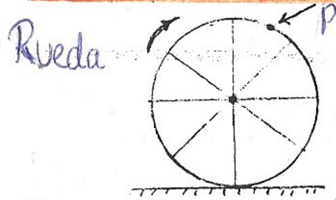
Caso 3: MCU en dos sistemas (S y S') en movimiento relativo, con $\bar{V}$ constante



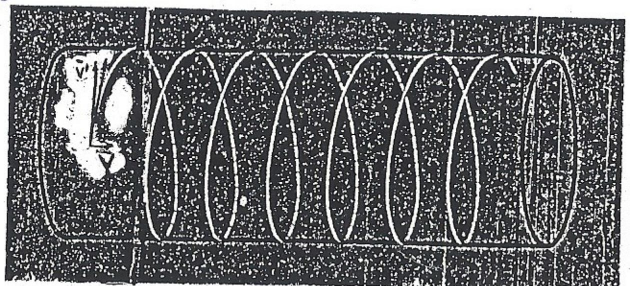


Figura 1

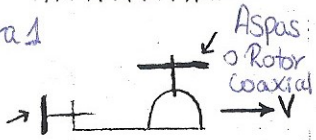
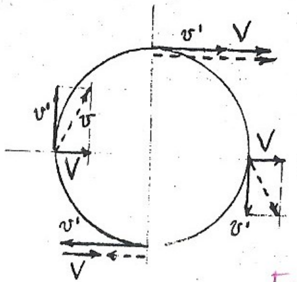
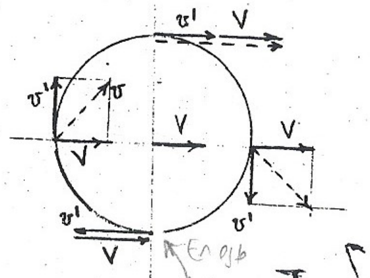


Figura 2

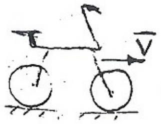




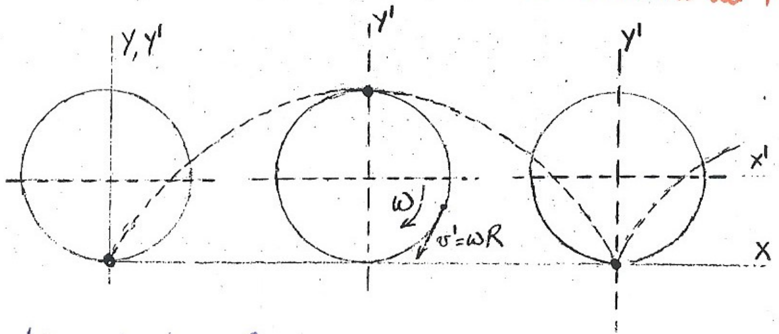
En ese punto,

Velocid. Puntual O Suma de vectores

Punto, su $\vec{v}$ es $\vec{0}$

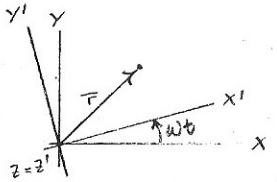


Más adelante estudiaremos el movimiento de rodadura del rígido.



Una de las formas de cicloide

3.17b Sistemas en Rotación Pura con ω constante



Eje Común

Sean dos sistemas coordinados $\Sigma(x,y,z)$ y $\Sigma'(x',y',z')$, pertenecientes a los Sistemas de Referencia S y S', respectivamente, como se ilustra en la figura. S' gira con ω constante alrededor de S', con el cual mantiene coincidiendo el eje z : z = z'

S' gira con respecto a S.

Es visualizado por 2 observadores.

De las transformaciones halladas en el ítem 2.12 se tiene que las transformaciones de posición se encuentran

$$\boxed{\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\omega t & \operatorname{sen}\omega t \\ -\operatorname{sen}\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad y \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\operatorname{sen}\omega t \\ \operatorname{sen}\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}}$$

Por definición: Medida de la rapidez con la cual cambia el vector.

