Finite Elements

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Agenda

1. The Laplace Equation
    - From 1d to 2d/3d
    - Regularity of the solution
    - Variational formulation
2. Finite Elements
    - Finite element meshes
    - Linear finite elements
    - From finite elements to linear systems

The agenda for today

- You studied the Laplace-problem in one dimension:

$$-\frac{\partial^2 u}{\partial^2 x}(x) = f(x) \text{ in } I = [a,b]$$

with

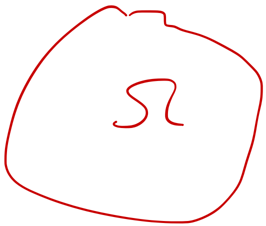
$$u(a) = u_a \text{ and } u(b) = u_b$$

- And you learned about finite elements for the Laplace problem

Today

- We learn how to extend this problem to 2d

$$-\frac{\partial^2 u}{\partial^2 x}(x,y) - \frac{\partial^2 u}{\partial^2 y}(x,y) = f(x,y) \text{ in } \Omega \subset \mathbb{R}^2$$

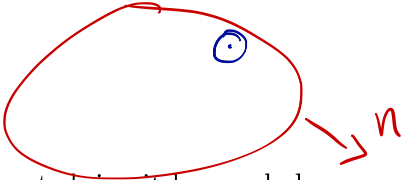


or 3d

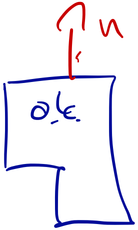
$$-\frac{\partial^2 u}{\partial^2 x}(x,y,z) - \frac{\partial^2 u}{\partial^2 y}(x,y,z) - \frac{\partial^2 u}{\partial^2 z}(x,y,z) = f(x,y,z) \text{ in } \Omega \subset \mathbb{R}^3$$

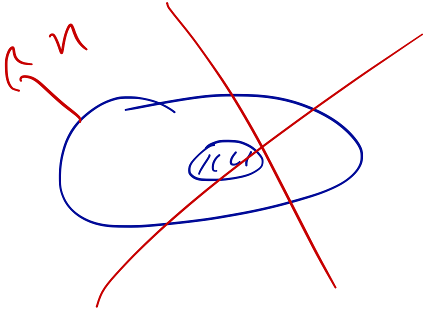
- What is Ω
- How do boundary values work in 2d and 3d?
- How do we do finite elements in 2d and 3d?
- What are the main differences between 1d and 2d?

- We call $\Omega \subset \mathbb{R}^d$ for $d = 1, 2, 3$ a **domain** if:
  1. Ω is open: for each point $x \in \Omega$ there is a small open interval (in 1d) or sphere (in 2d) or ball (in 3d) around x that is also completely in Ω. (The boundary does not belong to the domain)

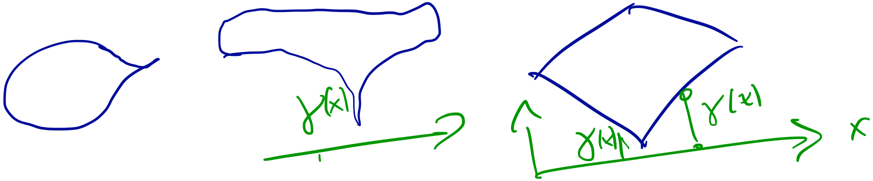


  2. Ω is connected, i.e. it has no holes

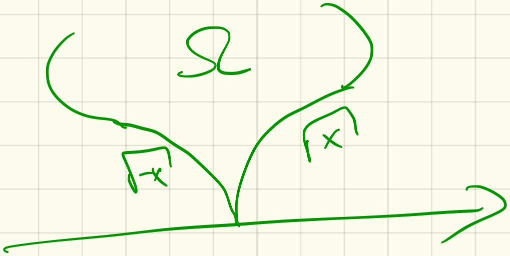




- We call $\Gamma = \partial\Omega$ the **boundary** of the domain Ω. We assume that the boundary is "nice"

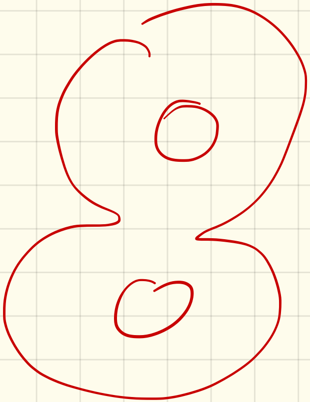


- By $\vec{n}$ we denote the **unit normal vector** (facing outwards) on the boundary



$\sqrt{|x|}$ not Lipschitz in $x = 0$

Boundary is not regular enough ☹



- Let $d = 2, 3$ and $\Omega \subset \mathbb{R}^d$ be the domain

- We define the **function space of differentiable functions**

$$C^m(\Omega) := \{f : \Omega \to \mathbb{R} \mid f(x_1, \ldots, x_d) \text{ is continuous and the first } m \text{ derivatives are continuous}\}$$

