Christian-Albrechts-Universität zu Kiel

OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

Teoría de elementos finitos y su implementación

Malte Braack[1],
Carolin Mehlmann[2], Thomas Richter[2]

[1] Mathematical Seminar, Kiel University
[2] Institute of Analysis and Numerics, University of Magdeburg

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Stationary case:

$$\begin{aligned} \sigma u + (\beta \cdot \nabla) u - \epsilon \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- Reaction term $\sigma \in L^\infty(\Omega)$
- Convection field $\beta \in L^\infty(\Omega)^d$, $\operatorname{div} \beta \in L^\infty(\Omega)$
- Diffusion $\epsilon \in \mathbb{R}_{>0}$

For existence and uniqueness:

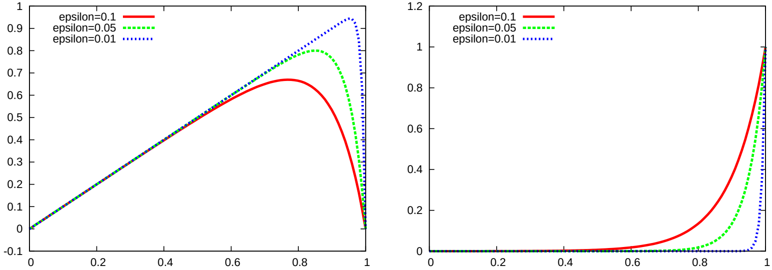
$$\sigma - \tfrac{1}{2} \operatorname{div} \beta \geq c \geq 0 \quad \text{a.e. in } \Omega$$

e.g., $\sigma \geq 0$ and $\operatorname{div} \beta = 0$.

3.1 Convection-Diffusion Equation in 1D

In intervall $I = (0,1)$ with diffusion parameter $\epsilon > 0$:

$$\begin{aligned} -\epsilon u'' + u' &= f \quad x \in I \\ u(0) = u_0\,,\ u(1) &= u_1 \end{aligned}$$



Examples for $\epsilon \in \{0.1, 0.05, 0.01\}$ with $f = 1$ (left) and $f = 0$ (right).:

- Boundary layers for $\epsilon \to 0$

Central differences for 1D convection-diffusion

- Central difference quotient for 1. derivatives (of 2. order) and 2. derivatives (of 2. order):

$$
\begin{aligned}
u'(x_i) &\approx \frac{1}{2h}(u_{i+1} - u_{i-1}) \\
u''(x_i) &\approx \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1})\,.
\end{aligned}
$$

- Linear system for approximate solution with $u_0 = u_N = 0$:

$$
\begin{pmatrix}
a & c & & & \\
b & \ddots & \ddots & & \\
 & \ddots & \ddots & \ddots & \\
 & & \ddots & \ddots & c \\
 & & & b & a
\end{pmatrix}
\begin{pmatrix}
u_1 \\ \vdots \\ \vdots \\ u_{N-1}
\end{pmatrix}
= F
$$

with $a = 2\epsilon/h^2$, and $b = -\epsilon/h^2 - 1/(2h)$, and $c = -\epsilon/h^2 + 1/(2h)$.

- Diagonal elements a are always positive

$$a = \frac{2\epsilon}{h^2} > 0$$

- Off-diagonal elements b, c are negative for $h < 2\epsilon$:

$$\begin{aligned} b &= -\frac{\epsilon}{h^2} - \frac{1}{2h} \leq c \\ c &= -\frac{\epsilon}{h^2} + \frac{1}{2h} < 0 \end{aligned}$$

- But for $h > 2\epsilon$ we obtain positive off-diagonal elements:

$$c = -\frac{\epsilon}{h^2} + \frac{1}{2h} > 0$$

$\rightarrow$ No M-matrix, no Discrete Maximum Principle (DMP) !

