0.19867E+03  0.19576E+03
```

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Example 7.3.3 (Radially Symmetric Heat Transfer in a Solid Cylinder)

Consider radially symmetric heat transfer in a solid circular cylinder of radius R_0 of Example 4.3.4. The governing equations of the problem are given by

$$-\frac{d}{dr}\left(2\pi kr\frac{dT}{dr}\right)=2\pi r g_0 \quad \text{for } 0<r<R_0 \tag{7.3.3a}$$

$$\left(2\pi kr\frac{dT}{dr}\right)\bigg|_{r=0}=0, \quad T(R_0)=T_0 \tag{7.3.3b}$$

with $k=20$ W/(m·°C), $g_0=2\times10^8$ W/m^3, $T_0=100$°C, and $R_0=0.01$ m. We have MODEL = 1, NTYPE = 0, and ITEM = 0 (for a steady-state solution), and the data is continuous (ICONT = 1) in the domain for a mesh of two quadratic elements (IELEM = 2, NEM = 2). The data are

$$a=2\pi kr\rightarrow a_0=0, \quad a_1=2\pi k; \quad b=0\rightarrow b_0=0.0, \quad b_1=0.0$$

$$c=0\rightarrow c_0=0.0, \quad c_1=0.0; \quad f=2\pi g_0 r\rightarrow f_0=0.0, \quad f_1=2\pi g_0, \quad f_2=0.0$$

Thus, we have [for values $k=20$ W/(m·°C), $g_0=2\times10^8$ W/m^3]

$$\text{AX0}=0.0, \quad \text{AX1}=125.6637, \quad \text{BX0}=0.0, \quad \text{BX1}=0.0$$

$$\text{CX0}=0.0, \quad \text{CX1}=0.0, \quad \text{FX0}=0.0, \quad \text{FX1}=12.5664\times10^8, \quad \text{FX2}=0.0$$

For two-element mesh of quadratic elements, the array {DX} and boundary information are given by

$$\text{DX}=\{0.0, 0.005, 0.005\}$$

$$\text{NSPV}=1, \quad \text{ISPV}(1,1)=5, \quad \text{ISPV}(1,2)=1, \quad \text{VSPV}(1)=100$$

The input file along with the modified output from FEM1D are presented in Boxes 7.3.5 and 7.3.6, respectively. Note that the finite element solution obtained with two quadratic elements is more accurate than the solution obtained with four linear elements, and it is essentially the same as the exact solution (see Table 4.3.2 for the results).

Box 7.3.5 Input data for radially symmetric problem of Example 7.3.3.

```
Example 7.3.3: Radially symmetric heat transfer in a cylinder (2 quadratic ele)

1 0 0                      MODEL, NTYPE, ITEM
2 2                        IELEM, NEM
1 1                        ICONT, NPRNT
0.0  0.005  0.005          DX(1) = X0; DX(2), etc. Ele. lengths
0.0  125.6637              AX0, AX1
0.0  0.0                   BX0, BX1
0.0  0.0                   CX0, CX1
0.0  12.56637E08   0.0     FX0, FX1, FX2
1                          NSPV
5 1  100.0                 ISPV(1, 1), ISPV(1, 2), VSPV(1)
0                          NSS V
0                          NN BC
0                          NMPC
```

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Box 7.3.6 Partial output for radially symmetric problem of Example 7.3.3.

```
OUTPUT from program FEM1D by J. N. REDDY

Example 7.3.3: Radially symmetric heat transfer in a cylinder (2 quadratic ele)

Element type (0, Hermite, >0, Lagrange)..=  2
     No. of deg. of freedom per node, NDF  ....=  1
     No. of elements in the mesh, NEM  .......=  2
     No. of total DOF in the model, NEQ  ......=  5
     No. of specified primary DOF, NSPV ......=  1
     No. of specified secondary DOF, NSSV....=  0
     No. of specified Newton B. C.: NNBC .....=  0

Element coefficient matrix, [ELK]:
 0.62832E+02  -0.83776E+02  0.20944E+02
-0.83776E+02   0.33510E+03 -0.25133E+03
 0.20944E+02  -0.25133E+03  0.23038E+03

Element source vector, {ELF}:
 0.00000E+00    0.10472E+05    0.52360E+04

SOLUTION (values of PVs) at the NODES:
 0.35000E+03    0.33437E+03    0.28750E+03    0.20937E+03    0.10000E+03
```

X	P. Variable	S. Variable
0.00000E-00	0.35000E+03	0.00000E+00
0.62500E-03	0.34902E+03	-0.24544E+03
0.12500E-02	0.34609E+03	-0.98175E+03
0.18750E-02	0.34121E+03	-0.22089E+04
0.25000E-02	0.33437E+03	-0.39270E+04
0.31250E-02	0.32559E+03	-0.61359E+04
0.37500E-02	0.31484E+03	-0.88357E+04
0.43750E-02	0.30215E+03	-0.12026E+05
0.50000E-02	0.28750E+03	-0.15708E+05
0.50000E-02	0.28750E+03	-0.15708E+05
0.56250E-02	0.27090E+03	-0.19880E+05
0.62500E-02	0.25234E+03	-0.24544E+05
0.68750E-02	0.23184E+03	-0.29698E+05
0.75000E-02	0.20937E+03	-0.35343E+05
0.81250E-02	0.18496E+03	-0.41479E+05
0.87500E-02	0.15859E+03	-0.48106E+05
0.93750E-02	0.13027E+03	-0.55223E+05
0.10000E-01	0.10000E+03	-0.62832E+05

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Next, we consider a couple of steady-state solid mechanics problems. The reader is asked to visit the corresponding examples from Chapter 4 to have a complete background of the problems discussed here.

Example 7.3.4 (Axial Deformation of a Pier)

Here we consider the axial deformation of a pier (Example 4.5.1). The problem is governed by the equations (see Fig. 7.3.3)

$$-\frac{d}{dx}\left[\frac{1}{4}E(1+x)\frac{du}{dx}\right]=6.25(1+x),\quad 0<x<2 \tag{7.3.4a}$$

$$\left[\frac{1}{4}E(1+x)\frac{du}{dx}\right]\Bigg|_{x=0}=-5,\quad u(2)=0 \tag{7.3.4b}$$

For this case, the data of the problem is MODEL = 1, NTYPE = 1, and ITEM = 0 with

$$a=\frac{1}{4}E(1+x)\rightarrow a_0=0.25E,\quad a_1=0.25E;\quad b=0\rightarrow b_0=0.0,\quad b_1=0.0$$

$$c=0\rightarrow c_0=0.0,\quad c_1=0.0;\quad f=6.25(1+x)\rightarrow f_0=6.25,\quad f_1=6.25,\quad f_2=0.0$$

For $E=28\times10^9$ N/m^2, the input data becomes

$$\text{AX0}=7\times10^9,\quad \text{AX1}=7\times10^9,\quad \text{BX0}=0.0,\quad \text{BX1}=0.0$$

$$\text{CX0}=0.0,\quad \text{CX1}=0.0,\quad \text{FX0}=6.25\times10^3,\quad \text{FX1}=6.25\times10^3,\quad \text{FX2}=0.0$$

Figure 7.3.3 The concrete pier problem of Example 7.3.4.

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For two-element mesh of linear elements, the array DX and boundary information are given by (NMPC = 0)

$$DX = \{0.0, 1.0, 1.0\}$$

$$NSPV = 1, \quad ISPV(1, 1) = 3, \quad ISPV(1, 2) = 1, \quad VSPV(1) = 0$$

$$NSSV = 1, \quad ISSV(1, 1) = 1, \quad ISSV(1, 2) = 1, \quad VSSV(1) = 5 \times 10^3$$

The input file and modified output of the problem are presented in Boxes 7.3.7 and 7.3.8, respectively.

Box 7.3.7 Input data for the pier problem of Example 7.3.4.