- We define the **Laplace operator**

$$\Delta : C^m(\Omega) \to C^{m-2}(\Omega), \quad \Delta := \sum_{i=1}^{d} \frac{\partial^{\cancel{d}\,2}}{\partial^{\cancel{d}\,2} x_i}$$

$$\Delta : C^2(\Omega) \to C(\Omega)$$

- Let $f \in C^0(\Omega)$ be the **right hand side function**
- Let $g \in C^0(\Gamma)$ be the **boundary value function**
- **Dirichlet problem:** we are looking for $u \in C^2(\Omega)$ such that

$$-\Delta u = f \text{ in } \Omega$$

  and on the boundary

$$u = g \text{ on } \Gamma$$

- **Neumann problem:** we are looking for $u \in C^2(\Omega)$ such that

$$-\Delta u = f \text{ in } \Omega$$

  and on the boundary

$$(\vec{n} \cdot \nabla) u = g \text{ on } \Gamma$$

  with the **normal derivative**

$$(\vec{n} \cdot \nabla) u = \vec{n}_1 \frac{\partial u}{\partial x}(x,y) + \vec{n}_2 \frac{\partial u}{\partial y}(x,y) \qquad \text{in 2d}$$

$$(\vec{n} \cdot \nabla) u = \vec{n}_1 \frac{\partial u}{\partial x}(x,y,z) + \vec{n}_2 \frac{\partial u}{\partial y}(x,y,z) + \vec{n}_3 \frac{\partial u}{\partial z}(x,y,z) \qquad \text{in 3d}$$

Where

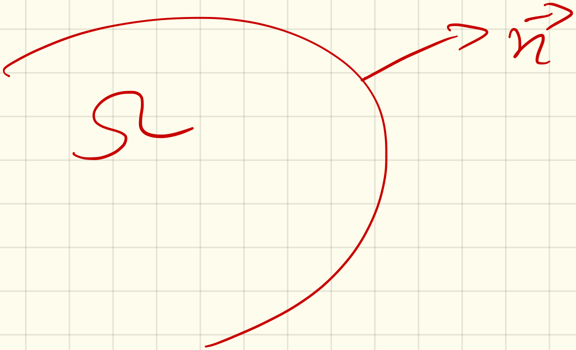
$$\Delta u = \frac{\partial^2 u}{\partial^2 x}(x,y) + \frac{\partial^2 u}{\partial^2 y}(x,y)$$

$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y) \\ \frac{\partial u}{\partial y}(x,y) \end{pmatrix}$$

and - in 3d

$$\Delta u = \frac{\partial^2 u}{\partial^2 x}(x,y,z) + \frac{\partial^2 u}{\partial^2 y}(x,y,z) + \frac{\partial^2 u}{\partial^2 z}(x,y,z)$$

$$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x}(x,y,z) \\ \frac{\partial u}{\partial y}(x,y,z) \\ \frac{\partial u}{\partial z}(x,y,z) \end{pmatrix}$$



Neumann:

$$(\vec{n} \cdot \nabla) u = g$$

$$\left[\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \right] u = g$$

$$n_1 \partial_x u + n_2 \partial_y u = g$$

- Let Ω be the unit-sphere

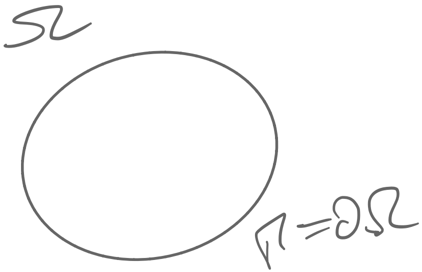
$$\Omega = \{\mathbf{x} = (x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$$

- Let $f \equiv 1$ and $g = 0$
- The solution to the Dirichlet problem

$$-\Delta u = 1 \text{ in } \Omega \text{ and } u = 0 \text{ on } \Gamma$$

is

$$u(x,y) = \frac{1 - x^2 - y^2}{4}$$



$$-\Delta u = -\partial_{xx} u - \partial_{yy} u$$

$$= \frac{2+2}{4} = 1$$

- Let Ω be the unit-square

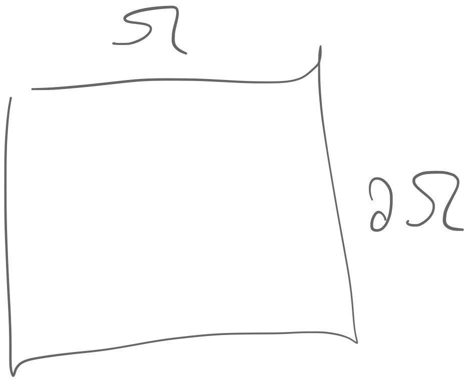
$$\Omega = \{\mathbf{x} = (x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1 \text{ and } 0 < y < 1\}$$

- Let $f \equiv 1$ and $g = 0$
- There is **no solution** to the Dirichlet problem

$$-\Delta u = 1 \text{ in } \Omega \text{ and } u = 0 \text{ on } \Gamma$$

which is 2 times differentiable

$$u \in C^2(\Omega)$$



- Assume that $u \in C^2(\Omega)$ is a solution to the Laplace problem

$$-\Delta u(x,y) = f(x,y) \text{ in } \Omega \text{ with } u = 0 \text{ on } \Gamma$$