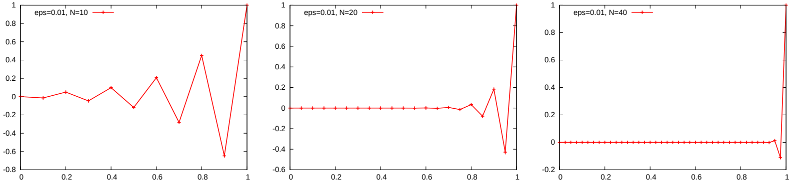
Good and bad news for central differences

Good properties:

- Formally of 2. order (in L^2-norm).
- DMP if $h < 2\epsilon$

Bad properties:

- No DMP if $h \geq 2\epsilon$
- The violation of a DMP leads to strong numerical artefacs:
- In many practical applications $h < 2\epsilon$ in unfeasable, because often $\epsilon \sim 10^{-8} - 10^{-5}$.



$\epsilon = 0.01$ and $N = 10, 20, 40$ (from left to right)

Alternatives for FDM:

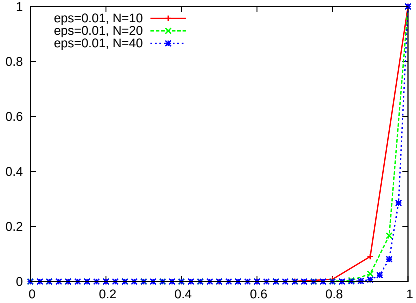
- Artificial diffusion
- Upwinding

Artificial diffusion

Idea: Ensure $\epsilon \geq 2h$ by changing to an augmented diffusion coefficient:

$$\epsilon_h \quad := \quad \max(2h, \epsilon)$$

Use ϵ_h instead of ϵ in the discrete equation.



Drawback: Very diffusive and only of 1. order in L^2-norm.

Upwinding in 1D

Idea: Use one-sided difference quotient for the first derivatives instead of a central difference quotient:

$$u'(x_i) \quad \approx \quad \frac{1}{h}(U_i - U_{i-1})$$

Resulting linear system:

$$bU_{i-1} + aU_i + cU_{i+1} \quad = \quad f_i$$

With positive diagonal elements a and negative off-diagonal elements b, c:

$$\begin{aligned} a &= \frac{2\epsilon}{h^2} + \frac{1}{h} \; > \; 0 \\ b &= -\frac{\epsilon}{h^2} - \frac{1}{h} \; < 0 \\ c &= -\frac{\epsilon}{h^2} \; < \; 0 \end{aligned}$$

M-matrix

Upwinding in 1D (cont'd)

More general convection-diffusion-reaction equation with varying coefficients:

$$-\epsilon u'' + bu' + cu \quad = \quad f \quad x \in I$$

The difference quotient depends on the direction of the convective part:

$$u'(x_i) \quad \approx \quad \begin{cases} h^{-1}(U_i - U_{i-1}) & \text{if } b(x_i) > 0 \\ h^{-1}(U_{i+1} - U_i) & \text{if } b(x_i) < 0 \end{cases}$$

Leads to DMP in 1D.

3.2 Scalar convection-diffusion problems in multi-dimensions

- Convection-diffusion problem:

$$\begin{aligned} \sigma u + (\beta \cdot \nabla) u - \epsilon \Delta u &= f \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- Variational formulation in $V = H_0^1(\Omega)$:

$$u \in V : \quad A(u, \phi) = \langle f, \phi \rangle \quad \forall \phi \in V$$

with bilinear form:

$$A(u, \phi) := (\sigma u + (\beta \cdot \nabla) u, \phi) + (\epsilon \nabla u, \nabla \phi)$$

- Existence and uniqueness of solutions due to Lax-Milgram.

Existence and uniqueness for scalar convection-diffusion problem

Theorem

Let $\sigma, \nabla \cdot \beta \in L^\infty(\Omega)$ and $\epsilon > 0$. If there exists a constant c_0, s.t.

$$\sigma - \frac{1}{2} div\, \beta \quad \geq \quad c_0 \geq 0 \quad a.e. \;\; in\; \Omega$$

the variational formulation features for each $f \in L^2(\Omega)$ a unique solution $u \in H_0^1(\Omega)$.

Proof.