```
Example 7.3.4: Axial deformation of a pier
1 0 0                          MODEL, NTYPE, ITEM
1 2                            IELEM, NEM
1 1                            ICONT, NPRNT
0.0     1.0     1.0            DX(1)=X0;  DX(2), etc. Ele. lengths
7.0E9     7.0E9                AX0, AX1
0.0     0.0                    BX0, BX1
0.0     0.0                    CX0, CX1
6.25E3    6.25E3    0.0        FX0, FX1, FX2
1                              NSPV
3 1    0.0                     ISPV(1,1), ISPV(1,2), VSPV(1)
1                              NSSV
1 1    5.0E3                   ISSV(1,1), ISSV(1,2), VSSV(1)
0                              NNBC
0                              NMPC
```

Box 7.3.8 Modified output for the pier problem of Example 7.3.4.

```
OUTPUT from program FEM1D by J. N. REDDY

Example 7.3.4: Axial deformation of a pier (2 linear elements)

 Boundary information on primary variables:
    3   1   0.00000E+00

 Boundary information on secondary variables:
    1   1   0.50000E+04

 Global coordinates of the nodes, {GLX}:
 0.00000 E+00   0.10000E+01   0.20000E+01
```

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(**Box 7.3.8** is continued from the previous page)

```
Coefficients of the differential equation:
    AX0=0.7000E+10      AX1=0.7000E+10
    BX0=0.0000E+00      BX1=0.0000E+00
    CX0=0.0000E+00      CX1=0.0000E+00
    FX0=0.6250E+04      FX1=0.6250E+04      FX2= 0.0000E+00

Element coefficient matrix, [ELK]:
 0.10500E+11 -0.10500E+11
-0.10500E+11  0.10500E+11

Element source vector, {ELF}:
 0.41667E+04   0.52083E+04

SOLUTION (values of PVs) at the NODES:
  0.21111E-05   0.12381E-05    0.00000E+00
```

Example 7.3.5 (Deformation of a Rotating Disk)

We wish to determine the deformation and stresses in a rotating homogeneous (solid) disk of thickness H and made of isotropic material (E, ν). The governing equation of the problem is (see Problem 4.37)

$$-\frac{d}{dr}\left[c\left(r\frac{du}{dr}+\nu u\right)\right]+c\left(\frac{u}{r}+\nu\frac{du}{dr}\right)=\rho\omega^2 r, \quad 0<r<R_0 \tag{7.3.5a}$$

$$c=\frac{E}{1-\nu^2} \tag{7.3.5b}$$

For this case, we have MODEL = 1, NTYPE = 1, and ITEM = 0. For a mesh of two quadratic elements (see Figure 7.3.4) (i.e., IEL = 2 and NEM = 2), we use ICONT = 1 and

$$\mathrm{DX}=\{0.0, 0.5R_0, 0.5R_0\}$$

where R_0 is the radius of the disk. Since we are seeking results in nondimensional form, we take

$$R_0=1.0, \quad E_1=E_2=E=1.0, \quad \nu_{12}=\nu=0.3, \quad \rho\omega^2=1$$

Figure 7.3.4 The rotating disk of Example 7.3.5.

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where E_1 is the modulus along the radial direction and E_2 is the modulus in the circumferential direction. This information is supplied to the program through the following variables:

$$\text{AX0}(=E_1)=1.0,\ \text{AX1}(=E_2)=1.0,\ \text{BX0}(=\nu_{12})=0.3,\ \text{BX1}(=H)=1.0$$

$$\text{CX0}=0.0,\ \text{CX1}=0.0,\ \text{FX0}=0.0,\ \text{FX1}=1.0,\ \text{FX2}=0.0$$

The boundary conditions are

$$u(0)=0 \text{ (by symmetry)}, \quad Q_1^1=-(r\sigma_r)_0=0 \text{ at } r=R_0 \text{ (stress-free condition)}$$

Since the secondary variable is homogeneous, there is no need to include it in the data (NSSV=0). We have NSPV = 1, NNBC = 0, and NMPC = 0 with

$$\text{ISPV}(1,1)=1,\ \text{ISPV}(1,2)=1,\ \text{VSPV}(1)=0.0$$

The input data and modified output for this problem are presented in Boxes 7.3.9 and 7.3.10, respectively.

Box 7.3.9 Input data for the rotating disk problem of Example 7.3.5.

```
Example 7.3.5: Deformation of a circulardisk (2 quadratic elements)

 1 1 0                    MODEL, NTYPE, ITEM
 2 2                      IELEM, NEM
 1 1                      ICONT, NPRNT
  0.0  0.50  0.50         DX(I)
  1.0  1.0                AX0, AX1
  0.3  1.0                BX0, BX1
  0.0  0.0                CX0, CX1
  0.0  1.0   0.0          FX0, FX1, FX2
 1                        NSPV
 1 1  0.0                 ISPV(1,1), ISPV(1, 2), VSPV(1)
 0                        NSSV
 0                        NNBC
 0                        NMPC
```

Box 7.3.10 Output for the rotating disk problem of Example 7.3.5.

```
OUTPUT from program FEM1D by J.N. REDDY

Example 7.3.5: Deformation of a circular disk (2 quadratic elements)

*** ANALYSIS OF MODEL 1, AND TYPE 1 PROBLEM ***

MODEL=1, NTYPE=0: A problem described by MODEL EQ. 1
MODEL=1, NTYPE=1: A circular DISK (PLANE STRESS)
MODEL=1, NTYPE>1: A circular DISK (PLANE STRAIN)
MODEL=2, NTYPE=0: A Timoshenko BEAM (RIE) problem
MODEL=2, NTYPE=1: A Timoshenko PLATE (RIE) problem
MODEL=2, NTYPE=2: A Timoshenko BEAM (CIE) problem
MODEL=2, NTYPE>2: A Timoshenko PLATE (CIE) problem
MODEL=3, NTYPE=0: A Euler-Bernoulli BEAM problem
MODEL=3, NTYPE>0: A Euler-Bernoulli Circular plate
```

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(**Box 7.3.10** is continued from the previous page)

```
Element type (0, Hermite,>0, Lagrange) ..= 2
No. of deg. of freedom per node, NDF ....= 1
No. of elements in the mesh, NEM .........= 2
No. of total DOF in the model, NEQ .......= 5
No. of specified primary DOF, NSPV ......= 1
No. of specified secondary DOF, NSSV ...= 0
No. of specified Newton B. C.: NNBC .....= 0

Boundary information on primary variables:
  1   1   0.00000E+00

Global coordinates of the nodes, {GLX}:
 0.00000E+00   0.25000E+00   0.50000E+00   0.75000E+00  0.10000E+01

 Element coefficient matrix, [ELK]:
  0.15018E+01    0.64565E-09     0.10761E-08
  0.64565E-09    0.43956E+01    -0.21978E+01
  0.10761E-08   -0.21978E+01     0.25275E+01

