Finite Elements

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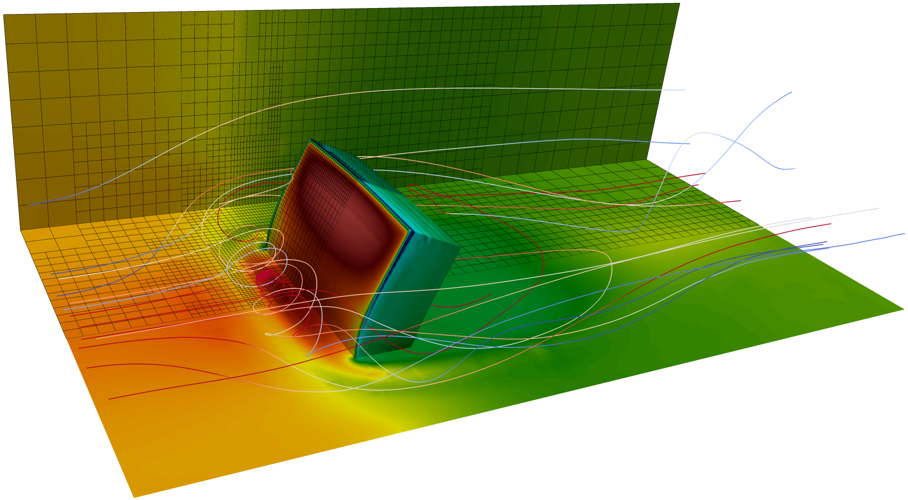
Agenda

1. The Stokes Equation
   - Models for flow problems
   - The Stokes equations
   - Finite Elements for the Stokes equation
2. The Navier-Stokes Equation
   - The Reynolds number - transport problem
   - Nonlinear problems

Computational fluid dynamics



- Flow around an obstacle
- The curves describe the path of particles

$M, \quad h = \frac{1}{M}$

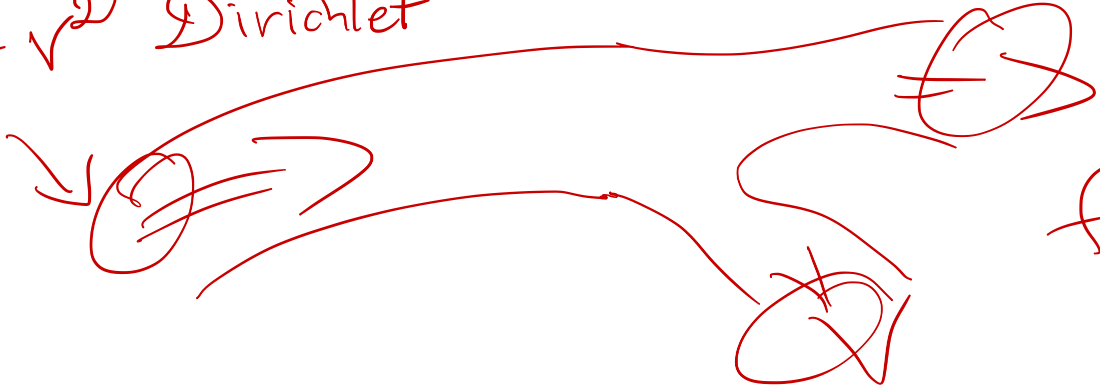
The Navier-Stokes equations

- Let Ω be a domain in 2d or 3d
- We find the velocity vector $\mathbf{v} : \Omega \to \mathbb{R}^d$

$$
\begin{aligned}
\rho\big(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}\big) - \operatorname{div} \boldsymbol{\sigma} &= \rho \mathbf{f} \\
\operatorname{div} \mathbf{v} &= 0
\end{aligned}
$$

- ρ is the density of the fluid
- $\mathbf{f}$ is a force acting on the fluid
- $\boldsymbol{\sigma}$ is the *stress tensor* describing the forces within the fluid (friction, pressure)

$v = v^D$ Dirichlet



$f = 0$

$$\boxed{\rho\big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\big)} - \operatorname{div}\boldsymbol{\sigma} = \rho\mathbf{f}$$
$$\operatorname{div}\mathbf{v} = 0$$

- This term describes the inertia of the fluid in motion
- It takes a force to influence a moving mass
- It is a *reaction-transport* problem

$+$

$$\rho\big(\partial_t \mathbf{v}^1 \mathbf{v}^1\partial_x\mathbf{v}^1 + \mathbf{v}^2\partial_y\mathbf{v}^1\big) + \cdots$$
$$\rho\big(\partial_t \mathbf{v}^2 \mathbf{v}^1\partial_x\mathbf{v}^2 + \mathbf{v}^2\partial_y\mathbf{v}^2\big) + \cdots$$

$+$

$$\rho\big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) - \boxed{\operatorname{div}\,\boldsymbol{\sigma}} = \rho\mathbf{f}$$
$$\operatorname{div}\,\mathbf{v} = 0$$

- This term describes the internal foces within the fluid
- The Navier-Stokes stress tensor is given by

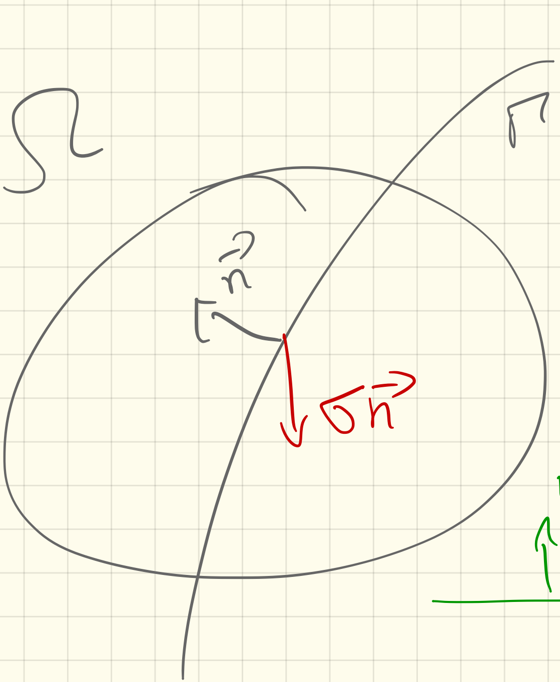
$$\boldsymbol{\sigma} = \rho\nu\big(\nabla\mathbf{v} + \nabla\mathbf{v}^T\big) - pI$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

- ν is the *viscosity* of the fluid
- p is the pressure

$$\nabla v = \begin{pmatrix} \partial_x v^1 & \partial_y v^1 & \partial_z v^1 \\ \partial_x v^2 & \partial_y v^2 & \partial_z v^2 \\ \partial_x v^3 & \partial_y v^3 & \partial_z v^3 \end{pmatrix}$$

$$\sigma = \rho\nu \begin{pmatrix} 2\partial_x v^1 & \partial_y v^1 + \partial_x v^2 & \partial_z v^1 + \partial_x v^3 \\ \partial_y v^1 + \partial_x v^2 & \partial_y v^2 & \partial_z v^2 + \partial_y v^3 \\ \partial_z v^1 + \partial_x v^3 & \partial_z v^2 + \partial_y v^3 & 2\partial_z v^3 \end{pmatrix} - \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

