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Lima, Oct. 31, 2019

Spring School on the Introduction on Numerical Modelling of Differential Equations – Programming Exercise 7

Exercise 7.1 [Poisson's equation – P2 elements]

(a) Implement P_2 finite elements to solve the problem from Exercise 6. Recapitulate quadratic shape functions for yourself by hand first.

(b) Go into the code `fem1d_quadratic.cc` and implement the necessary modifications.

(c) Implement a numerical quadrature rule in order to evaluate the integrals locally.

(d) Check your code using your 'physical intuition'. This means, does the code deliver results that are 'similar' to those from yesterday?
Hint: On purpose we do not perform a rigorous computational convergence analysis in this exercise because in 1D the finite element method is actually 'too simple' and would yield for point-wise errors exactly zero.

Remarks on quadratic elements:
First we define the discrete space

$$V_h = \{v \in C[0,1] |\ v|_{K_j} \in P_2\}.$$

The space V_h is composed by the basis functions:

$$V_h = \{\phi_0, \dots, \phi_{n+1}, \phi_{\frac{1}{2}}, \dots, \phi_{n+\frac{1}{2}}\}.$$

The dimension of this space is $\dim(V_h) = 2n + 1$. The mid-points represent degrees of freedom as the two edge points. For instance on each $K_j = [x_j, x_{j+1}]$ we have as well $x_{j+\frac{1}{2}} = x_j + \frac{h}{2}$, where $h = x_{j+1} - x_j$.
On the element $K^{(1)}$ (unit element), we have

$$\begin{aligned}
\phi_0(\xi) &= 1 - 3\xi + 2\xi^2,\\
\phi_{\frac{1}{2}}(\xi) &= 4\xi - 4\xi^2,\\
\phi_1(\xi) &= -\xi + 2\xi^2.
\end{aligned}$$

These basis functions fulfill the property $\phi_i(\xi_j) = \delta_{ij}$, for $i, j = 0, \frac{1}{2}, 1$. On the master element, a function has therefore the representation

$$u(\xi) = \sum_{j=0}^{1} u_j \phi_j(\xi) + u_{\frac{1}{2}} \phi_{\frac{1}{2}}(\xi).$$

Using these three shape functions we can now evaluate

$$A_{i,j} = \int_0^1 \phi_i' \phi_j' \, dx$$

and

$$b_j = \int_0^1 (-a) \phi_j \, dx$$

with the Simpson rule to obtain the *local* stiffness matrix

$$A = \frac{1}{h} \begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}$$

and the *local* right hand side

$$b = \frac{h}{6}(-a, -4a - a)^T$$

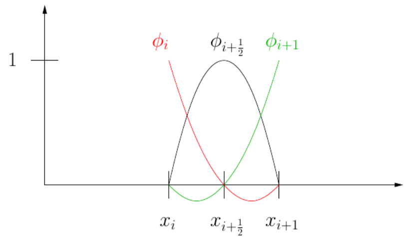

Figure 1: Example for quadratic elements.