1 P_1 Finite elements on triangles - tetrahedrons

Let $V = H_0^0(\Omega)$. We consider the Poisson problem $u \in H_0^1(\Omega)$

$$-\Delta u = f \text{ in } \Omega$$

and its weak formulation

1.1 Lecture 2: Accuracy of Finite Element Discretization

Outline for today:

1. A priori error estimates in H^1

$$\|\nabla(u - u_h)\|_{L^2(h)}.$$

C dependes of the solution. One is usually interest in the error

$$\|u - u_h\|_X = ?$$

for a certain norm $\|\cdot\|_X$.

A priori error estimates: Information about the error in terms of mesh size asymptotics, e.g, for P_1 or Q_1 elements

$$\begin{aligned}\|\nabla(u - u_h)\|_{L^2(h)} &\le ch|u|_{H^2(\Omega)}\\ \|u - u_h\|_{L^2(\Omega)} &\le ch^2|u|_{H^2(\Omega)}.\end{aligned}$$

A posteriori error estimates: Information about the error in terms of u_h, e.g,

$$\|\nabla(u - u_h)\|_{L^2(\Omega)}^2 \le \sum.$$

1.2 Galerkin Orthogonality

- Continuous problem with $A\colon V \times V \to \mathbb{R}$ bilinear:

$$u \in V\colon A(u, \phi) = (f, \phi) \quad \forall \phi \in V.$$

Most simple example:

$$A(u, \phi) = (\nabla u, \nabla \phi) = \int_\Omega \nabla u \nabla \phi \mathrm{d}x.$$

- Discrete problem:

$$u_h \in V_N \colon A\left(u_h, \phi\right) = (f, \phi) \quad \forall \phi \in V_h.$$

- Discretization error

$$e_h = .$$

Theorem 1 (Cea's lemma)**.** Suppose that the bilinear form $A \colon V \times V \to \mathbb{R}$ satisfies the conditions of Lax–Milgram's theorem (continuous and V-coercive with $\alpha_1, \alpha_2 > 0$). Further, let $V_h \subseteq V$ a subspace. Then:

$$\|u - u_h\|_V \leq \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V} .$$

1.3 Continuity and coercivity

Continuity: There exists $\alpha_1 \geq 0$ such that

$$A\left(u, \phi\right) \leq \alpha_1.$$

Proof. Let $v_h \in V_h$ be arbitrary,

$$\alpha_2 \|u - u_h\|_V^2 \leq A\left(u - u_h, u - u_h\right) \quad \text{(coercivity)}$$

□

- $V = H_0^1(\Omega)$.
- associated norm

$$\|u\|_V = \|\nabla u\|_\Omega = \left(\int_\Omega |\nabla u(x)|^2 \mathrm{d}x\right)^{1/2}.$$

- $\alpha_1 = \alpha_2 = 1$

$$\|\nabla\left(u - u_h\right)\|_\Omega = \inf_{v_h \in V_h} \|\nabla\left(u - v_h\right)\|_\Omega.$$

1.4 Interpolation error

- Let $I_h \colon V \to V_h$ be an arbitrary interpolation. Then

$$\begin{aligned}\|u - u_h\|_V &\leq \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V_h} \|u - v_h\|_V \\ &\leq \frac{\alpha_1}{\alpha_2} \|u - I_h u\|_V.\end{aligned}$$

- We only need do get and idea about the interpolation error

$$\|u - I_h u\|_V.$$

- Most simple is the nodal interpolation of continuous functions

$$I_h u(N) = u(N)$$

for nodes N of the mesh.

- But: Are $H^1(\Omega)$ functions continuous?

$$d = 1 \quad (\text{yes}) \qquad d \geq 2 \quad (\text{no}) \, .$$

1.5 More regular Sobolev functions

- Higher order Sobolev spaces of order $k \geq 1$:

$$H^k(\Omega) = .$$

1.6 H^2 functions are continuous

- For $d = 1$

$$H^1(\Omega) \subseteq C(\Omega).$$

- For $d = 2$ and $d = 3$

$$H^2(\Omega) \subseteq C(\Omega).$$

- If $\partial\Omega$ is Lipschitz ....