- Then, we can multiply this equation with a **test function** ϕ

$$\Rightarrow \qquad -\Delta u(x,y)\cdot\phi(x,y) = f(x,y)\cdot\phi(x,y) \text{ in } \Omega$$

- Then, we can **integrate over the domain**

$$\Rightarrow \qquad -\int_\Omega \Delta u(x,y)\cdot\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x,y)\cdot\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

- We assume that the test function is differentiable $\phi \in C^1(\Omega)$. Then, we can **integrate by parts**

$$\Rightarrow \qquad \int_\Omega \nabla u(x,y)\cdot\nabla\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_\Gamma (\vec{n}\cdot\nabla)u\cdot\phi\,\mathrm{d}s = \int_\Omega f(x,y)\cdot\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

- We assume that the test-function is zero on the boundary. Then

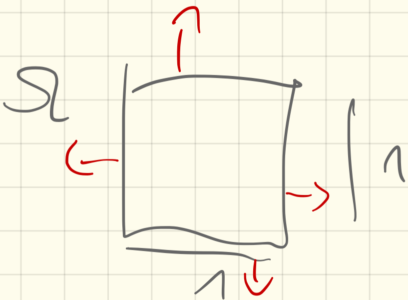
$u \in C^1(\Omega)$ $$\Rightarrow \qquad \int_\Omega \nabla u(x,y)\cdot\nabla\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x,y)\cdot\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$\forall \varphi \in C^1(\Omega)$ and $\varphi = 0$ on $\partial\Omega$

$$-\iint \partial_{xx} u \cdot \varphi \, dx\, dy - \iint \partial_{yy} u \, \varphi \, dx\, dy$$

$$= \iint \partial_x u \, \partial_x \varphi \, dx\, dy + \iint \partial_y u \cdot \partial_y \varphi \, dx\, dy$$

$$- \int_0^h \partial_x u \cdot \varphi \Big|_{x=0}^{x=1} dy - \int_0^h \partial_y u \cdot \varphi \Big|_{y=0}^{y=1} dx$$



$$= \iint \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} \cdot \begin{pmatrix} \partial_x \varphi \\ \partial_y \varphi \end{pmatrix} dx\, dy$$

$$- \int_{\partial\Omega} (n \cdot \nabla) u \cdot \varphi \, ds$$

- We call $u \in C^1(\Omega)$ with $u = 0$ on Γ which satisfies

$$\int_\Omega \nabla u(x,y) \cdot \nabla \phi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x,y) \cdot \phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for all text functions $\phi \in C^1(\Omega)$ with $\phi = 0$ on Γ the **weak solution to the Laplace equation**

- Is a weak solution also a **classical solution**?

$$\Rightarrow \qquad -\Delta u(x,y) = f(x,y) \text{ in } \Omega$$

- Assume that $u \in C^2(\Omega)$ is a solution to the Laplace problem

$$-\Delta u(x,y) = f(x,y) \text{ in } \Omega \text{ with } u = 0 \text{ on } \Gamma$$

- Then, we can multiply this equation with a **test function** ϕ

$$\Leftrightarrow \qquad -\Delta u(x,y)\cdot\phi(x,y) = f(x,y)\cdot\phi(x,y) \text{ in } \Omega$$

- Then, we can **integrate over the domain**

$$\Leftrightarrow \qquad -\int_\Omega \Delta u(x,y)\cdot\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x,y)\cdot\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$\Uparrow$ if $u \in C^2$

- We assume that the test function is differentiable $\phi \in C^1(\Omega)$. Then, we can **integrate by parts**

$$\Leftrightarrow \qquad \int_\Omega \nabla u(x,y)\cdot\nabla\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_\Gamma (\vec{n}\cdot\nabla)u\cdot\phi\mathrm{d}s = \int_\Omega f(x,y)\cdot\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

- We assume that the test-function is zero on the boundary. Then

$$\Leftrightarrow \qquad \int_\Omega \nabla u(x,y)\cdot\nabla\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x,y)\cdot\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

if the boundary is given by the graph of a function in C^2

then there exists a classical solution $u \in C^2(\Omega)$

- We introduce the L^2 scalar product

$$
\begin{aligned}
(u,\phi) &:= \int_\Omega u(x)\cdot\phi(x)\,\mathrm{d}x && \text{in 1d}\\
(u,\phi) &:= \int_\Omega u(x,y)\cdot\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y && \text{in 2d}\\
(u,\phi) &:= \int_\Omega u(x,y,z)\cdot\phi(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z && \text{in 3d}
\end{aligned}
$$

- Then, the **weak formulation** is to find $u \in \mathcal{V}$ as solution to

$$(\nabla u, \nabla\phi) = (f,\phi)$$

for all **test functions** $\phi \in V$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ for $d = 1, 2, 3$ be a domain and $f \in L^2(\Omega)$. Then, there exists a solution*

$$u \in \mathcal{V} := H_0^1(\Omega)$$

to the Laplace problem in variational formulation

$$(\nabla u, \nabla \phi) = (f, \phi).$$

- The space $H_0^1(\Omega)$ is the **Sobolev space** of functions:
  - $u \in L^2(\Omega)$, which means square integrable

$$\int_\Omega u^2 \, \mathrm{d}x < \infty$$

  - which have a first (that is the 1) **weak derivative** $\nabla u \in L^2(\Omega)^d$
  - and that are zero on the boundary (this is the 0) $u = 0$ on Γ
- Functions $u \in H_0^1(\Omega)$ have weak derivatives that can be integrated

$$\int_\Omega \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x < \infty$$

  but they are not necessarily continuous.