De la transformación de velocidades:

En S: $\bar{v} = \dfrac{d\bar{r}}{dt}$; en S': $\bar{v}' = \dfrac{d\bar{r}'}{dt}$

Desde la perspectiva de S, el cálculo de $\dot{\bar{r}}'$ realizado por S' para determinar $\bar{v}'$ es incompleto pues también debería considerarse la variación en el tiempo de la orientación de $\bar{r} = (x'\hat{i}' + y'\hat{j}')$. Entonces:

$$\frac{d\bar{r}}{dt} = \frac{d}{dt}(x'\hat{i}' + y'\hat{j}') = \left(\hat{i}'\frac{dx'}{dt} + \hat{j}'\frac{dy'}{dt}\right) + \left(x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt}\right)$$

Vector unitario constante

y usando $\dfrac{d\hat{i}'}{dt} = \omega\hat{j}'$ y $\dfrac{d\hat{j}'}{dt} = -\omega\hat{i}'$, se obtiene la transformación

$$\boxed{\bar{v} = \bar{v}' + \bar{\omega}\times\bar{r}}$$

$\hat{x}$ $\hat{y}$ en S

$\hat{i}'$, $\hat{j}'$ rota respecto a Tierra.

Consejo del profesor: Repetir el cálculo.

De la transformación de las aceleraciones:

Para hallar la transformación entre $\bar{a}$ y $\bar{a}'$ se procede de manera análoga:

En S: $\bar{a} = \dfrac{d\bar{v}}{dt}$ y En S': $\bar{a}' = \dfrac{d\bar{v}'}{dt}$

Incorporamos la variación de sus vectores unitarios. Luego del aparataje matemático.

Luego, de $\bar{v} = \bar{v}' + \underbrace{\bar{\omega}\times\bar{r}}$:

$$\frac{d\bar{v}}{dt} = \frac{d}{dt}(\bar{v}' + \bar{\omega}\times\bar{r}) \Rightarrow \bar{a} = \frac{d\bar{v}}{dt} = \frac{d\bar{v}'}{dt} + \bar{\omega}\times\bar{v}$$

$$\therefore \frac{d\bar{v}'}{dt} = \bar{a}' + \bar{\omega}\times\bar{v}' \quad y \quad \bar{\omega}\times\bar{v} = \bar{\omega}\times(\bar{v}' + \bar{\omega}\times\bar{r}) = \bar{\omega}\times\bar{v}' + \bar{\omega}\times(\bar{\omega}\times\bar{r})$$

Entonces, las aceleraciones se transforman según

Aceleración Coriolis, aceleración centrípeta

$$\boxed{\bar{a} = \bar{a}' + 2\bar{\omega}\times\bar{v}' + \bar{\omega}\times(\bar{\omega}\times\bar{r})}$$ (A. Finn, Vol I, pág. 123)

Velocidad en el sistema móvil

De manera que, una partícula que se mueve con $\bar{v}'$ y $\bar{a}'$ respecto de S' (que se encuentra en rotación pura con $\bar{\omega}$ constante respecto de S), en S tendrá una aceleración adicional la cual es convenientemente separada en dos componentes

Transf. de Lorentz

Apuntar a clase

i) aceleración centrípeta $\bar{\omega}\times(\bar{\omega}\times\bar{r})$, en la dirección radial

ii) aceleración Coriolis $2\bar{\omega}\times\bar{v}'$, en dirección perpendicular a $\bar{v}'$