 Element source vector, {ELF}:
  -0.20833E-02    0.25000E-01     0.18750E-01

 SOLUTION (values of PVs) at the NODES:
  0.00000E+00 0.70706E-01 0.13004E+00 0.16875E+00 0.17500E+00
```

X	Displacemnt	Radial Strs	Hoop Stress
0.00000E+00	0.00000E+00	0.33580E+00	0.10074E+00
0.62500E-01	0.18743E-01	0.42216E+00	0.42654E+00
0.12500E+00	0.36775E-01	0.40779E+00	0.41654E+00
0.18750E+00	0.54096E-01	0.39341E+00	0.40654E+00
0.25000E+00	0.70706E-01	0.37904E+00	0.39654E+00
0.31250E+00	0.86606E-01	0.36466E+00	0.38654E+00
0.37500E+00	0.10179E+00	0.35029E+00	0.37654E+00
0.43750E+00	0.11627E+00	0.33591E+00	0.36654E+00
0.50000E+00	0.13004E+00	0.32154E+00	0.35654E+00
0.50000E+00	0.13004E+00	0.32727E+00	0.35826E+00
0.56250E+00	0.14276E+00	0.28952E+00	0.34065E+00
0.62500E+00	0.15345E+00	0.25111E+00	0.32086E+00
0.68750E+00	0.16212E+00	0.21223E+00	0.29948E+00
0.75000E+00	0.16875E+00	0.17299E+00	0.27690E+00
0.81250E+00	0.17336E+00	0.13348E+00	0.25341E+00
0.87500E+00	0.17593E+00	0.93750E-01	0.22919E+00
0.93750E+00	0.17648E+00	0.53846E-01	0.20440E+00
0.10000E+01	0.17500E+00	0.13802E-01	0.17914E+00

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The exact displacement and stresses in the disc are given by

$$u(r) = \frac{(1-\nu)R_0^3}{8E}\left[(3+\nu)\frac{r}{R_0} - (1+\nu)\left(\frac{r}{R_0}\right)^3\right]\rho\omega^2$$

$$\sigma_{rr}(r) = \frac{(3+\nu)R_0^2}{8}\left[1 - \left(\frac{r}{R_0}\right)^3\right]\rho\omega^2 \tag{7.3.6}$$

$$\sigma_{\theta\theta}(r) = \frac{(3+\nu)R_0^2}{8}\left[1 - \frac{1+3\nu}{3+\nu}\left(\frac{r}{R_0}\right)^3\right]\rho\omega^2$$

The exact displacements at $x = 0.25$, $x = 0.5$, $x = 0.75$, and $x = 1.0$ are 0.0704, 0.1302, 0.1686, and 0.1750, respectively. The maximum values of stresses occur at $r = 0$, and they are $\sigma_{rr}(0) = \sigma_{\theta\theta}(0) = 0.4125$. The displacements obtained with two quadratic elements are in good agreement with the analytical solution. The postcomputed stresses at $r = 0$ are not accurate because of the singularity there:

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right), \quad \sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\nu\frac{du}{dr} + \frac{u}{r}\right) \tag{7.3.7}$$

The next example deals with a multipoint constraint problem of the type discussed in Example 4.6.3.

Example 7.3.6 (Multipoint Constraint Problem)

A rigid bar AB of length $L = 1.6$ m is hinged to a support at A and supported by two vertical bars attached at points C and D, as shown in Figure 7.3.5. Both bars have the same cross-sectional

Geometric constraints : $\frac{U_1}{0.5} = \frac{U_2}{1.2} = \frac{U_5}{1.6}$

Figure 7.3.5 A rigid member supported by cables (Example 7.3.6).

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area ($A = 16\ \text{mm}^2$) and are made of the same material with modulus $E = 200$ GPa. The lengths of the bars and the horizontal distances are shown in the figure. We wish to determine the elongations as well as tensile stresses in the bars.

The two geometric constraints are:

$$\frac{U_1}{0.5} = \frac{U_5}{1.6} \rightarrow 3.2U_1 - U_5 = 0, \quad \frac{U_2}{1.2} = \frac{U_5}{1.6} \rightarrow 1.333U_2 - U_5 = 0$$

Thus, we have [see Eq. (4.6.37)]

$$\beta_1^1 = 3.2,\ \beta_2^1 = -1.0,\ \beta_{12}^1 = 0.0; \quad \beta_1^2 = 1.333,\ \beta_2^2 = -1.0,\ \beta_{12}^2 = 0.0$$

In order to include the load P in the analysis, we introduce a node at B. Thus, we have NEM = 3 and NNM = 5 with $a = EA = 3.2 \times 10^6\ \text{m}^2$ for the bar elements and $a = 0$ for the third (rigid) element. It is sufficient to use linear finite elements to represent the bars. Thus, the data for the problem are MODEL = 1, NTYPE = 1, ITEM = 0, ICONT = 0. The boundary and constraint information are: NSPV = 2, ISPV(1,1) = 1, ISPV(1,2) = 1, VSPV(1) = 0, ISPV(2,1) = 2, ISPV(2, 2) = 1, VSPV(2) = 0, NSSV = 0, NNBC = 0, NMPC = 2, VMPC(1,1) = $\beta_1^1 = 3.2$, VMPC(1,2) = $\beta_2^1 = -1.0$, VMPC(1,3) = $\beta_{12}^1 = 0$, VMPC(1,4) = 0; VMPC(2,1) = $\beta_1^2 = 1.333$, VMPC(2,2) = $\beta_2^2 = -1.0$, VMPC(2,3) = $\beta_{12}^2 = 0$, and VMPC(2,4) = $P = 970$ N.

The input file and modified output of the problem are presented in Box 7.3.11. The displacements are $U_1 = 0.1$ mm, $U_2 = 0.24$ mm, and $U_5 = 0.32$ mm. The stresses are $\sigma_1 = 10^6(800/16) = 50$ MPa and $\sigma_2 = 10^6(960/16) = 60$ MPa.

Box 7.3.11 Input and edited output for the problem of Example 7.3.6.

```
Example 7.3.6: DEFORMATION OF A CONSTRAINED STRUCTURE
  1   0   0                    MODEL, NTYPE, ITEM
  1   3                        IELEM, NEM
  0   1                        ICONT, NPRNT
  5                            NNM

  1        3        0.4        NOD(1,J),GLX(1)
  3.2E6    0.0                 AX0, AX1          Data for
  0.0      0.0                 BX0, BX1          Element 1
  0.0      0.0                 CX0, CX1
  0.0      0.0      0.0        FX0, FX1, FX2

  2        4        0.8        NOD(2,J),GLX(2)
  3.2E6    0.0                 AX0, AX1          Data for
  0.0      0.0                 BX0, BX1          Element 2
  0.0      0.0                 CX0, CX1
  0.0      0.0      0.0        FX0, FX1, FX2

  1        5        1.6        NOD(3,J),GLX(3)
  0.0      0.0                 AX0, AX1          Data for
  0.0      0.0                 BX0, BX1          Element 3
  0.0      0.0                 CX0, CX1
  0.0      0.0      0.0        FX0, FX1, FX2
```

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(**Box 7.3.11** is continued from the previous page)

```
2                                   NS PV
3 1        0.0                      ISPV(1,1), ISPV(1,2), VSPV(1)
4 1        0.0                      ISpV(2,1), ISpV(2,2), VSpV(2)
0                                   NS SV
0                                   NN BC
2                                   NMPC
1 1        5 1        3.2           -1.0 0.0 0.0
2 1        5 1        1.33333       -1.0 0.0 970.0
                        IMC1(1,1),IMC1(1,2),IMC2(1,1),IMC2(1,2),
                                        (VMPC(1,I),I=1 to 4)
```

```
OUTPUT from program  FEM1D by J. N. REDDY

Multipoint constraint information:

 1      1      5      1
0.32000E+01        -0.10000E+01        0.00000E+00        0.00000E+00
 2      1      5      1
0.13333E+01        -0.10000E+01        0.00000E+00        0.97000E+03
```

```
Properties of Element =  1

   Element  length,  H  .......  = 0.4000E+00
   AX0 =  0.3200E+07      AX1 = 0.0000E+00
   BX0 =  0.0000E+00      BX1 = 0.0000E+00
   CX0 =  0.0000E+00      CX1 = 0.0000E+00
   FX0 =  0.0000E+00      FX1 = 0.0000E+00        FX2 = 0.0000E+00

Properties of Element =  2

   Element  length,  H  .......  = 0.8000E+00
   AX0 = 0.3200E+07       AX1 = 0.0000E+00
   BX0 = 0.0000E+00       BX1 = 0.0000E+00
   CX0 = 0.0000E+00       CX1 = 0.0000E+00
   FX0 = 0.0000E+00       FX1 = 0.0000E+00        FX2 = 0.0000E+00

Properties of Element =  3

   Element  length,  H  .......  = 0.1600E+01
   AX0 = 0.0000E+00       AX1 = 0.0000E+00
   BX0 = 0.0000E+00       BX1 = 0.0000E+00
   CX0 = 0.0000E+00       CX1 = 0.0000E+00
   FX0 = 0.0000E+00       FX1 = 0.0000E+00        FX2 = 0.0000E+00
```

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(**Box 7.3.11** is continued from the previous page)

