## 0.1 Existence and uniqueness for scalar convection-diffusion problem

### 0.1.1 Continuity

### 0.1.2 Coercivity

$$A(u,u) = (\sigma u + (\beta \cdot \nabla) u, u) + \varepsilon \|\nabla u\|^2.$$

Reformulation of the convective part:

$$((\beta \cdot \nabla) u, u) = \int_\Omega \nabla u \cdot () \, \mathrm{d}u.$$

## 0.2 Where is the problem for discrete (FE) schemes?

- Lax-Milgram also applies to the discrete standard FE scheme.
- Existence and uniqueness of discrete solutions.
- Cea's lemma:

$$\|u - u_h\| + \|\nabla (u - u_h)\| \le \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V_h} (\|u - v_h\| + \|\nabla(u - v_h)\|).$$

- But:

$$2.$$

## 0.3 Standard Galerkin formulation

Standard Galerkin formulation. Seek $u_h \in V_h$ such that

$$A(u_h, \phi) = (f, \phi) \, \forall \phi \in V_h.$$

## 0.4 Instabilities of the pure Galerkin formulation

Standard Galerkin formulation: Seek $u_h \in V_h$ such that

$$A(u_h, \phi) = (f, \phi) \, \forall \phi \in V_h.$$

A priori estimate in the norm

$$\|u_h\|.$$

## 0.5 Numerical diffusion with FEM

- Similar strategies as for FD possible with FEM: Adding the term
$$(h\nabla u_h, \nabla\phi)$$
to the bilinear form $A(\cdot,\cdot)$.
- Diffusion in streamline direction only: Adding
$$(h(\beta\cdot\nabla)u_h, (\beta\cdot\nabla)\phi)$$
Only 1. order!
- Accuracy of 1. order methods can we worse than pure Galer (see Brooks & Hughs).

## 0.6 3.4 Stabilized finite elements

$$A(u,\phi) = (\sigma u + (\beta\cdot\nabla)u, \phi) + (\varepsilon\nabla u, \nabla\phi).$$

- Stabilized form:
$$u_h \in V_h : A(u_h,\phi) + S_h(u_h,\phi) = (F_h,\phi)\,\forall\phi\in V_h.$$
- Such stabilization.

## 0.7 3.4 Streamline Upwind Petrov Galerkin (SUPG)

- Add a diffusion term in direction of the streamline, i.e. $\beta$
$$((\beta\cdot\nabla)u_h, ).$$

## 0.8 SUPG for convection-diffusion problems

- Idea: Add a consistent diffusion term in direction if the flow:
$$S_h(u_h,\phi) = \sum_{T\in T_h}.$$

## 0.9 Coercivity of the SUPG stabilization form

For $P_1$ elements ($u_h$ cell-wise linear)

$$S_h(u_h,u_h) = \sum_{T\in T_h}\delta_T\left(\|(\beta\cdot\nabla)_{u_j}\|_T^2 - \|\sigma u_h\|_T^2\right)$$

with

## 0.10 Optimal choice for $\delta_t$

Local Peclet number

$$Pe_T = \frac{h_T \|\beta\|_{L=T}}{\varepsilon}.$$

## 0.11 A priori estimate for SUPG

$$\|u\|_T^2 = \sigma \|u\|^2 + \varepsilon \|\nabla u\|^2 + \sum_{T \in T_h} \delta_T \| \left(\beta \cdot \nabla\right) \|.$$

## 0.12 Proof

- Aim: Bound the discretization error by a multiple of

$$H_h\left(u\right) = \sum_{T \in T_h} a_T h_T' |u|_{H^{r+1}}\left(T\right).$$

- Splitting the error in **interpolation error**.



- Individual bounds.
- e.g. for the Galerkin terms arising in $A\left(\eta_h, \xi_h\right)$.
- With a priori bounds in $\|\eta_g\|$, $\|\nabla \eta_h\|$.
- The order term $|S_h|$.

## 0.13 Effect of SUPG stabilization

- Diffusion $\varepsilon = 0.01$, convection $\beta = \left(1, 1\right)^T$.
- The SUPG formulation stabilizes much earlier.
- Over-and undershots still remain on coarse.

## 0.14 Drawbacks of SPUG

1. Troublemsome in the parabolic case (details belows).
2. Computation of second derivatives necessary for $r \geq 2$ and $Q_1$–elements on arbitrary quadrilateral (or hexahedral) elements.
3. .

## 0.15 3.4.2 SUPG for parabolic problems

- Time-dependent convection-diffusion system.

$$\partial_t u + (\beta \cdot \nabla)\, u - \varepsilon \Delta u = f \quad \text{in } U.$$

## 0.16 SUPG with and implicit Euler

With time step $\mathcal{T} = t_n - t_{n-1}$:

$$\left(\tau^{-1} u_n, \phi\right) + ((\beta \cdot \nabla)\, u, \phi)\,.$$

$$\sum_{T \in \mathcal{T}_h} \delta_T \left(\tau^{-1} u_n + (\beta \cdot \nabla)\, u + \cdots, (\beta \cdot \nabla)\, \phi\right)_T = \left(\tau^{-1} u_{n-1}, \phi\right) + \cdots\,.$$

- .

## 0.17 SUPG for convection-diffusion problems

- Idea: Add a consistent diffusion term in direction of the flow

$$S_h\,(u_h, \phi)\,.$$