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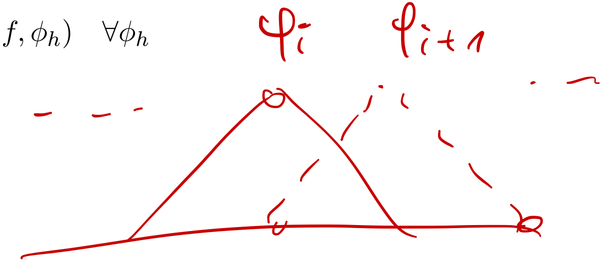
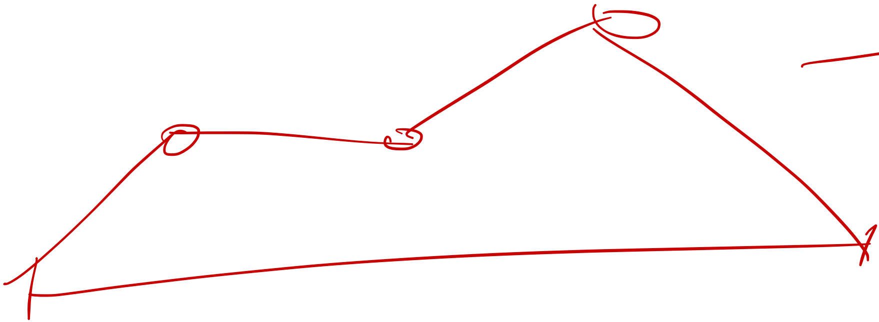
Steps for a finite element discretization

1. We discretize the domain Ω by a mesh Ω_h
2. On Ω_h we discretize the function space $\mathcal{V} = H_0^1(\Omega)$ by a finite element space V_h
3. We restrict the variational formulation to V_h

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h$$



4. We solve a linear system of equations



- We discretize the domain Ω by splitting it into simple **open elements**, e.g. triangles, quadrilaterals (in 2d) or tetrahedras, prisms, hexahedras, pyramids (in 3d)
- The **finite element mesh** Ω_h is the set of all **elements**

$$\Omega_h = \{T_1, T_2, \dots, T_N\}$$

- We make the following **structural assumptions**
  1. The union of all elements covers the domain

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{T}_i$$

  2. Two different elements never overlap

$$T_i \cap T_j = \emptyset \quad \forall i \neq j$$

  3. The closure of two elements can only overlap in a **corner vertex**, an **edge** or a **face**

$$\bar{T}_i \cap \bar{T}_j = \begin{cases} x & \text{a vertex} \\ e & \text{an edge} \\ f & \text{a face} \end{cases} \quad \forall i \neq j$$

Basic rule: *triangles should look like triangles, tetrahedras should look like tetrahedras, ...*

Shape regularity for triangular meshes:

- We call a mesh shape regular, if it holds for all $T \in \Omega_h$

$$\frac{\rho_T}{\mathrm{diam}(T)} < c,$$

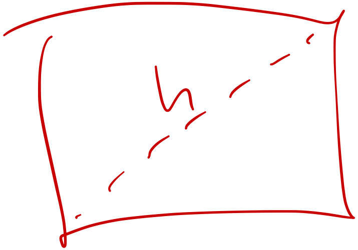
$h := \mathrm{diam}(T)$

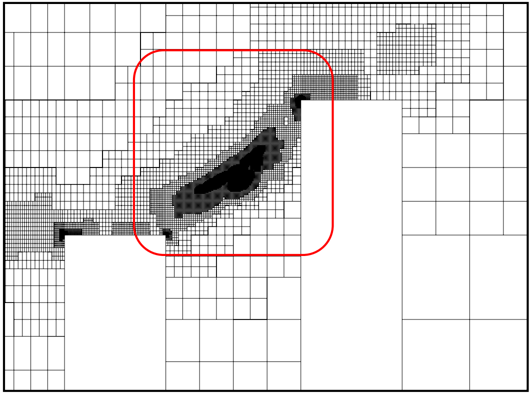
where ρ_T is the diameter of the largest circle in T and $\mathrm{diam}(T)$ the longest edge of T

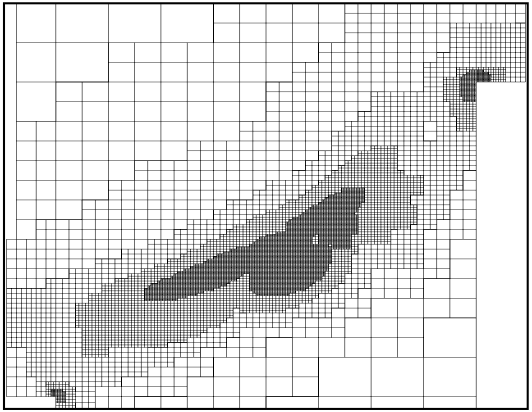
- Equivalent definition: All angles α in T are bound away from zero