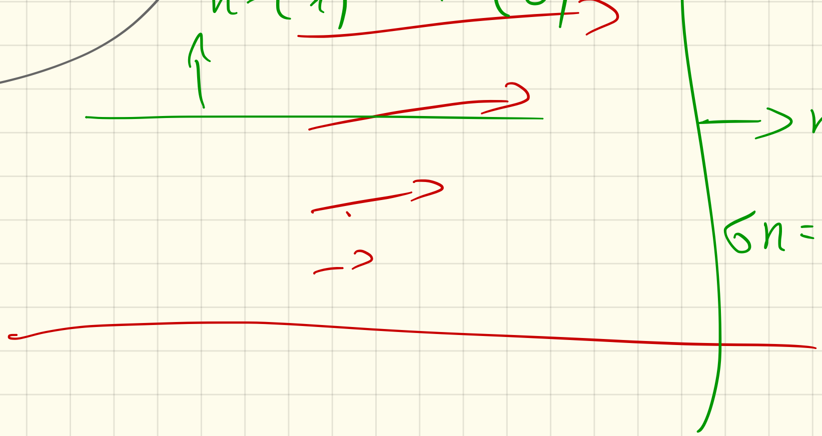


$$v = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

$$Tv = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \sigma n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

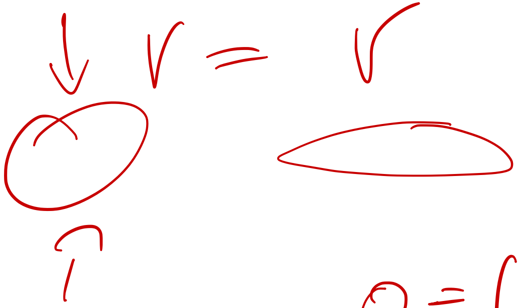


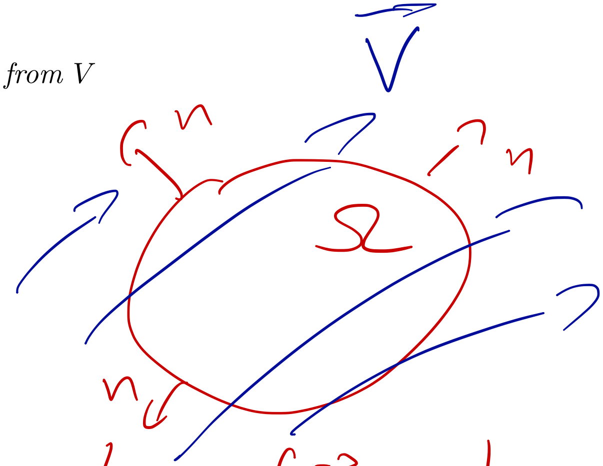
$$n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sigma n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\rho\big(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\big) - \operatorname{div}\boldsymbol{\sigma} = \rho\mathbf{f}$$
$$\boxed{\operatorname{div}\mathbf{v}} = 0$$

- This term describes the *incompressibility* of the fluid: it is not possible to change the volume
- Not every fluid is incompressible: water is incompressible but air can be compressed if the force is big
- Meaning of incompressibility
  If V is a volume then it holds: inflow to V equals outflow from V





$$0 = \int_\Omega \operatorname{div} v \, dx = \int_{\partial\Omega} \vec{n} \cdot v \, dx$$

Inflow = Outflow

- We skip the *inertia term*

$$-\nu\Delta\mathbf{v} + \nabla p = \rho\mathbf{f}$$
$$\text{div } \mathbf{v} = 0$$

- In 2d:

$$-\nu\Delta\mathbf{v}^1 + \partial_x p = \rho\mathbf{f}^1$$
$$-\nu\Delta\mathbf{v}^2 + \partial_y p = \rho\mathbf{f}^2$$
$$\partial_x\mathbf{v}^1 + \partial_y\mathbf{v}^2 = 0$$

  3 equations and 3 unknowns $(\mathbf{v}^1, \mathbf{v}^2, p)$

$$-\nu\Delta\mathbf{v} + \nabla p = \rho\mathbf{f}$$
$$\text{div } \mathbf{v} = 0$$

- Multiplication with test-functions $\phi = (\phi^1, \phi^2)$ and ξ and integration

$$\int_\Omega \big(-\nu\Delta\mathbf{v}^1 + \partial_x p\big)\phi^1 \, \mathrm{d}x = \int_\Omega \rho\mathbf{f}^1\phi^1 \, \mathrm{d}x$$
$$\int_\Omega \big(-\nu\Delta\mathbf{v}^2 + \partial_y p\big)\phi^2 \mathrm{d}x = \int_\Omega \rho\mathbf{f}^2\phi^2 \, \mathrm{d}x$$
$$\int_\Omega \big(\partial_x\mathbf{v}^1 + \partial_y\mathbf{v}^2\big)\xi \, \mathrm{d}x = 0$$

- Or, in shorter notation

$$-(\nu\Delta\mathbf{v}, \phi) + (\nabla p, \phi) = (\rho\mathbf{f}, \phi)$$
$$(\text{div } \mathbf{v}, \xi) = 0$$

- Integration by parts

$$(\nu\nabla\mathbf{v}, \nabla\phi) - (p, \text{div } \phi) = (\rho\mathbf{f}, \phi)$$
$$(\text{div } \mathbf{v}, \xi) = 0$$

$v = 0$ on $\partial\Omega$

- The theory of the Stokes equations is very difficult (compared to Laplace)
- But, we can simplify the problem:
  - If $(\mathbf{v}, p)$ is a solution to the Stokes equations its divergence is zero

$$\operatorname{div}\, \mathbf{v} = 0$$

  - We define the space of divergence free functions

$$\mathcal{V}_0 := \{\phi \in H_0^1(\Omega)^d \mid \operatorname{div}\, \cancel{\mathbf{v}}\,\varphi = 0\}$$

    Every solution $\mathbf{v}$ is in this space $\mathbf{v} \in \mathcal{V}_0$
  - Now, we restrict the variational formulation to this space

$$\mathbf{v} \in \mathcal{V}_0 : \qquad \begin{array}{c} (\nu\nabla\mathbf{v}, \nabla\phi) - (p, \overbrace{\operatorname{div}\, \phi}^{0}) = (\rho\mathbf{f}, \phi) \\ (\underbrace{\operatorname{div}\, \mathbf{v}}_{=0}, \xi) = 0 \end{array} \quad \forall \phi \in \mathcal{V}_0$$

  - The velocity is defined by a Laplace problem

$$\mathbf{v} \in \mathcal{V}_0 : \qquad (\nu\nabla\mathbf{v}, \nabla\phi) = \rho(\mathbf{f}, \phi) \quad \forall \phi \in \mathcal{V}_0$$

- If we have the velocity, the pressure is defined by the problem

$$p \in L^2(\Omega) \qquad (p, \mathrm{div}\ \phi) = (\nu \nabla \mathbf{v}, \nabla \phi) - (\rho \mathbf{f}, \phi) \quad \forall \phi \in H_0^1(\Omega)^2$$

Theorem: There exists a unique velocity $\mathbf{v} \in \mathcal{V}_0$ and a unique pressure $p \in L^2(\Omega)$ and the *inf-sup-condition* holds

$$\inf_{p \in L^2(\Omega)} \sup_{\mathbf{v} \in H_0^1(\Omega)^2} \frac{(p, \mathrm{div}\ \mathbf{v})}{\|p\| \cdot \|\nabla \mathbf{v}\|} \geq \gamma.$$

with the *inf-sup constant* $\gamma > 0$

Babuska-Brezzi-Condition

$-\Delta v + \nabla p = f$ Step 1 Existence of v

Step 2 pressure $\nabla p = f + \Delta v$

Difficult, as pressure in L^2, no derivative

Equivalent

(1) inf-sup $\quad \gamma \leq \inf \sup \dfrac{(p, \nabla \cdot \varphi)}{\|p\| \cdot \|\nabla \varphi\|}$

