0.1 The navier Stokes equation

$$\rho(\partial_t v + (v \cdot \nabla v)) - \div = \rho f \quad \div v = 0$$

- This term describes the inertia of the fluid in motion.
- It tkes a fornce to influence
- This term describes the *incompresibility* of the fluid, it is not possible to change the volume.
- Not every fluid is incompressible, water is incompressible buy air can be compressed if the force is big.
- Meaning of incompresiblility If V is a vvolume then it holds, inflow to V equals to outflow from V.
- THe term $\div\sigma$ describes the internal forces within the fluid. (friction)
- The Navier Stokes stress tensor is given by

$$\sigma = \rho\nu(\nabla v + \nabla v^T) - pI$$

- ν is the viscosity of the fluid
- ρ is the pressure.
- We skip the *inert term*

$$-\nu\Delta v + \nabla p = \rho f \quad \div v = 0$$

- In $2D$:

$$-\nu\Delta v^1 +$$

  $3d$ existes 4 equations.

$$-\nu\Delta v + \nabla p = \rho f \quad \div v = 0$$

- Multplicaiton with test functions $\phi = (\phi^1, \phi^2)$ and ξ and intgration

$$\int_\Omega (-\nu\Delta v^1 + \partial_t p)\phi^1 \mathrm{d}x = \int_\Omega \rho f^1 \phi^1 \mathrm{d}x$$

- The theory of the Stokes equations is very difficult (compared to Lplace)

- But, we can simplift the problem

  If (v, p) is a solution to the Stokes equation its divergence is zzero.

$$\div v = 0$$

  We define the space of divergence free functions

$$\mathcal{V}_h = \{\phi \in H_1^0(\Omega)^2 \mid \div \varphi = 0\}$$

  Every solution v in this space $v \in \mathcal{V}_h$

  Now, we restrict the variational formmulation to this space

$$v \in \mathcal{V}_0 \colon \quad (\nu \nabla v, \nabla \phi) = (\rho f, \phi) \forall \phi \in$$

- It we have the velocity, the pressure is define by the problem

$$p \in L^2(\Omega) \quad (p, \div \phi) = (\nu \nabla b, \nabla \phi) - (\rho f, \phi) \quad \forall \phi \in H_0^1(\Omega)^2$$

  Theorem 1. There exists a unique velocity $v \in \mathcal{V}_0$ and a unique pressure $p \in L^2(\Omega)$ and $\inf - \sup$ *condition* holds

$$\inf_{p \in L^2(\Omega)} \sup_{v \in H_0^1(\Omega)^2} \frac{(p, \div v)}{\|p\| \|\nabla v\|} \geq \gamma$$

  with the inf -sup constantan $\gamma > 0$

- We discretize the variant

- Stationary Navier Stokes equations

$$\rho(v \cdot \nabla v - \rho \nu \Delta v + \nabla \rho = \rho f \quad \div v = 0$$

- Divide by

0.2 Reynolds number

$$R = \frac{VL}{\gamma}$$

where V is the velocity, for example:

$$100 = \frac{v \cdot 10\,\mathrm{m}}{10^{-6}} = 1 \times 10^{-5} \cdot V$$

$\iff V = 10^{-7}\,\mathrm{m\,s^{-1}}$.

$$\mathrm{Re} = \frac{V \cdot L}{\nu}$$

- Length of submarine
$$L = 100\,\mathrm{m}$$
- Velocity of submarine
$$V = \frac{10\,\mathrm{m}}{s}$$
- Viscosity of eater
$$\nu = \frac{1.2 \times 10^{-6}\,\mathrm{m}}{s}$$
- Reynolds number
$$\mathrm{Re} = \frac{100 \cdot 10}{1.2 \times 10^{-6}} \approx 833000000$$
- If the Reynolds number is large we nned very fine meshes
$$h < \sqrt{\frac{1}{\mathrm{Re}}}$$
- This is not possible in reality:
$$h < \sqrt{\frac{1}{833000000}} \approx 0.0000035$$
- If the domain is $\Omega = (\frac{1}{0.000035})^2 \approx 24041828902976$ elemtns ($M = 28000$)
- This is too much, We must stabilize:
  - Artificial diffusion? Stable but too much diffusion
  - Streamline diffusion? Stable and good accuracy