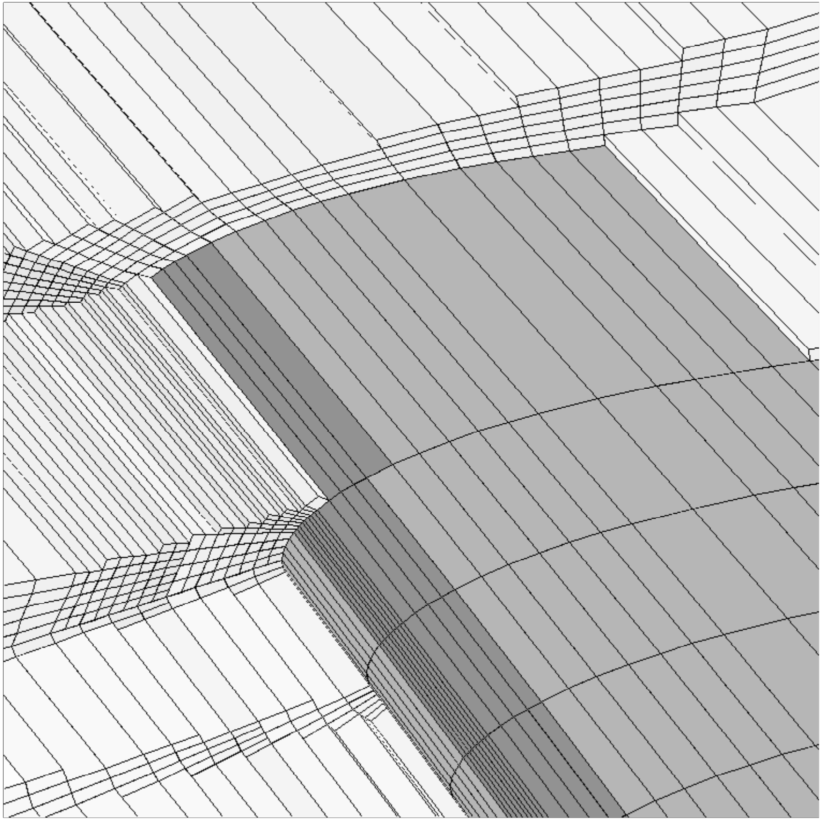
$$\alpha \geq \alpha_0 > 0$$

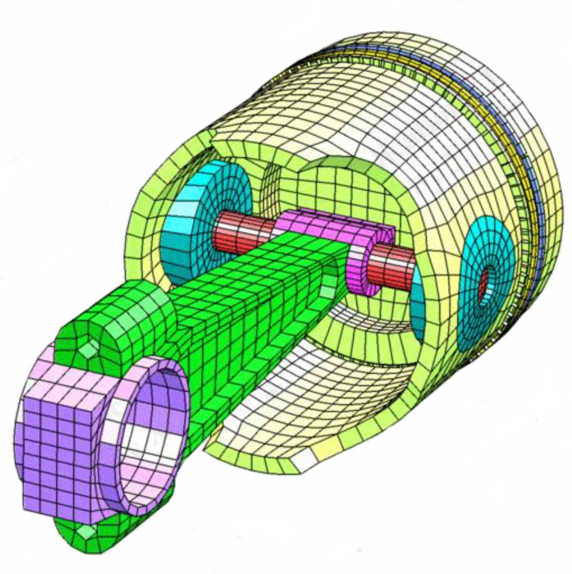
with a constant $\alpha_0 > 0$.

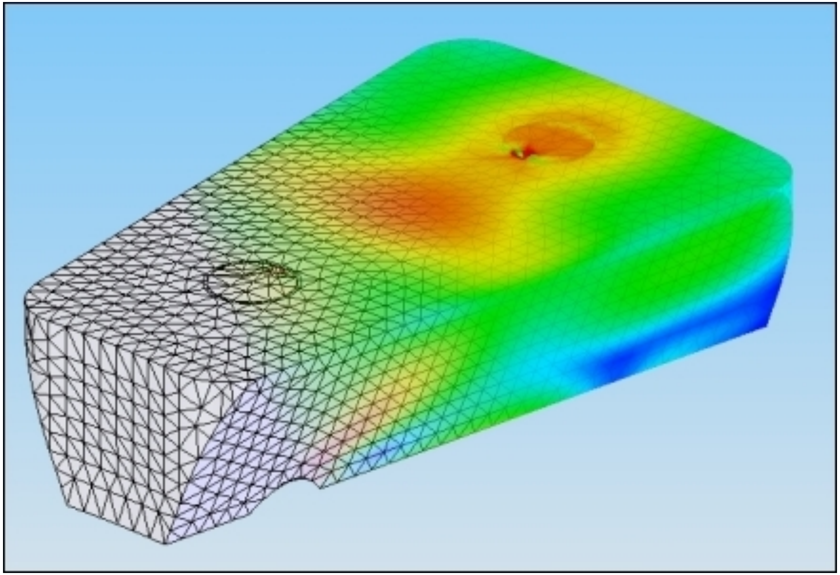












Local Finite Element space

- On every element $T \in \Omega_h$ define the basis functions of a simple polynomial space
- **linear finite elements** on triangles
- Triangle with the points $x^{(1)} = (0,0)$, $x^{(2)} = (h,0)$ and $x^{(3)} = (0,h)$