Continuity:

$$\begin{aligned} |A(u,\phi)| &\leq \sigma\|u\|\|\phi\| + \epsilon\|\nabla u\|\|\nabla\phi\| + \|\beta\|_{L^\infty(\Omega)}\|\nabla u\|\|\phi\| \\ &\leq C_{\epsilon,\beta,\sigma}\left(\|u\| + \|\nabla u\|\right)\left(\|\phi\| + \|\nabla\phi\|\right) \end{aligned}$$

With Poincare inequality we obtain continuity.

Coercivity:

$$A(u,u) \quad = \quad (\sigma u + (\beta\cdot\nabla)u, u) + \epsilon\|\nabla u\|^2$$

Reformulation of the convective part:

$$\begin{aligned}
((\beta\cdot\nabla)u, u) \quad &= \quad \int_\Omega \nabla u \cdot (\beta u)\, dx \\
&= \quad -(u, \operatorname{div}(\beta u)) + \int_{\partial\Omega} u^2(\beta\cdot n) \\
&= \quad -(u, (\operatorname{div}\beta)u) - (u, (\beta\cdot\nabla)u)
\end{aligned}$$

Hence

$$((\beta\cdot\nabla)u, u) \quad = \quad -\frac{1}{2}(u, u\operatorname{div}\beta)$$

$$(\sigma u + (\beta\cdot\nabla)u, u) = ((\sigma - \tfrac{1}{2}\operatorname{div}\beta), u^2) \quad \geq \quad c_0\|u\|^2$$

Therefore, we get for $c_0 > 0$:

$$A(u,u) \quad \geq \quad c_0\|u\|^2 + \epsilon\|\nabla u\|^2 \geq \epsilon\|\nabla u\|^2$$

The conditions for Lax-Milgram are fulfilled.

Where is the problem for discrete (FE) schemes ?

- Lax-Milgram also applies to the discrete standard FE scheme.
- Existence and uniqueness of discrete solutions.
- Cea's Lemma:

$$\|u - u_h\| + \|\nabla(u - u_h)\| \quad \leq \quad \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V_h} \left(\|u - v_h\| + \|\nabla(u - v_h)\| \right)$$

- But:

$$\frac{\alpha_1}{\alpha_2} \quad = \quad \frac{\sigma + \|\beta\|_{L^\infty(\Omega)} + \epsilon}{\min\{c_0, \epsilon\}} \sim \frac{1}{\epsilon} \qquad \text{for small } \epsilon$$

- E.g., for P_1-elements:

$$\|u - u_h\| + \|\nabla(u - u_h)\| \quad \sim \quad \frac{h}{\epsilon}$$

Large errors for small viscosities $\epsilon \ll 1$.

Standard Galerkin formulation: Seek $u_h \in V_h$ s.t.

$$A(u_h, \phi) \quad = \quad (f, \phi) \quad \forall \phi \in V_h$$

A priori estimate in the norm

$$\|u\|_\epsilon \quad := \quad \left(\epsilon \|\nabla u\|^2 + \|u\|^2\right)^{1/2} .$$

Theorem

Under the conditions of the previous Thm. and H^{r+1}-regularity of the solution u the P_r (or Q_r) FE solution of the CDR problem fulfills the following a priori estimate:

$$\|u - u_h\|_\epsilon \quad \leq \quad Ch^r |u|_{H^{r+1}(\Omega)}$$

with $C = C(\beta, \epsilon, \sigma, \Omega, \kappa)$.

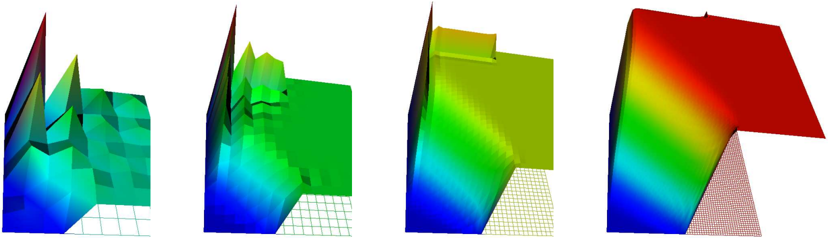
This is not satisfactory because e.g. for $r = 1$:

$$\|\nabla(u - u_h)\| \quad \sim \quad \frac{h}{\epsilon} |u|_{H^2(\Omega)}$$

and $\lim_{\epsilon \to 0} |u|_{H^2(\Omega)} = \infty$.