(2) the gradient : $-\nabla : L^2 \to V^0$ is surjective

(3) There exists a unique pressure

- We discretize the variational formulation

$$
\begin{aligned}
(\nu\nabla\mathbf{v},\nabla\phi)-(p,\operatorname{div}\,\phi)=(\rho\mathbf{f},\phi) \quad & \forall\phi\in H_0^1(\Omega)^d\\
(\operatorname{div}\,\mathbf{v},\xi)=0 \quad & \forall\xi\in L^2(\Omega)
\end{aligned}
$$

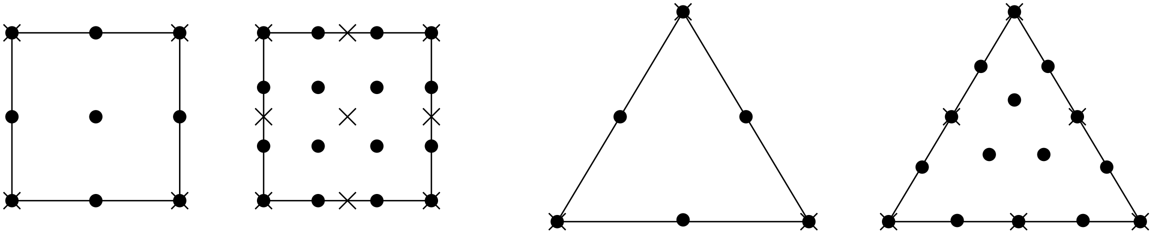
- Step 1: discretize the domain Ω by a mesh Ω_h
- Step 2: discretize the function spaces

$$V_h\subset H_0^1(\Omega)^d,\quad L_h\subset L^2(\Omega)$$

Step 3

$$(\nu\nabla v_h,\nabla\varphi_h)-(p_h,\operatorname{div}\varphi_h)=(f,\varphi_h)$$

$$(\operatorname{div} v_h,\xi_h)=0$$



- The *finite element pair* $V_h \times Q_h$ must satisfy the *discrete inf-sup condition*

$$\inf_{p_h \in L_h} \sup_{\mathbf{v}_h \in V_h} \frac{(p_h, \operatorname{div} \mathbf{v}_h)}{\|p_h\| \cdot \|\nabla \mathbf{v}_h\|} \geq \gamma_h.$$

with $\gamma_h \geq \gamma_0 > 0$.

Taylor-Hood : Velocity : Quadratic FE
Pressure : Linear FE

- Basis of V_h and L_h

$$V_h = < \phi_h^1, \phi_h^2, \ldots, \phi_h^N >$$
$$L_h = < \xi_h^1, \xi_h^2, \ldots, \xi_h^M >$$

- The solution $\mathbf{v}_h \in V$ and $p_h \in L_h$

$$\mathbf{v}_h = \sum_{i=1}^{N} \mathbf{v}_i \phi_h^i$$
$$p_h = \sum_{i=1}^{M} p_i \xi_h^i$$

- Usually

$$M \neq N$$

- Finite element solution

$$\sum_{j=1}^{N} \nu(\nabla\phi_h^j, \nabla\phi_h^i)\mathbf{v}_j - \sum_{j=1}^{M}(\xi_h^j, \nabla\cdot\phi_h^i)\mathbf{p}_j = \mathbf{f}_i := (\rho\mathbf{f}, \phi_h^i), \quad i = 1,\dots,N$$
$$\sum_{j=1}^{N}(\nabla\cdot\phi_h^j, \xi_h^i)\mathbf{v}_j = 0, \qquad i = 1,\dots,M$$

- Matrix

$$\mathbf{A}_{ij} = (\nabla\phi_h^j, \nabla\phi_h^i), \quad i \; = 1,\dots,N, j \; = 1,\dots,N$$
$$\mathbf{B}_{ij} = (\nabla\cdot\phi_h^j, \xi_h^i), \quad i \; = 1,\dots,M, j \; = 1,\dots,N$$

- Linear system of equations

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}$$

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2. The Navier-Stokes Equation
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- Stationary Navier-Stokes equations

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \rho\nu\Delta\mathbf{v} \overset{+\nabla p}{} = \rho\mathbf{f}$$
$$\operatorname{div}\,\mathbf{v} = 0$$

- Divide by $\rho\nu$

$$\frac{1}{\nu}(\mathbf{v}\cdot\nabla)\mathbf{v} - \Delta\mathbf{v} \overset{+\nabla p}{} = \frac{1}{\nu}\mathbf{f}$$
$$\operatorname{div}\,\mathbf{v} = 0$$

- The *Reynolds number* describes the character of the flow

$$Re = \frac{V\cdot L}{\nu},$$

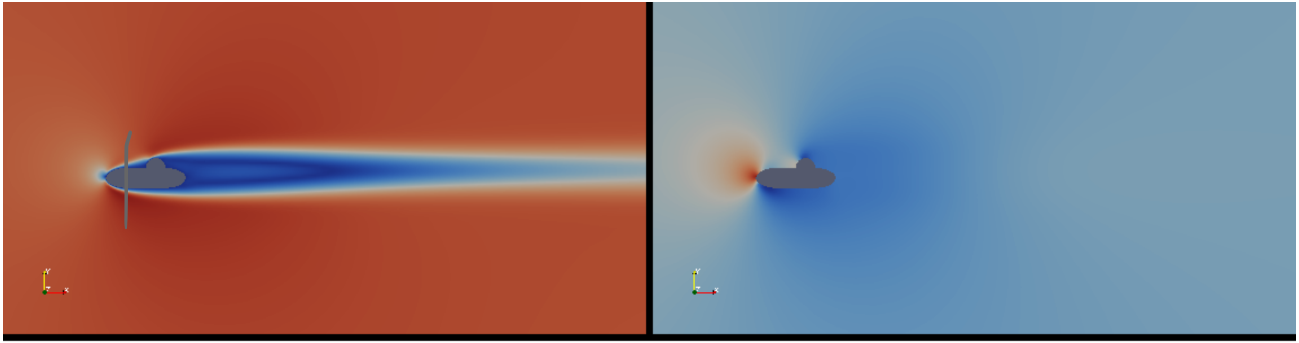
where V is the typical velocity and L the typical size of the domain

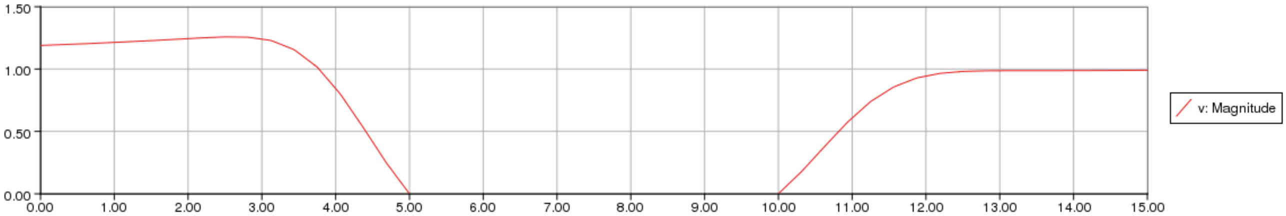
Stokes equation is good if $\nu \to \infty$