1.7 Nodal interpolation of H^2-functions

Hence, if $u \in H^2(\Omega)$, then it holds for the Poisson pb

$$\|\nabla(u - I_h)\|.$$

1.8 Structure to address the interpolation error

1. Location:

$$\|\nabla(u - I_h u)\|_\Omega^2 = \sum_{T \in T_h} \|\nabla(u - I_h u)\|_T^2.$$

2. Transformation to the reference cell:

$$\|\nabla \left(u - I_h\right)\|.$$

1.9 Step 2: Transformation to the reference cell

How to transform an expression as ($w = u - I_h u$)

$$\|\nabla (u - I_h u)\|_T^2 = \int_T |\nabla w(x)|^2 \mathrm{d}x$$

onto the reference triangle $\hat{T}$ by an affine linear transformation

$$\phi_T(\hat{x}) = x_0 + B_T \hat{x}.$$

Partial derivative

$$\frac{\partial w(x)}{\partial x_i} = \sum_{j=1}^{d}$$

Gradient oin T:

$$|\nabla w(x)|^2 \leq \|B_T^{-t}\|_F^2 |\hat{\nabla} \hat{w}(\hat{x})|^2$$

with Frobenius norm $\|B_T^{-t}\|$.

1.10 Step 3: Interpolation error on the reference cell

Theorem 2 (Bramble-Hilbert lemma)**.** Let $T \subset \mathbb{R}^d$ a Lipschitz domain, F a normed space, $\phi\colon H^m(T) \to F$.

1.11 Step 4: Backward transformation

$$|\hat{u}|_{H^2(\hat{T})} = |\det B_T|^{-1/2} \|B_T^t\|.$$

1.12 Geometrical parameters

Let $h_T =$ an outer radius, ρ_T an inner radius and $\kappa_T = \frac{h_T}{\rho_T}$ the aspect ratio.

- A family of meshes $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots$ is called **shape regular** if $\max_i \max_{T \in \mathcal{T}_i} \kappa_T \leq \kappa$.
- A family of meshes $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots$ is called **quasi uniform** if $\frac{\max}{\min} \leq \kappa$.

1.13 Spectral norm of transformation

For the spectral norm of the affin linear transformation ϕ_T

1.14 Step 5: Assembling together

$$\|\nabla\left(u-I_h u\right)\|_{\Omega}^2 \leq c \sum_{T\in\mathcal{T}_h} \|B_T^{-1}\|_F^2 \|B_T\|.$$

Error estimates in the L^2 norm

1.15 L^2 error estimates

Are you interested in a "weaker" norm

$$\|u-u_h\|_{L^2(\Omega)}$$

instead of

$$\|u-I_h u\|_{L^2(\Omega)} \leq c_\kappa \boldsymbol{h^2} |u| H^2\left(\Omega\right)$$

?

1.16 Duality argument

Aim: Derive error bound on

$$\|u-u_h\|_W$$

in a weaker norm, i.e. let W be an Hilbert space with continuous embedding $V \subseteq W$, i.e.

$$\|u\|_W.$$

- Due to the continuous embedding $V \subseteq W$ it holds

$$W' \subseteq V'.$$

- Hence, $g \in S \subset V'$ is a possible rhs in the dual problem:

$$z_g \in V: \quad A\left(\phi, z_g\right) = \langle g, \phi\rangle \quad \forall \phi \in V.$$

- Primal problem:

$$u \in V: \quad A\left(u, \phi\right) = \langle f, \phi\rangle.$$

1.17 Aubin-Nitsche trick

We arribe at:

Theorem 3. Let $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$, be a convex domain or a domain with C^2–boundary, $\{T_h\}$.

1.18 Higher order finite elements

- FEM or order $r \geq 1$:

$$P_r\left(\mathcal{T}_h\right) = \left\{\varphi\colon \Omega_h \to \mathbb{R}\colon \varphi\big|_T \in P_r \forall T \in \mathcal{T}_h\right\}.$$

1.19 Error estimate for higher order finite elements

Theorem 4. We consider the Poisson problem, discretized with P_r finite elements $(r \geq 1)$ on a family of shape regular meshes. If the solution u has regularity H^{r+1}, then

$$\|\nabla\left(u - I_h u\right)\|_\Omega \leq c_\kappa h^r \left|u\right| H^{r+1}.$$

1.20 Pro's and cons of higher order finite elements

Pro's:

- A better approximation property is expected due to better asymptotic behavior.
- Less degrees of freedom for a given accuracy.
- More local couplings in the stiffness matrix (can be advantageous for CPU reasons).

Contra:

- More regularity of the solution is necessary. Otherwise: reduction of accuracy/order of convergence.
- Stiffness matrix become more dense due to many couplings inside each elements.
- Robust linear solvers are usually more difficult.

1.21 Accuracy of Q_r elements

$$\varphi\left(x, y\right) = \sum_{i,j=0}^{r} \alpha_{ij} x^i y^j.$$

- The nodal interpolation $\hat{I}$ in the reference quadrilateral \hexahedral is exacts for polynomials of degree $\leq r$.

1.22 Summary of Lecture 2

- FE for continuous, coercive bilinear forms are quasi-optimal with respect to discretization error:

Abcedario.