0.1 Existence and uniqueness for scalar convection-difussion problem

Continuity

Coercivity

$$A(u,u) = (\sigma u + (\beta \cdot \nabla)u, u) + \varepsilon \|\nabla u\|^2$$

Reformulation of the convective part:

$$((\beta \cdot \nabla)u, u) = \int_\Omega \nabla u \cdot () \mathrm{d}u$$

0.2 Where is the problem for discrete (FE) schemes?

- Lax-Milgram also applies to the discrete standard FE scheme.
- Existence and uniqueness of discrete solutions.
- Cea's lemma:

$$\|u - u_h\| + \|\nabla(u - u_h)\| \leq \frac{\alpha_1}{\alpha_2} \inf_{v_h \in V_h} (\|u - v_h\| + \|\nabla(u - v_h)\|)$$

- But:

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0.3 Standard Galerkin formulation

Standard Galerkin formulation. Seek $u_h \in V_h$ such that

$$A(u_h, \phi) = (f, \phi) \forall \phi \in V_h$$

0.4 Instabilities of the pure Galerkin formulation

Standard Galerkin formulation: Seek $u_h \in V_h$ such that

$$A(u_h, \phi) = (f, \phi) \forall \phi \in V_h$$

A priori estimate in the norm

$$\|u_h\|$$

0.5 Numerical diffusion with FEM

- Similar strategies as for FD possible with FEM: Adding the term
$$(h\nabla u_h, \nabla\phi)$$
to the bilinear form $A(\cdot,\cdot)$.
- Diffusion in streamline direction only: Adding
$$(h(\beta\cdot\nabla)u_h, (\beta\cdot\nabla)\phi)$$
Only 1. order!
- Accuracy of 1. order methods can we worse than pure Galer (see Brooks & Hughs).

0.6 3.4 Stabilized finite elements

$$A(u,\phi) = (\sigma u + (\beta\cdot\nabla)u, \phi) + (\varepsilon\nabla u, \nabla\phi)$$

- Stabilized form:
$$u_h \in V_h : A(u_h,\phi) + S_h(u_h,\phi) = (F_h,\phi)\forall\phi \in V_h$$
- Such stabilization

0.7 3.4 Streamline Upwind Petrov Galerkin (SUPG)

- Add a diffusion term in direction of the streamline, i.e. β
$$((\beta\cdot\nabla)u_h,)$$

0.8 SUPG for convection-diffusion problems

- Idea: Add a consistent diffusion term in direction if the flow:
$$S_h(u_h,\phi) = \sum_{T\in T_h}$$

0.9 Coercivity of the SUPG stabilization form

For P_1 elements (u_h cell-wise linear)

$$S_h(u_h,u_h) = \sum_{T\in T_h} \delta_T(\|(\beta\cdot\nabla)_{u_j}\|_T^2 - \|\sigma u_h\|_T^2)$$

with

0.10 Optimal choice for δ_t

Local Peclet number

$$Pe_T = \frac{h_T \|\beta\|_{L=T}}{\varepsilon}$$

0.11 A priori estimate for SUPG

$$\|u\|_T^2 = \sigma \|u\|^2 + \varepsilon \|\nabla u\|^2 + \sum_{T \in T_h} \delta_T \|(\beta \cdot \nabla)\|$$

0.12 Proof

- Aim: Bound the discretization error by a multiple of

$$H_h(u) = \sum_{T \in T_h} a_T h'_T |u|_{H^{r+1}}(T)$$

- Splitting the error in **interpolation error**

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- Individual bounds
- e.g. for the Galerking terms arising in $A(\eta_h, \xi_h)$
- With a priori bounds in $\|\eta_g\|$, $\|\nabla$
  $eta_h\|$
- Rhe order term $|S_h|$

0.13 Effect of SUPG stabilization

- Diffusion $\varepsilon = 0.01$, convection $\beta = (1, 1)^T$
- The SUPG formulation stabilizes much earlier
- Over-and undershots still remain on coarse

0.14 Drawbacks of SPUG

1. Troublemsome in the parabolic case (details belows
2. Computation of second derivatives necessary for $r \geq 2$ and Q_1-elements on arbitrary quadrilateral (or herxahedral) elements.
3.

0.15 3.4.2 SUPG for parabolic problems

- Time-dependent convection-diffusion system

$$\partial_t u + (\beta \cdot \nabla)u - \varepsilon \Delta u = f \quad \text{in } U$$

0.16 SUPG with and implicit Euler

With time step $\mathcal{T} = t_n - t_{n-1}$:

$$(\tau^{-1} u_n, \phi) + ((\beta \cdot \nabla)u, \phi)$$

$$\sum_{T \in \mathcal{T}_h} \delta_T (\tau^{-1} u_n + (\beta \cdot \nabla)u + \cdots, (\beta \cdot \nabla)\phi)_T = (\tau^{-1} u_{n-1}, \phi) + \cdots$$

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0.17 SUPG for convection-diffusion problems

- Idea: Add a consistent diffusion term in direction of the flow

$$S_h(u_h, \phi)$$