Water $\nu \approx 10^{-6}$

Air $\nu \approx 10^{-5}$

Reynolds number $Re = 100$





$$Re = \frac{v \cdot L}{\nu}$$

$$100 = \frac{v \cdot 10m}{1 \cdot 10^{-6}} = 10^{5} \cdot v$$

$$\Leftrightarrow \quad v = 10^{-7} \frac{m}{s}$$

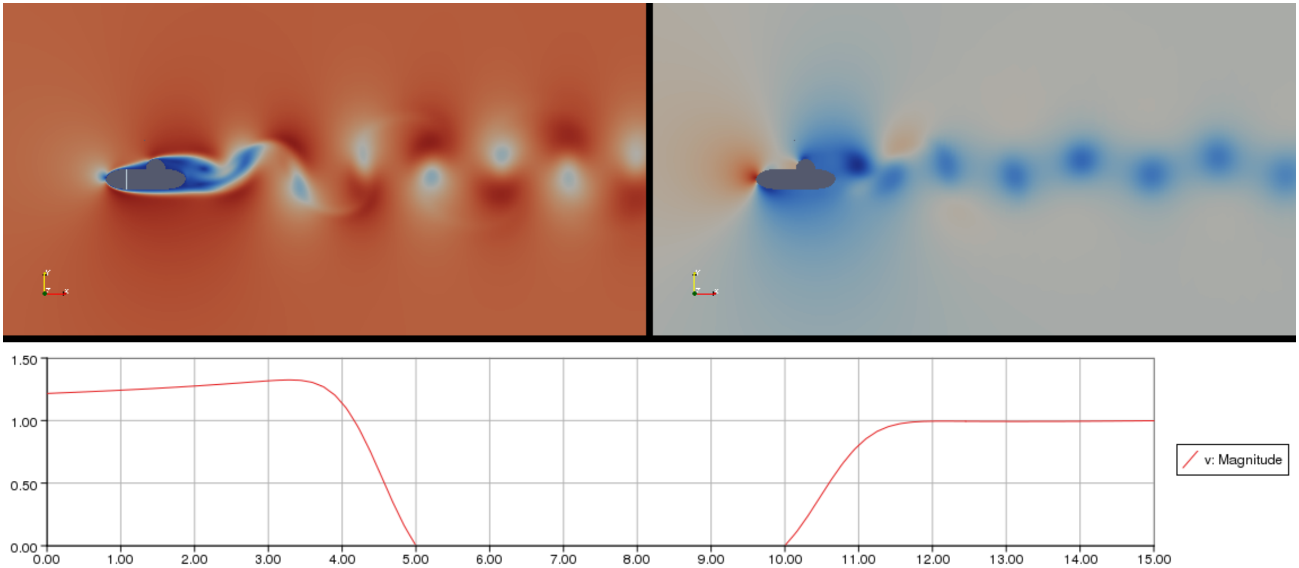
Fast Submarine

$L = 10\,m$

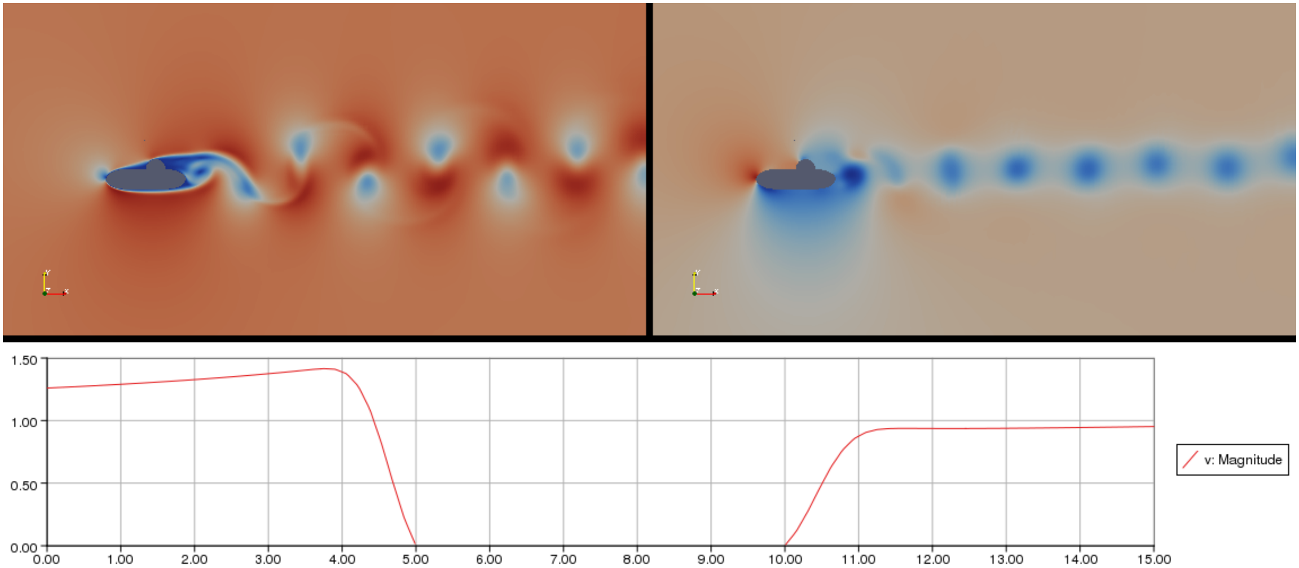
$V = 10\,\frac{m}{s}$

$$Re = \frac{10\,m \cdot 10\,\frac{m}{s}}{10^{-6}\,\frac{m^2}{s}} = \underline{\underline{10^8}}$$