Instabilities of the pure Galerkin formulation

- Diffusion coefficent $\epsilon = 0.01$, convection $\beta = (1,1)^T$.
- The Galerkin formulation stabilizes only on sufficient fine meshes:



- No DMP

$$\text{if } h > \frac{\epsilon}{2|\beta|} \quad \Longrightarrow \quad \text{unphysical oscillations}$$

- Convergence for P_1 (or Q_1) elements of pure Galerkin:

$$\|\nabla(u - u_h)\| \quad \sim \quad \frac{h}{\epsilon}|u|_{H^2(\Omega)}$$

- Galerkin formulation of first order term is equivalent to central difference scheme.

Numerical diffusion with FEM

- Similar strategies as for FD are possible with FEM: Adding the term

$$(h\nabla u_h, \nabla\phi)$$

  to the bilinear form $A(\cdot,\cdot)$.
- Diffusion in streamline direction only: Adding

$$(h(\beta\cdot\nabla)u_h, (\beta\cdot\nabla)\phi)$$

  Only 1. order !
- Accuracy of 1. order methods can be worse than pure Galerkin (see Brooks & Hughes)

Numerical schemes for convection-diffusion-reaction problems

- Still challenging for dominating convection, i.e. $\|\beta\| >> \epsilon$.
- Apperance of interior and boundary layers, small subregions with large gradients
- Width of layers are often smaller than the mesh size.
- Then, sufficient resolution of sharp gradients are not possible.
- Spurious nonphysical oszillations.
- Extensive research in the last decades.

$$A(u,\phi) \quad := \quad (\sigma u + (\beta \cdot \nabla)u, \phi) + (\epsilon \nabla u, \nabla \phi)$$

- Stabilized form:

$$u_h \in V_h : \quad A(u_h,\phi) + S_h(u_h,\phi) \quad = \quad \langle F_h, \phi \rangle \quad \forall \phi \in V_h$$

- Such stabilization is called **fully consistent**, if the strong solution u still fulfills the discrete equation.
- Pure Galerkin formulation for

$$S_h(u_h,\phi) = 0 \quad \text{and} \quad F_h = F$$

- Several choices are possible, e.g.
  - SUPG: Streamline upwind Petrov Galerkin
  - LPS: Local projection stabliization
  - EOS: edge oriented stabilization / IP: interior penalty
  - SOLD: Spurious Oszillation at Layer Disminishing methods
  - DG: Discontinuous Galerkin

3.4 Stabilized finite elements

$$A(u,\phi) \quad := \quad (\sigma u + (\beta\cdot\nabla)u,\phi) + (\epsilon\nabla u,\nabla\phi)$$

- Stabilized form:

$$u_h \in V_h : \quad A(u_h,\phi) + S_h(u_h,\phi) \quad = \quad \langle F_h,\phi\rangle \quad \forall\phi\in V_h$$

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- Pure Galerkin formulation for

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  - SUPG: Streamline Upwind Petrov Galerkin
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3.4.1 Streamline Upwind Petrov Galerkin (SUPG)

- Add a diffusion term in direction of the streamlines, i.e. β

$$((\beta \cdot \nabla) u_h, (\beta \cdot \nabla)\phi)$$

- choose a clever weighting s.t. a balance of sufficient diffusion and accuracy is obtained:

$$\sum_{T \in \mathcal{T}_h} \delta_T ((\beta \cdot \nabla) u_h, (\beta \cdot \nabla)\phi)_T$$

- Introduce further consistency terms:

$$\sum_{T \in \mathcal{T}_h} \delta_T (\sigma u_h + (\beta \cdot \nabla) u_h - \epsilon \Delta u_h, (\beta \cdot \nabla)\phi)_T$$

- No additional diffusion in crosswind direction.