```
Element coefficient matrix, [ELK]:

 0.80000E+07  -0.80000E+07
-0.80000E+07   0.80000E+07

Element coefficient matrix, [ELK]:

 0.40000E+07  -0.40000E+07
-0.40000E+07   0.40000E+07

Element coefficient matrix, [ELK]:

 0.00000E+00   0.00000E+00
 0.00000E+00   0.00000E+00

SOLUTION (values of PVs) at the NODES:

 0.10000E-03    0.24000E-03    0.00000E+00    0.00000E+00    0.32001E-03

Forces at the constrained points:

 0.80001E+03    0.95999E+03
```

The next two examples deal with beams and frames of Chapter 5.

Example 7.3.7 (Clamped and Spring-Supported Beam)

Consider the spring-supported cantilever beam shown in Figure 7.3.6. We solve the problem first using the Euler–Bernoulli beam element (MODEL=3, NTYPE=0, IELEM=0) and then using the Timoshenko beam element (MODEL=2, NTYPE=0 or 2, IELEM=1,2,or 3). Since the loading is discontinuous, we set ICONT=0. A minimum of two elements are required to model the problem (i.e., NEM=2).

If we take $EI = 1.0E6$ (i.e., 10^6 lb-ft.^{2}), then

$$GAK = \frac{E}{2(1+\nu)} BH\frac{5}{6} = \frac{EI}{1+\nu}\frac{5}{H^2}$$

For $L/H = 10$, we have $H = 1.0$ because $L = 10$ ft. For the choice $\nu = 0.25$, we have

$$GAK = 4EI/H^2 = 4 \times 10^6 \text{ lb}$$

Thus, we use

$$\text{AX0} = 0.0, \quad \text{AX1} = 0.0, \quad \text{BX0}(= EI) = 1.0 \times 10^6$$

$$\text{BX1} = 0.0, \quad \text{CX0} = 0.0, \quad \text{CX1} = 0.0$$

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Figure 7.3.6 The spring-supported beam of Example 7.3.7.

for the Euler–Bernoulli beam and

$$\text{AX}(= GAK) = 4.0 \times 10^6, \quad \text{AX1} = 0.0, \quad \text{BX0}(= EI) = 1.0 \times 10^6$$
$$\text{BX1} = 0.0, \quad \text{CX0} = 0.0, \quad \text{CX1} = 0.0$$

for the Timoshenko beam.

The distributed transverse load is zero in element 1, and it is

$$q(\bar{x}) = \frac{100}{6}\bar{x}$$

in element 2. Hence,

$$\text{FX0} = 0.0, \quad \text{FX1} = 0.0, \quad \text{FX2} = 0.0 \text{ in element 1}$$
$$\text{FX0} = 0.0, \quad \text{FX1} = 16.666, \quad \text{FX2} = 0.0 \text{ in element 2}$$

The global coordinates of nodes and the connectivity matrix entries for each element are obvious from the geometry. For the Euler–Bernoulli beam element, the number of nodes is always equal to two (NPE = 2), whereas for the Timoshenko beam element, the number of nodes depends on the degree of interpolation (or element type) selected:

$$\text{NPE} = \text{IELEM} + 1$$

The boundary conditions for this problem are

$$w(0) = 0, \quad \left(\frac{dw}{dx}\right)\bigg|_{x=0} = 0, \quad (V + kw)\bigg|_{x=L} = 0$$

where V is the shear force. Therefore, we have (NSSV=0)

$$\text{NSPV} = 2, \quad \text{ISPV}(1,1) = 1, \quad \text{ISPV}(1,2) = 1, \quad \text{VSPV}(1) = 0.0$$
$$\text{ISPV}(2,1) = 1, \quad \text{ISPV}(2,2) = 2, \quad \text{VSPV}(2) = 0.0$$
$$\text{NNBC} = 1, \quad \text{INBC}(1,1) = 3, \quad \text{INBC}(1,2) = 1$$
$$\text{VNBC}(1)\ (= k) = 1.0 \times 10^6 \quad (\text{for } \mu = 1) \text{ and } 0.0 \ (\text{for } \mu = 0.0)$$
$$\text{UREF}(1) = 0.0$$

The inputs and modified outputs for the Euler–Bernoulli and Timoshenko elements are presented in Boxes 7.3.12 and 7.3.13, respectively. Note that the Euler–Bernoulli element is a Hermite cubic element, whereas the Timoshenko element is only a linear element (a quadratic element may be used if desired).

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Box 7.3.12 Input and edited output files for the cantilever beam problem of Example 7.3.7 (Euler–Bernoulli beam element).

```
Example 7.3.7: CLAMPED AND SPRING-SUPPORTED BEAM (Euler-Bernoulli)
  3   0   0                          MODEL, NTYPE, ITEM
  0   2                              IELEM, NEM
  0   1                              ICONT, NPRNT
    3                                NNM
    1   2   4.0                      NOD(1,J),GLX(1)
    0.0     0.0                      AX0, AX1        |   Data for
    1.0E6   0.0                      BX0, BX1        |   Element 1
    0.0     0.0                      CX0, CX1        |
    0.0     0.0       0.0            FX0,FX1,FX2     |
    2   3   6.0                      NOD(2,J),GLX(2)
    0.0     0.0                      AX0, AX1        |   Data for
    1.0E6   0.0                      BX0, BX1        |   Element 2
    0.0     0.0                      CX0, CX1        |
    0.0     16.666667     0.0        FX0,FX1,FX2     |
  2                                  NSPV
  1     1   0.0                      ISPV(1,J), VSPV(1)
  1     2   0.0                      ISPV(2,J), VSPV(2)
  0                                  NSSV
  1                                  NNBC (with spring)
  3     1   1.0E4         0.0        INBC(1,1),INBC(1,2),VNBC(1),UREF
  0                                  NMPC
```

OUTPUT from program FEM1D by J. N. REDDY

```
Example 7.3.7: CLAMPED AND SPRING-SUPPORTED BEAM (Euler-Bernoulli)

    *** ANALYSIS OF MODEL 3, AND TYPE 0 PROBLEM ***
                (see the code below)

    MODEL=2, NTYPE=0: A Timoshenko BEAM (RIE) problem
    MODEL=2, NTYPE=1: A Timoshenko PLATE (RIE) problem
    MODEL=2, NTYPE=2: A Timoshenko BEAM (CIE) problem
    MODEL=2, NTYPE>2: A Timoshenko PLATE (CIE) problem
    MODEL=3, NTYPE=0: A Euler-Bernoulli BEAM problem

