0.1 The Navier-Stokes equation

$$\rho\left(\partial_t v + (v \cdot \nabla v)\right) - \operatorname{div} = \rho f \quad \operatorname{div} v = 0.$$

- This term describes the inertia of the fluid in motion.
- It takes a force to influence.
- This term describes the *incompresibility* of the fluid, it is not possible to change the volume.
- Not every fluid is incompressible, water is incompressible buy air can be compressed if the force is big.
- Meaning of incompresiblility. If V is a vvolume then it holds, inflow to V equals to outflow from V.
- The term $\operatorname{div}\sigma$ describes the internal forces within the fluid. (friction)
- The Navier–Stokes stress tensor is given by

$$\sigma = \rho\nu\left(\nabla v + \nabla v^T\right) - pI$$

- ν is the viscosity of the fluid.
- ρ is the pressure.
- We skip the *inert term*

$$-\nu\Delta v + \nabla p = \rho f \quad \operatorname{div} v = 0.$$

- In $2D$:

$$-\nu\Delta v^1 +$$

  $3d$ exists 4 equations.

$$-\nu\Delta v + \nabla p = \rho f \quad \operatorname{div} v = 0.$$

- Multiplication with test functions $\phi = \left(\phi^1, \phi^2\right)$ and ξ and integration

$$\int_\Omega \left(-\nu\Delta v^1 + \partial_t p\right)\phi^1 \mathrm{d}x = \int_\Omega \rho f^1 \phi^1 \mathrm{d}x.$$

- The theory of the Stokes equations is very difficult (compared to Lplace)

- But, we can simplift the problem

  If (v, p) is a solution to the Stokes equation its divergence is zero.

  $$\operatorname{div} v = 0.$$

  We define the space of divergence free functions

  $$\mathcal{V}_h = \left\{ \phi \in H_1^0 (\Omega)^2 \mid \operatorname{div} \varphi = 0 \right\}.$$

  Every solution v in this space $v \in \mathcal{V}_h$.

  Now, we restrict the variational formulation to this space

  $$v \in \mathcal{V}_0 \colon \quad (\nu \nabla v, \nabla \phi) = (\rho f, \phi) \,\forall \phi \in .$$

- It we have the velocity, the pressure is defined by the problem

  $$p \in L^2 (\Omega) \quad (p, \operatorname{div} \phi) = (\nu \nabla b, \nabla \phi) - (\rho f, \phi) \quad \forall \phi \in H_0^1 (\Omega)^2 .$$

  Theorem 1. There exists a unique velocity $v \in \mathcal{V}_0$ and a unique pressure $p \in L^2 (\Omega)$ and $\inf - \sup$ *condition* holds

  $$\inf_{p \in L^2(\Omega)} \sup_{v \in H_0^1(\Omega)^2} \frac{(p, \operatorname{div} v)}{\|p\| \|\nabla v\|} \geq \gamma$$

  with the inf–sup constant $\gamma > 0$.

- We discretize the variant.

- Stationary Navier Stokes equations

  $$\rho v \cdot \nabla v - \rho \nu \Delta v + \nabla \rho = \rho f \quad \operatorname{div} v = 0.$$

- Divide by.

0.2 Reynolds number

$$R = \frac{VL}{\gamma}$$

where V is the velocity, for example:

$$100 = \frac{v \cdot 10\,\mathrm{m}}{10^{-6}} = 1 \times 10^{-5} \cdot V$$

$\iff V = 10^{-7}\,\mathrm{m\,s^{-1}}.$

$$\mathrm{Re} = \frac{V \cdot L}{\nu}.$$

- Length of submarine
$$L = 100\,\mathrm{m}.$$
- Velocity of submarine
$$V = \frac{10\,\mathrm{m}}{s}.$$
- Viscosity of eater
$$\nu = \frac{1.2 \times 10^{-6}\,\mathrm{m}}{s}.$$
- Reynolds number
$$\mathrm{Re} = \frac{100 \cdot 10}{1.2 \times 10^{-6}} \approx 833000000.$$
- If the Reynolds number is large we nned very fine meshes
$$h < \sqrt{\frac{1}{\mathrm{Re}}}.$$
- This is not possible in reality:
$$h < \sqrt{\frac{1}{833000000}} \approx 0.0000035.$$
- If the domain is $\Omega = \left(\frac{1}{0.000035}\right)^2 \approx 24041828902976$ elements $(M = 28000)$.
- This is too much, We must stabilize:
  - Artificial diffusion? Stable but too much diffusion.
  - Streamline diffusion? Stable and good accuracy.