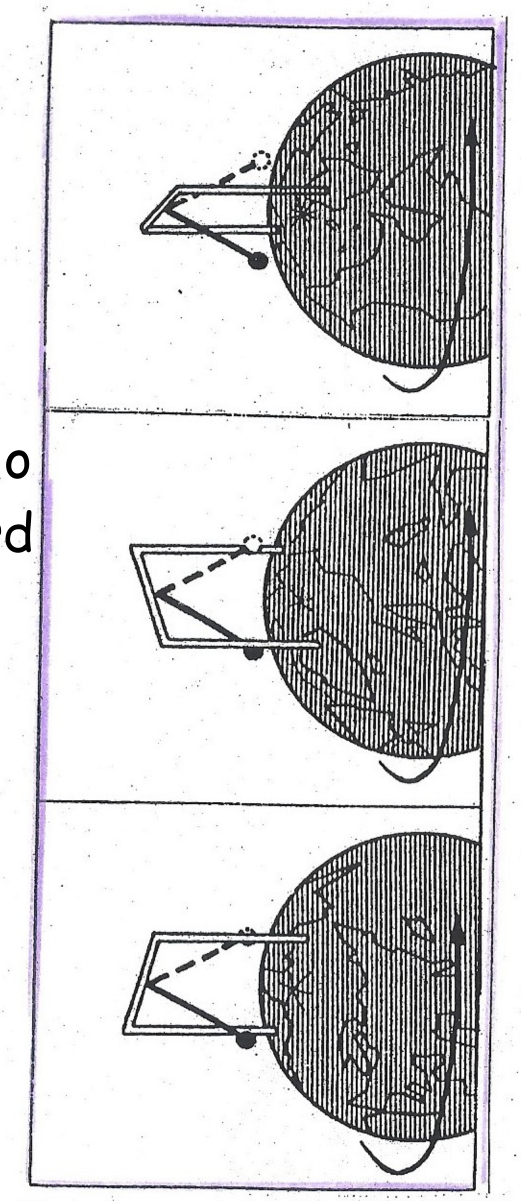
* 2 observadores en movimiento de rotación entre sí discrepan de sus lecturas.

Pregunta del profesor: Qué significa físicamente $\omega\times(\vec{\omega}\times\vec{r})$ responder verbalmente

Efect rotac de la rotación sobre la gravit Lorentz

$\bar{a}$ se manifiesta Colión

Al final del pdf, están las páginas 123-125 del libro de Marcelo Alonso y Edward Finn



Péndulo de Foucault

Foucault's pendulum

3.17.b Sistemas en rotación pura con ω constante

Sean dos sistemas coordenados $SC(x,y,z)$ y $SC'(x,y,z)$, pertenecientes a los sistemas de referencia S y S' respectivamente, como se ilustra en la figura escaneada.

S' gira con ω constante alrededor de S, con el cual mantiene coincidiendo el eje Z: $Z = Z'$.

De las transformaciones halladas con el ítem 2.12 se tiene que las transformaciones de posición se expresan:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -sin\,\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ y } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ sin\,\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

De las transformaciones de velocidades:

En S: $\vec{v} = \frac{d\vec{r}}{dx}$ En S': $\overrightarrow{v'} = \frac{d\vec{r}'}{dx}$

Desde la perspectiva de S, el cálculo de $\dot{\vec{r}}'$ realizado por S' para determinar $\vec{v}'$ es incompleto, pues también debería considerarse la variación en el tiempo de la orientación de $\vec{r} = (x'\hat{\imath}' + y'\hat{\jmath}')$. Entonces:

$$\frac{d\vec{r}}{dt} = \frac{d(x'\hat{\imath}' + y'\hat{\jmath}')}{dt} = \left(\hat{\imath}'\frac{dx'}{dt} + \hat{\jmath}'\frac{dy'}{dt}\right) + \left(x'\frac{d\hat{\imath}'}{dt} + y'\frac{d\hat{\jmath}'}{dt}\right)$$

Y usando $\frac{d\hat{\imath}'}{dt} = \omega\hat{\jmath}'$ y $\frac{d\hat{\jmath}'}{dt} = -\omega\hat{\imath}'$, se obtiene la transformación:

$$\boxed{\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}}$$

De la transformación de las aceleraciones:

Para hallar la transformación entre $\vec{a}$ y $\vec{a}'$ se procede de manera análoga:

En S: $\vec{a} = \frac{d\vec{v}}{dt}$ y En S': $\vec{a}' = \frac{d\vec{v}'}{dt}$

Luego, de $v = \vec{v}' + \vec{\omega} \times \vec{r}$

Además: $\vec{\omega} \times \vec{r} = \vec{V}$

$$\boxed{\frac{d\vec{v}}{dt} = \frac{d(\vec{v}' + \vec{\omega} \times \vec{r})}{dt} \Rightarrow \vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{v}'}{dt} + \vec{\omega} \times \vec{v}}$$

$$\therefore \frac{d\vec{v}'}{dt} = \vec{a} + \vec{\omega} \times \vec{v}' \qquad \text{y}$$

$$\vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{v}' + \vec{\omega} \times \vec{r}) \equiv \vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Entonces, las aceleraciones se transforman según:

$$\boxed{\vec{a} = \vec{a}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r})}$$

