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Tiling with Dominoes

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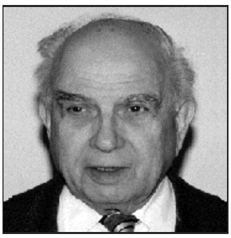

Nathan Mendelsohn (mendel@cc.umanitoba.ca) was born on April 14, 1917, and all of his education was obtained in Toronto, Canada. His early publications were in the areas of geometry and number theory. He later changed his research to combinatorics, specializing in block designs, orthogonal arrays, and bipartite graphs. At age 86, he still does research (although he claims to have slowed down considerably).

Introduction and definitions

A domino is a rectangular area of sides 2 units by 1 unit. This paper is concerned with the problems and properties of covering regions of the plane with non-overlapping dominoes. The regions considered in the purview of this paper may be described as follows. We take a rectangle r units by s units and rule it with unit squares, which we call *cells*. The cells are coloured black and white in checkerboard fashion, with the square in the upper left corner coloured black. The region is then modified by removing some cells. One such region appears in Figure 1. We call such a region a *pruned checkerboard*.

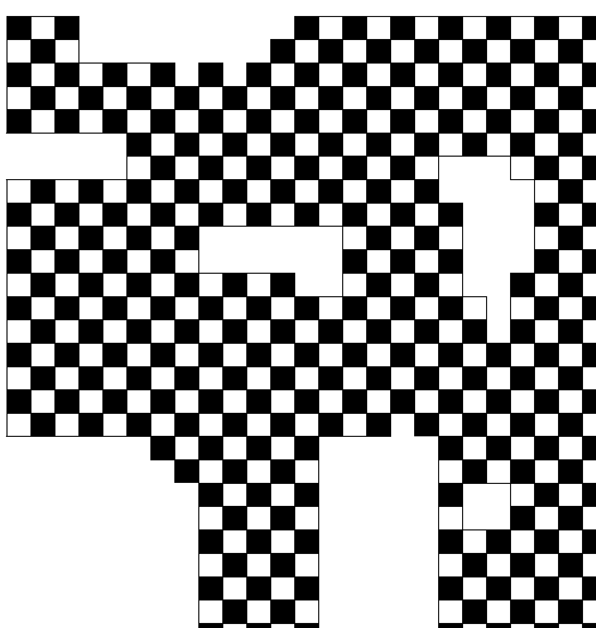

Figure 1.

An ancient puzzle

An ancient puzzle asks the following question. Suppose a square 8 units by 8 units has two diagonally opposite squares removed as shown in Figure 2. Can dominoes tile the resulting region?

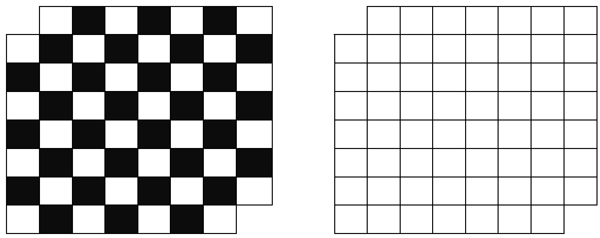

Figure 2.

First solution

From the checkerboard diagram, the region contains 30 black cells and 32 white cells. Since each domino covers 1 black and 1 white cell, tiling is impossible.

Second solution

When I was first shown the problem many years ago, it did not occur to me to colour the cells. The region itself had seven cells in the top and bottom rows and eight cells in the remaining rows. The same held for the columns. I proceeded to obtain information on how many dominoes pointed horizontally and how many vertically. The first count dealt with the vertical dominoes. If the region is tiled, the horizontal dominoes in the top row occupies an even number of cells. Hence, the cells in the top row that are not occupied by horizontal dominoes are odd in number. Thus there are an odd number of vertical dominoes between the first and second rows. Since the second row has eight cells, and an odd number are occupied by vertical dominoes coming down from the first row, there remain an odd number of cells in the second row. The same argument now shows there is an odd number of vertical dominoes from the second row to the third. Continuing this way, we see that there is an odd number of vertical dominoes between any pair of consecutive rows. Hence the total number of vertical dominoes is the sum of seven odd numbers, which is odd. In the same way, using columns instead of rows, there is an odd number of horizontal dominoes. Hence the total number of dominoes is even. Since there are 62 cells to cover, the number of dominoes required is 31, an odd number. Therefore, tiling is impossible.