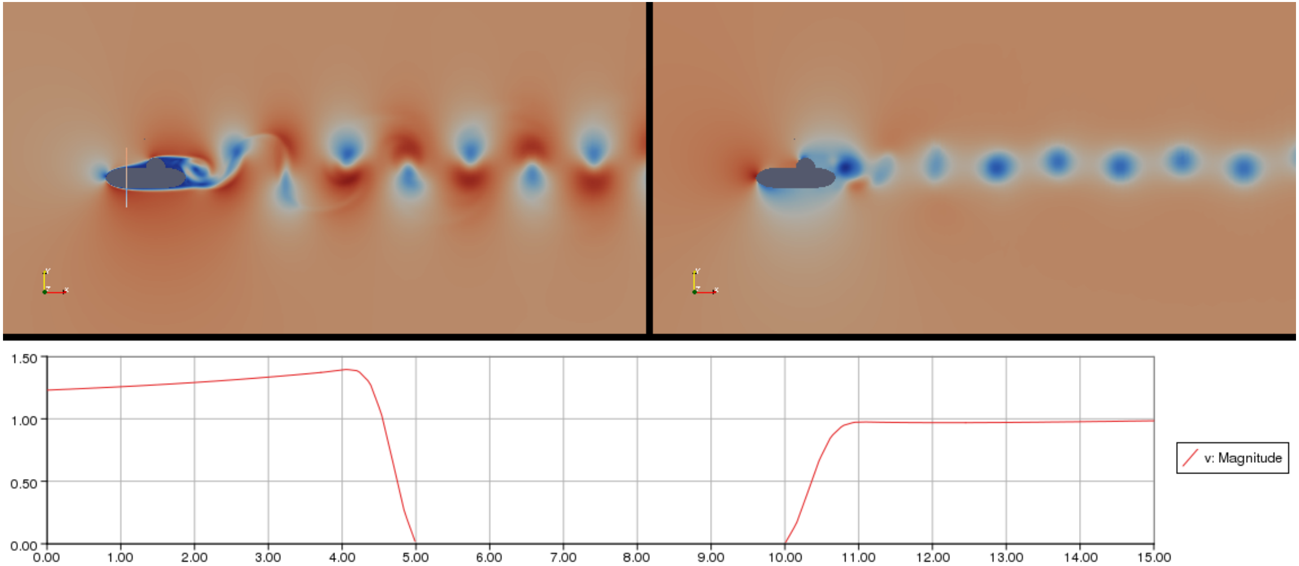
Reynolds number $Re = 200$



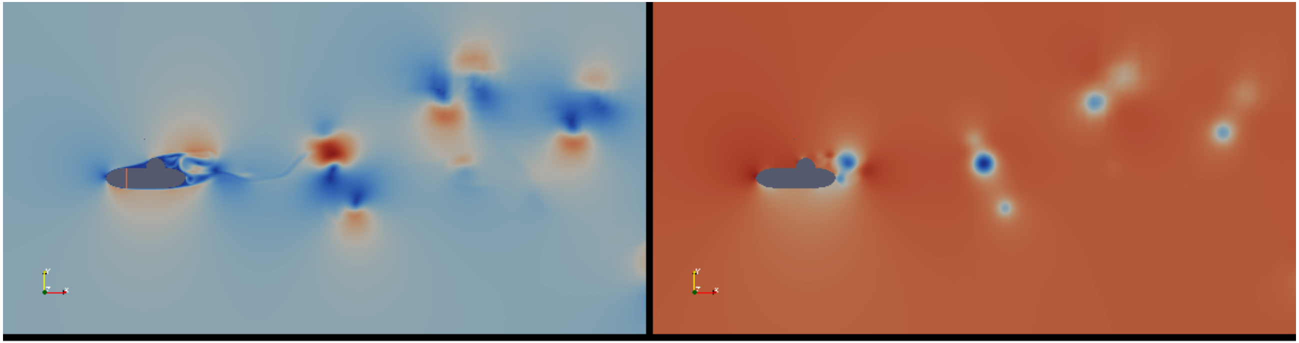
Reynolds number $Re = 400$

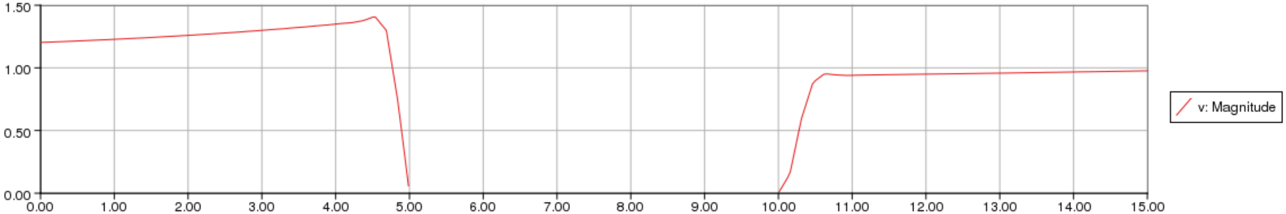


Reynolds number $Re = 800$

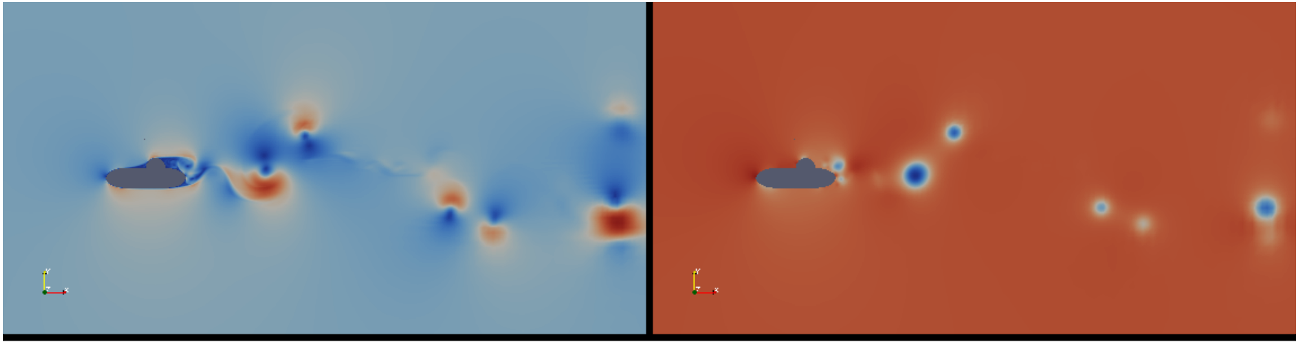


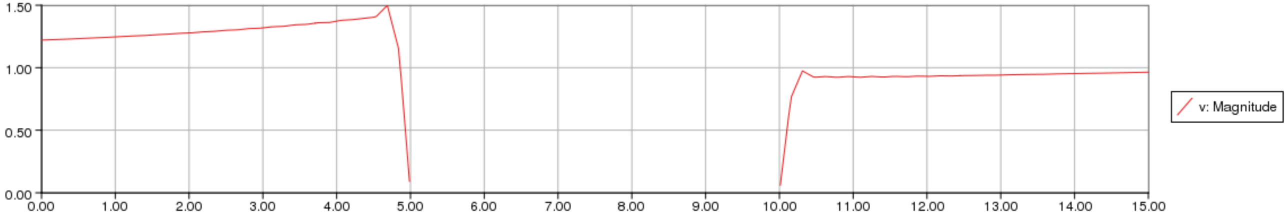
Reynolds number $Re = 3200$

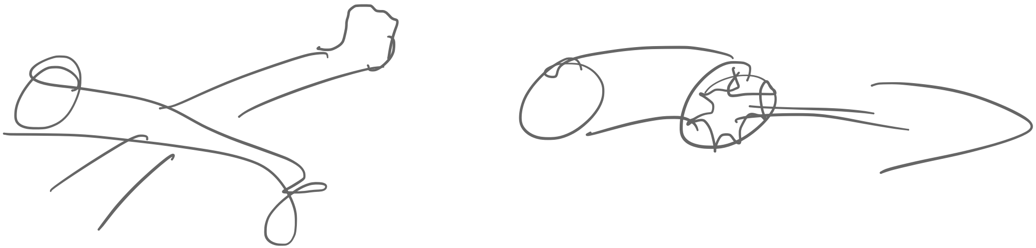




Reynolds number $Re = 12800$







$$Re = \frac{V \cdot L}{\nu}$$

- Length of submarine

$$L = 100m$$

- Velocity of submarine

$$V = \frac{10m}{s}$$

- Viscosity of water

$$\nu = \frac{1.2 \cdot 10^{-6} m^2}{s}$$

- Reynolds number

$$Re = \frac{100 \cdot 10}{1.2 \cdot 10^{-6}} \approx 833\,000\,000$$

$L_{model} = 2\,m$

$$\frac{L \cdot V}{\nu} = \frac{L_{model} \cdot V_{model}}{\nu}$$

$$V_{model} = \frac{V \cdot L}{L_{model}}$$

$$= \frac{10\,\frac{m}{s} \cdot 100m}{2\,m}$$

$$= 500\,\frac{m}{s}$$

We say

- If $Re < 50$ the flow is *stationary*
- If $50 < Re < 500$ the flow is *non-stationary* and *laminar*
- If $Re > 500$ the flow is *turbulent*