     Properties of Element = 1

          Element length, H . . . . . . . = 0.4000E+01
          AX0 =  0.0000E+00    AX1 = 0.0000E+00
          BX0 =  0.1000E+07    BX1 = 0.0000E+00
          CX0 =  0.0000E+00    CX1 = 0.0000E+00
          FX0 =  0.0000E+00    FX1 = 0.0000E+00    FX2 =  0.0000E+00
```

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(**Box 7.3.12** is continued from the previous page)

Properties of Element = 2

Element length, H . . . . . = 0.6000E+01
AX0 = 0.0000E+00 AX1 = 0.0000E+00
BX0 = 0.1000E+07 BX1 = 0.0000E+00
CX0 = 0.0000E+00 CX1 = 0.0000E+00
FX0 = 0.0000E+00 FX1 = 0.1667E+02 FX2 = 0.0000E+00

Element coefficient matrix, [ELK]:

0.18750E+06	-0.37500E+06	-0.18750E+06	-0.37500E+06
-0.37500E+06	0.10000E+07	0.37500E+06	0.50000E+06
-0.18750E+06	0.37500E+06	0.18750E+06	0.37500E+06
-0.37500E+06	0.50000E+06	0.37500E+06	0.10000E+07

Element coefficient matrix, [ELK]:

0.55556E+05	-0.16667E+06	-0.55556E+05	-0.16667E+06
-0.16667E+06	0.66667E+06	0.16667E+06	0.33333E+06
-0.55556E+05	0.16667E+06	0.55556E+05	0.16667E+06
-0.16667E+06	0.33333E+06	0.16667E+06	0.66667E+06

Element source vector, {ELF}:

0.90000E+02 -0.12000E+03 0.21000E+03 0.18000E+03

SOLUTION (values of PVs) at the NODES:
0.00000E+00 0.00000E+00 0.46272E-02 -0.19510E-02 0.16403E-01
-0.16985E-02

x is the local coord. if ICONT=0

x	Deflect.	Rotation	B. Moment	Shear Force
0.00000E+00	0.00000E+00	0.00000E+00	0.75969E+03	-0.13597E+03
0.50000E+00	0.92129E-04	-0.36285E-03	0.69171E+03	-0.13597E+03
0.10000E+01	0.35718E-03	-0.69171E-03	0.62372E+03	-0.13597E+03
0.15000E+01	0.77817E-03	-0.98657E-03	0.55574E+03	-0.13597E+03
0.20000E+01	0.13381E-02	-0.12474E-02	0.48775E+03	-0.13597E+03
0.25000E+01	0.20200E-02	-0.14743E-02	0.41977E+03	-0.13597E+03
0.30000E+01	0.28068E-02	-0.16672E-02	0.35178E+03	-0.13597E+03
0.35000E+01	0.36815E-02	-0.18261E-02	0.28380E+03	-0.13597E+03
0.40000E+01	0.46272E-02	-0.19510E-02	0.21582E+03	-0.13597E+03
0.00000E+00	0.46272E-02	-0.19510E-02	0.95815E+02	-0.45969E+02
0.75000E+00	0.61142E-02	-0.20099E-02	0.61338E+02	-0.45969E+02
0.15000E+01	0.76357E-02	-0.20430E-02	0.26862E+02	-0.45969E+02
0.22500E+01	0.91722E-02	-0.20502E-02	-0.76154E+01	-0.45969E+02
0.30000E+01	0.10705E-01	-0.20316E-02	-0.42092E+02	-0.45969E+02
0.37500E+01	0.12213E-01	-0.19871E-02	-0.76569E+02	-0.45969E+02
0.45000E+01	0.13679E-01	-0.19167E-02	-0.11105E+03	-0.45969E+02
0.52500E+01	0.15082E-01	-0.18205E-02	-0.14552E+03	-0.45969E+02
0.60000E+01	0.16403E-01	-0.16985E-02	-0.18000E+03	-0.45969E+02

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Box 7.3.13 Input and edited output files for the cantilever beam problem of Example 7.3.7 (Timoshenko beam element).

```
Example 7.3.7: CLAMPED AND SPRING-SUPPORTED BEAM (Timoshenko)
  2   0   0                          MODEL, NTYPE, ITEM
  1   2                              IELEM, NEM
  0   1                              ICONT, NPRNT
      3                              NNM
    1   2     4.0                    NOD(1,J), GLX(1)
    4.0E6     0.0                    AX0, AX1          |   Data for
    1.0E6     0.0       (L/H = 10)   BX0, BX1          |   Element 1
    0.0       0.0                    CX0, CX1          |
    0.0       0.0       0.0          FX0, FX1, FX2     |
    2   3     6.0                    NOD(2,J), GLX(2)
    4.0E6     0.0                    AX0, AX1          |   Data for
    1.0E6     0.0                    BX0, BX1          |   Element 2
    0.0       0.0                    CX0, CX1          |
    0.0       16.666667     0.0      FX0, FX1, FX2     |
  2                                  NSPV
  1   1       0.0                    ISPV(1,J), VSPV(1)
  1   2       0.0                    ISPV(2,J), VSPV(2)
  0                                  NSSV
  1                                  NNBC (with spring)
  3   1       1.0E4         0.0      INBC(1,1),INBC(1,2),VNBC(1),UREF
  0                                  NMPC
```

OUTPUT from program FEM1D by J. N. REDDY

```
Example 7.3.7: CLAMPED AND SPRING-SUPPORTED BEAM (Timoshenko)

Properties of Element = 2
            Element length, H . . . . . . . = 0.6000E+01
            AX0 = 0.4000E+07    AX1 = 0.0000E+00
            BX0 = 0.1000E+07    BX1 = 0.0000E+00
            CX0 = 0.0000E+00    CX1 = 0.0000E+00
            FX0 = 0.0000E+00    FX1 = 0.1667E+02  FX2 = 0.00

Element coefficient matrix, [ELK]:
    0.66667E+06     -0.20000E+07     -0.66667E+06     -0.20000E+07
   -0.20000E+07      0.61667E+07      0.20000E+07      0.58333E+07
   -0.66667E+06      0.20000E+07      0.66667E+06      0.20000E+07
   -0.20000E+07      0.58333E+07      0.20000E+07      0.61667E+07

Element source vector, {ELF}:
    0.10000E+03      0.00000E+00      0.20000E+03      0.00000E+00

 SOLUTION (values of PVs) at the NODES:

 0.00000E+00     0.00000E+00     0.38273E-02     -0.18473E-02     0.16727E-01
-0.24364E-02
```

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(**Box 7.3.13** is continued from the previous page)

x is the the local coord.if ICONT=0

x	Deflect.	Rotation	B. Moment	Shear Force
0.00000E+00	0.00000E+00	0.00000E+00	0.46182E+03	-0.38273E+04
0.50000E+00	0.47841E-03	-0.23091E-03	0.46182E+03	-0.29036E+04
0.10000E+01	0.95682E-03	-0.46182E-03	0.46182E+03	-0.19800E+04
0.15000E+01	0.14352E-02	-0.69273E-03	0.46182E+03	-0.10564E+04
0.20000E+01	0.19136E-02	-0.92364E-03	0.46182E+03	-0.13273E+03
0.25000E+01	0.23920E-02	-0.11545E-02	0.46182E+03	0.79091E+03
0.30000E+01	0.28705E-02	-0.13855E-02	0.46182E+03	0.17145E+04
0.35000E+01	0.33489E-02	-0.16164E-02	0.46182E+03	0.26382E+04
0.40000E+01	0.38273E-02	-0.18473E-02	0.46182E+03	0.35618E+04
0.00000E+00	0.38273E-02	-0.18473E-02	0.98182E+02	-0.12109E+04
0.75000E+00	0.54398E-02	-0.19209E-02	0.98182E+02	-0.91636E+03
0.15000E+01	0.70523E-02	-0.19945E-02	0.98182E+02	-0.62182E+03
0.22500E+01	0.86648E-02	-0.20682E-02	0.98182E+02	-0.32727E+03
0.30000E+01	0.10277E-01	-0.21418E-02	0.98182E+02	-0.32727E+02
0.37500E+01	0.11890E-01	-0.22155E-02	0.98182E+02	0.26182E+03
0.45000E+01	0.13502E-01	-0.22891E-02	0.98182E+02	0.55636E+03
0.52500E+01	0.15115E-01	-0.23627E-02	0.98182E+02	0.85091E+03
0.60000E+01	0.16727E-01	-0.24364E-02	0.98182E+02	0.11455E+04

Example 7.3.8 (Analysis of a Plane Truss)

Here, we consider the three-member plane truss shown in Figure 7.3.7 (see Example 4.6.1). A plane truss problem falls into MODEL = 4 and NTYPE = 0. To obtain the solution in nondimensional form, we take $E = 1$, $A = 1$, $P = 1$, and $L = 1$. The data are discontinuous (ICONT=0). The input and partial output are presented in Box 7.3.14.

Figure 7.3.7 The plane truss problem of Example 7.3.8.

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Box 7.3.14 Input and partial output for the plane truss problem of Example 7.3.8.

