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Spring School on the Introduction on Numerical Modelling of Differential Equations – Programming Exercise 4

Exercise 4.1 [The pendulum model – Theory]

Recapitulate the model for the pendulum

$$\frac{d^2\phi(t)}{dt^2} = -\frac{g}{\ell}\sin(\phi(t)) \qquad \forall t > t_0.$$

with the two initial conditions

$$\phi(0) = \phi_0, \qquad \frac{d\phi}{dt}(0) = \phi_0'.$$

For *small* deflection angle ϕ derive the *approximation*

$$\frac{d^2\phi(t)}{dt^2} = -\frac{g}{\ell}\phi(t)$$

Show that it has the general solution $\phi(t) = A\cos(\omega t)$ and determine the constants A, ω from the initial conditions

Exercise 4.2 [The pendulum model – Solver]

(a) **Recap, Method 1**

In the first method, begin by rewriting the second order ODE as a first order system

$$\frac{d\phi(t)}{dt} = u(t), \qquad\qquad \frac{d^2\phi(t)}{dt^2} = \frac{du(t)}{dt} = -\frac{g}{\ell}\sin(\phi(t)).$$

Replacing the derivatives by difference quotients

$$\begin{aligned}\frac{\phi(t+\Delta t) - \phi(t)}{\Delta t} &\approx \frac{d\phi(t)}{dt} = u(t),\\ \frac{u(t+\Delta t) - u(t)}{\Delta t} &\approx \frac{du(t)}{dt} = -\frac{g}{\ell}\sin(\phi(t)),\end{aligned}$$

yields the *one step* scheme

$$\begin{aligned}\phi^{n+1} &= \phi^n + \Delta t\, u^n & \qquad\qquad \phi^0 &= \phi_0\\ u^{n+1} &= u^n - \Delta t\,(g/\ell)\sin(\phi^n) & u^0 &= u_0\end{aligned}$$

Where ϕ^n approximates $\phi(n\Delta t)$ for a chosen Δt using recursion (*Euler*).

(b) **Recap, Method 2**

Now, we derive a method that directly approximates the second-order ODE. It uses a *central difference quotient* for the second derivative

$$\frac{\phi(t+\Delta t) - 2\phi(t) + \phi(t-\Delta t)}{\Delta t^2} \approx \frac{d^2\phi(t)}{dt^2} = -\frac{g}{\ell}\sin(\phi(t)).$$

Solving for $\phi(t+\Delta t)$ yields the *two step* scheme ($n \geq 2$):

$$\phi^{n+1} = 2\phi^n - \phi^{n-1} - \Delta t^2\,(g/\ell)\sin(\phi^n), \tag{1}$$

with the initial condition

$$\phi^0 = \phi_0, \qquad \phi^1 = \phi_0 + \Delta t\, u_0. \tag{2}$$

The starting value ϕ^1 is derived with one step of method 1.

(c) **Actual Task**

(i) Write a C++ program implementing methods 1 and 2 using a time step Δt that can be entered by the user. For the constants choose $\ell = 9.81$ and $g = 9.81$.

(ii) Write the results to a file, where every line contains

$$t_i \qquad \phi^i \qquad u^i.$$

(iii) you can visualize the results using `gnuplot` as follows
`plot "filename"  u 1:2`
where the x-axis uses the first column and the y-axis uses the second column.

Exercise 4.3 [The pendulum model – implementations]

(a) For method 1: choose an initial deflection angle $\phi_0 = 0.1$ and a time step $\Delta t = 0.1$ and compute the solution up to time 4.0. What do you observe?

(b) Repeat the experiment with successively smaller time steps, say 0.01, 0.001, 0.0001. What do you observe?

(c) Try to compute the solution for longer times with the small timesteps. What happens?

(d) Repeat the same experiments with method 2. Is there a difference?

(e) Compare the solution of the full model and the reduced model for different initial angles $\phi_0 = 0.1, 0.5, 3.0$. Use your favourite method and a timestep Δt that is small enough to avoid any visibly numerical error.

(f) Recapitulate the concepts stability, discretization error and modeling error in the light of the results of exercise 1.