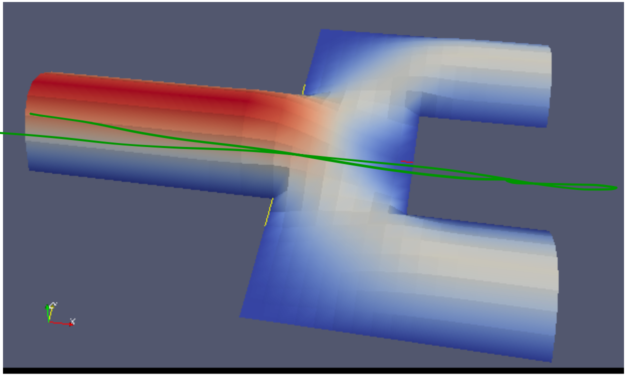
- If the Reynolds number is large $Re \gg 1$ the problem is *transport dominated*

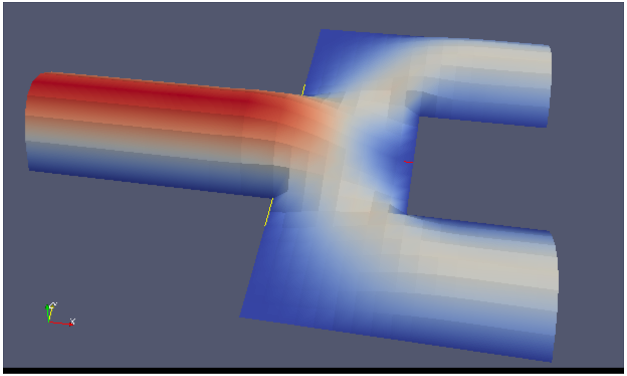
$$\begin{aligned} Re(\mathbf{v}\cdot\nabla)\mathbf{v} - \Delta\mathbf{v} &= Re\cdot\mathbf{f} \\ \operatorname{div}\,\mathbf{v} &= 0 \end{aligned} \quad\Leftrightarrow\quad \begin{aligned} (\mathbf{v}\cdot\nabla)\mathbf{v} - \frac{1}{Re}\Delta\mathbf{v} &= \mathbf{f} \\ \operatorname{div}\,\mathbf{v} &= 0 \end{aligned}$$

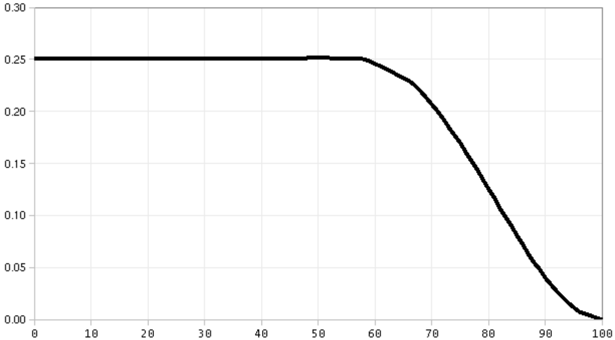
- Similar to the diffusion-transport problem (yesterday)
- We expect problems with *oscillations*