SUPG for convection-diffusion problems

- Idea: Add a consistent diffusion term in direction of the flow:

$$
\begin{aligned}
S_h(u_h, \phi) &:= \sum_{T \in \mathcal{T}_h} (\sigma u_h + (\beta \cdot \nabla) u_h - \epsilon \Delta u_h, \delta_h (\beta \cdot \nabla)\phi)_T \\
\langle F_h, \phi \rangle &:= \langle F, \phi \rangle + \sum_{T \in \mathcal{T}_h} (f, \delta_h (\beta \cdot \nabla)\phi)_T
\end{aligned}
$$

- SUPG is fully consistent: The strong solution u fulfills the discrete equation.
- The parameter δ_h is mesh size dependent, $\delta_T := \delta_h|_T$.

Coercivity of the SUPG stabilization form

For P_1 elements (u_h cell-wise linear):

$$S_h(u_h, u_h) = \sum_{T\in\mathcal{T}_h} (\sigma u_h + (\beta\cdot\nabla)u_h - \epsilon\Delta u_h, \delta_T(\beta\cdot\nabla)u_h)_T$$

Cell-wise estimation:

$$\begin{aligned} & (\sigma u_h + (\beta\cdot\nabla)u_h - \epsilon\Delta u_h, \delta_T(\beta\cdot\nabla)u_h)_T \\ = \; & \delta_T\|(\beta\cdot\nabla)u_h\|_T^2 + (\sigma u_h, \delta_T(\beta\cdot\nabla)u_h)_T \\ \geq \; & \delta_T\|(\beta\cdot\nabla)u_h\|_T^2 - \|\sigma u_h\|_T\delta_T\|(\beta\cdot\nabla)u_h\|_T \\ \geq \; & \delta_T\|(\beta\cdot\nabla)u_h\|_T^2 - \frac{\delta_T}{2}\|\sigma u_h\|_T^2 - \frac{\delta_T}{2}\|(\beta\cdot\nabla)u_h\|_T^2 \end{aligned}$$

We obtain the coercivity property:

$$S_h(u_h, u_h) \geq \frac{1}{2}\sum_{T\in\mathcal{T}_h} \delta_T\left(\|(\beta\cdot\nabla)u_h\|_T^2 - \|\sigma u_h\|_T^2\right)$$

Coercivity of the SUPG bilinear form

For P_1 elements (u_h cell-wise linear):

$$\begin{aligned} & A(u_h, u_h) + S_h(u_h, u_h) \\ \geq \quad & c_0\|u_h\|^2 + \epsilon\|\nabla u_h\|^2 + \frac{1}{2}\sum_{T\in\mathcal{T}_h} \delta_T \left(\|(\beta\cdot\nabla)u_h\|_T^2 - \|\sigma u_h\|_T^2\right) \\ \geq \quad & c'\|u_h\|^2 + \epsilon\|\nabla u_h\|^2 + \frac{1}{2}\sum_{T\in\mathcal{T}_h} \delta_T\|(\beta\cdot\nabla)u_h\|_T^2 \end{aligned}$$

with

$$c' \quad = \quad \|\sigma(1-\delta_h/2) - \tfrac{1}{2}\mathrm{div}\,\beta\|_{L^\infty(\Omega)}$$

By Lax-Milgramm we obtain:

Theorem

Under the condition that $\delta_h \leq 1$ and

$$div\,\beta \quad \leq \quad \sigma \qquad a.e.$$

the SUPG formulation of the convection-diffusion-reaction system with P_1 elements has a unique discrete solution $u_h \in P_1(\mathcal{T}_h)$.