(Incidentally, I was told by John Selfridge that this second solution was known to others although not published.)

Why do I produce two solutions to the puzzle? It is because I am interested in the question of which is the better solution. At first glance, it appears that the first solution is the better. It is much shorter and is easily understood by many people with virtually no knowledge of mathematics. But are there considerations that might judge the second solution to be the better one? Most mathematicians would agree that of two solutions to a problem, that which contains ideas applicable to other problems is superior to that which is a dead end. The first solution is a dead end with respect to tilings of other regions. For instance, suppose that instead of removing diagonally opposite cells a black cell and a white cell are removed. The first solution tells us nothing about whether or not tilings exist. The second solution, based on the counting of horizontal and vertical dominoes, might lead to interesting information. Let us apply the argument (of the second solution) in the particular case where the two top corner cells are removed. Using the same argument, it is immediately seen that in any tiling of the region the number of vertical dominoes is even and the number of horizontal dominoes is odd. Further reflection on the proof suggests it may be applicable to any pruned checkerboard. Let us see what the argument leads to.

First we assume that there is some tiling of the given region. (If there is no tiling of the region, the statements to be made are all true but vacuously.) We assume that the region has n rows. In what follows, we refer to the addition and subtraction of parities (oddness and evenness). To make this precise, we assign to oddness the value 1 and to evenness the value 0, and we assume the following common rules of addition: $1 + 1 = 0 + 0 = 0$, $1 + 0 = 0 + 1 = 1$, $1 = -1$.

Step 1. We count the parity of the number of vertical dominoes between the first and second rows. Let P be the parity of the number of cells in the first row. In any tiling, the horizontal dominoes in the first row occupy an even number of cells. The remaining cells still have parity P. Hence, the number of vertical dominoes between the first and second rows has parity P. This is independent of any tiling.

Step 2. Suppose the number of cells in the second row has parity Q. Since some of these are already occupied by vertical dominoes from the first to the second row, the number of free cells in the second row has parity $Q - P$. As in step 1, the number of vertical dominoes from the second row to the third has parity $Q - P$.

The argument of step 2 is iterated through rows 3 to $n - 1$. Hence we have the following result.

Theorem. *In any tiling of a pruned checkerboard the parity of the number of vertical dominoes from row r to row $r + 1$ is independent of the tiling.*

By adding the parities of vertical dominoes from row r to $r + 1$, for $r = 1, 2, \dots,$ $n - 1$, we obtain the following:

Corollary. *The parity of the number of vertical dominoes in a pruned checkerboard is independent of the tiling.*

The argument for horizontal dominoes is the same, using columns instead of rows.

The following table lists the parities in the case of rectangles. If the rectangle has both sides odd we assume that the cell in the upper-right corner is removed (the removal of some black square is necessary since such a rectangle has an odd number of cells).

rows	columns	horizontal dominoes	vertical dominoes
$2r$	$2s$	even	even
$4r$	$2s + 1$	even	even
$4r + 2$	$2s + 1$	even	odd
$4r + 1$	$4s + 1$	even	even
$4r + 1$	$4s + 3$	odd	even
$4r + 3$	$4s + 3$	odd	odd

A point of some interest is the following. In the case where both sides are odd, if, instead of the removal of the upper right cell, the black cell just to the lower left is removed, the parities are reversed.

Extension of the puzzle

A question which has been asked is whether an 8-by-8 checkerboard with a white and a black cell removed is always tileable. This question remained unanswered for many years until solved by Gomory. His proof appears in *Honsberger's Mathematical Gems 1* [**2**].

In what follows, we extend the problem to the following. Let R be a pruned checkerboard which can be tiled with dominoes. Find sufficient conditions on the region such that the removal of any two oppositely coloured cells always leaves a tileable region. This latter property, we sometimes refer to as the *removable property*.