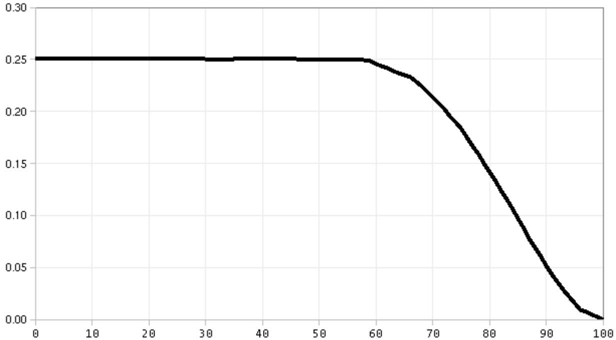
High Reynolds numbers

$Re = 100$ **and** $Re = 200$

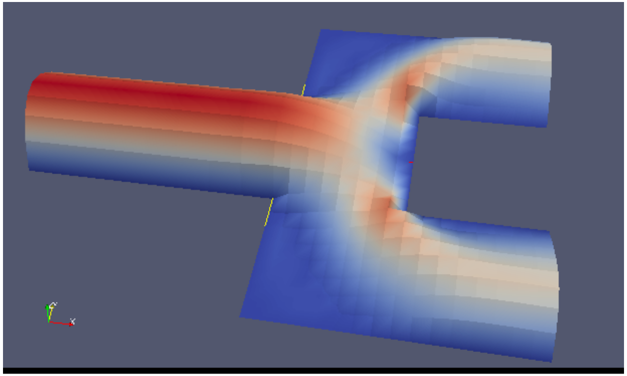


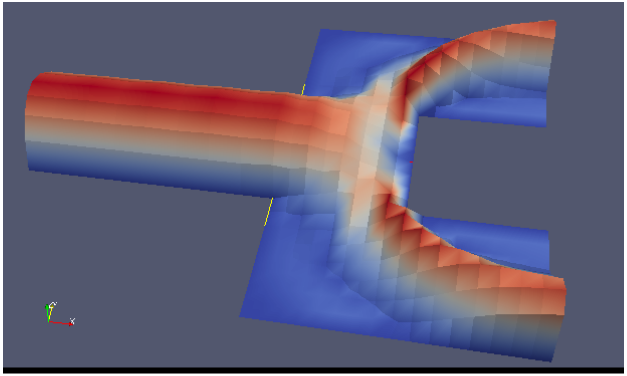


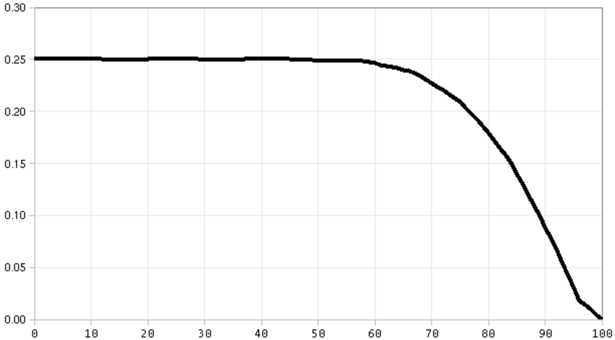


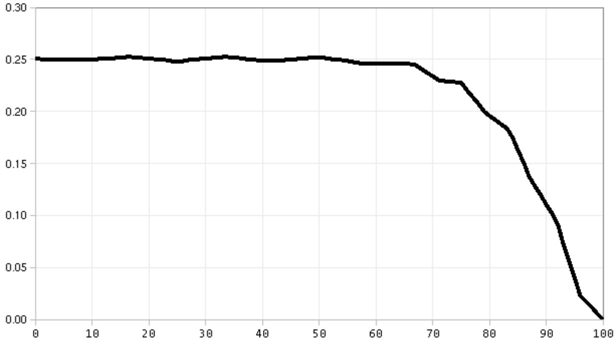


$Re = 400$ and $Re = 800$

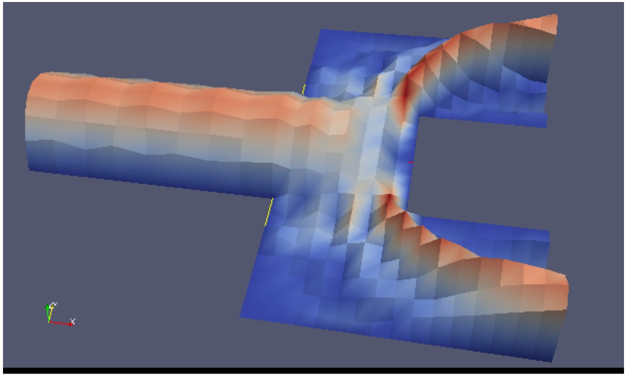


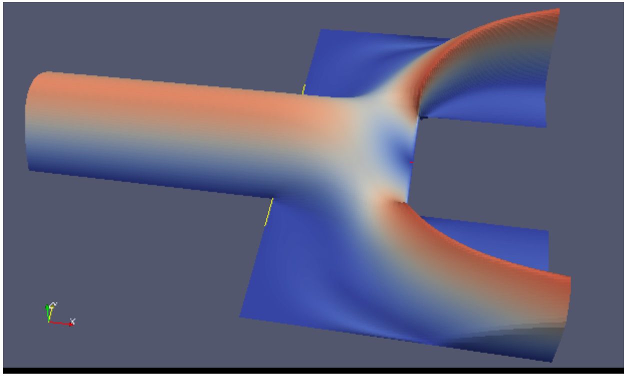


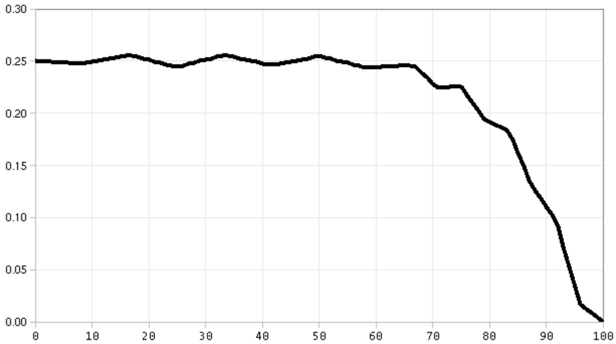


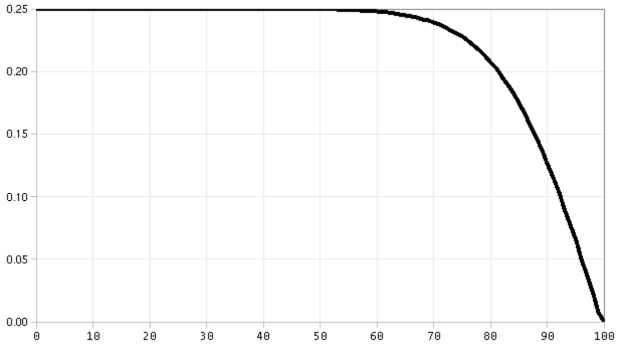


$Re = 800$ - fine mesh









- If the Reynolds number is large we need very fine meshes

$$h < \sqrt{\frac{1}{Re}}$$

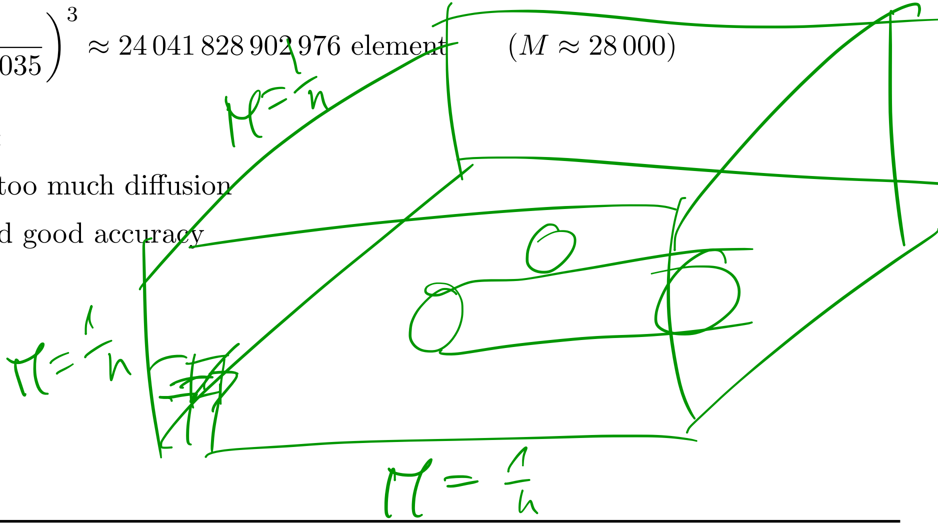
- This is not possible in reality:

$$h < \sqrt{\frac{1}{833\,000\,000}} \approx 0.000035$$

- If the domain is $\Omega = (0,1)^3$ we need

$$N = M^3 = \left(\frac{1}{0.000035}\right)^3 \approx 24\,041\,828\,902\,976 \text{ element} \qquad (M \approx 28\,000)$$

- This is too much. We must stabilize:
  - Artificial diffusion? Stable but too much diffusion
  - Streamline diffusion? Stable and good accuracy



$N = M^3$

- The Navier-Stokes equation is nonlinear

$$
\begin{aligned}
\rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \operatorname{div} \boldsymbol{\sigma} &= \rho \mathbf{f} \\
\operatorname{div} \mathbf{v} &= 0
\end{aligned}
$$

- Discretization of the nonlinear term

$$
\mathbf{v}_h = \sum_{j=1}^{N} \mathbf{v}_i \phi_h^i
$$

Insert in equation

$$
((\mathbf{v}_h \cdot \nabla)\mathbf{v}_h, \phi_h^i) = \left(\left(\sum_{j=1}^{N} \mathbf{v}_j \phi_h^j \cdot \nabla \right) \left(\sum_{k=1}^{N} \mathbf{v}_k \phi_h^k \right), \phi_h^i \right) = \sum_{j=1}^{N} \sum_{k=1}^{N} \left((\phi_h^j \cdot \nabla)\phi_h^k, \phi_h^i \right) \mathbf{v}_j \mathbf{v}_k
$$

- The nonlinearity is no matrix, but a *tensor*

$$
N_{ijk} = \left((\phi_h^j \cdot \nabla)\phi_h^k, \phi_h^i \right), \quad i, j, k = 1, \ldots, N.
$$

- If N is large, we cannot store N. Too much memory!

- Iteration: Initial guess $\mathbf{v}^0$. Iterate $\mathbf{v}^l \mapsto \mathbf{v}^{l+1}$.
- **Stokes iteration**