```
Example 7.3.8: ANALYSIS OF A PLANE TRUSS
 4 0 0                          MODEL, NTYPE, ITEM
 0 3                            IELEM, NEM
 0 1                            ICONT, NPRNT
   3                            NNM
 1.0   1.0   1.0   1.0   0.0  0.0     SE, SL, SA, CS, SN, HF
   2   3                        NOD(1, I)  (Element 1)
 1.0  1.414  1.0   0.707  0.707  0.0     same for Element 2
   1   3
 1.0  1.0   1.0   0.0  1.0
 0.0  0.0   0.0   0.0  0.0       Same for Element 3
   1   2                        0        NCON
   4                            NSPV
   1   1   0.0
   1   2   0.0                  ISPV, VSPV
   2   1   0.0
   2   2   0.0
   2                            NSSV
   3   1  -2.0                  ISSV, VSSV
   3   2   1.0
   0                            NNBC
   0                            NMPC
```

```
OUTPUT from program  FEM1D  by J. N. REDDY

SOLUTION (values of PVs) at the NODES:

  0.00000E+00  0.00000E+00  0.00000E+00  0.00000E+00  -0.30000E+01  -0.58289E+01

     Generalized forces in the element coordinates
  (second line gives the results in the global coordinates)
```

Ele	Force, H1	Force, V1	Force, H2	Force, V2
1	0.3000E+01	0.0000E+00	−0.3000E+01	0.0000E+00
	0.3000E+01	0.0000E+00	−0.3000E+01	0.0000E+00
2	−0.1414E+01	0.0000E+00	0.1414E+01	0.0000E+00
	−0.1000E+01	−0.1000E+01	0.1000E+01	0.1000E+01
3	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

Example 7.3.9 (Analysis of a Plane Frame)

This example deals with the two-member frame structure shown in Figure 7.3.8 (see Example 5.4.1). We shall analyze the frame for displacements and member forces using first

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Figure 7.3.8 The plane frame problem of Example 7.3.9.

the Euler–Bernoulli frame element (MODEL = 4, NTYPE = 1) and then using the Timoshenko frame element (MODEL = 4, NTYPE = 2). The input files of the problem using two Euler–Bernoulli elements and 2 (linear) Timoshenko beam elements are presented in Box 7.3.15.

Box 7.3.15 Input data for the plane frame problem of Example 7.3.9.

```
Example 7.3.9(a): ANALYSIS OF A PLANE FRAME (2 Euler-Bernoulli elements)
 4  1  0                          MODEL, NTYPE, ITEM
 0  2                             IELEM, NEM
 0  1                             ICONT, NPRNT
 3                                NNM
 0.3  1.0E6  144.0  10.0  10.0  0.0   -1.0  PR, SE, SL, SA, SI, CS, SN
 0.0  0.0138888  0.0 0.0 0.0 0.0      HF, VF, PF, XB, CST, SNT
 1    2                              NOD(1,J)
 0.3  1.0E6  180.0  10.0  10.0  0.8   -0.6  PR, SE, SL, SA, SI, CS, SN
 0.0  0.0    4.0 90.0  -0.6  0.8      HF, VF, PF, XB, CST, SNT
 2    3                              NOD(2,J)
                                0     NCON
 6                              NSPV
 1    1    0.0
 1    2    0.0
 1    3    0.0
 3    1    0.0                      ISPV, VSPV
 3    2    0.0
 3    3    0.0
 1                              NSSV
 2    2    2.0                  ISSV, VSSV
 0                              NNBC
 0                              NMPC
```

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(**Box 7.3.15** is continued from the previous page)

```
Example 7.3.9(b): ANALYSIS OF A PLANE FRAME (2 TIMOSHENKO elements)
 4 2 0                          MODEL, NTYPE, ITEM
 0 2                            IELEM, NEM
 0 1                            ICONT, NPRNT
 3                              NNM
 0.3  1.0E6  144.0  10.0  10.0  0.0  -1.0   PR, SE, SL, SA, SI, CS, SN
 0.0  0.0138888  0.0  0.0  0.0  0.0         HF, VF, PF, XB, CST, SNT
 1  2                                       NOD(1,J)
 0.3  1.0E6  180.0  10.0  10.0  0.8  -0.6   PR, SE, SL, SA, SI, CS, SN
 0.0  0.0      4.0  90.0  -0.6  0.8         HF, VF, PF, XB, CST, SNT
 2  3                                       NOD(2,J)
                                0           NCON
 6                              NSPV
 1  1  0.0
 1  2  0.0
 1  3  0.0
 3  1  0.0                      ISPV, VSPV
 3  2  0.0
 3  3  0.0
 1                              NSSV
 2  2  2.0                      ISSV, VSSV
 0                              NNBC
 0                              NMPC
```

The next two examples deal with the use of **FEM1D** for eigenvalue and time-dependent problems (see Chapter 6).

Example 7.3.10 (Natural Vibration of a Cantilever Beam)

Here, we consider natural vibrations of a cantilever beam (see Example 6.1.2). The input data for all variables is the same as in the static analysis. In addition, we must input c_{i0} and c_{i1}. For the Timoshenko beam theory, c_{i0} denotes the inertia ρA, and c_{i1} denotes the rotatory inertia ρI. The eigenvalue solver used in **FEM1D** requires the matrix $[B]$ in $[A]\{x\} = \lambda[B]\{x\}$ to be positive-definite. Hence, c_{i0} and c_{i1} should be nonzero, otherwise, the mass matrix coefficients associated with Ψ will be zero. When rotary inertia is omitted in the Timoshenko beam theory, a small value for the inertia coefficient is used in order to ensure positive-definiteness of the mass matrix.

The input files for the natural vibrations of the cantilever beam by the two types of elements are given in Box 7.3.16. The reader can investigate the convergence characteristics of the elements in improving the accuracy of the fundamental frequency with the use of **FEM1D**.

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Box 7.3.16 Input data files for beam vibration of Example 7.3.10.

```
Example 7.3.10(a): NATURAL VIBRATIONS OF A CANTILEVER BEAM (E-B; w/o RI)
 3 0 3                    MODEL, NTYPE, ITEM
 0 2                      IELEM, NEM
 1 1                      ICONT, NPRNT
  0.0   0.5   0.5         DX(I)
  0.0   0.0               AX0, AX1
  1.0   0.0               BX0, BX1
  0.0   0.0               CX0, CX1
  2                       NSPV
 1 1                      ISPV(1, J)
 1 2                      ISPV(2, J)
 0                        NNBC
 0                        NMPC
 1.0  8.3333-06           CT0, CT1
-----------------------------------------------------------------
Example 7.3.10(b): NATURAL VIB. OF A CANTILEVER BEAM (TIM element w/o RI)
 2 0 3                    MODEL, NTYPE, ITEM
 1 2                      IELEM, NEM
 1 1                      ICONT, NPRNT
 0.0   0.5   0.5          DX(I)
 4.0E4 0.0                AX0, AX1 (nu=0.25, H=0.01)
 1.0   0.0                BX0, BX1
 0.0   0.0                CX0, CX1
 2                        NSPV
 1 1                      ISPV(1,J)
 1 2                      ISPV(2,J)
 0                        NNBC
 0                        NMPC
 1.0  8.33333E-06         CT0, CT1 (L/H=100)
```

Example 7.3.11 (Transient Heat Transfer)

Consider the transient heat conduction in a plane wall. The governing equation is

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } 0 < x < 1 \tag{7.3.8}$$

The boundary conditions considered are

$$\textbf{Set 1:}\quad u(0,t) = 0 \quad \frac{\partial u}{\partial x}(1,t) = 0; \quad \textbf{Set 2:}\quad u(0,t) = 0, \quad \left(\frac{\partial u}{\partial x} + u\right)\bigg|_{x=1} = 0 \tag{7.3.9}$$

and the initial condition is taken to be

$$u(x,0) = 1 \tag{7.3.10}$$

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We have MODEL = 1, NTYPE = 0, ICONT = 1, NSPV = 1, and NSSV = 0; NNBC = 0 for Set 1 and NNBC = 1 for Set 2. The coefficients of the differential equations are ($a = 1.0$, $b = 0.0$, $c = 0.0$, $c_t = 1.0$, and $f = 0$)

$$AX0 = 1.0, \quad AX1 = 0.0, \quad BX0 = 0.0, \quad BX1 = 0.0, \quad CX0 = 0.0, \quad CX1 = 0.0$$

$$FX0 = 0.0, \quad FX1 = 0.0, \quad FX2 = 0.0, \quad CT0 = 1.0, \quad CT1 = 0.0$$

Box 7.3.17 Input files for the transient heat conduction problem of Example 7.3.11.