De manera que, una partícula que se mueve con $\vec{v}'$ y $\vec{a}'$ respecto de S'(que se encuentra en rotación pura con constante respecto de S), en S tendrá un aceleración adicional la cual es convenientemente separada en dos componentes.

i) Aceleración centrípeta $\vec{\omega} \times (\vec{\omega} \times \vec{r})$, en la dirección radial.

ii) Aceleración Coriolis $2\vec{\omega} \times \vec{v}'$, en la dirección perpendicular a $\vec{v}'$.

into $V' = V + v = 383$ m s^{-1}. For situation (c) we use Eq. (6.12), so that $V' = \sqrt{V^2 + v^2} = 358.9$ m s^{-1}. To the moving observer, the sound appears to propagate in a direction which makes an angle α' with the X'-axis such that

$$\tan \alpha' = \frac{V'_{y'}}{V'_{x'}} = \frac{V}{-v} = -15.32 \qquad \text{or} \qquad \alpha' = 93.7°.$$

Finally, in case (d), the direction of propagation of the sound in air is such that it appears to O' to be moving in the Y'-direction. Thus $V'_{x'} = 0$, $V'_{y'} = V'$, and $V'_{z'} = 0$. Therefore, using Eq. (6.10), we have $0 = V_x - v$ or $V_x = v$ and $V' = V_y$. Thus $V^2 = V_x^2 + V_y^2 = v^2 + V'^2$ or $V' = \sqrt{V^2 - v^2} = 357.1$ m s^{-1}. In this case sound propagates through the still air in a direction making an angle α with the X-axis such that

$$\tan \alpha = \frac{V_y}{V_x} = \frac{V'}{v} = 14.385 \qquad \text{or} \qquad \alpha = 86.0°.$$

6.4 Uniform Relative Rotational Motion

Let us now consider two observers O and O' rotating relative to each other but with no relative translational motion. For simplicity we shall assume that both O and O' are in the same region of space and that each uses a frame of reference attached to itself but with a common origin. For example, observer O, who uses the frame XYZ (Fig. 6–5), notes that the frame $X'Y'Z'$ attached to O' is rotating with angular velocity $\boldsymbol{\omega}$. To O', the situation is just the reverse; O' observes frame XYZ rotating with angular velocity $-\boldsymbol{\omega}$. The position vector $\boldsymbol{r}$ of particle A referred to XYZ is

$$\boldsymbol{r} = \boldsymbol{u}_x x + \boldsymbol{u}_y y + \boldsymbol{u}_z z, \tag{6.15}$$

and therefore the velocity of particle A as measured by O relative to its frame of reference XYZ is

$$\boldsymbol{V} = \frac{d\boldsymbol{r}}{dt} = \boldsymbol{u}_x \frac{dx}{dt} + \boldsymbol{u}_y \frac{dy}{dt} + \boldsymbol{u}_z \frac{dz}{dt}. \tag{6.16}$$

Similarly, the position vector of A referred to $X'Y'Z'$ is

$$\boldsymbol{r} = \boldsymbol{u}_{x'} x' + \boldsymbol{u}_{y'} y' + \boldsymbol{u}_{z'} z', \tag{6.17}$$