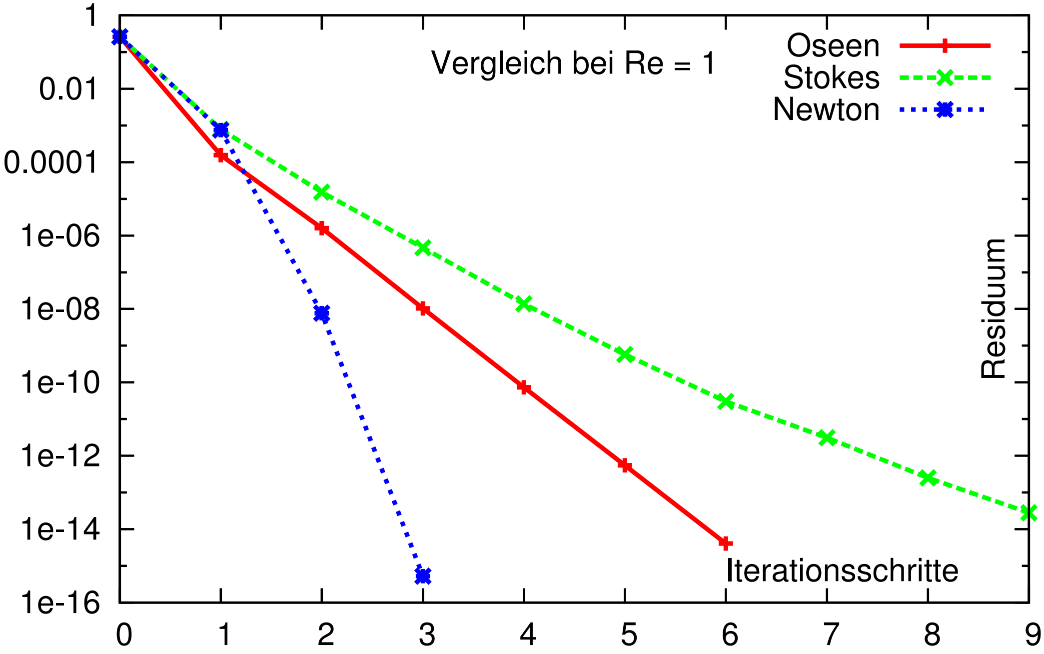
$$\begin{aligned} \rho\nu\Delta\mathbf{v}^{l+1} - \nabla p^{l+1} &= \rho\mathbf{f} - \rho(\mathbf{v}^l\cdot\nabla)\mathbf{v}^l \\ \operatorname{div}\,\mathbf{v}^{l+1} &= 0 \end{aligned}$$

- **Oseen iteration**

$$\begin{aligned} \rho(\mathbf{v}^l\cdot\nabla)\mathbf{v}^{l+1} + \rho\nu\Delta\mathbf{v}^{l+1} - \nabla p^{l+1} &= \rho\mathbf{f} \\ \operatorname{div}\,\mathbf{v}^{l+1} &= 0 \end{aligned}$$

- **Newton iteration**

$$\begin{aligned} \rho(\mathbf{v}^l\cdot\nabla)\mathbf{w}^{l+1} + \rho(\mathbf{w}^{l+1}\cdot\nabla)\mathbf{v}^l + \rho\nu\Delta\mathbf{w}^{l+1} - \nabla q^{l+1} &= \rho\mathbf{f} \\ \operatorname{div}\,\mathbf{w}^{l+1} &= 0 \end{aligned}, \qquad \mathbf{v}^{l+1} = \mathbf{v}^l + \mathbf{w}^{l+1}, \quad p^{l+1} = p^l + q^l$$


Vergleich bei Re = 1
Oseen
Stokes
Newton
Residuum
Iterationsschritte
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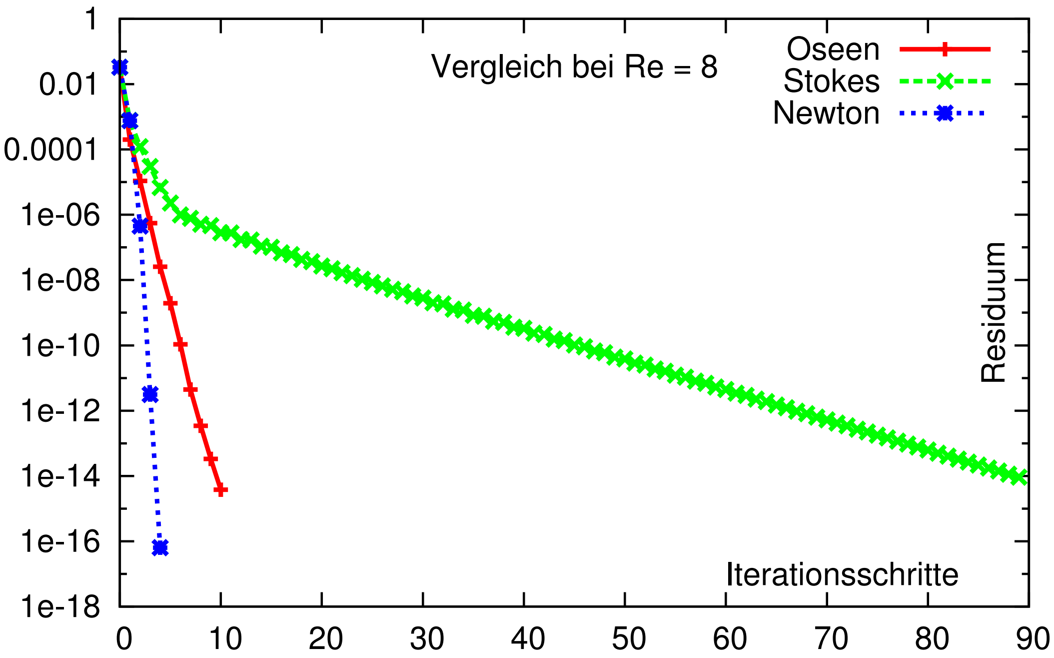
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
1e-14
1e-16
0
1
2
3
4
5
6
7
8
9


Vergleich bei Re = 8
Oseen
Stokes
Newton
Residuum
Iterationsschritte
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0.01
0.0001
1e-06
1e-08
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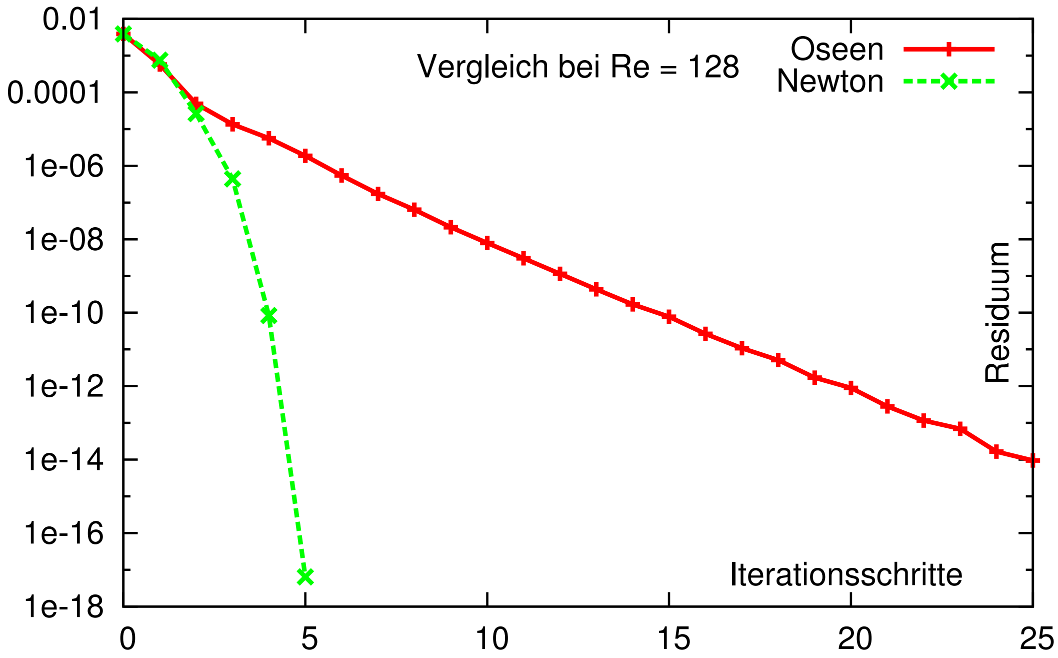
10
20
30
40
50
60
70
80
90


Vergleich bei Re = 128
Oseen
Newton
Residuum
Iterationsschritte
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1e-12
1e-14
1e-16
1e-18
0
5
10
15
20
25

Thank You!