Optimal choice for δ_T

Local Peclet number

$$Pe|_T \quad := \quad \frac{h_T \|\beta\|_{L^\infty(T)}}{\epsilon}$$

$Pe|_T > 1/2$: convection dominated region

$$\delta_T \quad := \quad \frac{h_T}{\|\beta\|_{L^\infty(T)}}$$

$Pe|_T \leq 1/2$: diffusion dominated region

$$\delta_T \quad := \quad \frac{h_T^2}{\epsilon}$$

Or

$$\delta_T \quad := \quad \min\left(\frac{h_T}{\|\beta\|_{L^\infty(T)}}, \frac{h_T^2}{\epsilon}, \frac{1}{\sigma}\right)$$

A priori estimate for SUPG

$$\|u\|_h^2 \quad := \quad \sigma\|u\|^2 + \epsilon\|\nabla u\|^2 + \sum_{T\in\mathcal{T}_h} \delta_T\|(\beta\cdot\nabla)u\|_T^2$$

Theorem

Under the smoothness assumption $u \in H^{r+1}(\Omega)$ it holds for P_r-elements

$$\|u - u_h\|_h \quad \leq \quad C\sum_{T\in\mathcal{T}_h} a_T h_T^r |u|_{H^{r+1}(T)}$$

with $a_T := (\epsilon + h_T\|\beta\|_{L^\infty(T)} + h_T^2\sigma)^{1/2}$.

E.g. for P_1- or Q_1-elements:

$$\|\nabla(u - u_h)\| \quad \sim \quad \left(1 + \frac{h}{\epsilon}\right)^{1/2} h|u|_{H^2}$$

plus additional control about streamline derivative.

For pure Galerkin remember: $\|\nabla(u - u_h)\| \sim h/\epsilon$.

Proof

- Aim: Bound the discretization error by a multiple of

$$H_h(u) \quad := \quad \sum_{T \in \mathcal{T}_h} a_T h_T^r |u|_{H^{r+1}(T)}$$

- Splitting the error in interpolation error and projection error:

$$u - u_h \quad = \quad (u - I_h u) + (I_h u - u_h)$$

- The interpolation error $\eta_h = u - I_h u$ is properly bounded by standard interpolation results:

$$\|u_h - I_h u\|_h \quad \leq \quad C H_h(u)$$

- The projection error $\xi_h := I_h u - u_h \in V_h$ can be bounded by help of the coercivity:

$$\alpha_2^{-1} \|\xi_h\|_h^2 \quad \leq \quad A(\xi_h, \xi_h) + S_h(\xi_h, \xi_h)$$

- Use Galerkin orthogonality:

$$A(u - u_h, \xi_h) + S_h(u - u_h, \xi_h) \quad = \quad 0$$

$$
\begin{aligned}
\alpha_2^{-1}|||\xi_h|||_h^2 &\leq A(\xi_h,\xi_h) - A(u-u_h,\xi_h) \\
&\quad + S_h(\xi_h,\xi_h) - S_h(u-u_h,\xi_h) \\
&= A(I_h u - u, \xi_h) + S_h(I_h u - u, \xi_h)
\end{aligned}
$$

- Individual bounds

$$
\begin{aligned}
|A(\eta_h,\xi_h)| &\leq H_h(u)|||\xi_h|||_h \\
|S_h(\eta_h,\xi_h)| &\leq H_h(u)|||\xi_h|||_h
\end{aligned}
$$

- e.g. for the Galerkin terms arising in $A(\eta_h,\xi_h)$:

$$
\begin{aligned}
|(\sigma\eta_h,\xi_h)| &\leq \sigma^{1/2}\|\eta_h\| \, |||\xi_h|||_h \\
|((\beta\cdot\nabla)\eta_h,\xi_h)| &= (\eta_h,(\beta\cdot\nabla)\xi_h) \\
&\leq \delta_h^{-1/2}\|\eta_h\| \, |||\xi_h|||_h \\
|\epsilon(\nabla\eta_h,\nabla\xi_h)| &\leq \epsilon^{1/2}\|\nabla\eta_h\| \, |||\xi_h|||_h
\end{aligned}
$$

- With a priori bounds on $\|\eta_h\|$, $\|\nabla\eta_h\|$:

$$
\begin{aligned}
A(-\eta_h, \xi_h) &\leq \underbrace{\left(\sigma^{1/2}\|\eta_h\| + \delta_h^{-1/2}\|\eta_h\| + \epsilon^{1/2}\|\nabla\eta_h\|\right)}_{(\sigma^{1/2}h+\delta_h^{-1/2}h+\epsilon^{1/2})h^r|u|_{H^{r+1}}} |||\xi_h|||_h \\
&\leq H_h(u)|||\xi_h|||_h
\end{aligned}
$$