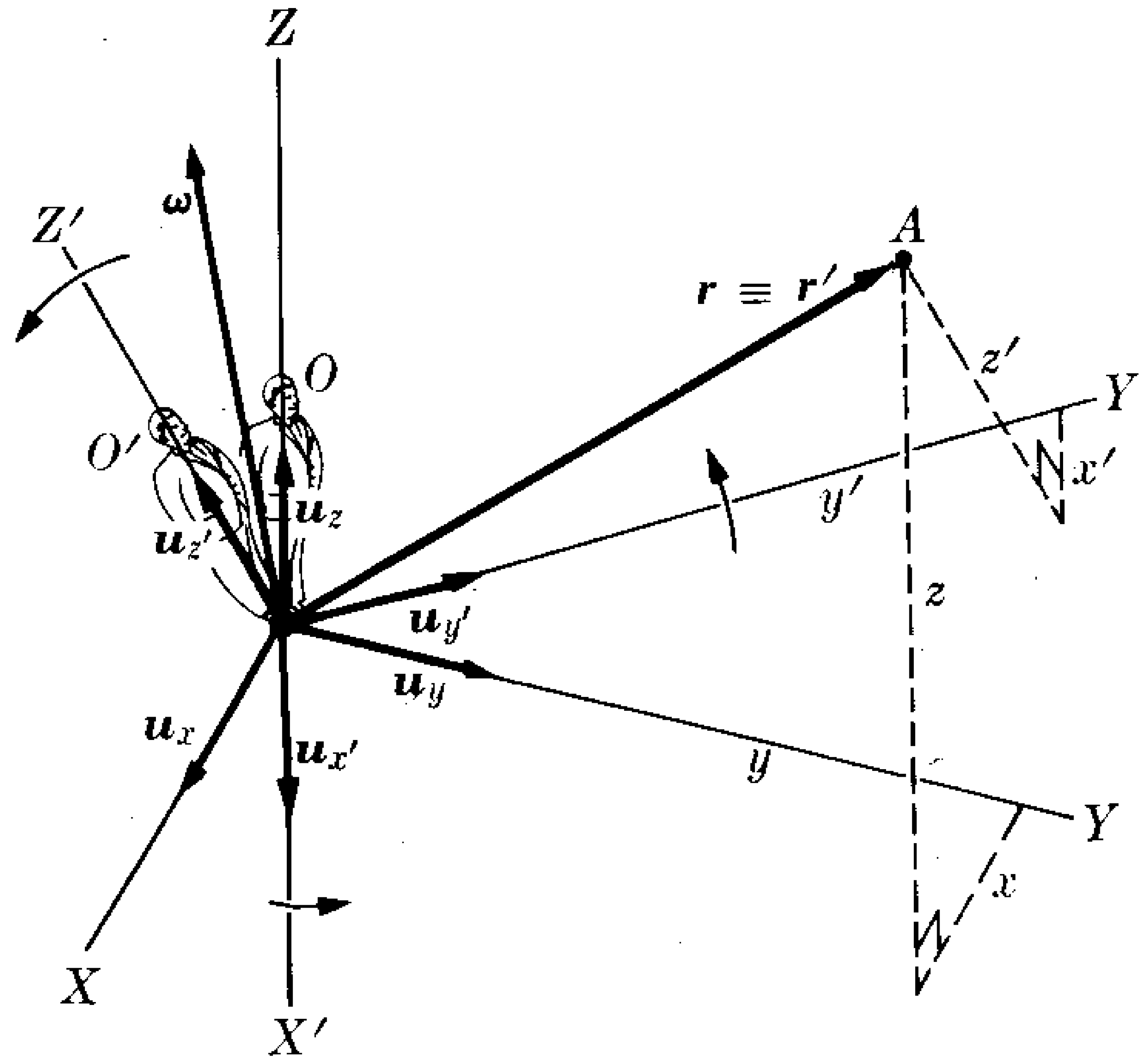


Fig. 6–5. Frames of reference in uniform relative rotational motion.

where, because the origins are coincident, the vector $\boldsymbol{r}$ is the same as in Eq. (6.15);

that is the reason why we have not written $\boldsymbol{r}'$. The velocity of A, as measured by O' relative to its own frame of reference $X'Y'Z'$, is

$$\boldsymbol{V}' = \boldsymbol{u}_{x'} \frac{dx'}{dt} + \boldsymbol{u}_{y'} \frac{dy'}{dt} + \boldsymbol{u}_{z'} \frac{dz'}{dt}. \tag{6.18}$$

