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Problem Set 6 – Gascoigne Workshop
Summer Term 2013

Stabilization of convection-diffusion equations

The basic equation for convection-diffusion processes is

$$\begin{aligned} -\epsilon\Delta u + \beta \cdot \nabla u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

Hereby, $\beta \in \mathbb{R}^2$ is the transport direction and $\epsilon \in \mathbb{R}$ the diffusion coefficient.

Problem 6.1:

At first, we implement the above problem for

$$\beta = (1,1), \quad g = 0, \quad \text{and} \quad \epsilon = 1.$$

Therefore, modify all necessary classes in Gascoigne. You can control your implementation by choosing a known solution.

Hint: Consider a function with homogeneous Dirichlet boundary conditions. You can also implement the exact solution in the class `MyExactSolution` and compare the error development under mesh refinement to find out if your program is working properly (Section *"Exact Solution and Evaluation of Errors"*).

Problem 6.2:

Set $f = 1$ for the right-hand side and test different values for

(i) the direction β,

(ii) the diffusion coefficient ϵ.

What can you observe? Now fix the value $\beta = (1,1)$ and solve the above problem for $\epsilon = 1$, $\epsilon = 0.01$, $\epsilon = 0.001$, and $\epsilon = 0.0001$. Save the corresponding results and compare them to each other.

Problem 6.3:

In this exercise, we add the so-called "artificial diffusion" term

$$s_{AD}(u, \varphi) = \alpha(\nabla u, \nabla \varphi),$$

to the weak formulation of the convection-diffusion equation. The parameter α is chosen as ratio between the mesh-size h and the diffusion parameter h^2/ϵ.

Implement this term and observe the behavior of the solution for different values of ϵ.

Hint: The cell size parameter h is accesible in the function `point` of the class `Equation`. See the script for details.

Problem 6.4:

Adding artificial diffusion smears out the original solution behavior. This can be avoided by adding add the diffusion only in the streamline direction. Therefore, the following terms must be added to the weak formulation

$$s_{SD}(u, \varphi) = \alpha \left(((\beta \cdot \nabla)u, (\beta \cdot \nabla)\varphi) - (f, (\beta \cdot \nabla)\varphi) \right),$$

with

$$\alpha = \min \left(\frac{h^2}{\epsilon}, \frac{h}{\|\beta\|_2} \right).$$

How does the solution change for different (small) diffusion parameter values?

Problem 6.5: (*optional*)

Compare the computations of the last three exercises to each other and explain the results. What are the benefits and the drawbacks of the different approaches. Compare the suggested *streamline diffusion* in **Problem 6.4** with the *streamline diffusion* methods suggested in the literature[1]. What are the differences? What happens when you omit the right-hand side term?

[1] e.g., H.-G. Roos, M. Stynes, L. Tobiska: *Numerical Methods for Singularly Pertubed Differential Equations.* 1996, Springer.