We attack this problem using concepts from graph theory. Informally, we define a graph as follows. A *graph* G is a system containing two sets P and E called *vertices* and *edges*. Every edge "joins" two vertices and every pair of vertices are joined by at most one edge. If two vertices are joined by an edge, we say they are *adjacent*. A *path* in a graph is a sequence $a_1, \ell_1, a_2, \ell_2, \ldots, a_n$, where $a_1, a_2, \ldots, a_n$ are distinct vertices and $\ell_1, \ell_2, \ldots, \ell_{n-1}$ are edges, with a_i and a_{i+1} joined by ℓ_i. If a_n and a_1 are joined by an edge ℓ_n and if ℓ_n is added to the sequence, the sequence is called a *circuit*. If a circuit contains every vertex of the graph, the circuit is called a *Hamiltonian* circuit.

We say that a path with n vertices is of *length* n, although it is more usual to call the length $n-1$. For a circuit there is no ambiguity in calling its *length* the number of vertices it contains.

If the vertices of a graph can be partitioned into two disjoint sets B and W and each edge of the graph joins a vertex from B with a vertex from W, we say the graph is *bipartite*. In such a graph we refer to a vertex from the set W as a white vertex and one from B as a black vertex. In any path the vertices alternate in colour. Hence, a path whose end vertices have the same colour is of odd length, while if the end vertices are of opposite colour its length is even. The length of a circuit is always even since the vertices of the circuit alternate in colour.

We define now a bipartite graph on a pruned checkerboard. The vertices for the graph are the cells of the region, with two vertices adjacent if the corresponding cells have a side in common. Calling the black and white cells black and white vertices, it is clear that the graph is bipartite.

Suppose the graph has a Hamiltonian circuit. Tile the corresponding region as follows. Take any two consecutive vertices of the circuit. They correspond to a black and a white cell which are either horizontally adjacent or vertically adjacent. Hence, these cells can be covered with a domino. The next two vertices of the circuit correspond also to two cells which can be covered with a domino. We continue this way moving around the circuit, and hence can cover all cells of the circuit, which correspond to the vertices. Since the circuit is Hamiltonian, we have covered the whole region with dominoes. It is easily seen that corners in the geometric circuit cause no obstruction to the tiling. If we were now to remove a black and a white cell from the region, the effect on the Hamiltonian circuit is to leave it with two even paths. Each of these can now be covered. We thus have the following result:

If the graph of a pruned checkerboard has a Hamiltonian circuit, then it can be tiled with dominos. If, furthermore, any black and any white cell are removed, the remaining region is also tileable.

Finding Hamiltonian circuits

There are a number of theorems that deal with finding of Hamiltonian circuits in graphs. In our examples we do not use these theorems. Rather, we build up a Hamiltonian circuit from smaller circuits.

Suppose in a graph there are two disjoint circuits C_1 and C_2. Suppose also C_1 and C_2 have consecutive vertices A, B and A', B' respectively and that A and A' are adjacent in the graph, as are B and B'. If we remove the edges joining A to B and A' to B' and replace them by edges joining A to A' and B to B', the result is the formation of a circuit containing all the vertices of both C_1 and C_2. We refer to this as the *splicing* of C_1 and C_2.

We now show that a Hamiltonian circuit exists in the following cases: (1) Any rectangle with at least one even side, and (2) any rectangle for which both sides are odd and from which any black cell is removed.

First let R be a rectangle of size $2 \times n$. In this case a Hamiltonian circuit is obtained by moving from cell to cell around the boundary.

Now let R be a rectangle of size $2r \times n$. We cut this up into r consecutive blocks of size $2 \times n$. In each of these blocks, take the Hamiltonian circuit and then splice consecutive circuits to form a Hamiltonian circuit of the whole rectangle.