```
Example 7.3.11(a): Transient heat conduction in a plane wall (Set 1)
 1 0 1                    MODEL, NTYPE, ITEM
 1 2                      IELEM, NEM
 1 1                      ICONT, NPRNT
  0.0  0.5  0.5           DX(I)
  1.0  0.0                AX0, AX1
  0.0  0.0                BX0, BX1
  0.0  0.0                CX0, CX1
  0.0  0.0  0.0           FX0, FX1, FX2
  1                       NSPV
  1 1  0.0                ISPV(I, J), VSPV(I)
  0                       NSSV
  0                       NNBC
  0                       NMPC
  1.0  0.0                CT0, CT1
  0.05 0.5  0.0           DT, ALFA, GAMA
  1    20   1             INCOND, NTIME, INTVL
  0.0  1.0  1.0           GU0(I)

Example 7.3.11(b): Transient heat conduction in a plane wall (Set 2)
 1 0 1                    MODEL, NTYPE, ITEM
 1 2                      IELEM, NEM
 1 1                      ICONT, NPRNT
  0.0  0.5  0.5           DX(I)
  1.0  0.0                AX0, AX1
  0.0  0.0                BX0, BX1
  0.0  0.0                CX0, CX1
  0.0  0.0  0.0           FX0, FX1, FX2
  1                       NSPV
  1 1   0.0               ISPV(I, J), VSPV(I)
  0                       NSSV
  1                       NNBC
  0                       NMPC
  5 1  1.0  0.0           INBC, VNBC, UREF
  1.0  0.0                CT0, CT1
  0.05 0.5  0.0           DT, ALFA, GAMA
  1   20   1              INCOND, NTIME, INTVL
  0.0  1.0  1.0           GU0(I)
```

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The boundary and initial conditions (since INCOND=1) are input as

$$\text{ISPV}(1,1)=1, \quad \text{ISPV}(1,2)=1, \quad \text{VSPV}(1)=0.0; \quad \text{GU0(I)}=\{0.0, 1.0, 1.0, \ldots\}$$

From the discussions of Example 6.2.1, we use $\Delta t = 0.05$ (DT=0.05) and print the solution for every time step (i.e., INTVL=1).

The input files of the problem for the two sets of boundary conditions are presented in Box 7.3.17 for a mesh of two linear elements.

7.4 SUMMARY

In this chapter three main items have been discussed: numerical integration of finite element coefficient matrices and vectors, numerical implementation of a typical finite element program and their contents, and applications of the finite element program **FEM1D.** The numerical evaluation of the coefficients is required because of (*a*) variable coefficients of the differential equations modeled and (*b*) special evaluation of the coefficients, as was required for the Timoshenko beam element with equal interpolation. The Newton–Cotes and Gauss–Legendre integration rules have been discussed. The integration rules require transformation of the integral expressions from the global coordinate system to a local coordinate system and interpolation of the global coordinate x. Depending on the relative degrees of interpolation of the geometry and the dependent variables, the formulations are classified as subparametric, isoparametric, and superparametric.

The three logical units—preprocessor, processor, and postprocessor—have been discussed. The contents of the processor, where most finite element calculations are carried out, have been considered in detail. Computer implementation for numerical evaluation of integral expressions, assembly of element coefficients, and imposition of boundary conditions have been discussed. A description of the input variables to the finite element computer program **FEM1D** has been presented, and application of **FEM1D** to problems of heat transfer and solid mechanics has been discussed. Fluid mechanics problems are similar in mathematical structure to heat transfer problems and hence not discussed in this chapter.

PROBLEMS

Section 7.1

In Problems 7.1–7.4, use the appropriate number of integration points, and verify the results with those obtained by the exact integration.

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7.1 Evaluate the integrals in Example 7.1.1 using the Newton–Cotes formula and Gauss–Legendre quadrature when ψ_i are the quadratic interpolation functions

$$\psi_1 = \left(1 - \frac{x - x_a}{x_b - x_a}\right)\left(1 - 2\frac{x - x_a}{x_b - x_a}\right) = -\frac{1}{2}\xi(1-\xi)$$

$$\psi_2 = 4\left(\frac{x - x_a}{x_b - x_a}\right)\left(1 - \frac{x - x_a}{x_b - x_a}\right) = (1-\xi^2)$$

$$\psi_3 = -\frac{x - x_a}{x_b - x_a}\left(1 - 2\frac{x - x_a}{x_b - x_a}\right) = \frac{1}{2}\xi(1+\xi)$$

7.2 Use Newton–Cotes integration formula to evaluate

$$K_{11} = \int_{x_a}^{x_b} \left(\frac{d^2\phi_1}{dx^2}\right)^2 dx, \quad G_{11} = \int_{x_a}^{x_b} (\phi_1)^2\, dx$$

where ϕ_i are the Hermite cubic interpolation functions [see Eqs. (5.2.12), (5.2.13*a*), and (5.2.13*b*)]. *Answer:* $r = 2$: $K_{11} = 12/h^3$ (exact), $G_{11} = 0.398148h$.

7.3 Use Gauss–Legendre quadrature to evaluate the integrals of Problem 7.2 for the case in which the interpolation functions ϕ_i are the fifth-order Hermite polynomials of Problem 5.4. *Answer:* $K_{11} = \frac{120}{7}h^3$, $G_{11} = \frac{181}{462}h$ (exact).

7.4 Repeat Problem 7.3 for the case in which the interpolation functions ϕ_i are the fifth-order Hermite polynomials of Problem 5.5. *Answer:* $K_{11} = \frac{5092}{35h_e^3}$, $G_{11} = \frac{523h_e}{3465}$ (exact).

Section 7.2

Computer Exercises (use FEM1D)

7.5 Solve the problem

$$-\frac{d}{dx}\left(k\frac{dT}{dx}\right) = g_0$$

$$\left(-k\frac{dT}{dx}\right)_{x=0} = Q_0, \quad \left[k\frac{dT}{dx} + \beta(T - T_\infty)\right]_{x=L} = 0$$

using two and four linear elements. Compare the results with the exact solution. Use the following data: $L = 0.02$ m, $k = 20$ W/(m ·°C), $g_0 = 10^6$ W/m^2, $Q_0 = 10^2$ W, $T_\infty = 50$°C, $\beta = 500$ W/(m ·°C).

7.6 Solve Problem 7.5 using two quadratic elements.

7.7 Solve the heat transfer problem in Example 4.3.3 (Set 1), using (*a*) four linear elements and (*b*) two quadratic elements (see Table 4.3.1).

7.8 Solve the axisymmetric problem in Example 4.3.4 using four quadratic elements and compare the solution with that obtained using eight linear elements and the exact solution of Table 4.3.2.

7.9 Solve the one-dimensional flow problem of Example 4.4.1 (Set 1), for $dP/dx = -24$, using eight linear elements (see Figure 4.4.1). Compare the finite element results with the exact solution $(4.4.20)_1$.

7.10 Solve the Couette flow problem in Example 4.4.1 (Set 2) using four quadratic elements. Compare the finite element solution with the exact solution.

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7.11 Solve Problem 4.10 (heat flow in a composite wall) using the minimum number linear finite elements.

7.12 Solve Problem 4.22 (axisymmetric problem of unconfined aquifer) using the minimum number of linear finite elements.

7.13 Solve Problem 4.25.

7.14 Solve Problem 4.27.

7.15 Solve Problem 4.35 using two linear elements.