In taking the derivative of Eq. (6.17), observer O' has assumed that his frame $X'Y'Z'$ is not rotating, and has therefore considered the unit vectors as constant in direction. However, observer O has the right to say that, to him, the frame $X'Y'Z'$ is rotating and therefore the unit vectors $\boldsymbol{u}_{x'}$, $\boldsymbol{u}_{y'}$, and $\boldsymbol{u}_{z'}$ are not constant in direction, and that in computing the time derivative of Eq. (6.17) one must write

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{u}_{x'} \frac{dx'}{dt} + \boldsymbol{u}_{y'} \frac{dy'}{dt} + \boldsymbol{u}_{z'} \frac{dz'}{dt} + \frac{d\boldsymbol{u}_{x'}}{dt} x' + \frac{d\boldsymbol{u}_{y'}}{dt} y' + \frac{d\boldsymbol{u}_{z'}}{dt} z'. \tag{6.19}$$

Now the endpoints of vectors $\boldsymbol{u}_{x'}$, $\boldsymbol{u}_{y'}$, and $\boldsymbol{u}_{z'}$ are (by assumption) in uniform circular motion relative to O, with angular velocity $\boldsymbol{\omega}$. In other words, $d\boldsymbol{u}_{x'}/dt$ is the velocity of a point at unit distance from O and moving with uniform circular motion with angular velocity $\boldsymbol{\omega}$. Therefore, using Eq. (5.48), we have,

$$\frac{d\boldsymbol{u}_{x'}}{dt} = \boldsymbol{\omega} \times \boldsymbol{u}_{x'}, \qquad \frac{d\boldsymbol{u}_{y'}}{dt} = \boldsymbol{\omega} \times \boldsymbol{u}_{y'}, \qquad \frac{d\boldsymbol{u}_{z'}}{dt} = \boldsymbol{\omega} \times \boldsymbol{u}_{z'}.$$

Accordingly, from Eq. (6.19) we may write

$$\begin{aligned} \frac{d\boldsymbol{u}_{x'}}{dt} x' + \frac{d\boldsymbol{u}_{y'}}{dt} y' + \frac{d\boldsymbol{u}_{z'}}{dt} z' &= \boldsymbol{\omega} \times \boldsymbol{u}_{x'}x' + \boldsymbol{\omega} \times \boldsymbol{u}_{y'}y' + \boldsymbol{\omega} \times \boldsymbol{u}_{z'}z' \\ &= \boldsymbol{\omega} \times (\boldsymbol{u}_{x'}x' + \boldsymbol{u}_{y'}y' + \boldsymbol{u}_{z'}z') \\ &= \boldsymbol{\omega} \times \boldsymbol{r}. \end{aligned} \tag{6.20}$$

Introducing this result in Eq. (6.19), and using Eqs. (6.16) and (6.18), we finally get

$$\boldsymbol{V} = \boldsymbol{V}' + \boldsymbol{\omega} \times \boldsymbol{r}. \tag{6.21}$$

This expression gives the relation between the velocities $\boldsymbol{V}$ and $\boldsymbol{V}'$ of A, as recorded by observers O and O' in relative rotational motion.

To obtain the relation between the accelerations, we proceed in a similar way. The acceleration of A, as measured by O relative to XYZ, is

$$\boldsymbol{a} = \frac{d\boldsymbol{V}}{dt} = \boldsymbol{u}_x \frac{dV_x}{dt} + \boldsymbol{u}_y \frac{dV_y}{dt} + \boldsymbol{u}_z \frac{dV_z}{dt}. \tag{6.22}$$

The acceleration of A, as measured by O' relative to $X'Y'Z'$, when he again ignores the rotation, is

$$\boldsymbol{a}' = \boldsymbol{u}_{x'} \frac{dV'_{x'}}{dt} + \boldsymbol{u}_{y'} \frac{dV'_{y'}}{dt} + \boldsymbol{u}_{z'} \frac{dV'_{z'}}{dt}. \tag{6.23}$$

When we differentiate Eq. (6.21) with respect to t, remembering that we are assuming that $\boldsymbol{\omega}$ is constant, we obtain

$$\boldsymbol{a} = \frac{d\boldsymbol{V}}{dt} = \frac{d\boldsymbol{V}'}{dt} + \boldsymbol{\omega} \times \frac{d\boldsymbol{r}}{dt}. \tag{6.24}$$

