# Chapter 1

# Laplace Equation

If does not have boundary conditions, the problem is ill. The equation needs two conditions.

It is very easy to partition the interval $[a, b]$

**Definition 1.** *We call $\Omega \subset \mathbb{R}^d$ for $d = 1, 2, 3$ a* ***domain*** *if*

1. *$\Omega$ is open, is connected. It has no holes. It must be smooth.*

**Definition 2.** *We call $\Gamma = \partial\Omega$ the* ***boundary*** *of the domain $\Omega$.*

**Definition 3.** *By $\hat{n}$ we denote the* ***unit normal vector*** *(facing outwards) on the boundary.*

**Definition 4.** *We define* ***function space of differentiable functions***

$$C^m * (\Omega) = \{f : \Omega \to \mathbb{R} | f(x_1.x_2, \dots, x_d)\}$$

**Definition 5.** *We define* ***laplace operator***

- Let $f \in C^0(\Omega)$ be the **right hand side function**.
- Let $g \in C^0(\Gamma)$ be the **boundary value function**.
- **Dirichtlet Problem** we are looking for $u \in C^2(\Omega)$ such that

  $$-\Delta u = f \text{ in}$$

- Let $\Omega$ be the unit sphere

  $$\Omega = \{\boldsymbol{x} = (x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$$

- Let $f = 1$ and $g = 0$
- There is **no solution** to the Dirichlet Problem

  $$-\Delta u = 1 \text{in} u = 0 \text{on} \Gamma.$$

  which is 2 times differentiable.

## 1.1 The variational formulation

- Assume that $u \in C^2(\Omega)$ is a solution to the Laplace problem

$$-\Delta u(x,y) = f(x,y) \text{ in } \Omega \text{ with } u = 0 \text{ on } \Gamma.$$

- The, we can multiply this equation with a **test function** $\phi$

$$-\Delta u(x,y) \cdot \phi(x,y) = f(x,y) \cdot \phi(x,y) \text{ in } \Omega$$

- The, we can **integrate by parts over the domain**

$$-\int_D \Delta u(x,y) \cdot \phi(x,y) \mathrm{d}x\mathrm{d}y = \int_\Omega f(x,y) \cdot \phi(x,y) \mathrm{d}x\mathrm{d}y$$

- We assume that the test function is differentiable $\phi \in C^1(\Omega)$. Then, we can **integrate by parts**

$$\int_\Omega \nabla u(x,y) \cdot \phi(x,y) \mathrm{d}x\mathrm{d}y - \int_\Gamma (\hat{n} \cdot \nabla) u \cdot \phi \mathrm{d}S = \int_\Omega f(x,y) \cdot \phi(x,y) \mathrm{d}x\mathrm{d}y$$

- We assume that the test function is zero on the boundary. Then

$$\int_\Omega \nabla u(x,y) \cdot \nabla\phi(x,y) \mathrm{d}x\mathrm{d}y = \int_\Omega f(x,y) \cdot \phi(x,y) \mathrm{d}x\mathrm{d}y.$$

If the boundary is given by he graph of a function in $C^2$, then there exists a classical solution $u \in C^2(\Omega)$.

- We introduce $L^2$ scalar product

$$(u, \phi) = \int_\Omega$$

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^d$ for $d = 1, 2, 3$ be a domain and $f \in L^2(\Omega)$. Then, there exists a solution*

$$u \in \nu = H_0^1(\Omega)$$

*to the Laplace problem in variational formulation*

# Chapter 2

# Finite Element Method

**Steps for a finit element discretization**

1. We discretize the domain $\Omega$ by a mesh $\Omega_h$.
2. On $\Omega_h$ we discretize the function space $\nu = H_0^1(\Omega)$ by a finite element space $V_h$.
3. We restrict the variational formulation to $V_h$

$$u_h \in V_h(\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h$$

4. We solve a linear system of equations.

## 2.1 Construction

- We discretize the domain $\Omega$ by splitting it into simple **open elements**, e.g, triangles, quadrilaterals (in $2D$) or tetrahedras, prisms, hexaedras, pyramids (in $3D$)
- The **finite element mesh** $\Omega_h$

## 2.2 Some examples

## 2.3 Shape assumption

**Local Finite Element space**

- On every element $T \in \Omega_h$ define the basis functions of a simple polynomial splace
- **bi-linear finite elments**
- Let $T$ be and quadrilateral with the points $x^{(1)} = (0, 0)$, $x^{(2)} = (h, 0)$, $x^{(3)} = (o, h)$, $x^{(4)} = (h, h)$.

- $\phi^{(1)}(x,y) = (1-\frac{x}{h})(1-\frac{y}{h})$, $\phi^{(2)}(x,y) = \frac{x}{h}(1-\frac{y}{h})$, $\phi^{(3)}(x,y) = (1-\frac{x}{h})\frac{y}{h}$, $\phi^{(1)}(x,y) = \frac{xy}{h^2}$
- The Lagrange basis of the finite element space is given as

$$V_h = \{\phi_h \in C(\Omega) | \phi|_T \in Q^1 = \text{span}(\phi_h^{(1)}, \phi_h^{(2)}, \phi_h^{(3)}, \phi_h^{(4)})\}$$

- The **Lagrange basis** of **nodal basis** is given by

$$V$$

- Starting point: weak formulation of Laplace equation

$$u \in \nu$$

- 

$$u_h \in V_h (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h$$

- The finite element is given by a local basis

$$V_h = \text{span}\{\phi_h^{(1)}, \dots, \phi_h^{(N)} \quad \forall i = 1, \dots, N\}$$

- We write the unknown solution $u_h \in V_h$

## 2.4 Assembling the matrix

- We must compute the matrix entries

$$A_{ij}(\nabla \phi_h^{(j)}, \nabla \phi_h^{(i)}) = \int_\Omega \nabla \phi_h^{(j)} \cdot \nabla \phi_h^{(i)} \mathrm{d}x = \sum_{T \subset \Omega_h} \int_T \nabla \phi_h^{(j)} \cdot \nabla \phi_h^{(i)} \mathrm{d}x$$

- For every **nodal**....
- We combine the result in a **stencil**

$$S = \begin{bmatrix} s_{31} & s_{32} & s_{33} \\ s_{21} & s_{22} & s23 \\ s_{11} & s_{12} & s_{13} \end{bmatrix}$$

- The finite element matrix on a small mesh with $16 = 4 \cdot 4$ nodes like

$$A = \frac{1}{3} \begin{bmatrix} 1 \end{bmatrix}$$

- The main difference between $1D$ and $2D$ (or $3D$)