7.16 Determine the forces and elongations in the wires AB and CD shown in Figure P7.16. Each wire has a cross-sectional area of $A = 0.03$ in.^{2} and modulus of elasticity $E = 30 \times 10^6$ psi.

Figure P7.16

7.17 Solve the problem of axisymmetric deformation of a rotating circular disk using four linear elements (see Example 7.3.5).

Figure P7.17

7.18–7.25 Solve Problems 5.7–5.14 using the minimum number of Euler–Bernoulli beam elements (*Note:* Numerous other beam problems can be found in books on mechanics of deformable solids).

7.26 Analyze Problem 7.22 (same as Problem 5.11) using the RIE Timoshenko element. Assume $\nu = 0.25$, $K_s = 5/6$, and $H = 0.1$ m (beam height). Use two, four and eight elements to see the convergence characteristics of the RIE (two-element model may yield results very far off from the Euler–Bernoulli beam solution).

7.27 Repeat Problem 7.26 using the CIE Timoshenko element.

7.28 Analyze a clamped circular plate under a uniformly distributed transverse load using the Euler–Bernoulli plate element. Investigate the convergence using two, four, and eight

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elements by comparing with the exact solution [from Reddy (2002)]

$$w(r) = \frac{q_0 a^4}{64D}\left[1 - \left(\frac{r}{a}\right)^2\right]^2$$

where $D = EH^3/12(1-\nu^2)$, q_0 is the intensity of the distributed load, a is the radius of the plate, H is its thickness, and ν is Poisson's ratio ($\nu = 0.25$). Tabulate the center deflection.

7.29 Repeat Problem 7.28 with the RIE Timoshenko plate element for $a/H = 10$. Use four and eight linear elements and two and four quadratic elements and tabulate the center deflection. Take $E = 10^7$, $\nu = 0.25$, and $K_s = 5/6$. The exact solution is [see page 403 of Reddy (2002)]

$$w(r) = \frac{q_0 a^4}{64D}\left[1 - \left(\frac{r}{a}\right)^2\right]^2 + \frac{q_0 a^2}{4K_s GH}\left[1 - \left(\frac{r}{a}\right)^2\right]$$

7.30 Repeat Problem 7.29 with the Timoshenko plate element (CIE) (and linear elements) for $a/H = 10$.

7.31 Consider an annular plate of outer radius a, and an inner radius b, and thickness H. If the plate is simply supported at the outer edge and subjected to a uniformly distributed load q_0 (see Fig. P7.31), analyze the problem using the Euler–Bernoulli plate element. Compare the four-element solution with the analytical solution [from Reddy (2002)]

$$w = \frac{q_0 a^4}{64D}\left\{-1 + \left(\frac{r}{a}\right)^4 + \frac{2\alpha_1}{1+\nu}\left[1 - \left(\frac{r}{a}\right)^2\right] - \frac{4\alpha_2\beta^2}{1-\nu}\log\left(\frac{r}{a}\right)\right\}$$

$$\alpha_1 = (3+\nu)(1-\beta^2) - 4(1+\nu)\beta^2\kappa, \quad \alpha_2 = (3+\nu) + 4(1+\nu)\kappa$$

$$\kappa = \frac{\beta^2}{1-\beta^2}\log\beta, \quad \beta = \frac{b}{a}, \quad D = \frac{EH^3}{12(1-\nu^2)}$$

where E is the modulus of elasticity, H the thickness, and ν Poisson's ratio. Take $E = 10^7$, $\nu = 0.3$, and $b/a = 0.25$.

Figure P7.31

7.32 Repeat Problem 7.31 with (a) four linear elements and (b) two quadratic Timoshenko (RIE) elements for $a/H = 10$.

7.33–7.47 Analyze the truss problems in Figures P4.38–P4.44 and frame problems in Figures P5.28–P5.35.

7.48 Consider the axial motion of an elastic bar, governed by the second-order equation

$$EA\frac{\partial^2 u}{\partial x^2} = \rho A\frac{\partial^2 u}{\partial t^2} \quad \text{for } 0 < x < L$$

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with the following data: length of bar $L = 500$ mm, cross-sectional area $A = 1$ mm^2, modulus of elasticity $E = 20,000$ N/mm^2, density $\rho = 0.008$ kg/mm^3, boundary conditions

$$u(0,t) = 0, \quad EA\frac{\partial u}{\partial x}(L,t) = 1$$

and zero initial conditions. Using 20 linear elements and $\Delta t = 0.002$ s, determine the axial displacement and plot the displacement as a function of position along the bar for $t = 0.8$ s.

7.49 Consider the following nondimensionalized differential equation governing the plane wall transient:

$$-\frac{\partial^2 T}{\partial x^2} + \frac{\partial T}{\partial t} = 0 \quad \text{for } 0 < x < 1$$

with boundary conditions $T(0,t) = 1$ and $T(1,t) = 0$, and initial condition $T(x,0) = 0$. Solve the problem using eight linear elements. Determine the critical time step; solve the problem using the Crank–Nicholson method and $\Delta t = 0.002$ s.

Note: Modify program **FEM1D** to solve Problems 7.50–7.52 (solutions to these problems are not presented here for obvious reasons).

7.50 Consider a simply supported beam of length L subjected to a point load

$$P(t) = \begin{cases} P_0 \sin \frac{\pi t}{\tau} & \text{for } 0 \le t \le \tau \\ 0 & \text{for } t \ge \tau \end{cases}$$

at a distance c from the left end of the beam (assumed to be at rest at $t = 0$). The transverse deflection $w(x,t)$ is given by [see Harris and Crede (1961), pp. 8–53]

$$w(x,t) = \begin{cases} \dfrac{2P_0L^3}{\pi^4 EI}\displaystyle\sum_{i=1}^{\infty}\frac{1}{i^4}\sin\frac{i\pi c}{L}\sin\frac{i\pi x}{L}\left[\frac{1}{1 - T_i^2/4\tau^2}\left(\sin\frac{\pi t}{\tau} - \frac{T_i}{2\tau}\sin\omega_i t\right)\right], & 0 \le t \le \omega\tau \\ \dfrac{2P_0L^3}{\pi^4 EI}\displaystyle\sum_{i=1}^{\infty}\frac{1}{i^4}\sin\frac{i\pi c}{L}\sin\frac{i\pi x}{L}\left[\frac{\frac{T_i}{\tau}\cos\frac{\pi\tau}{T_i}}{T_i^2/4\tau^2 - 1}\sin\omega_i\left(t - \frac{1}{2}\tau\right)\right], & t \ge \tau \end{cases}$$

where

$$T_i = \frac{2\pi}{\omega_i} = \frac{2L^2}{i^2\pi}\sqrt{\frac{A\rho}{EI}} = \frac{T_1}{i^2}$$

Use the data $P_0 = 1000$ lb, $\tau = 20 \times 10^{-6}$ s, $L = 30$ in., $E = 30 \times 10^6$ lb/in.^{2}, $\rho = 733 \times 10^{-6}$ lb/in.^{3}, $\Delta t = 10^{-6}$ s, and assume that the beam is of square cross section of 0.5 in by 0.5 in. Using five Euler–Bernoulli beam elements in the half-beam, obtain the finite element solution and compare with the series solution at midspan for the case $c = \frac{L}{2}$.

7.51 Repeat Problem 7.50 for $c = \frac{1}{4}L$ and eight elements in the full span.

7.52 Repeat Problem 7.50 for $P(t) = P_0$ at midspan and eight elements in the full span.

7.53 Consider a cantilevered beam with a point load P_0 at the free end (Fig. P7.53). Using the data of Problem 7.50, find the finite element solution for the transverse deflection using eight Euler–Bernoulli beam elements.

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Figure P7.53

$E = 30 \times 10^6$ psi, $\nu = 0.3$,
$\rho = 733 \times 10^{-6}$ lb/in.^{3},
$L = 30$ in., $P_0 = 10^3$ lbs,
$A = 0.25$ in.^{2}, $\Delta t = 10^{-6}$ s

Figure P7.54

7.54 Repeat Problem 7.50 for a clamped beam with the load at the midspan (see Fig. P7.54).

7.55 Repeat Problem 7.54 using four linear Timoshenko beam elements. Use $\nu = 0.3$.

7.56 Repeat Problem 7.54 using two quadratic Timoshenko beam elements. Use $\nu = 0.3$.

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