- The other term $|S_h(\eta_h, \xi_h)|$ is bounded as well:

$$
|S_h(\eta_h, \xi_h)| \leq \left(\sum_{T\in\mathcal{T}_h} \delta_T \|\sigma\eta_h + (\beta\cdot\nabla)\eta_h - \epsilon\Delta\eta_h\|_T^2\right)^{1/2} |||\xi_h|||_h
$$

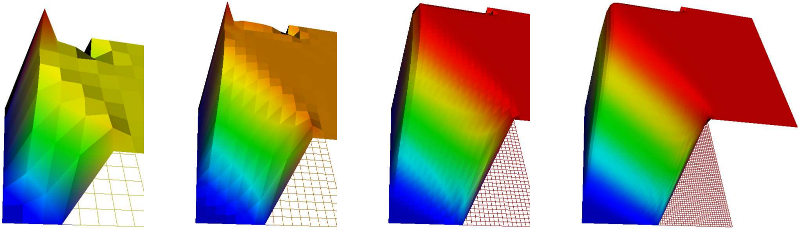
For $r = 1$ we deduce e.g.

$$
\begin{aligned}
\delta_T^{1/2}\|\epsilon\Delta\eta_h\|_T &\leq \delta_T^{1/2}\|\epsilon\Delta u\|_T \\
&\leq \delta_T^{1/2}\epsilon|u|_{H^2(T)} \\
&\leq h_T\epsilon^{1/2}|u|_{H^2(T)}
\end{aligned}
$$

For $r > 1$ an inverse estimate is needed.

Effect of SUPG stabilization

- Diffusion $\epsilon = 0.01$, convection $\beta = (1, 1)^T$.
- The SUPG formulation stabilizes much earlier.
- Over- and undershots still remain on coarse meshes:



- Unphysical oszillations still are problematic (positivity of physical quantities, shocks in compressible flows, deterioration of nonlinear problems)

Drawbacks of SUPG

1. Troublesome in the parabolic case (details below)
2. Computation of second derivatives necessary for $r \geq 2$ and Q_1-elements on arbitrary quadrilateral (or hexahedral) elements.
3. In the context of optimization: discretization and optimization do not commute.

3.4.2 SUPG for parabolic problems

- Time-dependent convection-diffusion system:

$$\begin{aligned} \partial_t u + (\beta\cdot\nabla)u - \epsilon\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \\ u\big|_{t=0} &= u_0 \quad \text{in } \Omega \end{aligned}$$

- Variational form:

$$(\partial_t u, \phi) + A(u,\phi) = (f,\phi)$$

with L^2-scalar product $(\cdot,\cdot)$ in space-time.

- Add consistent SUPG term:

$$S_h(u,\phi) := \sum_{T\in\mathcal{T}_h} \delta_T(\partial_t u + (\beta\cdot\nabla)u - \epsilon\Delta u, (\beta\cdot\nabla)\phi)_T$$

- The term $(\partial_t u, (\beta\cdot\nabla)\phi)$ is troublesome for discrete time derivative.

SUPG with an implicit Euler

With time step $\tau := t_n - t_{n-1}$:

$$
\begin{aligned}
&(\tau^{-1}u_n, \phi) + ((\beta\cdot\nabla)u, \phi) + (\epsilon\nabla u, \nabla\phi) + \\
&\sum_{T\in\mathcal{T}_h} \delta_T(\tau^{-1}u_n + (\beta\cdot\nabla)u + \ldots, (\beta\cdot\nabla)\phi)_T \quad = \quad (\tau^{-1}u_{n-1}, \phi) + \ldots
\end{aligned}
$$

- Not only the mass matrix scales with $1/\tau$.
- Additional terms may become dominant for small τ.

Remedy 1 in practise:

- Surpress the blue consistency terms.
- No optimal convergence analysis.

Remedy 2 in practise:

- Use different stabilization: LPS, EOS, ...