$$\phi^{(1)}(x,y) = 1 - \frac{x}{h} - \frac{y}{h}, \quad \phi^{(2)}(x,y) = \frac{x}{h}, \quad \phi^{(3)}(x,y) = \frac{y}{h}$$

- We have basis functions on every triangle $T \in \Omega_h$
- We combine them to a global function space

$$V_h := \{\phi_h \in C(\bar{\Omega}) \mid \phi\Big|_T \in P^1 := \operatorname{span}\left(\phi_h^{(1)}, \phi_h^{(2)}, \phi_h^{(3)}\right)\}$$

- This is called the **Lagrange basis** or **nodal basis**. It holds

$$\phi_h^{(i)} \in V_h : \quad \phi_h^{(i)}\Big|_T \in P^1, \quad \phi_h^{(i)}(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

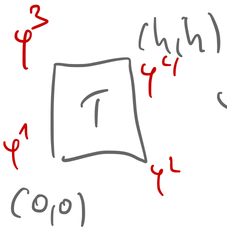
- Assume that the mesh elements $T \in \Omega_h$ are quadrilaterals
- **bi-linear finite elements:**
- Let T be a quadrilateral with the points $x^{(1)} = (0,0)$, $x^{(2)} = (h,0)$, $x^{(3)} = (0,h)$, $x^{(4)} = (h,h)$.

$V_h \subset H^1_0(\Omega)$

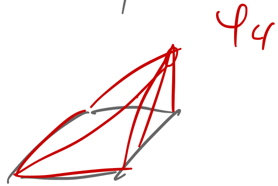
$\varphi_h \in V_h$ ① is continous on Ω

② on each element T

$\varphi_h|_T$ is polynomial



$\varphi_h|_T \in Q^1 = \langle 1, x, y, xy \rangle$



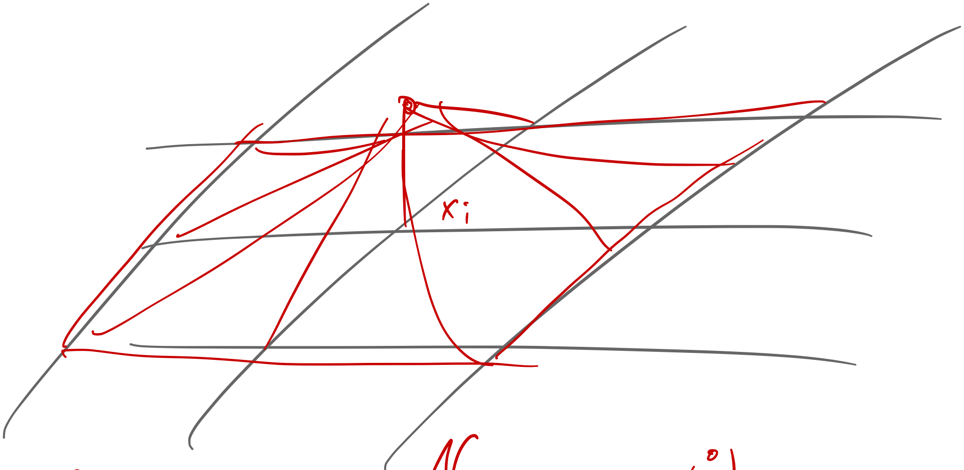
$$\phi^{(1)}(x,y) = \left(1 - \frac{x}{h}\right)\left(1 - \frac{y}{h}\right), \quad \phi^{(2)}(x,y) = \frac{x}{h}\left(1 - \frac{y}{h}\right), \quad \phi^{(3)}(x,y) = \left(1 - \frac{x}{h}\right)\frac{y}{h}, \quad \phi^{(4)}(x,y) = \frac{xy}{h^2}$$

- The Lagrange basis of the finite element space is given as

$$V_h := \{\phi_h \in C(\bar{\Omega}) \mid \phi\big|_T \in Q^1 := \operatorname{span}\left(\phi_h^{(1)}, \phi_h^{(2)}, \phi_h^{(3)}, \phi_h^{(4)}\right)\}$$

- The **Lagrange basis** or **nodal basis** is given by

$$\phi_h^{(i)} \in V_h : \quad \phi_h^{(i)}\big|_T \in Q^1, \quad \phi_h^{(i)}(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$



$$u_h \in V_h \qquad u_h = \sum_{i=1}^{N} u_i \varphi_h^{(i)} \Rightarrow u_h(x_i) = u_i$$