Now let R be a rectangle of size $2r + 1 \times 2s + 1$ with a black cell removed. We first show that a $3 \times 2r + 1$ rectangle has a Hamiltonian circuit. For a 3×3 rectangle the black cells are the corner cells and the middle cell. In the case where the middle cell is removed, we get a Hamiltonian circuit by moving around the boundary. In the case where a corner cell is removed, we get such a circuit by starting with the middle cell and moving to one of the cells contiguous to the removed cell; now there is exactly one way to move from cell to cell to get the circuit. Now, consider a 3×5 board with a black cell removed. Either the first two or the last two columns (or both) do not contain the deleted cell. Hence, the rectangle can be broken up into a 3×3 rectangle with a deleted cell and a 3×2 rectangle.

Splicing the circuit of the 3×3 deleted square with that of the 3×2 rectangle yields a Hamiltonian circuit of the 3×5 deleted rectangle. It is clear that the process can be iterated, yielding the fact that the graph of a $3 \times 2r + 1$ rectangle with a deleted cell always has a Hamiltonian circuit. (Note that the iteration could be expressed as a mathematical induction).

This is extended to a $2s + 1 \times 2r + 1$ deleted rectangle as follows. Start with a $5 \times 2r + 1$ deleted rectangle. Either the first two rows or the last two rows do not contain the deleted cell. Choosing these two rows, we divide the configuration into a $3 \times 2r + 1$ deleted rectangle and a $2 \times 2r + 1$ rectangle. By splicing, we obtain a Hamiltonian circuit for the $5 \times 2r + 1$ deleted rectangle. Iterating the process yields a Hamiltonian circuit for an arbitrary $2s + 1 \times 2r + 1$ rectangle with one black cell removed.

Many other examples of Hamiltonian circuits obtained with the use of splicing have been found. We describe a few of them here; our list is far from exhaustive.

First we give some cases for which a Hamiltonian circuit does not exist. These cases require some additional graph theory definitions. A graph has an *isthmus* if it consists of two circuits which are joined by a single path. Such a graph does not have a Hamiltonian circuit.

A *tree* is a connected graph which has no circuits. A graph which consists of a circuit together with a number of trees each of which has exactly one vertex on the circuit does not have a Hamiltonian circuit.

Some positive results are:

1. If a region consists of two abutting rectangles whose sides are both even and the abutment is at least two cells, then a Hamiltonian circuit can be drawn by splicing the Hamiltonian circuits of the two rectangles. (The concept of abutment can be made clear by looking at Figure 3).
2. If a region consists of two abutting rectangles each of which has both its sides odd and the abutment is an odd number of cells greater than or equal to three, then the Hamiltonian circuit is drawn in the following way. The region is cut up into three rectangles two of which are overhanging and the remaining one is a long one containing the abutting cells. Each of these rectangles has an even side so that the Hamiltonian circuits on these rectangles can be spliced into one circuit covering the entire region. This is illustrated in Figure 3. Figure 3 also shows a case where the abutment is two cells across. In this case, as in any with an even abutment, all of the corner cells are black so that there are more black than white cells, making a tiling impossible.
3. Occasionally one can work backwards, by unsplicing a Hamiltonian cycle of a large region. In this case, after unsplicing a large Hamiltonian circuit into two disjoint circuits and then removing the cells of one of these circuits, we obtain a Hamiltonian circuit of the altered region.

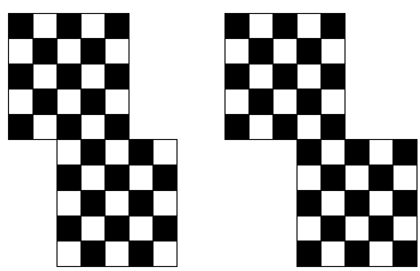

Figure 3.

An advantage of the Hamiltonian approach is that once the circuit is found, the tiling after the removal of two cells is immediate.

If we are only interested in tilings without the removable property, a large literature exists. This includes algorithms for finding matchings in bipartite graphs, which goes beyond the scope of this paper. In our references we give three sources dealing with matchings, namely Brualdi [**1**], Dulmage and Mendelsohn [**3**] and Hall [**4**].

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Solution to cryptic crossword clue (page 92): Perfect number