Now, since $\boldsymbol{V}' = \boldsymbol{u}_{x'}V'_{x'} + \boldsymbol{u}_{y'}V'_{y'} + \boldsymbol{u}_{z'}V'_{z'}$, we obtain by differentiation

$$\frac{d\boldsymbol{V}'}{dt} = \boldsymbol{u}_{x'}\frac{dV'_{x'}}{dt} + \boldsymbol{u}_{y'}\frac{dV'_{y'}}{dt} + \boldsymbol{u}_{z'}\frac{dV'_{z'}}{dt} + \frac{d\boldsymbol{u}_{x'}}{dt}V'_{x'} + \frac{d\boldsymbol{u}_{y'}}{dt}V'_{y'} + \frac{d\boldsymbol{u}_{z'}}{dt}V'_{z'}.$$

The first three terms are just $\boldsymbol{a}'$, as given by Eq. (6.23), and the last three, by a procedure identical to that used to derive Eq. (6.20), are $\boldsymbol{\omega} \times \boldsymbol{V}'$. That is, by substituting the appropriate quantities into Eq. (6.20), we have

$$\boldsymbol{\omega} \times \boldsymbol{u}_{x'}V'_{x'} + \boldsymbol{\omega} \times \boldsymbol{u}_{y'}V'_{y'} + \boldsymbol{\omega} \times \boldsymbol{u}_{z'}V'_{z'} = \boldsymbol{\omega} \times (\boldsymbol{u}_{x'}V'_{x'} + \boldsymbol{u}_{y'}V'_{y'} + \boldsymbol{u}_{z'}V'_{z'}) = \boldsymbol{\omega} \times \boldsymbol{V}'.$$

Therefore $d\boldsymbol{V}'/dt = \boldsymbol{a}' + \boldsymbol{\omega} \times \boldsymbol{V}'$. Also from Eqs. (6.16) and (6.21), $d\boldsymbol{r}/dt = \boldsymbol{V} = \boldsymbol{V}' + \boldsymbol{\omega} \times \boldsymbol{r}$, so that

$$\boldsymbol{\omega} \times \frac{d\boldsymbol{r}}{dt} = \boldsymbol{\omega} \times (\boldsymbol{V}' + \boldsymbol{\omega} \times \boldsymbol{r}) = \boldsymbol{\omega} \times \boldsymbol{V}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}).$$

Substituting both results in Eq. (6.24), we finally obtain

$$\boldsymbol{a} = \boldsymbol{a}' + 2\boldsymbol{\omega} \times \boldsymbol{V}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}). \tag{6.25}$$

This equation gives the relation between the accelerations $\boldsymbol{a}$ and $\boldsymbol{a}'$ of A as recorded by observers O and O' in uniform relative rotational motion. The second term, $2\boldsymbol{\omega} \times \boldsymbol{V}'$, is called the *Coriolis acceleration*. The third term is similar to Eq. (5.59) and corresponds to a *centripetal acceleration*. Both the Coriolis and centripetal accelerations are the result of the relative rotational motion of the observers. In the next section we shall illustrate the use of these relations.

6.5 Motion Relative to the Earth

One of the most interesting applications of Eq. (6.25) is the study of a body's motion relative to the earth. As indicated in Example 5.10, the angular velocity of the earth is $\omega = 7.292 \times 10^{-5}$ rad s^{-1}. Its direction is that of the axis of rotation of the earth. Consider a point A on the earth's surface (Fig. 6–6). Let us call $\boldsymbol{g}_0$ the acceleration of gravity as measured by a nonrotating observer at A. Then $\boldsymbol{g}_0$ corresponds to $\boldsymbol{a}$ in Eq. (6.25). Solving Eq. (6.25) for $\boldsymbol{a}'$, we obtain the acceleration measured by an observer rotating with the earth:

$$\boldsymbol{a}' = \boldsymbol{g}_0 - 2\boldsymbol{\omega} \times \boldsymbol{V}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}). \tag{6.26}$$