- Starting point: weak formulation of Laplace equation

$$u \in \mathcal{V} \quad (\nabla u, \nabla \phi) = (f, \phi) \quad \forall \phi \in \mathcal{V}$$

- We discretize the **trial functions** $u_h \in V_h \subset \mathcal{V}$ and the **test functions** $\phi_h \in V_h \subset \mathcal{V}$

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h.$$

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h. \tag{1}$$

- The finite element space is given by a local basis

$$V_h = \operatorname{span}\{\phi_h^{(1)}, \dots, \phi_h^{(N)}\}$$

- We split (1) into N equations

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h^{(i)}) = (f, \phi_h^{(i)}) \quad \forall i = 1, \dots, N \tag{2}$$

- We write the unknown solution $u_h \in V_h$ as

$$u_h(x,y) = \sum_{j=1}^{N} \mathbf{u}_j \phi_h^{(j)}(x,y)$$

and insert this notation into (3)

$$\sum_{j=1}^{N} \overbrace{(\nabla \phi_h^{(j)}, \nabla \phi_h^{(i)})}^{=: A_{ij}} \mathbf{u}_j = \overbrace{(f, \phi_h^{(i)})}^{=: f_i} \quad \forall i = 1, \dots, N \tag{3}$$

- This is equivalent to a **linear system of equations**

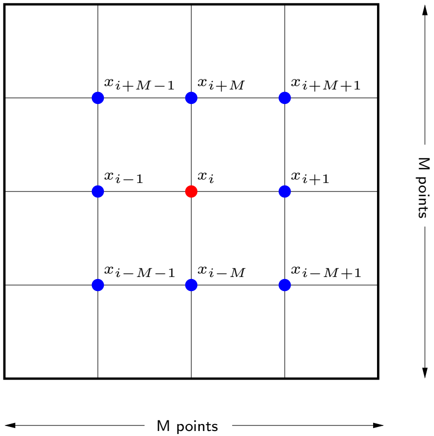
$$\mathbf{A}\mathbf{u} = \mathbf{f}, \quad \mathbf{A}_{ij} := (\nabla \phi_h^{(j)}, \nabla \phi_h^{(i)}), \quad \mathbf{f}_i := (f, \phi_h^{(i)})$$

Assembling the matrix

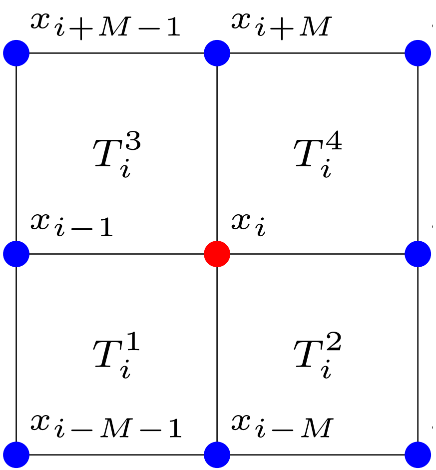
- We must compute the matrix entries

$$A_{ij} = (\nabla\phi_h^{(j)}, \nabla\phi_h^{(i)}) = \int_\Omega \nabla\phi_h^{(j)} \cdot \nabla\phi_h^{(i)}\,\mathrm{d}x = \sum_{T\in\Omega_h} \int_T \nabla\phi_h^{(j)} \cdot \nabla\phi_h^{(i)}\,\mathrm{d}x$$

- Each **nodal basis function** $\phi_h^{(i)}$ is non-zero only in the four quadrilaterals touching x_i
- The product $\nabla\phi_h^{(j)} \cdot \nabla\phi_h^{(i)}$ is non-zero only on elements T that have both points x_i and x_j in common



- Regular mesh with $N = M \cdot M$ nodes
- The test function $\phi_h^{(i)}$ couples with itself and 8 further testfunctions
- The matrix elements A_{ij} must only be computed in 4 elements



- We first compute all couplings in every element T_k for $k = 1, 2, 3, 4$

$$a_{ij}^T := \int_T \nabla \phi_h^{(i)} \cdot \nabla \phi_h^{(j)} \, \mathrm{d}x$$

- Then, we put it all together in the global matrix

$$\begin{aligned} A_{i,i} &= a_{i,i}^{T_1} + a_{i,i}^{T_2} + a_{i,i}^{T_3} + a_{i,i}^{T_4} \\ A_{i,i+1} &= \underbrace{a_{i,i+1}^{T_1}}_{=0} + a_{i,i+1}^{T_2} + \underbrace{a_{i,i+1}^{T_3}}_{=0} + a_{i,i+1}^{T_4} = a_{i,i+1}^{T_2} + + a_{i,i+1}^{T_4} \\ A_{i,i+M+1} &= a_{i,i+M+1}^{T_4} \\ &\vdots \end{aligned}$$

Example Let $T_i^4 = (0, h) \times (0, h)$

$$\phi_h^{(i)} = \left(1 - \frac{x}{h}\right)\left(1 - \frac{y}{h}\right) \qquad \phi_h^{(i+1)} = \frac{x}{h}\left(1 - \frac{y}{h}\right)$$

$$\phi_h^{(i+M)} = \left(1 - \frac{x}{h}\right)\frac{y}{h} \qquad \phi_h^{(i+1+M)} = \frac{xy}{h^2}$$

- We combine the result in a **stencil**

$$S = \begin{bmatrix} s_{31} & s_{32} & s_{33} \\ s_{21} & s_{22} & s_{23} \\ s_{11} & s_{12} & s_{13} \end{bmatrix}$$

- The entries have the following meaning:

$$\begin{aligned} \mathbf{A}_{i,i-M-1} &= s_{11}, & \mathbf{A}_{i,i-M} &= s_{12}, & \mathbf{A}_{i,i-M+1} &= s_{13} \\ \mathbf{A}_{i,i-1} &= s_{21}, & \mathbf{A}_{i,i} &= s_{22}, & \mathbf{A}_{i,i+1} &= s_{23} \\ \mathbf{A}_{i,i+M-1} &= s_{31}, & \mathbf{A}_{i,i+M} &= s_{32}, & \mathbf{A}_{i,i+M+1} &= s_{33} \end{aligned}$$

- For the Laplace problem with bi-linear finite elements the stencil is given by

$$S = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

- For the Laplace problem with linear finite elements on triangles the stencil would be given by

$$S_{tria} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$S_{1d} = [-1 \quad 2 \quad -1]$$

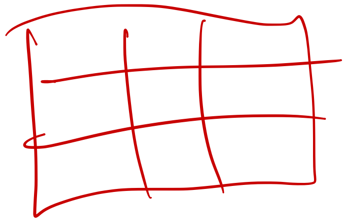
- The finite element matrix on a small mesh with $16 = 4 \cdot 4$ nodes looks like

$$\mathbf{A} = \frac{1}{3}\begin{pmatrix}
8 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 8 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 8 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 8 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 8 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & -1 & 8 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & -1 & 8 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & -1 & 8 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 8 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 8 & -1 & 0 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 8 & -1 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 8 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 8 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 8 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 8 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 8
\end{pmatrix}$$

- But: all the nodes $x_i \in \Gamma$ on the boundary are Dirichlet-Nodes. Here we want so set

$$u(x_i) = 0$$

- We must modify the matrix!



Matrix with Dirichlet values

- The finite element matrix on a small mesh with $16 = 4 \cdot 4$ nodes looks like

$$
\mathbf{A} = \frac{1}{3}\begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & -1 & 8 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & -1 & 8 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 8 & -1 & 0 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 8 & -1 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}
$$

- We also have to compute the right hand side

$$\mathbf{f}_i = (f, \phi_h^{(i)}) = \int_\Omega f \phi_h^{(i)} \,\mathrm{d}x = \sum_{T \in \Omega_h} \int_T f \phi_h^{(i)} \,\mathrm{d}x$$

- The right hand side $\mathbf{f}_i$ is only computed in those elements T that touch the basis function $\phi_h^{(i)}$
- Only 4 elements in 2d

Problem! If f is a general function we cannot compute the integral analytically!

- Computation of the integrals by numerical quadrature

$$\int_T f(x,y)\phi_h^{(i)}(x,y) \,\mathrm{d}x \,\mathrm{d}y \approx \sum_{k=1}^{q_N} \omega_k f(x_k, y_k) \phi_h^{(i)}(x_k, y_k)$$

where:

q_N is the number of quadrature points

ω_k is the quadrature weight

x_k, y_k is the quadrature point

- A typical numerical quadrature rule is the 2×2-*point Gauss rule*

$$\int_0^h \int_0^h g(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \frac{h^2}{4}\Big(g\big(\frac{1}{2}-\frac{1}{\sqrt{3}},\frac{1}{2}-\frac{1}{\sqrt{3}}\big)+g\big(\frac{1}{2}+\frac{1}{\sqrt{3}},\frac{1}{2}-\frac{1}{\sqrt{3}}\big) + g\big(\frac{1}{2}-\frac{1}{\sqrt{3}},\frac{1}{2}+\frac{1}{\sqrt{3}}\big)+g\big(\frac{1}{2}+\frac{1}{\sqrt{3}},\frac{1}{2}+\frac{1}{\sqrt{3}}\big)\Big)$$

- It is very accurate and efficient.

- For all nodes $x_i \in \Gamma$ on the boundary we modify the right hand side vector $\mathbf{f}_i$ such that (out example uses $g(x) = 0$)

$$\mathbf{f}_i = g(x_i) = 0$$

- Then, the i-th line of the linear system is

$$(\mathbf{A}\mathbf{u})_i = \mathbf{f}_i \quad \Leftrightarrow \quad 1 \cdot \mathbf{u}_i = g_i \quad \Leftrightarrow \quad \mathbf{u}_i = 0.$$

- The main difference between 1d and 2d (or 3d) is the domain
  In 1d we have only intervals
- In 2d, the solution is usually not regular
- We first discretize the domain, then we set up the finite element space
- We must integrate the matrix and the right hand side
- Assembling the matrix is easy if we use the stencil, the right hand side often needs numerical quadrature

Thanks